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CREDIT RISK MODELLING UNDER THE JARROW AND TURBULL MODEL

AN EMPIRICAL APPROACH

Dissertação no âmbito do Mestrado em Métodos Quantitativos em Finanças, orientada pela Professora Doutora Ana Margarida Monteiro e pelo Professor Doutor Rui Pascoal e apresentada ao Departamento de Matemática da Faculdade de Ciências e Tecnologia e à Faculdade de Economia.

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Abstract

Probability of default refers to the likelihood that a borrower will fail to meet debt obligations, resulting in loan or bond default. This estimation can be achieved through structural or reduced form approaches in credit risk analysis. This dissertation aims to explore the estimation of default probabilities using the Jarrow and Turnbull model a reduced form approach, employing both simulated and market data for two well-known companies: *Google* (GOOGL) and *Farfetch* (FTCH), aiming to extract valuable information regarding market dynamics and also historical events.

Moreover, to glean pertinent insights into bond dynamics, we used zero-coupon yields derived from the Svensson approach employed by central banks to infer zero-coupon default-free bond prices. Besides this, in order to try to extract relevant information regarding market data, we analyzed the coupon maturities and cash flows for both companies. First, we study the efficiency of the estimation method of the Jarrow and Turnbull model by examining the differences between estimated and simulated prices, then compare the estimates for both companies and understand the differences between them as well as for different bond maturities but only for *Google*.

By digging into the data and studying different scenarios, we have gained some valuable insights into what drives default risk and how it affects bond prices. Armed with this knowledge, investors can make better decisions when navigating the ups and downs of the bond market.

Keywords: Probability of default, Merton model, Defaultable and nondefaultable bonds, Jarrow and Turnbull model, Structural and reduced credit risk models

Resumo

A probabilidade de incumprimento (*default*) refere-se à probabilidade de um devedor falhar em cumprir as obrigações de dívida, resultando em *default* de empréstimo ou de obrigações. Esta estimativa pode ser alcançada por meio de abordagens estruturais ou na forma reduzida na análise de risco de crédito. Esta dissertação tem como objetivo explorar a estimativa das probabilidades de incumprimento (*default*) usando o modelo de Jarrow e Turnbull, uma abordagem na forma reduzida, empregando tanto dados simulados quanto de mercado para empresas como a *Google* (GOOGL) e *Farfetch* (FTCH), visando extrair informações valiosas sobre dinâmicas de mercado e também eventos históricos.

Além disso, para obter perceções pertinentes sobre a dinâmica das obrigações, utilizamos taxas de cupão zero derivados da abordagem de Svensson empregue pelos bancos centrais para inferir os preços das obrigações sem risco de *default* de cupão zero. A fim de tentar extrair informações relevantes sobre os dados de mercado, analisamos os vencimentos dos cupões e os fluxos de caixa de ambas as empresas. Primeiramente, estudamos a eficiência do método de estimação do modelo de Jarrow e Turnbull examinando as diferenças entre preços estimados e simulados, em seguida, comparamos as estimativas para ambas as empresas e percebemos as diferenças entre elas, bem como diferentes maturidades de obrigações mas apenas para a *Google*.

Ao analisar os dados e estudar diferentes cenários, obtivemos resultados importantes sobre o que impulsiona o risco de *default* e como é que isso afeta os preços das obrigações. Munidos desses conhecimentos, os investidores podem tomar melhores decisões ao analisar os altos e baixos do mercado de obrigações.

Palavras-Chave: Probabilidade de incumprimento (*default*), Modelo de Merton, Obrigações com e sem *default*, Modelo de Jarrow e Turnbull, Modelos de risco de crédito estruturais e reduzidos

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Chapter 1

Introduction

In an era dominated by financial uncertainties and economic complexity, the ability to predict and manage credit risk has become a cornerstone in the finance field, specially in the development of financial markets. A crucial example of a financial market is the derivatives market. This market provides a dynamic field where financial instruments are traded. It is an opportunity for investors: to hedge against adverse price movements in their existing portfolios; to control a larger position with a relatively small amount of capital; or, eventually to generate additional income with certain derivatives, like covered call options.

Credit risk is related to the risk that a borrower may fail to meet its financial obligations, resulting in a loss for the lender or investor. This risk is inherent in various financial transactions where one party lends credit to another, such as loans, bonds, or credit derivatives. Understanding and managing credit risk is crucial for financial institutions, investors, and lenders, to ensure the stability and sustainability of financial markets.

Risk management ensures the protection of financial resources, property, and intellectual capital against unforeseen events that could lead to losses, like the 2008 subprime crisis. Moreover, it guarantees that organizations operate within the bounds of applicable laws, reducing the likelihood of legal and regulatory consequences as well as allowing for more accurate forecasting, strategic planning, and resource allocation. Measuring risk, contributes to a positive perception, attracting investors and ensuring sustained financial support.

This dissertation presents an analysis of credit risk modelling, delving into two primary approaches: structural and reduced form models. The intricacy between market dynamics and the potential for default sets the stage for an examination of these models, each offering unique insights and grappling with distinct challenges.

A major discovery in option pricing theory was accomplished by Black-Scholes (1973) [4] and Merton (1973) [22] by proposing an option pricing formula. This groundbreaking mathematical formula used for pricing European-style options, was the cornerstone for developments in option pricing, formally known as the Black–Scholes–Merton (BSM) model. The model has left an indelible mark on the world of finance, fundamentally shaping how traders approach the pricing and hedging of derivatives.

The BSM model assumes that the underlying asset price follows a geometric Brownian motion, with constant volatility. In reality, volatility can fluctuate, but the BSM model simplifies by treating it as a fixed parameter. Reflecting however that implied volatility is non-constant across exercise prices and option's maturities (Jondeau et al., 2007) [17]. This indicates that the BSM assumption imposing the distribution of stock prices to follow a log-normal distribution, does not appear to occur in the markets. The model considers that financial markets are efficient and there are no opportunities for arbitrage.

Structural models, exemplified by the Merton (1974) [23] model that uses the option pricing formula, offer a unique approach evaluating the likelihood of default for a firm by establishing a direct link between its financial structure and default risk. At the foundation of these models, lies the concept of an asset-liability relationship, where firm's assets are contrasted against its liabilities. The model assumes that default occurs when the firm's assets falls below a predetermined value. In essence, structural models view default as a consequence of a firm's financial structure, providing a clear economic interpretation of credit risk.

The Merton Model, an outstanding structural model, assumes continuous trading and perfect capital markets, factors that underline its foundation in option pricing theory. The model formulates default probability by comparing the firm's market value to its liabilities. In spite of their assumptions, structural models have proven beneficial in credit risk management, offering insights into a firm's creditworthiness and informing decisions related to lending, investing, and portfolio management. Even if they are considered to be simple, structural models remain fundamental in understanding the interplay between a firm's financial health and its perceptivity of propension to default.

However, reduced form models take a different approach by focusing on the unpredictability of default events, steering away from detailed assumptions about a firm's financial structure over time. In contrast to structural models, reduced form models, like those proposed by Jarrow and Turnbull (1995) [16] or Duffie and Singleton (1997)[7], treat default as an exogenous process determined by external factors. These models don't focus on the firm's financial intricacies but instead examine the hazard rate or intensity of default over time. While they avoid some complexities, reduced form models excel in capturing the dynamic and often unpredictable nature of credit risk derived from the assessment of the risk of default on the bond market.

The Jarrow and Turnbull model provides a framework for understanding and predicting default events. This model focuses on the timing and probability of default. It employs a Poisson process with a steady intensity to model the occurrence of defaults. The key innovation lies in its treatment of default as an exogenous event, influenced by external factors such as market conditions. Although they are criticized for the lack of explicit economic interpretation, the Jarrow and Turnbull model flexibility and ability to incorporate diverse factors makes them indispensable tools for understanding and managing credit risk in dynamic market conditions.

In this dissertation it is analyzed the default intensity on simulated and market prices, using the Frühwirth and Sögner (2001) [8] approach and the Svensson (1994) [27] model. The Svensson model is used to estimate the instantaneous forward rates as well as yield rates and consequently obtain the zero-coupon bond prices, which constitute the discount factor that is multiplied by the expected value of the defaultable payoff to achieve the risky bond prices. To examine the reliability to recover the default intensity, simulated data obtained from the Jarrow and Turnbull model was used. Market bond prices were also considered for *Google*, renowned for its cutting-edge technology and artificial intelligence contributions, and *Farfetch*, emerging as one of the most shorted companies in the past year. *Google*, a tech giant, leads the industry with innovative advancements, while *Farfetch*, a distinctive player, has drawn attention for its notable short positions in recent times.

This dissertation unfolds across six chapters. Chapter 2, corresponds to the literature review. In Chapter 3, we present fundamental concepts related to options and the BSM option pricing model as well as structural form models, providing a concise overview of both the Merton and KMV models. In Chapter 4, we discern the zero-coupon bond prices and deduce the Jarrow and Turnbull model, pioneering the reduced form approach for pricing bonds with default. Chapter 5, navigates through the empirical results analysis, employing the Jarrow and Turnbull model to scrutinize simulated data, extending the analysis subsequently to market data. The culmination in Chapter 6 involves a thoughtful commentary on the results obtained throughout the dissertation.

Chapter 2

Literature Review

Some default models use market data to predict the ocurrence of a default event. These models were created to analyze the probability of a corporate or sovereign entity defaulting on its credit obligation. There are two different types of models: structural and reduced form models. The structural form was inspired by Black and Scholes (1973) [4] and Merton (1973) [22] using the option pricing theory by linking the firm's asset variables and the risk of default. The Merton (1974) [23] model assumes that default occurs when the liabilities of the firm are higher than the firm's market value. This model formulates a probability of default by comparing the value of the firm to its debt.

The breakthrough achieved by Merton assumes that: bankruptcy is only possible at the maturity of the debt; the firm's market value follows a geometric Brownian motion; perfect liquidity in capital markets; the risk-free interest rate is constant over time and non-stochastic; asset trading continuous in time. This approach has been very important for several credit risk management models in the financial industry such as CreditMetrics and Moody's KMV model. The KMV model is based on an empirical estimator of default probability, named the Estimated Default Frequency (EDF) developed by Kealhofer, McQuown and Vasicek (1989) and is essentially characterized by Merton's bond pricing model assumptions. However, the Merton's model has some disadvantages, such as the over simplified capital structure it endures, and the limitation that default only be possible at the maturity of the debt, among others.

Hence, there has been attempts to reduce this shortcomings on the original Merton model, resulting in various extensions to the model, incorporating more realistic assumptions. To overcome the premise of default only at maturity, Black and Cox (1976) [3] introduced first passage models, allowing a firm to default at any time before maturity. First passage models take into account the default as a barrier option, implicating that default occurs when the asset value falls below a certain default barrier.

Another model that expands the Black and Cox (1976) model is Longstaff and Schwartz (1992) [21]. This framework is based on a two-factor model. The two factors are the instantaneous riskless interest rate and volatility. They introduced a stochastic volatility factor giving the model reasonable dynamics of the factors or state variables. According to Jones, Mason and Rosenfeld (1984) [18] there is empirical evidence that the introduction of stochastic interest rates can improve the performance of the original Merton (1974) model, as they account for the relationship between the firm's asset value and the short rate.

One other crucial extension of the Merton (1974) model is the compound option approach introduced by Geske (1979) [12]. The Geske approach suggests a less simple capital structure by stating that equity is seen as a compound call option on the firm's asset, with a strike price equal to the coupon payments.

Although these previous models have proven very useful addressing the qualitatively important characteristics of pricing credit risks, they have been less efficacious in practical applications. Firm's capital structures are usually very complex.

To overcome this issue, the reduced form models were introduced. They do not pay attention to the firm-specific variables, modeling default based on a hazard-rate process. In this case, recovery rate and probability of default depend on other variables than the anatomical features of the firm such as volatility. Default is assumed to be unpredictable, therefore driven by exogenous variables extracted from market details. These can be the time when some information is revealed about court decisions, political events or technological advances which could wipe out the firm. For instance, a new scientific discovery that could make the production technology outdated or it might lower the demand for the firm's product. A political event might have the same outcome in critical regions. In these cases, default can occur even if the firm was in a good state before the prompt event. Default will be considered to happen at a random time τ, not exclusively related to the operating of the firm.

Jarrow and Turnbull (1995) [16] consider the case in which the default is modeled by a Poisson process with steady intensity and a known payoff at default. They use stochastic rates, but the processes for bankruptcy and the payoff on the risky debt conditional on default are exclusively exogenous. In addition, they suppose that the market is complete and there are no arbitrage opportunities, and all assets can be valued at the riskless rate under the risk-neutral density. It is straightforward to use this model when pricing formulas for risky bonds and options on interest rate sensitive stocks. An excellent feature provided by the Jarrow and Turnbull model is the existence of explicit pricing formulas for risky bonds and for options on interest rate sensitive stocks which facilitates implementation and calibration (Frühwirth and Sögner, 2001) [8]. In spite of its simplicity, the Jarrow and Turnbull model proves to be effective in some situations as revealed by the empirical studies of Houweling and Vorst (2002)[15].

In a similar way, Duffie and Singleton (1997) [7] assume a multi-factor square-root process just as a Cox-Ingersoll-Ross process for the riskless interest rate and a Poisson process for default with state dependent values for the hazard rate and the loss in default (Cooper and Martin, 2006). The valuation under the risk-neutral probability measure can be completed by discounting the non-defaultable payoff on the debt by a discount rate that is adapted for the parameters of the default process.

The main difference between these two reduced form models is mainly in the parameterization of the recovery rate, refering to how we model, after a firm defaults, the value that remains from a debt instrument has left. While Jarrow and Turnbull (1995) recognize that the market value of a bond at default is the same as an external given fraction of an equivalent default-free bond, Duffie and Singleton (1997) assume that the model produces the term structure of credit spreads, showing how credit risk varies across different maturities. This model assumes that the expect loss at default is exogenous, suggesting that the recovery rate is independent from the defaultable claim value. Moreover, unlike the Jarrow and Turnbull model, the Duffie and Singleton model claims that recovery is paid immediately upon default and equals a fraction of what the bond is worth right before to default.

The relationship between the probability of default and recovery rate has been one of the main topics of the research for many years. The default on these models is driven by the state of the economy. The corresponding economic conditions are supposed to bring about the probability of default to increase and recovery rates to decrease. Therefore, the correlation between recovery rate and default is triggered by the common dependence on the state variable. Default rates and recovery rates are negatively correlated (Frye, 2000a and 2000b)[9][10]. The articles using intensity-based models normally build the cross-sections, that is data collected by observing many subjects at a single point or period of time, and specify them externally, usually built on credit ratings, to derive the estimates. It is common to use daily data to extract the parameters based on the pre-specified cross-sections (Frühwirth and Sögner, 2001) [8].

Structural models are very useful for specialists in the credit management areas and credit portfolio because of its easy economic interpretations, while reduced form models, due to its flexibility, have achieved great importance among the credit trading field. As seen previously, structural models suppose that the theorist has full knowledge of the firm's balance sheet evolution producing a predictable default. However, in reduced form models the theorist only has information based on the market value, resulting in an unpredictable default. These models cannot form a link between firm value and corporate default, as the structural models can. Overall, the choice between a reduced form or structural model should lean on the real intentions for the model and its advantages or disadvantages when comparing with real world scenarios (Arora, Bohn and Zhu 2005)[1].

Chapter 3

Structural models

In this chapter, we provide a comprehensive exploration of essential concepts related to options, the BSM option pricing model and structural form models, offering a succinct overview of both the Merton and KMV models, elucidating key principles and methodologies in credit risk applications.

3.1 Basic Concepts

Options

An option is "a contract that provides the owner the right to buy and sell shares at a later date or within a certain period at a particular price" (Cambridge Online Dictionary, 2023a). Hence, an option is a type of guarantee for investors in the stock market. The investor has the right to buy or sell the underlying asset on the maturity date at a certain price also known as strike price. In addition, when analyzing the right to buy/sell the underlying asset, call options give the investor the right to purchase an asset at a strike price either before or at the maturity date, while put options allow the investor the right to buy it. Typically, a call option will only be exercised, if the strike price is below the market value before or at the maturity date whereas a put option will be exercised if the strike price is above the market value. Options can be separated in two types depending on the exercising date. American options can be exercised before or at the maturity date, while European options are only exercised at the maturity date. The most common ones are the American options having a longer period for exercising comparing to European options.

Admit that an investor buys an European call option with a strike price of 100€, the current asset price is 98€, and the maturity is 7 months. If the stock price at maturity is lower than 100€, then the investor will not exercise it. In opposition, if the stock price is higher than 100€ at maturity, then the investor will exercise it. For an European put option with the same data, the investor will only exercise it if the asset price at maturity is lower than 100€. It is important to exemplify an European option regarding its payoffs from the perspective of the investor of the option (long) position. The option's payoff is assumed to be the profit for the investor in case of exercise, so the price paid for the option (premium) is not covered in that illustration, see Table 3.1. S_t denotes the price of the underlying asset at time t, K is the strike price and T the maturity date of the option. Therefore, the payoffs on the expiration date of a long and a short position in call and put options are presented on the following table:

	Long	Short
Call	$max(S_T - K, 0)$	$-max(S_T - K, 0) = \min(K - S_T, 0)$
Put	$max(K - S_T, 0)$	$-max(K - S_T, 0) = \min(S_T - K, 0)$

Table 3.1 Call and put option payoffs

The payoff at expiry for European long call options is the maximum of $S_t - K$ and zero. As stated in the generic table, if the stock price is lower than the strike price ($S_t < K$), the profit will, naturally, be zero, since max $(S_t - K, 0) = 0$. In fact, if the stock price at maturity is higher than the strike price, the profit will be positive symbolizing $S_t - K$, which signifies that the investor only pays K instead of S_t. These figures bellow represent the payoff diagrams of the call and put options. Based on two counterparts, the buyer and seller, in this case the seller of the option requires a compensation payment when the contract is established. In the best-case scenario for the seller, the option buyer lets the option expire. In the worst-case scenario, the seller has to give a compensation payment.

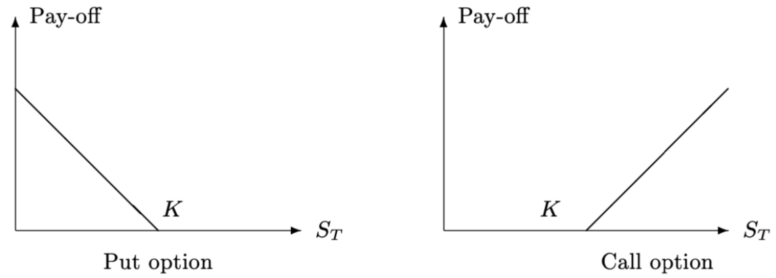


Figure 3.1 Put and Call option

Besides the stock price and the strike price, the Black-Scholes formula also depends on the risk-free interest rate. This rate is well known for being the best rate of return possible that does not involve taking a risk. Normally, the risk-free interest rate for a specific period is taken to be the return on government bonds. Also, option prices depend on volatility, which is the variation of a trading price series over time usually, measured by the standard deviation of the asset's return. Some stocks are relatively stable so they have a lower volatility and others can be unstable with a higher level of uncertainty and therefore more volatile. Thus, it is expected that the stock may have either an extremely high or low value on the maturity date. Even if the stock price is much below the strike price of a call option it makes no difference because the option will not be exercised. If the stock price is above the strike price at maturity, the option will be exercised, and the payoff will be $S_t - K$. Hence, an option with high volatility is more likely to attain a higher profit whenever the price rises.

Black-Scholes-Merton Model

The Black-Scholes-Merton model (BSM) allows to calculate the value of a stock option. Black and Scholes (1973) and Merton (1973) proposed a formula to the stock price by considering a geometric Brownian motion. The dynamic of the stock price over time is crucial. It can be expressed as:

$$dS_t = S_t \mu dt + S_t \sigma dW_t, \tag{3.1}$$

where dS_t denotes the instantaneous price change, W_t is a Wiener process, μ is the expected return, σ is the volatility of the price process and t is the time. The parameters μ and σ are constant over time. The model assumes no arbitrage opportunities.

With the pricing dynamic for the underlying asset, the next step is inferring the pricing dynamic for the derivative asset. By considering $f(S_t, t)$ the price of a derivative asset, it is introduced the notation $f_{SS} = \frac{\partial^2 f}{\partial S^2}$, $f_S = \frac{\partial f}{\partial S}$, $f_t = \frac{\partial f}{\partial t}$.

By applying Itô's formula, the price dynamics of a derivative asset is given by:

$$df = \left(\frac{1}{2} \sigma^2 S_t^2 f_{SS} + \mu S_t f_S + f_t \right) dt + \sigma S_t f_S dW_t. \tag{3.2}$$

Next, it is considered a portfolio with one unit of the derivative asset and a short position of f_S units in the underlying asset. The portfolio value is $J_t = f - f_S S_t$, with a pricing dynamic of $dJ_t = df - f_S dS_t$. From (3.1) and (3.2) it will come:

$$\begin{aligned} dJ_t &= \left(\frac{1}{2} \sigma^2 S_t^2 f_{SS} + \mu S_t f_S + f_t - \mu S_t f_S \right) dt + \sigma S_t f_S dW_t - \sigma S_t f_S dW_t - f_S S_t \mu dt \\ &= \left(\frac{1}{2} \sigma^2 S_t^2 f_{SS} + f_t \right) dt + 0 dW_t. \end{aligned} \tag{3.3}$$

Given the stock price dynamics in (3.1) and considering $f(S_t, t) = log(S_t)$.

By applying the Itô's lemma as in (3.3), it will appear:

$$dlog(S_t) = \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dW_t. \tag{3.4}$$

Integrating (3.4) from t to T gives

$$log(S_T) = log(S_t) + \left(\mu - \frac{1}{2} \sigma^2 \right) (T - t) + \sigma (W_T - W_t).$$

Considering S_T the underlying asset price at time t, it follows that

$$log(S_T) \sim N \left(log(S_t) + \left(\mu - \frac{1}{2} \sigma^2 \right) \tau, \sigma^2 \tau \right), \tag{3.5}$$

where $\tau = T - t$ and S_T has a log-normal distribution.

The BSM pricing formulas for European call and put options from (3.3) are:

$$C(S_t, t) = S_t N(d_1) - K e^{-r\tau} N(d_2) \tag{3.6}$$

and

$$P(S_t,t) = Ke^{-r\tau}N(-d_2) - S_tN(-d_1) \tag{3.7}$$

with

$$d_1 = \frac{log(S_t/K) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \tag{3.8}$$

$$d_2 = \frac{log(S_t/K) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau}. \tag{3.9}$$

where r denotes the risk-free interest rate, N is the cumulative standard normal probability distribution function, $C(S_t,t)$ is the price of a call option and $P(S_t,t)$ the price of a put option.

3.2 Merton (1974) – Distance to Default

Starting from the study of Black and Scholes (1973) and Merton (1973), in this section of the chapter it will be presented a generalization of the Merton model, that seeks to estimate the probability of default of a given firm. The first approach proposed by Merton (1974) considers that the firm's assets value, V_t, follows a geometric Brownian motion (3.1):

$$dVt = rV_tdt + V_t\sigma dW_t, \tag{3.10}$$

where V_t represents the firm's assets value in t and r the risk-free rate.

Defining E_t as the equity value of the firm at time t and K as the nominal value of debt with maturity at time T. Therefore, at maturity T the equity value can be seen as a call option on the total value of the firm's assets with a strike price equal to the debt value (Merton, 1974):

$$E_t = max(V_t - K, 0). \tag{3.11}$$

In this way, Merton's distance to default model (Merton–DD model) propose to estimate the default features of a specific firm. The model stipulates that at time t the equity value of the firm, E_t, satisfies the equation (3.6):

$$E_t = V_tN(d_1) - Ke^{-r\tau}N(d_2), \tag{3.12}$$

with,

$$P(V_t,t) = Ke^{-r\tau}N(-d_2) - S_tN(-d_1), \tag{3.13}$$

with,

$$d_1 = \frac{log(V_t/K) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \tag{3.14}$$

$$d_2 = \frac{log(V_t/K) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau}. \tag{3.15}$$

Merton assumes that a given firm defaults if the value of its assets falls to a certain level. Namely, when assets value is lower than the debt value. Under the assumptions of the Merton-DD model, it is considered that the volatility associated with equity, σ_E, is directly related to the volatility of the value of the assets, σ_V. Based on the market structure it follows:

$$\sigma_E = \frac{V_t}{E_t} \times \frac{\partial E_t}{\partial V_t} \sigma_V. \tag{3.16}$$

From the Black and Scholes option pricing formula (3.12), $\partial E_t / \partial V_t = N(d_1)$. Thus, the above expression can be written as:

$$\sigma_E = \frac{V_t}{E_t} N(d_1) \sigma_V. \tag{3.17}$$

The Merton-DD model consider the equations (3.12) and (3.17), aiming to determine the value of the firm's assets, V_t, and its volatility, σ_V. Finally, the following system of two equations is solved:

$$\begin{cases} E_t = V_t N(d_1) - K e^{-r\tau} N(d_2) \\ \sigma_E = \frac{V_t}{E_t} N(d_1) \sigma_V, \end{cases} \tag{3.18}$$

where d_1 and d_2 is given by (3.14) and (3.15), respectively.

The first step to implement the Merton-DD model is to estimate the volatility of equity from data associated with options volatilities or historical reports of equity. The second step is to define the maturity of the debt and its nominal value. Then, it is necessary to set the value of the risk-free interest rate and the value of equity. Thus, the variables of the problem are V_t and σ_V. They can be obtained by solving the system (3.18).

The Distance to Default (DD) is defined by (Vassalou and Xing, 2004)[28]:

$$DD_t = \frac{log\left(V_t / K\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \tag{3.19}$$

where DD_t is equal to d_2 represented in equation (3.15). The implied probability of default, also denominated by Expected Default Frequency (EDF) is

$$P_{default} = N\left(-\frac{log(V_t/K) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right) = N(-DD_t). \tag{3.20}$$

If $V_t < K$ than there is a probability of default.

3.2.1 Moody's KMV model

The most effective default measurement derives from models that use both market prices and financial statements. The model was first introduced in the late 80's by KMV[1] corporation, a leading provider of quantitative credit analysis tools. The model is now maintained and developed on a continuous basis by Moody's KMV, a division of Moody's Analytics. Moody's Analytics acquired KMV in 2002.

[1] KMV Corporation is a financial technology firm pioneering the use of structural models for credit valuation. Founded in 1989 in San Francisco by Stephen Kealhofer, John Andrew McQuown and Oldrich Vasicek.

This model assumes that the firm's equity is an option with the default point acting as the absorbing barrier for the firm's asset value. If the asset value hits the default point, the firms is assumed to default (Crosbie and Bohn, 2003) [6]. In this way, the dynamic between asset value and liabilities can be captured without having to use a more complex model characterizing a firm's liability process.

The KMV model is an extensive modification of the original Merton's approach, since it incorporates more realistic assumptions and empirical observations that better reflect real-world default dynamics. This is the main advantage of this model: it provides both up-to-date view of a firm's value and a timely warning of changes in credit risk (Sun et al., 2012)[26].

The main output of the KMV model is the EDF credit measure, which is the probability of default, during the forthcoming year, or years. The EDF value requires equity prices and some items from financial statements as inputs. The probability of default is then computed as a function of the firm's capital structure, the volatility of the asset returns and the current asset value.

According to Crosbie and Bohn (2003), there are three essential parts for determining the default probability of a firm:

- **The market value of the firm's assets**: the present value of the future free cash flows produced by the assets of the firms discounted back at the appropriate discount rate.
- **Asset risk**: the unpredictablility of the asset value risk, which measure the firm's business and industry risk. Given that the value of the firm's assets is an estimate, it is unpredictable, and therefore, it should be analised in the context of asset risk or firm's business.
- **Leverage**: the extent of the firm's contractual liabilities. A relevant measure of the firm's leverage is the book value of liabilities relative to the market value of assets, since that represents the amount that the firm must repay.

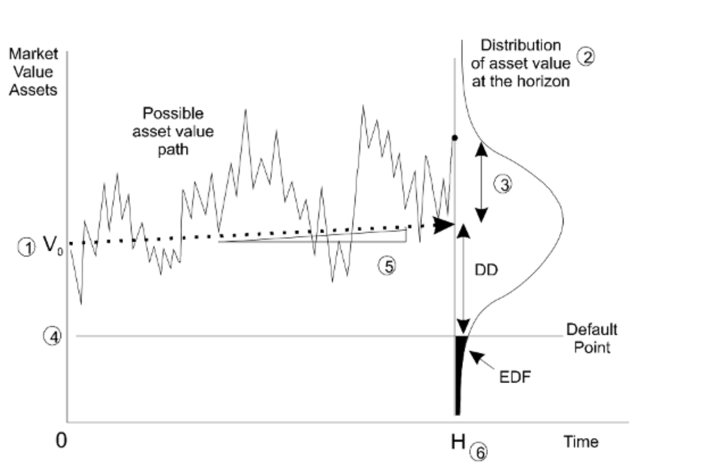


Figure 3.2 Frequency distribution of a firm's asset value at the horizon of time H and probability of default, Crosbie and Bohn (2003)

The firm's probability of default increases as the current market value of the firm's assets decreases, the volatility of the firm's assets increases or the amount of liabilities increases. The firm defaults when the market value of it's assets is insufficient to repay the liabilities, which happens when the market value of the firm's assets falls below the default point. Hence, the firm's default probability is the probability that the asset value will fall below that default point, represented by the black area (EDF value) below the default point in Figure (3.2) .

Overall, Crosbie and Bohn (2003) considered that many firms do not default when their assets attain the book value of their total liabilities. Normally, the default point extends somewhere from the total liabilities to the current liabilities.

Although these previous models have proven very useful addressing the qualitatively important characteristics of pricing credit risks, it has been less efficacious in practical applications. The absence of efficacity is due to the difficulty of modelling realistic boundary conditions. These boundaries include the case under which default occurs, and when it occurs, the division of the value of the firm among claimants. Firm's capital structures are usually very complex and priority rules are often violated. To overcome these disadvantages, an alternative reduced form model has been developed.

Chapter 4

Bond pricing models

In this chapter, we delve into nondefaultable and defaultable bonds. First, we explore zero-coupon bonds and how to estimate them with the Svensson parameters then we deduce the Jarrow and Turnbull model, along with the Frühwirth and Sögner empirical approach, which is a representative example of reduced form models.

4.1 Nondefaultable and Defaultable Bonds

4.1.1 Nondefaultable Bonds

A zero-coupon bond with maturity date T is a contract warranting to pay 1 monetary unit to the investor at time T (Brigo and Mercurio, 2001)[5]. Denote the time t price of a default-free bond that pays off a dollar at time T as $P(t,T)$. A coupon bond with maturity T delivers payments in $[0,T]$ and provides the bondholder with a deterministic cash flow.

After defining a zero-coupon and coupon bonds, some assumptions are made so that it is assured the existence of a bond market sufficiently deep: there is a frictionless market for all bonds of all maturities, $P(t,t) = 1$ in order to avoid arbitrage and $P(t,T)$ is differentiable with respect to T for all fixed t (Björk, 1998)[2].

Zero-coupon bond prices serve as fundamental benchmarks in interest-rate theory, as they encapsulate the essence of interest rates. It is crucial to grasp the underlying characteristics and components, which refer to the day-count convention to be applied and the type of composition such as simply compounded, of interest rates when transitioning between zero-coupon bond prices and interest rates, ensuring a comprehensive understanding of both concepts.

First, the money market account process is given by:

$$B_t = \exp\left(\int_0^t r(s)ds\right). \tag{4.1}$$

The money market account can be seen as a strategy of instantaneously reinvesting at the current short rate (Kiesel and Bingham, 1998)[24]. $r(t)$ is the interest rate at which a person can borrow money for an infinitely small period of time starting from time t. Based on Brigo and Mercurio [5], with the assumption of a general interest rate process which affects the nature of the bank account numeraire

B, $D(t,T)$ represents the discount factor between two time instants t and T. It is the value at time t that is identical to one unit of currency payable at T. It is given by:

$$D(t,T) = \frac{B(t)}{B(T)} = \exp\left(-\int_t^T r(s)ds\right). \tag{4.2}$$

In many derivatives pricing applications, essentially when applying the Black and Scholes formula in equity or foreign exchange markets, r is assumed to be a deterministic function of time, so that both the bank account $B(t)$ and the discount factor $D(t,T)$ at any time in the future are deterministic function of time as well (Brigo and Mercurio, 2001)[5]. It becomes fundamental to let go of the deterministic assumption and begin modelling the interest rate r, through a stochastic process because of its capacity to influence interest rate derivatives. Hence, $B(t)$ and $D(t,T)$ will also be stochastic processes.

Now it is possible to establish a relationship between the discount factor $D(t,T)$ and the zero-coupon bond price $P(t,T)$. $D(t,T)$ can be named the equivalent amount at t of one monetary unit at T, obtained through a bank account and in general is random, while $P(t,T)$ can be described as the value of a bond that pays one monetary unit at T and is deterministic.

Assume there is a numeraire N and a measure Q_N, equivalent to the initial probability Q_0 such that the price of any traded asset S relative to N is a martingale under the equivalent (only if $N = B$) measure Q_N given by (Geman et al., 1995) [11]:

$$\frac{S_t}{N_t} = E^N\left(\frac{S_T}{N_T}|F_t\right), \tag{4.3}$$

where F_t denotes the information at t given by a σ-algebra.

If r is deterministic, then D is also deterministic, and consequently $D(t,T) = P(t,T)$ for each pair (t,T). However, if r is stochastic, then $D(t,T)$ becomes a random quantity depending on the future evolution of rates r between t and T, whilst $P(t,T)$ is the time $t-value$ of a contract with payoff at time t. $P(t,T)$ and $D(t,T)$ are closely linked due to the fact that $P(t,T)$ can actually be the one for which, if the numeraire is the money account, viewed as the expectation of the random variable $D(t,T)$ under a risk neutral measure as follow (Brigo and Mercurio, 2001)[5]:

$$\frac{P(t,T)}{B_t} = E^Q\left(\frac{P(T,T)}{B_T}|F_t\right) \tag{4.4}$$

that is,

$$\frac{P(t,T)}{B_t} = E^Q\left(\frac{1}{B_T}|F_t\right) \Leftrightarrow P(t,T) = E^Q\left(\frac{B_t}{B_T}|F_t\right). \tag{4.5}$$

Replacing with the result in equation (4.2) it will come,

$$P(t,T) = E^Q[D(t,T)|F_t]. \tag{4.6}$$

When the numeraire process N is given by the zero-coupon bond price $P(t,T)$ we have:

$$\frac{S_t}{P(t,T)} = E^T\left(\frac{S_T}{P(T,T)}|F_t\right), \tag{4.7}$$

where $P(T,T) = 1$.
It is possible to denote the new measure, called T-forward measure, as Q_T and E^T represents the expectation under Q^T. Hence,

$$S_t = P(t,T)E^T(S_T|F_t). \tag{4.8}$$

In this context, forward rates $F(t;S;T)$ represent the interest rates that can be secured today at time t for an investment in a future period of time (from S to T). These rates are established based on the prevailing term structure of discount factors. A forward rate is formally established through a financial instrument known as a forward rate agreement (FRA). Let's delve into its definition.

A FRA is a contract with three time moments: the current time t, the expiry time $S > t$ and the maturity time $T > S$. At maturity, a permanent payment based on a permanent rate K, set at t is exchanged against a volatile payment based on the rate $L(S,T)$ resetting in S with maturity T. In this case, $L(S,T)$, named after the Libor rate, is the simply compounded spot interest rate of a zero-coupon bond set at S with maturity at T.

Allow N to be the contract nominal amount. Properly, at time T one sustains $N\tau(S,T)\times K$ units of currency and pays the value $N\tau(S,T)\times L(S,T)$. The value of the contract at maturity is given by:

$$N\tau(S,T)(K - L(S,T)), \tag{4.9}$$

where both rates have the same day-count convention. If L is bigger or smaller than K at time S, the contract value will be negative or positive, respectively. However, if L is equal to K, the contract value will be null, suggesting a neutral outcome (Brigo and Mercurio, 2001)[5].

The price of the contract at time t is then as followed:

$$P(t,T)N\tau(S,T)(K - L_t(S,T)) = \tag{4.10}$$
$$= P(t,T)N\tau(S,T)K - P(t,T)N\tau(S,T)L_t(S,T)) \tag{4.11}$$
$$= P(t,T)N\tau(S,T)K - P(t,T)N\tau(S,T)\frac{1 - P_t(S,T)}{\tau(S,T)P_t(S,T)} \tag{4.12}$$
$$= P(t,T)N\tau(S,T)K - P(t,T)N\left(\frac{1}{P_t(S,T)} - 1\right) \tag{4.13}$$
$$= P(t,T)N\tau(S,T)K - N\frac{P(t,T)}{P_t(S,T)} + NP(t,T) \tag{4.14}$$
$$= N[P(t,T)\tau(S,T)K - P(t,S) + P(t,T)], \tag{4.15}$$

where $L(S,T)$ is the simply compounded spot interest rate at maturity T and $L_t(S,T)$ is the expectation of $L(T,S)$ at t. In the absence of arbitrage opportunities, $(1 + L(t,T)) = (1 + L(t,S))(1 + L_t(S,T))$ guarantees consistency in the discount term structure. Likewise, $P_t(S,T)$ is the expectation of $P(S,T)$ at t.

When discussing a simple compounding rate for $L(S,T)$, it signifies the consistent rate at which an investment, starting with $P(t,S)$ units of currency at time t, must be made to yield one unit of currency at maturity. This occurs when accrual happens proportionally to the investment duration.

Analytically,

$$P(t,S)(1+L(t,S)\tau(t,S)) = 1, \tag{4.16}$$

and the bond price will come:

$$P(t,S) = \frac{1}{(1+L(t,S)\tau(t,S))}. \tag{4.17}$$

There is a particular value of K, for which the contract is fair at time t, that is the contract value is 0, at t. This value is with simple composition the forward interest rate prevailing at time t for the expiry $S > t$ and maturity $T > S$ denoted by $F(t;S,T)$ and is given by:

$$F(t;S,T) := \frac{1}{\tau(S,T)}\left(\frac{P(t,S)}{P(t,T)} - 1\right), \tag{4.18}$$

or

$$F(t;S,T) = \frac{1}{T-S}\frac{P(t,S)-P(t,T)}{P(t,T)} = \frac{P(t,S)-P(t,T)}{T-S}\frac{1}{P(t,T)}, \tag{4.19}$$

making $\tau(S,T) = T - S$, specially when T is small.
When maturity tends to expiry, the instantaneous forward rate is obtained. It will come:

$$f(t,S) = \lim_{T\to S^+} F(t;S,T) = -\lim_{T\to S^+} \frac{P(t,S)-P(t,T)}{T-S}\frac{1}{P(t,T)} = \tag{4.20}$$

$$= -\frac{\partial P(t,S)}{\partial S}\frac{1}{P(t,S)} \tag{4.21}$$

$$= -\frac{\partial lnP(t,S)}{\partial S}. \tag{4.22}$$

Along with some simple transformations to the equation $f(t,S)$:

$$f(t,S) = -\frac{\partial lnP(t,S)}{\partial S} \tag{4.23}$$

$$\Leftrightarrow f(t,S)\mathrm{d}S = -\mathrm{d}lnP(t,S) \tag{4.24}$$

$$\Leftrightarrow -\int_t^S f(t,u)\,du = lnP(t,S) \tag{4.25}$$

$$\Leftrightarrow P(t,S) = e^{-\int_t^S f(t,u)du}. \tag{4.26}$$

Now that $F(t;S,T)$ is deduced, it is possible to justify why the measure Q_T is named forward measure.

Every simply compounded forward rate spanning a time interval ending in T is a martingale under the T-forward measure, for example,

$$E^T(F(t;S,T)|F_u) = F(u;S,T), \tag{4.27}$$

for each $0 \le u \le t \le S \le T$.

Specifically, the forward rate spanning between S and T is the Q_T - expectation of the future simply compounded spot rate at time S for the maturity T,

$$E^T(L(S,T)|F_t) = F(t;S,T), \tag{4.28}$$

for each $0 \leq u \leq t \leq S \leq T$.
From equation (4.19), we will obtain the relative price given by:

$$\frac{P(t,S)-P(t,T)}{T-S}\frac{1}{P(t,T)} = \frac{[P(t,S)-P(t,T)]/(T-S)}{P(t,T)} = F(t;S,T), \tag{4.29}$$

is a martingale under such a measure.

$$E^T(L(S,T)|F_t) = E^T(F(t;S,T)|F_t) = F(t;S,T). \tag{4.30}$$

Then, immediately follows from the equality $F(t;S,T) = L(S,T)$. This can also be applied to instantaneous rates as we will demonstrate later on.

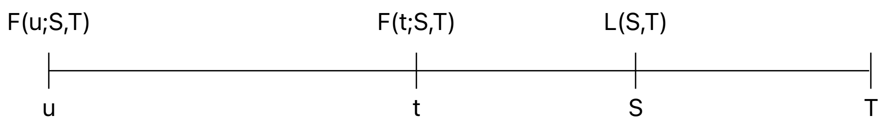


Figure 4.1 Forward interest rate timeline

Observing this figure, it is possible to understand better how the forward interest rate timeline appears. Hence, the instantaneous short rate at time t is given by:

$$r(t) = f(t,t). \tag{4.31}$$

The instantaneous forward rate $f(t,T)$ is equal to the conditional expectation of the instantaneous short rate in the future conditioned by the simply compounded forward rate in t.

According to Brigo and Mercurio, the instantaneous forward interest rate can be seen as a forward interest rate at time t for which maturity is arbitrarily near its expiry T and is an essential concept in modelling interest rates, as in absence of arbitrage opportunities, the interest rate approach is based on the evolution of f (Heath, Jarrow and Morton, 2002)[14].

On the other hand, the yield of zero-coupon bonds is a constant rate with continuous reinvestment at which $P(t,T)$ invested in t allows to obtain one monetary unit at T. The yield curve is a plot of interest rates, denoted by $i(t,T)$, as a function of distinct maturities T as assumed by Svensson [27], so that $P(t,T)$ will follow as:

$$P(t,T) = e^{-(T-t)i(t,T)}. \tag{4.32}$$

From (4.26) and (4.32), it follows that,

$$(T-t)i(t,T) = \int_t^T f(t,u)du \Leftrightarrow i(t,T) = \frac{\int_t^T f(t,u)du}{T-t}, \tag{4.33}$$

considering now T and not S as in (4.26).
From (4.3), with $N = B$,

$$\frac{S_t}{B_t} = E^B\left[\frac{S_T}{B_T}|F_t\right], \tag{4.34}$$

and therefore,

$$S_t = E^B[D(t,T)S_T|F_t]. \tag{4.35}$$

4.1.2 Defaultable Bonds

According to Lando (2004) [20], the default time τ is defined as:

$$\tau = \{t : \int_0^t \lambda(X_s)\, ds \geq E_1\}, \tag{4.36}$$

where X is a process of state variables with values in $\mathbb{R}^d$ defined on the probability space, $\lambda(X)$ is a stochastic intensity for the jump time τ and E_1 is an exponential random variable with mean 1.

A risky bond at time 0, supposing zero recovery, has a payoff:

$$\mathbb{1}_{\tau>T}(T) = \begin{cases} 1, & \text{if } \tau > T \\ 0, & \text{if } \tau \leq T \end{cases}. \tag{4.37}$$

As demonstrated in (4.35), its price is given by:

$$P_d(t,T) = E[exp(-\int_0^T r(X_s)ds)\mathbb{1}_{\tau>T}(T)|F_t] \tag{4.38}$$

$$= E[E[exp(-\int_0^T r(X_s)ds)\mathbb{1}_{\tau>T}(T)|G_T]|F_t] \tag{4.39}$$

$$= E[exp(-\int_0^T r(X_s)ds)E[\mathbb{1}_{\tau>T}(T)|G_T]|F_t] \tag{4.40}$$

$$= E[exp(-\int_0^T r(X_s)ds)exp(-\int_0^T \lambda(X_s)ds)|F_t] \tag{4.41}$$

$$= E[exp(-\int_0^T (r+\lambda)(X_s)ds)|F_t], \tag{4.42}$$

where expectation is computed under the risk neutral measure, and G_T is a null or positive filtration generated by X, for example, $G_T = \sigma\{X_s; 0 \leq s \leq T\}$ (Lando, 2004)[20]. To start to define $P_d(t,T)$, it is used equation (4.5).

In equation (4.39), it is applied the tower law while in (4.40) the expectation of the default-free zero-coupon bond price is transferred to the indicator of the default intensity because the rest of the equation is notable in T. Moreover, equation (4.42) is explained by:

$$Q(\tau > T|G_T) = Q(\int_t^T \lambda(X_s)ds < E_1|G_T) = exp(-\int_t^T \lambda(X_s)ds). \quad (4.43)$$

The last expression will then come as it is represented because, in the $Exp(1)$ law:

$$Q(E_1 > K) = \underset{|\lambda=1}{e^{-\lambda K}} = e^{-K}. \quad (4.44)$$

Following the argument given by Privault (2023) [25],

$$P_d(t,T) = E[exp(-\int_t^T r_s + \lambda_s)\, ds|F_t] \quad (4.45)$$

$$= E[exp(-\int_t^T r_s ds)|F_t]E^T[exp(-\int_t^T \lambda_s ds)|F_t] \quad (4.46)$$

$$= P(t,T)Q^T(\tau > T|G_t), \quad (4.47)$$

where $P(t,T)$ is the numeraire process under the T-forward measure and E^T is the expectation under the T-forward measure.

According to Frühwirth and Sögner [8], the Jarrow and Turnbull model hypothesis is based on a Poisson process for the occurrence of default, under the equivalent martingale measure with intensity $\lambda := \mu\lambda_1$, where μ may be defined as a (constant) market price of default risk influenced by the degree of risk aversion of market participants and λ_1 as the risk premium per risk unit. The default intensity λ estimated by the Jarrow and Turnbull model is not a pure default intensity. Nonetheless, it captures a lot of factors, such as additional risk factors for companies that potentially create a barrier between corporate and government bonds such as liquidity, tax and systematic risk differences (Duffie and Singleton, 1997)[7].

With everything previously in mind, in the Jarrow and Turnbull model the payoff of the risky bond is given by:

$$e(u) = \begin{cases} 1, & \text{if } u < \tau \\ \delta, & \text{if } \ u \geq \tau \end{cases}. \quad (4.48)$$

The price of this default-risky zero-coupon bond, $v(t,u)$ is given by:

$$v(t,u) = P(t,u)E_u[e(u)|F_t] \quad (4.49)$$

$$= \begin{cases} P(t,u)[exp(-\lambda(u-t)) + \delta(1 - exp(-\lambda(u-t)))], if\ t < \tau \\ P(t,u)\delta, if\ t \geq \tau \end{cases}, \quad (4.50)$$

where $P(t,u)$ is the time t price of a default-free zero-coupon bond with maturity u, Q^u the u-forward measure, δ the recovery rate and $E_u[e(u)|F_t]$ the relative price at u of a default-risky to a default-free zero-coupon bond.

As previously illustrated, Privault (2023) denote the price of a default-risky zero-coupon bond as $P_d(t,T)$, whilst Frühwirth and Sögner (2001) chose $v(t,u)$ as the label for the same concept but with a maturity u instead of T.

The equation (4.50) results from (4.49) since

$$E_u[e(u)] = \begin{cases} [exp(-\lambda(u-t)) + \delta(1-exp(-\lambda(u-t)))], if\ t < \tau \\ \delta, if\ t \geq \tau. \end{cases} \tag{4.51}$$

Remark that in the previous analysis, there was no recovery with a payoff equal to $\mathbb{1}_{\tau>T}(u)$ while now there is a recovery δ and a payoff equal to $e(u)$. Hence, (4.49) is a generalization of (4.47).

With all this, the second factor in the expression (4.49) corresponds to the relative price at t of a default-risky zero-coupon bond. As seen previously, it can be the discount factor for default risk. The equation shows that the default-free term structure, the default intensity and the recovery rate are the inputs to compute $v(t,u)$.

In a similar way as for, the price of a default-free coupon bond the time t price of a default-risky coupon bond with maturity T, $B(t,T)$ is a linear combination of its cash flows $C_B(u)$ given by:

$$B(t,T) = \sum_{u=t}^{T} v(t,u)C_B(u), \tag{4.52}$$

where the coefficient of $C_B(u)$, $v(t,u)$, is the price of a monetary unit at u, with partial recovery δ in case of default.

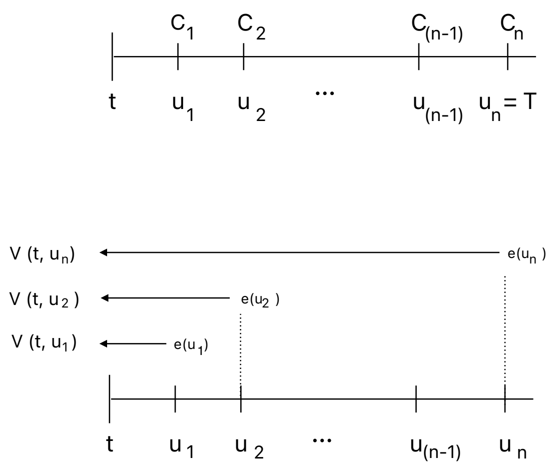


Figure 4.2 B(t,T) construction

In Figure (4.2), C_u are the different coupons during the period time between t and maturity T. With this, $v(t,u)$ represents the bond price with payoff $e(u)$, in u. $v(t,u) \times C_u$ is the bond price with payoff $e(u) \times C_u$ in u.

In this way, in $v(t,u)$ with different maturities u_1, u_2, etc., payoff $e(u)$ is priced. In this case, if there is default, the payoff is the recovery rate, which is a rate that always varies between 0 and 1. In sum:

- $v(t,u_i)$ is the bond price with payoff $e(u_i)$ in u_i;
- $v(t,u_i) \times C_i$ is the bond price with payoff $C_i \times e(u_i)$ in u_i.

To implement the Jarrow and Turnbull model, $P(t,u)$ along with the estimation of the default process parameters (λ,δ) are needed.

Chapter 5

Empirical analysis

In this chapter, we describe how the estimation of the different default intensities can be achieved and analyze the issues arising with the estimation. To estimate the default intensity, it is necessary to previously compute the default-free zero-coupon bond prices $P(t,u)$. Hence, it must be executed by a two-step procedure (Frühwirth and Sögner, 2001) [8]. First, measure the default-free zero-coupon bond prices using the default-free term structure, according to the Svensson (1994) [27] model. We assumed that the term structures of interest rates estimated accurately reflect the genuine term structures. With this assumption in mind, we then use non-linear least squares to estimate the default intensity, which we denote as λ.

Using the Jarrow and Turnbull (JT) model from Chapter 4, bond pricing only requires $P(t,u)$ from the default-free area, the term structure time series satisfies. This is one of the main advantages in the practical implementation of this credit risk model compared to more complex models. This analysis was conducted with the aid of MATLAB, using daily market bond prices for *Google* and *Farfetch*, the last one being one of the most shorted companies by the end of 2023.

We are also using a recovery rate fixed at $\delta = 0.5$, since this estimation is derived from the US market and there are differences between bankruptcy legislations in different countries. It is consistent with the Basle 2 provisions of a loss-given-default for bank loans of 50% independently of the country. Furthermore, Kim and Zhao (2001) [19] state that there is no reason to assume that the 50% rate does not hold for other countries, thus, when implementing their default risk model, they also use a 50% recovery rate. In the same way, Houweling and Vorst (2002) [15] use a 50% recovery rate for bonds (and credit default swaps) outside the US market.

The estimation of λ at time t, is determined by:

$$\hat{\lambda} = \underset{\lambda}{argmin} \sum_{i=1}^{n} [B_i^{obs} - B_i(\lambda)]^2. \tag{5.1}$$

The expression on the right-hand represents the sum of squares of daily pricing errors between the theoretical JT model prices and the observed prices. In this way, through the square we can get the shortest possible distance between the two. For this, we used time-series data (over a period of time). We performed estimations using data from several days (time-series) where n is the number of days used in the estimation procedure and applied restrictions to the parameter λ for a range from 0 to 15.

5.1 Simulated Data analysis

In this approach, market data yet in a simulated environment is used to ensure that the estimation processes can effectively present accurate estimations when comparing between different versions of the model considered. By adopting a nonlinear objective function, where there is no certainty of its convexity, the implementation of nonlinear constraints and also the statistical challenges, resulting from the estimation of the parameters, the simulated environment allows a first test for reliability of the procedures.

In this section, we present a theoretical analysis of the Jarrow and Turnbull model, considering values for the following parameters: n=5 and $\delta = 0.5$.

For this simulation, we generate theoretical bond prices based on the equations (4.49) and (4.52) of the JT model, equation (5.1) and given values for the default intensity, respectively. We consider a bond with seven years of maturity, with semi-annual coupon maturities on datasets and zero-coupon yield data retrieved from the Federal Reserve System (Gürkaynak, Sack and Wright, 2007) [13], when evaluating the default-risky bond prices. We intend to replicate certain conditions of the *Google* bond such as, bond with an issue date on August 14th, 2020 and a maturity of seven years (August 15th, 2027), coupon maturities paid semiannually and cash flows from August 14th to September 1st, 2023 and consequently convert it to a 360-day calendar. For the sake of simplification, the 4-year zero-coupon yield was taken into account for this analysis due to the difference between the bond maturity and the time considered. This is a reasonable assumption given that the rates do not change too much.

As the zero-coupon bond prices are directly affected by the continuously compounded yield, the more it is increased the lower will be the zero-coupon bond prices and vice-versa. Now we will see how the default intensity can affect the default-risky coupon bond prices $v(t,u)$ and the defaultable bond prices $B(t,T)$. When a bond is quoted in database around 85, it means that the bond is trading at approximately 85% of its face value. For example, a commonly used face value for a bond is $1,000, so a bond priced at 85 is trading at $850 $(85 \times 1000/100)$ and that is why $B(t,T)$ is expressed as this last formula for the rest of the analysis. Below, we will examine data corresponding to three consecutive weeks.

In this first estimation, we will generate simulated risky bond prices for five days (a week) based on different default intensities, λ. After, we will compare them with these estimated prices, obtained based on these simulated prices. For that, we will consider that λ is equal to 0.01, 0.05, 0.1, 0.3 and 0.5.

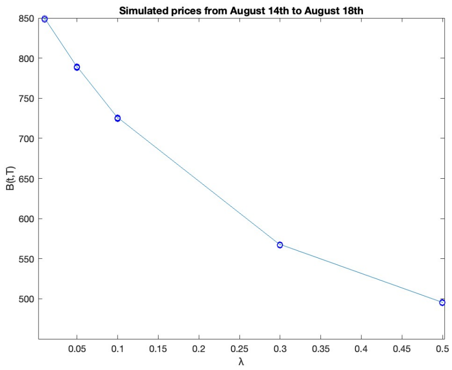
Observing Table 5.1, estimated and simulated bond prices are exactly the same, which denotes that the estimation method of the Jarrow and Turnbull model is accurately capturing the behaviour of the underlying data and that it is performing well in predicting the prices based on the given input parameters. This alignment between estimated and simulated bond prices can provide confidence in the reliability of the estimation method predictions and may suggest that it is reliable. We also observed that the two next weeks had the same outlook.

λ	**Estimated Prices**	**Simulated Prices**
0.01	849.604	849.604
	849.954	849.954
	848.157	848.157
	848.651	848.651
	849.879	849.879
0.05	788.958	788.958
	789.322	789.322
	787.693	787.693
	788.190	788.190
	789.368	789.368
0.1	725.529	725.529
	725.900	725.900
	724.438	724.438
	724.931	724.931
	726.050	726.050
0.3	567.226	567.226
	567.564	567.564
	566.473	566.473
	566.906	566.906
	567.827	567.827
0.5	495.496	495.496
	495.779	495.779
	494.816	494.816
	495.183	495.183
	495.977	495.977

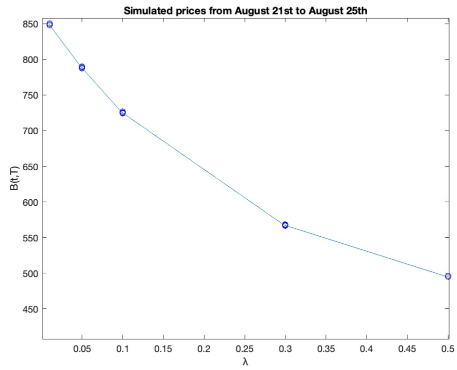
Table 5.1 Comparison between simulated bond prices replicating the bond and market conditions and the corresponding estimated prices from August 14th to August 18th, 2023

When the default intensity increases, it means a higher risk of default for the bond issuer. Consequently, it can lead to a decrease in bond prices as demonstrated in the Tables. For the first week of observations, between August 14th and August 18th, bond prices tend to decrease as well as in the second week of observations. However, in the second week of observations, between August 21st and August 25th, the zero-coupon yield curve was higher so bond prices were lower than in the first week. In the third week of observations, between August 28th and September 1st, the zero-coupon yield curve was the lowest of all the three week observations translating into higher bond prices.

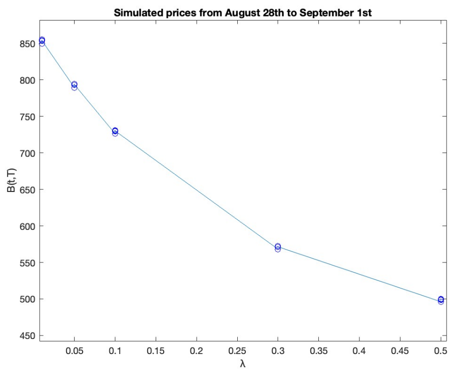
In summary, the close alignment between estimated and simulated bond prices suggests that the estimation method accurately reflects underlying market dynamics. As default intensity rises, indicating heightened default risk for bond issuers, bond prices tend to decrease. This inverse relationship underscores the significance of monitoring default intensity in assessing bond market performance and risk.



(a) Simulated prices in the conditions from August 14th to August 18th



(b) Simulated prices in the conditions from August 21st to August 25th



(c) Simulated prices in the conditions from August 28th to September 1st

Figure 5.1 Analysis of simulated prices across three weeks for different λ values

These figures demonstrate the impact of the different default intensities on simulated bond prices, $B(t,T)$. They provide a graphical representation of how changes in default intensity influence the behavior of $B(t,T)$ with time. Once again, they reveal that heightened default intensity corresponds to a notable decrease in the value of $B(t,T)$, thereby expressing on the inverse relationship between default intensity and the valuation of financial assets. Moreover, as default intensity increases, indicating a higher risk of default for the bond issuer, bond prices tend to decrease. This inverse relationship reflects investors' perception of increased risk associated with holding the bonds, leading to a decrease in their market value as described by the model.

5.2 Market Data analysis

5.2.1 Data set description

In this section, we illustrate the estimation of the default intensity using data from *Google* (GOOGL) and *Farfetch* (FTCH). *Google* is an American multinational technology company focusing essentially on artifical intelligence, computer software, online advertising and search engine technology. *Google's* parent company *Alphabet Inc.* is one of the five Big Tech companies, alongside *Amazon*, *Apple Inc*, *Meta* and *Microsoft*. Differently, *Farfetch* is an e-commerce company focused on luxury clothing and beauty products. It operates as a digital marketplace that sells products from several hundred brands, boutiques and department stores from around the world.

For each sample, we considered the arithmetic mean of bid and ask, callable bond prices for *Google* and convertible bond prices for *Farfetch*. The zero-coupon yield data is also collected from the Federal Reserve System (Gürkaynak, Sack and Wright, 2007) [13] consistent with the Svensson model [27].

The sample data was taken from the EIKON database for the *Google* bond prices that correspond to observations from August, 2021 to August, 2023, with an issue date on August 14th, 2020 and a maturity of seven years (August 15th, 2027) for the first and second scenario. It is also necessary to consider values related to the coupon maturities, the cash flows, the zero-coupon yield, so that we can calculate the zero-coupon bond prices, the time when each bond is obtained and the risky bond prices associated. From the historical and financial knowledge of the company, one would expect a low value for the default intensity.

Additionally, the par value is the face value of a bond and determines the bond or fixed-income instrument's redemption value as well as the dollar value of coupon payments.The market price of a bond may fluctuate above or below its par value, contingent on factors like prevailing interest rates and the bond's creditworthiness. Bonds are typically issued in standard denominations of \$1,000, representing a common par value in the market, therefore this is the value considered in our analysis.

In this case, we have an annual coupon of 0.8%, paid semi-annually. This indicates that, the cash flow will be equal to:

$$C = \frac{0.8}{100} \times 1000 \times \frac{1}{2} = \$4, \tag{5.2}$$

where \$4 is the cash flow paid semi-annually and in the last semester it will be the par value \$1,000 as the reimbursement, plus \$4.

For the two following scenarios, the maturity is as mentioned before of seven years, which signifies that if the coupons are paid semiannually, we will have fourteen coupons in total when retrieving the estimated default intensities. Moreover, for the sake of simplification, the zero-coupon yield data will be recovered as that of the arithmetic mean yield for the coupon maturities, from the day considered in the analysis. For example, if the data is collected for 2021 and the maturity date of the bond is 2027, then it will be the arithmetic mean yield for maturities between one and six years.

For *Farfetch* bond prices, the sample data was also taken from the EIKON database that correspond to observations from August 28th to September 8th, 2023, with an issue date on May 1st, 2020 and a maturity of seven years (May 1st, 2027). We will consider the same parameters obtained, as previously described but applied specifically to this company.

Therefore, the coupon is paid semiannually with an annual coupon of 0.375%. This indicates that the cash flow, will be equal to \$1.875 and the final one will be added to the par value, \$1,000.

The next table, summarizes the information on the dates.

	Starting date for data	Underlying asset	Maturity date (T)	Issue date (t_0)	
	August 16th, 2021	GOOGL	August 15th, 2027	August 14th, 2020	
	August 28th, 2023	FTCH	May 1st, 2027	May 1st, 2020	

Table 5.2 Dates set information for GOOGL and FTCH (first and second scenario)

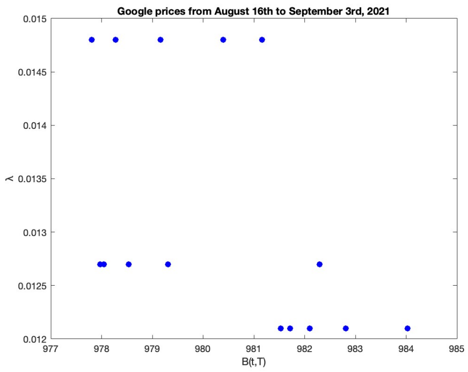
5.2.2 λ estimation results

In this section, we present the estimates of the default intensity when implementing the Jarrow and Turnbull model and (5.1), applied to *Google* and *Farfetch*. We will consider three different scenarios: the first one will retrieve the estimated default intensity based on 3 weeks of August, 2021, February, 2022 and August, 2022 for *Google*; the second one to compare *Google* and *Farfetch* between August 28th and September 8th, 2023, or two weeks observations and lastly to compare one week of observations from August 14th to August 18th, 2023 with different bond maturities dates but only for *Google*.

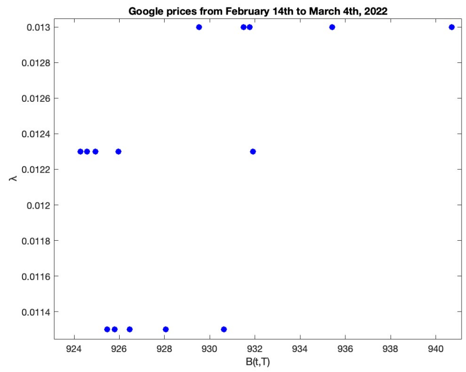
GOOGL and FTCH data

For the first scenario, in Figure (5.2) and Table 5.3, we can detect very different default intensities estimations when observing the various periods. From August 16th to September 3rd, 2021 (5.2a, prices are being traded at a value never below \$977 meaning that the default intensity should be low just considering these prices. We obtain an estimated λ=0.0148, for the first week of observations and the other two, λ=0.0127 and λ=0.0121, respectively. Due to COVID-19 pandemic, a recession did occur affecting the zero-coupon yield and showing percentages near 0. Historically, we observed a similar event as the one during the subprime crisis. On August 28th, 2019, the ten-year/two-year spread briefly went negative. The U.S. economy suffered a two-month recession in February and March of 2020 amid the outbreak of the COVID-19 pandemic. The COVID-19 economic shock still affected the months after March, threatening with a potential recession in the economy and consequently having effects on the yield curve.

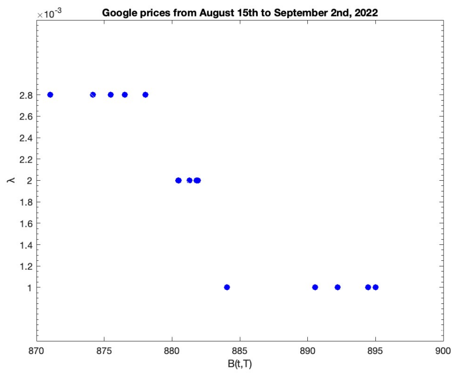
From February 14th to March 4th, 2022 (5.2b), prices declined comparing to the previous year, but never below \$900 which is still considerably high. Lastly, from August 15th to September 2nd, 2022 (5.2c), we denote another decrease in *Google* bond prices already below \$900. The discount factor in the model can be a potential cause for this decrease given that the default intensity is also decreasing. It is an uncommon event from the supposedly inverse relationship between bond prices and default intensity. However, during this period the economy started to boom after the pandemic recession giving higher expectations to investors. Hence, we will have the lowest default intensity of the three weeks observations, with λ=0.0010 in the first week.



(a) *Google* prices from August 16th to September 3rd, 2021



(b) *Google* prices from February 14th to March 4th, 2022



(c) *Google* prices from August 15th to September 2nd, 2022

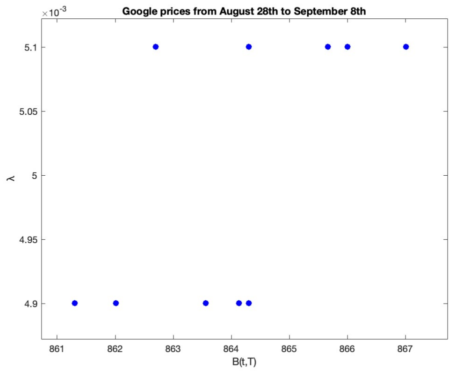
Figure 5.2 Analysis of *Google* bond prices across three weeks and the estimated λ values

Starting date	1st week	2nd week	3rd week
August 16th, 2021	$\lambda = 0.0148$	$\lambda = 0.0127$	$\lambda = 0.0121$
February 14th, 2022	$\lambda = 0.0113$	$\lambda = 0.0123$	$\lambda = 0.0130$
August 15th, 2022	$\lambda = 0.0010$	$\lambda = 0.0020$	$\lambda = 0.0028$

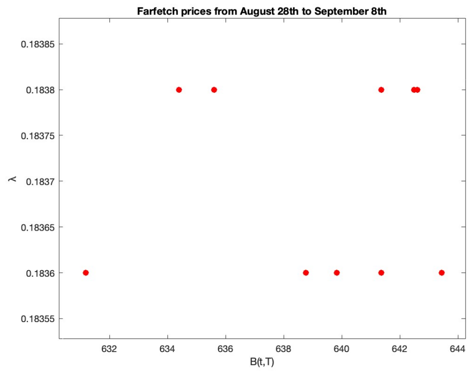
Table 5.3 Estimated default intensities results for different periods for *Google*, 2027 maturity bond

Historically, *Alphabet Inc.* has maintained strong credit ratings and issued bonds with favorable terms, reflecting market confidence in the company's financial strength and creditworthiness. As a result, the bond prices of *Alphabet Inc.* have often been perceived as relatively high compared to bonds issued by other companies, indicating that investors may be willing to pay higher prices for their bonds due to their regarded safety and stability, as reflected in the company's strong financial performance, diversified revenue streams, and dominant market position in the technology sector. Due to the same reason, high demand for *Alphabet Inc.* bonds may contribute to their relatively high prices in the secondary market.

For the second scenario, we used data estimates from August 28th to September 8th, 2023 or two week observations to compare how *Google* and *Farfetch* bond prices affected the estimation of the different λ values.



(a) *Google* prices from August 28th to September 8th



(b) *Farfetch* prices from August 28th to August 8th

Figure 5.3 Comparison between *Google* and *Farfetch* bond prices across two weeks and the different estimated λ values

Company	**1st week**	**2nd week**
Google	λ=0.0051	λ=0.0049
Farfetch	λ=0.1838	λ=0.1836

Table 5.4 Estimated default intensities results between August 28th and September 8th, 2023

Observing Figure (5.3), we verify significant differences between the two companies' bond prices. While *Google* prices are always above \$860, *Farfetch* can not surpass \$645. This is the starting point of *Farfetch*'s struggles to become profitable and to secure product because top luxury brands do not want to sell through third parties, preferring to maintain control and avoid the discounting that online retailers rely on to bring in clients. Even the credit rating agencies, such as Moody's, cut the company's rating to *Caa*2, deep in junk territory, quoting deepening worries about *Farfetch*'s financial situation. They had an enormous booming during the pandemic because of online shopping and luxury spending. But as lockdown lavishness retrenched and investors turned away from unprofitable technology companies, Farfetch's cash burn and the cracks in its business model were laid bare.

This is why it is possible to notice high default intensities in Table 5.4 for both first and second week of observations. Contrarily, *Google* is maintaining a steady pace with bond prices around $860 and low default intensities, yet higher than the ones observed in August 2022 (5.3b). *Alphabet Inc.* has always maintained a high credit rating from major rating agencies such as Moody's, Standard & Poor's, and Fitch Ratings. *Alphabet Inc.* has been viewed favorably by credit rating agencies due to its strong financial performance, diversified revenue streams, and leading position in the technology industry.

For the third and last scenario, we will analyze how the different default intensities estimation changes when we select different bond maturities. In this case, *Alphabet Inc.* issued a set of bonds with different maturities. The ones used in the previous analysis had a maturity of seven years, from 2020 until 2027. The next ones are still issued in 2020 but with maturities of five and ten years (2025 and 2030). We selected data estimates from August 14th to August 18th, 2023 or one week observations. Let T be the maturity measured since 2020 and $T^* = T - t$ the maturity since 2023 (t=3).

The next table, summarizes the information on the dates.

GOOGL bonds	**Starting date for data**	**Maturity date (T)**	**Issue date (t_0)**	
$T^* = 2$	August 14th, 2023	August 15th, 2025	August 14th, 2020	
$T^* = 7$	August 14th, 2023	August 15th, 2030	August 14th, 2020	

Table 5.5 Dates set information for GOOGL bonds

After estimating the different default intensities, we noticed distinct bond prices when the bond maturity increases. While for $T^* = 2$, *Google* prices are well above $900, in $T^* = 7$ they are below $800. The relationship between bond maturity and bond prices is a fundamental concept in fixed-income markets, as it provides insights into the potential risks and returns associated with different bond investments over time.

As for the default intensity, it is, as observed in Table 5.6, increasing as the bond maturity also increases. A greater uncertainty of default over time is linked to higher default intensities. Moreover, it can also be explained by higher discount factor associated to the zero-coupon bond prices.

$T^* = 2$	$T^* = 4$	$T^* = 7$
$\lambda = 0.0042$	$\lambda = 0.0045$	$\lambda = 0.0507$

Table 5.6 Estimated default intensities results for different maturities

To conclude the analysis of this scenario, we will obtain conditional probabilities using the different default intensities estimates for the three maturities. First, the probability of not occurring default is given by:

$$P(\tau > T^*) = 1 - P(\tau \leq T^*), \tag{5.3}$$

where τ denotes the default time and $T^* = T - t$. If $\tau > T^*$, it means that the default did not yet occur so it is in surviving, if $\tau \leq T^*$, then there is default until maturity T^*.

The probability of surviving up to T_i^* years given that it has survived up to T_{i-1}^* will be, generally:

$$P(\tau > T_i^* | \tau > T_{i-1}^*) = \frac{P(\tau > T_i^* \cap \tau > T_{i-1}^*)}{P(\tau > T_{i-1}^*)} = = \frac{P(\tau > T_i^*)}{P(\tau > T_{i-1}^*)} = \frac{1 - P(\tau \leq T_i^*)}{1 - P(\tau \leq T_{i-1}^*)}. \tag{5.4}$$

Recalling that $\tau \sim Exp(\lambda)$, it comes

$$P(\tau > T^*) = e^{-\lambda T^*}. \tag{5.5}$$

As seen in Table (5.6), λ=0.0042 for $T^* = 2$ and λ=0.0045 for $T^* = 4$, which means that the probabilities of default will be, approximately, $1 - e^{-0.0042\times 2} = 0.00836$ and $1 - e^{-0.0045\times 4} = 0.01782$, respectively. If we apply this to equation (5.4), then it will follow:

$$\widehat{P}(\tau > 4 | \tau > 2) = \frac{1 - 0.01782}{1 - 0.00836} \approx 0.9905, \tag{5.6}$$

where $\widehat{P}$ denotes the estimated probability.

This signifies that there is a 99,05% probability that *Google* will not default until 2027, considering that it did not until 2025.

In a similar way, it is also possible to obtain the probability of default until $T^* = 4$, considering that it did not occur until $T^* = 2$. Generally, it is given by:

$$\widehat{P}(\tau \leq 4 | \tau > 2) = 1 - \widehat{P}(\tau > 4 | \tau > 2) \approx 1 - 0.9905 = 0.0095, \tag{5.7}$$

there is a 0.95% probability that *Google* will default until 2027, considering that it did not until 2025, which is a higher probability when comparing with the difference between default intensities for these two maturities. We can apply the same example for $T^* = 7$, considering that default did not occur until 2027. We already deduced the probability of default for $T^* = 4$, and for $T^* = 7$ it will be $1 - e^{-0.0507\times 7} = 0.29653$. Using again equation (5.4) it will come:

$$\widehat{P}(\tau > 7 | \tau > 4) = \frac{1 - 0.29653}{1 - 0.01782} \approx 0.7159, \tag{5.8}$$

meaning that there is a 71.59% probability that *Google* will not default until 2030, considering that it did not until 2027. Consequently, there is a 28.41% probability that *Google* will default until 2030, considering that it did not until 2027.

Even if *Alphabet Inc.*, the parent company of *Google*, is very unlikely to default due to its financial strength, debt levels, and business outlook, there is an increased uncertainty when considering a longer time horizon that is why we notice a higher probability of default in $T^* = 7$.

Chapter 6

Conclusion

The Jarrow and Turnbull model has emerged as a crucial model within the area of default probability estimation in bond markets. Its significance lies not only in providing a framework for assessing default risks but also in offering insights into the complex dynamics that underlie default events. One of the model's key contributions is its recognition of the inherent unpredictability of default. By acknowledging this uncertainty, the model prompts analysts to consider a wide array of factors that may influence probabilities of default, including economic conditions and market sentiment. This nuanced approach is crucial for accurately assessing default risks and making informed investment decisions.

Moreover, the Jarrow and Turnbull model facilitates a deeper understanding of how probabilities of default evolve over time. Through sophisticated mathematical techniques and an estimation method, analysts can model the dynamics of default intensity and assess its sensitivity to changes in market conditions. The model also offers valuable insights into the comparative effectiveness of different evaluation techniques. By comparing evaluation formulas based on numeraire change techniques and Poisson default process modelling, analysts can identify the strengths and weaknesses of each approach and refine their estimation methods accordingly. This empirical approach to model evaluation enhances the robustness and accuracy of probability of default estimates, thereby improving risk management practices within the bond market.

Furthermore, empirical analysis of market trends provides valuable context. For example, the sharp decline in bond prices since 2022, potentially driven by rising interest rates, has significant implications for default risk. There was an unexpected relationship between bond prices and default. When comparing default intensities across different companies and industries it offers valuable insights into sector-specific risk factors. During our analysis, the observed increase in default intensities from August, 2022 to August, 2023 for *Google* may be attributed to macroeconomic trends such as rising inflation, while high default intensities for companies like *Farfetch* in August, 2023 may be driven, not only by inflation, but also by company-specific factors such as falling bond prices.

Finally, an examination of default intensities across different maturities for *Google*, reveals important insights into the temporal dynamics of default risk. The observed trend of higher default intensities for longer maturities underscores the influence of time-related factors such as default uncertainty and discount factor dynamics. By understanding these temporal variations in default risk, investors can tailor their investment strategies to better align with their risk tolerance and investment horizon.

In summary, the Jarrow and Turnbull model, a reduced form model, succinctly captures the market's sensitivity to a company's default risk through intensity estimation. By addressing the inherent unpredictability of default events and employing dynamic modeling techniques validated empirically, this model proves invaluable in assessing default risk and informing investment decisions. Leveraging the insights provided by the Jarrow and Turnbull model enables investors to navigate the complexities of the bond market with greater efficacy, empowering them to pursue their investment objectives with heightened confidence.

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