1 2 9 0

UNIVERSIDADE D
COIMBRA

Rui Rodrigues de Abreu Fernandes Prezado

Some Aspects of Descent Theory and Applications

Tese no âmbito do Programa Inter-Universitário de Doutoramento em Matemática orientada pela Professora Doutora Maria Manuel Pinto Lopes Ribeiro Clementino e pelo Doutor Fernando Lucatelli Nunes e apresentada ao Departamento de Matemática da Faculdade de Ciências e Tecnologia da Universidade de Coimbra.

Setembro de 2023

Some aspects of descent theory and applications

Rui Rodrigues de Abreu Fernandes Prezado

1 2 9 0 UNIVERSIDADE D COIMBRA

U. PORTO

UC|UP Joint PhD Program in Mathematics

Programa Interuniversitário de Doutoramento em Matemática

PhD Thesis | Tese de Doutoramento

September 2023

Acknowledgements

I am forever indebted to Maria Manuel Clementino and Fernando Lucatelli Nunes. Not only for your guidance and your teachings, but also for all the opportunities you have provided for me through this experience, as well as providing encouragement and support.

I would like to extend this gratitude to the jury members of my Ph. D. defense: Jorge Picado, Marcelo Fiore, Peter Gothen, Dirk Hofmann, Diana Rodelo, Manuela Sobral. Thank you for reading the thesis with care, and for the fascinating discussion that followed.

I am also very thankful for the opportunity to co-author with Lurdes Sousa, and I hope this bears future occasions for collaboration.

I would like to thank the people at the Centre of Mathematics of the University of Coimbra (CMUC), particularly the Algebra, Logic and Topology research group which taught me many things about research in category theory.

For being kindly patient and very helpful, I would like to thank Rute Andrade.

For warmly hosting me during my stay at UCLouvain, a special thank you to Marino Gran and Tim Van der Linden.

To André, Carlos and Eli, I thank each of you for the valuable friendships, and for all the conservations we have had about mathematics and other general nonsense.

Thanks to Mercedes and Silvana, I was supplied plenty of cold beers and hot coffee. I am grateful for all the humourous moments at the "Bar de Matemática" we had, and I hope more are to come.

Finally, I would like to thank my family for always being supportive and present. In particular: my parents, Maria de Fátima and Jaime, thank you for nurturing my passion for mathematics; my partner, Liliana, thank you for your endless love and patience, which I will always treasure; my companions, Boris and Maria.

The financial supports from Fundação para a Ciência e Tecnologia (FCT) through the grant PD/BD/150461/2019, and from Centro de Matemática da Universidade de Coimbra (CMUC) funded by the Portuguese Government through FCT/MCTES are gratefully acknowledged.

The support from the Oberwolfach Foundation Fellowship, funded by the Mathematisches Forschunginstitut Oberwolfach (MFO), as well as the opportunity for the research stay at MFO, are gratefully acknowledged as well.

Abstract

This thesis is an exposition of the author's contribution on effective descent morphisms in various categories of generalized categorical structures. It consists of: Chapter 1, where an elementary description of descent theory and the content of each remaining chapter is provided, supplemented with references; Chapter 2, consisting of various descent theoretical definitions and results employed in the remainder of this work; four chapters, each corresponding to an article written by the author during the period of his PhD studies.

In Chapter 3, we describe conditions for which a $\mathcal{V}$-functor is an effective descent morphism in the category $\mathcal{V}$-Cat of $\mathcal{V}$-categories, where $\mathcal{V}$ is a cartesian monoidal category with finite limits. Since these conditions rely on understanding (effective) descent morphisms in the free coproduct completion $\mathsf{Fam}(\mathcal{V})$ of the category $\mathcal{V}$, we also carried out a study of such morphisms. We show how these results may be applied to describe the effective descent $\mathcal{V}$-functors for the categories $\mathcal{V} = \mathsf{CHaus}$ of compact Hausdorff spaces and $\mathcal{V} = \mathsf{Stn}$ of Stone spaces. The main reference of this chapter is the single-authored article *On effective descent $\mathcal{V}$-functors and familial descent morphisms*, published in the *Journal of Pure and Applied Algebra, vol. 228, n. 5, 2024*.

We study effective descent morphisms for generalized multicategories internal to a category $\mathcal{V}$ with finite limits in Chapter 4, proposing two approaches to obtain their description. The first approach relies on depicting the category $\mathsf{Cat}(T,\mathcal{V})$ of T-categories internal to $\mathcal{V}$ as a 2-dimensional limit, which provides a method of studying their effective descent morphisms. The second approach extends Ivan Le Creurer's techniques on internal categories to the setting of generalized internal multicategories. As a consequence of this work, we provide conditions for functors between internal multicategories to be of effective descent, as well as for functors between internal graded categories (by an internal monoid), internal operadic multicategories and "enhanced" multicategories. The main reference for this chapter is the article *Descent for internal multicategory functors*, published in *Applied Categorical Structures, vol. 31, n. 11, 2023*, with Fernando Lucatelli Nunes.

Furnished with the results for effective descent morphisms in internal generalized multicategories, Chapter 5 aims to extend these results to the setting of *enriched* generalized multicategories – the so-called $(T,\mathcal{V})$-categories. This is accomplished by extending the embedding of "enriched $\to$ internal" categories to the setting of generalized multicategories, via a broad notion of *change-of-base* for generalized categorical structures, which we specialize to our setting. We discuss the conditions under which the embedding $(\overline{T},\mathcal{V})$-$\mathsf{Cat} \to \mathsf{Cat}(T,\mathcal{V})$ exists and whether it reflects effective descent morphisms. Finally, we show these results can be applied to the enriched counterparts of the multicategories considered in Chapter 4. More precisely, we obtain descriptions of the effective descent functors between enriched multicategories, enriched graded categories, enriched operadic multicategories, and the discrete counterparts to the "enhanced" multicategories. The main reference

for this chapter is the article *Generalized multicategories: change-of-base, embedding and descent, arXiv:2309.08084, DMUC preprints 23-29*, under review, with Fernando Lucatelli Nunes.

Chapter 6 considers the techniques used by Sobral to study effective descent functors with respect to the fibration of discrete opfibrations under a new perspective. More specifically, we first highlight the relationship between the Cauchy completion of $\mathscr{V}$-enriched categories and the $\mathscr{V}$-fully faithful lax epimorphisms: the latter are precisely those $\mathscr{V}$-functors that induce an equivalence on the Cauchy completions. Second, we show that the study of effective descent functors with respect to a suitable pseudofunctor $\mathsf{Cat}^{\mathsf{op}} \to \mathsf{CAT}$ can be simplified via formal methods. Combining these two ideas, we confirm that Sobral's characterization can be extended, showing the same conditions also characterize the effective descent morphisms with respect to the fibration of *split opfibrations*. The main reference for this chapter is the article *Cauchy completeness, lax epimorphisms and effective descent for split fibrations*, published in *Bulletin of the Belgian Mathematical Society – Simon Stevin, vol. 30, n. 1, 2023*, with Fernando Lucatelli Nunes and Lurdes Sousa.

Resumo

Esta tese é uma exposição das contribuições do autor sobre morfismos de descida efetiva em várias categorias de estruturas categoriais generalizadas. É consistido por: Capítulo 1, onde é fornecida uma descrição elementar sobre a teoria de descida e o conteúdo dos demais capítulos, complementado com referências bibliográficas; Capítulo 2, composto por várias definições e vários resultados da teoria de descida empregues na restante obra; quatro Capítulos, correspondendo a cada artigo escrito pelo autor durante o período dos seus estudos doutorais.

No Capítulo 3, descrevemos condições para que um $\mathscr{V}$-functor seja de um morfismo de descida efetiva na categoria $\mathscr{V}$-Cat de $\mathscr{V}$-categorias, onde $\mathscr{V}$ é uma categoria monoidal cartesiana com limites finitos. Como estas condições dependem do conhecimento dos morfismos de descida (efetiva) no completamento livre $\mathsf{Fam}(\mathscr{V})$ da categoria $\mathscr{V}$ para coprodutos, também se desempenhou um estudo de tais morfismos. Demostramos como estes resultados podem ser aplicados para descrever os $\mathscr{V}$-functores de descida efetiva para as categorias $\mathscr{V} = \mathsf{CHaus}$ de espaços de Hausdorff compactos e $\mathscr{V} = \mathsf{Stn}$ de espaços de Stone. A referência principal deste capítulo é o artigo de autoria única *On effective descent $\mathscr{V}$-functors and familial descent morphisms*, publicado no *Journal of Pure and Applied Algebra, vol. 228, n. 5, 2024*.

Estudamos morfismos de descida efetiva para multicategorias generalizadas internas a uma categoria $\mathscr{V}$ com limites finitos no Capítulo 4, propondo duas abordagens para obter a sua descrição. A primeira abordagem recorre a uma descrição da categoria $\mathsf{Cat}(T,\mathscr{V})$ de T-categorias internas a $\mathscr{V}$ como um limite de dimensão 2, que proporciona um método para o estudo dos seus morfismos de descida efetiva. A segunda abordagem extende as técnicas de Ivan Le Creurer para a descrição de morfismos de descida efetiva em categorias internas para o contexto das multicategorias internas generalizadas. Como consequência destas descrições, fornecemos condições para que functores entre multicategorias internas sejam de descida efetiva, tal como functores entre categorias graduadas internas (por um monóide interno), multicategorias operádicas internas, e multicategorias "aprimoradas". A referência principal para este capítulo é o artigo *Descent for internal multicategory functors*, publicado em *Applied Categorical Structures, vol. 31, nº 11, 2023*, com Fernando Lucatelli Nunes.

Munido com os resultados sobre morfismos de descida efetiva em multicategorias generalizadas internas, o Capítulo 5 pretende extender estes resultados para o contexto das multicategorias generalizadas *enriquecidas* – as ditas $(T,\mathscr{V})$-categorias. Isto foi concretizado através de uma extensão da imersão de categorias "enriquecidas $\to$ internas" para o ambiente das multicategorias generalizadas, através de uma noção abrangente de *mudança de base* para estruturas categoriais generalizadas, que especializamos para o nosso contexto. São também discutidas as condições sob as quais é possível levar a cabo uma tal extensão, e quando é, se $(\overline{T},\mathscr{V})$-$\mathsf{Cat} \to \mathsf{Cat}(T,\mathscr{V})$ reflete morfismos de descida efetiva. Finalmente, demostra-se que estes resultados podem ser aplicados às multicategorias en-

riquecidas associadas às consideradas no Capítulo 4. Mais precisamente, obtemos descrições para os morfismos de descida efetiva entre multicategorias enriquecidas, categorias graduadas enriquecidas, multicategorias operádicas enriquecidas, e os análogos discretos das multicategorias "aprimoradas". A referência principal para este capítulo é o artigo *Generalized multicategories: change-of-base, embedding and descent, arXiv:2309.08084, pré-publicações DMUC 23-29*, sob revisão, com Fernando Lucatelli Nunes.

O Capítulo 6 considera as técnicas utilizadas por Sobral no estudo de morfismos de descida efetiva em relação ao fibrado dos opfibrados discretos sob uma nova perspetiva. Mais especificamente, realçamos, em primeiro lugar, a relação entre o completamento de Cauchy para categorias enriquecidas em $\mathscr{V}$ e os epimorfismos lassos $\mathscr{V}$-plenamente fiéis: estes últimos são precisamente os $\mathscr{V}$-functores que induzem uma equivalência nos completamentos de Cauchy. Em segundo lugar, mostramos que o estudo de morfismos de descida efetiva em relação a um pseudofunctor $\mathsf{Cat}^{\mathsf{op}} \to \mathsf{CAT}$ pode ser simplificado através de métodos formais. Combinando estas duas ideias, confirmamos que a caracterização de Sobral pode ser extendida, mostrando que as mesmas condições também caracterizam os morfismos de descida efetiva em relação ao pseudofunctor de *opfibrados cindidos*. A referência principal para este capítulo é o artigo *Cauchy completeness, lax epimorphisms and effective descent for split fibrations*, publicado em *Bulletin of the Belgian Mathematical Society – Simon Stevin, vol. 30, nº 1, 2023*, com Fernando Lucatelli Nunes e Lurdes Sousa.

Table of contents

1 Introduction **1**

1.1 Descent theory with respect to the basic bifibration . . . 1

1.2 Descent theory with respect to a pseudofunctor . . . 5

2 A primer on descent theory **9**

2.1 Lax descent category . . . 10

2.1.1 Universal property . . . 11

2.2 Effective descent morphisms . . . 13

2.2.1 Beck-Chevalley condition . . . 13

2.3 Basic bifibration . . . 14

2.3.1 Descent theory in categorical structures . . . 18

2.4 Bifibration of split opfibrations . . . 19

3 Enriched $\mathscr{V}$-functors **21**

3.1 Properties of the free coproduct completion . . . 22

3.2 Embedding $\mathscr{V}\text{-}\mathsf{Cat} \to \mathsf{Fam}(\mathscr{V})\text{-}\mathsf{Cat}$ . . . 25

3.3 Familial descent and effective descent morphisms . . . 27

3.4 Examples . . . 31

3.4.1 Meet semilattices . . . 32

3.4.2 Regular categories . . . 34

4 Generalized internal multicategory functors **37**

4.1 Reflexive T-graphs . . . 39

4.2 Internal T-categories . . . 41

4.3 Effective descent morphisms via bilimits . . . 43

4.4 A direct description of effective descent morphisms . . . 46

4.5 Application to graded, operadic and enhanced multicategories . . . 50

5 Generalized enriched multicategory functors **53**

5.1 $(T,\mathscr{V})$-categories . . . 54

5.2 Change-of-base and embedding . . . 56

5.3 Reflection of effective descent morphisms . . . 58

5.4 Scope of the findings . . . 60

5.4.1 Classical multicategories . . . 61

5.4.2 Cartesian and cocartesian multicategories 61
5.4.3 Graded, operadic and symmetric multicategories 61

6 Fibration of split opfibrations 63
6.1 Fully faithful morphisms and lax epimorphisms 63
6.2 Enriched Cauchy completion 64
6.3 Descent for the bifibration of split fibrations 67

References 69

Chapter 1

Introduction

Descent theory was first established in [23, 20, 18, 22] in the context of algebraic geometry, aiming to generalize the solution of the following problem: describe the commutative ring homomorphisms $R \to S$ for which the extension-of-scalars functor $R\text{-}\mathsf{Mod} \to S\text{-}\mathsf{Mod}$ is well-behaved. A more recent account of this problem, studied in a broader context, can be found in [32].

Descent theory has since found various applications and connections with other areas of mathematics, namely:

- the theory of monads [4], [42], [46],
- two-dimensional limits and coherence, [36], [43], [45],
- algebraic topology [8], [11],
- Janelidze-Galois theory [27], [7],
- topology [54], [14].

It is often useful to depict descent theory as a higher dimensional analogue of *sheaf theory*, as in [30, Introduction]. The *gluing condition*, described in terms of an equalizer of a parallel pair of functions on sets (*sheaf condition*), is replaced by *descent data*, described in terms of a descent object of a suitable diagram of categories (*descent condition*).

1.1 Descent theory with respect to the basic bifibration

The fundamental setting begins with a category $\mathscr{C}$ with pullbacks, a morphism $p\colon e \to b$, and considers the following *change-of-base* adjunction

$$\mathscr{C} \downarrow b \;\underset{p^*}{\overset{p_!}{\rightleftarrows}}\; \mathscr{C} \downarrow e, \qquad p_! \dashv p^*$$

where $\mathscr{C} \downarrow x$ is the comma category whose objects are the morphisms in $\mathscr{C}$ with codomain x, also called *bundles* over x. The descent problem, in this setting, can be stated as follows: describe the morphisms $p\colon e \to b$ where bundles over b admit a presentation as bundles over e plus some structure, specified by p^*, satisfying coherence conditions – this structure is the so-called *descent data*.

The *category* $\mathsf{Desc}(p)$ *of descent data of p* may be presented as the category T^p-Alg of T^p-algebras, where T^p is the monad induced by the change-of-base adjunction, by the Bénabou-Roubaud theorem [4]. Hence, we may consider the *Eilenberg-Moore factorization* of the pullback functor p^* as follows:

$$\begin{array}{ccc} \mathscr{C} \downarrow b & \xrightarrow{\quad p^* \quad} & \mathscr{C} \downarrow e \\ & {\scriptstyle \mathscr{K}^p} \searrow \quad \nearrow & \\ & \mathsf{Desc}(p) & \end{array}$$

Therefore, the descent problem is reduced to the question of whether the comparison functor $\mathscr{K}^p \colon \mathscr{C} \downarrow b \to \mathsf{Desc}(p)$ is an equivalence. When this is the case, we say that p is an *effective descent morphism*: these morphisms, the main object of study of this work, are precisely the solutions to the descent problem. Therefore it is informative to obtain descriptions of such morphisms. In a pursuit of such descriptions, it is useful to consider the notions of *descent* and *almost descent* morphism. We say that p is

- p is a descent morphism if $\mathscr{K}^p$ is fully faithful,
- p is an almost descent morphism in $\mathscr{K}^p$ is faithful.

These refinements will be useful in our description of effective descent morphisms in categorical structures.

If $\mathscr{C}$ has all finite limits, then

- the descent morphisms are precisely the pullback-stable regular epimorphisms, and
- the almost descent morphisms are precisely the pullback-stable epimorphisms,

but, in general[1], the effective descent morphisms seldom have an elementary description. In fact, our story begins with the classical example of this phenomenon: the characterization of effective descent morphisms in the category Top of topological spaces, first given in [54], shows how involved such a description can get.

In [40], we find another example of the prominently challenging problem of studying effective descent morphisms. This work studies these morphisms in categories of essentially algebraic theories internal to a category $\mathscr{B}$ with finite limits. In particular, given a functor $p \colon x \to y$ of categories internal to $\mathscr{B}$, Le Creurer shows that p is effective for descent if[2].

(I) $p_1 \colon x_1 \to y_1$ is an effective descent morphism in $\mathscr{B}$,
(II) $p_2 \colon x_2 \to y_2$ is a descent morphism in $\mathscr{B}$,
(III) $p_3 \colon x_3 \to y_3$ is an almost descent morphism in $\mathscr{B}$,

where $p_n \colon x_n \to y_n$ is the component of p on the object of the n-tuples of composable morphisms (or *n-chains*). Moreover, when $\mathscr{B}$ is lextensive and has a (regular epi, mono)-factorization system, it was also verified that these criteria are necessary.

[1] If $\mathscr{C}$ is either Barr-exact [2] or locally cartesian closed, the effective descent morphisms are precisely the regular epimorphisms.

[2] Le Creurer required the component of p on objects, $p_0 \colon x_0 \to y_0$ to be effective for descent, but this was shown to be redundant in [52, Lemma A.3].

The effective descent morphisms in the category of finite ordered sets (equivalent to the category of finite topological spaces) were studied in [28], proving that a morphism $p\colon x \to y$ between finite ordered sets is effective for descent if and only if for all $a, b, c \in y$ with $a \leqslant b \leqslant c$ there exist $a', b', c' \in x$ with $a' \leqslant b' \leqslant c'$ such that $a = pa'$, $b = pb'$ and $c = pc'$. We point out the similarity of this condition with (II), as any ordered set is a category. This insight of "chain-surjectivity" led [14] to show that a suitable restatement of these conditions provided a neat perspective on the characterization of the effective descent morphisms in Top, which arise naturally once topological spaces are incarnated as a *generalized categorical structure*. Inspired by this perspective, [11, 13, 12, 15] studied the description problem of effective descent morphisms in various notions of *spaces*; when $\mathscr{V}$ is a suitable quantale, these are the so-called $(T, \mathscr{V})$*-categories*, which were introduced in a more general setting in [16]. These categories of $(T, \mathscr{V})$-categories are a notion of *enriched* generalized categorical structure, of which the category Top of topological spaces is an example. This corroborates the perspective of Lawvere, given in [39], that *fundamental structures*, such as topological spaces, are *categorical* in nature.

The insight of [40, 28, 14] supports the intuition that effective descent morphisms of categorical structures have a natural description in terms of the "chain-surjectivity" conditions (I), (II) and (III). As an example, via his study on the commutativity of bilimits, Lucatelli Nunes obtained [45, Theorem 9.10], where it was shown that the embedding of the category $\mathscr{V}$-Cat of $\mathscr{V}$-categories into the category $\mathsf{Cat}(\mathscr{V})$ of categories internal to $\mathscr{V}$

$$\mathscr{V}\text{-}\mathsf{Cat} \to \mathsf{Cat}(\mathscr{V}) \tag{1.1}$$

reflects effective descent functors, when $\mathscr{V}$ is a suitable lextensive, cartesian monoidal category. Together with the "chain-surjectivity" criteria of Le Creurer, we recover a list of criteria for such enriched $\mathscr{V}$-functors to be effective for descent.

This is where the work of the author comes in. It was shown in [51] that

$$\mathscr{V}\text{-}\mathsf{Cat} \to \mathsf{Fam}(\mathscr{V})\text{-}\mathsf{Cat}$$

reflects effective descent morphisms, when $\mathscr{V}$ is a category with finite limits ([51, Lemma 3.1]), where $\mathsf{Fam}(\mathscr{V})$ is the category of *families* of objects of $\mathscr{V}$, also known as the *free coproduct completion* of $\mathscr{V}$. The category $\mathsf{Fam}(\mathscr{V})$ is a suitable lextensive category, so that [45, Theorem 9.10] can be applied to the embedding $\mathsf{Fam}(\mathscr{V})\text{-}\mathsf{Cat} \to \mathsf{Cat}(\mathsf{Fam}(\mathscr{V}))$, confirming that it reflects effective descent morphisms. Thus, by reflecting along the composite

$$\mathscr{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{Fam}(\mathscr{V})\text{-}\mathsf{Cat} \longrightarrow \mathsf{Cat}(\mathsf{Fam}(\mathscr{V})),$$

the "chain–surjectivity" criteria of [40] allow us to describe the effective descent morphisms in $\mathscr{V}$-Cat in terms of morphisms in $\mathsf{Fam}(\mathscr{V})$ ([51, Theorem 3.3]), extending [45, Theorem 9.10] to all categories $\mathscr{V}$ with finite limits.

The conditions stated in [51, Theorem 3.3] for effective descent morphisms in $\mathscr{V}$-Cat are stated in terms of conditions on morphisms in $\mathsf{Fam}(\mathscr{V})$. This naturally prompts the study of effective descent morphisms in free coproduct completions, which was also carried out in [51]. Via these results, we can show that, when $\mathscr{V}$ is a frame, we obtain one implication of the main results of [12] ([51, Theorem 4.7]), effectively confirming that the criterion set forth by [28] are implied by the criteria of [40], confirming that both approaches to seemingly unrelated descent problems have the same underlying ideas.

The work of [14, 11, 13, 15] regards effective descent morphisms of $(T,\mathcal{V})$*-categories*, when $\mathcal{V}$ is a quantale. The conditions described therein can also be described via similar "chain-surjectivity" conditions, evidencing that this perspective on effective descent morphisms goes beyond plain categorical structures.

Multicategories are the most fundamental example of a generalized categorical structure. An illustrative example of which is the multicategory Vect of vector spaces and *multilinear* maps, that is, functions $f\colon V_1 \times \ldots \times V_n \to W$ from a finite list of vector spaces $V_1, \ldots, V_n$ to a vector space W, which are linear in each component:

$$f(v_1, \ldots, v_i + \lambda w_i, \ldots, v_n) = f(v_1, \ldots, v_i, \ldots, v_n) + \lambda f(v_1, \ldots, w_i, \ldots, v_n),$$

where v_j is a vector in V_j for each $j = 1, \ldots, n$, w_i is a vector in V_i, and λ is a scalar. This definition includes $n = 0$; in which case f consists of a vector in W.

Thus, multicategories generalize categories in the sense that the domain of a morphism consists of a finite string of objects, with an adequate notion of composition of morphisms, as well as identity morphisms, satisfying suitable associativity and unity laws. A more thorough introduction to these objects can be found in Chapter 4, along with references for further study.

More general notions of "multicategory" can be obtained by varying the "shape" of the domain of a morphism. In the case of categories, the "shape" is just an object, while in the multicategory case, the "shape" is a finite string of objects. As we have mentioned, topological spaces are generalized categorical structures. In this case, for a topological space X, the domains of the morphisms are *ultrafilters* on its underlying set of objects (points), and a morphism $\mathfrak{x} \to x$ is the *assertion* that the ultrafilter $\mathfrak{x}$ converges to the point x.[3] This perspective on topological spaces can be traced back to [3].

As in the case of plain categories, we have a notion of *internal* generalized multicategories, first considered in [9], and more recently in [24], as well as a notion of *enriched* generalized multicategories, which are the $(T,\mathcal{V})$-categories of [16]. Our goal is to obtain an uniform description of the effective descent morphisms on both accounts of generalized categorical structure.

Towards such a description, the work carried out in [52] is our first step, where we studied the effective descent morphisms of T*-categories*[4] *internal to* $\mathcal{V}$. Therein, we showed that, if $\mathcal{V}$ is a category with finite limits, then any functor of T-categories internal to $\mathcal{V}$ satisfying a suitable notion of "chain-surjectivity" conditions is an effective descent morphism ([52, Theorem 5.3]), extending the result of [40] to the setting of internal generalized categorical structures. In particular, our results provide insight into the effective descent functors of (ordinary) multicategories internal to any category with finite limits, such as Set or Top.

Based on [52], and inspired by the techniques of [45], the work developed in [53] considers the problem of reflecting effective descent morphisms along a suitable embedding "enriched → internal" in the setting of generalized multicategories, analogous to (1.1). Therein, a notion of *change-of-base*

[3]This is analogous to the notion that a ordered set X can be viewed as a category, whose morphisms $x \to y$ are the assertions that "x is related to y".

[4]Here, T is a suitable monad on a $\mathcal{V}$ with finite limits which models the shape of the domains of the morphisms.

for generalized multicategories was developed [53, Theorem 5.2], and it was shown that, under suitable conditions (for instance, when $\mathscr{V}$ is lextensive), the natural generalization of (1.1) to the setting of generalized multicategories

$$(\overline{T},\mathscr{V})\text{-}\mathsf{Cat} \to \mathsf{Cat}(T,\mathscr{V}) \tag{1.2}$$

is an embedding (Theorem 9.2) and reflects effective descent morphisms. Thus, by applying the results of [52], we obtain a description of effective descent morphisms for enriched generalized multicategories in terms of "chain-surjectivity" conditions (Theorem 10.5).

The topic of obtaining the relationship between the work of [14, 11, 13, 15] and [53] regarding effective descent morphisms of enriched generalized multicategories is still the subject of on-going work.

1.2 Descent theory with respect to a pseudofunctor

The descent problem in [23] was stated in a more general setting. For each object x in $\mathscr{C}$, we replace the category of bundles $\mathscr{C}\downarrow x$ by a category Fx of "structures" over x, and for each morphism $p\colon e\to b$, we have a change-of-base functor $p^*\colon Fb\to Fe$, replacing the pullback functor. Together with coherent isomorphisms $\mathsf{id}_x^*\cong \mathsf{id}_{Fx}$ and $(r\circ p)^*\cong p^*\circ r^*$ for a morphism $r\colon b\to c$, F defines a *pseudofunctor* $\mathscr{C}^{\mathsf{op}}\to\mathsf{CAT}$.

As was the case for the basic bifibration, we can also define a category $\mathsf{Desc}_F(p)$ of F-descent data for p for any pseudofunctor F, for which we obtain a factorization of p^* – the *F-descent factorization*:

$$\begin{array}{ccc} Fb & \xrightarrow{\quad p^* \quad} & Fe \\ & {\scriptstyle \mathscr{K}_F^p}\searrow \quad \nearrow & \\ & \mathsf{Desc}_F(p) & \end{array} \tag{1.3}$$

We say that p is an *effective F-descent* (F-descent) morphism if $\mathscr{K}_F^p$ is an equivalence (fully faithful).

The main objects of study of [56] are the (effective) descent morphisms with respect to the pseudofunctor

$$F=\mathsf{CAT}(-,\mathsf{Set})\colon \mathsf{Cat}^{\mathsf{op}}\to\mathsf{CAT}$$

of discrete opfibrations. Therein, Sobral has shown that any functor $p\colon e\to b$ between small categories has a factorization

$$\begin{array}{ccc} b & \xleftarrow{\quad p \quad} & e \\ & {\scriptstyle \phi}\nwarrow \quad \swarrow & \\ & k_p & \end{array} \tag{1.4}$$

whose image via $\mathsf{CAT}(-,\mathsf{Set})$ is equivalent to the F-descent factorization (1.3) with $F=\mathsf{CAT}(-,\mathsf{Set})$. Consequently, we have an equivalence $\theta\colon \mathsf{Desc}_F(p)\simeq\mathsf{CAT}(k_p,\mathsf{Set})$ such that $\mathscr{K}_F^p=\mathsf{CAT}(\phi,\mathsf{Set})\circ\theta$.

In this way, the relevance of the notion of *lax epimorphism* becomes transparent. We say that a functor $f\colon c\to d$ between small categories is a *lax epimorphism* [1] if $\mathsf{CAT}(f,\mathsf{Set})$ is fully faithful. Thus, it follows that p is a $\mathsf{CAT}(-,\mathsf{Set})$-descent morphism if and only if ϕ is a lax epimorphism [56, Theorem 1]. Moreover p is an effective $\mathsf{CAT}(-,\mathsf{Set})$-descent morphism if and only if $\mathsf{CAT}(\phi,\mathsf{Set})$ is an equivalence. It was shown in [56, Theorem 2] that this is the case if and only if ϕ is a *fully faithful* lax epimorphism.

The work developed in [47] aims to give a systematic view on the observations of [56], aiming to apply them in other contexts. More specifically, we confirm that the characterization given in [56] can be plainly extended to further characterize the (effective) descent morphisms with respect to the pseudofunctor $\mathsf{CAT}(-,\mathsf{Cat})\colon \mathsf{Cat}^{\mathsf{op}} \to \mathsf{CAT}$ of *split opfibrations*, see [47, Theorem 3.2].

We begin by considering a factorization as in (1.4) where k_p is the *lax codescent category* for the kernel pair of p. When a pseudofunctor $\mathscr{F}\colon \mathsf{Cat}^{\mathsf{op}} \to \mathsf{CAT}$ preserves lax descent categories, it follows that $\mathscr{K}_p^F$ is equivalent to $\mathscr{F}\phi$, reducing the study of whether p is an (effective) F-descent morphism to the study of ϕ.

The relationship of (fully faithful) lax epimorphisms with copresheaf categories influenced the study of their relationship with the *Cauchy completion* of a category. This work was carried out in the $\mathscr{V}$-enriched context, as it is suitable for future considerations, and to do so, we consider the notion of $\mathscr{V}$-fully faithful functors, and $\mathscr{V}$-lax epimorphisms, as studied in [48]. We have shown that the following are equivalent, for a $\mathscr{V}$-functor $p\colon e \to b$ between small $\mathscr{V}$-categories:

– p is a $\mathscr{V}$-fully faithful lax epimorphism,
– the functor $p^*\colon \mathfrak{C}b \to \mathfrak{C}b$ on the Cauchy completions induced by p is an equivalence,
– the change-of-base functor $p^*\colon \mathscr{V}\text{-}\mathsf{CAT}(b,\mathscr{V}) \to \mathscr{V}\text{-}\mathsf{CAT}(e,\mathscr{V})$ is an equivalence,

provided $\mathscr{V}$ is a suitable monoidal category.

We obtain the main result of [47], Theorem 3.2, by applying our characterization of $\mathscr{V}$-fully faithful lax epimorphisms when $\mathscr{V} = \mathsf{Set},\mathsf{Cat}$ to the formal considerations pertaining to the factorization (1.4).

Outline

In Chapter 2 we provide a concise introduction to descent theory. We begin by recalling the 2-dimensional limit known as *lax descent category* [42], and stating its 2-dimensional universal property in Section 2.1. Afterwards, we proceed to establish the fundamental notion of this thesis: that of *effective descent morphism* with respect to a pseudofunctor $\mathscr{C}^{\mathsf{op}} \to \mathsf{CAT}$, in Section 2.2, where we also give some remarks about the *Beck-Chevalley condition*. In Section 2.3, we focus on the *basic bifibration*, fixing several pieces of notation and describing the fundamental descent-theoretical results present in [52], [51] and [53].

Chapter 3, which covers the work done in [51], aims to study effective descent morphisms in $\mathscr{V}$-Cat for a cartesian monoidal category $\mathscr{V}$ with finite limits. Our first goal is to establish that

Theorem 1.1. $\mathscr{V}\text{-}\mathsf{Cat} \to \mathsf{Fam}(\mathscr{V})\text{-}\mathsf{Cat}$ *reflects effective descent morphisms.*

Theorem 1.1 is obtained via a series of observations on pseudopullbacks and the fact that the enrichment 2-functor preserves these 2-dimensional limits. Moreover, since $\mathsf{Fam}(\mathscr{V})$ is a suitable lextensive category, we conclude, by [45, Theorem 9.11], that $\mathsf{Fam}(\mathscr{V})\text{-}\mathsf{Cat} \to \mathsf{Cat}(\mathsf{Fam}(\mathscr{V}))$ reflects effective descent morphisms. Thus, if F is a $\mathscr{V}$-functor, we can verify whether it is effective for descent in terms of (effective, almost) descent morphisms in $\mathsf{Fam}(\mathcal{V})$ (Theorem 3.10). In turn, this motivated us to study the (effective) descent morphisms in the free coproduct completions of categories with finite limits. We apply these results when $\mathcal{V}$ is

- a (co)complete Heyting lattice, establishing the connection between the "chain-surjectivity" ideas from [40] and [28],
- a regular category, such as the categories CHaus of compact Hausdorff spaces and Stn of Stone spaces,

This aforementioned connection between [40] and [28] helps solidify our intuition and general understanding of the problem of effective descent in categorical structures. More precisely, we recover one implication of [12, Theorem 2.5] for Heyting lattices $\mathcal{V}$, which, when taking $\mathcal{V} = 2$, also recovers the "chain surjectivity" of [28], confirming the link with the approach of [40].

In Chapter 4, covering the work done in [52], we begin with an overview of the notion of generalized internal multicategories, studied in [9] and [24]. After illustrating the approach carried out in Section 4.2 for the simpler setting of *reflexive T-graphs*, we provide a description of the category $\mathsf{Cat}(T,\mathcal{V})$ of *internal T-categories* via a 2-dimensional limit, via which we describe the effective descent morphisms, Theorem 4.10. This is one of the approaches; we give a second approach to the study of effective descent morphisms in $\mathsf{Cat}(T,\mathcal{V})$ via direct calculation, closely following the ideas of [40, Chapter 3], Theorem 4.13. We finish the chapter by applying our results to various sorts of generalized multicategory, among them are the graded, operadic and enhanced multicategories.

In Chapter 5, covering the relevant definitions and the descent theoretical results of [53], we revisit (a slight generalization of) the notion of $(T,\mathcal{V})$-categories defined in [16], under the terminology *enriched $(T,\mathcal{V})$-categories*. Afterwards, we give a few concise remarks regarding the change-of-base functor "enriched $\to$ internal" in the context of generalized multicategories, mentioning only the most fundamental definitions to fix the notation, and ideas for the results. Having established the embedding under suitable conditions, we then study the problem of reflection of effective descent morphisms, and, via Theorem 4.13, we obtain our main result, Theorem 5.6, providing a description of the effective descent morphisms in the enriched generalized multicategory setting. The chapter ends with some brief comments on the scope of our results, and we list some examples.

In Chapter 6, covering the work of [47], the goal is to study the effective descent morphisms with respect to the bifibration of *split opfibrations*. The main observation is that the results of [56] on effective descent morphisms for the bifibration of *discrete opfibrations* carry over exactly to our setting (Theorem 6.11). Indeed, we verify that a functor is of effective $\mathsf{CAT}(-,\mathsf{Cat})$-descent if and only if it is of effective $\mathsf{CAT}(-,\mathsf{Set})$-descent. These results are obtained via a study of the relationship between the Cauchy completion of a category and the fully faithful, lax epimorphisms; this study was carried out in the enriched setting, alluding to future work.

List of publications

The present thesis is based on the work of the following four papers:

[47] F. Lucatelli Nunes, R. Prezado and L. Sousa. Cauchy completeness, lax epimorphisms and effective descent for split fibrations. *Bull. Belg. Math. Soc. Simon Stevin*, 30(1):130–139, (2023).

[51] R. Prezado. On effective descent $\mathscr{V}$-functors and familial descent morphisms. *J. Pure Appl. Algebra*, 228(5) (2024).

[52] R. Prezado and F. Lucatelli Nunes. Descent for internal multicategory functors. *Appl. Categor. Struct.*, 31(11) (2023).

[53] R. Prezado, F. Lucatelli Nunes. Generalized multicategories: change-of-base, embedding and descent. *DMUC preprints* 23–29, under review.

Chapter 2

A primer on descent theory

This chapter aims to give a concise introduction to classical descent theory, under a categorical point of view, as well as to uniformize the notation and gather the preliminary results from [52], [47], [51] and [53] pertaining to descent theory.

We begin by reviewing the 2-dimensional limit known as *lax descent category* [57, p. 177], [44, 42, 46] in Section 2.1, which is the fundamental notion encompassing the idea of coherence, and we state its universal property. Part of the work developed in Chapter 3 is done directly under the perspective of the lax descent category, particularly in Section 3.3.

The fundamental notion to our work, that of *effective descent morphism* with respect to a pseudofunctor $F\colon \mathscr{C}^{\mathsf{op}} \to \mathsf{CAT}$, is presented in Section 2.2. We also provide a few remarks on the *Beck-Chevalley* condition (introduced in [4]), and its importance in the relationship between monadicity and descent theory, as evidenced by the *Bénabou-Roubaud theorem*.

In Section 2.3, we begin our study of descent theory with respect to the *basic bifibration*

$$\mathscr{C} \downarrow -\colon \mathscr{C}^{\mathsf{op}} \to \mathsf{CAT}$$

associated to a category $\mathscr{C}$ with pullbacks. We establish the preliminary descent theoretical tools which are employed in Chapters 4 and 5, as well as most of Chapter 3, namely, classical results regarding the *reflection* of effective descent morphisms along an embedding $\mathscr{C} \to \mathscr{D}$ (Propositions 2.4 and Corollary 2.5), descent-theoretical results via bilimits (Propositions 2.6 and 2.7), as well as an original result, [51, Lemma 2.5], regarding preservation of descent morphisms (Lemma 2.8). We also recall results from [40] and [45] regarding effective descent morphisms of categorical structures as our starting point (Theorems 2.10 and 2.12).

We finish this preliminary chapter with Section 2.4, where we give some remarks about the descent theory with respect to the *bifibration of split opfibrations* $\mathsf{CAT}(-,\mathsf{Cat})\colon \mathsf{Cat}^{\mathsf{op}} \to \mathsf{CAT}$. Chapter 6 studies descent theory with respect to this pseudofunctor, which is an important example of a bifibration that does not satisfy the Beck-Chevalley condition.

2.1 Lax descent category

Throughout this work, we let CAT be the 2-category of (large) categories, functors and natural transformations, and we let $i\colon \mathsf{Cat} \to \mathsf{CAT}$ be the full sub-2-category of small categories.

The present definition of lax descent category follows the approach of [42, Section 1] of considering the 2-dimensional limit of a pseudofunctor $\Delta_3 \to \mathsf{CAT}$, as opposed to the approach via a 2-functor $\Delta_{\mathsf{str}} \to \mathsf{CAT}$ carried out in [46], where Δ_{str} is a strict replacement of Δ_3.

To fix the notation, we briefly recall the definition of the truncated cosimplicial diagram Δ_3 and of a pseudofunctor. We define the category Δ_3 to be generated by the following diagram

$$1 \underset{d_0}{\overset{d_1}{\rightrightarrows}} 2 \underset{d_0}{\overset{d_2}{\rightrightarrows}} 3, \qquad s_0\colon 2 \to 1, \quad d_1\colon 2 \to 3$$

with relations

$$s_0 \circ d_1 = \mathsf{id}_1, \qquad d_2 \circ d_1 = d_1 \circ d_1,$$
$$s_0 \circ d_0 = \mathsf{id}_1, \qquad d_0 \circ d_0 = d_1 \circ d_0,$$
$$d_2 \circ d_0 = d_0 \circ d_1.$$

Let $\mathscr{C}$ be a category. A *pseudofunctor* $F\colon \mathscr{C} \to \mathsf{CAT}$ consists of

- a function $\mathsf{ob}\,F\colon \mathsf{ob}\,\mathscr{C} \to \mathsf{ob}\,\mathsf{CAT}$,
- a function $F_{x,y}\colon \mathscr{C}(x,y) \to \mathsf{CAT}(Fx,Fy)$, for each pair of objects x,y in $\mathscr{C}$,
- a natural isomorphism $\mathsf{e}^F_x\colon \mathsf{id}_{Fx} \to F(\mathsf{id}_x)$ for each object x in $\mathscr{C}$,
- a natural isomorphism $\mathsf{m}^F_{f,g}\colon Fg \circ Ff \to F(g \circ f)$ for each pair of morphisms $f\colon x \to y$, $g\colon y \to z$ in $\mathscr{C}$,

such that the following diagrams commute

$$\begin{array}{ccc} Ff & \xrightarrow{\mathsf{e}^F_y \cdot Ff} & F(\mathsf{id}_y) \circ Ff \\ & \searrow{=} & \downarrow \mathsf{m}^F_{f,\mathsf{id}_y} \\ & & Ff \end{array} \qquad \begin{array}{ccc} Ff & \xrightarrow{Ff \cdot \mathsf{e}^F_x} & Ff \circ F(\mathsf{id}_x) \\ & \searrow{=} & \downarrow \mathsf{m}^F_{\mathsf{id}_x,f} \\ & & Ff \end{array}$$

$$\begin{array}{ccc} Fh \circ Fg \circ Ff & \xrightarrow{\mathsf{m}^F_{g,h} \cdot Ff} & F(h \circ g) \circ Ff \\ {\scriptstyle Fh \cdot \mathsf{m}^F_{f,g}} \downarrow & & \downarrow \mathsf{m}^F_{f,h\circ g} \\ Fh \circ F(g \circ f) & \xrightarrow[\mathsf{m}^F_{g\circ f,h}]{} & F(h \circ g \circ f) \end{array}$$

for all morphisms $f\colon x \to y$, $g\colon y \to z$ and $h\colon z \to w$.

For a pseudofunctor $F\colon \Delta_3 \to \mathsf{CAT}$, we write the underlying diagram as

$$F1 \underset{d^F_0}{\overset{d^F_1}{\rightrightarrows}} F2 \underset{d^F_0}{\overset{d^F_2}{\rightrightarrows}} F3, \qquad s^F_0\colon F2 \to F1, \quad d^F_1\colon F2 \to F3$$

and we define the following natural isomorphisms:

$$\upsilon_1^F = {\mathsf{m}^F_{d_1,s_0}}^{-1} \circ \mathsf{e}^F_1 \colon \mathsf{id} \to s_0^F d_1^F, \qquad \theta_{01}^F = {\mathsf{m}^F_{d_1,d_1}}^{-1} \circ \mathsf{m}^F_{d_1,d_2} \colon d_2^F d_1^F \to d_1^F d_1^F,$$
$$\upsilon_0^F = {\mathsf{m}^F_{d_0,s_0}}^{-1} \circ \mathsf{e}^F_1 \colon \mathsf{id} \to s_0^F d_0^F, \qquad \theta_{02}^F = {\mathsf{m}^F_{d_0,d_2}}^{-1} \circ \mathsf{m}^F_{d_1,d_0} \colon d_2^F d_0^F \to d_0^F d_1^F,$$
$$\theta_{12}^F = {\mathsf{m}^F_{d_0,d_1}}^{-1} \circ \mathsf{m}^F_{d_0,d_0} \colon d_1^F d_0^F \to d_0^F d_0^F.$$

The *lax descent category* of a pseudofunctor $F\colon \Delta_3 \to \mathsf{CAT}$ is a category $\mathsf{Desc}(F)$ whose objects, called *lax F-descent data*, are pairs (x,ϕ) where x is an object in $F1$ and $\phi\colon d_1^F(x) \to d_0^F(x)$ is a morphism in $F2$ satisfying the following *reflexivity* condition

$$\begin{array}{ccc} & x & \\ {\scriptstyle \upsilon_1^F}\swarrow & & \searrow{\scriptstyle \upsilon_0^F} \\ s_0^F d_1^F(x) & \xrightarrow[s_0^F(\phi)]{} & s_0^F d_0^F(x) \end{array} \tag{2.1}$$

and *transitivity* condition

$$\begin{array}{ccccc} & d_1^F d_1^F(x) & \xrightarrow{d_1^F(\phi)} & d_1^F d_0^F(x) & \\ {\scriptstyle \theta_{01}^F}\nearrow & & & & \searrow{\scriptstyle \theta_{12}^F} \\ d_2^F d_1^F(x) & & & & d_0^F d_0^F(x) \\ {\scriptstyle d_2^F(\phi)}\searrow & & & & \nearrow{\scriptstyle d_0^F(\phi)} \\ & d_2^F d_0^F(x) & \xrightarrow[\theta_{02}^F]{} & d_0^F d_1^F(x) & \end{array} \tag{2.2}$$

Given lax F-descent data (x,ϕ), (y,ψ), a morphism $\omega\colon (x,\phi) \to (y,\psi)$ in $\mathsf{Desc}(F)$ of lax F-descent data consists of a morphism $\omega\colon x \to y$ such that the following diagram commutes:

$$\begin{array}{ccc} d_1^F(x) & \xrightarrow{\phi} & d_0^F(x) \\ {\scriptstyle d_1^F(\omega)}\downarrow & & \downarrow{\scriptstyle d_0^F(\omega)} \\ d_1^F(y) & \xrightarrow[\psi]{} & d_0^F(y) \end{array}$$

2.1.1 Universal property

Associated to the lax descent category, we have a forgetful functor $X\colon \mathsf{Desc}(F) \to F1$ given on objects by $(x,\phi) \mapsto x$, and a natural transformation $\Phi\colon d_1^F X \to d_0^F X$ given by $\Phi_{(x,\phi)} = \phi$. This pair (X,Φ) makes the following diagrams commute

$$\begin{array}{ccc} & X & \\ {\scriptstyle \upsilon_1^F \cdot X}\swarrow & & \searrow{\scriptstyle \upsilon_0^F \cdot X} \\ s_0^F d_1^F X & \xrightarrow[s_0^F \cdot \Phi]{} & s_0^F d_0^F X \end{array}$$

$$\begin{array}{ccccccc} & & d_1^F d_1^F X & \xrightarrow{d_1^F \cdot \Phi} & d_1^F d_0^F X & & \\ & {\scriptstyle \theta_{01}^F \cdot X}\nearrow & & & & \searrow{\scriptstyle \theta_{12}^F \cdot X} & \\ d_2^F d_1^F X & & & & & & d_0^F d_0^F X \\ & {\scriptstyle d_2^F \cdot \Phi}\searrow & & & & \nearrow{\scriptstyle d_0^F \cdot \Phi} & \\ & & d_2^F d_0^F X & \xrightarrow[\theta_{02}^F \cdot X]{} & d_0^F d_1^F X & & \end{array}$$

which is just a restatement of the conditions (2.1) and (2.2).

If (Y,Ψ) is a pair where $Y: \mathscr{A} \to F1$ is a functor and $\Psi: d_1^F Y \to d_0^F Y$ is a natural transformation satisfying

$$\begin{array}{ccc} & Y & \\ {\scriptstyle \upsilon_1^F \cdot Y}\swarrow & & \searrow{\scriptstyle \upsilon_0^F \cdot Y} \\ s_0^F d_1^F Y & \xrightarrow[s_0^F \cdot Y]{} & s_0^F d_0^F Y \end{array} \tag{2.3}$$

$$\begin{array}{ccccccc} & & d_1^F d_1^F Y & \xrightarrow{d_1^F \cdot \Psi} & d_1^F d_0^F Y & & \\ & {\scriptstyle \theta_{01}^F \cdot Y}\nearrow & & & & \searrow{\scriptstyle \theta_{12}^F \cdot Y} & \\ d_2^F d_1^F Y & & & & & & d_0^F d_0^F Y \\ & {\scriptstyle d_2^F \cdot \Psi}\searrow & & & & \nearrow{\scriptstyle d_0^F \cdot \Psi} & \\ & & d_2^F d_0^F Y & \xrightarrow[\theta_{02}^F \cdot Y]{} & d_0^F d_1^F Y & & \end{array} \tag{2.4}$$

then there is a unique $G: \mathscr{A} \to \mathsf{Desc}(F)$ such that

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{Y} & F1 \\ & {\scriptstyle G}\searrow \quad \nearrow{\scriptstyle X} & \\ & \mathsf{Desc}(F) & \end{array} \tag{2.5}$$

commutes, and $\Psi = \Phi \cdot G$.

Let (Z,Ξ) be another pair where $Z: \mathscr{A} \to F1$ is a functor and $\Xi: d_1^F Z \to d_0^F Z$ is a natural transformation satisfying (2.3) and (2.4), and let $H: \mathscr{A} \to \mathsf{Desc}(F)$ be the unique functor such that $Z = XH$ and $\Xi = \Phi \cdot H$. For any natural transformation $\Gamma: Y \to Z$ such that

$$\begin{array}{ccc} d_1^F Y & \xrightarrow{\Psi} & d_0^F Y \\ {\scriptstyle d_1^F \cdot \Gamma}\downarrow & & \downarrow{\scriptstyle d_0^F \cdot \Gamma} \\ d_1^F Z & \xrightarrow[\Xi]{} & d_0^F Z \end{array}$$

commutes, there is a unique $\hat{\Gamma}: G \to H$ such that $X \cdot \hat{\Gamma} = \Gamma$.

2.2 Effective descent morphisms

Let $\mathscr{C}$ be a category with pullbacks, and $p\colon e \to b$ be a morphism in $\mathscr{C}$. The *kernel pair* of p, given by the pullback

$$\begin{array}{ccc} p \times_b p & \xrightarrow{d_1} & e \\ {\scriptstyle d_0}\big\downarrow & \ulcorner & \big\downarrow{\scriptstyle p} \\ e & \xrightarrow[p]{} & b \end{array} \tag{2.6}$$

induces an *equivalence relation* internal to $\mathscr{C}$, given by the following diagram

$$p \times_b p \times_b p \quad \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \quad p \times_b p \quad \substack{\xrightarrow{d_1} \\ \longleftarrow \\ \xrightarrow[d_0]{}} \quad e, \tag{2.7}$$

which we denote by $\mathsf{Ker}(p)\colon \Delta_3^{\mathsf{op}} \to \mathscr{C}$.

For any pseudofunctor $F\colon \mathscr{C}^{\mathsf{op}} \to \mathsf{CAT}$, we let $f^* = Ff$ for every morphism f in $\mathscr{C}$, and $\overline{F}_p = F \circ \mathsf{Ker}(p)^{\mathsf{op}}$. We write

$$\mathsf{Desc}_F(p) = \mathsf{Desc}(\overline{F}_p) \tag{2.8}$$

for the category of lax descent data of $\overline{F}_p$. Moreover, we observe that the kernel pair (2.6) induces a pair $(p^*\colon Fb \to Fe,\ \Omega\colon d_1^* p^* \to d_0^* p^*)$ satisfying (2.3) and (2.4), where $\Omega = {\mathsf{m}^F_{p,d_0}}^{-1} \circ \mathsf{m}^F_{p,d_1}$. Thus, by the universal property (2.5), we obtain a unique functor $\mathscr{K}^p_F\colon Fb \to \mathsf{Desc}_F(p)$ such that the following diagram commutes

$$\begin{array}{ccccc} Fb & & \xrightarrow{\quad p^* \quad} & & Fe \\ & {\scriptstyle \mathscr{K}^p_F}\searrow & & \nearrow{\scriptstyle \mathscr{U}^p_F} & \\ & & \mathsf{Desc}_F(p) & & \end{array} \tag{2.9}$$

and $\Omega = \Phi \cdot p^*$, where $(\mathscr{U}^p_F, \Phi)$ is the pair associated to the lax descent category of $\overline{F}_p$. We call (2.9) the *F-descent factorization* of p.

We now reach the most important definition in this work. A morphism p in a category $\mathscr{C}$ with pullbacks is said to be

- an *almost F-descent morphism* if $\mathscr{K}^p_F$ is faithful,
- an *F-descent morphism* if $\mathscr{K}^p_F$ is fully faithful,
- an *effective F-descent morphism* if $\mathscr{K}^p_F$ is an equivalence.

2.2.1 Beck-Chevalley condition

The *Beck-Chevalley condition*, introduced in [4], underlies the relationship between monadicity and descent theory. We will give a brief remark on this topic; we recall that a pseudofunctor $F\colon \mathscr{C}^{\mathsf{op}} \to \mathsf{CAT}$ is said to be a *bifibration* if, for every morphism f in $\mathscr{C}$, the functor f^* has a left adjoint, which we denote by $f_! \dashv f^*$, and whose unit and counit are denoted by η^f and ε^f.

In this context, a bifibration is said to satisfy the *Beck-Chevalley condition* if, for all pullback squares in $\mathscr{C}$

$$\begin{array}{ccc} w & \xrightarrow{h} & x \\ {\scriptstyle k}\big\downarrow & \ulcorner & \big\downarrow{\scriptstyle f} \\ y & \xrightarrow[g]{} & z \end{array}$$

the following natural transformation

$$h_!k^* \xrightarrow{h_!k^*\eta^g} h_!k^*g^*g_! \xrightarrow{h_!\theta g_!} h_!h^*f^*g_! \xrightarrow{\varepsilon^h f^* g_!} f^*g_!$$

is a natural isomorphism, where $\theta\colon k^*g^* \to h^*f^*$ is the induced isomorphism by the commutative square (2.2.1).

Theorem 2.1 (Bénabou-Roubaud [4])**.** *Let $F\colon \mathscr{C}^{\mathsf{op}} \to \mathsf{CAT}$ be a bifibration satisfying the Beck-Chevalley condition. For a morphism $p\colon e \to b$ in $\mathscr{C}$, we write T^p for the monad induced by the adjunction $p_! \dashv p^*$.*

The F-descent factorization (2.9) *of p is equivalent to the Eilenberg-Moore factorization of p^*:*

$$\begin{array}{ccc} Fb & \xrightarrow{\quad p^* \quad} & Fe \\ & {\scriptstyle \mathscr{K}^p}\searrow \quad \nearrow{\scriptstyle \mathscr{U}^p} & \\ & T^p\text{-}\mathsf{Alg} & \end{array} \tag{2.10}$$

so, in particular, we have an equivalence of categories

$$\mathsf{Desc}_F(p) \simeq T^p\text{-}\mathsf{Alg},$$

and the following are equivalent:

(i) *p is an effective F-descent morphism (resp. F-descent morphism).*
(ii) *p^* is monadic (resp. premonadic).*

This result was generalized in [45, Theorem 7.4], via the study of commutativity of bilimits, and the main result of [42] (Theorem 4.7) confirms that (i) $\implies$ (ii), for *all* bifibrations (that is, not necessarily satisfying the Beck-Chevalley condition).

2.3 Basic bifibration

Let $\mathscr{C}$ be a category with pullbacks. For each morphism $f\colon x \to y$, we have a functor $f_!\colon \mathscr{C}\downarrow x \to \mathscr{C}\downarrow y$ given on objects by $g \mapsto f\circ g$. For each morphism $h\colon z \to y$, we consider the following pullback diagram:

$$\begin{array}{ccc} f^*(z) & \xrightarrow{\varepsilon^f_h} & z \\ {\scriptstyle f^*(h)}\big\downarrow & \ulcorner & \big\downarrow{\scriptstyle h} \\ x & \xrightarrow[f]{} & y \end{array}$$

We observe that $\varepsilon^f_h\colon f\circ f^*(h)\to h$ is a morphism in $\mathscr{C}\downarrow y$, whose universal property borne out of the pullback diagram guarantees that the assignment $f^*\colon \mathscr{C}\downarrow y\to\mathscr{C}\downarrow x$ defines a functor right adjoint to $f_!$.

Together with the canonical isomorphisms $\mathsf{id}^*_x\cong\mathsf{id}_{\mathscr{C}\downarrow x}$ and $f^*g^*\cong(g\circ f)^*$, we obtain the *basic bifibration*

$$\mathscr{C}\downarrow -\colon \mathscr{C}^{\mathsf{op}}\to\mathsf{CAT}$$
$$x\mapsto\mathscr{C}\downarrow x$$
$$f\colon x\to y\mapsto f^*\colon\mathscr{C}\downarrow y\to\mathscr{C}\downarrow x$$

which provides the context for our study of descent theory in Chapters 3, 4, and 5. In this setting, we write $\mathsf{Desc}(p)$ instead of $\mathsf{Desc}_{\mathscr{C}\downarrow -}(p)$ for the category of lax descent data, and we say "(effective/almost) descent morphism" instead of "(effective/almost) $(\mathscr{C}\downarrow -)$-descent morphism".

When $\mathscr{C}$ is a category with finite limits, it can be shown (see [30, 2.4], [40, Corollary 0.3.5], [29, Theorem 3.4], [52, Proposition 2.1]) via Beck's monadicity theorem that

- p is an almost descent morphism $\iff$ p is a pullback-stable epimorphism,
- p is a descent morphism $\iff$ p is a pullback-stable regular epimorphism.

While it is true that descent morphisms are effective for descent when $\mathscr{C}$ is Barr-exact [2] or locally cartesian closed, in general, effective descent morphisms are challenging to describe. For instance, we note the characterization of [54] for $\mathscr{C}=\mathsf{Top}$, or the characterization of [40] for $\mathscr{C}=\mathsf{Cat}(\mathscr{V})$, for suitable categories $\mathscr{V}$.

Nevertheless, the study of effective descent morphisms can be approached directly, by studying the (essential) image of (fully faithful) comparison functors $\mathscr{K}^p$ for descent morphisms p. Thus, the following elementary observation regarding the (essential) image of $\mathscr{K}^p$ is of particular interest for our work.

Proposition 2.2 ([52, Corollary 2.3]). *The comparison functor $\mathscr{K}^p$ is essentially surjective if and only if, for all descent data (a,γ), there exists a morphism $f\colon w\to b$ such that $p^*f\cong a$ in $\mathscr{C}\downarrow x$, and*

$$\varepsilon^p_f\circ\gamma=\varepsilon^p_f\circ\varepsilon^p_{p\circ a}. \tag{2.11}$$

Proof. We begin by noting that $\mathscr{K}^pf=(p^*f,p^*_{\varepsilon^p_f})$ is a lax descent datum satisfying (2.11), by naturality.

Conversely, if (p^*f,γ) satisfies (2.11), then

$$\gamma=p^*\varepsilon^p_f\circ\eta^p_{p^*f}\circ\gamma=p^*\varepsilon^p_f\circ p^*\gamma\circ\eta^p_{p^*(p\circ p^*f)}=p^*\varepsilon^p_f\circ p^*\varepsilon^p_{p\circ a}\circ\eta^p_{p^*(p\circ p^*f)}=p^*\varepsilon^p_f,$$

hence $(p^*f,\gamma)=\mathscr{K}^pf$. □

Remark 2.3. We should point out that Proposition 2.2 is often implicitly used in the study of effective descent morphisms (for instance, [40, Proposition 3.2.4]). Moreover, it can be shown that this result also holds in the general context for a pseudofunctor $F\colon\mathscr{C}^{\mathsf{op}}\to\mathsf{CAT}$, so its applicability in descent arguments does not rely on the Beck-Chevalley condition.

Another fruitful strategy, undertaken by both [54] and [40], and justified by the following Proposition 2.4, is to suitably embed the category whose effective descent morphisms we wish to study in a larger one in which those are well-understood.

Proposition 2.4 ([30, 2.7])**.** *Let $U\colon \mathscr{C} \to \mathscr{D}$ be a fully faithful, pullback-preserving functor between categories with pullbacks, and let $p\colon x \to y$ be a morphism such that Up is an effective descent morphism. Then p is an effective descent morphism if and only if, for all pullback diagrams of the form*

$$\begin{array}{ccc} Uv & \longrightarrow & w \\ \big\downarrow & \ulcorner & \big\downarrow \\ Ux & \xrightarrow[Up]{} & Uy \end{array} \tag{2.12}$$

we have $w \cong Uz$ for some object z of $\mathscr{C}$.

Throughout our study of effective descent morphisms in categorical structures, we have found the following particular instance of Proposition 2.4 to be particularly useful:

Corollary 2.5 ([52, Corollary 2.5])**.** *Let $U\colon \mathscr{C} \to \mathscr{D}$ be a fully faithful, pullback-preserving functor between categories with pullbacks. If for every effective descent morphism $g\colon Ux \to z$ there exists an isomorphism $z \cong Uy$, then U reflects effective descent morphisms.*

Proof. We recall that effective descent morphisms are stable under pullback. Thus, if (2.12) is a pullback square, and Up is an effective descent morphism, then so is $Uv \to w$ by pullback-stability. By hypothesis, there exists an isomorphism $w \cong Uy$, whence we conclude that p is effective for descent by Proposition 2.4. □

As one of the byproducts of the study of commutativity of bilimits carried out in [45], Lucatelli Nunes provides a description of the effective descent morphisms for the bilimit of a diagram of categories with pullbacks and pullback-preserving functors, in terms of the (effective) descent morphisms of the categories in the underlying diagram. A particularly important consequence is that this provides a second, widely applicable approach to the study of effective descent morphisms, as exhibited by the following Propositions:

Proposition 2.6 ([45, Theorem 1.6, Corollary 9.6])**.** *If we have a pseudopullback diagram of categories with pullbacks and pullback-preserving functors*

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{F} & \mathscr{B} \\ {\scriptstyle G}\big\downarrow & \cong & \big\downarrow{\scriptstyle H} \\ \mathscr{C} & \xrightarrow[K]{} & \mathscr{D} \end{array}$$

and a morphism f in $\mathscr{A}$ such that

- *Ff and Gf are effective descent morphisms, and*
- *$KGf \cong HFf$ is a descent morphism,*

then f is an effective descent morphism.

Proposition 2.7 ([45, Theorem 9.2])**.** *If we have a pseudoequalizer diagram of categories with pullback and pullback-preserving functors*

$$\mathscr{A} \xrightarrow{F} \mathscr{B} \underset{H}{\overset{G}{\rightrightarrows}} \mathscr{C}$$

and a morphism f in $\mathscr{A}$ such that

- *Ff is an effective descent morphism, and*
- *$GFf \cong HFf$ is a descent morphism,*

then f is an effective descent morphism.

To apply Proposition 2.6, the following result is useful:

Lemma 2.8 ([51, Lemma 2.5])**.** *Let $H\colon \mathscr{B} \to \mathscr{D}$ be a functor between categories with finite limits. If H preserves coequalizers, reflects pullbacks, and has a fully faithful left adjoint $L\colon \mathscr{D} \to \mathscr{B}$, then H preserves descent morphisms.*

Proof. Let $h\colon x \to y$ be a descent morphism in $\mathscr{B}$, and we consider the following pullback diagram:

$$\begin{array}{ccc} (Hh)^*z & \xrightarrow{\varepsilon_\phi} & z \\ {\scriptstyle (Hh)^*\phi}\downarrow & \ulcorner & \downarrow{\scriptstyle \phi} \\ Hx & \xrightarrow[Hh]{} & Hy \end{array}$$

Since the unit $\mathrm{id} \to HL$ is an isomorphism, the following is a pullback diagram as well:

$$\begin{array}{ccc} HL(Hh)^*z & \xrightarrow{HL\varepsilon_\phi} & HLz \\ {\scriptstyle H\psi^\sharp}\downarrow & \ulcorner & \downarrow{\scriptstyle H\phi^\sharp} \\ Hx & \xrightarrow[Hh]{} & Hy \end{array}$$

where $\phi^\sharp\colon Lz \to y$ is obtained from ϕ via the hom-isomorphism $\mathscr{D}(z, Hy) \cong \mathscr{B}(Lz, y)$, and, likewise, $\psi^\sharp\colon L(Hh)^*z \to x$ is obtained from $\psi = (Hh)^*\phi$.

Since H reflects pullbacks, we conclude $L\varepsilon_\phi$ is a regular epimorphism. This property is preserved by H, as it preserves coequalizers. Thus, we conclude that Hh is a pullback-stable regular epimorphism, as desired. □

Remark 2.9. The applications of Lemma 2.8 we have in mind are

- the underlying object-of-objects functor $(-)_0\colon \mathsf{Cat}(\mathscr{V}) \to \mathscr{V}$,
- the canonical fibration $\mathsf{Fam}(\mathscr{V}) \to \mathsf{Set}$,
- and the underlying object-of-objects functor $(-)_0\colon \mathsf{Cat}(T, \mathscr{V}) \to \mathscr{V}$.

Each functor has fully faithful left and right adjoints, and therefore satisfies the hypotheses of Lemma 2.8. Hence, we conclude that each functor preserves descent morphisms.

Thanks to the first functor, we can obtain the conclusion of [45, Theorem 9.11] for $\mathcal{V}$ a lextensive category such that the functor $-\cdot 1\colon \mathsf{Set} \to \mathcal{V}$[1], is fully faithful, without assuming $\mathcal{V}$ has a (regular epi, mono)-factorization system, using precisely the same proof.

The second functor plays an important role in Chapter 3 in obtaining effective descent morphisms in $\mathcal{V}$-Cat, in the more general setting of a cartesian monoidal category $\mathcal{V}$ with finite limits; this is Theorem 3.10.

The third functor plays a similar role in Chapter 5, to obtain effective descent morphisms in a suitable category of generalized enriched multicategories. The statement of this result is given by Theorem 5.6.

2.3.1 Descent theory in categorical structures

In the context of effective descent morphisms in categorical structures, the results of Le Creurer [40] are the cornerstone upon which we obtain our own. Indeed, since enriched categorical structures can be embedded into an internal setting, under suitable conditions, the general strategy is to first study the effective descent morphisms for internal categorical structures directly (such results are given by Theorems 2.10 and 4.13), and then apply Propositions 2.4 and 2.6 to study whether the embedding reflects the effective descent morphisms back to the enriched setting.

Theorem 2.10 ([40, Corollary 3.3.1])**.** *Let $\mathcal{V}$ be a category with finite limits, and $p\colon \mathcal{C} \to \mathcal{D}$ be a functor of categories internal to $\mathcal{V}$. If*

- $p_0\colon \mathcal{C}_0 \to \mathcal{D}_0$ *is an effective descent morphism,*
- $p_1\colon \mathcal{C}_1 \to \mathcal{D}_1$ *is an effective descent morphism,*
- $p_2\colon \mathcal{C}_2 \to \mathcal{D}_2$ *is a descent morphism, and*
- $p_3\colon \mathcal{C}_3 \to \mathcal{D}_3$ *is an almost descent morphism,*

then p is an effective descent morphism in $\mathsf{Cat}(\mathcal{V})$.

Remark 2.11. Let $\mathcal{C}$ be a category internal to $\mathcal{V}$. Since Theorem 2.10 is stated in terms of various epimorphic conditions on the objects on composable tuples of morphisms, it is convenient for $\mathcal{C}_n$ to denote the object of *n-chains* of $\mathcal{C}$, therefore allowing "n-chain" to be synonymous with "composable n-tuple of morphisms".

In this vein, let $\mathcal{W}$ be a monoidal category, and let $\mathcal{D}$ be an enriched $\mathcal{W}$-category. For objects x_i on $\mathcal{D}$ for $i = 0, 1, 2, 3$, we let

$$\mathcal{D}(x_0,x_1,x_2) = \mathcal{D}(x_1,x_2) \otimes \mathcal{D}(x_0,x_1), \quad \text{and} \quad \mathcal{D}(x_0,x_1,x_2,x_3) = \mathcal{D}(x_1,x_2,x_3) \otimes \mathcal{D}(x_0,x_1).$$

Likewise, if $F\colon \mathcal{W} \to \mathcal{X}$ is a lax monoidal functor, we write

$$\mathsf{m}^F\colon (F_!\mathcal{D})(x_0,x_1,x_2) \to F(\mathcal{D}(x_0,x_1,x_2))$$

for the "shortened" version of the comparison morphism for the tensor product.

[1]This is the left adjoint to $\mathcal{V}(1,-)\colon \mathcal{V} \to \mathsf{Set}$

As an example of a reflection result, we have the following consequence of [45, Theorem 9.11] and Theorem 2.10 (see also Remark 2.9):

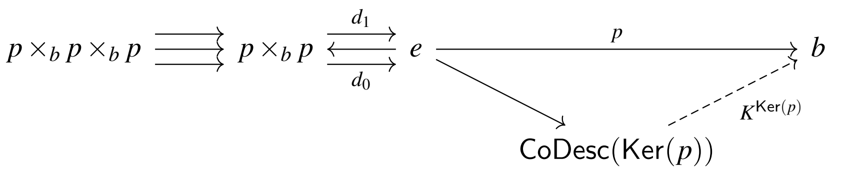
Theorem 2.12 ([45, Theorem 9.11])**.** *Let $\mathscr{V}$ be a lextensive category, and assume that $-\cdot 1\colon \mathsf{Set} \to \mathscr{V}$ is fully faithful. An enriched $\mathscr{V}$-functor $F\colon \mathscr{C} \to \mathscr{D}$ such that*

– *F_0 is surjective,*
– *$F_1 \cdot 1\colon \sum_{x_i} \mathscr{C}(x_0,x_1) \to \sum_{y_i} \mathscr{D}(y_0,y_1)$ is an effective descent morphism,*
– *$F_2 \cdot 1\colon \sum_{x_i} \mathscr{C}(x_0,x_1,x_2) \to \sum_{y_i} \mathscr{D}(y_0,y_1,y_2)$ is a descent morphism,*
– *$F_3 \cdot 1\colon \sum_{x_i} \mathscr{C}(x_0,x_1,x_2,x_3) \to \sum_{y_i} \mathscr{D}(y_0,y_1,y_2,y_3)$ is an almost descent morphism,*

is an effective descent morphism in $\mathscr{V}$-Cat.

2.4 Bifibration of split opfibrations

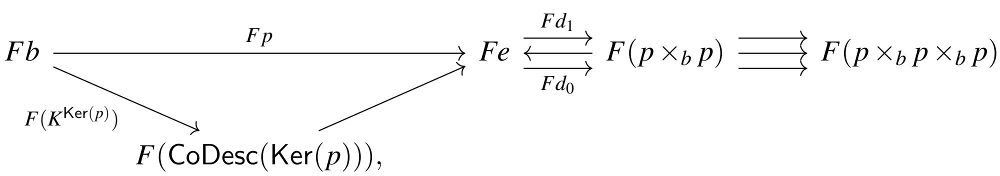
Let $\mathbb{A}$ be a 2-category with 2-pullbacks and lax codescent objects[2]. Just like in Section 2.3, we consider the kernel pair $\mathsf{Ker}(p)$ (2.7) of a morphism $p\colon e \to b$ in $\mathbb{A}$. Since $\mathbb{A}$ has lax codescent objects, there is a unique $K^{\mathsf{Ker}(p)}\colon \mathsf{CoDesc}(\mathsf{Ker}(p)) \to b$ making the triangle in Diagram (2.13) commute:



(2.13)

Lemma 2.13 ([47, Lemma 1.1])**.** *If a 2-functor $F\colon \mathbb{A}^{\mathsf{op}} \to \mathsf{CAT}$ preserves lax descent objects, then a morphism $p\colon e \to b$ is of effective F-descent (F-descent) if and only if $F(K^{\mathsf{Ker}(p)})$ is an equivalence (fully faithful).*

Proof. When such a 2-functor F is composed with Diagram (2.13), we obtain



(2.14)

and we observe that $F(\mathsf{CoDesc}(\mathsf{Ker}(p))) \simeq \mathsf{Desc}(F(\mathsf{Ker}(p)))$, from which our result follows. □

Naturally, the representable 2-functors $\mathbb{A}(-,a)\colon \mathbb{A}^{\mathsf{op}} \to \mathsf{CAT}$ for each object a preserve lax descent objects. The bifibrations F, F_D of split, respectively discrete, opfibrations are obtained by composing $\mathsf{CAT}(-,\mathsf{Cat})$, respectively $\mathsf{CAT}(-,\mathsf{Set})$, with the inclusion $\mathsf{Cat} \to \mathsf{CAT}$.

We remark that these bifibrations do not satisfy the Beck-Chevalley condition, as the conclusion of the Bénabou-Roubaud theorem [4] does not hold for F nor F_D. Indeed, [56, Remark 7] gives an example of a functor p such that $F_D p$ is monadic, but p is not an effective F_D-descent morphism.

[2] This is the notion dual to *lax descent object*, which can be described in any 2-category, via the universal property in Subsection 2.1.1.

We provide another example, given in [47, Remark 3.3]: let $p\colon 1 \to b$ be a functor, where 1 is the terminal category. The functor $Fp = \mathsf{CAT}(p, \mathsf{Cat})$ is monadic if and only if p is (essentially) surjective, but, as a consequence of Theorem 6.11, p is an effective F-descent morphism if and only if p is an equivalence.

Chapter 3

Enriched $\mathscr{V}$-functors

Effective descent morphisms for the category $\mathscr{V}$-Cat of $\mathscr{V}$-categories were studied in [12, Section 5] when $\mathscr{V}$ is a (co)complete symmetric monoidal closed thin category, and in [45, Theorem 9.11], when $\mathscr{V}$ is a lextensive, cartesian monoidal category such that the copower functor $- \cdot 1 \colon \mathsf{Set} \to \mathscr{V}$ is fully faithful. Despite the different approaches to the problem, in both works, the conditions for a $\mathscr{V}$-functor F to be an effective descent morphism in $\mathscr{V}$-Cat are expressed in terms of surjectivity of F in chains of hom-objects.

The goal of this chapter is to prove that the same conditions remain sufficient for a $\mathscr{V}$-functor to be effective for descent when $\mathscr{V}$ is a cartesian monoidal category with finite limits, placing the results of [12, Section 5] (when $\mathscr{V}$ is cartesian monoidal) and [45] on common ground. We shall prove that if a $\mathscr{V}$-functor F is

- an effective descent morphism on hom-objects,
- a descent morphism on 2-chains of hom-objects,
- an almost descent morphism on 3-chains of hom-objects,

in a suitable sense, then F is an effective descent morphism in $\mathscr{V}$-Cat (Theorem 3.10).

There are two key ideas for this result: if $\mathscr{V}$ is a category with finite limits, then

- $\mathsf{Fam}(\mathscr{V})$ is a lextensive category, and $- \cdot 1 \colon \mathsf{Set} \to \mathsf{Fam}(\mathscr{V})$ is fully faithful (Proposition 3.1), so that we may apply Theorem 2.12 to obtain a description of the effective descent morphisms in $\mathsf{Fam}(\mathscr{V})$-Cat, and
- the direct image $\eta_! \colon \mathscr{V}\text{-}\mathsf{Cat} \to \mathsf{Fam}(\mathscr{V})\text{-}\mathsf{Cat}$ of $\eta \colon \mathscr{V} \to \mathsf{Fam}(\mathscr{V})$ reflects effective descent morphisms.

Thus, we conclude that the composite functor reflects effective descent morphisms as well. We go over the first idea in Section 3.1, where we review the central properties of the free coproduct completion of a category. The second idea is obtained via Proposition 2.6, so, in Section 3.2 we study the relevant pseudopullbacks and their preservation by the enrichment 2-functor $\mathscr{V} \mapsto \mathscr{V}\text{-}\mathsf{Cat}$.

Via the description of the effective descent morphisms in $\mathsf{Fam}(\mathscr{V})$-Cat given by Lucatelli Nunes's result (Theorem 2.12), we can state the conditions for a $\mathscr{V}$-functor F to be an effective descent morphism in $\mathscr{V}$-Cat in terms of (effective, almost) descent conditions on the underlying morphisms of F on chains of hom-objects. Since these refer to morphisms in the category $\mathsf{Fam}(\mathscr{V})$, this motivates the study of (effective) descent morphisms in the free coproduct completion of a category, which is carried out in Section 3.3.

We highlight the relationship between the results of [12] and [45] in Section 3.4, where we apply our results on (effective) descent morphisms in $\mathsf{Fam}(\mathscr{V})$ to study effective descent morphisms in $\mathscr{V}$-Cat for special families of categories $\mathscr{V}$. Among such categories, we draw our attention to the category CHaus of compact Hausdorff spaces and the category Stn of Stone spaces, giving a description of effective descent CHaus-functors and effective descent Stn-functors.

3.1 Properties of the free coproduct completion

Let $\mathscr{V}$ be a category. The *free coproduct cocompletion* of $\mathscr{V}$, denoted $\mathsf{Fam}(\mathscr{V})$, consists of

- objects which given by set-indexed families $(X_j)_{j\in J}$ of objects X_j in $\mathscr{V}$,
- morphisms $(X_j)_{j\in J} \to (Y_k)_{k\in K}$ which are given by a function $f\colon J \to K$, and a set-indexed family of morphisms $(\phi_j\colon X_j \to Y_{fj})_{j\in J}$, with ϕ_j in $\mathscr{V}$,

with suitable identities and composition law. It may also be obtained via the Grothendieck construction [22] of the pseudofunctor $\mathsf{Set}^{\mathsf{op}} \to \mathsf{CAT}$ given by $X \mapsto \mathscr{V}^X$ on objects and $f \mapsto f^*$ on morphisms, whose fibration we denote by $P\colon\ \mathsf{Fam}(\mathscr{V}) \to \mathsf{Set}$.

We recall the following properties:

Proposition 3.1 ([51, Lemma 3.2])**.** *Let $\mathscr{V}$ be a category.*

(a) $\mathsf{Fam}(\mathscr{V})$ *is extensive.*
(b) If $\mathscr{V}$ has a terminal object, $-\cdot 1\colon \mathsf{Set} \to \mathsf{Fam}(\mathscr{V})$ *is fully faithful.*
(c) If $\mathscr{V}$ has finite limits, so does $\mathsf{Fam}(\mathscr{V})$.

Proof. Property (a) is well-known, and is present in [10, Proposition 2.4], for instance. Moreover, it was verified that (b) holds in [7, Proposition 6.2.1]. Property (c) is also well-known; see [21, 25, 7]. Nevertheless, to illustrate the mechanics of $\mathsf{Fam}(\mathscr{V})$, we will revisit the arguments. Recall that

- Set has all (finite) limits,
- $\mathscr{V}^X$ has all finite limits, given componentwise in $\mathscr{V}$,
- the change-of-base functor $f^*\colon \mathscr{V}^Y \to \mathscr{V}^X$ preserves finite limits, for every function $f\colon X \to Y$,

so, if $\mathscr{A}\colon \mathscr{J} \to \mathsf{Fam}(\mathscr{V})$ is a finite diagram, with $\mathscr{A}_j = (P\mathscr{A}_j, x_j)$, we consider the limit cone $\lambda_j\colon\ \mathsf{lim}(P\mathscr{A}) \to P\mathscr{A}j$, and we define a diagram

$$\begin{aligned}\Phi\colon\ \mathscr{J} &\to \mathscr{V}^{\mathsf{lim}(P\mathscr{A})}\\ j &\mapsto \lambda_j^*(x_j)\end{aligned}$$

and since $\mathscr{V}^{\mathsf{lim}(P\mathscr{A})}$ has finite limits, $\mathsf{lim}\,\Phi$ exists. To verify that $\mathsf{lim}\,\mathscr{A} \cong (\mathsf{lim}\,P\mathscr{A}, \mathsf{lim}\,\Phi)$, given a cone $(\gamma_j\colon b \to P\mathscr{A}_j, \zeta_j\colon w \to \gamma_j^* x_j)$, we let $\omega\colon b \to \mathsf{lim}\,P\mathscr{A}$ be the unique function such that $\gamma_j = \lambda_j \circ \omega$, and we observe that $\omega^*(\mathsf{lim}\,\Phi) \cong \mathsf{lim}\,\omega^*\Phi$, since ω^* preserves limits. Then, there exists a unique $\xi\colon w \to \omega^*(\mathsf{lim}\,\Phi)$ such that $\zeta_j = \mathsf{m}^{\mathscr{V}^-}_{\lambda_j,\omega} \circ \omega^*\phi_j \circ \xi$, where $\phi_j\colon\ \mathsf{lim}\,\Phi \to x_j$ is the limit cone of Φ, and $\mathsf{m}^{\mathscr{V}^-}_{\lambda_j,\omega}\colon \omega^*\lambda_j^* \cong (\lambda_j \circ \omega)^*$ is the natural comparison isomorphism of the pseudofunctor.

Indeed, we have $(\gamma_j, \zeta_j) = (\lambda_j, \phi_j) \circ (\omega, \xi)$, and if $(\gamma_j, \zeta_j) = (\lambda_j, \phi_j) \circ (\theta, \chi)$, then $\gamma_j = \lambda_j \circ \theta$ which confirms $\theta = \omega$, and $\zeta_j = \mathsf{m}^{\mathscr{V}-}_{\lambda_j,\omega} \circ \omega^* \phi_j \circ \chi$, confirming $\chi = \xi$. □

In particular, it follows that

Corollary 3.2 ([51, p. 10])**.** *The functor* $\mathsf{Fam}(\mathscr{V})\text{-}\mathsf{Cat} \to \mathsf{Cat}(\mathsf{Fam}(\mathscr{V}))$ *reflects effective descent morphisms.*

Proof. Since $\mathsf{Fam}(\mathscr{V})$ is lextensive and $- \cdot 1 \colon \mathsf{Set} \to \mathsf{Fam}(\mathscr{V})$ is fully faithful by Proposition 3.1, we may apply Theorem 2.12. □

This result was the original motivation to study the problem of whether $\mathscr{V}\text{-}\mathsf{Cat} \to \mathsf{Fam}(\mathscr{V})\text{-}\mathsf{Cat}$ reflects effective descent morphisms as well. In this direction, we recall from [60] that the inclusion $\mathscr{V} \to \mathsf{Fam}(\mathscr{V})$ is a 2-cartesian natural transformation:

Proposition 3.3 ([60, 5.15 Proposition])**.** *For any functor* $F \colon \mathscr{V} \to \mathscr{W}$*, the following diagram*

$$\begin{array}{ccc} \mathscr{V} & \xrightarrow{\eta} & \mathsf{Fam}(\mathscr{V}) \\ {\scriptstyle F}\downarrow & \ulcorner & \downarrow{\scriptstyle \mathsf{Fam}(F)} \\ \mathscr{W} & \xrightarrow[\eta]{} & \mathsf{Fam}(\mathscr{W}) \end{array} \tag{3.1}$$

is a 2-pullback.

In particular, we note there is a unique functor $! \colon \mathscr{V} \to 1$, and $\mathsf{Fam}(!) \colon \mathsf{Fam}(\mathscr{V}) \to \mathsf{Fam}(1) \simeq \mathsf{Set}$ is the fibration associated to $\mathsf{Fam}(\mathscr{V})$. Thus, by [58], (3.1) is a pseudopullback when $\mathscr{W} \simeq 1$.

The last preliminary result we shall need is the following reformulation of [7, Proposition 6.1.5]:

Proposition 3.4. *Let* $\mathscr{A}$ *be a category with coproducts. If* $\mathscr{C}$ *is a full subcategory of* $\mathscr{A}$ *such that*

(i) for all objects a *of* $\mathscr{A}$*, there exists a set* J *and objects* c_j *of* $\mathscr{C}$ *for each* $j \in J$ *such that* $a \cong \sum_{j\in J} c_j$*,*
(ii) for all objects x*,* d_k *of* $\mathscr{C}$ *for each* $k \in K$*, any morphism* $x \to \sum_{k\in K} d_k$ *factors uniquely via* $\iota_k \colon d_k \to \sum_{k\in K} d_k$ *for some* $k \in K$*,*

then we have an equivalence $\mathsf{Fam}(\mathscr{C}) \simeq \mathscr{A}$*.*

Under the above hypotheses, we conclude that $\mathscr{A}$ is an extensive category, and that $\mathscr{C}$ (essentially) consists of the *connected* objects of $\mathscr{A}$ – that is, those objects $a \in \mathscr{A}$ such that $\mathscr{A}(a, -)$ preserves coproducts.

Proof. We have a coproduct functor

$$\sum \colon \mathsf{Fam}(\mathscr{C}) \to \mathscr{A} \tag{3.2}$$

defined on objects by $(c_j)_{j\in J} \mapsto \sum_{j\in J} c_j$. We observe that (3.2) is essentially surjective if and only if (i) holds.

On morphisms, (3.2) is given by the composite

$$\begin{aligned}\mathsf{Fam}(\mathscr{C})((c_j)_{j\in J},(d_k)_{k\in K}) &\cong \prod_{j\in J}\sum_{k\in K}\mathscr{C}(c_j,d_k)\\ &\cong \prod_{j\in J}\sum_{k\in K}\mathscr{A}(c_j,d_k)\\ &\to \prod_{j\in J}\mathscr{A}\big(c_j,\sum_{k\in K}d_k\big)\\ &\cong \mathscr{A}\big(\sum_{j\in J}c_j,\sum_{k\in K}d_k\big)\end{aligned}$$

where $\mathscr{A}(c_j,d_k)\to\mathscr{A}\big(c_j,\sum_{k\in K}d_k\big)$ is given by $\mathscr{A}(c_j,\iota_k)$. We observe that (ii) holds if and only if

$$\sum_{k\in K}\mathscr{A}(x,d_k)\to\mathscr{A}\big(x,\sum_{k\in K}d_k\big)$$

is an isomorphism, so we conclude that (3.2) is fully faithful if and only if (ii) holds. □

One consequential application of Proposition 3.4 is that it allows us to reduce the study of (effective, almost) descent morphisms in $\mathsf{Fam}(\mathscr{V})$ to the study of *covers*, that is, morphisms in $\mathsf{Fam}(\mathscr{V})$ of the form $\phi\colon (X_j)_{j\in J}\to Y$.

Lemma 3.5. *Let $\mathscr{E}$ be a pullback-stable class of morphisms in $\mathsf{Fam}(\mathscr{V})$ that is closed under coproducts (as a full subcategory of $[2,\mathscr{V}]$). We have an equivalence of categories $\mathscr{E}\simeq\mathsf{Fam}(\mathscr{E}_{\mathsf{conn}})$, where $\mathscr{E}_{\mathsf{conn}}\subseteq\mathscr{E}$ is the subclass of covers in $\mathscr{E}$; that is, those morphisms of the form $\phi\colon (X_i)_{j\in J}\to Y$.*

Proof. If $(f,\psi)\colon (X_j)_{j\in J}\to(Y_k)_{k\in K}$ is in $\mathscr{E}$, we consider the following pullback

$$\begin{array}{ccc}(X_j)_{j\in f^*k} & \xrightarrow{\psi|_k} & Y_k\\ \downarrow & \ulcorner & \downarrow\\ (X_j)_{j\in J} & \xrightarrow[(f,\psi)]{} & (Y_k)_{k\in K}\end{array}$$

for each $k\in K$. By pullback stability, we find that $\psi|_k\in\mathscr{E}_{\mathsf{conn}}$ for all $k\in K$, and $\psi\cong\sum_{k\in K}\psi|_k$.

If $\phi\colon (V_i)_{i\in I}\to W$ is in $\mathscr{E}_{\mathsf{conn}}$, and we have a commutative square

$$\begin{array}{ccc}(V_i) & \xrightarrow{(g,\chi)} & (X_j)_{j\in J}\\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle(f,\psi)}\\ W & \xrightarrow[(k,\omega)]{} & (Y_k)_{k\in K}\end{array}\tag{3.3}$$

then $f(g(i))=k$, so that $g(i)\in f^*k$ for all $i\in I$, and hence (3.3) factors uniquely as

$$\begin{array}{ccccc}(V_i) & \xrightarrow{(g,\chi)} & (X_j)_{j\in f^*k} & \xrightarrow{\iota_k} & (X_j)_{j\in J}\\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi|_k} & & \downarrow{\scriptstyle(f,\psi)}\\ W & \xrightarrow[\omega]{} & Y_k & \xrightarrow[\iota_k]{} & (Y_k)_{k\in K}\end{array}$$

so we may apply Proposition 3.4. □

3.2 Embedding $\mathscr{V}$-Cat $\to$ Fam($\mathscr{V}$)-Cat

In [19, Corollary 3.8], the authors have shown that the enrichment 2-functor $(-)$-Cat: Bicat $\to$ 2-CAT preserves all weighted, connected 2-limits, where Bicat is the 2-category of bicategories, pseudofunctors and icons. Denoting by SymCat the 2-category of symmetric monoidal categories, monoidal functors and monoidal natural transformations, we obtain the following corollary:

Proposition 3.6 ([19, Corollary 3.8])**.** *The enrichment 2-functor* SymCat $\to$ CAT *preserves pseudopullbacks.*

Proof. Pseudopullbacks are Cat-connected limits. □

It is known that a morphism $p\colon a \to b$ in a 2-category is fully faithful if and only if $p \downarrow p \simeq 2 \pitchfork a$[1]. However, it was shown in [19] that powers and comma objects are *not* Cat-connected limits. Despite this, we can still obtain the following result:

Lemma 3.7 ([51, Lemma 2.2])**.** *The enrichment 2-functor* SymCat $\to$ CAT *preserves fully faithful functors.*

Proof. Let $F\colon \mathscr{V} \to \mathscr{W}$ be a fully faithful monoidal functor, and let $\mathscr{C}$, $\mathscr{D}$ be $\mathscr{V}$-categories. A $\mathscr{W}$-functor $\Psi\colon F_!\mathscr{C} \to F_!\mathscr{D}$ consists of

- A function $\mathsf{ob}\,\Psi\colon \mathsf{ob}\,\mathscr{C} \to \mathsf{ob}\,\mathscr{D}$ on objects,
- A morphism $\Psi_{x,y}\colon F\mathscr{C}(x,y) \to F\mathscr{D}(\Psi x, \Psi y)$ in $\mathscr{W}$ for each pair $x, y \in \mathsf{ob}\,\mathscr{C}$.

We claim that we have a $\mathscr{V}$-functor $\Phi\colon \mathscr{C} \to \mathscr{D}$ given $\mathsf{ob}\,\Phi = \mathsf{ob}\,\Psi$ and $\Phi_{x,y}$ is the unique morphism $\mathscr{C}(x,y) \to \mathscr{D}(\Psi x, \Psi y)$ such that $F\Phi_{x,y} = \Psi_{x,y}$ for all $x, y \in \mathsf{ob}\,\mathscr{C}$, by full faithfulness of F.

Recalling that $\mathsf{e}^{F_!\mathscr{X}} = F\,\mathsf{e}^{\mathscr{X}} \circ \mathsf{e}^F$ and $\mathsf{c}^{F_!\mathscr{X}} = F\,\mathsf{c}^{\mathscr{X}} \circ \mathsf{m}^F$ for any $\mathscr{V}$-category $\mathscr{X}$, we note that the following diagrams commute

$$
\begin{array}{ccc}
 & I & \\
 & \downarrow{\scriptstyle \mathsf{e}^F} & \\
 & FI & \\
{\scriptstyle F\mathsf{e}^{\mathscr{C}}}\swarrow & & \searrow{\scriptstyle F\mathsf{e}^{\mathscr{D}}} \\
F\mathscr{C}(x,y) & \xrightarrow[F\Phi_{x,y}]{} & F\mathscr{D}(\Phi x, \Phi y)
\end{array}
\qquad
\begin{array}{ccc}
(F_!\mathscr{C})(x,y,z) & \xrightarrow{(F\Phi)_{x,y,z}} & (F_!\mathscr{D})(\Phi x, \Phi y, \Phi z) \\
\downarrow{\scriptstyle \mathsf{m}^F} & & \downarrow{\scriptstyle \mathsf{m}^F} \\
F(\mathscr{C}(x,y,z)) & \xrightarrow{F(\Phi_{x,y,z})} & F(\mathscr{D}(\Phi x, \Phi y, \Phi z)) \\
\downarrow{\scriptstyle F\mathsf{c}^{\mathscr{C}}} & & \downarrow{\scriptstyle F\mathsf{c}^{\mathscr{D}}} \\
F\mathscr{C}(x,z) & \xrightarrow[F\Phi_{x,z}]{} & F\mathscr{D}(\Phi x, \Phi z)
\end{array}
$$

since Ψ is a $\mathscr{W}$-functor. Since e^F and m^F are invertible, and F is fully faithful, we deduce that Φ must be a $\mathscr{V}$-functor, as desired. □

[1] $f \downarrow g$ is the *comma object* of two morphisms f, g in a 2-category $\mathbb{A}$. $\mathscr{C} \pitchfork x$ is the *power object* of an object x in $\mathbb{A}$ by a category $\mathscr{C}$.

Lemma 3.8 ([51, p. 9])**.** *Let $\mathscr{V}$ be a category with finite limits. We have a pseudopullback diagram*

$$\begin{array}{ccc} \mathscr{V}\text{-}\mathsf{Cat} & \xrightarrow{\eta_!} & \mathsf{Fam}(\mathscr{V})\text{-}\mathsf{Cat} \\ \downarrow & \cong & \downarrow \\ \mathsf{Set} & \longrightarrow & \mathsf{Set}\text{-}\mathsf{Cat} \end{array} \tag{3.4}$$

of categories with pullbacks and pullback-preserving functors.

Proof. We observe that (3.4) is the composite of the enrichment 2-functor with the diagram

$$\begin{array}{ccc} \mathscr{V} & \xrightarrow{\eta} & \mathsf{Fam}(\mathscr{V}) \\ \downarrow & \ulcorner & \downarrow \\ 1 & \longrightarrow & \mathsf{Set} \end{array}$$

which is a 2-pullback by [60, Proposition 5.15], and since $\mathsf{Fam}(\mathscr{V}) \to \mathsf{Set}$ is an (iso)fibration, it is in fact a pseudopullback by [58]. Now the result follows by Proposition 3.6. ☐

Theorem 3.9 ([51, Lemma 3.1])**.** *If $\mathscr{V}$ has finite limits, $\eta_! \colon \mathscr{V}\text{-}\mathsf{Cat} \to \mathsf{Fam}(\mathscr{V})\text{-}\mathsf{Cat}$ reflects effective descent morphisms.*

Proof. Let $F \colon \mathscr{C} \to \mathscr{D}$ be a $\mathscr{V}$-functor such that $\eta_! F$ is an effective descent $\mathsf{Fam}(\mathscr{V})$-functor. Since $\mathsf{Fam}(\mathscr{V}) \to \mathsf{Set}$ has fully faithful left and right adjoints, the same holds for $\mathsf{Fam}(\mathscr{V})\text{-}\mathsf{Cat} \to \mathsf{Set}\text{-}\mathsf{Cat}$, since enrichment is a 2-functor which preserves fully faithful functors by Lemma 3.7.

Thus, $\mathsf{Fam}(\mathscr{V})\text{-}\mathsf{Cat} \to \mathsf{Set}\text{-}\mathsf{Cat}$ maps $\eta_! F$ to a descent functor, which, in turn, is reflected along $\mathsf{Set} \to \mathsf{Set}\text{-}\mathsf{Cat}$ to its underlying function on objects, which must be surjective. Since surjections are effective descent morphisms in Set, we may apply Proposition 2.6 to conclude that F is an effective descent functor. ☐

Now, by applying Lucatelli Nunes's criteria for the effective descent morphisms (Theorem 2.12) to Theorem 3.9, we obtain (see also Lemma 3.5):

Theorem 3.10 ([51, Theorem 3.3])**.** *If $\mathscr{V}$ is a category with finite limits, and $F \colon \mathscr{C} \to \mathscr{D}$ is a $\mathscr{V}$-functor such that*

(I) $F \colon (\mathscr{C}(x_0,x_1))_{x_i \in F^* y_i} \to \mathscr{D}(y_0,y_1)$ is an effective descent morphism,
(II) $F \times F \colon (\mathscr{C}(x_0,x_1,x_2))_{x_i \in F^* y_i} \to \mathscr{D}(y_0,y_1,y_2)$ is a descent morphism, and
(III) $F \times F \times F \colon (\mathscr{C}(x_0,x_1,x_2,x_3))_{x_i \in F^* y_i} \to \mathscr{D}(y_0,y_1,y_2,y_3)$ is an almost descent morphism

in the category $\mathsf{Fam}(\mathscr{V})$ for all $y_0, y_1, y_2, y_3 \in \mathscr{V}$, then F is an effective descent morphism in $\mathscr{V}\text{-}\mathsf{Cat}$.

Unlike Le Creurer's and Lucatelli Nunes's results, which describe the effective descent morphisms in $\mathsf{Cat}(\mathscr{V})$ and $\mathscr{V}$-Cat in terms of morphisms in $\mathscr{V}$, the description we provide for the effective descent morphisms of $\mathscr{V}$-Cat (Theorem 3.10) relies on understanding (effective, almost) descent morphisms in $\mathsf{Fam}(\mathscr{V})$, prompting their study.

3.3 Familial descent and effective descent morphisms

By Lemma 3.5, we learn that when studying (effective, almost) descent morphisms in $\mathsf{Fam}(\mathscr{V})$, it is enough to the consider covers $\phi\colon (X_j)_{j\in J} \to Y$.

There is not much to say about (pullback-stable) epimorphisms in this generality; they are, tautologically, the jointly epimorphic covers (preserved by pullbacks).

Regarding (effective) descent morphisms, it is useful to compute the kernel pair of a cover $\phi\colon (X_j)_{j\in J} \to Y$; we use the construction described in Proposition 3.1. For each $j,k \in J$, we have a pullback diagram

$$\begin{array}{ccc} \phi_j \times_Y \phi_k & \xrightarrow{\delta_{1,j,k}} & X_j \\ {\scriptstyle \delta_{0,j,k}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \phi_j} \\ X_k & \xrightarrow[\phi_k]{} & Y \end{array} \tag{3.5}$$

so the kernel pair of ϕ is given by

$$(\phi_j \times_Y \phi_k)_{j,k\in J\times J} \underset{(d_0,\delta_0)}{\overset{(d_1,\delta_1)}{\rightrightarrows}} (X_j)_{j\in J}$$

where $d_i\colon J\times J \to J$ discards the ith component, for $i = 0$, 1.

We also define a category $\mathscr{D}_J$ with set of objects $J + J^2$, and for each $j,k \in J$, two different morphisms $(j,k) \to j$ and $(j,k) \to k$.

For each cover $\phi\colon (X_j)_{j\in J} \to Y$, we define a diagram $K_\phi\colon \mathscr{D}_J \to \mathscr{V}$, mapping

- $(j,k) \to j$ to $\delta_{1,j,k}\colon \phi_j \times_Y \phi_k \to X_j$
- $(j,k) \to k$ to $\delta_{0,j,k}\colon \phi_j \times_Y \phi_k \to X_k$,

We recall that in a category with finite limits, regular epimorphisms are the coequalizers of their kernel pairs. With that in mind, we may obtain the following result:

Lemma 3.11 ([51, Lemma 4.1]). *A cover ϕ is a (pullback-stable) regular epimorphism if and only if* $\mathsf{colim}\, K_\phi \cong Y$ *(and the colimit is stable).*

Proof. We shall assume J is non-empty throughout. We have a natural isomorphism

$$[\rightrightarrows, \mathsf{Fam}(\mathscr{V})](\ker\phi, \Delta_{(Z_k)_{k\in K}}) \cong \sum_{k\in K} [\mathscr{D}_J, \mathscr{V}](K_\phi, \Delta_{Z_k}),$$

and to verify this, note that the composite of $\ker\phi$ with $\mathsf{Fam}(\mathscr{V}) \to \mathsf{Set}$ is given by

$$J\times J \underset{d_0}{\overset{d_1}{\rightrightarrows}} J,$$

whose coequalizer is terminal. Since the data for a natural transformation $\ker\phi \to \Delta_{(Z_k)_{k\in K}}$ consists of a morphism $(f,\psi)\colon (X_j)_{j\in J} \to (Z_k)_{k\in K}$ such that

$$(f,\chi)\circ(d_1,\delta_1) = (f,\psi)\circ(d_0,\delta_0),$$

there exists a unique $k \in K$ such that $fj = k$ for all $j \in J$, and $\chi_j \circ \delta_{1,i,j} = \chi_i \circ \delta_{0,i,j}$ for all $i, j \in J$. This data corresponds to a unique natural transformation $K_\phi \to \Delta_{Z_k}$.

So, if K_ϕ has a colimit, then

$$\begin{aligned}\mathsf{Fam}(\mathscr{V})(\operatorname{colim} K_\phi, (Z_k)_{k\in K}) &\cong \sum_{k\in K} \mathscr{V}(\operatorname{colim} K_\phi, Z_k) \\ &\cong \sum_{k\in K} [\mathscr{D}_J, \mathscr{V}](K_\phi, \Delta_{Z_k}) \\ &\cong [\rightrightarrows, \mathsf{Fam}(\mathscr{V})](\ker\phi, \Delta_{(Z_k)_{k\in K}}),\end{aligned}$$

hence $\ker\phi$ has a colimit and $\operatorname{colim}\ker\phi \cong \operatorname{colim} K_\phi$. Conversely, if $\ker\phi$ has a colimit, then its underlying set is the coequalizer of

$$J \times J \overset{d_1}{\underset{d_0}{\rightrightarrows}} J,$$

which is the terminal object, hence $\ker\phi$ is connected and can be identified with an object of $\mathscr{V}$. Thus,

$$\begin{aligned}\mathscr{V}(\operatorname{colim}(\ker\phi), Z) &\cong \mathsf{Fam}(\mathscr{V})(\operatorname{colim}(\ker\phi), Z) \\ &\cong [\rightrightarrows, \mathsf{Fam}(\mathscr{V})](\ker\phi, \Delta_Z) \\ &\cong [\mathscr{D}_J, \mathscr{V}](K_\phi, \Delta_Z),\end{aligned}$$

so we conclude that $\operatorname{colim}\ker\phi \cong \operatorname{colim} K_\phi$.

If the colimit of K_ϕ is stable under pullback, for each $j \in J$ we write

$$\begin{array}{ccc} V_j & \xrightarrow{\psi_j} & Z \\ {\scriptstyle \phi_j^*(\omega)}\downarrow & \ulcorner & \downarrow{\scriptstyle \omega} \\ X_j & \xrightarrow[\phi_j]{} & Y \end{array}$$

for the pullback of ϕ_j and a morphism $\omega\colon Z \to Y$ for each $j \in J$, so that $Z \cong \operatorname{colim} K_\psi$, hence $Z \cong \operatorname{colim}\ker\psi$. So, if we have a morphism $\xi : (W_l)_{l\in L} \to Y$, we can do this procedure for ξ_l for each $l \in L$, and then take the coproduct of the results, confirming the pullback stability of the colimit of $\ker\phi$.

Conversely, if the colimit of $\ker\phi$ is stable under pullback, for any morphism $\omega\colon Z \to Y$, we may consider the pullback

$$\begin{array}{ccc} (V_j) & \xrightarrow{\psi} & Z \\ {\scriptstyle \phi_j^*(\omega)}\downarrow & \ulcorner & \downarrow{\scriptstyle \omega} \\ X_j & \xrightarrow[\phi_j]{} & Y \end{array}$$

and since $Y \cong \operatorname{colim}\ker\psi \cong \operatorname{colim} K_\psi$, we immediately conclude that the colimit of K_ϕ is stable under pullback. $\square$

To understand effective descent morphisms in $\mathsf{Fam}(\mathscr{V})$, we recall that lax descent data for a cover $\phi\colon (X_j)_{j\in J} \to Y$ consists of

– a morphism $(p,\pi)\colon (W_k)_{k\in K} \to (X_j)_{j\in J}$ in $\mathsf{Fam}(\mathscr{V})$,
– and a morphism $(\gamma,\Gamma)\colon D_1^*(p,\pi) \to D_0^*(p,\pi)$ in $\mathsf{Fam}(\mathscr{V}) \downarrow (\phi_i \times_Y \phi_j)_{i,j\in J\times J}$

satisfying reflexivity (2.1) and transitivity (2.2) conditions, namely,

– $\nu_0 = S_0^*(\gamma,\Gamma)\circ \nu_1$,
– $\theta_{01}\circ D_1^*(\gamma,\Gamma)\circ\theta_{12} = D_0^*(\gamma,\Gamma)\circ\theta_{02}\circ D_2^*(\gamma,\Gamma)$,

Since $\mathsf{Fam}(\mathscr{V}) \to \mathsf{Set}$ preserves pullbacks, we recover descent data (p,π) for the unique morphism $J\to 1$, implying that $K\cong I\times J$ for some set I, and, under this isomorphism, we have $p\cong d_0\colon I\times J\to J$. Moreover, we note that the underlying Set-pullbacks of $D_i\colon (\phi_j\times_Y \phi_k)_{j,k\in J\times J} \to (X_j)_{j\in J}$ for $i=0,1$ are given by

$$\begin{array}{ccc} K\times J & \xrightarrow{d_1} & K \\ {\scriptstyle p\times\mathsf{id}}\downarrow & \ulcorner & \downarrow{\scriptstyle p} \\ J\times J & \xrightarrow[d_1]{} & J \end{array} \qquad \begin{array}{ccc} J\times K & \xrightarrow{d_0} & K \\ {\scriptstyle \mathsf{id}\times p}\downarrow & \ulcorner & \downarrow{\scriptstyle p} \\ J\times J & \xrightarrow[d_0]{} & J \end{array}$$

and since $(\mathsf{id}\times p)\circ\gamma = p\times\mathsf{id}$, we deduce that γ is isomorphic to the function

$$I\times J\times J \to J\times I\times J$$
$$(i,j,k)\mapsto (j,i,k),$$

and the reflexivity and transitivity conditions are given in components by

$$\begin{array}{ccc} & W_{ij} & \\ {\scriptstyle \nu_{1,j}}\swarrow & & \searrow{\scriptstyle \nu_{0,j}} \\ \sigma_{0,j}^*\delta_{1,j,j}^*(\pi_{i,j}) & \xrightarrow[\sigma_{0,j}^*(\Gamma_{i,j,j})]{} & \sigma_{0,j}^*(\delta_{0,j,j}^*(\pi_{i,j}) \end{array}$$

$$\begin{array}{ccccc} & \delta_{1,j,k,l}^*(\delta_{1,j,l}^*(\pi_{i,j})) & \xrightarrow{\delta_{1,j,k,l}^*(\Gamma_{i,j,l})} & \delta_{1,j,k,l}^*(\delta_{0,j,l}^*(\pi_{i,l})) & \\ {\scriptstyle \theta_{12,j,k,l}(\pi_{i,j})}\nearrow & & & & \searrow{\scriptstyle \theta_{01,j,k,l}(\pi_{i,l})} \\ \delta_{2,j,k,l}^*(\delta_{1,j,l}^*(\pi_{i,j})) & & & & \delta_{0,j,k,l}^*(\delta_{0,j,l}^*(\pi_{i,l})) \\ {\scriptstyle \delta_{2,j,k,l}^*(\Gamma_{i,j,k})}\searrow & & & & \nearrow{\scriptstyle \delta_{0,j,k,l}^*(\Gamma_{i,k,l})} \\ & \delta_{2,j,k,l}^*(\delta_{0,j,l}^*(\pi_{i,k})) & \xrightarrow[\theta_{02,j,k,l}(\pi_{i,k})]{} & \delta_{0,j,k,l}^*(\delta_{1,j,l}^*(\pi_{i,k})) & \end{array} \qquad (3.6)$$

for each $i\in I$, $j,k,l\in J$. This observation allows us to prove the following result:

Lemma 3.12 ([51, Lemma 4.2])**.** *Let $\phi\colon (X_j)_{j\in J}\to Y$ be a cover in $\mathsf{Fam}(\mathscr{V})$. We have an equivalence*

$$\mathsf{Desc}(\phi) \simeq \mathsf{Fam}(\mathsf{Desc}_{\mathsf{conn}}(\phi)), \qquad (3.7)$$

where $\mathsf{Desc}_{\mathsf{conn}}(\phi)$ is the full subcategory of $\mathsf{Desc}(\phi)$ consisting of the connected objects.

Proof. If $(p,\pi)\colon (W_{i,j})_{i,j\in I\times J} \to (X_j)_{j\in J}$ and (γ,Γ) is a lax descent datum for ϕ as given above, then for each $i\in I$, we define $W_{i,-} = (W_{i,j})_{j\in J}$, as well as a morphism $(\iota_i,\mathsf{id})\colon W_{i,-}\to (W_{i,j})_{i,j\in I\times J}$, where $\iota_i(j) = (i,j)$.

We note that the composites $(p,\pi)\circ(\iota_i,\mathsf{id}) = (\mathsf{id},\pi_{i,-})$, and $(\mathsf{id},\Gamma_{i,-.-})\colon D_1^*(\mathsf{id},\pi_{i,-}) \to D_0^*(\mathsf{id},\pi_{i,-})$ constitute a lax descent datum for ϕ, for each $i \in I$. Indeed, Diagrams (3.3) and (3.6) commute for each fixed $i \in I$, confirming reflexivity and transitivity for each component.

Thus, the lax descent datum (p,π), (γ,Γ) is the coproduct of the lax descent data $(\mathsf{id},\pi_{i,-})$, $(\mathsf{id},\Gamma_{i,-})$ for each $i \in I$.

Now, we let

$$(\mathsf{id},\xi)\colon (V_j)_{j\in J} \to (X_j)_{j\in J}, \qquad (\mathsf{id},\Xi)\colon (\mathsf{id},\delta_1^*\circ\xi) \to (\mathsf{id},\delta_0^*\circ\xi)$$

be a connected lax descent datum, and let $(g,\chi)\colon (V_j)_{j\in J} \to (W_{i,j})_{i,j\in I\times J}$ be a morphism such that $(d_0,\pi)\circ(g,\chi) = (\mathsf{id},\xi)$ and the following diagram

$$\begin{array}{ccc} D_1^*(\mathsf{id},\xi) & \xrightarrow{(\mathsf{id},\Xi)} & D_0^*(\mathsf{id},\xi) \\ {\scriptstyle D_1^*(g,\chi)}\big\downarrow & & \big\downarrow{\scriptstyle D_0^*(g,\chi)} \\ D_1^*(d_0,\pi) & \xrightarrow[(\gamma,\Gamma)]{} & D_0^*(d_0,\pi) \end{array}$$

commutes. Since $p\circ g = \mathsf{id}$, we conclude that $g(j) = (h(j),j)$ for a function $h\colon J \to I$, and since $\gamma(h(j),j,k) = (j,h(k),k)$ for all $j,k \in J$, we conclude that h is constant; let $i \in I$ be its value, so that $g = \iota_i$. We obtain a factorization

$$(V_j)_{j\in J} \xrightarrow{(\mathsf{id},\chi_{i,-})} (W_{i,j})_{j\in J} \xrightarrow{(\iota_i,\mathsf{id})} (W_{i,j})_{i,j\in I\times J},$$

so, we may apply Proposition 3.4 to conclude our proof. □

In fact, by noticing that the underlying objects, morphisms and properties for each connected descent datum lie in a slice category of a fiber of $\mathsf{Fam}(\mathscr{V}) \to \mathsf{Set}$, we deduce that:

Corollary 3.13. *The category* $\mathsf{Desc}_{\mathsf{conn}}(\phi)$ *is the lax descent category of the following diagram:*

$$\mathscr{V}^J \downarrow (X_j)_{j\in J} \underset{D_0^*}{\overset{D_1^*}{\underset{\leftarrow S_0^* -}{\rightrightarrows}}} \mathscr{V}^{J\times J} \downarrow (\phi_i\times\phi_j)_{i,j\in J\times J} \underset{D_0^*}{\overset{D_2^*}{\underset{- D_1^* \to}{\rightrightarrows}}} \mathscr{V}^{J\times J\times J} \downarrow (\phi_i \times_Y \phi_j \times_Y \phi_k)_{i,j,k\in J\times J\times J}. \qquad (3.8)$$

Remark 3.14. This corollary provides us with another point of view; let $\mathscr{D}_J^+$ be the category whose set of objects is given by $J+J^2+J^3$, containing $\mathscr{D}_J$ as a subcategory, such that for each triple $i,j,k \in J$, we have three distinct morphisms $(i,j,k)\to(j,k)$, $(i,j,k)\to(i,k)$ and $(i,j,k)\to(i,j)$, such that the following diagrams commute:

$$\begin{array}{ccc} (i,j,k) & \longrightarrow & (i,j) \\ \downarrow & & \downarrow \\ (i,k) & \longrightarrow & i \end{array} \qquad \begin{array}{ccc} (i,j,k) & \longrightarrow & (j,k) \\ \downarrow & & \downarrow \\ (i,j) & \longrightarrow & j \end{array} \qquad \begin{array}{ccc} (i,j,k) & \longrightarrow & (i,k) \\ \downarrow & & \downarrow \\ (j,k) & \longrightarrow & k \end{array}$$

and for each $j \in J$, a morphism $j \to (j,j)$ such that both of the composites below are the identity:

$$j \longrightarrow (j,j) \rightrightarrows j$$

We define a diagram $K_\phi^+ \colon \mathscr{D}_J^+ \to \mathscr{V}$, extending K_ϕ, where $(i,j,k) \mapsto \phi_i \times_Y \phi_j \times_Y \phi_k$, and the morphisms from objects in J^3 to J^2 are mapped to the respective projections, while the morphisms from J to J^2 are mapped to the respective diagonals $\sigma_{0,i} \colon X_i \to \phi_i \times_Y \phi_i$. It can be shown that the 2-limit of the composite

$$(\mathscr{D}_J^+)^{\mathsf{op}} \xrightarrow{(K_\phi^+)^{\mathsf{op}}} \mathscr{V}^{\mathsf{op}} \xrightarrow{\mathscr{V}\downarrow -} \mathsf{Cat}$$

is equivalent to $\mathsf{Desc}_{\mathsf{conn}}(\phi)$; by taking products of categories, we recover Diagram (3.8).

Theorem 3.15 ([51, Theorem 4.3])**.** *Let $\phi \colon (X_j)_{j\in J} \to Y$ be a cover in $\mathsf{Fam}(\mathscr{V})$. The following are equivalent:*

(i) ϕ is an effective descent morphism.
(ii) We have an equivalence $\mathscr{V} \downarrow Y \simeq \mathsf{Desc}_{\mathsf{conn}}(\phi)$.

Proof. First, we observe that $\mathsf{Fam}(\mathscr{V} \downarrow Y) \simeq \mathsf{Fam}(\mathscr{V}) \downarrow Y$, since for any morphism $\phi \colon (W_j)_{i\in J} \to Y$, we have $\phi = \coprod_{j\in J} \phi_j$ as objects in $\mathsf{Fam}(\mathscr{V}) \downarrow Y$, and if we have a commutative triangle

$$\begin{array}{ccc} W & \xrightarrow{(f,\omega)} & (X_j)_{j\in J} \\ & {\scriptstyle\psi}\searrow \quad \swarrow{\scriptstyle\phi} & \\ & Y & \end{array}$$

then there exists a unique $j \in J$ (given by f) factoring $(f,\omega) = (\iota_j, \mathsf{id}) \circ \omega$ uniquely, so we may apply Theorem 3.4.

Since the comparison functor $\mathscr{K}^{\mathsf{Ker}(\phi)} \colon \mathsf{Fam}(\mathscr{V}) \downarrow Y \to \mathsf{Desc}(\phi)$ preserves connected objects, we obtain an equivalence

$$\mathscr{K}^{\mathsf{Ker}(\phi)} \simeq \mathsf{Fam}(\mathscr{K}_{\mathsf{conn}}^{\mathsf{Ker}(\phi)}), \tag{3.9}$$

where $\mathscr{K}_{\mathsf{conn}}^{\mathsf{Ker}(\phi)} \colon \mathscr{V} \downarrow Y \to \mathsf{Desc}_{\mathsf{conn}}(\phi)$ is the restriction of $\mathscr{K}^{\mathsf{Ker}(\phi)}$ to the connected objects.

We have (i) $\implies$ (ii), since we have (3.9), and Fam reflects equivalences, as the unit is 2-cartesian. Conversely, (ii) $\implies$ (i) follows immediately by (3.9). □

3.4 Examples

The study of descent morphisms in $\mathsf{Fam}(\mathscr{V})$ for certain categories $\mathscr{V}$ with finite limits inspired us to highlight the following specialization of Lemma 3.11:

Lemma 3.16. *Let $\phi \colon (X_j)_{j\in J} \to Y$ be a cover in $\mathsf{Fam}(\mathscr{V})$ such that for all $j \in J$, ϕ_j is a monomorphism; that is, $(X_j)_{j\in J}$ is a family of subobjects of Y. If the kernel pair of ϕ has a coequalizer, then $\mathsf{colim}\, K_\phi \cong \bigvee_{j\in J} X_j$ as a subobject of Y, where K_ϕ is given as in Lemma 3.11.*

Proof. Let $\xi \colon \mathsf{colim}\, K_\phi \to Y$ be the unique morphism such that $\phi_j = \xi \circ q_j$ for all $j \in J$, where $q \colon (X_j)_{j\in J} \to \mathsf{colim}\, K_\phi$ is the coequalizer, which is pullback-stable by hypothesis.

It is enough to prove that ξ is a monomorphism. We consider the following diagram in $\mathsf{Fam}(\mathscr{V})$

$$\begin{array}{ccccc}
(X_j \wedge X_k)_{j,k\in J\times J} & \longrightarrow & (\pi_0^*(X_j))_{j\in J} & \xrightarrow{\pi_0|_j} & (X_j)_{j\in J} \\
\downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow q \\
(\pi_1^*(X_j))_{j\in J} & \longrightarrow & \xi \times_Y \xi & \xrightarrow{\pi_0} & \operatorname{colim} K_\phi \\
\downarrow \pi_1|_j & \ulcorner & \pi_1 \downarrow & \ulcorner & \downarrow \xi \\
(X_j)_{j\in J} & \xrightarrow[q]{} & \operatorname{colim} K_\phi & \xrightarrow[\xi]{} & Y
\end{array}$$

whose squares are pullbacks. Since q is pullback-stable, it follows that $(X_j \wedge X_k)_{j,k\in J\times J} \to (\pi_i^*(X_j))_{j\in J}$ is a regular epimorphism for $i = 0,1$. Its kernel pair is the kernel pair of $(X_j \wedge X_k)_{j,k\in J\times J} \to (X_j)_{j\in J}$, hence $\pi_i|_j \colon \pi_i^*(X_j) \cong X_j$ is an isomorphism for all $j \in J$ and $i = 0,1$. We observe that q^* is conservative, so π_i is an isomorphism for $i = 0,1$. But π_0, π_1 is the kernel pair of ξ, so it must be a monomorphism.

If W is a subobject of Y such that $X_j \leqslant W$ for all j, then the above observation (with W replacing Y) also confirms that $\operatorname{colim} K_\phi \leqslant W$. Thus, $\operatorname{colim} K_\phi \cong \bigvee_j X_j$ in the (thin) category of subobjects of Y. □

Thus, if a cover $\phi\colon (X_j)_{j\in J} \to Y$ of monomorphisms is a descent morphism in $\mathsf{Fam}(\mathscr{V})$, we conclude that $Y \cong \bigvee_{j\in J} X_j$, a perspective that is helpful when $\mathscr{V}$ is thin or regular.

3.4.1 Meet semilattices

Let $\mathscr{V}$ be a thin category. A morphism $\phi\colon (X_j)_{j\in J} \to Y$ in $\mathsf{Fam}(\mathscr{V})$ is the assertion that "for all $j \in J$, we have $X_j \leqslant Y$". Therefore, we simply write $(X_j)_{j\in J} \leqslant Y$ in this setting.

A thin category $\mathscr{V}$ is said to be a *meet semilattice with a top element* if $\mathscr{V}$ is a thin category with finite limits, which are called *(finite) meets* in this context.

Lemma 3.17 ([51, Lemma 4.4])**.** *Let $\mathscr{V}$ be a meet semilattice with a top element, and let $(X_j)_{j\in J} \leqslant Y$ be a cover.*

(a) It is an epimorphism if and only if J is non-empty.
(b) If it is an epimorphism, then it is pullback-stable.
(c) It is a regular epimorphism if and only if $\bigvee_{j\in J} X_j \cong Y$.
(d) If it is a regular epimorphism, it is pullback-stable if and only if we have

$$Z \cong \bigvee_{j\in J} Z \wedge X_j. \tag{3.10}$$

for all $Z \leqslant Y$.
(e) If it is a descent morphism, and $\mathscr{V} \downarrow Y$ is (co)complete, then it is effective for descent.

Proof. If $(X_j)_{j\in J} \leqslant Y$, then for all $j \in J$, $X_j \leqslant Y$ is an epimorphism. Thus, we conclude that $(X_j)_{j\in J} \leqslant Y$ is an epimorphism if and only if the underlying function $J \to 1$ is an epimorphism, which is the case if and only if J is non-empty, confirming (a).

If $(X_j)_{j\in J}$ is an epimorphism, then for all $Z \leqslant Y$, we have $(X_j \wedge Z)_{j\in J} \leqslant Z$, which is still an epimorphism, as J is non-empty, proving (b).

We also have (c) as a consequence of Lemma 3.11, and the condition for pullback-stability is precisely (3.10), giving (d).

Let $\phi\colon (X_j)_{j\in J} \leqslant Y$ be a pullback-stable regular epimorphism. A connected descent datum for ϕ is given by subobjects $W_j \leqslant X_j$ for each $j \in J$ such that $W_j \wedge X_i \cong X_j \wedge W_i$ for all $i, j \in J$; the reflexivity and transitivity conditions are automatically satisfied, as $\mathscr{V}$ is thin. If $\mathscr{V} \downarrow Y$ has joins, we let $Z \cong \bigvee_{j\in J} W_j$. We have

$$X_j \wedge Z \leqslant \bigvee_{i\in J} X_j \wedge W_i \cong \bigvee_{i\in J} W_j \wedge X_i \cong W_j,$$

and since $W_j \leqslant X_j \wedge Z$ by definition, we conclude that $\mathscr{K}^{\mathsf{Ker}(\phi)}_{\mathsf{conn}} \colon \mathscr{V} \downarrow Y \to \mathsf{Desc}_{\mathsf{conn}}(\phi)$ is essentially surjective, and therefore ϕ is an effective descent morphism. □

A bounded meet semilattice $\mathscr{V}$ is said to be a *Heyting lattice*[2] if $\mathscr{V}$ is cartesian closed; that is, the functor $A \wedge -$ has a right adjoint functor for all objects A. As a corollary, we obtain:

Corollary 3.18 ([51, Corollary 4.5]). *If $\mathscr{V}$ is a Heyting semilattice, regular epimorphisms in* $\mathsf{Fam}(\mathscr{V})$ *are pullback-stable.*

Proof. Since $Z \wedge -$ preserves joins, (3.10) is always satisfied. □

Corollary 3.19. *If $\mathscr{V}$ is a (co)complete lattice, pullback-stable regular epimorphisms in* $\mathsf{Fam}(\mathscr{V})$ *are effective for descent.*

Proof. For all Y, $\mathscr{V} \downarrow Y$ is cocomplete. □

Combining both of the previous results yields:

Corollary 3.20 ([51, Corollary 4.6]). *If $\mathscr{V}$ is a (co)complete Heyting (semi)lattice, regular epimorphisms in* $\mathsf{Fam}(\mathscr{V})$ *are effective for descent.*

So, we recover one implication of [12, Theorem 5.4]:

Theorem 3.21 ([51, Theorem 4.7]). *Let $\mathscr{V}$ be a (co)complete Heyting lattice, and let $F\colon \mathscr{C} \to \mathscr{D}$ be a $\mathscr{V}$-functor. If*

$$\bigvee_{x_i \in F^* y_i} \mathscr{C}(x_0, x_1, x_2) \cong \mathscr{D}(y_0, y_1, y_2) \tag{3.11}$$

for all $y_0, y_1, y_2 \in \mathscr{D}$, then F is an effective descent $\mathscr{V}$-functor.

Proof. By hypothesis, (II) is satisfied, and since F is surjective on objects (consider (3.11) with $y_0 = y_1 = y_2$), it follows by Lemma 3.17 that condition (III) holds. Moreover, if we consider (3.11) with $y_1 = y_2$, so that $\mathscr{D}(y_1, y_2) \cong 1$, we have

$$\mathscr{D}(y_0, y_1) \cong \bigvee_{x_i \in F^* y_i} \mathscr{C}(x_0, x_1, x_2) \leqslant \bigvee_{x_i \in F^* y_i} \mathscr{C}(x_0, x_1),$$

and since $\mathscr{C}(x_0, x_1) \leqslant \mathscr{D}(y_0, y_1)$ for all $x_i \in F^* y_i$, we conclude that $(\mathscr{C}(x_0, x_1))_{x_i \in F^* y_i} \leqslant \mathscr{D}(y_0, y_1)$ is a regular epimorphism in $\mathsf{Fam}(\mathscr{V})$, which is effective for descent by Corollary 3.20, guaranteeing (I). Thus, Theorem 3.10 can be applied to conclude that F is effective for descent. □

[2] Also known as *implicative semilattices* [50] and *Brouwerian semilattices* [35].

Remark 3.22. As alluded to in the Introduction, Theorem 3.10 confirms the link between the idea of "chain-surjectivity" conditions of [40] and the "chain-surjectivity" of [28], as evidenced by Theorem 3.21.

3.4.2 Regular categories

The same ideas work here, if we employ the (regular epi, mono)-factorization system of a regular category.

Lemma 3.23 ([51, Lemma 4.8]). *Let $\mathscr{V}$ be a regular category, and let $\phi\colon (X_j)_{j\in J} \to Y$ be a cover. For each $j \in J$, we consider the (regular epi,mono)-factorization of ϕ_j, given by*

$$X_j \xrightarrow{\pi_j} M_j \xrightarrow{\iota_j} Y, \tag{3.12}$$

where π_j is a descent morphism, and ι_j is a monomorphism for all $j \in J$. Thus, we consider the cover $\iota\colon (M_j)_{j\in J} \to Y$.

(a) ϕ is a (pullback-stable) epimorphism if and only if ι is a (pullback-stable) epimorphism.
(b) ϕ is a (pullback-stable) regular epimorphism if and only if ι is a (pullback-stable) regular epimorphism.
(c) If π_j is an effective descent morphism for all $j \in J$, then ϕ is effective for descent if and only if ι is effective for descent.

Proof. The factorization (3.12) gives a factorization $\phi = \iota \circ (\mathsf{id}, \pi)$ in $\mathsf{Fam}(\mathscr{V})$, and since π_j is a descent morphism for all j, (id, π) is a coproduct of descent morphisms, therefore it is a descent morphism in $\mathsf{Fam}(\mathscr{V})$. Thus, we obtain (a) and (b) by composition and cancellation [29, Propositions 1.3 and 1.5].

Moreover, if π_j is effective for descent for all $j \in J$, then so is (id, π), and (c) follows by [31, Theorem 4.5], as the basic bifibration respects the BED (see [31, 4.4, 4.6]). □

Thus, the study of (effective) descent covers in $\mathsf{Fam}(\mathscr{V})$ can be reduced to the study of (effective) descent covers of monomorphisms (and effective descent morphisms in $\mathscr{V}$). When applied to the study of effective descent morphisms in $\mathscr{V}$-Cat, we obtain:

Theorem 3.24 ([51, Theorem 4.9]). *Let $\mathscr{V}$ be a regular category, let $F\colon \mathscr{C} \to \mathscr{D}$ be a $\mathscr{V}$-functor, and consider the hom-covers*

$$F = (F_{x_0,x_1})_{x_i\in F^*y_i}\colon (\mathscr{C}(x_0,x_1))_{x_i\in F^*y_i} \to \mathscr{D}(y_0,y_1)$$
$$F\times F = (F_{x_0,x_1,x_2})_{x_i\in F^*y_i}\colon (\mathscr{C}(x_0,x_1,x_2))_{x_i\in F^*y_i} \to \mathscr{D}(y_0,y_1,y_2)$$
$$F\times F\times F = (F_{x_0,x_1,x_2,x_3})_{x_i\in F^*y_i}\colon (\mathscr{C}(x_0,x_1,x_2,x_3))_{x_i\in F^*y_i} \to \mathscr{D}(y_0,y_1,y_2,y_3)$$

and their respective (regular epi, mono)-factorizations

$$F_{x_0,x_1} = I_{x_0,x_1} \circ P_{x_0,x_1},$$
$$F_{x_0,x_1,x_2} = I_{x_0,x_1,x_2} \circ P_{x_0,x_1,x_2},$$
$$F_{x_0,x_1,x_2,x_3} = I_{x_0,x_1,x_2,x_3} \circ P_{x_0,x_1,x_2,x_3}.$$

If

(a) P_{x_0,x_1} *is an effective descent morphism for all* x_0, x_1*,*
(b) $(I_{x_0,x_1})_{x_i \in F^* y_i}$ *is an effective descent morphism,*
(c) $(I_{x_0,x_1,x_2})_{x_i \in F^* y_i}$ *is a descent morphism, and*
(d) $(I_{x_0,x_1,x_2,x_3})_{x_i \in F^* y_i}$ *is an almost descent morphism,*

then F *is an effective descent morphism in* $\mathscr{V}$-Cat.

Proof. By Lemma 3.23,

– conditions (a) and (b) together guarantee (I),
– conditions (c) and (d) respectively guarantee (II) and (III),

so that Theorem 3.10 can be applied. □

The above list of conditions can be further reduced if $\mathscr{V}$ satisfies extra properties. For instance, if $\mathscr{V}$ is Barr-exact, or locally cartesian closed, then descent morphisms are effective descent morphisms, so condition (a) is redundant.

More specifically, a CHaus-functor $F\colon \mathscr{C} \to \mathscr{D}$ satisfying (b), (c) and (d) is an effective descent morphism in CHaus-Cat, as CHaus is an Barr-exact category [49]. Moreover, it is shown therein that Stn is a regular category, thus if a Stn-functor satisfies all of the hypotheses of Theorem 3.24, then it is an effective descent Stn-functor.

Chapter 4

Generalized internal multicategory functors

A *multicategory* is a categorical structure which models the notion of "multimorphisms": morphisms which map from a (possibly empty) finite string of inputs to a single output. The quintessential example of such a structure is the multicategory Vect of vector spaces over some field $\mathbb{F}$, and *multilinear maps*. A multilinear map $f\colon (V_1,\ldots,V_n) \to W$ has a finite string $V_1,\ldots,V_n$ of vector spaces as the domain, and a vector space W as codomain. It consists of a function

$$f\colon V_1 \times \ldots \times V_n \to W$$

which is linear in each component:

$$f(v_1,\ldots,v_i+\lambda w,\ldots,v_n) = f(v_1,\ldots,v_i,\ldots,v_n) + \lambda f(v_1,\ldots,w,\ldots,v_n),$$

where $v_j \in V_j$ for all $j \in \{1,\ldots,n\}$, $w \in V_i$ for each $i \in \{1,\ldots,n\}$, and $\lambda \in \mathbb{F}$. In case $n=0$, a multilinear map $f\colon () \to W$ is simply a vector $f \in W$, or equivalently, a linear map $f\colon \mathbb{F} \to W$. In case $n=1$, a multilinear map $f\colon (V_1) \to W$ is an ordinary linear map.

As is the case with categories, multicategories also have an adequate *composition operation*. In the case of Vect, if we have a finite string of multilinear maps $g_1,\ldots,g_n$ given by

$$g_j\colon (U_{j1},\ldots,U_{jk_j}) \to V_j,$$

then we have the composite multilinear map

$$f\circ(g_1,\ldots,g_n)\colon (U_{11},\ldots,U_{1k_1},\ldots,U_{n1},\ldots,U_{nk_n}) \to W$$

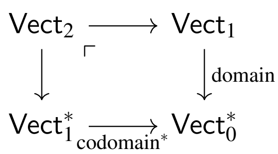
whose underlying function is given by $f\circ(g_1\times\ldots\times g_n)$. Of course, the identity linear map $\mathsf{id}_W\colon (W) \to W$ is a (multi)linear map, and these satisfy suitable associativity and unit laws. Naturally, if we consider the multilinear maps whose domain is a string of length 1, we precisely recover the ordinary category of vector spaces and linear maps.

The notion of multicategory can be traced back to [38, p. 103], where it was developed for the purpose of studying deductive systems in logic, and it has since found applications in algebraic topology and higher category theory. A comprehensive introduction to these categorical structures is given in [41].

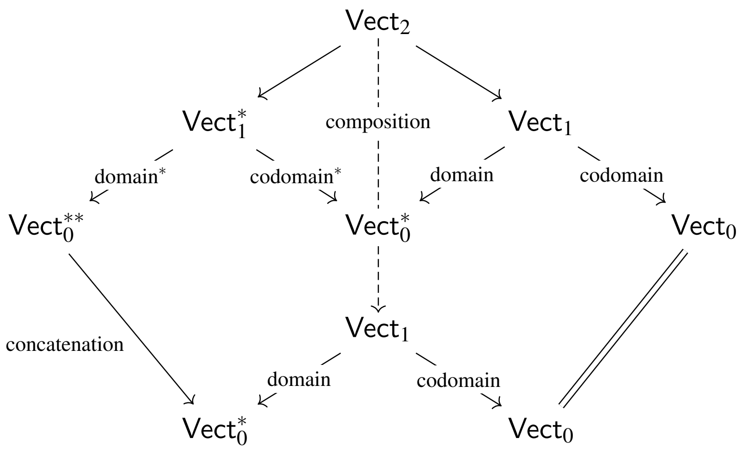
The composition operation of Vect carries an underlying structure on the domains, given by *concatenation* of strings, as does the identity multilinear map, *casting* each vector space as a string of vector spaces of length 1. These operations are well modeled by the multiplication and unit natural transformations for the *free monoid monad* $(-)^*$ on Set. Indeed, the multicategory of vector spaces may be described by a span of functions

$$\{\text{set of vector spaces}\}^* \xleftarrow{\text{domain}} \{\text{multilinear maps}\} \xrightarrow{\text{codomain}} \{\text{set of vector spaces}\}$$

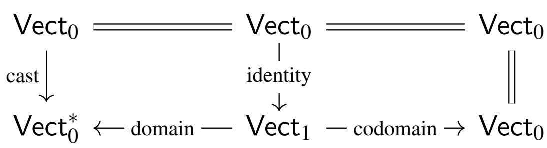
and the identity and composition operation, as well as the associativity and unit laws, can be described diagrammatically as well, via the monad structure of $(-)^*$, and its properties. Let Vect_0 be the set of vector spaces, and Vect_1 be the set of multilinear maps. We obtain the set Vect_2 of "multicomposable" pairs of multilinear maps via the pullback



so that the composition operation is given by



and the identity maps are given by



This diagrammatic description lends itself to the "internalization" of the notion of multicategory to any category $\mathcal{V}$ with pullbacks, provided we also replace the free monoid monad on Set by an arbitrary monad $T = (T, m, e)$ on $\mathcal{V}$. Incidentally, this also allows for the "shape" of the domain to be more general than "finite strings". Indeed, these ideas gave rise to *T-catégories*, first defined in [9], and later studied by [24] when T is a *cartesian* monad. In these works, generalized multicategories are defined to be monads in the bicategory $\mathsf{Span}_T(\mathcal{V})$.

The main theme of this thesis is to obtain a unified perspective on the effective descent morphisms in generalized categorical structures, and the purpose of this chapter, covering the work done in [52], is to provide an understanding of these morphisms in the category $\mathsf{Cat}(T,\mathcal{V})$ of *T-categories* internal to $\mathcal{V}$. We will undertake two approaches.

Our first approach to the study of effective descent morphisms in $\mathsf{Cat}(T,\mathcal{V})$ can be summed up in four steps:

- we construct $\mathsf{Cat}(T,\mathcal{V})$ as a 2-equalizer of a diagram of categories of essentially algebraic theories internal to $\mathcal{V}$,
- we recall from [40, Proposition 3.2.4] that effective descent morphisms in essentially algebraic theories internal to $\mathcal{V}$ can be described via descent conditions on the underlying data,
- we recall the description of effective descent morphisms of a pseudoequalizer (isoinserter) via Proposition 2.7,
- we confirm that the embedding of $\mathsf{Cat}(T,\mathcal{V})$ into the associated pseudoequalizer reflects effective descent morphisms.

Section 4.1 illustrates the tools and techniques used for the construction of $\mathsf{Cat}(T,\mathcal{V})$ in a simpler T-structure, that of *reflexive T-graphs*, and the full construction is carried out in Section 4.2. Afterwards, Section 4.3 is devoted to confirming that effective descent morphisms are reflected along the embedding of $\mathsf{Cat}(T,\mathcal{V})$ into the associated pseudoequalizer.

In Section 4.4, we provide a second method to obtain a description of the effective descent functors of T-categories. Here, we employ the ideas of [40] to extend his results to our setting, by directly studying the "sketch" of these generalized multicategories.

4.1 Reflexive T-graphs

Let $T = (T, m, e)$ be a monad on a category $\mathcal{V}$ with pullbacks. For the purpose of studying effective descent functors of T-categories, we obtain sharper results by describing $\mathsf{Cat}(T,\mathcal{V})$ as a 2 - equalizer of a suitable diagram of categories. Before providing such a description, we will first consider a simpler T-structure as a guiding example.

For a pointed endofunctor $T = (T, e)$ on $\mathcal{V}$, a *reflexive T-graph* x consists of

- an object x_0 of *objects*,
- an object x_1 of *arrows*,
- a *domain* morphism $d_1 \colon x_1 \to T x_0$,
- a *codomain* morphism $d_0 \colon x_1 \to x_0$,
- a *loop* morphism $s_0 \colon x_0 \to x_1$,

which must satisfy $d_0 \circ s_0 = \mathsf{id}$ and $d_1 \circ s_0 = e$. We note that this data can be organized in the following diagram:

$$\begin{array}{ccc} x_0 & \overset{s_0}{\underset{d_0}{\rightleftarrows}} & x_1 \\ & \searrow^{e} & \downarrow d_1 \\ & & Tx_0 \end{array} \tag{4.1}$$

Moreover, a morphism of reflexive T-graphs $f\colon x \to y$ consists of

– an *object* morphism $f_0\colon x_0 \to y_0$,
– an *arrow* morphism $f_1\colon x_1 \to y_1$,

satisfying $d_0 \circ f_1 = f_0 \circ d_0$, $d_1 \circ f_1 = Tf_0 \circ d_1$ and $f_1 \circ s_0 = s_0 \circ f_0$. These form a category $\mathsf{RGrph}(T,\mathcal{V})$, with componentwise composition and identities. We observe that these are the *T-graphes pointés* of [9].

We take this opportunity to remark that reflexive T-graphs allow us to draw conclusions about the descent theory of categorical structures:

Lemma 4.1 ([52, Lemma A.3])**.** *Let $\mathcal{E}$ be a class of epimorphisms in* $\mathsf{RGrph}(T,\mathcal{V})$ *such that*

– *$\mathcal{E}$ contains all split epimorphisms,*
– *$\mathcal{E}$ is closed under composition,*
– *if $g \circ f, f \in \mathcal{E}$, then $g \in \mathcal{E}$ (right cancellation),*

and let $f\colon x \to y$ be a morphism of reflexive T-graphs. If $f_1 \in \mathcal{E}$, then $f_0 \in \mathcal{E}$.

Proof. In any reflexive T-graph x, the codomain morphism $d_0\colon x_1 \to x_0$ is a split epimorphism since $d_0 \circ s_0 = \mathsf{id}$, so that $d_0 \in \mathcal{E}$. If $f_1 \in \mathcal{E}$, then we have $d_0 \circ f_1 = f_0 \circ d_0 \in \mathcal{E}$ by closure under composition, and $f_0 \in \mathcal{E}$ by right cancellation. □

Of particular interest are the classes $\mathcal{E}$ given by the (effective, almost) descent morphisms, which satisfy each of the properties. Taking $T = \mathsf{id}$, we observe that the condition that p_0 is an effective descent morphism is redundant in Theorem 2.10, and can be omitted.

Returning to our main point, we observe that the category $\mathsf{RGrph}(T,\mathcal{V})$ can be described by a (2-)equalizer of diagram categories as well: we consider the graph

$$\mathcal{G} = \begin{array}{ccc} x_0 & \overset{s_0}{\underset{d_0}{\rightleftarrows}} & x_1 \\ & \searrow^{e_0} & \downarrow d_1 \\ & & x_0' \end{array}$$

with relations $d_1 \circ s_0 = e_0$ and $d_0 \circ s_0 = \mathsf{id}$, and we consider the diagram category $[\mathcal{G},\mathcal{V}]$, together with functors

$$\mathcal{V} \xleftarrow{x_0^*} [\mathcal{G},\mathcal{V}] \xrightarrow{e_0^*} [2,\mathcal{V}]$$

induced by the inclusions $x_0 \to \mathcal{G}$ and $(x_0 \xrightarrow{e_0} y_0) \to \mathcal{G}$, where $2 = (\cdot \to \cdot)$.

We recall that any natural transformation $\phi\colon F \to G$ of functors $\mathscr{C} \to \mathscr{D}$ is precisely determined by a functor $\phi^\sharp\colon \mathscr{C} \to [2,\mathscr{D}]$, which satisfies $\mathsf{ev}_0 \circ \phi^\sharp = F$ and $\mathsf{ev}_1 \circ \phi^\sharp = G$, where $\mathsf{ev}_j\colon [\mathscr{J},\mathscr{D}] \to \mathscr{D}$ is the evaluation functor.

For example, if we take the point $e\colon \mathsf{id} \to T$ of the endofunctor T, $e^\sharp\colon \mathscr{V} \to [2,\mathscr{V}]$ is a functor which satisfies $\mathsf{ev}_0 \circ e^\sharp = \mathsf{id}$ and $\mathsf{ev}_1 \circ e^\sharp = T$. With this notation, we obtain the following statement:

Lemma 4.2. *We have a 2-equalizer diagram*

$$\mathsf{RGrph}(T,\mathscr{V}) \longrightarrow [\mathscr{G},\mathscr{V}] \underset{e^\sharp \circ x_0^*}{\overset{e_0^*}{\rightrightarrows}} [2,\mathscr{V}] \tag{4.2}$$

Proof. From the condition imposed by the 2-equalizer in CAT we obtain the full subcategory of $[\mathscr{G},\mathscr{V}]$ whose diagrams are of the form (4.1). □

4.2 Internal T-categories

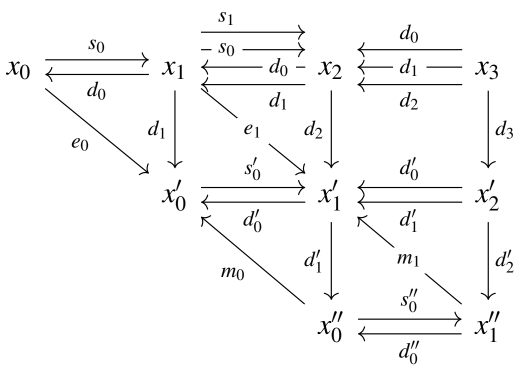
Let $T = (T,m,e)$ be a monad on a category $\mathscr{V}$ with finite limits. Recall that T is said to be *cartesian* if T preserves pullbacks and

$$\begin{array}{ccc} x & \xrightarrow{e_x} & Tx \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle Tf} \\ y & \xrightarrow[e_y]{} & Ty \end{array} \qquad\qquad \begin{array}{ccc} TTx & \xrightarrow{m_x} & Tx \\ {\scriptstyle TTf}\downarrow & \lrcorner & \downarrow{\scriptstyle Tf} \\ TTy & \xrightarrow[m_y]{} & Ty \end{array}$$

are pullback squares for all $f\colon x \to y$.

The category $\mathsf{Cat}(T,\mathscr{V})$ of *T-categories* was defined diagrammatically in [9, I.1] for general monads T, and this is the definition we will use throughout this chapter. However, Burroni also observed that the category of T-categories can equivalently be defined as the category of monads for the (proarrow) equipment $\mathsf{Span}_T(\mathscr{V})$ of T*-spans*, for T a cartesian monad; indeed, this is precisely how T-categories were defined in [24].

Here, we shall verify that $\mathsf{Cat}(T,\mathscr{V})$ can be given via a 2-equalizer involving the category of $\mathscr{V}$-models for a finite limit sketch $\mathscr{S}$. Its underlying graph is given by



(4.3)

with the following relations[1]

[1] We point out the resemblance of these relations with the cosimplicial identities.

- $s_1 \circ s_0 = s_0 \circ s_0 \colon x_0 \to x_2$,
- $d_{1+i} \circ s_i = e_i \colon x_i \to x'_i$,
- $d_i \circ s_j = \mathsf{id} \colon x_i \to x_i$,
- $d_2 \circ s_0 = s'_0 \circ d_1 \colon x_1 \to x'_1$,
- $d_0 \circ s_1 = s_0 \circ d_0 \colon x_1 \to x_1$,
- $d'_0 \circ s'_0 = \mathsf{id} \colon x'_0 \to x'_0$,
- $d_{1+i} \circ d_{1+i} = m_i \circ d'_{1+i} \circ d_{2+i} \colon x_{2+i} \to x'_i$,
- $d_{1+i} \circ d_0 = d'_0 \circ d_{2+i} \colon x_{2+i} \to x_i$,
- $d'_j \circ d_{2+i} = d_{1+i} \circ d_j \colon x_{2+i} \to x'_i$,
- $d_0 \circ d_1 = d_0 \circ d_0 \colon x_2 \to x_0$,
- $d_j \circ d_{1+i} = d_i \circ d_j \colon x_3 \to x_1$,
- $d'_1 \circ d'_0 = d''_0 \circ d'_2 \colon x'_2 \to x''_0$,
- $d'_0 \circ d'_1 = d'_0 \circ d'_0 \colon x'_2 \to x'_0$,

and limit cones

$$\begin{array}{ccc} x_2 & \xrightarrow{d_0} & x_1 \\ {\scriptstyle d_2}\downarrow & \ulcorner & \downarrow{\scriptstyle d_1} \\ x'_1 & \xrightarrow[d'_0]{} & x'_0 \end{array} \qquad \begin{array}{ccc} x_3 & \xrightarrow{d_0} & x_2 \\ {\scriptstyle d_3}\downarrow & \ulcorner & \downarrow{\scriptstyle d_2} \\ x'_2 & \xrightarrow[d'_0]{} & x'_1 \end{array} \qquad \begin{array}{ccc} x'_2 & \xrightarrow{d'_0} & x'_1 \\ {\scriptstyle d'_2}\downarrow & \ulcorner & \downarrow{\scriptstyle d'_1} \\ x''_1 & \xrightarrow[d''_0]{} & x''_0 \end{array} \tag{4.4}$$

with $i = 0, 1$ and $j \leqslant i$. We let $\mathsf{Mod}(\mathscr{S}, \mathscr{V})$ be the category of $\mathscr{V}$-models for the sketch $\mathscr{S}$. Moreover, abusing notation, we will also denote by $\mathscr{S}$ the category freely generated by the underlying graph of $\mathscr{S}$ modulo the given relations.

Remark 4.3 (Objects of n-chains)**.** As in Remark 2.11, it is convenient to denote x_2 and x_3 to be the objects of *2-chains* and *3-chains* of morphisms respectively, for an internal (T-)category x.

Lemma 4.4 ([52, Lemma 3.1])**.** *If T preserves pullbacks, then we have a 2-equalizer diagram*

$$\mathsf{Cat}(T, \mathscr{V}) \longrightarrow \mathsf{Mod}(\mathscr{S}, \mathscr{V}) \underset{d^0}{\overset{d^1}{\rightrightarrows}} [\mathscr{S}_T, \mathscr{V}] \times [2, \mathscr{V}] \tag{4.5}$$

where the functors d^1, d^0 are respectively given by

$$\mathsf{Mod}(\mathscr{S}, \mathscr{V}) \longrightarrow [\mathscr{S}, \mathscr{V}] \xrightarrow{(I_T^*, I_{d'_1}^*)} [\mathscr{S}_T, \mathscr{V}] \times [2, \mathscr{V}]$$

$$\mathsf{Mod}(\mathscr{S}, \mathscr{V}) \longrightarrow [\mathscr{S}, \mathscr{V}] \xrightarrow{(I_{s_0,d_0}^*, I_{d_1}^*)} [\mathscr{S}_{s_0,d_0}, \mathscr{V}] \times [2, \mathscr{V}] \xrightarrow{(T,m,e)_! \times T_!} [\mathscr{S}_T, \mathscr{V}] \times [2, \mathscr{V}]$$

and $I_T \colon \mathscr{S}_T \to \mathscr{S}$, $I_{d'_1} \colon 2 \to \mathscr{S}$, $I_{s_0,d_0} \colon \mathscr{S}_{s_0,d_0} \to \mathscr{S}$ and $I_{d_1} \colon 2 \to \mathscr{S}$ are the subcategories of $\mathscr{S}$ respectively determined by the subgraphs

$$\begin{array}{ccccc} x_0 & \xrightarrow{e_0} & x'_0 & \xleftarrow{m_0} & x''_0 \\ {\scriptstyle d'_0}\uparrow\downarrow{\scriptstyle s_0} & & {\scriptstyle d'_0}\uparrow\downarrow{\scriptstyle s'_0} & & {\scriptstyle d''_0}\uparrow\downarrow{\scriptstyle s''_0} \\ x_1 & \xrightarrow[e_1]{} & x'_1 & \xleftarrow[m_1]{} & x''_1 \end{array}, \qquad x'_1 \xrightarrow{d'_1} x''_0, \qquad x_0 \underset{d'_0}{\overset{s_0}{\rightleftarrows}} x_1, \qquad x_1 \xrightarrow{d_1} x'_0,$$

the functor $T_!\colon [2,\mathscr{V}] \to [2,\mathscr{V}]$ *is given by the direct image of* T*, and* $(T,m,e)_!\colon [\mathscr{S}_{s_0,d_0},\mathscr{V}] \to [\mathscr{S}_T,\mathscr{V}]$ *is given by*

$$\begin{array}{ccc} a & & a \xrightarrow{\;e_a\;} Ta \xleftarrow{\;m_a\;} TTa \\ g\Big\uparrow\Big\downarrow f & \mapsto & g\Big\uparrow\Big\downarrow f \qquad Tg\Big\uparrow\Big\downarrow Tf \qquad TTg\Big\uparrow\Big\downarrow TTf \; . \\ b & & b \xrightarrow[\;e_b\;]{} Tb \xleftarrow[\;m_b\;]{} TTb \end{array}$$

Moreover, if T *is cartesian, then* $\mathsf{Cat}(T,\mathscr{V})$ *has pullbacks, and the inclusion* $\mathsf{Cat}(T,\mathscr{V}) \to \mathsf{Mod}(\mathscr{S},\mathscr{V})$ *preserves them.*

Proof. The objects of the 2-equalizer are precisely those diagrams of the form

$$\begin{array}{ccccccc} x_0 & \overset{s_0}{\underset{d_0}{\rightleftarrows}} & x_1 & \overset{s_1,\,s_0}{\underset{d_0,\,d_1}{\rightleftarrows}} & x_2 & \overset{d_0,\,d_1,\,d_2}{\longleftarrow} & x_3 \\ & \searrow^{e_{x_0}} & \Big\downarrow d_1 & \searrow^{e_{x_1}} & \Big\downarrow d_2 & & \Big\downarrow d_3 \\ & & Tx_0 & \overset{Ts_0}{\underset{Td_0}{\rightleftarrows}} & Tx_1 & \overset{d'_0,\,d'_1}{\longleftarrow} & x'_2 \\ & & & \nwarrow^{m_{x_0}} & \Big\downarrow Td_1 & \nwarrow^{m_{x_1}} & \Big\downarrow d'_2 \\ & & & & TTx_0 & \xleftarrow[TTd_0]{} & TTx_1 \end{array} \tag{4.6}$$

satisfying the relations imposed by $\mathscr{S}$, such that the following squares

$$\begin{array}{ccc} x_2 & \xrightarrow{d_0} & x_1 \\ d_2\Big\downarrow & \ulcorner & \Big\downarrow d_1 \\ Tx_1 & \xrightarrow[Td_0]{} & Tx_0 \end{array} \qquad \begin{array}{ccc} x_3 & \xrightarrow{d_0} & x_2 \\ d_3\Big\downarrow & \ulcorner & \Big\downarrow d_2 \\ Tx_2 & \xrightarrow[Td_0]{} & Tx_1 \end{array} \qquad \begin{array}{ccc} x'_2 & \xrightarrow{d'_0} & Tx_1 \\ d'_2\Big\downarrow & \ulcorner & \Big\downarrow Td_1 \\ TTx_1 & \xrightarrow[TTd_0]{} & TTx_0 \end{array} \tag{4.7}$$

are pullback diagrams; compare (4.6) with [9, Figure 1]. Since T preserves pullbacks, the rightmost pullback diagram in (4.7) can be replaced by

$$\begin{array}{ccc} Tx_2 & \xrightarrow{Td_0} & Tx_1 \\ Td_2\Big\downarrow & \ulcorner & \Big\downarrow Td_1 \\ TTx_1 & \xrightarrow[TTd_0]{} & TTx_0 \end{array}$$

When T is cartesian, both d^1 and d^0 preserve pullbacks, hence (4.5) can be seen as a 2-equalizer in the category of categories with pullbacks and pullback-preserving functors. □

4.3 Effective descent morphisms via bilimits

Let $T = (T,m,e)$ be a cartesian monad on a category $\mathscr{V}$ with pullbacks. We begin by observing that:

Lemma 4.5 ([52, Lemma A.1]). *Let* $\mathscr{P}$ *be a pullback-stable class of morphisms of* $\mathscr{V}$*.* T *creates such morphisms in its essential image.*

Proof. Let f be a morphism such that $Tf \in \mathscr{P}$. Since the naturality squares for m and e at f are pullbacks, we conclude that $TTf, f \in \mathscr{P}$. □

We now consider the sketch $\mathscr{S}$ constructed in Section 4.2.

Lemma 4.6 ([52, Proposition 4.2])**.** *Let $p\colon x \to y$ be a morphism in $\mathsf{Mod}(\mathscr{S},\mathscr{V})$. If*

- $p_0, p_0', p_0'', p_1, p_1', p_1''$ *are effective descent morphisms,*
- p_2, p_2' *are descent morphisms, and*
- p_3 *is an almost descent morphism,*

then p is an effective descent morphism.

Proof. The proof rests in describing $\mathsf{Mod}(\mathscr{S},\mathscr{V})$ as the category of $\mathscr{V}$-models for a suitable essentially algebraic theory; then the result is an immediate consequence of [40, Proposition 3.2.4].

Indeed, we obtain a sketch $\overline{\mathscr{S}}$, Morita equivalent to $\mathscr{S}$, from the essentially algebraic theory $\mathscr{A}$ defined by

- sorts $x_0, x_1, x_0', x_1', x_0'', x_1''$,
- total operations given by the arrows of the full subgraph of the underlying graph of $\mathscr{S}$ consisting of the aforementioned sorts,
- partial operations given by $d_1\colon x_1 \times x_1' \to x_1$, $d_1'\colon x_1' \times x_1'' \to x_1'$, and the limit cones of $\mathscr{S}$, with x_2 and x_2' respectively replaced by $x_1 \times x_1'$ and $x_1' \times x_1''$, give the equations for these partial operations,
- the remaining equations are given by the underlying relations of $\mathscr{S}$, with x_2, x_3 and x_2' replaced by $x_1 \times x_1'$, $x_1 \times x_1' \times x_1''$ and $x_1' \times x_1''$, respectively.

The sketch $\overline{\mathscr{S}}$ constructed from $\mathscr{A}$ via the procedure described in [40, 3.2.1] contains $\mathscr{S}$ as a "subsketch", containing extra limit cones for the formal products $x_1 \times x_1'$, $x_1' \times x_1''$ and $x_1 \times x_1' \times x_1''$, which are used to construct the limit cones of x_2, x_2' and x_3, as well as the (derived) partial operations and equations.

□

Denoting by $\mathsf{PsEq}(F,G)$ the *pseudoequalizer* of a pair of functors $F,G\colon \mathscr{A} \to \mathscr{B}$, we obtain:

Lemma 4.7 ([52, Lemma 3.3])**.** *The induced inclusion $\mathsf{Cat}(T,\mathscr{V}) \to \mathsf{PsEq}(d^1,d^0)$ is full and preserves pullbacks.*

Proof. We recall that a morphism $f\colon (x,\zeta) \to (y,\xi)$ of $\mathsf{PsEq}(d^1,d^0)$ is a morphism $f\colon x \to y$ in $\mathsf{Mod}(\mathscr{S},\mathscr{V})$ satisfying $\xi \circ d^0 f = d^1 f \circ \zeta$. Thus, when ζ and ξ are identities, we precisely obtain a T-category functor.

Since d^1 and d^0 are pullback-preserving functors between categories with pullbacks, it follows that $\mathsf{PsEq}(d^1,d^0)$ has pullbacks, and $\mathsf{PsEq}(d^1,d^0) \to \mathsf{Mod}(\mathscr{S},\mathscr{V})$ creates them. We also note that $\mathsf{Cat}(T,\mathscr{V}) \to \mathsf{Mod}(\mathscr{S},\mathscr{V})$ preserves pullbacks as well, concluding the proof. □

Lemma 4.8 ([52, Theorem 3.5])**.** *If a morphism $p\colon (x,\mathsf{id}) \to (y,\theta)$ in $\mathsf{PsEq}(d^1,d^0)$ is a componentwise epimorphism, then $(y,\theta) \cong (z,\mathsf{id})$ for a T-category z.*

Proof. By hypothesis, we have $d^0 p = \theta \circ d^1 p$. Writing $\theta = (\theta_T, \theta_d)$, we obtain equations

$$\begin{aligned} p_i &= \theta_{T,i} \circ p_i & Tp_1 &= \theta_{d,1} \circ p'_1 \\ Tp_i &= \theta'_{T,i} \circ p'_i & Tp'_0 &= \theta_{d,0} \circ p''_0 \\ TTp_i &= \theta''_{T,i} \circ p''_i, \end{aligned}$$

for $i = 0, 1$ from which we deduce

$$\theta_{T,i} = \mathsf{id}, \qquad \theta''_{T,0} = T\theta'_{T,0} \circ \theta_{d,0}, \qquad \theta'_{T,1} = \theta_{d,1},$$

since p is a componentwise epimorphism.

We claim (y, θ) is isomorphic to a T-category. The construction of the T-category presented below is similar to [52, Lemma 3.4]. This T-category has underlying reflexive T-graph

$$Ty_0 \xleftarrow{\theta'_{T,0}} y'_0 \xleftarrow{d_1} y_1 \underset{s_0}{\overset{d_0}{\rightleftarrows}} y_0,$$

and we observe that

$$\begin{array}{ccccc} y_2 & \xrightarrow{d_2} & y'_1 & \xrightarrow{\theta'_{T,1}} & Ty_1 \\ {\scriptstyle d'_0}\downarrow & & \downarrow{\scriptstyle d'_0} & & \downarrow{\scriptstyle Td_0} \\ y_1 & \xrightarrow[d_1]{} & y'_0 & \xrightarrow[\theta'_{T,0}]{} & Ty_0 \end{array} \qquad \begin{array}{ccccc} y_3 & \xrightarrow{d_3} & y'_2 & = & y'_2 \\ {\scriptstyle d'_0}\downarrow & & \downarrow{\scriptstyle d'_0} & & \downarrow{\scriptstyle \theta'_{T,1} \circ d'_0} \\ y_2 & \xrightarrow[d_2]{} & y'_1 & \xrightarrow[\theta'_{T,1}]{} & Ty_1 \end{array}$$

are pullback diagrams; the left squares are a pullback, and the right squares commute, and their parallel sides are isomorphisms, as θ is an isomorphism.

Likewise,

$$\begin{array}{ccccc} y'_2 & \xrightarrow{d''_2} & y''_1 & \xrightarrow{\theta''_{T,1}} & TTy_1 \\ {\scriptstyle d'_0}\downarrow & & \downarrow{\scriptstyle d'_0} & & \downarrow{\scriptstyle TTd_0} \\ y'_1 & \xrightarrow{d'_1} & y''_0 & \xrightarrow{\theta''_{T,0}} & TTy_0 \\ {\scriptstyle \theta_{d,1}}\downarrow & & \downarrow{\scriptstyle \theta_{d,0}} & & \| \\ Ty_1 & \xrightarrow[Td'_1]{} & Ty'_0 & \xrightarrow[T\theta'_{T,0}]{} & TTy_0 \end{array}$$

is a pullback diagram, since the top left square is a pullback, and the remaining squares, whose parallel sides are isomorphisms, are commutative, and therefore pullback diagrams.

Now, we are left with verifying that the relations hold. It is enough to verify relations of morphisms with (co)domain Ty_0, Ty_1, TTy_0, TTy_1, as the remaining hold by definition. We have

$$(\theta'_{T,i} \circ d_{1+i}) \circ s_i = \theta'_{T,i} \circ e_i = e_{x_i} \circ \theta_{T,i} = e_{x_i},$$

$$(\theta'_{T,1} \circ d_2) \circ s_0 = \theta'_{T,1} \circ s'_0 \circ d_1 = Ts_0 \circ (\theta'_{T,0} \circ d_1),$$

$$Td_0 \circ Ts_0 = T(d_0 \circ s_0) = \mathsf{id},$$

$$\begin{aligned}\theta'_{T,0} \circ d_1 \circ d_1 &= \theta'_{T,0} \circ m_0 \circ d'_1 \circ d_2 \\ &= m_{x_0} \circ \theta''_{T,0} \circ d'_1 \circ d_2 \\ &= m_{x_0} \circ T\theta'_{T,0} \circ \theta_{d,0} \circ d'_1 \circ d_2 \\ &= m_{x_0} \circ T\theta'_{T,0} \circ Td_1 \circ \theta_{d,1} \circ d_2 \\ &= m_{x_0} \circ T(\theta'_{T,0} \circ d_1) \circ (\theta'_{T,1} \circ d_2),\end{aligned}$$

$$\theta'_{T,1} \circ d_2 \circ d_2 = \theta'_{T,1} \circ m_1 \circ d'_2 \circ d_3 = m_{x_1} \circ (\theta''_{T,1} \circ d'_2) \circ d_3,$$

$$\theta'_{T,i} \circ d_{1+i} \circ d_j = \theta'_{T,i} \circ d'_j \circ d_{2+i} = \begin{cases} (\theta'_{T,1} \circ d'_1) \circ d_3 & i = j = 1 \\ Td_0 \circ (\theta'_{T,i} \circ d_{2+i}) & j = 0 \end{cases} \qquad i, j = 0, 1, \quad j \leqslant i$$

$$T(\theta'_{T,0} \circ d_1) \circ \theta'_{T,1} \circ d'_0 = T\theta'_{T,0} \circ \theta_{d,0} \circ d'_1 \circ d'_0 = \theta''_{T,0} \circ d''_0 \circ d'_2 = TTd_0 \circ (\theta''_{T,1} \circ d'_2),$$

$$Td_0 \circ \theta'_{T,1} \circ d'_1 = \theta'_{T,0} \circ d'_0 \circ d'_1 = \theta'_{T,0} \circ d'_0 \circ d'_0 = Td_0 \circ \theta'_{T,1} \circ d'_0,$$

and this concludes the proof. □

Corollary 4.9 ([52, Lemma 4.4]). *The embedding* $\mathsf{Cat}(T, \mathscr{V}) \to \mathsf{PsEq}(d^1, d^0)$ *reflects effective descent morphisms.*

Proof. Since any effective descent morphism is an epimorphism, this result is an immediate consequence of Lemma 4.8 and Corollary 2.5. □

Theorem 4.10 ([52, Theorem 4.5]). *A functor* $p\colon x \to y$ *of* T*-categories is an effective descent morphism in* $\mathsf{Cat}(T, \mathscr{V})$*, provided that*

- Tp_1 *is an effective descent morphism,*
- Tp_2 *is a descent morphism, and*
- p_3 *is an almost descent morphism.*

Proof. By Lemma 4.5, if Tp_2 is a descent morphism, then so is p_2, and if Tp_1 is an effective descent morphism, then so are p_1 and TTp_1. Moreover, by Lemma 4.1, we may also deduce that p_0, Tp_0 and TTp_0 are effective descent morphisms.

These conditions, and the fact that p_3 is an almost descent morphism, guarantee that p is an effective descent morphism in $\mathsf{Mod}(\mathscr{S}, \mathscr{V})$, and the morphism $d^1 p = d^0 p$ is a descent morphism, as these are determined componentwise.

Thus, we conclude that (p, id) is an effective descent morphism in $\mathsf{PsEq}(d^1, d^0)$ by Proposition 2.7, and, by Corollary 4.9, so is p. □

4.4 A direct description of effective descent morphisms

We return to the setting of Burroni's T*-catégories*, where $\mathscr{V}$ is any category with finite limits, and T is a monad on $\mathscr{V}$, not necessarily cartesian. We confirm that the arguments of Le Creurer for effective descent morphisms of essentially algebraic theories can be applied just as well to T*-catégories*.

Throughout, we assume that $p\colon x \to y$ is a functor of T-categories. We recall that since p is a reflexive T-graph morphism, if $\mathscr{E}$ is the class of (effective/almost) descent morphisms in $\mathscr{V}$ and $p_1 \in \mathscr{E}$, then $p_0 \in \mathscr{E}$ as well by Lemma 4.1.

Lemma 4.11 ([52, Lemma 5.1]). *If p_1 is a (pullback-stable) epimorphism in $\mathscr{V}$, then so is p in* $\mathsf{Cat}(T,\mathscr{V})$.

Proof. If $q,r\colon y \to z$ are functors such that $q \circ p = r \circ p$, then $q_i \circ p_i = r_i \circ p_i$ for $i = 0,1$, so $q_i = r_i$, implying $q = r$. We conclude that p is an epimorphism.

Since pullbacks in $\mathsf{Cat}(T,\mathscr{V})$ are calculated componentwise, our claim is verified. □

Lemma 4.12 ([52, Lemma 5.2]). *If p_1 is a (pullback-stable) regular epimorphism, and p_2 is a (pullback-stable) epimorphism in $\mathscr{V}$, then p is a (pullback-stable) regular epimorphism in* $\mathsf{Cat}(T,\mathscr{V})$.

Proof. We consider the kernel pair of p:

$$\begin{array}{ccc} k & \xrightarrow{r} & x \\ {\scriptstyle s}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ x & \xrightarrow[p]{} & y \end{array} \tag{4.8}$$

If $q\colon x \to z$ is a functor such that $r \circ q = s \circ q$, then, when p_i is a regular epimorphism for $i = 0,1$, there exists a unique $t_i\colon y_i \to z_i$ making the triangle of Diagram (4.9) commute

$$\begin{array}{ccccc} k_i & \overset{r_i}{\underset{s_1}{\rightrightarrows}} & x_i & \xrightarrow{p_i} & y_i \\ & & & {\scriptstyle q_i}\searrow & \downarrow{\scriptstyle t_i} \\ & & & & z_i \end{array} \tag{4.9}$$

for $i = 0,1$. The morphisms t_0, t_1 define a functor $t\colon y \to z$ of T-categories; indeed, we note that, by the universal property, $t_2 \circ (g,f) = (t_1 \circ g, Tt_1 \circ f)$, from which we deduce $q_2 = t_2 \circ p_2$. Then, the following calculations

$$\begin{aligned} t_1 \circ d_1 \circ p_2 &= t_1 \circ p_1 \circ d_1 = q_1 \circ d_1 = d_1 \circ q_2 = d_1 \circ t_2 \circ p_2, \\ d_i \circ t_1 \circ p_1 &= d_i \circ q_1 = T^i q_0 \circ d_i = T^i t_0 \circ T^i p_0 \circ d_i = T^i t_0 \circ d_i \circ p_1 \end{aligned}$$

plus the fact that p_1, p_2 are epimorphisms confirm our claim. Naturally, $t\colon y \to z$ is the unique functor making the triangle below commute

$$\begin{array}{ccccc} k & \overset{r}{\underset{s}{\rightrightarrows}} & x & \xrightarrow{p} & y \\ & & & {\scriptstyle q}\searrow & \downarrow{\scriptstyle t} \\ & & & & z, \end{array}$$

for if $l\colon y \to z$ were another such functor, we would deduce that $l_i = t_i$ by uniqueness given at (4.9) for $i = 0,1$, so $l = t$.

Again, by componentwise calculation of pullbacks of $\mathsf{Cat}(T,\mathscr{V})$, we conclude that if p_1 and p_2 are pullback-stable, then so is p. □

Theorem 4.13 ([52, Theorem 5.3]). *If T preserves pullbacks, and*

- $p_1 \colon x_1 \to y_1$ *is an effective descent morphism,*
- $p_2 \colon x_2 \to y_2$ *is a descent morphism, and*
- $p_3 \colon x_3 \to y_3$ *is an almost descent morphism*

in $\mathscr{V}$, then p is an effective descent morphism in $\mathsf{Cat}(T,\mathscr{V})$.

Proof. By Lemma 4.12, our hypotheses guarantee that $\mathscr{K}^p \colon \mathsf{Cat}(T,\mathscr{V}) \downarrow y \to \mathsf{Desc}(p)$ is fully faithful. Hence, our goal is to confirm $\mathscr{K}^p$ is essentially surjective, and we shall do so via Proposition 2.2.

We consider the kernel pair (4.8) of p, and we let $(a \colon v \to x, \gamma \colon r \times_x a \to v)$ be a discrete fibration (internal to $\mathsf{Cat}(T,\mathscr{V})$) over $\mathsf{Ker}(p)$.

If p_i is an effective descent morphism for $i = 0,1$, we obtain an equivalence $\mathscr{K}^{p_i} \colon \mathscr{V} \downarrow y_i \to \mathsf{Desc}(p_i)$ for $i = 0,1$. Thus, by Proposition 2.2 there exist $b_i \colon w_i \to y_i$ and $h_i \colon v_i \to w_i$ such that

$$\begin{array}{ccc} v_i & \xrightarrow{\;h_i\;} & w_i \\ {\scriptstyle a_i}\big\downarrow & \ulcorner & \big\downarrow{\scriptstyle b_i} \\ x_i & \xrightarrow[\;p_i\;]{} & y_i \end{array} \tag{4.10}$$

is a pullback diagram, satisfying

$$h_i \circ \gamma_i = h_i \circ \varepsilon_{p_i \circ a_i} \tag{4.11}$$

for $i = 0,1$.

We claim that w is a T-category, h_0, h_1 define a functor $h \colon v \to w$, and b_0, b_1 define a functor $b \colon w \to y$. To do so, we consider the kernel pairs of h_0 and h_1; given that p_0 and p_1 are descent morphisms, h_0 and h_1 are the coequalizers of their kernel pairs. Moreover, since T preserves kernel pairs, we obtain

$$\begin{array}{ccccc} u_1 & \rightrightarrows & v_1 & \xrightarrow{\;h_1\;} & w_i \\ {\scriptstyle d_i}\big\downarrow & & \big\downarrow{\scriptstyle d_i} & & \big\downarrow{\scriptstyle d_i} \\ T^i u_0 & \rightrightarrows & T^i v_0 & \xrightarrow[\;h_0\;]{} & T^i w_0 \end{array}$$

which provides the T-graph structure of w. With this, we obtain the following cospan of cospans

$$\begin{array}{ccccc} Tw_1 & \xrightarrow{\;Td_0\;} & Tw_0 & \xleftarrow{\;d_1\;} & w_1 \\ {\scriptstyle Tb_1}\big\downarrow & & \big\downarrow{\scriptstyle b_0} & & \big\downarrow{\scriptstyle b_1} \\ Ty_1 & \xrightarrow{\;Td_0\;} & Ty_0 & \xleftarrow{\;d_1\;} & y_1 \\ {\scriptstyle Tp_1}\big\uparrow & & \big\uparrow{\scriptstyle p_0} & & \big\uparrow{\scriptstyle p_1} \\ Tx_1 & \xrightarrow{\;Td_0\;} & Tx_0 & \xleftarrow{\;d_1\;} & x_1 \end{array} \tag{4.12}$$

and since T preserves pullbacks, the horizontal and vertical pullbacks of (4.12) are, respectively

$$\begin{array}{c} w_2 \xrightarrow{\;b_2\;} y_2 \xleftarrow{\;p_2\;} x_2 \\[2em] Tv_1 \xrightarrow{\;Td_0\;} Tv_0 \xleftarrow{\;d_1\;} v_1 \end{array} \tag{4.13}$$

so, by commutativity of limits, the cospans (4.13) have isomorphic pullbacks. Since the pullback of the last span defines v_2, we obtain the following pullback diagram:

$$\begin{array}{ccc} v_2 & \xrightarrow{h_2} & w_2 \\ {\scriptstyle a_2}\downarrow & \ulcorner & \downarrow{\scriptstyle b_2} \\ x_2 & \xrightarrow[p_2]{} & y_2 \end{array} \tag{4.14}$$

Analogously, we deduce that

$$\begin{array}{ccc} v_3 & \xrightarrow{h_3} & w_3 \\ {\scriptstyle a_3}\downarrow & \ulcorner & \downarrow{\scriptstyle b_3} \\ x_3 & \xrightarrow[p_3]{} & y_3 \end{array} \tag{4.15}$$

is a pullback diagram as well.

If p_2 is also a descent morphism, we conclude via (4.14) that h_2 is a regular epimorphism as well. So, we may consider its kernel pair as well, to obtain

$$\begin{array}{ccccc} u_0 & \rightrightarrows & v_0 & \xrightarrow{h_0} & w_0 \\ {\scriptstyle s_0}\downarrow & & \downarrow{\scriptstyle s_0} & & \downarrow{\scriptstyle s_0} \\ u_1 & \rightrightarrows & v_1 & \xrightarrow{h_1} & w_1 \\ {\scriptstyle d_1}\uparrow & & \uparrow{\scriptstyle d_1} & & \uparrow{\scriptstyle d_1} \\ u_2 & \rightrightarrows & v_2 & \xrightarrow{h_2} & w_2 \end{array}$$

which give the identity and composition structure morphisms for w. Indeed, under the hypothesis w is a T-category, we can already conclude that $h\colon v \to w$ is a functor.

To prove this hypothesis, note that if p_3 is also an almost descent morphism, then so is h_3 by (4.15). Now, we note that we have

$$\begin{aligned} d_1 \circ s_0 \circ h_0 &= Th_0 \circ d_1 \circ s_0 = Th_0 \circ e = e \circ h_0 \\ d_0 \circ s_0 \circ h_0 &= h_0 \circ d_0 \circ s_0 = h_0 \\ d_1 \circ d_1 \circ h_2 &= Th_0 \circ d_1 \circ d_1 = Th_0 \circ m \circ Td_1 \circ d_2 = m \circ Td_1 \circ d_2 \circ h_2 \\ d_0 \circ d_1 \circ h_2 &= h_0 \circ d_0 \circ d_1 = h_0 \circ d_0 \circ d_0 = d_0 \circ d_0 \circ h_2 \\ d_1 \circ s_i \circ h_1 &= h_1 \circ d_1 \circ s_i = h_1 \circ s_0 \circ d_0 = s_0 \circ d_0 \circ h_1 \\ d_1 \circ d_2 \circ h_3 &= h_1 \circ d_1 \circ d_2 = h_1 \circ d_1 \circ d_1 = d_1 \circ d_1 \circ h_3 \end{aligned}$$

so, by cancellation, we conclude that w is a T-category.

Finally, we must confirm b_0, b_1 define a functor $b\colon w \to y$. We recall that h_0, h_1, h_2 are epimorphisms, and we observe that

$$\begin{aligned} b_1 \circ d_1 \circ h_2 &= b_1 \circ h_1 \circ d_1 = p_1 \circ a_1 \circ d_1 = d_1 \circ p_2 \circ a_2 = d_1 \circ b_2 \circ h_2 \\ d_i \circ b_1 \circ h_1 &= d_i \circ p_1 \circ a_1 = T^i p_0 \circ T^i a_0 \circ d_i = T^i b_0 \circ T^i h_0 \circ d_i = T^i b_0 \circ d_i \circ h_1, \end{aligned}$$

confirming that the properties for a functor are satisfied, via cancellation.

Since (4.10) is a pullback diagram for $i = 0, 1$, we obtain a pullback diagram

$$\begin{array}{ccc} v & \xrightarrow{h} & w \\ {\scriptstyle a}\downarrow & \ulcorner & \downarrow{\scriptstyle b} \\ x & \xrightarrow[p]{} & y \end{array}$$

and we have

$$h \circ \gamma = h \circ \varepsilon_{p\circ a} \tag{4.16}$$

as a consequence of (4.11) for $i = 0, 1$. This concludes our proof, by Proposition 2.2. □

4.5 Application to graded, operadic and enhanced multicategories

Let $\mathscr{V}$ be a category with finite limits, and T a cartesian monad on $\mathscr{V}$. We recall that we have an equivalence

$$\mathsf{Cat}(T, \mathscr{V}) \downarrow x \simeq \mathsf{Cat}(T_x, \mathscr{V} \downarrow x_0) \tag{4.17}$$

for any internal $(T, \mathscr{V})$-category x, where T_x is a cartesian monad on $\mathscr{V} \downarrow x_0$ induced by x; see [41, Section 6.2]. Since $\mathscr{C} \downarrow x \to \mathscr{C}$ creates (effective, almost) descent morphisms, we can obtain results about effective descent morphisms in $\mathsf{Cat}(T_x, \mathscr{V} \downarrow x_0)$ by studying those of $\mathsf{Cat}(T, \mathscr{V})$.

To illustrate this, let $\mathscr{V}$ be a category with finite limits. For an internal category $\mathscr{C}$ in $\mathscr{V}$, the monad on $\mathscr{V} \downarrow \mathscr{C}_0$ induced by the identity monad on $\mathscr{V}$ is denoted $\mathscr{C} \times_{\mathscr{C}_0} -$. Via the equivalence (4.17), the category of $\mathscr{C}$-graded categories internal to $\mathscr{V}$ is the category of internal functors over $\mathscr{C}$:

$$\mathsf{Cat}(\mathscr{C} \times_{\mathscr{C}_0} -, \mathscr{V} \downarrow \mathscr{C}_0) \simeq \mathsf{Cat}(\mathscr{V}) \downarrow \mathscr{C}.$$

Hence, the study of effective descent functors of graded internal categories reduces to the study of effective descent functor of internal categories. Of particular importance is the case $\mathscr{C}_0 \cong 1$; that is, when $\mathscr{C}$ is an internal $\mathscr{V}$-monoid.

Now, let $\mathscr{V}$ be a lextensive category. The free monoid monad $\mathfrak{M}$ on $\mathscr{V}$, given on objects by

$$X \mapsto \sum_{n \in \mathbb{N}} X^n,$$

is a cartesian monad, hence we may consider the category of *multicategories* internal to $\mathscr{V}$, given by $\mathsf{MultiCat}(\mathscr{V}) = \mathsf{Cat}(\mathfrak{M}, \mathscr{V})$. By Theorem 4.13, a multicategory functor $p\colon x \to y$ internal to $\mathscr{V}$ is an effective descent morphism in $\mathsf{MultiCat}(\mathscr{V})$, provided that

- $p_1 \colon x_1 \to y_1$ is an effective descent morphism in $\mathscr{V}$,
- $p_2 \colon x_2 \to y_2$ is a descent morphism in $\mathscr{V}$,
- $p_3 \colon x_3 \to y_3$ is an almost descent morphism in $\mathscr{V}$.

If we have an internal $\mathscr{V}$-operad $\mathfrak{O}$ (an internal $\mathscr{V}$-multicategory with terminal object-of-objects), the monad $T_{\mathfrak{O}}$ induced by the free monoid monad is given by

$$X \mapsto \sum_{n \in \mathbb{N}} \mathfrak{O}_n \times X^n,$$

and is cartesian as well. We define the objects of $\mathsf{Cat}(T_{\mathfrak{O}},\mathscr{V})$ to be the *operadic* multicategories internal to $\mathscr{V}$. Via (4.17), we obtain

$$\mathsf{Cat}(T_{\mathfrak{O}},\mathscr{V}) \simeq \mathsf{MultiCat}(\mathscr{V}) \downarrow \mathfrak{O},$$

so a morphism of operadic multicategories is effective for descent if it is so as a morphism on the underlying internal multicategories, which we have described above.

Finally, we consider the 2-monad $\mathsf{Fam}_{\mathsf{fin}}$ on Cat, the *free finite coproduct completion* 2-monad. This is one of the central objects of study in [60], where it was shown to be a cartesian (2-)monad. Hence, we may consider $\mathsf{Fam}_{\mathsf{fin}}$-categories internal to Cat, and we can obtain a description of the effective descent morphisms of $\mathsf{Cat}(\mathsf{Fam}_{\mathsf{fin}},\mathsf{Cat})$ via Theorem 4.13.

Moreover, we have a cartesian 2-natural transformation $\mathfrak{S} \to \mathsf{Fam}_{\mathsf{fin}}$ (see [60, Example 7.5]), where $\mathfrak{S}$ is the free symmetric strict monoidal category 2-monad, and the objects of $\mathsf{Cat}(\mathfrak{S},\mathsf{Cat})$ were called *enhanced symmetric multicategories* in [41, p. 212], whose study of effective descent functors reduces to the previous case.

By analogy, we may take the objects of $\mathsf{Cat}(\mathsf{Fam}_{\mathsf{fin}},\mathsf{Cat})$ to be the enhanced *cocartesian* multicategories, and analogously, we have the free finite product completion 2-monad $\mathsf{Fam}^*_{\mathsf{fin}}$ on Cat, defined on objects by $\mathscr{A} \mapsto \mathsf{Fam}_{\mathsf{fin}}(\mathscr{A}^{\mathsf{op}})^{\mathsf{op}}$, which is also cartesian, and the objects of $\mathsf{Cat}(\mathsf{Fam}^*_{\mathsf{fin}},\mathsf{Cat})$ may be called enhanced *cartesian* multicategories[2].

For an ordinary functor $F : \mathscr{C} \to \mathscr{D}$ of categories, we have that

- F is an almost descent functor if F is surjective on morphisms,
- F is a descent functor if F is surjective on 2-chains,
- F is an effective descent functor if F is surjective on 3-chains,

so, for T a cartesian monad on Cat (such as one of $\mathsf{Fam}_{\mathsf{fin}}, \mathsf{Fam}^*_{\mathsf{fin}}, \mathfrak{S}$), a functor $P\colon \mathscr{X} \to \mathscr{Y}$ of T-categories is effective for descent if

- $P_1 : \mathscr{X}_1 \to \mathscr{Y}_1$ is surjective on 3-chains,
- $P_2 : \mathscr{X}_2 \to \mathscr{Y}_2$ is surjective on 2-chains,
- $P_3 : \mathscr{X}_3 \to \mathscr{Y}_3$ is surjective on morphisms.

[2] These are related to the *multi-sorted Lawvere theories* – see [17], or Subsection 5.4.2

Chapter 5

Generalized enriched multicategory functors

Equipped with the description of effective descent morphisms in $\mathsf{Cat}(T,\mathscr{V})$ given in Theorem 4.13, our goal is to study effective descent morphisms in a category of enriched generalized multicategories, by suitably embedding it into a category of internal generalized multicategories, and studying whether the effective descent morphisms are reflected by this embedding. In short, we aim to generalize the approach of [45, Theorem 9.11] for the embedding $\mathscr{V}\text{-}\mathsf{Cat} \to \mathsf{Cat}(\mathscr{V})$ to the setting of generalized multicategorical structures.

Such an embedding is constructed via a suitable notion of *change-of-base* for generalized multicategories. This work was carried out in [53] from a general point-of-view; here, we recount the details that are relevant in the study of effective descent functors for enriched generalized multicategories.

The notion of enriched $(T,\mathscr{V})$-categories was introduced in [16], under the terminology $(T,\mathscr{V})$-categories, as a suitable notion of *lax algebras*. In Section 5.1, we will provide the definition in a slightly more general setting, as done in [55] and [26] (when $\mathscr{V}$ is a suitable quantale), as well as [17]. We consider a monad $T = (T,m,e)$ on $\mathscr{V}\text{-}\mathsf{Mat}$ in $\mathsf{Equip}_{\mathsf{lax}}$ – the 2-category of *equipments*, *lax functors* and *icons*, and the enriched $(T,\mathscr{V})$-categories are defined as a suitable notion of lax algebra. The original setting of [16] is recovered when T is a *normal* lax monad.

Working in the 2-category $\mathsf{Equip}_{\mathsf{lax}}$, in Section 5.2, we review the lifting of the functor $-\cdot 1\colon \mathsf{Set} \to \mathscr{V}$ to a functor $-\cdot 1\colon \mathscr{V}\text{-}\mathsf{Mat} \to \mathsf{Span}(\mathscr{V})$ in Proposition 5.1, and we describe the monad $\overline{T}$ on $\mathscr{V}\text{-}\mathsf{Mat}$, given by reflecting the monad T on $\mathsf{Span}(\mathscr{V})$ along $-\cdot 1$. The results of [53] on change-of-base for generalized categorical structures then confirm that we obtain an embedding

$$-\cdot 1\colon (\overline{T},\mathscr{V})\text{-}\mathsf{Cat} \to \mathsf{Cat}(T,\mathscr{V}), \tag{5.1}$$

under a suitable condition (Theorem 5.2).

In Section 5.3, we study the problem of reflecting effective descent morphisms along the embedding (5.1). We confirm that, under a second suitable condition, every effective descent morphism is reflected (Lemma 5.5). Then, via our description of the effective descent morphisms in $\mathsf{Cat}(T,\mathscr{V})$, we obtain Theorem 5.6, the main result of this chapter.

We finish this chapter with Section 5.4, where we provide our applications in the study of effective descent morphisms in categorical structures, after briefly discussing whether the two extra conditions introduced in Sections 5.2 and 5.3 affect the scope of the available examples.

5.1 $(T,\mathscr{V})$-categories

Throughout this chapter, $\mathscr{V}$ is assumed to be a lextensive, cartesian monoidal category. We recall that $\mathscr{V}$ is necessarily distributive; see [10, Proposition 4.5].

The equipment of $\mathscr{V}$-matrices, denoted $\mathscr{V}$-Mat, is defined in [5]; we simply recall that a $\mathscr{V}$-matrix r is a family $(r(x,y))_{x,y\in X\times Y}$ of objects in $\mathscr{V}$ indexed by $X\times Y$. Such a $\mathscr{V}$-matrix shall be denoted as $r\colon X\to Y$. If $s\colon Y\to Z$ is another $\mathscr{V}$-matrix, the composite $s\cdot r\colon X\to Z$ is given at x,z by

$$(s\cdot r)(x,z)=\sum_{y\in Y} s(y,z)\times r(x,y),$$

and for every set X, the unit $\mathscr{V}$-matrix $1_X\colon X\to X$ is given at x,y by the terminal object if $x=y$, and the initial object if $x\neq y$.

We let $\mathsf{Equip}_{\mathsf{lax}}$ be the 2-category of *equipments*, *lax functors*, and *icons* [37], and we let $T=(T,m,e)$ be a monad on $\mathscr{V}$-Mat in the 2-category $\mathsf{Equip}_{\mathsf{lax}}$. We remark that T has an underlying monad on Set, and it can be shown that T is its *lax extension*, under the terminology of [16, p. 18], provided we relax the condition $T1_r=1_{Tr}$ for $\mathscr{V}$-matrices r. This was observed in [26, Subsection 1.13] when $\mathscr{V}$ is a quantale, and in [17, Appendix B].

The data for such a lax monad T consists of

- a set TX for each set X,
- a $\mathscr{V}$-matrix $Tr\colon TX\to TY$ for each $\mathscr{V}$-matrix $r\colon X\to Y$,
- for each set X, a family of comparison morphisms

  $$\mathsf{e}^T_X\colon 1\to (T1_X)(\mathfrak{x},\mathfrak{x})$$

  indexed by $\mathfrak{x}\in TX$,
- for each pair of $\mathscr{V}$-matrices $r\colon X\to Y$, $s\colon Y\to Z$, a family of comparison morphisms

  $$\mathsf{m}^T_{r,s}\colon (Ts\cdot Tr)(\mathfrak{x},\mathfrak{z})\to (T(s\cdot r))(\mathfrak{x},\mathfrak{z})$$

  indexed by $\mathfrak{x}\in TX$, $\mathfrak{z}\in TZ$,
- for each $\mathscr{V}$-matrix $r\colon X\to Y$, a family of morphisms

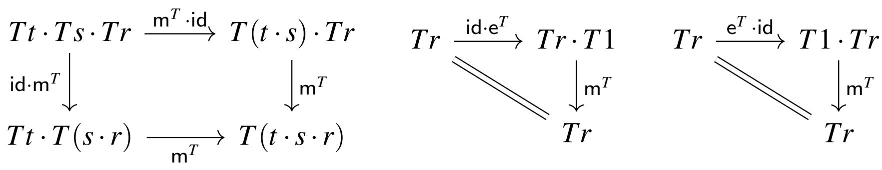
  $$e_{r,x,y}\colon r(x,y)\to (Tr)(e(x),e(y))$$

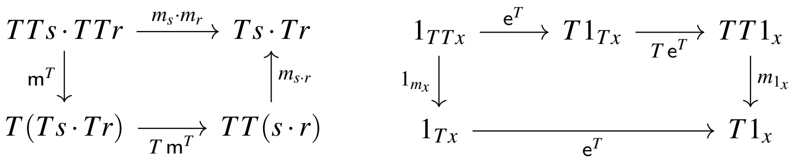
  indexed by $x\in X$, $y\in Y$,
- for each $\mathscr{V}$-matrix $r\colon X\to Y$, a family of morphisms

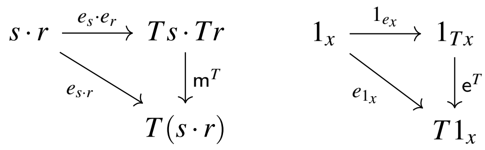
  $$m_{r,\mathfrak{x},\mathfrak{y}}\colon (TTr)(\mathfrak{x},\mathfrak{y})\to (Tr)(m(\mathfrak{x}),m(\mathfrak{y}))$$

indexed by $\mathfrak{x} \in TTX$, $\mathfrak{y} \in TTY$.

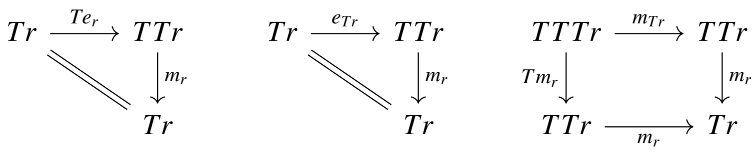
satisfying the following coherence conditions, where we omit the indexing elements, as well as the associator and unitor isomorphisms for convenience:







for each 3-chain of $\mathscr{V}$-matrices r, s, t, as well as the following associativity and identity conditions:



where we have also omitted the indexing elements.

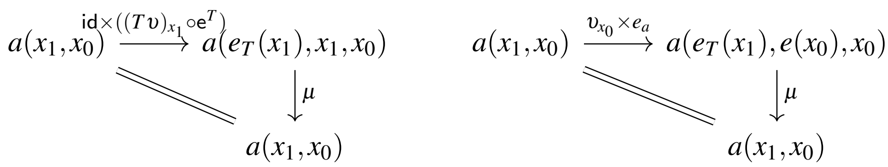
An enriched $(T,\mathscr{V})$-category is a quadruple (X,a,υ,μ), where X is a set, $a\colon TX \to X$ is a $\mathscr{V}$-matrix, υ is a family of morphisms $\upsilon_x\colon 1 \to a(e(x),x)$ indexed by $x \in X$, and μ is a family of morphisms

$$\mu_{x_0,x_1,x_2}\colon a(x_2,x_1,x_0) \to a(m(x_2),x_0)$$

indexed by $x_i \in T^iX$, where we define

$$a(x_2,x_1,x_0) = a(x_1,x_0) \times (Ta)(x_2,x_1) \quad \text{and} \quad a(x_3,x_2,x_1,x_0) = a(x_2,x_1,x_0) \times (TTa)(x_3,x_2)$$

for $x_i \in T^iX$. These families satisfy the following identity and associativity laws:



$$\begin{array}{ccc} a(x_3,x_2,x_1,x_0) & \xrightarrow{\mathsf{id}\times(T\mu\circ\mathsf{m}^T)} & a((Tm)(x_3),x_1,x_0) \\ {\scriptstyle \mu\times m_a}\downarrow & & \downarrow{\scriptstyle \mu} \\ a((mT)(x_3),m(x_2),x_0) & \xrightarrow[\mu]{} & a((m\circ mT)(x_3),x_0) \end{array}$$

5.2 Change-of-base and embedding

Let $\mathscr{V}$ be a lextensive category. The following diagram depicts the adjunction fundamental to our study of viewing enriched generalized multicategories as internal generalized multicategories:

$$\mathsf{Set} \underset{\mathscr{V}(1,-)}{\overset{-\cdot 1}{\rightleftarrows}} \mathscr{V} \quad (\perp) \tag{5.2}$$

We shall denote the counit of this adjunction by $\hat{\varepsilon}$. As was done in [45, Theorem 9.11], we will assume for the remainder of this chapter that $-\cdot 1$ is fully faithful, so that Set may be the seen as the full subcategory of $\mathscr{V}$ consisting of the *discrete objects*. As observed in [59, Lemma 2.2.1] (when $\mathscr{V}$ is a presheaf category), we have:

Proposition 5.1 ([53, Remark 7.4, Lemma 7.6])**.** *We have an adjunction*

$$\mathscr{V}\text{-}\mathsf{Mat} \underset{\mathscr{V}(1,-)}{\overset{-\cdot 1}{\rightleftarrows}} \mathsf{Span}(\mathscr{V}) \quad (\perp) \tag{5.3}$$

in the 2-category $\mathsf{Equip}_{\mathsf{lax}}$*. Moreover,* $-\cdot 1\colon \mathscr{V}\text{-}\mathsf{Mat} \to \mathsf{Span}(\mathscr{V})$ *is fully faithful, and the underlying adjunction on the categories of objects is precisely* (5.2).

Spans $a\colon X \to Y$ in a category $\mathscr{V}$ are denoted by a diagram such as:

$$X \xleftarrow{\;l_a\;} M_a \xrightarrow{\;r_a\;} Y$$

So, if $a\colon X \to Y$ is a span, the $\mathscr{V}$-matrix $\mathscr{V}(1,a)$ is given at $x \in \mathscr{V}(1,X)$, $y \in \mathscr{V}(1,Y)$ by the pullback

$$\begin{array}{ccc} \mathscr{V}(1,a)(x,y) & \longrightarrow & 1 \\ \downarrow & \ulcorner & \downarrow{\scriptstyle x,y} \\ M_a & \xrightarrow[l_a,r_a]{} & X\times Y \end{array}$$

while if $r\colon S \to T$ is a $\mathscr{V}$-matrix, the span $r \cdot 1$ is given by the coproduct of

$$\begin{array}{ccc} & r(x,y) & \\ \swarrow & & \searrow \\ 1 & & 1 \end{array} \qquad \text{indexed by} \qquad \begin{array}{ccc} & S \times T & \\ {\scriptstyle d_1}\swarrow & & \searrow{\scriptstyle d_0} \\ S & & T \end{array}$$

Now, we let $T = (T, m, e)$ be a cartesian monad on $\mathscr{V}$. By [24, Proposition A.2], T induces a pseudomonad on $\mathsf{Span}(\mathscr{V})$, which may be seen as a monad in $\mathsf{Equip}_{\mathsf{lax}}$, which we also denote by T; see [17, Example A.6].

Via (5.3), the monad T on $\mathsf{Span}(\mathscr{V})$ induces a monad $\overline{T} = (\overline{T}, \overline{m}, \overline{e})$ on $\mathscr{V}$-Mat (see [53, Proposition 8.1]), where

- $\overline{T}a = \mathscr{V}(1, T(a \cdot 1))$,
- $\overline{m}_a = \mathscr{V}(1, m_{a\cdot 1} \circ T\hat{\varepsilon}_{T(a\cdot 1)})$
- $\overline{e}_a = \mathscr{V}(1, e_{a\cdot 1}) \circ \hat{\eta}_a$,

for each $\mathscr{V}$-matrix a. This monad is the lax extension of the monad (also denoted $\overline{T}$) induced by T on Set via (5.2). Under suitable conditions, the category of enriched $(\overline{T}, \mathscr{V})$-categories is embedded in the category of internal $(T, \mathscr{V})$-categories, which are our objects of interest in this Chapter.

Theorem 5.2 ([53, Lemma 9.1, Theorem 9.2])**.** *If $\hat{\varepsilon}_{T(-\cdot 1)}$ is a cartesian natural transformation, we have an adjunction*

$$(\overline{T}, \mathscr{V})\text{-}\mathsf{Cat} \underset{\mathscr{V}(1,-)}{\overset{-\cdot 1}{\rightleftarrows}} \mathsf{Cat}(T, \mathscr{V}) \qquad \bot \tag{5.4}$$

and the functor $- \cdot 1\colon (\overline{T}, \mathscr{V})\text{-}\mathsf{Cat} \to \mathsf{Cat}(T, \mathscr{V})$ is fully faithful.

The embedding functor $- \cdot 1\colon (\overline{T}, \mathscr{V})\text{-}\mathsf{Cat} \to \mathsf{Cat}(T, \mathscr{V})$ is given on an enriched $(\overline{T}, \mathscr{V})$-category (X, a, υ, μ) by the span

$$\begin{array}{ccccc} & & & M_{a\cdot 1} & \\ & & {\scriptstyle l_{a\cdot 1}}\swarrow & & \searrow{\scriptstyle r_{a\cdot 1}} \\ & & \overline{T}X \cdot 1 & & X \cdot 1 \\ & {\scriptstyle \hat{\varepsilon}_{T(X\cdot 1)}}\swarrow & & & \\ T(X \cdot 1) & & & & \end{array}$$

with unit $\hat{\upsilon}$ given by

$$\begin{array}{ccc} 1 & \xrightarrow{\upsilon_x \cdot 1} & a(e(x), x) \cdot 1 \\ {\scriptstyle \hat{\eta}_x}\downarrow & & \downarrow{\scriptstyle \iota_{e(x),x}} \\ X \cdot 1 & \underset{\hat{\upsilon}}{\dashrightarrow} & M_{a\cdot 1} \end{array}$$

via the universal property of the coproduct, and the composition $\hat{\mu}$ is given by

$$\begin{array}{ccc} a(x_2,x_1,x_0)\cdot 1 & \xrightarrow{\mu_{x_2,x_1,x_0}\cdot 1} & a(m(x_2),x_0)\cdot 1 \\ \downarrow & & \downarrow \\ \sum_{x_i\in T^iX} a(x_2,x_1,x_0)\cdot 1 & \underset{\hat{\mu}}{\dashrightarrow} & M_{a\cdot 1} \end{array}$$

via the universal property of the coproduct. The hypothesis that $\hat{\varepsilon}_{T(-\cdot 1)}$ is a cartesian natural transformation ensures that the following diagram

$$\begin{array}{ccc} \sum_{x_i\in T^iX} \overline{T}a(x_2,x_1)\cdot 1 & \longrightarrow & \overline{T}X\cdot 1 \\ {\scriptstyle \hat{\varepsilon}_{T(a\cdot 1)}}\downarrow & \ulcorner & \downarrow{\scriptstyle \hat{\varepsilon}_{T(X\cdot 1)}} \\ TM_{a\cdot 1} & \xrightarrow[Tr_{a\cdot 1}]{} & T(X\cdot 1) \end{array}$$

is a pullback square, by [53, Lemma 8.3], thereby guaranteeing that

$$\sum_{x_i\in T^iX} a(x_2,x_1,x_0)$$

is the object of 2-chains of $(X,a,\upsilon,\mu)\cdot 1$.

If $(f,\phi)\colon (X,a,\upsilon,\mu)\to(Y,b,\upsilon,\mu)$ is an enriched $(\overline{T},\mathcal{V})$-functor, $(f,\phi)\cdot 1$ is given on objects by $f\cdot 1\colon X\cdot 1\to Y\cdot 1$, and $\phi\cdot 1\colon M_{a\cdot 1}\to M_{b\cdot 1}$ on morphisms.

5.3 Reflection of effective descent morphisms

Having established all the necessary notation, we can now proceed to study the effective descent morphisms in $(\overline{T},\mathcal{V})$-Cat, by studying whether the embedding $-\cdot 1\colon (\overline{T},\mathcal{V})\text{-Cat}\to \mathsf{Cat}(T,\mathcal{V})$ reflects effective descent morphisms. The key idea, developed by the next result, is that we must guarantee that the full inclusion $(\overline{T},\mathcal{V})\text{-Cat}\to\mathsf{Cat}(T,\mathcal{V})$ consists precisely of those internal $(T,\mathcal{V})$-categories with a discrete object of objects, which could not be guaranteed in general.

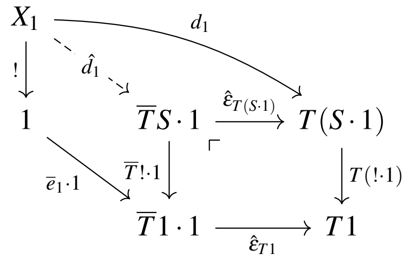
Lemma 5.3 ([53, Lemma 10.2])**.** *Let X be an internal $(T,\mathcal{V})$-category whose object-of-objects is discrete; that is, $X_0\cong S\cdot 1$ for a set S.*

If we let a be the span in $\mathcal{V}$ given by the underlying T-graph of X, as depicted in (5.5)*,*

$$\begin{array}{ccccc} & & X_1 & & \\ & {\scriptstyle d_1}\swarrow & & \searrow{\scriptstyle d_0} & \\ TX_0 & & & & X_0 \end{array} \tag{5.5}$$

then $\hat{\varepsilon}_a$ is a split epimorphism. Moreover, if $\hat{\varepsilon}_{T1}\colon \overline{T}1\cdot 1\to T1$ is a monomorphism, then $\hat{\varepsilon}_a$ is an isomorphism.

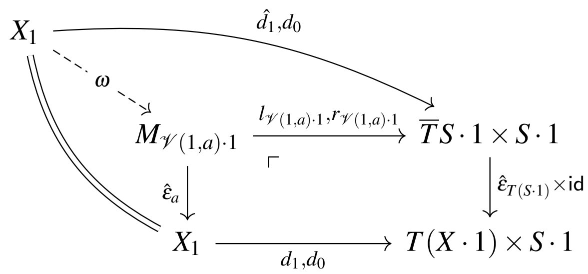
Proof. We may assume that $X_0 = S \cdot 1$. Our first step is to notice that $d_1 \colon X_1 \to T(S \cdot 1)$ factors uniquely through $\hat{\varepsilon}_{T(S\cdot 1)}$; we have $e_1 = \hat{\varepsilon}_{T1} \circ (\overline{e}_1 \cdot 1)$ (by definition of $\overline{e}$), so there exists a unique $\hat{d_1}$, depicted by dashed morphism in (5.6)



(5.6)

making the adjacent diagrams commute.

We may conclude that there is a unique morphism $\omega \colon X_1 \to M_{\mathscr{V}(1,a)\cdot 1}$ such that $\hat{\varepsilon}_a \circ \omega = \mathsf{id}$ (confirming that $\hat{\varepsilon}_a$ is a split epimorphism) and $(\hat{d_1}, d_0) = (d_1, d_0) \circ \omega$, as depicted in (5.7)[1]



(5.7)

Moreover, if $\hat{\varepsilon}_{T1}$ is a monomorphism, then, by the pullback square in (5.6), so is $\hat{\varepsilon}_{T(S\cdot 1)}$, and by the pullback square in (5.7), we conclude $\hat{\varepsilon}_a$ is a monomorphism. Therefore, $\hat{\varepsilon}_a$ must be invertible. □

As a corollary, we conclude that the enriched $(\overline{T}, \mathscr{V})$-categories are precisely the internal T-categories whose object of objects is discrete. More precisely, we have:

Lemma 5.4 ([53, Theorem 10.3]). *If $\hat{\varepsilon}_{T1}$ is a monomorphism, we have a pseudopullback diagram*

$$\begin{array}{ccc} (\overline{T}, \mathscr{V})\text{-}\mathsf{Cat} & \xrightarrow{-\cdot 1} & \mathsf{Cat}(T, \mathscr{V}) \\ \downarrow & \cong & \downarrow \\ \mathsf{Set} & \xrightarrow[-\cdot 1]{} & \mathscr{V} \end{array}$$

of categories with pullbacks and pullback-preserving functors.

Proof. The objects of the pseudopullback are triples (S, X, ϕ) where S is a set, X is an internal $(T, \mathscr{V})$-category, and ϕ is an isomorphism $\phi \colon S \cdot 1 \to X$. By Lemma 5.3, it follows that $\hat{\varepsilon}_a$ is invertible, where a is the span given by the underlying T-graph of X, as in (5.5).

By general remarks about change-of-base adjunctions between horizontal lax algebras given in [53, Section 6], this implies that X is isomorphic to an enriched $(\overline{T}, \mathscr{V})$-category. □

Everything is set up to apply the results about effective descent morphisms in bilimits from Chapter 2, which gives the following reflection result:

[1] It should be noted that $\hat{\varepsilon}_a$ is *defined* by this pullback square, see [53, (2.5)], noting that $\mathscr{V}$ is lextensive.

Lemma 5.5 ([53, Lemma 10.4])**.** *If $\hat{\varepsilon}_{T1}$ is a monomorphism, then $-\cdot 1\colon (\overline{T},\mathscr{V})\text{-}\mathsf{Cat} \to \mathsf{Cat}(T,\mathscr{V})$ reflects effective descent morphisms.*

Proof. We follow the same approach as Theorem 3.9. As stated in Remark 2.9, the functor $\mathsf{Cat}(T,\mathscr{V}) \to \mathscr{V}$ preserves descent morphisms, which are reflected by $-\cdot 1\colon \mathsf{Set} \to \mathscr{V}$.

Since descent morphisms in Set are effective for descent, we may apply Proposition 2.6 and Lemma 5.4 to conclude our result. □

Now, we can apply our knowledge of effective descent morphisms in $\mathsf{Cat}(T,\mathscr{V})$ to obtain the main result of this chapter:

Theorem 5.6 ([53, Theorem 10.5])**.** *Let $(f,\phi)\colon (X,a,\upsilon,\mu) \to (Y,b,\upsilon,\mu)$ be a functor of enriched $(\overline{T},\mathscr{V})$-categories. If $\hat{\varepsilon}_{T1}$ is a monomorphism, and if*

- *$((f,\phi)\cdot 1)_1$ is an effective descent morphism,*
- *$((f,\phi)\cdot 1)_2$ is a descent morphism,*
- *$((f,\phi)\cdot 1)_3$ is an almost descent morphism,*

then (f,ϕ) is an effective descent morphism in $(\overline{T},\mathscr{V})$-Cat.

Proof. By Theorem 4.13, the above conditions guarantee that $(f,\phi)\cdot 1$ is an effective descent morphism in $\mathsf{Cat}(T,\mathscr{V})$. Since $\hat{\varepsilon}_{T1}$ is a monomorphism, we may apply Lemma 5.5 to conclude that (f,ϕ) is an effective descent morphism in $(\overline{T},\mathscr{V})$-Cat. □

5.4 Scope of the findings

Our main result holds in the context of a lextensive category $\mathscr{V}$ such that $-\cdot 1\colon \mathsf{Set} \to \mathscr{V}$ is fully faithful. These properties are enjoyed by the categories Cat, Top, any connected Grothendieck topos, and any free coproduct completion $\mathsf{Fam}(\mathscr{B})$ of a category $\mathscr{B}$ with finite limits. Moreover, we have required two more hypotheses:

(a) $\hat{\varepsilon}_{T(-\cdot 1)}$ is a cartesian natural transformation.
(b) $\hat{\varepsilon}_{T1}$ is a monomorphism.

We can verify that (b) holds when

- the terminal object is a *separator*, that is, when $\mathscr{V}(1,-)$ is faithful, so that $\hat{\varepsilon}$ is a componentwise monomorphism. This is the case for Cat, Top, any *hyperconnected* Grothendieck topos (by definition), but not the case for Grph nor Fam(Set)[2],
- T is *discrete*, that is $\hat{\varepsilon}_{T1}$ is an isomorphism. This is the case when T is the free monoid monad on any category $\mathscr{V}$ under the above conditions, but not when T is the free category monad on Grph.

Further discussion may be found in [53, Section 10].

On the other hand, we have found no examples of cartesian monads T on $\mathscr{V}$ that do not satisfy (a). This is discussed at length in [53, Section 8]; here we merely recall the results required to discuss our examples.

[2] These are Grothendieck toposes which are not hyperconnected.

5.4.1 Classical multicategories

The free monoid monad $\mathfrak{M}$ on $\mathscr{V}$ satisfies (a) and (b) [53, Lemma 8.7 and Subsection 10.2]. In this case, we let $\mathscr{V}\text{-}\mathsf{MultiCat} = (\overline{\mathfrak{M}}, \mathscr{V})\text{-}\mathsf{Cat}$ be the category of *enriched $\mathscr{V}$-multicategories*.

If $(f, \phi) \colon (X, a, \upsilon, \mu) \to (Y, b, \upsilon, \mu)$ is a functor of enriched $\mathscr{V}$-multicategories such that

$$\phi \colon \sum_{x_i \in (\mathfrak{M}^i f)^*(y_i)} a(x_1, x_0) \to b(y_1, y_0)$$

is an effective descent morphism,

$$\phi \times \mathfrak{M}\phi \colon \sum_{x_i \in (\mathfrak{M}^i f)^*(y_i)} a(x_2, x_1, x_0) \to b(y_2, y_1, y_0)$$

is a descent morphism, and

$$\phi \times \mathfrak{M}\phi \times \mathfrak{M}^2\phi \colon \sum_{x_i \in (\mathfrak{M}^i f)^*(y_i)} a(x_3, x_2, x_1, x_0) \to b(y_3, y_2, y_1, y_0)$$

is an almost descent morphism, for all $y_i \in \mathfrak{M}^i Y$ with $i = 0, 1, 2, 3$, then (f, ϕ) is an effective descent morphism in $\mathscr{V}\text{-}\mathsf{MultiCat}$, by Theorem 5.6.

5.4.2 Cartesian and cocartesian multicategories

The free finite coproduct completion $\mathsf{Fam}_{\mathsf{fin}}$ on Cat satisfies (a) [53, Subsection 8.5]. Since the terminal category is a separator in Cat, we conclude that (b) holds. Thus, by Lemma 5.4, $(\overline{\mathsf{Fam}}_{\mathsf{fin}}, \mathsf{Cat})\text{-}\mathsf{Cat}$ is the category of enhanced cocartesian multicategories with a discrete object-of-objects. For this reason, we refer to its objects as cocartesian multicategories. Likewise, $(\overline{\mathsf{Fam}}^*_{\mathsf{fin}}, \mathsf{Cat})\text{-}\mathsf{Cat}$ is the category of cartesian multicategories[3].

Via the description of effective descent morphisms for functors of enhanced multicategories, given in Section 4.5 we obtain a description of the effective descent functors for cartesian and cocartesian multicategories.

5.4.3 Graded, operadic and symmetric multicategories

Let S, T be endofunctors on $\mathscr{V}$, and let $\alpha \colon S \to T$ be a cartesian natural transformation. If T satisfies (a), so does S. Thus, it follows that (a) is satisfied by $\mathscr{V}$-operadic monads, as well as the free symmetric strict monoidal category monad $\mathfrak{S}$, when $\mathscr{V} = \mathsf{Cat}$.

However, we do not guarantee that every $\mathscr{V}$-operadic monad satisfies (b) in general. It certainly is true if the terminal object of $\mathscr{V}$ is a separator, and if $\mathfrak{O}$ is a $\mathscr{V}$-operad such that $\mathfrak{O}_n$ is discrete for all $n \in \mathbb{N}$, then the $\mathscr{V}$-operadic monad $\mathfrak{M}_{\mathfrak{O}}$ induced by $\mathfrak{O}$ is discrete, so the property is also satisfied in this setting. We call such $\mathscr{V}$-operads *discrete*.

If the $\mathscr{V}$-operad $\mathfrak{O}$ is discrete, we let $(\overline{\mathfrak{M}}_{\mathfrak{O}}, \mathscr{V})\text{-}\mathsf{Cat}$ be the category of *enriched $\mathfrak{O}$-categories*. Via the results of 4.5, and Theorem 5.6, we obtain a description for the effective descent functors of

[3]These are a *wide* subcategory of the category of multi-sorted Lawvere theories – the morphisms between cartesian multicategories are precisely the "degree one" morphisms between Lawvere theories. See also [17, Example 4.17].

enriched $\mathfrak{O}$-categories. In particular, we also obtain the *enriched graded multicategories* by a discrete $\mathscr{V}$-monoid M, and a description of the respective effective descent functors.

Since the terminal category is a separator in Cat, it follows that (b) is satisfied for $\mathfrak{S}$. Arguing as we did in the case of (co)cartesian multicategories, we let $(\overline{\mathfrak{S}}, \mathsf{Cat})$-$\mathsf{Cat}$ be the category of *symmetric multicategories*, for which we also obtain a description of the respective effective descent functors.

Chapter 6

Fibration of split opfibrations

The bifibration $F_D = \mathsf{CAT}(-,\mathsf{Set})\colon \mathsf{Cat}^{\mathsf{op}} \to \mathsf{CAT}$ of discrete opfibrations is the main object of study in [56]. Therein, the functors $p\colon E \to B$ of (effective) F_D-descent were characterized. Indeed, p is a F_D-descent morphism if and only if p is a *lax epimorphism*, while p is an effective F_D-descent morphism if and only if p is a *fully faithful* lax epimorphism.

Our goal for this chapter, covering the work done in [47], is to show that the results of [56] for discrete opfibrations can be applied just as well to other settings. Among them, we are able to characterize the effective F-descent morphisms for the bifibration $F = \mathsf{CAT}(-,\mathsf{Cat})\colon \mathsf{Cat}^{\mathsf{op}} \to \mathsf{CAT}$ of split opfibrations. We confirm that a functor p is of (effective) F-descent if and only if it is of (effective) F_D-descent (Theorem 6.11).

We begin Section 6.1 by recalling the notions of fully faithful morphism and lax epimorphism in a 2-category $\mathbb{A}$, as well as a couple of relevant results. Then we restrict our attention to the setting of enriched categories, recalling from [48] the notions of $\mathscr{V}$-fully faithful and $\mathscr{V}$-lax epimorphic $\mathscr{V}$-functors, for $\mathscr{V}$ a complete and cocomplete symmetric monoidal closed category, and comparing them with the notions of fully faithful morphism and lax epimorphism in the 2-category $\mathscr{V}$-Cat of *small* $\mathscr{V}$-categories.

6.1 Fully faithful morphisms and lax epimorphisms

Let $\mathbb{A}$ be a 2-category. A morphism $f\colon x \to y$ is said to be

- *fully faithful* if $\mathbb{A}(w,f)\colon \mathbb{A}(w,x) \to \mathbb{A}(w,y)$ is fully faithful for all w,
- *lax epimorphic* if $\mathbb{A}(f,z)\colon \mathbb{A}(y,z) \to \mathbb{A}(x,z)$ is fully faithful for all z.

A comprehensive study of lax epimorphisms in a 2-category is carried out in [48]. We shall recall some fundamental aspects.

If we have an adjunction $f \dashv g$ in $\mathbb{A}$, the following are equivalent:

- the unit of $f \dashv g$ is invertible,
- g is fully faithful,
- f is a lax epimorphism.

Codually, it follows that the following are equivalent:

– the counit of $f \dashv g$ is invertible,
– g is a lax epimorphism,
– f is fully faithful.

From this, we may conclude that:

Proposition 6.1 ([47, p. 134])**.** *If a morphism f has a left or right adjoint, and is a fully faithful lax epimorphism, then f is an equivalence.*

For the remainder of this chapter, we assume that $\mathscr{V}$ is a complete and cocomplete, symmetric monoidal closed category. We now restrict our scope to the 2-category $\mathbb{A} = \mathscr{V}\text{-}\mathsf{Cat}$ of small $\mathscr{V}$-categories, $\mathscr{V}$-functors and $\mathscr{V}$-natural transformations; this setting was studied in [48, Section 5]. We will recall the pertinent definitions and results therein.

A $\mathscr{V}$-functor $p \colon E \to B$ between small $\mathscr{V}$-categories is said to be

– $\mathscr{V}$-fully faithful if $p \colon E(a,b) \to B(pa,pb)$ is an isomorphism for all objects a,b,
– a $\mathscr{V}$-lax epimorphism if the $\mathscr{V}$-functor

$$\mathscr{V}\text{-}\mathsf{Cat}[p,C] \colon \mathscr{V}\text{-}\mathsf{Cat}[B,C] \to \mathscr{V}\text{-}\mathsf{Cat}[E,C]$$

is $\mathscr{V}$-fully faithful for all small $\mathscr{V}$-categories C [48, Definition 5.4].

Proposition 6.2 ([48, Lemma 5.1])**.** *If $p \colon E \to B$ is a $\mathscr{V}$-fully faithful $\mathscr{V}$-functor, then p is a fully faithful morphism in $\mathscr{V}$-Cat. The converse holds if p has a left or right adjoint.*

Proposition 6.3 ([48, Theorem 5.6])**.** *Let $p \colon E \to B$ be a $\mathscr{V}$-functor. The following are equivalent:*

– *p is a lax epimorphism in $\mathscr{V}$-Cat,*
– *p is a $\mathscr{V}$-lax epimorphism,*
– *the functor $\mathscr{V}\text{-}\mathsf{CAT}(p,\mathscr{V}) \colon \mathscr{V}\text{-}\mathsf{CAT}(B,\mathscr{V}) \to \mathscr{V}\text{-}\mathsf{CAT}(E,\mathscr{V})$ is fully faithful.*

The result analogous to Proposition 6.3 for $\mathscr{V}$-fully faithful $\mathscr{V}$-functors, despite not being a consequence of duality, can be shown to hold as well:

Lemma 6.4 ([47, Proposition 2.4])**.** *A $\mathscr{V}$-functor $p \colon E \to B$ is $\mathscr{V}$-fully faithful if and only if the functor $\mathscr{V}\text{-}\mathsf{CAT}(p,\mathscr{V})$ is a lax epimorphism.*

Proof. We have an adjunction $\mathsf{Lan}_p \dashv \mathscr{V}\text{-}\mathsf{CAT}(p,\mathscr{V})$. From previous remarks, $\mathscr{V}\text{-}\mathsf{CAT}(p,\mathscr{V})$ is a lax epimorphism if and only if Lan_p is fully faithful.

By the enriched Yoneda lemma, p is $\mathscr{V}$-fully faithful if and only if Lan_p is fully faithful (see [34, Proposition 4.23]). ☐

6.2 Enriched Cauchy completion

Let $\mathscr{C}$ be a (possibly large) $\mathscr{V}$-category. An object x on $\mathscr{C}$ is said to be *tiny* (also called *absolutely presentable* in [6] and *small-projective* in [34]) if the representable $\mathscr{V}$-functor $\mathscr{C}(x,-) \colon \mathscr{C} \to \mathscr{V}$ preserves colimits.

The *Cauchy completion* of a small $\mathscr{V}$-category X is the full sub-$\mathscr{V}$-category of tiny objects of $\mathscr{V}\text{-}\mathsf{CAT}[X^{\mathsf{op}},\mathscr{V}]$, and is denoted by $\mathfrak{C}X$. We observe that, in general, $\mathfrak{C}X$ is not a small $\mathscr{V}$-category; for instance, let $\mathscr{V}$ be the category of complete lattices. In [34, Section 5.5], it was shown that $\mathfrak{C}I$ is not small, where I is the unit $\mathscr{V}$-category. Hence, we will assume for the remainder of this chapter that:

$$\mathfrak{C}X \text{ is a small } \mathscr{V}\text{-category for all small } \mathscr{V}\text{-categories } X. \tag{6.1}$$

This property holds for many base categories $\mathscr{V}$ of our interest, such as Cat, Set, and any small quantale. More generally, it was shown in [33] that if the underlying category of $\mathscr{V}$ is locally presentable, then (6.1) holds.

The following result confirms that $\mathscr{V}$-equivalences preserve tiny objects:

Lemma 6.5 ([47, Lemma 2.1]). *Let $F \dashv G\colon \mathscr{D} \to \mathscr{C}$ be a $\mathscr{V}$-adjunction of (possibly large) $\mathscr{V}$-categories. If G preserves colimits, then F preserves tiny objects.*

Proof. If a is tiny, then $\mathscr{D}(Fa,-) \cong \mathscr{C}(a,G(-))$ is a composite of functors that preserve colimits, hence Fa is tiny. ☐

A consequence of Lemma 6.5 is that for any $\mathscr{V}$-functor $p\colon X \to Y$, we may define $\mathfrak{C}p\colon \mathfrak{C}X \to \mathfrak{C}Y$ by restricting the enriched left Kan extension $\mathsf{Lan}_p\colon \mathscr{V}\text{-}\mathsf{CAT}[X,\mathscr{V}] \to \mathscr{V}\text{-}\mathsf{CAT}[Y,\mathscr{V}]$ to the tiny objects. Indeed, we have a chain of $\mathscr{V}$-adjunctions

$$\mathsf{Lan}_p \dashv \mathscr{V}\text{-}\mathsf{CAT}[p,\mathscr{V}] \dashv \mathsf{Ran}_p$$

which confirms that $\mathscr{V}\text{-}\mathsf{CAT}[p,\mathscr{V}]$ preserves colimits, so Lan_p preserves tiny objects.

By the enriched Yoneda lemma, we readily confirm that any representable $\mathscr{V}$-presheaf is tiny,

$$\begin{aligned}\mathscr{V}\text{-}\mathsf{CAT}[X^{\mathsf{op}},\mathscr{V}](X(-,x),\mathsf{colim}(W,F)) &\cong \mathsf{colim}(W,F)x\\ &\cong \mathsf{colim}(W,F(-,x))\\ &\cong \mathsf{colim}(W,\mathscr{V}\text{-}\mathsf{CAT}[X^{\mathsf{op}},\mathscr{V}](X(-,x),F)),\end{aligned}$$

so that the Yoneda $\mathscr{V}$-embedding $\mathfrak{y}\colon X \to \mathscr{V}\text{-}\mathsf{CAT}[X^{\mathsf{op}},\mathscr{V}]$ restricts to a $\mathscr{V}$-functor $\eta_X\colon X \to \mathfrak{C}X$. It also follows that for any $\mathscr{V}$-functor $p\colon X \to Y$, we have a $\mathscr{V}$-natural isomorphism $\eta \circ p \cong \mathfrak{C}p \circ \eta$, and that $\mathscr{V}\text{-}\mathsf{CAT}(\eta,\mathscr{V})$ is an equivalence.

This allows us to highlight the relationship between Cauchy completion and fully faithful morphisms/lax epimorphisms:

Lemma 6.6 ([47, Lemma 2.2]). *If $p\colon X \to Y$ is a $\mathscr{V}$-functor, then the induced functor*

$$\mathscr{V}\text{-}\mathsf{CAT}(p,\mathscr{V})\colon \mathscr{V}\text{-}\mathsf{CAT}(Y,\mathscr{V}) \to \mathscr{V}\text{-}\mathsf{CAT}(X,\mathscr{V})$$

is fully faithful (respectively, a lax epimorphism) if and only if $\mathscr{V}\text{-}\mathsf{CAT}(\mathfrak{C}p,\mathscr{V})$ is fully faithful (respectively, a lax epimorphism).

Proof. We observe that the following diagram commutes up to isomorphism:

$$\begin{array}{ccc} \mathscr{V}\text{-}\mathsf{CAT}(\mathfrak{C}Y,\mathscr{V}) & \xrightarrow{\mathscr{V}\text{-}\mathsf{CAT}(\eta_Y,\mathscr{V})} & \mathscr{V}\text{-}\mathsf{CAT}(Y,\mathscr{V}) \\ {\scriptstyle \mathscr{V}\text{-}\mathsf{CAT}(\mathfrak{C}p,\mathscr{V})}\downarrow & \cong & \downarrow{\scriptstyle \mathscr{V}\text{-}\mathsf{CAT}(p,\mathscr{V})} \\ \mathscr{V}\text{-}\mathsf{CAT}(\mathfrak{C}X,\mathscr{V}) & \xrightarrow[\mathscr{V}\text{-}\mathsf{CAT}(\eta_X,\mathscr{V})]{} & \mathscr{V}\text{-}\mathsf{CAT}(X,\mathscr{V}) \end{array}$$

Since the rows are equivalences, the result follows. □

We immediately conclude that:

Corollary 6.7 ([47, Proposition 2.3])**.** *The following are equivalent for a $\mathscr{V}$-functor $p\colon X \to Y$:*

(i) p is a lax epimorphism,
(ii) $\mathfrak{C}p$ is a lax epimorphism,
(iii) $\mathscr{V}\text{-}\mathsf{CAT}(p,\mathscr{V})$ is fully faithful.
(iv) $\mathscr{V}\text{-}\mathsf{CAT}(\mathfrak{C}p,\mathscr{V})$ is fully faithful.

Proof. The equivalences (i) $\iff$ (iii) and (ii) $\iff$ (iv) were obtained in Proposition 6.3, and the equivalence (iii) $\iff$ (iv) follows by Lemma 6.6. □

Corollary 6.8 ([47, Proposition 2.4])**.** *The following are equivalent for a $\mathscr{V}$-functor $p\colon X \to Y$:*

(i) p is $\mathscr{V}$-fully faithful,
(ii) $\mathfrak{C}p$ is $\mathscr{V}$-fully faithful,
(iii) $\mathscr{V}\text{-}\mathsf{CAT}(p,\mathscr{V})$ is a lax epimorphism.
(iv) $\mathscr{V}\text{-}\mathsf{CAT}(\mathfrak{C}p,\mathscr{V})$ is a lax epimorphism.

Proof. The equivalences (i) $\iff$ (iii) and (ii) $\iff$ (iv) were obtained in Proposition 6.4, and we obtain the equivalence (iii) $\iff$ (iv) via Lemma 6.6. □

Combining Corollaries 6.7 and 6.8, we obtain

Theorem 6.9 ([47, Theorem 2.5])**.** *The following are equivalent for a $\mathscr{V}$-functor $p\colon X \to Y$:*

(i) p is a $\mathscr{V}$-fully faithful lax epimorphism,
(ii) $\mathfrak{C}p$ is a $\mathscr{V}$-fully faithful lax epimorphism,
(iii) $\mathscr{V}\text{-}\mathsf{CAT}(p,\mathscr{V})$ is a fully faithful lax epimorphism,
(iv) $\mathfrak{C}p$ is an equivalence,
(v) $\mathscr{V}\text{-}\mathsf{CAT}(p,\mathscr{V})$ is an equivalence.

Proof. Corollaries 6.7 and 6.8 guarantee that (i) $\iff$ (ii) $\iff$ (iii).

As any equivalence is a fully faithful lax epimorphism, we have (iv) $\implies$ (ii), and since equivalences fix tiny objects, we conclude (v) $\implies$ (iv).

Finally, we note that we have an adjunction $\mathsf{Lan}_p \dashv \mathscr{V}\text{-}\mathsf{CAT}(p,\mathscr{V})$, so we obtain (iii) $\implies$ (v) by Proposition 6.1. □

6.3 Descent for the bifibration of split fibrations

Lemma 6.10 ([47, Proposition 3.1]). *A functor $p\colon E \to B$ between small categories is fully faithful (respectively, a lax epimorphism) if and only if* $\mathsf{CAT}(p, \mathsf{Cat})$ *is a lax epimorphism (respectively, fully faithful).*

Proof. We consider the fully faithful functor $J\colon \mathsf{Set} \to \mathsf{Cat}$, which has left and right adjoint functors. Hence, its direct image $J_!\colon \mathsf{Set}\text{-}\mathsf{CAT} \to \mathsf{Cat}\text{-}\mathsf{CAT}$ is a fully faithful 2-functor, and has left and right 2-adjoints. By [48, Remark 2.8 and Lemma 2.10], we conclude that $J_!$ creates fully faithful morphisms and lax epimorphisms.

Hence, we conclude from Corollary 6.8 (respectively, Corollary 6.7) that $J_! p$ is fully faithful (a lax epimorphism) if and only if $\mathsf{Cat}\text{-}\mathsf{CAT}(J_! p, \mathsf{Cat}) \cong \mathsf{CAT}(p, \mathsf{Cat})$ is a lax epimorphism (fully faithful). □

Theorem 6.11 ([47, Theorem 3.2]). *For a functor $p\colon E \to B$ between small categories, we consider the lax codescent factorization:*

$$\begin{array}{ccc} E & \xrightarrow{\quad p \quad} & B \\ & \searrow \quad \nearrow_{\mathscr{K}^{\mathsf{Ker}(p)}} & \\ & \mathsf{CoDesc}(\mathsf{Ker}(p)) & \end{array}$$

The following are equivalent:

(i) *p is an effective F-descent morphism (respectively, F-descent morphism),*
(ii) $\mathsf{CAT}(\mathscr{K}^{\mathsf{Ker}(p)}, \mathsf{Cat})$ *is an equivalence (respectively, fully faithful),*
(iii) *p is an effective F_D-descent morphism (respectively, F_D-descent morphism),*
(iv) $\mathsf{CAT}(\mathscr{K}^{\mathsf{Ker}(p)}, \mathsf{Set})$ *is an equivalence (respectively, fully faithful),*
(v) $\mathscr{K}^{\mathsf{Ker}(p)}$ *is a fully faithful lax epimorphism (respectively, a lax epimorphism),*
(vi) $\mathfrak{C}\mathscr{K}^{\mathsf{Ker}(p)}$ *is an equivalence (respectively, a lax epimorphism),*

Proof. By Theorem 6.9 (respectively, Corollary 6.7), we deduce that (v) $\iff$ (vi) $\iff$ (iv).

Since both F and F_D preserve lax descent objects, we conclude by Lemma 2.13 that (i) $\iff$ (ii) and (iii) $\iff$ (iv).

Finally, by Lemma 6.10 we obtain (ii) $\iff$ (v). □

References

[1] J. Adámek, R. el Bashir, M. Sobral, and J. Velebil. On functors which are lax epimorphisms. *Theory Appl. Categ.*, 8(20):509–521, 2001.

[2] M. Barr. Exact categories. In *Exact Categories and Categories of Sheaves*, volume 236 of *Lecture Notes in Mathematics*, pp. 1–120. Springer, 1971.

[3] M. Barr. *Relational algebras*. In *Lecture Notes in Mathematics*. Volume 137. Springer, Berlin, 1970, pp. 39–55.

[4] J. Bénabou and J. Roubaud. Monades et descente. *C. R. Acad. Sci.* A, (270):96–98, 1970.

[5] R. Betti, A. Carboni, R. Street, and R. Walters. Variation through enrichment. *J. Pure Appl. Algebra*, 29:109–127, 1983.

[6] F. Borceux and D. Dejean. Cauchy completion in category theory. *Cah. Topol. Géom. Différ. Catég.*, 15(4):133–146, 1986.

[7] F. Borceux and G. Janelidze. *Galois Theories*, number 72 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2001.

[8] R. Brown and G. Janelidze. Van Kampen theorems for categories of covering morphisms in lextensive categories. *J. Pure Appl. Algebra*, 119:255–263, 1997.

[9] A. Burroni. T–catégories (catégories dans un triple). *Cah. Topol. Géom. Différ. Catég.*, 3(12):215–321, 1971.

[10] A. Carboni, S. Lack, and R. F. C. Walters. Introduction to extensive and distributive categories. *J. Pure Appl. Algebra*, 94:145–158, 1993.

[11] M. M. Clementino and D. Hofmann. Descent morphisms and a van Kampen theorem in categories of lax algebras. *Topology Appl.*, 159:2310–2319, 2012.

[12] M. M. Clementino and D. Hofmann. Effective descent morphisms in categories of lax algebras. *Appl. Categ. Structures*, 12(5):413–425, 2004.

[13] M. M. Clementino and D. Hofmann. The rise and fall of $\mathscr{V}$-functors. *Fuzzy Sets and Systems*, 321:29–49, 2017.

[14] M. M. Clementino and D. Hofmann. Triquotient maps via ultrafilter convergence. *Proc. Amer. Math. Soc.*, 130(11):3423–3431, 2002.

[15] M. M. Clementino and G. Janelidze. A note on effective descent morphisms of topological spaces and relational algebras. *Topology Appl.*, 158:2431–2436, 2011.

[16] M. M. Clementino and W. Tholen. Metric, topology and multicategory – a common approach. *J. Pure Appl. Algebra*, (179):13–47, 2003.

[17] G. S. H. Cruttwell and M. Shulman. A unified framework for generalized multicategories. *Theory Appl. Categ.*, 24(21):580–655, 2010.

[18] M. Demazure. *Topologies et faisceaux, Exposé IV*. In *Schemas en Groupes I (SGA3)*. Volume 151. Lecture Notes in Mathematics. Springer, Berlin, 1970, pp. 159–250.

[19] S. Fujii and S. Lack. The oplax limit of an enriched category. *Theory Appl. Categ.*, 2024. To appear.

[20] J. Giraud. *Méthode de la descente*, volume 2 of *Mémoires de la Société Mathématique de France*. Société Mathématique de France, 1964.

[21] J. W. Gray. Fibred and cofibred categories. In S. Eilenberg, D. K. Harrison, H. Rörhl, and S. Mac Lane, editors, *Proceedings of the Conference on Categorical Algebra*, pp. 21–83. Springer, 1966.

[22] A. Grothendieck. *Catégories fibrées et descente*. In *Revêtements Étales et Groupe Fondamental (SGA1)*. Volume 224. Lecture Notes in Mathematics. Springer, Berlin, 1971, pp. 145–194.

[23] A. Grothendieck. Technique de descente et théorèmes d'existence en géometrie algébrique. i. généralités. descente par morphismes fidèlement plats. In *Séminaire N. Bourbaki*, number 5, pp. 299–327. Société Mathématique de France, 1960.

[24] C. Hermida. Representable multicategories. *Adv. Math.*, 151:164–225, 2000.

[25] C. Hermida. Some properties of Fib as a fibred 2-category. *J. Pure Appl. Algebra*, 134:83–109, 1999.

[26] D. Hofmann, G. Seal, and W. Tholen. *Lax algebras*. In *Monoidal topology*. D. Hofmann, G. Seal, and W. Tholen, editors. Volume 153. Encyclopedia of Mathematics and Its Applications. Cambridge University Press, 2014, pp. 145–283.

[27] G. Janelidze. Pure Galois theory in categories. *J. Algebra*, 132:270–286, 1990.

[28] G. Janelidze and M. Sobral. Finite preorders and topological descent I. *J. Pure Appl. Algebra*, 175:187–205, 2002.

[29] G. Janelidze, M. Sobral, and W. Tholen. *Beyond Barr exactness: effective descent morphisms*. In *Categorical Foundations: Special Topics in Order, Topology, Algebra and Sheaf Theory*. Maria Cristina Pedicchio and Walter Tholen, editors. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2004, pp. 359–406.

[30] G. Janelidze and W. Tholen. Facets of descent, I. *Appl. Categ. Structures*, 2(3):245–281, 1994.

[31] G. Janelidze and W. Tholen. Facets of descent, II. *Appl. Categ. Structures*, 5(3):229–248, 1997.

[32] G. Janelidze and W. Tholen. Facets of descent, III. Monadic descent for rings and algebras. *Appl. Categ. Structures*, 12(5–6):461–477, 2004.

[33] S. Johnson. Small Cauchy completions. *J. Pure Appl. Algebra*, 62(1):35–45, 1989.

[34] G. M. Kelly. *Basic Concepts of Enriched Category Theory*, number 10 in Repr. Theory Appl. Categ. Cambridge University Press, 2005.

[35] P. Köhler. Brouwerian semilattices. *Trans. Amer. Math. Soc.*, 268:103–126, 1981.

[36] S. Lack. Codescent objects and coherence. *J. Pure Appl. Algebra*, 175(1–3):223–241, 2002. Special volume celebrating the 70th birthday of Professor Max Kelly.

[37] S. Lack. Icons. *Appl. Categ. Structures*, 18(3):289–307, 2010.

[38] J. Lambék. *Deductive systems and categories II. Standard constructions and closed categories*. In *Category Theory, Homology Theory and their Applications I.* P.J. Hilton, editor. Volume 86. Lecture Notes in Mathematics. Springer, Berlin, Heidelberg, 1969.

[39] F. W. Lawvere. Metric spaces, generalized logic, and closed categories. *Rend. Sem. Mat. Fis. Milano*, (43):135–166, 1973.

[40] I. Le Creurer. *Descent of Internal Categories*. PhD thesis, Université Catholique de Louvain, 1999.

[41] T. Leinster. *Higher Operads, Higher Categories*, volume 298 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 2004.

[42] F. Lucatelli Nunes. Descent data and absolute Kan extensions. *Theory Appl. Categ.*, 37(18):530–561, 2021.

[43] F. Lucatelli Nunes. On biadjoint triangles. *Theory Appl. Categ.*, 31(9):217–256, 2016.

[44] F. Lucatelli Nunes. On lifting of biadjoints and lax algebras. *Categ. Gen. Algebr. Struct. Appl.*, 9(1):29–58, 2018.

[45] F. Lucatelli Nunes. Pseudo-Kan extensions and descent theory. *Theory Appl. Categ.*, 33(15):390–448, 2018.

[46] F. Lucatelli Nunes. Semantic factorization and descent. *Appl. Categ. Structures*, 30:1393–1433, 2022.

[47] F. Lucatelli Nunes, R. Prezado, and L. Sousa. Cauchy completeness, lax epimorphisms and effective descent for split fibrations. *Bull. Belg. Math. Soc. Simon Stevin*, 30(1):130–139, 2023.

[48] F. Lucatelli Nunes and L. Sousa. On lax epimorphisms and the associated factorization. *J. Pure Appl. Algebra*, 226(12):107–126, 2022.

[49] V. Marra and L. Reggio. A characterization of the category of compact Hausdorff spaces. *Theory Appl. Categ.*, 35(51):1871–1906, 2020.

[50] W. C. Nemitz. Implicative semi-lattices. *Trans. Amer. Math. Soc.*, 117:128–142, 1965.

[51] R. Prezado. On effective descent $\mathscr{V}$-functors and familial descent morphisms. *J. Pure Appl. Algebra*, 228(5), 2024.

[52] R. Prezado and F. Lucatelli Nunes. Descent for internal multicategory functors. *Appl. Categ. Structures*, 31(11), 2023.

[53] R. Prezado and F. Lucatelli Nunes. Generalized multicategories: change-of-base, embedding and descent. arXiv:2309:08084, 2023.

[54] J. Reiterman and W. Tholen. Effective descent maps of topological spaces. *Topology Appl.*, 57:53–69, 1994.

[55] G. J. Seal. Canonical and op-canonical lax algebras. *Theory Appl. Categ.*, 14:221–243, 2005.

[56] M. Sobral. Descent for discrete (co)fibrations. *Appl. Categ. Structures*, 12:527–535, 2004.

[57] R. Street. Limits indexed by category-valued 2-functors. *J. Pure Appl. Algebra*, 8(2):149–181, 1976.

[58] R. Street and A. Joyal. Pullbacks equivalent to pseudopullbacks. *Cah. Topol. Géom. Différ. Catég.*, 34(2):153–156, 1993.

[59] D. Verity. *Enriched Categories, Internal Categories and Change of Base*. PhD thesis, Cambridge University, 1992.

[60] M. Weber. Familial 2-functors and parametric right adjoints. *Theory Appl. Categ.*, 18(22):665–732, 2007.