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Analysis of Equations of Motion of Inextensible Strings and Networks

Tese no âmbito do Programa Interuniversitário de Doutoramento em Matemática, orientada pelo Professor Doutor Dmitry Vorotnikov e apresentada ao Departamento de Matemática da Faculdade de Ciências e Tecnologia da Universidade de Coimbra.

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Abstract

In this thesis we study the equations of overdamped motion of inextensible networks and strings under the action of different external forces. These problems can be expressed as systems of PDE that involve unknown Lagrange multipliers and non-standard boundary conditions related to the freely moving junctions. These problems can also be formally interpreted as gradient flows on certain submanifolds of the Otto-Wasserstein space of probability measures.

The first model that we study is the overdamped motion of inextensible triods under the gravitational force. The triod is a network that consists of three strings that meet at a common point (junction), and the other ends are fixed at three distinct points. We prove global existence of generalized solutions. We observe that our approach is also applicable to the overdamped fall of a single inextensible string with the ends fixed at two distinct spatial points.

The next model under consideration is the uniformly compressing mean curvature flow for a shrinking θ-network. A θ-network is a network consisting of 3 strings that has two junction points at the end and at the beginning of each string. We show that the problem can be normalized in a smart way. The renormalized system that can be viewed as an overdamped motion of an inextensible θ-network repelled from the origin by the external force equal to the radius-vector. Our model does not require any version of the Herring condition. Invoking the normalized model, we prove global existence of generalized solutions.

The last model is the overdamped motion of an inhomogeneous inextensible strings with the whip boundary conditions. We prove global existence of generalized solutions to this problem and study its long-time behavior. We show the exponential decay of the relative energy of the system and the convergence to the equilibrium.

Resumo

O tema central desta tese é o estudo do movimento superamortecido de redes inextensíveis e cordas, quando submetidas a diferentes forças externas. A modelação deste tipo de fenómenos resulta em sistemas de equações com derivadas parciais envolvendo multiplicadores de Lagrange e condições de fronteira não habituais, relacionadas com o movimento livre das junções. Este tipo de problemas também pode ser analisado recorrendo a fluxos de gradiente em certas subvariedades do espaço de medidas de probabilidade de Otto-Wasserstein.

O primeiro problema analisado é o movimento superamortecido de tríodos inextensíveis sujeitos à força gravitacional. O tríodo é uma rede que em consiste em três cordas que se encontram num ponto comum (junção), e cujas extremidades estão fixas em três pontos distintos. Para este problema particular provámos a existência de soluções globais generalizadas. A técnica de análise proposta também pode ser aplicada à queda superamortecida de uma única corda inextensível, cujas extremidades estão fixas em dois pontos distintos.

De seguida, analisámos o problema de uma rede-θ em retração, recorrendo a um modelo baseado no fluxo de curvatura média com compressão uniforme. Uma rede-θ consiste em 3 cordas com pontos de junção no início e no fim de cada corda. Uma normalização adequada permite ver este problema como o movimento superamortecido de uma rede-θ inextensível repelida da origem por uma força externa igual ao raio-vetor. O nosso modelo não necessita nenhuma condição do tipo Herring. Recorrendo ao problema normalizado, provámos a existência de soluções globais generalizadas.

Por fim, considerámos o movimento superamortecido de cordas inextensíveis e não homogéneas com condições de fronteira do tipo "chicote". Além da demonstração da existência de soluções globais generalizadas, provámos ainda o decaimento exponencial da energia relativa do sistema e a convergência para o equilíbrio a longo prazo.

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Chapter 1

Introduction

1.1 The main objects of our study

The equations of overdamped motion of inextensible strings and networks subject to different external forces will be studied in this thesis. An *inextensible string* is defined (cf. [3]) to be the one for which the stretch is constrained to be equal to 1, whatever system of forces is applied to it. As in [56], some authors refer to it as a *chain* which is a long but very thin material that is inextensible but completely flexible, and hence mathematically described as a rectifiable curve of fixed length. Dynamics of pipes, flagella, chains, or ribbons of rhythmic gymnastics, mechanism of whips, and galactic motion are only a few phenomena and applications that can be related to inextensible strings (see [11, 21, 27] for more details). An *inextensible network* is a union of several inextensible strings that meet at some of their endpoints called *junctions* with different boundary conditions and external forces. The study of inextensible networks from the mathematical perspective was started a long time ago by Chebyshev [26] and Rivlin [58], aiming at modelling textile fabrics. The length of different inextensible strings can vary but just for simplicity we assume all of them to be 1.

The general full dynamical equations describing an inextensible network read

$$\begin{cases} \rho \partial_{tt} \eta^i = \partial_s \left(\sigma^i \partial_s \eta^i \right) + \psi^i, \\ |\partial_s \eta^i| = 1, \\ \eta^i (0,s) = \alpha^i (s), \partial_t \eta^i (0,s) = \beta^i (s), \end{cases} \tag{1.1}$$

subject to appropriate boundary conditions and external forces $\psi^i = \psi^i \left(s, \eta, \partial_s \eta \right)$. This problem has different physical meanings for different boundary conditions and different external forces. Here $\eta^i = \eta^i(t,s) \in \mathbb{R}^d$, $i = 1, ..., n$, and $n \in \mathbb{N}$, is the position vector at time $t \geq 0$ of the particle that is labelled by the arc length parameter $s \in [0,1]$ and belongs to the i-th inextensible string; $\mathbb{R}^d$, $d \in \mathbb{N}$, $d > 1$ is the ambient space (the physically relevant cases are $d = 3$ and to a lesser extent $d = 2$). The evolving network is allowed to self-intersect, and the subtle issue whether the embeddedness of the network is preserved (this might be particularly challenging for $d = 2$) lies beyond our scope. For each i, the scalar function $\sigma^i = \sigma^i(t,s)$ is the Lagrange multiplier (that is often referred to as the *tension*) coming from the inextensibility of the i-th arm. The number n indicates the number of strings inside the system. In our work, we consider the cases $n = 3$ and $n = 1$, the former case corresponds to a *triod* or a θ*-network* and the latter is an inextensible string. A detailed justification of system (1.1) is given later. The inextensibility of the strings is manifested by the equality $|\partial_s \eta^i| = 1$. The initial data are α^i and β^i. Finally, $\rho = \rho(s)$ is the density of the matter in the string. We include ρ in (1.1) for the sake of generality. This allows us to model inhomogeneous cords. However, in Chapters 2 and 3 we study homogeneous networks, so $\rho \equiv 1$ there. Only in Chapter 4 we consider genuinely inhomogeneous strings.

In this thesis, we do not analyze the full dynamical system (1.1). The reason why we do not do this is that working with the full dynamical system is extremely hard. We instead focus on the overdamped problems that look like

$$\begin{cases} \partial_t \eta^i = \partial_s \left(\sigma^i \partial_s \eta^i \right) + \psi^i, \\ |\partial_s \eta^i| = 1, \\ \eta^i(0,s) = \alpha^i(s). \end{cases} \tag{1.2}$$

The main difference in comparison with (1.1) is that we have just one time derivative instead of two. For particular ψ, we will manage to write system (1.2) as a gradient flow in a certain infinite-dimensional ambient space and a suitable energy. The derivation and physical meaning of the gradient flow systems (1.2) will be discussed later.

This kind of equations can be studied as networks or with single inextensible strings. This thesis contains both cases. For inextensible networks, depending on the number of strings in the object and the number of junction points present, one can have many different boundary conditions. We study two different boundary cases for inextensible networks with $n = 3$ strings in Chapter 2 and Chapter 3. To give the reader a flavor, we first list some well known boundary conditions for a single inextensible string ($n = 1$):

- Two fixed ends:

$$\eta(t,0) = \alpha(0) \text{ and } \eta(t,1) = \alpha(1).$$

This boundary condition will be revisited at the end of the Chapter 2 (Remark 2.8).

- Periodic boundary conditions:

$$\eta(t,s) = \eta(t,s+1) \text{ and } \sigma(t,s) = \sigma(t,s+1).$$

- Two free ends:

$$\sigma(t,0) = \sigma(t,1) = 0.$$

- Whip boundary conditions (one end is free and one end is fixed):

$$\sigma(t,0) = 0 \text{ and } \eta(t,1) = 0.$$

This condition naturally arises in applications: a bullwhip, a pendulum etc. and this is the most studied case in the literature, see below. We study this case with inhomogeneous inextensible strings in Chapter 4.

The thesis has particularly been influenced by [61] (that studied the overdamped dynamics of a falling whip) and [62] (that dealt with the "uniformly compressing" counterpart of the mean curvature flow for loops).

1.2 Related problems

In this section, we review some other problems that are related to the problems that we study.

The model that has much in common with our overdamped string models is the Muskat problem (also known as the incompressible porous medium equation) that received a lot of attention during the last decade, see [12, 16–18, 64] and the references therein. In a nutshell, the Muskat equations describe an overdamped motion of an incompressible and inhomogeneous fluid subject to some external forces (in particular, gravitational force). In comparison with our flow (1.2), the incompressibility of the Muskat is similar to our inextensibility and the inhomogeneity of the Muskat corresponds to the fact that inextensible strings and networks do not occupy the whole space as well as, even more literally, to inhomogeneity of the strings in Chapter 4. Apart from that, the Muskat and our problems are overdamped motions and gradient flows. The Muskat system with gravity involves an unknown pressure p, density ρ and velocity field v and reads

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ v = \nabla p + \rho g, \\ \operatorname{div} v = 0. \end{cases}$$

The first equation in above system represents the continuity equation, the second one represents the Darcy Law, and the last equation represents the incompressibility of the fluid. This problem is quite difficult and the global existence of unique solutions for arbitrary data is unknown even in 2D. The most studied situation is when ρ takes one of the two values 0 and 1 (and hence is a priori discontinuous).

In [31], Jerrard and Smets study the motion of a curve in the binormal direction driven by the curvature. Notably, the inextensibility condition is preserved by that flow. They introduce weak (varifold) formulation for the binormal curvature flow in $\mathbb{R}^3$ and a global existence theorem for the defined setup. This of course includes networks. This model is related to Heisenberg's magnetism, nonlinear Schrödinger equations and fluid dynamics, see [30]. The equation of binormal curvature flow is expressed as

$$\partial_t \gamma = \partial_s \gamma \times \partial_{ss} \gamma, \tag{1.3}$$

for a smooth family $(\gamma_t)_{t\in I}$ of curves in $\mathbb{R}^3$ with arc-length parameterization $\gamma : I \times \mathbb{R} \to \mathbb{R}^3$. Here $t \in I$ is the time variable, $s \in \mathbb{R}$ is the arc-length parameter, and $\times$ denotes the vector product in $\mathbb{R}^3$. The arc-length parametrization condition

$$|\partial_s \gamma(t,s)| = 1$$

is indeed compatible with equation (1.3), since

$$\partial_t\left(|\partial_s\gamma|^2\right) = 2\partial_s\gamma\times\partial_{st}\gamma = 2\partial_s\gamma\cdot\left(\partial_s\gamma\times\partial_{sss}\gamma\right) = 0$$

whenever the first equation is satisfied, at least for sufficiently smooth solutions. In particular, closed curves evolved by the binormal curvature flow equation all have constant length. In more geometric terms, equation (1.3) takes its name from its equivalent form

$$\partial_t\gamma = \kappa b$$

where κ and b are the curvature function and the binormal vector field along γ_t, respectively.

There also exists an important link between inextensible string equations and optimal transport. We refer to Villani's book [68] as a reference about optimal transport theory. Otto [52] introduced a Riemannian submersion of the space of diffeomorphisms onto the Wasserstein space of probability measures, which is crucial for the recent developments in the optimal transport theory. The Riemannian manifold $\mathscr{A}$ (compare with our formulas (2.4) and (3.4)) of volume preserving embeddings can be viewed, cf. [62], as a submanifold of the Otto-Wasserstein space of probability measures [52, 68, 69] from the optimal transport theory (this in particular implies that the geodesic distance on $\mathscr{A}$ does not vanish, being bounded from below by the Wasserstein distance, which is in stark contrast with the underlying geometry of the mean curvature, Willmore and similar flows, cf. [8, 9, 41–43]). One can obtain the space of " inextensible strings" or, more generally, surfaces via restriction of this submersion.

Let us now illustrate the connection to optimal transport using some probabilistic language. Assume for simplicity that we have just one single string and ignore the boundary conditions. Then the equations of motion read

$$\begin{cases}\partial_{tt}\eta = \partial_s\left(\sigma\partial_s\eta\right), \\ |\partial_s\eta| = 1.\end{cases} \tag{1.4}$$

We recall that a triple $(\Omega, \mathscr{U}, P)$ is called a probability space provided Ω is any set, $\mathscr{U}$ is a σ-algebra of subsets of Ω, and P is a probability measure on $\mathscr{U}$. A (stochastic) process η is a map $[0,T]\times\Omega\to\mathbb{R}^n$. For simplicity, assume that $T = 1$. We denote by η_t the corresponding random variable at time t. The probabilistic interpretation of the Monge-Kantorovich optimal transport problem with quadratic cost [8] is

$$\min_{\substack{\text{law of } \eta_0 \text{ is } \mu_0,\\ \text{law of } \eta_1 \text{ is } \mu_1}} \mathbb{E}\left(\int_0^1 |\partial_t\eta_t|^2 dt\right), \tag{1.5}$$

where μ_0 and μ_1 are given probability measures on $\mathbb{R}^n$, and $\mathbb{E}$ denotes the expected value. Assume that μ_0 and μ_1 are just Hausdorff-like measures concentrated on given initial and final curves η_0 and η_1, cf. the "geodesic" formulation as in Section 2.2. More precisely, we can define the corresponding measures by duality as follows: $\int\phi(x)\,d\mu_i := \int\phi\left(\eta^i(s)\right)ds$. Now, assume that $\Omega = [0,1]$, $\mathscr{U}$ is the algebra of Borel sets, P is the Lebesgue measure ds. Then at least formally the solutions of our equations (1.4) are the optimizers of

$$\min_{\substack{\text{law of } \eta_0 \text{ is } \mu_0, \\ \text{law of } \eta_1 \text{ is } \mu_1, \\ |\partial_s \eta|=1, \\ \eta \text{ is differentiable w.r.t. } t,s}} \mathbb{E}\left(\int_0^1 |\partial_t \eta_t|^2 dt\right). \tag{1.6}$$

The basic difference between (1.5) and (1.6) is the inextensibility constraint $|\partial_s \eta| = 1$. However, this constraint is not convex, and this complicates the things dramatically.

In the case of networks consisting of three arms we replace $\Omega = [0,1]$ by the quotient topological space obtained from $[0,1] \cup [0,1] \cup [0,1]$ by gluing all zeros (triod) or gluing all zeros and separately all ones (θ-network).

Observe also that the integral $\int_0^1 \eta(t,s)\,ds$ is the expected value of the process at time t and simultaneously the center of the mass of the curve. Moreover, the variance of the process at time t is given by

$$\mathbb{V}(\eta_t) := \mathbb{E}\left(|\eta_t|^2\right) - |\mathbb{E}(\eta_t)|^2. \tag{1.7}$$

In particular, if the center of the mass is fixed at the origin, the second term vanishes.

Finally, let us describe the link to the motion of incompressible fluids. It is known that the equations of motion of finite-dimensional mechanical systems governed by Newtonian mechanics can be interpreted as the geodesic equations of a Riemannian metric on the configuration space. Arnold in his famous paper [4] considers the space of velocity fields of an incompressible fluid as the Lie algebra of the infinite dimensional Lie group of volume preserving diffeomorphisms. He proves that the geodesic equations of the space of volume preserving diffeomorphisms are the incompressible Euler equations of fluid dynamics. The book of Arnold and Khesin [6] develops these studies of fluid dynamics and its topology. Molitor in [45] and Bauer, Michor and Muller in [8] study the relation between Euler equations of incompressible fluids and motion of incompressible membranes of arbitrary dimension. The inextensible strings can be regarded as a particular case ($m = 1$) of m-dimensional *incompressible membranes* (in other words, of volume preserving immersions), cf. [10, 45]. The opposite borderline case $m = d$ tallies with Arnold's formalism [4, 6] for ideal incompressible fluids or rather, even more specifically, with the motion of fluid patches in $\mathbb{R}^d$, which has recently been studied [36] from a similar perspective. However, in Arnold's case ($m = d$) the manifold has a Lie group structure, which allows one to work in the corresponding Lie algebra (i.e., in the mechanical language, to use the Eulerian coordinates). It has been observed recently [29] that although a full Lie group stucture is available for few PDE, in some applications it can be replaced by a relevant Lie grupoid structure. It seems to be the case for the inextensible strings and networks, but this is completely beyond our scope.

1.3 History of the mathematical analysis of inextensible strings

Let us now give a historical overview of the research related to mathematical analysis of motion of inextensible strings[1]. The study of an inextensible string goes back to the dawn of mathematical

[1] There is not much literature about analysis of inextensible networks, see the end of this section for the details.

analysis. Let us start with a brief summary of pioneer works, for further details see [3, 67]. The mechanics of inextensible strings was launched in the beginning of sixteenth century. First studies were done to find the shape of the curve of the stationary problem. Until the first half of nineteenth century, the following four static problems were studied: the suspension bridge whose shape was found as a parabola, the catenary problem whose shape was found as hyperbolic cosine, the problem of gravitational attraction to a fixed point that has the shape of a circle, and the velaria model whose shape was found as a circle as well. Let us specify the classical static problems using modern notation:

- Suspension bridge
$$0 = \partial_s(\sigma\partial_s\eta) + A\partial_s\eta \text{ where } A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
- Catenary
$$0 = \partial_s(\sigma\partial_s\eta) + g$$
- Gravitational attraction to a fixed point
$$0 = \partial_s(\sigma\partial_s\eta) - \frac{\eta}{|\eta|^3}$$
- Velaria
$$0 = \partial_s(\sigma\partial_s\eta) + B\partial_s\eta \text{ where } B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Of course, all these settings should be completed by the inextensibility constraint $|\partial_s\eta| = 1$.

The question of discovering the equation of the catenary (hanging chain) was asked by Leonardo da Vinci but he had wrong assertions about its mathematical character. In 1638, Galileo thought that he found the shape of the catenary as parabola, but he was incorrect, which is acceptable for his time because at that time the exponential function was not discovered yet. The catenary has the shape of hyperbolic cosine. Galileo had worked with a uniform inextensible string. In 1691, Joh. Bernoulli, Leibniz and Huygens found the equation for the curve of catenary, Leibniz obtained its graph in the form of hyperbolic cosine. This was the one of the earliest successes in calculus of variations. One of the known studies was done by Stevin in 1608, he did not study the catenary but another similar model which is the suspension bridge. The suspension bridge problem describes the equilibrium of the string suspended at the both ends and subject to the distributed load that is uniform along the horizontal coordinate. Meanwhile, Beeckman in 1615, Huygens in 1646 and Pardies in 1673 showed that the curve is actually a parabola for the suspension bridge problem.

Huygens had some errors in his first works on this subject as saying that velaria is a parabola but then he corrected his mistakes. Velaria problem is the shape of a sail under the pressure of the wind. In the model for velaria the force is the normal force of constant absolute value. In 1675 Hooke observed that a (uniform inextensible) moment free arch that supports its own weight is obtained by turning the catenary upside down. Joh. Bernoulli found that the shape of an inextensible string gravitationally attracted to a fixed point is a circle.

We now describe some studies on evolutionary problems of inextensible strings and related areas. Only few results about general well-posedness of inextensible string equation are known. One of the few existence results that we have is given by Reeken. He approaches to problem by using chains. In his papers [56, 57], he studies the infinite string with gravity when the initial values are near to trivial stable stationary solution. He explains the difficulty of studying this equation and gives possible approaches to the problem apart of his well-posedness result. Dickey [21] considers the two dimensional dynamic behavior of inextensible string. He studies some possible solutions rewriting the equation of inextensible string in polar coordinates in which the radius vector depends only on the polar angle. He works with the whip boundary conditions, and he makes some change of variables and finds that the equation admits some wave solutions, but with his change of variables, he loses the boundary conditions, so his solutions do not satisfy the boundary conditions. In the last part of his paper, he studies the link between galactic motions and the inextensible string.

Preston[54] studies the motion of inextensible string with whip boundary conditions in the absence of gravity, he also approximates the string with chains. He extends the curve and obtains new boundary condition. This new boundary condition is a special case of the whip boundary condition, he obtains the same value at both ends for displacement and tension. He proves local existence and uniqueness in a weighted Sobolev space defined for the energy. In another article [55], he studies the geometric aspects of the space of arcs parameterized by unit speed in the L^2-metric. He proves that the space of arcs is a submanifold of the space of all curves and the orthogonal projection exists but is not smooth, and as a consequence he gets a Riemannian exponential map that is continuous and even differentiable but not C^1. We will use the manifolds introduced in [55].

Preston and Saxton in [53] study the geodesics of the H^1 Riemannian metric on the space of inextensible curves. They use the results in [55] to show that the geodesic equation is C^∞ in a Banach topology which implies that there is a smooth Riemannian exponential map. In addition, they give global-in-time solutions for a special case. They have an extra term in their partial differential equation in contrast to the usual inextensible string equation, which is a fourth order mixed derivative term, and they work in the absence of gravity. The extra term changes the equation that the tension satisfies. The extra term reduces the mathematical difficulties.

McMillen and Goriely in [40] study one of the most interesting phenomena of whips. They see whips as unique objects due to the crack that they produce in certain movement. It is explained why this crack is a sonic bomb. Since it is an article regarding to an observation rather than pure mathematical analysis, they have added different parameters as the material of whip, the radius of whip etc. In [40], we see a wave type approach to whips. They show by asymptotic analysis that a wave traveling along the whip increases its speed as the radius decreases. Also, there is a numerical scheme to support their experimental and mathematical results. They use the whip boundary conditions, and they give importance to the movement of the hand that moves the string.

Vorotnikov and Şengül in [60] rewrite the problem as a hyperbolic conservation law with discontinuous flux. They work with the whip boundary conditions. They show the non-negativity of the tension and they use change of variables to transform their equation to a system, meanwhile they have difficulties since the tension causes some singularities. The non-negativity of the tension is an important fact for their work. After modifying the new system, they work with the hyperbolic conservation laws. They show the existence of the generalized Young measure solutions. Vorotnikov and Shi [61] observe that the mean curvature flow is a gradient flow on a Riemannian structure with a degenerate geodesic distance which was shown by Michor and Mumford in [41]. They introduce a new related gradient flow with respect to non a degenerate distance. The new flow is obtained by orthogonal projection of the mean curvature on the tangent bundle of the infinite-dimensional submanifold and it can be seen as the formal gradient flow in a submanifold of the Wasserstein space of probability measures. In 1D case, their new mean curvature flow is an overdamped motion of an inextensible loop. In [62], they study the gradient flow of the potential energy on a similar infinite-dimensional Riemannian manifold, akin to [55], which is the model for overdamped motion of a falling inextensible string. They show the exponential decay of the solution to the equilibrium after proving the existence of solutions. Their solutions satisfy the new system and the relaxed constraint, here relaxed constraint is about the length of the string. They construct a suitable family of approximating gradient flows on the a flat Hilbert space. Also, they observe that the system has non-unique trajectories.

Various elastic flows of inextensible strings were studied in [32, 35, 47–51, 70]. The presence of elastic forces contributes towards non-degenerate parabolicity of the flows and helps to overcome the difficulties caused by the Lagrange multipliers related to the inextensibility constraint.

Let us give some literature review about inextensible networks. As the study of networks is a huge area of research, we should restrict ourselves to the mathematics of motion of inextensible networks. As we already said above, the mathematics of inextensible networks goes back to Chebyshev [26] and Rivlin [58]. Aside from [46], we are however not aware of any investigation of evolutionary behavior of inextensible networks. Our work thus seems to be one of the first contributions to this particular field. On the other hand, there has been a major recent activity on well-posedness of geometric flows describing time-evolving extensible networks, see [19, 20, 24, 25, 33, 37, 38] and the survey [39]. Whereas the authors of [33, 37, 38] deal with variants of the mean curvature flow for networks, [46] and the other mentioned articles consider *elastic flows* (interpolations between the mean curvature flow and the Willmore flow). The main technical difficulties that appear in the study of networks in contrast with the evolution of single strings are due to the rather non-standard boundary conditions at the junction points.The literature on flows of networks cited above is concerned with variational evolution driven by “intrinsic” energies (related to the length or curvature).

1.4 Summary of the thesis

In this section, we write down the main results of each chapter with a brief summary and we give an outline of the thesis.

In Chapter 2 we study the overdamped equations of motion of inextensible triods under the gravitational force. The results of this chapter are published in [65]. The triod is a network that consists of three strings that meet at a common point (junction), and the other ends are fixed at three distinct points of $\mathbb{R}^d$, $d > 1$. The junction is moving in an unknown way. One of the main difficulties of such systems is caused by unusual boundary conditions at the junction. Because of that many of the estimates that were used in [61] fail to be generalizable to our setting. This in particular applies to the crucial L^∞ estimate in the spirit of Ladyzhenskaya, Solonnikov and Uraltseva, cf. [34]. We will manage to overcome these difficulties and to prove novel and more refined estimates by leveraging the gradient flow structure of the approximation problem much more thoroughly than in [61]. This will be combined with careful observations involving geometric properties of triods, the behaviour of the curvature and some convexity argument. Apart from that, in [61] the existence of C^∞-smooth solutions to the approximation problem was derived from Amann's theory, cf. [1]. It is not applicable here anymore (again due to the boundary conditions), so we will solve our approximate problem by the theory of abstract evolution equations with pseudomonotone maps, cf. [59], which we briefly recall in the Appendix A.

The equations of motion for a triod overdamped by a heavily dense environment that we analyze in Chapter 2 read

$$\begin{cases} \partial_t \eta^i = \partial_s \left(\sigma^i \partial_s \eta^i \right) + g, \\ |\partial_s \eta^i| = 1, \\ \eta^1 (t,0) = \eta^2 (t,0) = \eta^3 (t,0), \\ \eta^i (t,1) = \alpha^i (1), \\ \sigma^1 \partial_s \eta^1 + \sigma^2 \partial_s \eta^2 + \sigma^3 \partial_s \eta^3 = 0 \text{ at } s = 0 \text{ for all } t, \\ \eta^i (0,s) = \alpha^i (s). \end{cases} \tag{1.8}$$

Here, g represents the gravity and it is the external force. We assume that $|g| = 1$ for definiteness. In Chapter 2, first we introduce the original equations of motion of a triod, and then derive them using the least action principle. We also observe that they represent Newton's equations on a certain infinite-dimensional submanifold of the Otto-Wasserstein space. Then we derive the overdamped problem that has a gradient flow structure on that submanifold (the gradient flow is driven by the potential energy). After this we present an approximation system, which has a gradient flow structure driven by a suitable approximating/penalizing energy on a Hilbert space. We obtain various a priori bounds for this system, some of which are more direct and other are more refined. We manage to apply the theory of evolution equations with pseudomonotone maps from the book of Roubicek [59] for regularity and solvability of the approximation system and we use [7] for the decrement of the gradient flows. Then we manage to pass to the limit in order to return to the original problem. Here and below we use the shortcut $\mathfrak{Q}_\infty := (0,\infty) \times (0,1)$. The main result of Chapter 2 is the following one.

Theorem (Global existence of generalized solutions)**.** *For every initial data $\alpha^i \in W^{1,\infty}(0,1)^d$, $i = 1,2,3$, meeting the assumptions of Remark 2.3, there exists a generalized solution to* (1.8) *in $\mathfrak{Q}_\infty$. Moreover, those solutions satisfy $\sigma^i(t,s) \geq 0$ for almost every $(t,s) \in \mathfrak{Q}_\infty$.*

In particular, this approach is applicable to the overdamped fall of a single inextensible string with the ends fixed at two distinct spatial points (it suffices to observe that such a string can be viewed as a degenerate "triod" with one arm having zero length); remember that [61] studied the case of one free and one fixed end (i.e., a "whip"). More precisely, we have the following result.

Proposition. *Given* $\alpha \in W^{1,\infty}(0,1)^d$ *satisfying* $|\alpha(0)-\alpha(1)|<1$, $|\partial_s \alpha(s)|=1$ *a.e. in* $(0,1)$, *there exists a generalized solution to*

$$\begin{cases} \partial_t \eta = \partial_s \left(\sigma \partial_s \eta\right) + g, \\ |\partial_s \eta| = 1, \\ \eta(t,0) = \alpha(0), \quad \eta(t,1) = \alpha(1), \\ \eta(0,s) = \alpha(s). \end{cases} \tag{1.9}$$

in $\mathfrak{Q}_\infty$. *Moreover,* $\sigma(t,s) \geq 0$ *for almost every* $(t,s) \in \mathfrak{Q}_\infty$.

In Chapter 3 we study the uniformly compressing mean curvature flow for a shrinking θ-network. A θ-network is a network consisting of 3 strings that has two junction points at the end and at the beginning of each string. Furthermore, the junction points are not attached to any spatial point.

Remember that binormal curvature flow [31] preserves the uniform parametrization. On the other hand the classical mean curvature flow destroys it, which is unwelcome and may cause computational instabilities [44, 63]. In Chapter 3 we develop the ideas of [62] and consider a model of a shrinking θ-network that preserves the uniform parametrization. We show that the problem can be normalized in a smart way. Another advantage of our approach is that it works without assuming any version of the Herring condition. The latter one means that the arms of the network meet with equal angles $(2\pi/3)$ at the junctions. This restrictive geometric condition is usually assumed by the authors who study mean curvature flow of networks since it prevents instabilities, cf. [33, 37, 38]. The treatment of initial configurations that do not satisfy the Herring condition requires much more involved techniques and only local results have been recently obtained, cf. [28].

We exploit the gradient flow structure to derive some evolution properties of the length and the variance of the curves. After a suitable change of variables, we introduce a renormalized system that can be viewed as an overdamped motion of an inextensible θ-network repelled from the origin by the external force equal to the radius-vector. Namely, the renormalized problem reads

$$\begin{cases} \partial_t \xi^i = \partial_s \left(\sigma^i \partial_s \xi^i\right) + \xi^i, \\ |\partial_s \xi^i| = 1, \\ \xi^1(t,0) = \xi^2(t,0) = \xi^3(t,0), \\ \xi^1(t,1) = \xi^2(t,1) = \xi^3(t,1), \\ \sigma^1 \partial_s \xi^1 + \sigma^2 \partial_s \xi^2 + \sigma^3 \partial_s \xi^3 = 0 \text{ at } s = 0,1 \text{ for all } t, \\ \xi^i(0,s) = \beta^i(s). \end{cases} \tag{1.10}$$

Chapter 3 starts with a tedious derivation of the system above from the a variant of the mean curvature flow for networks that preserves the uniform parametrization. The renormalized system can be viewed as a gradient flow on an infinite-dimensional submanifold of the Otto-Wasserstein space. We then define an approximation system for the renormalized system and find different bounds. The theories of evolution equations with pseudomonotone maps and that of the Hilbertian gradient flows as well as new a priori bounds are used similarly as before in order to obtain the following existence result.

Theorem (Global existence of generalized solutions)**.** *For every initial data* $\beta^i \in W^{1,\infty}(0,1)^d$, $i = 1,2,3$, *meeting the assumptions of Remark 3.3, there exists a generalized solution to system* (1.10) *in* $\mathfrak{Q}_\infty$. *Moreover, those solutions satisfy* $\sigma^i(t,s) \geq 0$ *for almost every* $(t,s) \in \mathfrak{Q}_\infty$.

In the last Chapter 4 our model consists of a single inextensible string that is however inhomogeneous. We thus study the equations of overdamped motion of an inhomogeneous inextensible strings with the *whip* boundary conditions. Vorotnikov and Shi obtained similar results in [61] for homogeneous whips, and here we study the inhomogeneous whips. The inhomogeneity of the string means that the density of the string is not constant but is a function $\rho = \rho(s)$, and this leads to the following system:

$$\begin{cases} \partial_t \eta = \partial_s\left(\sigma \partial_s \eta\right) + \rho g, \\ |\partial_s \eta| = 1, \\ \eta(t,1) = 0 \text{ and } \sigma(t,0) = 0, \\ \eta(0,s) = \alpha(s). \end{cases} \tag{1.11}$$

This system describes the motion equations of inhomogeneous whips overdamped by the heavily dense environment. The idea for allowing the density to vary is also inspired by the Muskat problem and porous media. In this connection, it is important that ρ is not assumed to be continuous in s and it is allowed to take value 0 for some $s \in [0,1]$.

Chapter 4 starts with deriving the full dynamical equations for the inhomogeneous inextensible whips and then we obtain the overdamped flow. Afterwards, we define an approximating system that is a Hilbertian gradient flow. Contrary to the previous chapters, we do not employ the evolution equations by pseudomonotone maps but we use the theory by Amann [1] in order to obtain existence results for the approximating system and, moreover, an idea from the book [34] of Ladyzenskaja, Solonikov and Ural'ceva is used for our main estimate here. Furthermore, in this chapter, we show the exponential decay of the relative energy of the system and the convergence to the equilibrium. Here are the two main results of the chapter:

Theorem (Global existence of generalized solutions)**.** *For every* $\alpha \in W^{1,\infty}(0,1)^d$ *with* $\eta(1) = 0$, $|\partial_s \alpha| \leq 1$ *for a.e.* $s \in (0,1)$ *and given* $0 \leq \rho(s) \leq 1$ *in* $L^\infty\left((0,1)\right)$ *there exists a generalized solution to* (1.11) *in* $\mathfrak{Q}_\infty$. *Moreover, those solutions satisfy* $\sigma(t,s) \geq 0$ *for almost* $(t,s) \in \mathfrak{Q}_\infty$.

Theorem (Exponential decay of the energy)**.** *Let* (η,σ) *be a generalized solution in the sense of Definition 4.4. Assume that* $\sigma \geq 0$ *almost everywhere in* $\mathfrak{Q}_\infty$ *and* $\tilde{C}s^{-2} \geq \sigma_\infty^{-1}$ *for a positive constant* $\tilde{C}$. *Then there exists a universal constant* $c_0 > 0$ *such that*

$$\tilde{\mathscr{E}}(t) \leq e^{-c_0 t}\tilde{\mathscr{E}}(0), \quad t \in [0,\infty).$$

Here, $\tilde{\mathscr{E}}(t)$ *is the relative energy* (4.45) *and* $\sigma_\infty(s) := \int_0^s \rho$.

We refer to Remark 4.3 for a discussion of the technical assumptions of this theorem.

Remark 1.1. The proof of the last theorem is heavily relying on Hardy's inequality. The proof strategy cannot be easily adapted to the string with two fixed ends[2], let alone the triod, since that would require to substitute Hardy's inequality with a troublesome non-standard variant of the Poincaré inequality. Moreover, to the best of our knowledge, the explicit characterization of the steady state with the least potential energy is not available for the triod.

The exponential decay of the relative energy implies the exponential convergence of the generalized solution to the stationary solution $(\eta_\infty, \sigma_\infty) = \left((1-s)\,g, \int_0^s \rho\right)$ in $L^2(0,1)$ as $t \to \infty$.

Corollary. *Let (η, σ) be a generalized solution in the sense of Definition 4.4 with $\sigma \geq 0$ and $\tilde{C}s^{-2} \geq \sigma_\infty^{-1}$ for a positive constant $\tilde{C}$. Then there exist universal constants $C_0, c_0 > 0$ such that for all $t \in [0, \infty)$*

$$||\eta(t,\cdot) - \eta_\infty(\cdot)||^2_{L^2(0,1)} \leq C_0 \tilde{\mathscr{E}}(0)\, e^{-c_0 t}.$$

Notation

In this subsection, we introduce the main notation used in the sequel.

- $\Omega := (0,1)$,
- $\mathfrak{Q}_t := (0,t) \times \Omega$ for $t \in (0,\infty]$,
- $d > 1$ stands for the dimension of the space where the dynamics of the networks occurs,
- $g \in \mathbb{R}^d$ is a constant gravity vector, $|g| = 1$,
- $\mathbf{g}(s) := (g, g, g) \in \mathbb{R}^{3d}$,
- $\mathscr{M}\,(\mathscr{M}_{loc})$ stands for spaces of finite (locally finite) Radon measures,
- C stands for a generic positive constant.
- We will need the following function $F_\varepsilon(\kappa) := \varepsilon\kappa + \frac{\kappa}{\sqrt{\varepsilon + |\kappa|^2}}$, $\kappa \in \mathbb{R}^d, \varepsilon > 0$,
- and its inverse $G_\varepsilon(\tau) := (F_\varepsilon)^{-1}(\tau), \tau \in \mathbb{R}^d$.

[2]The corresponding steady state is the catenary.

Chapter 2

Triod

2.1 Introducing and setting the problem

In this chapter we study the overdamped equations of motion of inextensible triods under the gravitational force. The triod is a network that consists of three strings that meet at a common point (junction), and the other ends are fixed at three distinct points of $\mathbb{R}^d$, $d > 1$. The junction is moving in an unknown way and thus constitutes a kind of a free boundary. We repeat that we will only consider the triods that are inextensible.

We investigate the gradient flow of an "extrinsic" energy, namely, the potential energy determined by an external force (gravity), with respect to a suitable geometry, cf. [61]. As we explain below, this models the overdamped motion of a falling inextensible network (triod). The results in this chapter are published in [65].

In Section 2.2 we derive the following full system of equations of motion of a triod under the action of gravity:

$$\begin{cases} \partial_{tt}\eta^i = \partial_s\left(\sigma^i\partial_s\eta^i\right) + g, \\ |\partial_s\eta^i| = 1, \end{cases} \tag{2.1}$$

subject to the boundary conditions

$$\begin{cases} \eta^1(t,0) = \eta^2(t,0) = \eta^3(t,0), \\ \eta^i(t,1) = \alpha^i(1), \\ \sigma^1\partial_s\eta^1 + \sigma^2\partial_s\eta^2 + \sigma^3\partial_s\eta^3 = 0 \text{ at } s = 0 \text{ for all } t. \end{cases} \tag{2.2}$$

If we just know the initial configuration, we are given the initial conditions

$$\eta^i(0,s) = \alpha^i(s),\ \partial_t\eta^i(0,s) = \beta^i(s). \tag{2.3}$$

Here $\eta^i = \eta^i(t,s) \in \mathbb{R}^d$, $i = 1,2,3$, is the position vector at time $t \geq 0$ of the particle that is labelled by the arc length parameter s and belongs the ith arm of the triod. For each i, the scalar function $\sigma^i = \sigma^i(t,s)$ is the Lagrange multiplier (that is often referred to as the *tension*) coming from the inextensibility of the i-th arm (the nature of these multipliers will become more transparent later).

Finally, g is a constant gravity vector for which we assume w.l.o.g. that $|g| = 1$, and $\alpha^i(s)$, $\beta^i(s)$ determine the initial dynamical configuration of the triod. Note that $s = 1$ corresponds to the fixed ends, and $s = 0$ corresponds to the (moving) junction.

From the geometrical point of view, a natural inifinite-dimensional configuration manifold for the evolving inextensible triods is

$$\mathscr{A} = \{\eta = (\eta^1, \eta^2, \eta^3) : \eta^i \in H^2\left(0,1;\mathbb{R}^d\right), \\ \eta^1(0) = \eta^2(0) = \eta^3(0),\ \eta^i(1) = \alpha^i(1),\ |\partial_s \eta^i(s)| = 1\, \forall s \in [0,1]\} \tag{2.4}$$

viewed as a submanifold of $L^2(0,1;\mathbb{R}^{3d})$ (and hence equipped with a weak Riemannian metric). Observe that the tangent space at a "point" η is

$$T_\eta \mathscr{A} = \{v = (v^1, v^2, v^3) : v^i \in H^2\left(0,1;\mathbb{R}^d\right), \\ v^1(0) = v^2(0) = v^3(0),\ v^i(1) = 0,\ \partial_s \eta^i(s) \cdot \partial_s v^i(s) = 0\}. \tag{2.5}$$

Note that we never employ Einstein's summation convention. Then (2.1), (2.2) is at least formally equivalent to Newton's equation

$$\nabla_{\dot\eta} \dot\eta = -\nabla_{\mathscr{A}} \mathscr{E}(\eta). \tag{2.6}$$

Here

$$\mathscr{E}(\eta) := \sum_{i=1}^{3} \int_0^1 -g \cdot \eta^i(s)\, ds \tag{2.7}$$

is the potential energy of a triod, cf. Remark 2.2.

If the fall of the triod is overdamped by a heavily dense environment, the equations of motion (2.1) become

$$\begin{cases} \partial_t \eta^i = \partial_s\left(\sigma^i \partial_s \eta^i\right) + g, \\ |\partial_s \eta^i| = 1. \\ \eta^i(0,s) = \alpha^i(s). \end{cases} \tag{2.8}$$

We refer to the next section for the details of the derivation.

Remark 2.1 (Initial velocity is not prescribed in the overdamped regime). Note carefully that the first equation in (2.8) is a first order equation w.r.t. time, and the initial data do not contain the initial velocity anymore. This is natural because in the overdamped regime the initial velocity is determined by the friction.

Remark 2.2 (Computation of the gradient). Our problem (2.8), (2.2) can be realized as the gradient flow of the potential energy $\mathscr{E}$ on the manifold $\mathscr{A}$, i.e.,

$$\frac{d}{dt}\eta = -\nabla_{\mathscr{A}} \mathscr{E}(\eta). \tag{2.9}$$

In order to see that (2.6) is formally equivalent to (2.1), (2.2), we need to carry out a formal computation of $-\nabla_{\mathscr{A}}\mathscr{E}$. Fix a reference network $\eta \in \mathscr{A}$. Firstly, it is clear that $-\nabla_{L^2}\mathscr{E}(\eta) = g$. Hence, by some basic Riemannian geometry [22],

$$-\nabla_{\mathscr{A}}\mathscr{E}(\eta) = P_\eta g, \tag{2.10}$$

where $P_\eta g$ is the orthogonal projection of g onto the tangent space $T_\eta\mathscr{A}$ that was defined in (2.5). Assume that the following system of ODE

$$\partial_{ss}\sigma^i - |\partial_{ss}\eta^i|\sigma^i = 0 \tag{2.11}$$

with the initial/terminal conditions

$$\sum_{i=1}^{3}\sigma^i\partial_s\eta^i = 0 \text{ at } s=0, \tag{2.12}$$

$$\partial_s\sigma^i\partial_s\eta^i + \sigma^i\partial_{ss}\eta^i + g \text{ does not depend on } i \text{ at } s=0, \tag{2.13}$$

$$\partial_s\sigma^i\partial_s\eta^i + \sigma^i\partial_{ss}\eta^i = -g \text{ at } s=1, \tag{2.14}$$

is solvable for σ^i, $i=1,2,3$. We claim that

$$P_\eta g := \left(g+\partial_s\left(\sigma^1\partial_s\eta^1\right), g+\partial_s\left(\sigma^2\partial_s\eta^2\right), g+\partial_s\left(\sigma^3\partial_s\eta^3\right)\right) \tag{2.15}$$

fulfills the relevant conditions for the image of g under the orthogonal projection, namely, $P_\eta g \in T_\eta g$ and $g - P_\eta g$ is L^2-orthogonal to any $\upsilon^i \in T_\eta\mathscr{A}$. Indeed, differentiating the constraints $|\partial_s\eta^i|^2 = 1$ we find

$$\partial_s\eta^i\cdot\partial_{ss}\eta^i = 0,$$
$$\partial_s\eta^i\cdot\partial_{sss}\eta^i = -|\partial_{ss}\eta^i|^2.$$

Hence,

$$\partial_s\left(P_\eta g\right)^i\cdot\partial_s\eta^i = \partial_{ss}\sigma^i - |\partial_{ss}\eta^i|^2\sigma^i = 0.$$

It is easy to see that we have $P_\eta g(1) = 0$ by (2.14) componentwise. Moreover, by (2.13), $(P_\eta g)^i(0)$ does not depend on i. We have proved that $P_\eta g \in T_\eta\mathscr{A}$. Finally, for any $\upsilon^i \in T_\eta\mathscr{A}$, we obtain after summing the terms and integration by parts

$$\begin{aligned}\sum_{i=1}^{3}\int_0^1\left(g-\left(P_\eta g\right)^i\right)\cdot\upsilon^i ds &= \sum_{i=1}^{3}\int_0^1 -\partial_s\left(\sigma^i\partial_s\eta^i\right)\cdot\upsilon^i ds\\ &= \sum_{i=1}^{3}\int_0^1 \sigma^i\partial_s\eta^i\cdot\partial_s\upsilon^i ds - \sum_{i=1}^{3}\sigma^i\partial_s\eta^i\cdot\upsilon^i\Big|_{s=0}^{s=1}.\end{aligned}$$

It follows from the definition of $T_\eta\mathscr{A}$ in (2.5) and equality (2.12) that both terms vanish, and the claim (2.15) follows.

This formally justifies that the PDE form of the gradient flow (2.9) is (2.8), (2.2). A similar but slightly amended argument formally implies that (2.6) is equivalent to (2.1), (2.2).

In light of the discussion in the Introduction (see also [54, 55, 60, 66]) equation (2.1) has much in common with the Euler equation of ideal incompressible fluid. In the same spirit, the overdamped equation (2.8) is comparable to the Muskat problem (also known as the incompressible porous medium equation) that received a lot of attention during the last decade, see [16–18, 64] and the references therein.

In this chapter, we are interested in constructing global in time solutions to (2.8), (2.2). We deal with generalized solutions, which allows us to consider not necessarily smooth but merely rectifiable triods.

Observe that $\mathbf{g}(s) := (g, g, g) \in L^2(\Omega; \mathbb{R}^{3d})$.

Remark 2.3 (Initial data). We fix once and for all Lipschitz initial data $\alpha^i \in W^{1,\infty}(\Omega)^d$, $i = 1, 2, 3$, satisfying the compatibility conditions

$$\alpha^1(0) = \alpha^2(0) = \alpha^3(0) = 0 \tag{2.16}$$

and

$$|\partial_s \alpha^i(s)| = 1 \text{ a.e. in } \Omega. \tag{2.17}$$

Since (2.17) is only required to hold almost everywhere, the arms of the triod can have shape of any rectifiable curve at the initial moment. Note that we have also w.l.o.g. assumed that the junction is located at the origin at the initial moment. We will moreover assume that the arms of the triod are not fully straight at the initial moment which means that $|\alpha^i(1)| < 1$ (since the length of each arm is equal to 1), $i = 1, 2, 3$.

Observe that $\mathscr{A}$, being a formal submanifold of the Otto-Wasserstein space, see [62] for similar claims, is a metric space with a non-degenerate (Riemannian) distance. Nevertheless, $\mathscr{A}$ is neither a complete metric space nor a geodesic space. Accordingly, the theory of gradient flows in metric spaces, cf. [2, 69], does not sound to be applicable to well-posedness of our flow (2.9).

To achieve our goal of this chapter, we will follow the strategy suggested by Shi and Vorotnikov [61] for the evolution of a single string. It basically consists of approximation of the original gradient flow on $\mathscr{A}$ by suitable gradient flows on the flat ambient space $L^2(\Omega; \mathbb{R}^{3d})$. The idea is to derive uniform estimates for the approximation problem that would allow us to pass to the limit and to show that the limiting functions are solutions to (2.8), (2.2). However, because of the complicated boundary conditions (2.2), many of the estimates that were used in [61] fail to be generalizable to our setting. This in particular applies to the crucial L^∞ estimate in the spirit of Ladyzhenskaya, Solonnikov and Uraltseva, cf. [34]. We will manage to overcome these difficulties and to prove novel and more refined estimates by leveraging the gradient flow structure of the approximation problem much more thoroughly than in [61]. This will be combined with careful observations involving geometric properties of triods, the behaviour of the curvature and some convexity argument.

Apart from that, in [61] the existence of C^∞-smooth solutions to the approximation problem was immediate from Amann's theory, cf. [1]. It is not applicable here anymore (again due to the boundary conditions), so we will solve our approximate problem by the theory of abstract evolution equations with pseudomonotone maps, cf. [59].

Our results still hold for the overdamped dynamics of a falling single cord with two fixed ends, see Remark 2.8 and Proposition 2.22.

2.2 Derivation of the full dynamical system and of the gradient flow

In this section, we show that system (2.1)-(2.2) can be view as a manifestation of the celebrated physical principle of least action [5, 23]. This section does not contain any analysis but is only intended to derive the full dynamical system and the overdamped one (i.e., the gradient flow). This section is heuristic and we do not claim full mathematical rigor.

We define the *action functional* $S(\eta)$ for the system (2.1)-(2.2) as the time integral of the difference between the total kinetic energy $K(t) := \sum_{i=1}^{3}\int_0^1 \frac{1}{2}|\partial_t\eta^i|^2 ds$ and the total potential energy $P(t) := \sum_{i=1}^{3}\int_0^1 -g\cdot\eta^i ds$,

$$S(\eta) = \int_0^T K(t) - P(t)dt = \sum_{i=1}^{3}\int_{\Omega_T}\left(\frac{1}{2}|\partial_t\eta^i|^2 + g\cdot\eta^i\right)dsdt. \tag{2.18}$$

Consider the following set of *triods* with fixed initial and final configurations:

$$\begin{aligned}\mathfrak{A} := \{\eta = (\eta^1,\eta^2,\eta^3) : \eta^i \in C^1\left(\bar{\Omega};\mathbb{R}^d\right) : |\partial_s\eta^i|^2 = 1, \eta^i(T,s) = \eta^i_T(s),\\ \eta^1 = \eta^2 = \eta^3 \text{ at } s = 0, \eta^i(1) = \alpha^i(1)\ , \eta^i(0,s) = \alpha^i(s)\forall i\},\end{aligned} \tag{2.19}$$

and let us look for minimizers of the functional S within the constraint set $\mathfrak{A}$. We claim that for each local constrained minimizer η there are scalar functions σ^i, $i = 1,2,3$, such that the tuple $\left(\eta^i,\sigma^i\right)$ satisfies (2.1)-(2.2).

Indeed, take any local minimizer η. Let ε be a positive small parameter. Let $h^i = h^i(\varepsilon)$ be arbitrary elements of $C^1\left(\bar{\Omega},\mathbb{R}^d\right)$, satisfying the following conditions

$$h^1 = h^2 = h^3 \text{ at } s = 0,\ h^i(t,1) = 0, \tag{2.20}$$

$$h^i(0,s) = 0,\ h^i(T,s) = 0, \tag{2.21}$$

$$2\partial_s h^i\cdot\partial_s\eta^i + \varepsilon|\partial_s h^i|^2 = 0. \tag{2.22}$$

We claim that

$$\eta + \varepsilon h \in \mathfrak{A}. \tag{2.23}$$

It suffices to check the $|\partial_s(\eta^i + \varepsilon h^i)|^2 = 1$, the other conditions are obvious. Indeed,

$$|\partial_s(\eta^i + \varepsilon h^i)|^2 = |\partial_s\eta^i|^2 + 2\varepsilon\partial_s\eta^i\cdot\partial_s h^i + \varepsilon^2|\partial_s h^i|^2 = 1$$

due to (2.22).

Since η is a constrained minimizer, we have the following inequality

$$\begin{aligned}&\sum_{i=1}^{3}\int_{\Omega_T}\left(\frac{1}{2}|\partial_t\eta^i + \varepsilon\partial_t h^i|^2 + g\cdot\left(\eta^i + \varepsilon h^i\right)\right)dsdt\\ &\qquad \geq \sum_{i=1}^{3}\int_{\Omega_T}\left(\frac{1}{2}|\partial_t\eta^i|^2 + g\cdot\eta^i\right)dsdt.\end{aligned} \tag{2.24}$$

Dividing by ε, we can recast this in the form

$$\sum_{i=1}^{3}\int_{\Omega_T}\left(\partial_t\eta^i\cdot\partial_t h^i+\frac{1}{2}\varepsilon|\partial_t h^i|^2+g\cdot h^i\right)dsdt\geq 0. \tag{2.25}$$

Let $\varepsilon\to 0$ to obtain

$$\sum_{i=1}^{3}\int_{\Omega_T}\left(\partial_t\eta^i\cdot\partial_t h^i+g\cdot h^i\right)dsdt\geq 0. \tag{2.26}$$

Observe that the condition (2.22) as $\varepsilon\to 0$ becomes

$$\partial_s h^i\cdot\partial_s\eta^i=0. \tag{2.27}$$

The possibility of replacing h^i by $-h^i$ in (2.26) without violating the constraints (2.20), (2.21) and (2.27) allows us to have equality in (2.26):

$$\sum_{i=1}^{3}\int_{\Omega_T}\left(\partial_t\eta^i\cdot\partial_t h^i+g\cdot h^i\right)dsdt=0. \tag{2.28}$$

Now, we apply integration by parts to equality (2.28) and obtain

$$\sum_{i=1}^{3}\int_{\Omega_T}\left(\partial_{tt}\eta^i-g\right)\cdot h^i dsdt=0 \tag{2.29}$$

for all h^i satisfying (2.20), (2.21), (2.27). Denote

$$Z^i(t,s):=\int_0^s\left(\partial_{tt}\eta^i(t,\zeta)-g\right)d\zeta.$$

We rewrite (2.29)

$$\sum_{i=1}^{3}\int_{\Omega_T}\partial_s Z^i\cdot h^i dsdt=0$$

and apply integration by parts

$$\sum_{i=1}^{3}\int_{\Omega_T}Z^i\cdot\partial_s h^i dsdt=0 \tag{2.30}$$

for all h^i satisfying (2.20), (2.21), (2.27). By a Hilbertian duality argument, it is possible to deduce from (2.30) that there exists a measurable scalar function $\sigma^i(t,s)$ for each $i=1,2,3$ such that $Z^i=\sigma^i\partial_s\eta^i$. We only need to recover the boundary condition $\sum_{i=1}^{3}\sigma^i\partial_s\eta^i=0$ at point $s=0$ for all t. By the definition of Z^i we have the following equality for all $i=1,2,3$

$$\partial_{tt}\eta^i-g=\partial_s\left(\partial_s\eta^i\sigma^i\right).$$

We multiply the last equality with generic h^i that satisfies (2.20), (2.21), (2.27), integrate over spatial and time variables and make a summation w.r.t. i to infer

$$\sum_{i=1}^{3}\int_{\Omega_t}\left(\partial_{tt}\eta^i-g\right)\cdot h^i dsdt=\sum_{i=1}^{3}\int_{\Omega_t}\partial_s\left(\sigma^i\partial_s\eta^i\right)\cdot h^i dsdt \tag{2.31}$$

We integrate by parts in the right-hand side and obtain

$$\sum_{i=1}^{3}\int_{\Omega_t}\left(\partial_{tt}\eta^i-g\right)\cdot h^i ds dt=\sum_{i=1}^{3}\int_{\Omega_t}-\left(\sigma^i\partial_s\eta^i\right)\cdot\partial_s h^i ds dt$$
$$+\sum_{i=1}^{3}\int_0^T\left(\sigma^i\partial_s\eta^i\cdot\bar{h}\right)(t,0)\,dt.$$

Here, $\bar{h}(t,0):=h^1(t,0)=h^2(t,0)=h^3(t,0)$ is used for a shortcut for the perturbation of the junction. Notice that the left-hand side of the last equality vanishes due to (2.29), and the first term on the right-hand side of the equality is zero due to (2.27). We end up with

$$0=\sum_{i=1}^{3}\int_0^T\left(\sigma^i\partial_s\eta^i\cdot\bar{h}\right)(t,0)\,dt,$$

which implies the missing boundary condition

$$\sum_{i=1}^{3}\partial_s\eta^i\sigma^i=0 \text{ at } s=0 \text{ for all } t\in[0,T].$$

Let us now derive a model of motion for a triod overdamped by a heavily dense environment. Assume that the triod is subject to a frictional force $f_d^i=c\partial_t\eta^i$ for each $i=1,2,3$ (here c is the damping coefficient) and a gravity force f_g^i. In view of the previous discussion, this is governed by the system

$$\begin{cases}\partial_{tt}\eta^i(t,s)=\partial_s\left(\zeta^i(t,s)\,\partial_s\eta^i(t,s)\right)+f_g^i-f_d^i\\ |\partial_s\eta^i(t,s)|=1.\end{cases}$$

Assume that the gravity is of the same order as the damping, that is, $f_g^i=cg$ for some constant vector g. We divide the equations by c, and letting $\sigma^i=\zeta^i/c$ and $c\to\infty$ we formally deduce

$$\begin{cases}\partial_t\eta^i(t,s)=\partial_s\left(\sigma^i(t,s)\,\partial_s\eta^i(t,s)\right)+g\\ |\partial_s\eta^i(t,s)|=1.\end{cases}$$

We complement the system with the initial/boundary conditions

$$\begin{cases}\eta^1(t,0)=\eta^2(t,0)=\eta^3(t,0),\\ \eta^i(t,1)=\alpha^i(1),\\ \sigma^1\partial_s\eta^1+\sigma^2\partial_s\eta^2+\sigma^3\partial_s\eta^3=0 \text{ at } s=0 \text{ for all } t,\\ \eta^i(0,s)=\alpha^i(s).\end{cases}$$

Note that this is a first order equation w.r.t. time, so only the initial configuration α is needed do be prescribed.

Another way to feel the link between the original system (2.1) and the gradient flow (2.8) is to employ the quadratic change of time [13]. Indeed, introducing the "new time" $\theta:=t^2/2$ in (2.1), we derive

$$\begin{cases} \partial_\theta \eta^i + 2\theta \partial_{\theta\theta}\eta^i = \partial_s\left(\sigma^i \partial_s \eta^i\right) + g \\ |\partial_s \eta^i| = 1. \end{cases} \tag{2.32}$$

For small θ we have $\theta \partial_{\theta\theta}\eta^i \approx 0$, and we obtain (2.8).

2.3 The approximation problem

Let us now describe the way of approximation of our gradient flow that we are going to employ in order to prove the main result of this chapter (Theorem 2.21). Similar approaches (as well as the notation) will be used in the other chapters.

We begin with some heuristics. Consider the extra variables $\kappa^i = \sigma^i \partial_s \eta^i$, $i = 1,2,3$. Then our system (1.2) can at least formally be rewritten as

$$\begin{cases} \partial_t \eta^i = \partial_s \kappa^i + g, \\ \kappa^i = \sigma \partial_s \eta^i, \\ \sigma^i = \kappa^i \cdot \partial_s \eta^i. \end{cases} \tag{2.33}$$

More precisely, the constraints $|\partial_s \eta^i| = 1$ yield $|\kappa^i| = |\sigma|$ and $\kappa^i = \mathrm{sgn}(\sigma)|\kappa^i|\partial_s \eta^i$. We make the ansatz $\sigma^i \geq 0$ (that will be a posteriori justified) and infer $\kappa^i \cdot \partial_s \eta^i = \sigma^i$. Note that we formally have $\partial_s \eta^i = \frac{\kappa^i}{|\kappa^i|}$, thus the map $\kappa^i \mapsto \partial_s \eta^i$ is not a diffeomorphism. To overcome this issue, we fix $\varepsilon \in (0,1)$ and introduce the auxiliary functions

$$F_\varepsilon : \mathbb{R}^d \to \mathbb{R}^d,\ F_\varepsilon(\kappa) := \varepsilon \kappa + \frac{\kappa}{\sqrt{\varepsilon + |\kappa|^2}} \tag{2.34}$$

and

$$G_\varepsilon(\tau) := (F_\varepsilon)^{-1}(\tau).$$

Approximating the relations $\kappa^i \mapsto \partial_s \eta^i$ and $\partial_s \eta^i \mapsto \kappa^i$ by F_ε and G_ε, respectively, will be used to obtain different approximation systems.

Remark 2.4. Let us make an elementary observation that is very important in the sequel. The Euclidean norm $|F_\varepsilon(\kappa)|$ depends only on $|\kappa|$ and is an increasing function of $|\kappa|$. If $|\kappa| = 1$, then by simple calculation $|F_\varepsilon(\kappa)| > 1$. Consequently, if $|\tau| \leq 1$, then $|G_\varepsilon(\tau)| < 1$.

Now, we write some computations to show the bound for ∇G_ε; firstly we compute

$$\nabla F_\varepsilon^i = \frac{\partial F_\varepsilon^i}{\partial \kappa^j} = \varepsilon \delta_{ij} + \frac{\varepsilon \delta_{ij} + |\kappa|^2 \delta_{ij} - \kappa^i \kappa^j}{\left(\sqrt{\varepsilon + |\kappa|^2}\right)^3},$$

where δ_{ij} is the usual Kronecker delta and $i, j = 1...d$. To obtain estimates for eigenvalues of the matrix ∇F_ε^i, we test ∇F_ε^i by $\forall \xi \in \mathbb{R}^d$, then we have the following inequalities

$$\left(\Lambda_\varepsilon^{-1}(\kappa)\right)|\xi|^2 \leq \nabla F_\varepsilon \xi \cdot \xi \leq \left(\lambda_\varepsilon^{-1}(\kappa)\right)|\xi|^2$$

where

$$\Lambda_{\varepsilon}^{-1}(\kappa) = \varepsilon + \frac{\varepsilon}{\left(\sqrt{\varepsilon + |\kappa|^2}\right)^3}$$
$$\lambda_{\varepsilon}^{-1}(\kappa) = \varepsilon + \frac{\varepsilon + |\kappa|^2}{\left(\sqrt{\varepsilon + |\kappa|^2}\right)^3}.$$

We use the usual theory for eigenvalues and inverse matrices. By explicit computations done above we conclude that ∇G_{ε} is positive-definite and furthermore we infer the estimate

$$\lambda_{\varepsilon}(\tau)|\xi|^2 \le \nabla G_{\varepsilon}(\tau)\xi\cdot\xi \le \Lambda_{\varepsilon}(\tau)|\xi|^2, \ \ \forall \xi \in \mathbb{R}^d, \tau \in \mathbb{R}^d, \tag{2.35}$$

where Λ_{ε} and λ_{ε} satisfy

$$\begin{aligned} \frac{1}{\varepsilon + \varepsilon^{-1/2}} \le \lambda_{\varepsilon}(\tau) &:= \frac{1}{\varepsilon + \left(\varepsilon + |G_{\varepsilon}(\tau)|^2\right)^{-1/2}} \\ \Lambda_{\varepsilon}(\tau) &:= \frac{\varepsilon^{-1}}{1 + \left(\varepsilon + |G_{\varepsilon}(\tau)|^2\right)^{-3/2}} \le \varepsilon^{-1}. \end{aligned} \tag{2.36}$$

Using all that properties and definitions, we can proceed with approximation problem defined below. Approximation of the relations $\kappa^i \mapsto \partial_s \eta^i$ and $\partial_s \eta^i \mapsto \kappa^i$ by F_{ε} and G_{ε}, respectively, leads from (2.8), (2.2) to the problem

$$\partial_t \eta_{\varepsilon}^i = \partial_s \left(G_{\varepsilon}\left(\partial_s \eta_{\varepsilon}^i\right)\right) + g, \quad i = 1,2,3, \tag{2.37}$$

with the following initial and boundary conditions:

$$\begin{aligned} \eta_{\varepsilon}^i(0,s) &= \alpha^i(s), \\ \eta_{\varepsilon}^i(t,1) &= \alpha^i(1), \\ \eta_{\varepsilon}^1(t,0) = \eta_{\varepsilon}^2(t,0) &= \eta_{\varepsilon}^3(t,0), \\ \sum_{i=1}^{3} G_{\varepsilon}\left(\partial_s \eta_{\varepsilon}^i\right) &= 0 \text{ at } s = 0 \text{ for all } t. \end{aligned} \tag{2.38}$$

Motivated by the original system (2.33), given a solution η_{ε} to the approximation problem (2.37), (2.38) we define

$$\kappa_{\varepsilon}^i := G_{\varepsilon}\left(\partial_s \eta_{\varepsilon}^i\right), \ \ \sigma_{\varepsilon}^i := G_{\varepsilon}\left(\partial_s \eta_{\varepsilon}^i\right)\cdot \partial_s \eta_{\varepsilon}^i. \tag{2.39}$$

Observe from the definition of G_{ε} that there exists a bounded smooth positive scalar function γ_{ε} such that $G_{\varepsilon}(\tau) = \gamma_{\varepsilon}(|\tau^2|)\tau$, $\tau \in \mathbb{R}^d$. In particular, this implies for triods with inextensible strings that

$$\sigma_{\varepsilon}^i \ge 0. \tag{2.40}$$

Moreover, γ_{ε} is bounded away from 0 and ∞ (not uniformly w.r.t. ε). Let Γ_{ε} be the primitive of γ_{ε} with $\Gamma_{\varepsilon}(0) = 0$. Set

$$Q_{\varepsilon}(\tau) := \frac{1}{2}\Gamma_{\varepsilon}(|\tau|^2).$$

Observe that

$$\nabla Q_\varepsilon(\tau) = G_\varepsilon(\tau). \tag{2.41}$$

Moreover, Q_ε can be computed explicitly for the approximation system (2.37) :

$$Q_\varepsilon(\tau) = \varepsilon\left(\frac{|G_\varepsilon(\tau)|^2}{2} - \frac{1}{\sqrt{\varepsilon + |G_\varepsilon(\tau)|^2}}\right) + \sqrt{\varepsilon}. \tag{2.42}$$

By Remark 2.4, $Q_\varepsilon(\tau) << 1$ if $|\tau| \leq 1$.

We define the associated "total energy" of the approximation problem (2.37), (2.38) by

$$\begin{cases} \mathscr{E}_\varepsilon(\eta) := \sum_{i=1}^3 \left(\int_0^1 Q_\varepsilon\left(\partial_s \eta^i\right) ds + \int_0^1 (-g)\cdot \eta^i ds\right) \\ \text{for } \eta \in AC^2(\Omega;\mathbb{R}^{3d}) \text{ satisfying } \eta(1) = \alpha(1),\ \eta^1(0) = \eta^2(0) = \eta^3(0); \\ +\infty \text{ for any } \eta \in L^2(\Omega;\mathbb{R}^{3d}) \text{ except those above.} \end{cases} \tag{2.43}$$

Then (2.37), (2.38) can at least formally be interpreted as a gradient flow, with respect to the flat Hilbertian structure of $L^2(\Omega;\mathbb{R}^{3d})$, that is driven by this functional, i.e.

$$\frac{d}{dt}\eta = -\nabla_{L^2(\Omega;\mathbb{R}^{3d})}\mathscr{E}_\varepsilon(\eta), \quad \eta(0) = \alpha.$$

We will return to this issue in the next sections.

2.4 Evolution by pseudomonotone maps and solvability of the approximation problem

For the existence of the solution to the approximation problem, we use the theory of abstract evolution equations involving pseudomonotone maps. We prefer this approach (instead of directly employing the theory of gradient flows in Hilbert spaces, cf. [7, 15]) because it automatically gives us the regularity of solution that is required.

Our goal of this section is show the existence and regularity of solutions to the approximation problem. We use the theory described in Appendix A.

Remembering that $\alpha^i(s)$ is the initial condition, it is convenient to rewrite our approximation problem (2.37)-(2.38) with the help of the simple transformation,

$$\xi^i(t,s) := \eta_\varepsilon^i(t,s) - \alpha^i(s),$$

arriving at

$$\begin{cases} \partial_t \xi^i - \partial_s \left(G_\varepsilon \left(\partial_s \left(\xi^i + \alpha^i \right) \right) \right) = g, \quad i = 1,2,3, \\ \xi^1 (t,0) = \xi^2 (t,0) = \xi^3 (t,0), \\ \xi^i (t,1) = 0, \\ \xi^i (0,s) = 0, \\ \sum_{i=1}^3 G_\varepsilon \left(\partial_s \left(\xi^i + \alpha^i \right) \right) (t,0) = 0. \end{cases} \tag{2.44}$$

Let us recast this system in the form of the Cauchy problem (A.2). Let $H = L^2 \left(\Omega; \mathbb{R}^{3d} \right)$ be the Hilbert space of triples with the natural scalar product. Consider the set

$$V := \{ u = \{u^i\} \in AC^2 \left(\overline{\Omega}; \mathbb{R}^{3d} \right) \text{ such that } u^i(1) = 0 \text{ and } u^1(0) = u^2(0) = u^3(0) \}.$$

It is a separable reflexive Banach space with the norm inherited from H^1. Define a seminorm on V by $|\{u^i\}|_V := \|\{\partial_s u^i\}\|_H$. The required Poincaré inequality obviously holds. Let $A : V \to V^*$ be the mapping that is defined by duality as follows:

$$\langle A(\xi), \zeta \rangle = \sum_{i=1}^{3} \int_0^1 G_\varepsilon \left(\partial_s \left(\xi^i + \alpha^i \right) \right) \cdot \partial_s \zeta^i ds. \tag{2.45}$$

Then (2.44) rewrites as

$$\frac{d}{dt} \xi + A(\xi(t)) = \mathbf{g}, \ \xi(0) = 0. \tag{2.46}$$

Note that the last equality of (2.44) is hidden in the duality in (2.45).

In order to check that Theorem A.8 is applicable to (2.46) we need to prove several auxiliary statements. For the sake of readability, we will omit the subscript ε coming from the approximation problem.

Lemma 2.1. *The mapping A satisfies the inequality*

$$\langle A(\xi_1) - A(\xi_2), \xi_1 - \xi_2 \rangle \geq c_0 |\xi_1 - \xi_2|_V^2$$

for some constant $c_0 > 0$ (depending on ε) and any $\xi_1, \xi_2 \in V$.

Proof. Define $A^i : H^1 \left(\Omega; \mathbb{R}^d \right) \to \left(H^1 \left(\Omega; \mathbb{R}^d \right) \right)^*$ by

$$\langle A^i \left(\xi^i \right), \zeta \rangle = \int_0^1 G_\varepsilon \left(\partial_s \left(\xi^i + \alpha^i \right) \right) \cdot \partial_s \zeta^i ds.$$

Throughout the rest of the proof, we omit the index i to avoid heavy notation in A^i, ξ_1^i, ξ_2^i and α^i. With this convention, it suffices to prove that

$$\langle A(\xi_1) - A(\xi_2), \xi_1 - \xi_2 \rangle \geq c_0 \|\partial_s (\xi_1 - \xi_2)\|_{L^2}^2.$$

We compute

$$\langle A\left(\xi_1\right)-A\left(\xi_2\right),\xi_1-\xi_2\rangle=\int_\Omega\left[G\left(\partial_s\left(\xi_1+\alpha\right)\right)-G\left(\partial_s\left(\xi_2+\alpha\right)\right)\right]\cdot\partial_s\left(\left(\xi_1+\alpha\right)-\left(\xi_2+\alpha\right)\right)ds. \tag{2.47}$$

Let us denote $\mu:=G\left(\partial_s\left(\xi_1+\alpha\right)\right)$ and $\gamma:=G\left(\partial_s\left(\xi_2+\alpha\right)\right)$. Now, we use the relation between F and G and conclude that $F\left(\mu\right)=\partial_s\left(\xi_1+\alpha\right)$ and $F\left(\gamma\right)=\partial_s\left(\xi_2+\alpha\right)$. We can rewrite the right-hand side of (2.47) as

$$\begin{aligned}&\int_\Omega(\mu-\gamma)\cdot\left(F\left(\mu\right)-F\left(\gamma\right)\right)ds\\&=\int_\Omega(\mu-\gamma)\cdot\left(\varepsilon\left(\mu-\gamma\right)+\frac{\mu}{\sqrt{\varepsilon+|\mu|^2}}-\frac{\gamma}{\sqrt{\varepsilon+|\gamma|^2}}\right)ds\\&\geq\int_\Omega\varepsilon|\mu-\gamma|^2ds\end{aligned}$$

because the map $r\mapsto\frac{r}{\sqrt{\varepsilon+r^2}}$ is a gradient of a convex function. Observe that

$$|F\left(\mu\right)-F\left(\gamma\right)|\leq(\varepsilon+\varepsilon^{-1/2})|\mu-\gamma|$$

by the mean value theorem since the operator norm of the matrix $\nabla F(r)$ is bounded from above by $\varepsilon+\varepsilon^{-1/2}$, cf. (2.36).

Thus we conclude that

$$\langle A\left(\xi_1\right)-A\left(\xi_2\right),\xi_1-\xi_2\rangle\geq\varepsilon\int_\Omega|\mu-\gamma|^2ds\geq c_0\int_\Omega|F(\mu)-F(\gamma)|^2ds=c_0\|\partial_s(\xi_1-\xi_2)\|_{L^2}^2.$$

□

Corollary 2.2. *The mapping A is monotone.*

Proof. It is clear from Lemma 2.1. □

Corollary 2.3. *The mapping A is semicoercive.*

Proof. Employing Lemma 2.1 and Cauchy-Schwarz inequality, we see that

$$\begin{aligned}\langle A\left(\xi\right),\xi\rangle&=\langle A\left(\xi\right)-A\left(0\right),\xi\rangle+\langle A\left(0\right),\xi\rangle\\&\geq|\xi|_V^2+\langle A\left(0\right),\xi\rangle\\&=|\xi|_V^2+\sum_{i=1}^3\int_0^1G\left(\partial_s\alpha^i\right)\cdot\partial_s\xi^ids\\&\geq|\xi|_V^2-\left\|\left\{G\left(\partial_s\alpha^i\right)\right\}\right\|_H|\xi|_V\\&\geq|\xi|_V^2-c_2|\xi|_V,\end{aligned}$$

where c_2 is a positive constant depending on α. □

Lemma 2.4. *The mapping A is bounded.*

Proof. Indeed,

$$\begin{aligned}\langle A(\xi),\zeta\rangle &= \sum_{i=1}^{3}\int_0^1 G\left(\partial_s\left(\xi^i+\alpha^i\right)\right)\cdot\partial_s\zeta^i ds\\ &\leq \left\|\left\{G\left(\partial_s\left(\xi^i+\alpha^i\right)\right)\right\}\right\|_H |\zeta|_V\\ &\lesssim \left\|\left\{\partial_s\left(\xi^i+\alpha^i\right)\right\}\right\|_H |\zeta|_V\\ &\leq |\xi+\alpha|_V\|\zeta\|_V.\end{aligned}$$

(We have used sublinearity of G). Since $|\alpha|_V$ is finite, this implies that $\|A(\xi)\|_{V^*}$ is bounded provided $\|\xi\|_V$ is bounded. □

Lemma 2.5. *The mapping A is radially continuous.*

Proof. Fix $\xi,\zeta\in V$ and let $\tau_n\to\tau$ be a sequence. Then it is easy to see that

$$\sum_{i=1}^{3}G\left(\partial_s\left(\xi^i+\tau_n\zeta^i+\alpha^i\right)\right)(x)\cdot\partial_s\zeta^i(x)\to\sum_{i=1}^{3}G\left(\partial_s\left(\xi^i+\tau\zeta^i+\alpha^i\right)\right)(x)\cdot\partial_s\zeta^i(x)$$

a.e. in Ω. The claim will follow from Lebesgue's dominated convergence theorem if there is a function in $L^1(\Omega)$ that dominates the left-hand side. But it is indeed the case since we can leverage sublinearity of G to estimate

$$\left|\sum_{i=1}^{3}G\left(\partial_s\left(\xi^i+\tau_n\zeta^i+\alpha^i\right)\right)\cdot\partial_s\zeta^i\right|\leq C|\partial_s\left(\xi+\tau_n\zeta+\alpha\right)|\cdot|\partial_s\zeta|\leq C(|\partial_s\xi|^2+|\partial_s\zeta|^2+|\partial_s\alpha|^2),$$

and the right-hand side is L^1 by the assumption. □

Invoking Lemma A.4, we get the following corollary.

Corollary 2.6. *The mapping* **A** *is pseudomonotone.*

We can now legitimately use Theorem A.8 in order to solve (2.46).

Corollary 2.7. *Given α as in Remark 2.3, the system* (2.46) *has a solution $\xi=\{\xi^i\}\in W^{1,\infty}(0,T;H)\cap AC^2\left([0,T];V\right)$ that is understood in the same sense as in Theorem A.8.*

Returning back to the variable η and leveraging elementary properties of G_ε and ∇G_ε, we get the existence of approximate solutions.

Corollary 2.8. *Given α as in Remark 2.3, there exists a solution $\eta=\eta_\varepsilon$ to* (2.37)-(2.38) *in Ω_T that belongs to the following regularity class:*

$$\eta^i\in W^{1,\infty}\left(0,T;L^2(\Omega)\right)^d\cap AC^2\left([0,T];AC^2\left(\overline{\Omega}\right)\right)^d,$$

$$\partial_s\eta^i \in AC^2\left([0,T];L^2(\Omega)\right)^d,$$

$$\kappa^i := G_\varepsilon(\partial_s\eta^i) \in L^\infty\left(0,T;L^2(\Omega)\right)^d,$$

$$\nabla G_\varepsilon(\partial_s\eta^i) \in L^\infty\left(0,T;L^\infty(\Omega)\right)^d,$$

$$\partial_t\eta^i \in L^\infty\left(0,T;L^2(\Omega)\right)^d \cap L^2\left(0,T;H^1(\Omega)\right)^d,$$

$$\partial_s\kappa^i = \partial_s\left(G_\varepsilon\left(\partial_s\eta^i\right)\right) \in L^\infty\left(0,T;L^2(\Omega)\right)^d \cap L^2\left(0,T;H^1(\Omega)\right)^d,$$

$$\partial_{ss}\eta^i \in L^\infty\left(0,T;L^2(\Omega)\right)^d.$$

Note that the norms of the solution $\eta = \eta_\varepsilon$ in the corresponding spaces above may depend on ε. At this stage we cannot infer an L^∞ estimate on $\partial_s\eta$ (even ε-dependent) because we do not control $\partial_s\eta^i$ on $\partial\Omega$. Anyway, we will manage to establish a related bound in Corollary 2.16.

It is straightforward to see that $\eta = \eta_\varepsilon$ from Corollary 2.8 coincides with the unique solution of the gradient flow

$$\frac{d}{dt}\eta \in -\partial_{L^2(\Omega;\mathbb{R}^{3d})}\mathscr{E}_\varepsilon(\eta) \tag{2.48}$$

in the sense of Theorem B.2 in Appendix B, where the driving functional $\mathscr{E}_\varepsilon$ was defined in (2.43). This in particular implies that $t \mapsto \mathscr{E}_\varepsilon\left(\eta(t)\right)$ is a continuous and non-increasing function.

2.5 Uniform estimates of the approximate solutions

In this section we derive various uniform (in ε) estimates for the approximation solutions η^i_ε obtained in Corollary 2.8. These bounds are crucial for passing to the limit in Section 2.6. In the sequel, C will always stand for a constant independent of ε. For the sake of readability, we drop the dependence on ε in the subscripts and write $\eta^i = \eta^i_\varepsilon$, $G = G_\varepsilon$, $\alpha^i = \alpha^i_\varepsilon$, etc., until the proof of Lemma 2.15.

Lemma 2.9 (Energy estimate)**.** *Let $\eta = \{\eta^i\}$ be a solution of the approximation problem* (2.37)-(2.38) *in $\mathfrak{Q}_T$ as constructed in Corollary 2.8. Then*

$$\mathscr{E}(\alpha) + \sum_{i=1}^{3}\left(\int_{\mathfrak{Q}_T} |\partial_t\eta^i|^2 + |\nabla G\left(\partial_s\eta^i\right)\partial_{ss}\eta^i|^2\, ds dt\right) + \|\eta\|^2_{L^\infty\left(0,T;L^1(\Omega)\right)} \le C. \tag{2.49}$$

Here the constant may only depend on α and T, but not on ε.

Proof. We first establish a uniform bound (w.r.t. ε) on the initial energies. Indeed, since $|\partial_s\alpha^i(s)| = 1$, Remark 2.4 implies that $|G(\partial_s\alpha^i(s))| < 1$, and using the explicit definition of Q given in (2.42), we get that the first terms (for each i) in the expansion

$$\mathscr{E}(\alpha) = \sum_{i=1}^{3}\left(\int_0^1 Q\left(\partial_s\alpha^i(s)\right)ds + \int_0^1 (-g)\cdot\alpha^i(s)\,ds\right)$$

are uniformly bounded. The second terms are obviously uniformly bounded.

Let us estimate the remaining terms in the left-hand side of the energy estimate (2.49). Take the $L^2(\Omega)$-inner product of (2.37) and $\partial_t \eta^i$ and integrate over $\mathfrak{Q}_\mathrm{t}$, $\mathrm{t} \in (0,T]$. We obtain

$$\sum_{i=1}^{3} \int_{\mathfrak{Q}_\mathrm{t}} |\partial_t \eta^i|^2 \, dsdt = \sum_{i=1}^{3} \int_{\mathfrak{Q}_\mathrm{t}} \partial_s G\left(\partial_s \eta^i\right) \cdot \partial_t \eta^i \, dsdt + \sum_{i=1}^{3} \int_{\mathfrak{Q}_\mathrm{t}} g \cdot \partial_t \eta^i \, dsdt$$

Then, we perform an integration by parts and also integrate the last term over time, ending up with

$$\begin{aligned}
\sum_{i=1}^{3} \int_{\mathfrak{Q}_\mathrm{t}} |\partial_t \eta^i|^2 \, dsdt &= -\sum_{i=1}^{3} \int_{\mathfrak{Q}_\mathrm{t}} G\left(\partial_s \eta^i\right) \cdot \partial_{st} \eta^i \, dsdt + \sum_{i=1}^{3} \int_{\Omega} g \cdot \eta^i(\mathrm{t}) \, ds - \sum_{i=1}^{3} \int_{\Omega} g \cdot \alpha^i \, ds \\
&\quad + \sum_{i=1}^{3} \int_0^{\mathrm{t}} \underbrace{G\left(\partial_s \eta^i\right) \cdot \partial_t \eta^i}_{\text{at } s=1} dt - \sum_{i=1}^{3} \int_0^{\mathrm{t}} \underbrace{G\left(\partial_s \eta^i\right) \cdot \partial_t \eta^i}_{\text{at } s=0} dt \\
&= -\sum_{i=1}^{3} \int_{\mathfrak{Q}_\mathrm{t}} G\left(\partial_s \eta^i\right) \cdot \partial_{st} \eta^i \, dsdt + \sum_{i=1}^{3} \int_{\Omega} g \cdot \eta^i(\mathrm{t}) \, ds - \sum_{i=1}^{3} \int_{\Omega} g \cdot \alpha^i \, ds \\
&\quad + \sum_{i=1}^{3} \int_0^{\mathrm{t}} \underbrace{\underbrace{G\left(\partial_s \eta^i(1)\right) \cdot \partial_t \alpha^i(1)}_{\partial_t \alpha^i = 0}}_{=0} dt - \sum_{i=1}^{3} \int_0^{\mathrm{t}} \underbrace{\underbrace{G\left(\partial_s \eta^i(0)\right) \cdot \partial_t \bar{\eta}}_{\sum_{i=1}^3 G\left(\partial_s \eta^i(0)\right) = 0}}_{=0} dt.
\end{aligned}$$

Here $\bar{\eta}(t)$ denotes the spatial position of the junction. Consequently,

$$\sum_{i=1}^{3} \int_{\mathfrak{Q}_\mathrm{t}} |\partial_t \eta^i|^2 \, dsdt = -\sum_{i=1}^{3} \int_{\mathfrak{Q}_\mathrm{t}} G\left(\partial_s \eta^i\right) \cdot \partial_{st} \eta^i \, dsdt + \sum_{i=1}^{3} \int_{\Omega} g \cdot \eta^i(\mathrm{t}) \, ds - \sum_{i=1}^{3} \int_{\Omega} g \cdot \alpha^i \, ds. \quad (2.50)$$

For the first term on the right-hand side, we observe that

$$G(\partial_s \eta^i) \cdot \partial_{st} \eta^i = \partial_t Q\left(\partial_s \eta^i\right), \quad (2.51)$$

cf. (2.41), where Q is defined as in (2.42). In view of (2.51), (2.50) becomes

$$\begin{aligned}
\sum_{i=1}^{3} \int_{\mathfrak{Q}_\mathrm{t}} |\partial_t \eta^i|^2 \, dsdt + \sum_{i=1}^{3} \int_{\Omega} Q\left(\partial_s \eta^i\right)(\mathrm{t}, \cdot) + \int_{\Omega} (-\mathbf{g}) \cdot \eta\,(\mathrm{t}, \cdot)\, ds \\
= \sum_{i=1}^{3} \int_{\Omega} Q\left(\partial_s \alpha^i(s)\right) ds + \int_{\Omega} (-\mathbf{g}) \cdot \alpha(s)\, ds,
\end{aligned}$$

whence

$$\sum_{i=1}^{3} \int_{\mathfrak{Q}_\mathrm{t}} |\partial_t \eta^i|^2 \, dsdt + \mathscr{E}\left(\eta(\mathrm{t})\right) = \mathscr{E}(\alpha). \quad (2.52)$$

Using $Q \geq 0$ and the definition of $\mathscr{E}$, we derive that

$$\mathscr{E}\left(\eta(\mathrm{t})\right) \geq -\|\eta(\mathrm{t})\|_{L^1(\Omega)} \|\mathbf{g}\|_{L^\infty(\Omega)}. \quad (2.53)$$

Hence, employing Jensen's inequality, we can estimate

$$\frac{1}{3}\|\eta(\mathrm{t})\|^2_{L^1(\Omega)} \le \sum_{i=1}^{3}\|\eta^i(\mathrm{t})\|^2_{L^1(\Omega)} = \sum_{i=1}^{3}\left(\int_{\Omega}|\eta^i(\mathrm{t})|\,ds\right)^2$$
$$= \sum_{i=1}^{3}\left(\int_{\Omega}|\alpha^i(s)|\,ds + \int_{\Omega_\mathrm{t}}\partial_t|\eta^i|\,dsdt\right)^2 \le 2\|\alpha\|^2_{L^2(\Omega)} + 2\sum_{i=1}^{3}\left(\int_{\Omega_\mathrm{t}}|\partial_t\eta^i|\,dsdt\right)^2$$
$$\le 2\|\alpha\|^2_{L^2(\Omega)} + 2\mathrm{t}\sum_{i=1}^{3}\int_{\Omega_\mathrm{t}}|\partial_t\eta^i|^2\,dsdt \le 2\|\alpha\|^2_{L^2(\Omega)} + 2T\mathscr{E}(\alpha) + 2T\|\eta(\mathrm{t})\|_{L^1(\Omega)}\|\mathbf{g}\|_{L^\infty(\Omega)}.$$

Simple algebra implies that $\|\eta(\mathrm{t})\|_{L^1(\Omega)}$ is uniformly bounded. Consequently, letting $\mathrm{t} = T$ we conclude that $\sum_{i=1}^{3}\int_{\Omega_T}|\partial_t\eta^i|^2\,dsdt$ is uniformly bounded. On the other hand, from the equality $\partial_s\left(G\left(\partial_s\eta^i\right)\right) = \partial_t\eta^i - g$ we deduce

$$\sum_{i=1}^{3}\int_{\Omega_T}\|\nabla G(\partial_s\eta^i)\partial_{ss}\eta^i|^2\,dsdt = \sum_{i=1}^{3}\int_{\Omega_T}|\partial_s\left(G(\partial_s\eta^i)\right)|^2\,dsdt$$
$$\le 2\sum_{i=1}^{3}\int_{\Omega_T}|\partial_t\eta^i|^2\,dsdt + 6\int_{\Omega_T}|g|^2\,dsdt \le C.$$

We have used Jensen's inequality and the fact that $\mathbf{g} = (g,g,g)$. □

In view of (2.53) we simultaneously proved the following.

Corollary 2.10. *The energy of the approximation problem $\mathscr{E}\left(\eta(t)\right)$ is bounded from below for all $t \in [0,T]$ uniformly in ε.*

Since $\eta^i(0) = \alpha^i$ does not depend on ε, the uniform regularity can immediately be improved. Namely, $\eta(t,\cdot) = \alpha + \int_0^t\partial_t\eta$ and the second term on the right-hand side is uniformly bounded in $L^\infty\left(0,T;L^2(\Omega)\right)$ by (2.49). So we get the following corollary.

Corollary 2.11. *The norm $||\eta^i||_{L^\infty\left(0,T;L^2(\Omega)\right)}$ is uniformly bounded with respect to ε.*

For the the subsequent family of estimates will need to bound the time away from zero by some constant $\delta > 0$.

Lemma 2.12. *Given $\delta > 0$, the norm $||\partial_t\eta^i||_{L^\infty\left(\delta,T;L^2(\Omega)\right)}$ is bounded uniformly in ε.*

Proof. By Theorem B.2, the right derivative $\partial_t^+\eta$ exists for all times, and the expression $||\partial_t^+\eta(t)||^2_{L^2(\Omega)}$ is non-increasing in time. Using [7, formula (17.79)], we obtain

$$\mathscr{E}(\alpha) - \mathscr{E}\left(\eta(\delta)\right) \ge \limsup_{h\searrow 0}\mathscr{E}\left(\eta(h)\right) - \mathscr{E}\left(\eta(\delta)\right)$$
$$= \int_0^\delta ||\partial_t\eta(t)||^2_{L^2(\Omega)}dt$$
$$\ge \int_0^\delta ||\partial_t^+\eta(\delta)||^2_{L^2(\Omega)}$$
$$= \delta||\partial_t^+\eta(\delta)||^2_{L^2(\Omega)}.$$

By (2.49) and Corollary 2.10, the left-hand side is bounded from above uniformly in ε. Hence, $||\partial_t^+\eta^i(\delta)||_{L^2(\Omega)} \le C/\delta$.

Since $||\partial_t^+\eta(t)||_{L^2(\Omega)}$ is non-increasing in time, we infer that

$$||\partial_t\eta^i||_{L^\infty\left(\delta,T;L^2(\Omega)\right)} = ||\partial_t^+\eta^i||_{L^\infty\left(\delta,T;L^2(\Omega)\right)}$$

is bounded uniformly in ε. □

We now derive uniform bounds for κ that were defined in (2.39). We start with the following lemma.

Lemma 2.13. *For fixed $\delta > 0$, $\partial_s\kappa^i$ and the product $|\kappa^i||\partial_{ss}\eta^i - \varepsilon\partial_s\kappa^i|$ are bounded in $L^\infty\left(\delta,T;L^2(\Omega)\right)$ uniformly with respect to ε, $i = 1,2,3$.*

Proof. By Lemma 2.12, we know that $||\partial_t\eta^i||_{L^\infty\left(\delta,T;L^2(\Omega)\right)} \le C$. Since $\partial_t\eta^i = \partial_s\kappa^i + g$, we infer that $\partial_s\kappa^i$ is bounded in $L^\infty\left(\delta,T;L^2(\Omega)\right)$ uniformly with respect to ε. We differentiate both sides of the equality

$$\partial_s\eta^i = F_\varepsilon(\kappa^i) = \varepsilon\kappa^i + \frac{\kappa^i}{\sqrt{\varepsilon + |\kappa^i|^2}}$$

with respect to s to get

$$\partial_{ss}\eta^i = \varepsilon\partial_s\kappa^i + \frac{\partial_s\kappa^i}{\sqrt{\varepsilon+|\kappa^i|^2}} - \frac{\kappa^i\left(\partial_s\kappa^i\cdot\kappa^i\right)}{(\varepsilon+|\kappa^i|^2)^{3/2}}.$$

We multiply this equality by $\sqrt{\varepsilon+|\kappa^i|^2}$ and deduce

$$\partial_{ss}\eta^i\sqrt{\varepsilon+|\kappa^i|^2} = \varepsilon\partial_s\kappa^i\sqrt{\varepsilon+|\kappa^i|^2} + \partial_s\kappa^i - \frac{\kappa^i\left(\partial_s\kappa^i\cdot\kappa^i\right)}{\varepsilon+|\kappa^i|^2}.$$

We reorganize the equality above to obtain

$$\left(\partial_{ss}\eta^i - \varepsilon\partial_s\kappa^i\right)\sqrt{\varepsilon+|\kappa^i|^2} = \partial_s\kappa^i - \frac{\kappa^i\left(\partial_s\kappa^i\cdot\kappa^i\right)}{\varepsilon+|\kappa^i|^2}.$$

The right-hand side is bounded in $L^\infty\left(\delta,T;L^2(\Omega)\right)$ uniformly with respect to ε, hence so is the left-hand side. Consequently, $|\kappa^i||\partial_{ss}\eta^i - \varepsilon\partial_s\kappa^i|$ is bounded in $L^\infty\left(\delta,T;L^2(\Omega)\right)$ uniformly with respect to ε. □

Lemma 2.14. *Let Υ be a finite set in $\mathbb{R}^d$. Assume that there exists a point in the convex hull of Υ such that the distance between it and Υ is greater than or equal to 1. Then the radius of the smallest enclosing ball for Υ is greater than or equal to 1.*

Proof. Translating the origin if necessary, we may assume that the origin belongs to the convex hull of Υ, and Υ does not intersect with the open unit ball centered in the origin. It suffices to prove that there is no $p \in \mathbb{R}^d$ with $|y-p| < 1$ for any $y \in \Upsilon$. Indeed, if such p exists, then $y \cdot p \geq \frac{1}{2}|y|^2 - \frac{1}{2}|y-p|^2 > 0$. Since the origin belongs to the convex hull of Υ, we infer $0 > 0$, a contradiction. □

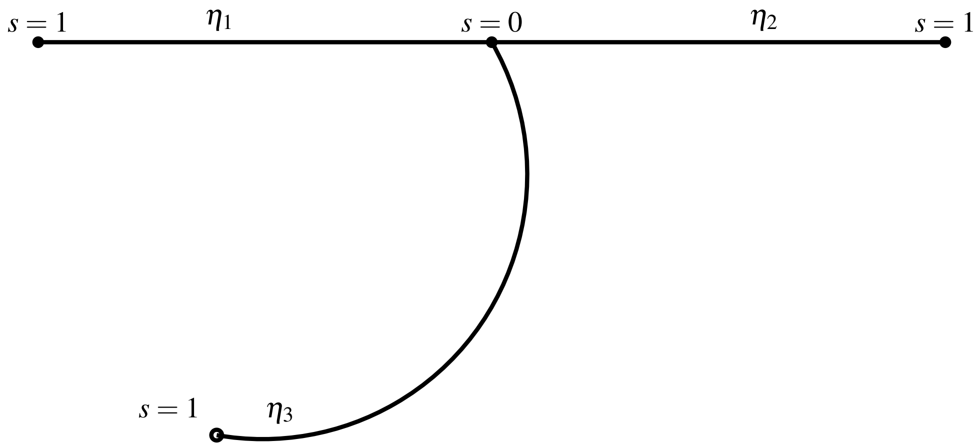

Fig. 2.1 Symbolic depiction of the 1st scenario in Lemma 2.15: two arms of the triod tend to the straight position

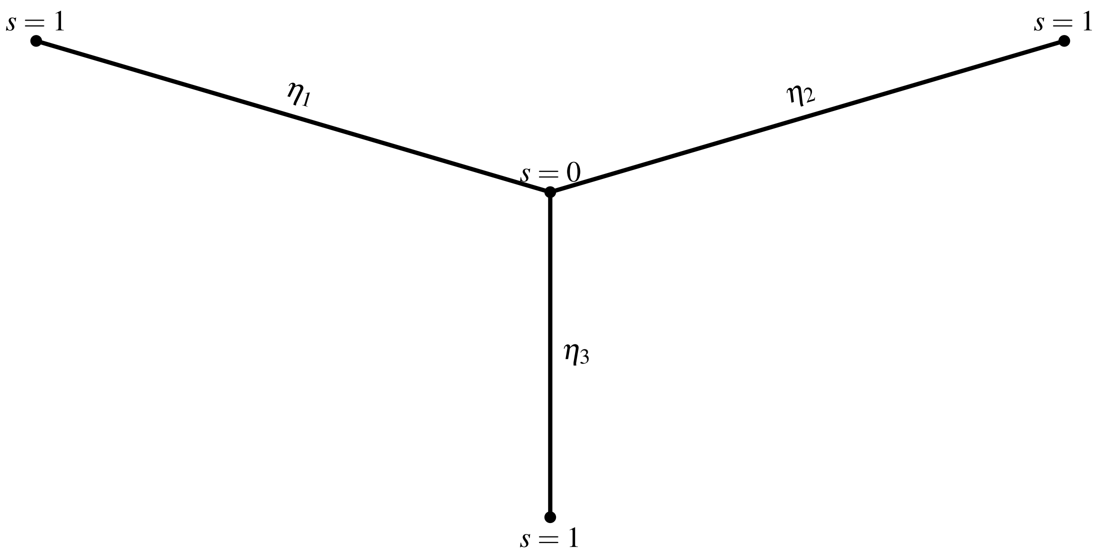

Fig. 2.2 Symbolic depiction of the 2nd scenario in Lemma 2.15: all the arms of the triod tend to the straight position

Now we assemble all the ingredients to get the crucial L^∞ bounds for κ and $\partial_s \eta$.

Lemma 2.15. *Given $\delta > 0$, the norm $||\kappa^i||_{L^\infty\left(\delta,T;L^\infty(\Omega)\right)}$ is uniformly bounded with respect to ε.*

Proof. From now on, we do not omit the subscript ε. However, in this proof we decided to swap the sub- and superindices for the sake of convenience and readability.

Step 1. We argue by contradiction. Assume that there is a sequence $\varepsilon^n \to 0$ such that

$$||\kappa_1^{\varepsilon^n}||_{L^\infty\left(\delta,T;L^\infty(\Omega)\right)} \to +\infty.$$

Here, without loss of generality, we have chosen the generic i to be equal to 1. By the regularity of $\partial_s \kappa^{\varepsilon^n}$ and $\partial_{ss}\eta^{\varepsilon^n}$ there exists a set $\mathfrak{T}_n$ of full measure in $[\delta,T]$ such that $\kappa_i^{\varepsilon^n}(t)$ and $\eta_i^{\varepsilon^n}(t)$ are C^1-smooth in $\overline{\Omega}$ whereas $\partial_{ss}\eta_i^{\varepsilon^n}(t) \in L^2(\Omega)$ for every i and every $t \in \mathfrak{T}_n$. Furthermore, by Lemma 2.13 without loss of generality we can assume that $\partial_s \kappa_i^{\varepsilon^n}(t)$ and $|\kappa_i^{\varepsilon^n}(t,\cdot)||\partial_{ss}\eta_i^{\varepsilon^n}(t,\cdot) - \varepsilon^n \partial_s \kappa_i^{\varepsilon^n}(t,\cdot)|$ are bounded in $L^2(\Omega)$ uniformly w.r.t. n and $t \in \mathfrak{T}_n$. Let $\mathfrak{T} := \cap_{n\in\mathbb{N}}\mathfrak{T}_n$. Then there is a sequence $(t^n,s^n) \in \mathfrak{T}\times\Omega$ such that $|\kappa_1^{\varepsilon^n}(t^n,s^n)| \to +\infty$ as $n \to \infty$. Thus,

$$\kappa_1^{\varepsilon^n}\left(t^n,s\right) = \underbrace{\kappa_1^{\varepsilon^n}\left(t^n,s^n\right)}_{\to+\infty} + \underbrace{\int_{s^n}^{s} \partial_\xi \kappa_1^{\varepsilon^n}\left(t^n,\xi\right)\partial\xi}_{\leq C}$$

when $n \to \infty$. Accordingly, $|\kappa_1^{\varepsilon^n}(t^n)| \to +\infty$ uniformly in s.

Step 2. By the boundary conditions,

$$\sum_{i=1}^{3} \kappa_i^{\varepsilon^n}(t^n,0) = 0. \tag{2.54}$$

By the previous step, $|\kappa_1^{\varepsilon^n}(t^n,0)| \to +\infty$. Hence we have two possible scenarios symbolically pictured in Figures 2.1 and 2.2, respectively. The first option is $|\kappa_2^{\varepsilon^n}(t^n,0)| \to +\infty$ and $|\kappa_3^{\varepsilon^n}(t^n,0)| \leq C$ as $n \to +\infty$ (up to swapping the second and the third arms). The second one is $|\kappa_2^{\varepsilon^n}(t^n,0)| \to +\infty$ and $|\kappa_3^{\varepsilon^n}(t^n,0)| \to +\infty$ as $n \to +\infty$.

Step 3. We start by examining the second scenario. An argument similar to the one of Step 1 shows that $|\kappa_i^{\varepsilon^n}(t^n)| \to +\infty$ uniformly in s, $i = 1,2,3$. Since $t^n \in \mathfrak{T}$, we know that

$$|\kappa_i^{\varepsilon^n}\left(t^n,\cdot\right)||\partial_{ss}\eta_i^{\varepsilon^n}\left(t^n,\cdot\right) - \varepsilon\partial_s\kappa_i^{\varepsilon^n}\left(t^n,\cdot\right)|$$

is uniformly bounded in $L^2(\Omega)$. Hence,

$$|\partial_{ss}\eta_i^{\varepsilon^n}\left(t^n,\cdot\right) - \varepsilon^n\partial_s\kappa_i^{\varepsilon^n}\left(t^n,\cdot\right)| \to 0$$

in $L^2(\Omega)$ as $n \to +\infty$. On the other hand, $\partial_s\kappa_i^{\varepsilon^n}(t^n)$ is uniformly bounded in $L^2(\Omega)$, whence

$$|\varepsilon^n\partial_s\kappa_i^{\varepsilon^n}\left(t^n\right)| \to 0$$

in $L^2(\Omega)$. We conclude that $|\partial_{ss}\eta_i^{\varepsilon^n}(t^n,\cdot)| \to 0$ in $L^2(\Omega)$ as $n \to +\infty$. By Remark 2.4, $|\kappa_i^{\varepsilon^n}(t^n,\cdot)| \geq 1$ implies $|\partial_s\eta_i^{\varepsilon^n}(t^n,\cdot)| \geq 1$ (assuming n to be large enough).

Step 4. The idea now is to compare the triangle formed by the points $p_i^n := \eta_i^{\varepsilon^n}(t^n,0) + \partial_s \eta_i^{\varepsilon^n}(t^n,0)$ with the fixed triangle[1] formed by $\eta_i^{\varepsilon^n}(t^n,1) = \alpha_i(1)$, $i = 1,2,3$. Observe that

$$|\partial_s \eta_i^{\varepsilon^n}(t^n,0) - \partial_s \eta_i^{\varepsilon^n}(t^n,\xi)| = \left| \int_0^\xi \partial_{ss} \eta_i^{\varepsilon^n}(t^n)\, ds \right|$$
$$\leq \int_0^\xi |\partial_{ss} \eta_i^{\varepsilon^n}(t^n)|\, ds \leq \sqrt{\int_0^1 |\partial_{ss} \eta_i^{\varepsilon^n}(t^n)|^2\, ds} \to 0 \text{ uniformly in } \xi \text{ as } n \to \infty.$$

Hence,

$$|p_i^n - \alpha_i(1)| = |\eta_i^{\varepsilon^n}(t^n,0) - \eta_i^{\varepsilon^n}(t^n,1) + \partial_s \eta_i^{\varepsilon^n}(t^n,0)|$$
$$= \left| \int_0^1 \partial_s \eta_i^{\varepsilon^n}(t^n,0) - \partial_s \eta_i^{\varepsilon^n}(t^n,s)\, ds \right| \to 0 \text{ as } n \to \infty.$$

It follows from our assumptions (Remark 2.3) that the radius of the smallest enclosing ball of the three points $\alpha_i(1)$ is less than 1. Since the radius of the smallest enclosing ball is a continuous function of the points of a set, it follows that the radius of the smallest enclosing ball of the three points p_i^n is less than 1 for n sufficiently large. Since the junction point $\eta_i^{\varepsilon^n}(t^n,0)$ does not depend on i, the radius of the smallest enclosing ball of the three points $\tilde{p}_i^n := \partial_s \eta_i^{\varepsilon^n}(t^n,0)$ is the same as the previous one. By Step 3, $|\tilde{p}_i^n| \geq 1$. Moreover, since $\sum_{i=1}^3 \kappa_i^{\varepsilon^n}(t^n,0) = 0$ and $\tilde{p}_i^n = F_{\varepsilon^n}(\kappa_i^{\varepsilon^n}(t^n,0))$, we conclude that the convex hull of $\{\tilde{p}_i^n\}$ contains the origin. We arrive at a contradiction because by Lemma 2.14 the radius of the smallest enclosing ball of $\{\tilde{p}_i^n\}$ must be greater than or equal to 1.

Step 5. We now study the first scenario. Define p_i^n and $\tilde{p}_i^n$ as in Step 4. The plan is to look at the angle θ_n between the position vectors of $\tilde{p}_1^n$ and $\tilde{p}_2^n$ and to obtain a contradiction from that.

We first show that θ_n cannot tend to π. Indeed, mimicking the arguments of Steps 3 and 4, we can prove that for $i = 1,2$ one has $|\partial_s \eta_i^{\varepsilon^n}(t^n,\cdot)| \geq 1$ with n large enough, $|\partial_{ss} \eta_i^{\varepsilon^n}(t^n,\cdot)| \to 0$ in $L^2(\Omega)$ and

$$|p_i^n - \alpha_i(1)| \to 0 \text{ as } n \to \infty.$$

Hence,

$$|\tilde{p}_1^n - \tilde{p}_2^n| = |p_1^n - p_2^n| \to |\alpha_1(1) - \alpha_2(1)| < 2.$$

Since we have $|\tilde{p}_1^n| \geq 1, |\tilde{p}_2^n| \geq 1$, the angle θ_n cannot converge to π.

Now take the wedge product of relation (2.54) with the vector

$$\frac{1}{|\partial_s \eta_1^{\varepsilon^n}(t^n,0)||\kappa_2^{\varepsilon^n}(t^n,0)|} \partial_s \eta_1^{\varepsilon^n}(t^n,0)$$

to obtain

$$\frac{\kappa_2^{\varepsilon^n}(t^n,0)}{|\kappa_2^{\varepsilon^n}(t^n,0)|} \wedge \frac{\partial_s \eta_1^{\varepsilon^n}(t^n,0)}{|\partial_s \eta_1^{\varepsilon^n}(t^n,0)|} + \frac{\kappa_3^{\varepsilon^n}(t^n,0)}{|\kappa_2^{\varepsilon^n}(t^n,0)|} \wedge \frac{\partial_s \eta_1^{\varepsilon^n}(t^n,0)}{|\partial_s \eta_1^{\varepsilon^n}(t^n,0)|} = 0.$$

[1] Both triangles can be degenerate.

Since $|\kappa_2^{\varepsilon^n}(t^n,0)| \to +\infty$ and $|\kappa_3^{\varepsilon^n}(t^n,0)| \leq C$, the second term converges to 0. Consequently,

$$|\sin\theta_n| = \left| \frac{\partial_s \eta_2^{\varepsilon^n}(t^n,0)}{|\partial_s \eta_2^{\varepsilon^n}(t^n,0)|} \wedge \frac{\partial_s \eta_1^{\varepsilon^n}(t^n,0)}{|\partial_s \eta_1^{\varepsilon^n}(t^n,0)|} \right| = \left| \frac{\kappa_2^{\varepsilon^n}(t^n,0)}{|\kappa_2^{\varepsilon^n}(t^n,0)|} \wedge \frac{\partial_s \eta_1^{\varepsilon^n}(t^n,0)}{|\partial_s \eta_1^{\varepsilon^n}(t^n,0)|} \right| \to 0$$

as $n \to \infty$.

To obtain a contradiction, it remains to observe that θ_n cannot tend to 0. Indeed, taking the scalar product of relation (2.54) with

$$\frac{1}{|\partial_s \eta_1^{\varepsilon^n}(t^n,0)||\kappa_2^{\varepsilon^n}(t^n,0)|} \partial_s \eta_1^{\varepsilon^n}(t^n,0)$$

we get

$$\frac{\kappa_1^{\varepsilon^n}(t^n,0)}{|\kappa_2^{\varepsilon^n}(t^n,0)|} \cdot \frac{\partial_s \eta_1^{\varepsilon^n}(t^n,0)}{|\partial_s \eta_1^{\varepsilon^n}(t^n,0)|} + \frac{\kappa_2^{\varepsilon^n}(t^n,0)}{|\kappa_2^{\varepsilon^n}(t^n,0)|} \cdot \frac{\partial_s \eta_1^{\varepsilon^n}(t^n,0)}{|\partial_s \eta_1^{\varepsilon^n}(t^n,0)|} + \frac{\kappa_3^{\varepsilon^n}(t^n,0)}{|\kappa_2^{\varepsilon^n}(t^n,0)|} \cdot \frac{\partial_s \eta_1^{\varepsilon^n}(t^n,0)}{|\partial_s \eta_1^{\varepsilon^n}(t^n,0)|} = 0. \quad (2.55)$$

The first term is equal to $\frac{\sigma_1^{\varepsilon^n}}{|\kappa_2^{\varepsilon^n}(t^n,0)||\partial_s \eta_1^{\varepsilon^n}(t^n,0)|} \geq 0$ by (2.39) and (2.40). The third term converges to 0. Accordingly, the second term, which is equal to $\cos\theta_n$, cannot tend to 1. □

Corollary 2.16. *Given $\delta > 0$, the norm $||\eta_{\varepsilon^n}^i||_{L^\infty(\delta,T;W^{1,\infty}(\Omega))}$ is uniformly bounded with respect to ε.*

Proof. Since $\partial_s \eta_{\varepsilon^n}^i = F_{\varepsilon^n}(\kappa_{\varepsilon^n}^i)$ and the sequence $\{\varepsilon_n\}$ is bounded, Lemma 2.15 yields a uniform L^∞ bound for $\partial_s \eta^i$. By Lemma 2.9, $||\eta^i||_{L^\infty(\delta,T;L^1(\Omega))}$ is also uniformly bounded with respect to ε, and the claim follows by the mean value theorem. □

Lemma 2.17. *Given $\delta > 0$, the norm $\|\sigma_{\varepsilon^n}^i\|_{L^\infty(\delta,T;H^1(\Omega))}$ is bounded uniformly in ε.*

Proof. In view of Lemma 2.15 and Corollary 3.20, the $L^\infty(\delta,T;L^\infty(\Omega))$-bound for σ immediately follows from the equality $\sigma^i = \partial_s \eta^i \cdot \kappa^i$. Differentiating this equality w.r.t. s we obtain

$$\partial_s \sigma^i = \partial_s \kappa^i \cdot \partial_s \eta^i + \kappa^i \cdot \partial_{ss} \eta^i.$$

We estimate the two terms on the right-hand side separately. Firstly, a uniform $L^\infty\left(\delta,T;L^2(\Omega)\right)$ bound for $\partial_s \kappa^i$ has been already established, cf. Lemma 2.13. This together with Corollary 3.20 implies the uniform boundedness of $\partial_s \kappa^i \cdot \partial_s \eta^i$ in $L^\infty\left(\delta,T;L^2(\Omega)\right)$.

Now, we estimate $\kappa^i \cdot \partial_{ss} \eta^i$. From the explicit expression of λ_{ε^n} in (2.36), for $\tau \in \mathbb{R}^d$ we have

$$\lambda_{\varepsilon^n}(\tau) = \frac{\sqrt{\varepsilon^n + |G_{\varepsilon^n}(\tau)|^2}}{\varepsilon^n \sqrt{\varepsilon^n + |G_{\varepsilon^n}(\tau)|^2} + 1} \geq \frac{|G_{\varepsilon^n}(\tau)|}{\varepsilon^n |G_{\varepsilon^n}(\tau)| + 1}.$$

Thus,

$$\begin{aligned} |G_{\varepsilon^n}\left(\partial_s \eta^i\right)||\partial_{ss} \eta^i| &\leq \left(\varepsilon^n |G_{\varepsilon^n}\left(\partial_s \eta^i\right)| + 1\right) |\lambda_{\varepsilon^n} \partial_{ss} \eta^i| \\ &\leq \left(\varepsilon^n |\kappa^i| + 1\right) |\nabla G_{\varepsilon^n}\left(\partial_s \eta^i\right) \partial_{ss} \eta^i|. \end{aligned}$$

By Lemma 2.15, $|\kappa^i|$ is uniformly bounded in $L^\infty\left(\delta,T;L^\infty(\Omega)\right)$, whence

$$|\kappa^i\cdot\partial_{ss}\eta^i|\le|G_{\varepsilon^n}\left(\partial_s\eta^i\right)||\partial_{ss}\eta^i|\le C|\nabla G_{\varepsilon^n}\left(\partial_s\eta^i\right)\partial_{ss}\eta^i|=C|\partial_s\kappa^i|.$$

Since the right-hand side is uniformly bounded in $L^\infty\left(\delta,T;L^2(\Omega)\right)$, so is the left-hand side and, consequently, the spatial derivative $\partial_s\sigma^i$ itself. □

2.6 Existence of generalized solutions

We are now at the position to define generalized solutions to the original problem (2.8), (2.2) and to prove their existence.

Definition 2.18. *Given initial data $\alpha^i\in W^{1,\infty}(\Omega)^d$ as in Remark 2.3, we call a pair $\left(\eta^i,\sigma^i\right)$ a generalized solution to* (2.8), (2.2) *in $\mathfrak{Q}_\infty$ if*

(i)
- $\eta^i\in L^\infty_{loc}\left((0,\infty;W^{1,\infty}(\Omega)\right)^d\cap C_{loc}\left((0,\infty);C\left(\overline{\Omega}\right)\right)^d\cap AC^2_{loc}\left([0,\infty);L^2(\Omega)\right)^d$,
- $\partial_t\eta^i\in L^\infty_{loc}\left((0,\infty);L^2(\Omega)\right)^d\cap L^2_{loc}\left([0,\infty);L^2(\Omega)\right)^d$,
- $\sigma^i\in L^\infty_{loc}\left((0,\infty);AC^2(\Omega)\right)$,
- $\sigma^i\partial_s\eta^i\in L^\infty_{loc}\left((0,\infty);AC^2(\Omega)\right)^d$.

(ii) Each pair $\left(\eta^i,\sigma^i\right)$ satisfies for a.e. $(t,s)\in\mathfrak{Q}_\infty$

$$\partial_t\eta^i(t,s)=\partial_s\left(\sigma^i(t,s)\,\partial_s\eta^i(t,s)\right)+g, \tag{2.56}$$

$$\sigma^i(t,s)\left(|\partial_s\eta^i(t,s)|^2-1\right)=0, \tag{2.57}$$

$$|\partial_s\eta^i(t,s)|\le 1, \tag{2.58}$$

as well as the initial conditions

$$\eta^i(0,s)=\alpha^i(s)$$

and the boundary conditions

$$\eta^1(t,0)=\eta^2(t,0)=\eta^3(t,0),$$
$$\eta^i(t,1)=\alpha^i(1),$$
$$\sum_{i=1}^3\sigma^i(t,0)\,\partial_s\eta^i(t,0)=0.$$

(iii) The solutions η^i satisfy the energy dissipation inequality

$$\sum_{i=1}^3\int_\Omega|\partial_t\eta^i(t,s)|^2\,ds\le\sum_{i=1}^3\int_\Omega g\cdot\partial_t\eta^i(t,s)\,ds \tag{2.59}$$

for a.e. $t \in (0,\infty)$.

Remark 2.5. Note that (2.57), (2.58) is a minor relaxation of the non-convex constraint

$$|\partial_s \eta^i(t,s)| = 1. \tag{2.60}$$

However, this is not a banal convexification of the constraint since (2.57) is still not convex. The new constraints (2.57), (2.58) naturally appear from the (η,σ,κ)-formulation (2.33), cf. (2.62) in the proof below. Moreover, if a generalized (in the sense of Definition 2.18) solution (η,σ) is C^2-smooth, then it automatically satisfies the strong constraint (2.60). On the other hand we claim that any generalized solution (η,σ) with $\eta \in C^1\left(\overline{\mathfrak{Q}_\infty}\right) \cap C^2(\mathfrak{Q}_\infty)$ and $|\partial_s \alpha^i| = 1$ is a solution to (2.8), (2.2). Now we prove the claim (it is independent of choice of i): it is enough to show that the open set $U := \{(t,s) \in \mathfrak{Q}_\infty : |\partial_s \eta^i(t,s)| < 1\}$ is empty, we argue by contradiction. Suppose not, then $\sigma^i = 0$ a.e. in U due to the constraint $\sigma^i\left(|\partial_s \eta| - 1\right)$ this implies that $\partial_t \eta^i = g$ hence $\partial_{st}\eta^i = 0$ in U. For each $(t_0,s_0) \in U$, let $t_1 = \inf\{t \geq 0 : (t,t_0) \times \{s_0\} \subset U\}$. If $t_1 = 0$ then

$$|\partial_s \eta^i(t_1,s_0)| = 1$$

due to our assumption about α^i and if $t_1 > 0$ then the above equality holds by the continuity of $\partial_s \eta^i$. From $\partial_{st}\eta^i = 0$ in U and the up to the boundary C^1 continuity of η^i, we deduce that

$$|\partial_s \eta^i(t_0,s_0)| = |\partial_s \eta^i(t_1,s_0)| = 1$$

and this completes the proof. Finally, we emphasize that (2.59) is not direct consequence of (2.56), (2.57) and (2.58).

As in [61], our generalized solutions are, generally speaking, not unique. Yet this has nothing to do with the fact that we slightly relaxed the constraint (2.60). As a matter of fact, non-uniqueness can persist even if the strong constraint (2.60) is imposed, cf. [61, Remark 6.5].

For convenience, we first pass to the limit on finite time intervals. In what follows, we use the shortcut $\mathfrak{Q}_T^* := (\delta,T) \times \Omega$.

Proposition 2.19. *Fix $T > 0$ and a small $\delta > 0$. Let η_ε be a solution to* (2.37) *in $\mathfrak{Q}_T$ with the initial/boundary conditions* (2.38) *as constructed in Section 2.4. Let $\left(\kappa^i,\sigma\right)$ be defined as in* (2.39)*. Then (up to selecting a subsequence ε^n) there exists a limit $(\eta^i,\sigma^i,\kappa^i)$ such that as $\varepsilon \to 0$ we have*

$\eta_\varepsilon^i \to \eta^i$ *weakly* in* $L^\infty\left(\delta,T;W^{1,\infty}(\Omega)\right)^d$*, strongly in* $C\left(\overline{\mathfrak{Q}_T^*}\right)^d$ *and weakly in* $L^2(\mathfrak{Q}_T)^d$*,*

$\partial_t \eta_\varepsilon^i \to \partial_t \eta^i$ *weakly-* in* $L^\infty\left(\delta,T;L^2(\Omega)\right)^d$ *and weakly in* $L^2(\mathfrak{Q}_T)^d$*,*

$\sigma_\varepsilon \to \sigma$ *weakly-* in* $L^\infty\left(\delta,T;H^1(\Omega)\right)$*,*

$\kappa_\varepsilon^i \to \kappa^i$ *weakly-* in* $L^\infty\left(\delta,T;H^1(\Omega)\right)$*.*

The limit satisfies the relation

$$\kappa^i = \sigma \partial_s \eta^i \in L^\infty\left(\delta,T;H^1(\Omega)\right)$$

and solves (2.8)-(2.2) *in* $\mathfrak{Q}_T^*$ *in the sense that*

$$\partial_t \eta^i = \partial_s \left(\sigma \partial_s \eta^i\right) + g \text{ a.e. in } \mathfrak{Q}_T^*,$$
$$\sigma \left(|\partial_s \eta^i|^2 - 1\right) = 0 \text{ a.e. in } \mathfrak{Q}_T^*,$$
$$\eta^i(t,1) = \alpha^i(1),$$
$$\eta^1(t,0) = \eta^2(t,0) = \eta^3(t,0),$$
$$\sum_{i=1}^{3} \kappa^i = 0 \text{ at } s = 0 \text{ for a.e. } t \in (\delta, T).$$

Remark 2.6. At this stage we don't discuss the validity of the initial condition $\eta^i(0,s) = \alpha^i(s)$ that is postponed until Remark 2.7.

Proof. The weak compactness results for η_ε^i, σ_ε^i and κ_ε^i follow immediately from the estimates above. By the Aubin-Lions-Simon theorem,

$$L^\infty\left(\delta, T; W^{1,\infty}(\Omega)\right) \cap W^{1,\infty}\left(\delta, T; L^2(\Omega)\right) \subset C\left([\delta, T]; C\left(\overline{\Omega}\right)\right) \tag{2.61}$$

and the embedding is compact, which implies strong compactness of η_ε^i in $C\left([\delta, T]; C\left(\overline{\Omega}\right)\right)$.

Let us show that

$$\kappa^i = \sigma^i \partial_s \eta^i, \quad \sigma^i = \kappa^i \cdot \partial_s \eta^i \tag{2.62}$$

a.e. in $\mathfrak{Q}_T^*$. Since both sides of the equalities (2.62) are integrable on $\mathfrak{Q}_T^*$, it suffices to prove (2.62) in the sense of the distributions, i.e., that for any $\phi^i \in L^2\left(\delta, T; H_0^1(\Omega)\right)$

$$\sum_{i=1}^{3} \int_{\mathfrak{Q}_T^*} \kappa^i \phi^i ds dt = -\sum_{i=1}^{3} \int_{\mathfrak{Q}_T^*} \sigma^i \eta^i \partial_s \phi^i ds dt - \sum_{i=1}^{3} \int_{\mathfrak{Q}_T^*} \partial_s \sigma^i \eta^i \phi^i ds dt \tag{2.63}$$
$$\sum_{i=1}^{3} \int_{\mathfrak{Q}_T^*} \sigma^i \phi^i ds dt = -\sum_{i=1}^{3} \int_{\mathfrak{Q}_T^*} \kappa^i \cdot \eta^i \partial_s \phi^i ds dt - \sum_{i=1}^{3} \int_{\mathfrak{Q}_T^*} \partial_s \kappa^i \cdot \eta^i \phi^i ds dt. \tag{2.64}$$

Firstly, applying integration by parts to the equality $\sigma_\varepsilon^i = \kappa_\varepsilon^i \cdot \partial_s \eta_\varepsilon^i$ we obtain

$$\sum_{i=1}^{3} \int_{\mathfrak{Q}_T^*} \sigma_\varepsilon^i \phi^i = -\sum_{i=1}^{3} \int_{\mathfrak{Q}_T^*} \kappa_\varepsilon^i \cdot \eta_\varepsilon^i \partial_s \phi^i ds dt - \sum_{i=1}^{3} \int_{\mathfrak{Q}_T^*} \partial_s \kappa_\varepsilon^i \cdot \eta_\varepsilon^i \phi^i ds dt,$$

and due to the strong compactness property of $\{\eta_\varepsilon\}$ given above we can pass to the limit to get (2.64). We now claim

$$\lim_{\varepsilon \to 0} \left|\kappa_\varepsilon^i\right| \left|\left|\partial_s \eta_\varepsilon^i\right|^2 - 1\right| = 0 \tag{2.65}$$

uniformly in $\mathfrak{Q}_T^*$. Before proving the claim we show how (2.63) follows from (2.65). Indeed, with (2.65) in our hand and noting that $\kappa_\varepsilon^i |\partial_s \eta_\varepsilon^i|^2 = \left(\kappa_\varepsilon^i \cdot \partial_s \eta_\varepsilon^i\right) \partial_s \eta_\varepsilon^i = \sigma_\varepsilon^i \partial_s \eta_\varepsilon^i$ we have for each $i = 1, 2, 3$

$$\lim_{\varepsilon \to 0} ||\sigma_\varepsilon^i \partial_s \eta_\varepsilon^i - \kappa_\varepsilon^i||_{L^\infty(\mathfrak{Q}_T^*)} = 0. \tag{2.66}$$

In particular, for any $\phi^i \in L^2\left(\delta, T; H_0^1(\Omega)\right)$

$$\lim_{\varepsilon\to 0}\sum_{i=1}^{3}\int_{\mathfrak{Q}_T^*} \kappa_\varepsilon^i \phi^i dsdt = \lim_{\varepsilon\to 0}\sum_{i=1}^{3}\int_{\mathfrak{Q}_T^*} \sigma_\varepsilon^i \partial_s \eta_\varepsilon^i \phi^i dsdt.$$

An integration by parts applied to the integral on the right-hand side gives

$$\lim_{\varepsilon\to 0}\sum_{i=1}^{3}\int_{\mathfrak{Q}_T^*} \kappa_\varepsilon^i \phi^i dsdt = \lim_{\varepsilon\to 0}\sum_{i=1}^{3}\int_{\mathfrak{Q}_T^*} \left(-\sigma_\varepsilon^i \eta_\varepsilon^i \partial_s \phi_\varepsilon^i - \partial_s \sigma_\varepsilon^i \eta_\varepsilon^i \phi^i\right) dsdt.$$

This together with the compactness properties established above yields (2.63).

We now provide a proof of (2.65). By the definition of F_ε in (2.34),

$$\begin{aligned}
|\partial_s \eta_\varepsilon^i| - 1 &= \left|F_\varepsilon\left(\kappa_\varepsilon^i\right)\right| - 1\\
&= \varepsilon|\kappa_\varepsilon^i| + \frac{|\kappa_\varepsilon^i|}{\sqrt{\varepsilon + |\kappa_\varepsilon^i|^2}} - 1\\
&= \varepsilon|\kappa_\varepsilon^i| - \frac{\varepsilon}{\sqrt{\varepsilon + |\kappa_\varepsilon^i|^2}\left(\sqrt{\varepsilon + |\kappa_\varepsilon^i|^2} + |\kappa_\varepsilon^i|\right)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
|\kappa_\varepsilon^i|\left||\partial_s \eta_\varepsilon^i|^2 - 1\right| &= \left||\partial_s \eta_\varepsilon^i| + 1\right|\left|\varepsilon|\kappa_\varepsilon^i|^2 - \frac{\varepsilon|\kappa_\varepsilon^i|}{\sqrt{\varepsilon + |\kappa_\varepsilon^i|^2}\left(\sqrt{\varepsilon + |\kappa_\varepsilon^i|^2} + |\kappa_\varepsilon^i|\right)}\right|\\
&\le \left||\partial_s \eta_\varepsilon^i| + 1\right|\left(\varepsilon|\kappa_\varepsilon^i|^2 + \sqrt{\varepsilon}\right).
\end{aligned}$$

This together with uniform L^∞ bounds on $\partial_s \eta_\varepsilon^i$ and κ_ε^i yields (2.65).

Passing to the limit in $L^2\left(\mathfrak{Q}_T^*\right)$ in $\partial_t \eta_\varepsilon^i = \partial_s \kappa_\varepsilon^i + g$ and using (2.62), we obtain $\partial_t \eta^i = \partial_s\left(\sigma^i \partial_s \eta^i\right) + g$. To get $\sigma^i\left(|\partial_s \eta^i|^2 - 1\right) = 0$, it suffices to express κ^i from the first of the equalities (2.62) and substitute the result into second one.

Due to the strong uniform convergence of η_ε^i, we have $\alpha^i(1) = \eta_\varepsilon^i(t,1) \to \eta^i(t,1)$, whence $\eta^i(t,1) = \alpha^i(1)$ for all $t \in [\delta, T]$. The condition $\eta_\varepsilon^1(t,0) = \eta_\varepsilon^2(t,0) = \eta_\varepsilon^3(t,0)$ similarly passes to the limit. To check the validity of the boundary condition at $s = 0$ for κ, we swap the variables t and s, noting that κ_ε^i are uniformly bounded and weakly-* converging in $H^1(0,1;L^\infty(\delta,T))$. Employing, for instance, [71, Corollary 2.2.1], we get

$$H^1(0,1;L^\infty(\delta,T)) = AC^2([0,1];L^\infty(\delta,T)). \tag{2.67}$$

Hence, by the Aubin-Lions-Simon theorem, the embedding

$$H^1(0,1;L^\infty(\delta,T)) \subset C([0,1];H^{-1}(\delta,T))$$

is compact, whence we may assume that $\kappa_\varepsilon^i \to \kappa^i$ strongly in $C([0,1];H^{-1}(\delta,T))$. Thus,

$$0=\sum_{i=1}^{3}\kappa_\varepsilon^i(\cdot,0)\to\sum_{i=1}^{3}\kappa^i(\cdot,0)$$

in $H^{-1}(\delta,T)$. Due to (2.67), $\sum_{i=1}^3 \kappa^i(\cdot,0)=0$ in $L^\infty(\delta,T)$. □

Remark 2.7 (Initial conditions). By the Aubin-Lions-Simon theorem, the embedding

$$H^1(0,T;L^2(\Omega))\subset C([0,T];H^{-1}(\Omega))$$

is compact. Since η_ε^i (w.l.o.g.) converge weakly in $H^1(0,T;L^2(\Omega))$ we can pass to the limit in the initial conditions to obtain $\eta^i(0,\cdot)=\alpha^i$ in $H^{-1}(\Omega)$. However, since $H^1(0,T;L^2(\Omega)) = AC^2(0,T;L^2(\Omega))$, the initial conditions actually hold in $L^2(\Omega)$.

Proposition 2.20. *Let (η,σ) be the limiting solution obtained in Proposition 2.19. Then*

(i) $|\partial_s\eta^i(t,s)|\le 1$ *for a.e.* $(t,s)\in\mathfrak{Q}_T^*$*;*

(ii) $\sigma^i\ge 0$ *for a.e.* $(t,s)\in\mathfrak{Q}_T^*$*;*

(iii) (2.59) *holds for a.a.* $t\in(\delta,T)$.

Proof. (i) We observe that the set

$$\mathscr{B}=\left\{\zeta\in L^2\left(\mathfrak{Q}_T^*\right)^d:|\zeta(t,s)|\le 1 \text{ for a.e }.(t,s)\in\mathfrak{Q}_T^*\right\}$$

is weakly closed in $L^2\left(\mathfrak{Q}_T^*\right)^d$. Remember that $F_\varepsilon\left(\kappa_\varepsilon^i\right)=\varepsilon\kappa_\varepsilon^i+\frac{\kappa_\varepsilon^i}{\sqrt{\varepsilon+|\kappa_\varepsilon^i|^2}}=\partial_s\eta_\varepsilon^i$. Now as $\varepsilon\to 0$ the first term goes to zero in $L^2\left(\mathfrak{Q}_T^*\right)^d$ and we notice that the remaining term belongs to $\mathscr{B}$, this implies that the weak limit $\partial_s\eta^i$ belongs to $\mathscr{B}$.

(ii) Remember the definition of $\sigma_\varepsilon^i=G_\varepsilon\left(\partial_s\eta_\varepsilon^i\right)\cdot\partial_s\eta_\varepsilon^i$: for each $\varepsilon>0$ we have $\sigma_\varepsilon^i=\kappa_\varepsilon^i\cdot\partial_s\eta_\varepsilon^i=\kappa_\varepsilon^i\cdot F_\varepsilon\left(\kappa_\varepsilon^i\right)$. Just using elementary properties of F_ε we immediately obtain that $\sigma_\varepsilon^i\ge 0$. It follows that passing to the limit in σ_ε^i we obtain $\sigma^i\ge 0$.

(iii) In this proof we have many upper and lower indexes, for this reason we do not write the upper index i to show with which arm of the triod we are working i.e. $\eta^i=\eta$, $\sigma^i=\sigma$ etc., however we still put the sum symbol to emphasize it. We of course tacitly assume that there is a sequence $\varepsilon_j\searrow 0$ and solutions $\{(\eta_j,\sigma_j)\}$ to the corresponding approximation problems (2.37)-(2.38) that tend to (η,σ). In order to prove (2.59), we start by multiplying both sides of the equation for η_j with $\partial_t\eta_j\varphi$, where $\varphi=\varphi(t)\in C_c^\infty((0,\infty))$ is a non-negative cut-off function:

$$\sum_{i=1}^{3}\int_{\mathfrak{Q}_\infty}|\partial_t\eta_j|^2\varphi dsdt=\sum_{i=1}^{3}\int_{\mathfrak{Q}_\infty}\partial_s G_j\left(\partial_s\eta_j\right)\partial_t\eta_j\varphi dsdt+\sum_{i=1}^{3}\int_{\mathfrak{Q}_\infty}g\cdot\partial_t\eta_j\varphi dsdt.$$

Here $G_j=G^{\varepsilon_j}$. Since the last term is linear and the first term is quadratic (hence lower-semicontinuous w.r.t. $j\to\infty$), we only need to show that second term $\sum_{i=1}^3\int_{\mathfrak{Q}_\infty}\partial_s G_j\left(\partial_s\eta_j\right)\cdot\partial_t\eta_j\varphi dsdt$ tends to 0 as $j\to\infty$.

After one integration by parts in the space variable and recalling $\kappa_j = G_j\left(\partial_s \eta_j\right)$, it suffices to show that

$$\sum_{i=1}^{3} \int_{\Omega_\infty} \kappa_j \cdot \partial_{ts} \eta_j \varphi dsdt \to 0 \tag{2.68}$$

Observe now that

$$\begin{aligned}
\kappa \cdot \partial_{st} \eta &= \kappa \cdot \partial_t \left(\varepsilon \kappa + \frac{\kappa}{\sqrt{\varepsilon + |\kappa|^2}} \right) \\
&= \varepsilon \kappa \partial_t \kappa + \kappa \cdot \left(\frac{\partial_t \kappa}{\sqrt{\varepsilon + |\kappa|^2}} - \frac{\kappa\left(\kappa \cdot \partial_t \kappa\right)}{\left(\sqrt{\varepsilon + |\kappa|^2}\right)^3} \right) \\
&= \varepsilon \kappa \partial_t \kappa + \varepsilon \frac{\kappa \cdot \partial_t \kappa}{\left(\sqrt{\varepsilon + |\kappa|^2}\right)^3} \\
&= \varepsilon \frac{d}{dt} \left(\frac{|\kappa|^2}{2} - \frac{1}{\sqrt{\varepsilon + |\kappa|^2}} \right)
\end{aligned}$$

Here we omit the index j for clarity. From the equality $\kappa_j \partial_{st} \eta_j = \varepsilon_j \frac{d}{dt} \left(\frac{|\kappa_j|^2}{2} - \frac{1}{\sqrt{\varepsilon_j + |\kappa_j|^2}} \right)$ we deduce

$$\sum_{i=1}^{3} \int_{\Omega_\infty} \kappa_j^i \cdot \partial_{st} \eta_j^i \varphi dsdt = -\varepsilon_j \sum_{i=1}^{3} \int_{\Omega_\infty} \left(\frac{|\kappa_j^i|^2}{2} - \frac{1}{\sqrt{\varepsilon_j + |\kappa_j^i|^2}} \right) \frac{d}{dt} \varphi dsdt$$

From Proposition 2.19, κ_j is uniformly bounded in $L^\infty\left(\delta, T; H^1(\Omega)\right)^{3d}$. W.l.o.g. φ is assumed to vanish outside (δ, T). Passing to the limit $j \to \infty$ we obtain (2.68).

□

Theorem 2.21 (Global existence of generalized solutions)**.** *For every initial configuration* $\alpha^i \in W^{1,\infty}(\Omega)^d$, $i = 1,2,3$, *meeting the assumptions of Remark 2.3, there exists a generalized solution to* (2.8)*,* (2.2) *in* Ω_∞*. Moreover, those solutions satisfy* $\sigma^i(t,s) \geq 0$ *for almost every* $(t,s) \in \Omega_\infty$.

Proof. Let δ_k, T_k be two sequences with $\delta_k \searrow 0$ and $T_k \nearrow \infty$ as $k \to \infty$. For each k fixed let $\{\left(\eta_k^\varepsilon, \sigma_k^\varepsilon\right)\}_\varepsilon$ be solutions to approximation problems (2.37)-(2.38) in $[0, T_k) \times (0,1)$. By Proposition 2.19 and a standard diagonal argument we can obtain a subsequence $\left(\eta_j, \sigma_j\right) := \left(\eta_{k_j}^{\varepsilon_{k_j}}, \sigma_{k_j}^{\varepsilon_{k_j}}\right)$ such that $\eta_j \to \eta$ weakly* in $L^\infty_{loc}\left(0,\infty; W^{1,\infty}(\Omega)\right)^{3d}$, strongly in $C_{loc}(\Omega_\infty)^{3d}$ and weakly in $L^2_{loc}\left([0,\infty); L^2(\Omega)\right)^{3d}$, $\partial_t \eta_j \to \partial_t \eta$ weakly-$*$ in $L^\infty_{loc}\left(0,\infty; L^2(\Omega)\right)^{3d}$ and weakly in $L^2_{loc}\left([0,\infty); L^2(\Omega)\right)^{3d}$, $\sigma_j \to \sigma$ weakly-* in $L^\infty_{loc}\left(0,\infty; H^1(\Omega)\right)^3$. By Propositions 2.19 and 2.20 and Remark 2.7, the limit (η, σ) is a generalized solution in the sense of Definition 2.18.

□

Remark 2.8 (Single cord with two fixed ends). The results of the chapter, mutatis mutandis, are valid for the overdamped fall of a single inextensible string with the ends fixed at two distinct spatial points (it suffices to observe that such a string can be viewed as a degenerate "triod" with one arm having zero length); remember that [61] studied the case of one free and one fixed end (i.e., a "whip"). More precisely, we have the following result.

Proposition 2.22. *Given $\alpha \in W^{1,\infty}(\Omega)^d$ satisfying $|\alpha(0)-\alpha(1)|<1$, $|\partial_s\alpha(s)|=1$ a.e. in Ω, there exists a generalized solution to*

$$\begin{cases} \partial_t\eta = \partial_s\left(\sigma\partial_s\eta\right)+g, \\ |\partial_s\eta| = 1, \\ \eta(t,0)=\alpha(0), \quad \eta(t,1)=\alpha(1), \\ \eta(0,s)=\alpha(s). \end{cases} \tag{2.69}$$

in $\mathfrak{Q}_\infty$. Moreover, $\sigma(t,s)\geq 0$ for almost every $(t,s)\in\mathfrak{Q}_\infty$.

Remark 2.9. The stationary solution of system (2.69) attracted attention of researchers at least since 17th century. It has the shape of catenary, in other words, a piece of the graph of the function cosh. It is plausible that the solutions of system (2.69) should approach the catenary in the long time regime, however, we do not have a rigorous proof, cf. Remark 1.1.

Chapter 3

θ-Network

3.1 Introducing and setting the problem

In this chapter we consider a variant of the mean curvature flow for θ-networks that preserves the uniform parametrization of the "arms" of the network. By a θ-network we mean an object that consists of three strings (arms of the θ-network) that meet at two common points (junctions). The junctions, similarly to the previous chapter, are moving in an unknown way. For simplicity, we chose to have 3 strings, but the results are valid for any finite number of strings in an object with the same structure.

Remember that binormal curvature flow [31] preserves the uniform parametrization. On the other hand the classical mean curvature flow destroys it. In this chapter we develop the ideas of [62] and consider a model of a shrinking θ-network that preserves the uniform parametrization. We will see that the problem can be normalized in a smart way. Another advantage of our approach is that works without assuming any version of the Herring condition. The latter means that the arms of the network meet with equal angles at every junction (thus the angle is $2\pi/n$ where n is the number of the strings meeting at a junction). This restrictive geometric condition is usually assumed by the authors who study mean curvature flow of networks since it prevents instabilities, cf. [33, 37, 38]. The treatment of initial configurations that do not satisfy the Herring condition requires much more involved techniques and only local results have been recently obtained, cf. [28].

We will derive and justify the following variant of the mean curvature flow for networks that preserves the uniform parametrization

$$\begin{cases} \partial_t \eta^i(t,s) = \rho^{-2}(t)\left(\partial_s\left(\tilde{\sigma}^i(t,s)\,\partial_s\eta^i(t,s)\right)\right), \quad i=1,2,3, \\ |\partial_s\eta^i(s)| = \rho(t), \\ \sum_{i=1}^3 \int_0^1 \tilde{\sigma}^i ds = 1 \text{ for all } t, \\ \eta^1=\eta^2=\eta^3 \ \text{ at } \ s=0,1 \text{ for all } t, \\ \sum_{i=1}^3 \tilde{\sigma}^i\partial_s\eta^i = 0 \text{ at } s=0,1 \text{ for all } t, \\ \eta^i(0,s) = \alpha^i(s), \end{cases} \tag{3.1}$$

see Subsection 3.2.1 for the details. Note that the extra variable ρ as well as the both sides of the third equation in (3.1) do not depend on s.

The uniformly compressing mean curvature flow (3.1) is difficult and nonlocal, but after a suitable change of variables we can equivalently rewrite it as a renormalized system that can be viewed as an overdamped motion of an inextensible θ-network repelled from the origin by the external force equal to the radius-vector. The main attention will thus be focused on the following normalized model:

$$\begin{cases} \partial_t \xi^1 = \partial_s \left(\sigma^1 \partial_s \xi^1\right) + \xi^1, \\ \partial_t \xi^2 = \partial^s \left(\sigma^2 \partial_s \xi^2\right) + \xi^2, \\ \partial_t \xi^3 = \partial_s \left(\sigma^3 \partial_s \xi^3\right) + \xi^3, \\ |\partial_s \xi^i(t,s)| = 1 \ \text{ for each } \ i, \\ \xi^1 = \xi^2 = \xi^3 \ \text{ at } \ s = 0,1 \text{ for all } t, \\ \sum_{i=1}^{3} \sigma^i \partial_s \xi^i = 0 \ \text{ at } \ s = 0,1 \text{ for all } t, \\ \xi^i(0,s) = \beta^i(s) \end{cases} \tag{3.2}$$

The notation $\xi^i(t,s) \in \mathbb{R}^d$, $d > 1$, is the position of the string number $i = 1,2,3$ at time $t \in \mathbb{R}^+$ and at the point corresponding to $s \in [0,1]$, and $\sigma^i(t,s)$ is the tension at the same point that can again be seen as a Lagrange multiplier. We keep the notation $\xi = \left(\xi^1, \xi^2, \xi^3\right)$ similarly to the previous chapter and also $\Omega := (0,1)$. Each string ξ^i has fixed length 1. Let us emphasize again this can be viewed as overdamped motion of an inextensible θ-network subject to the external force is $\psi = \xi$.

To avoid any possible confusion between the usage of (ξ, σ) and $(\eta, \tilde{\sigma})$ let us point out that in this chapter we start working with $(\eta, \tilde{\sigma}, \rho)$, and the unknown variables of the normalized system (3.2) are (ξ, σ). The transformation is given by (3.18), (3.21).

Observe that in comparison with the previous chapter, we do not derive the gradient flow (3.2) from the full dynamical system but obtain it as a renormalization of the uniformly compressing mean curvature flow.

3.2 Obtaining the flow

In this section we derive the uniformly compressing mean curvature flow of networks (3.1) and the renormalized system (3.2). In the subsequent manipulations, in order to fix the ideas and avoid avalanche of technicalities, we remain formal and fully concentrate on the geometry of the problem, neglecting regularity issues.

3.2.1 Uniformly compressing flow of networks

Firstly, we present the infinite-dimensional manifolds which are the ambient spaces of our gradient flows. For $d \geq 2$, let the space $\mathscr{K}$ be defined as follows.

$$\mathscr{K} := \left\{ \eta = (\eta^1, \eta^2, \eta^3) : \eta^i : \Omega \to \mathbb{R}^d, \ \sum_{i=1}^{3} \int_0^1 \eta^i ds = 0 \right\}.$$

As a linear vector space, it can be viewed as a smooth manifold with tangent space

$$T_\eta \mathscr{K} = \Big\{ w = (w^1, w^2, w^3) : w^i : \Omega \to \mathbb{R}^d : \sum_{i=1}^{3} \int_0^1 w^i(s) \cdot e^k ds = 0 \; \forall k = 1, \dots, d \Big\}, \; \eta \in \mathscr{K}.$$

Here $\{e^k\}$ denotes the canonical basis in $\mathbb{R}^d$.

Let

$$L : \mathscr{K} \to \mathbb{R}, \; L(\eta) := \sum_{i=1}^{3} \int_0^1 |\partial_s \eta^i| ds \tag{3.3}$$

be the length functional. We define

$$\mathscr{A} := \Big\{ \eta \in \mathscr{K} : |\partial_s \eta^i(s)| = \rho(\eta) := L(\eta)/3 > 0 \text{ for all } s \in \Omega \text{ and } i = 1,2,3, \\ \eta^1(0) = \eta^2(0) = \eta^3(0), \eta^1(1) = \eta^2(1) = \eta^3(1) \Big\}, \tag{3.4}$$

which is a smooth submanifold of $\mathscr{K}$ with the tangent space

$$T_\eta \mathscr{A} = \Big\{ w = (w^1, w^2, w^3) : w^i : \Omega \to \mathbb{R}^d : \\ \partial_s w^i(s) \cdot \partial_s \eta^i(s) \text{ does not depend on } s \text{ and on } i, \\ \text{and } \sum_{i=1}^{3} \int_0^1 w^i(s) \cdot e^k ds = 0 \; \forall k = 1, \dots, d, \\ w^1(0) = w^2(0) = w^3(0), w^1(1) = w^2(1) = w^3(1) \Big\}, \; \eta \in \mathscr{A}.$$

We endow $\mathscr{K}$ with the Riemannian metric

$$\langle v, w \rangle_{T_\eta \mathscr{K}} := \sum_{i=1}^{3} \int_0^1 v^i(s) \cdot w^i(s) |\partial_s \eta^i(s)| ds,$$

which is invariant under reparametrization and is compatible with the mean curvature flow, cf. [42, 43]. It induces a metric on $\mathscr{A}$:

$$\langle v, w \rangle_{T_\eta \mathscr{A}} := \sum_{i=1}^{3} \int_0^1 v^i(s) \cdot w^i(s) \rho(\eta) ds. \tag{3.5}$$

Remark 3.1. We emphasize that the equality $\rho(\eta) = |\partial_s \eta^i|$ allows us to have $\partial_s \eta^i \cdot \partial_{ss} \eta^i = 0$ and $\partial_s \eta^i \cdot \partial_{sss} \eta^i = -|\partial_{ss} \eta^i|^2$ pointwise for each $i = 1,2,3$.

We are interested in the formal gradient flow of the length functional $L(\eta)$, $\eta \in \mathscr{A}$, under the metric (3.5):

$$\frac{d}{dt} \eta = -\nabla_{\mathscr{A}} L(\eta(t)). \tag{3.6}$$

As explained in [62], this can be viewed as a uniformly compressing variant of the mean curvature flow ([62] deals with a flow of single loop; our flow is a natural generalization). In order to derive the

PDE formulation of (3.6), we compute the orthogonal projection from $T_\eta\mathscr{K}$ onto $T_\eta\mathscr{A}$ with respect to the metric (3.5).

Remark 3.2. Here we implicitly assume that all arms of the evolving network have the same length (this is not fundamental and in principle can be omitted but simplifies the calculations below). Note also that here and below we fix the center of mass of the evolving network at the origin. This does not affect the generality of the shape evolution because L is translation-invariant in the sense $L(\eta) = L(T_v(\eta))$, where $T_v(\eta^1,\eta^2,\eta^3) := (\eta^1+v,\eta^2+v,\eta^3+v)$, for any vector $v\in\mathbb{R}^d$.

Lemma 3.1. *Let $\eta\in\mathscr{A}$. The orthogonal projection $P_\eta : T_\eta\mathscr{K}\to T_\eta\mathscr{A}$ is (at least formally) given by*

$$P_\eta(z) = \left(z^1-\partial_s\left(\sigma^1\partial_s\eta^1\right), z^2-\partial_s\left(\sigma^2\partial_s\eta^2\right), z^3-\partial_s\left(\sigma^3\partial_s\eta^3\right)\right),$$

where $\sigma^i : [0,1]\to\mathbb{R}$ solve the system of ODE

$$\rho^2\partial_{ss}\sigma^i - |\partial_{ss}\eta^i|^2\sigma^i = \partial_s z^i\cdot\partial_s\eta^i + const(i) \tag{3.7}$$

$$\sum_{i=1}^{3}\int_0^1 \sigma^i(s)\,ds = 0, \tag{3.8}$$

$$\sum_{i=1}^{3}\sigma^i\partial_s\eta^i = 0 \text{ at } s=0,1, \tag{3.9}$$

$$\partial_s\sigma^i\partial_s\eta^i + \sigma^i\partial_{ss}\eta^i - z^i \text{ does not depend on } i \text{ at } s=0,1. \tag{3.10}$$

Proof. (1) We first show that for any σ^i satisfying (3.8), (3.9) and (3.10), the vector field $\left(\partial_s\left(\sigma^1\partial_s\eta^1\right), \partial_s\left(\sigma^2\partial_s\eta^2\right), \partial_s\left(\sigma^3\partial_s\eta^3\right)\right)$ is orthogonal to $T_\eta\mathscr{A}$. Indeed, given any $w\in T_\eta\mathscr{A}$, by integrating by parts we obtain

$$\begin{aligned}\langle\partial_s(\sigma\partial_s\eta), w\rangle_{T_\eta\mathscr{K}} &= \sum_{i=1}^{3}\rho\int_0^1 \partial_s\left(\sigma^i\partial_s\eta^i\right)\cdot w^i\,ds,\\ &= \sum_{i=1}^{3}\rho\sigma^i\partial_s\eta^i\cdot w^i|_{s=0}^{s=1} - \sum_{i=1}^{3}\rho\int_0^1\sigma^i\partial_s\eta^i\cdot\partial_s w^i ds\\ &= 0.\end{aligned}$$

Here we used the properties of the $T_\eta\mathscr{A}$, (3.8) and (3.9).

(2) Next we show $w = (w^1,w^2,w^3)\in T_\eta\mathscr{A}$ where $w^i = z^i - \partial_s\left(\sigma^i\partial_s\eta^i\right)$ for any $z\in T_\eta\mathscr{K}$. It follows from (3.10) that $w^1(0)=w^2(0)=w^3(0), w^1(1)=w^2(1)=w^3(1)$. Moreover, $\sum_{i=1}^3\int_0^1 w^i ds = 0$ due to (3.9). It remains to check the condition $\partial_s w^i\cdot\partial_s\eta^i = const(i)$. Indeed, by easy calculations, we have

$$\partial_s w^i\cdot\partial_s\eta^i = \partial_s z^i\cdot\partial_s\eta^i - \partial_{ss}\sigma^i|\partial_s\eta^i|^2 - 2\partial_s\sigma^i\partial_{ss}\eta^i\cdot\partial_s\eta^i - \sigma^i\partial_{sss}\eta^i\cdot\partial_s\eta^i.$$

By Remark 3.1, we end up with

$$\partial_s w^i\cdot\partial_s\eta^i = \partial_s z^i\cdot\partial_s\eta^i - \partial_{ss}\sigma^i\rho^2 + \sigma^i|\partial_{ss}\eta^i|^2$$

which is a different constant for each $i=1,2,3$ by (3.7). □

We now compute the $\mathscr{A}$-gradient of L. Let $\eta \in \mathscr{A}$. By the definition of the gradient and $\mathscr{A}$ being the submanifold of $\mathscr{K}$ we have

$$\nabla_{\mathscr{A}} L(\eta) = P_\eta \left(\nabla_{\mathscr{K}} L(\eta) \right).$$

Some calculus of variations applied to (3.3) shows that the $L^2(\Omega;\mathbb{R}^{3d})$-variation of $L(\eta)$ is

$$\begin{aligned} \delta L(\eta) = &- \left(\partial_s \left(\frac{\partial_s \eta^1}{|\partial_s \eta^1|} \right), \partial_s \left(\frac{\partial_s \eta^2}{|\partial_s \eta^2|} \right), \partial_s \left(\frac{\partial_s \eta^3}{|\partial_s \eta^3|} \right) \right) \\ &+ \left(\frac{\partial_s \eta^1}{|\partial_s \eta^1|} \left(\delta_0(s-1) - \delta_0(s) \right), \frac{\partial_s \eta^2}{|\partial_s \eta^2|} \left(\delta_0(s-1) - \delta_0(s) \right), \frac{\partial_s \eta^3}{|\partial_s \eta^3|} \left(\delta_0(s-1) - \delta_0(s) \right) \right), \end{aligned} \tag{3.11}$$

where δ_0 is the Dirac delta function. In particular, since L is translation-invariant by Remark 3.2, we formally have $\delta L(\eta) \in T_\eta \mathscr{K}$; this claim can also be derived directly from (3.11). By the very definition of the gradient,

$$\begin{aligned} \langle \nabla_{\mathscr{K}} L(\eta), w \rangle_{T_\eta \mathscr{K}} &= \int_0^1 \delta L(\eta)(s) \cdot w(s)\, ds \\ &= -\sum_{i=1}^{3} \int_0^1 \partial_s \left(\frac{\partial_s \eta^i}{|\partial_s \eta^i|} \right)(s) \cdot w^i(s)\, ds + \sum_{i=1}^{3} \left(\frac{\partial_s \eta^i}{|\partial_s \eta^i|}(1) \cdot w^i(1) - \frac{\partial_s \eta^i}{|\partial_s \eta^i|}(0) \cdot w^i(0) \right), \end{aligned}$$

for every $w \in T_\eta \mathscr{K} \simeq \mathscr{K}$. We conclude that

$$\begin{aligned} \nabla_{\mathscr{K}} L(\eta) = &- \left(\frac{1}{|\partial_s \eta^1|} \partial_s \frac{\partial_s \eta^1}{|\partial_s \eta^1|}, \frac{1}{|\partial_s \eta^2|} \partial_s \frac{\partial_s \eta^2}{|\partial_s \eta^2|}, \frac{1}{|\partial_s \eta^3|} \partial_s \frac{\partial_s \eta^3}{|\partial_s \eta^3|} \right) \\ &+ \left(\frac{\partial_s \eta^1}{|\partial_s \eta^1|^2} \left(\delta_0(s-1) - \delta_0(s) \right), \frac{\partial_s \eta^2}{|\partial_s \eta^2|^2} \left(\delta_0(s-1) - \delta_0(s) \right), \frac{\partial_s \eta^3}{|\partial_s \eta^3|^2} \left(\delta_0(s-1) - \delta_0(s) \right) \right). \end{aligned} \tag{3.12}$$

Note that the components of the first term in (3.12) are exactly the curvature vectors of the arms of the network (which is in agreement with the classical mean curvature flow), and the second term vanishes everywhere but at the junctions.

Since $\eta \in \mathscr{A}$, (3.12) simplifies to

$$\begin{aligned} \nabla_{\mathscr{K}} L(\eta) &= - \left(\frac{\partial_{ss} \eta^1}{\rho^2}, \frac{\partial_{ss} \eta^2}{\rho^2}, \frac{\partial_{ss} \eta^3}{\rho^2} \right) + \left(\frac{\partial_s \eta^1}{\rho^2}, \frac{\partial_s \eta^2}{\rho^2}, \frac{\partial_s \eta^3}{\rho^2} \right) \left(\delta_0(s-1) - \delta_0(s) \right) \\ &= -\partial_s \left(\frac{\mathbf{1}_\Omega(s) \partial_s \eta^1}{\rho^2}, \frac{\mathbf{1}_\Omega(s) \partial_s \eta^2}{\rho^2}, \frac{\mathbf{1}_\Omega(s) \partial_s \eta^3}{\rho^2} \right). \end{aligned}$$

By Lemma 3.1, the orthogonal projection of the negative gradient in $\mathscr{K}$ to the tangent space $T_\eta \mathscr{A}$ is

$$P_\eta\left(-\nabla_{\mathscr{K}} L(\eta)\right)$$
$$= \rho^{-2}(t)\left(\partial_s\left(\left(\mathbf{1}_\Omega - \sigma^1\right)(t,s)\,\partial_s\eta^1(t,s)\right),\right.$$
$$\left.\partial_s\left(\left(\mathbf{1}_\Omega - \sigma^2\right)(t,s)\,\partial_s\eta^2(t,s)\right), \partial_s\left(\left(\mathbf{1}_\Omega - \sigma^3\right)(t,s)\,\partial_s\eta^3(t,s)\right)\right),$$

where $\sigma^i : [0,1] \to \mathbb{R}$ satisfies

$$\rho^2\partial_{ss}\sigma^i - \sigma^i|\partial_{ss}\eta^i|^2 = \partial_{sss}\eta^i\cdot\partial_s\eta^i + const(i) = -|\partial_{ss}\eta^i|^2 + const(i),\quad \sum_{i=1}^3\int_0^1 \sigma^i ds = 0,$$

as well as (3.9) and

$$\partial_s\sigma^i\partial_s\eta^i + \sigma^i\partial_{ss}\eta^i - \partial_{ss}\eta^i + \partial_s\eta^i\left(\delta_0(s-1) - \delta_0(s)\right) \text{ does not depend on } i \text{ at } s = 0,1. \tag{3.13}$$

Letting $\tilde{\sigma}^i := \mathbf{1}_\Omega - \sigma^i$, we get the following expression for the gradient flow (3.6):

$$\partial_t\eta^i(t,s) = \rho^{-2}(t)\left(\partial_s\left(\tilde{\sigma}^i(t,s)\,\partial_s\eta^i(t,s)\right)\right). \tag{3.14}$$

This should be complemented by the following constraints that follow from the considerations above (note carefully that $\tilde{\sigma}^i = -\sigma^i$ at $s = 0,1$):

$$\sum_{i=1}^3 \tilde{\sigma}^i\partial_s\eta^i = 0 \text{ at } s = 0,1,\quad \sum_{i=1}^3\int_0^1 \tilde{\sigma}^i ds = 1. \tag{3.15}$$

The remaining constraints can be recovered from the condition $\eta(t) \in \mathscr{A}$. We have thus derived system (3.1).

It is sometimes useful to write the obtained flow in the weak form as follows. Test (3.14) by a smooth evolving curve $\gamma(t,s) = (\gamma^1,\gamma^2,\gamma^3)(t,s) \in \mathbb{R}^{3d}$ that satisfies

$$\gamma^1(0) = \gamma^2(0) = \gamma^3(0), \gamma^1(1) = \gamma^2(1) = \gamma^3(1).$$

After an integration by parts we arrive at

$$\sum_{i=1}^3\int_0^1 \partial_t\eta^i(t,s)\cdot\gamma^i(t,s)\,ds = \rho^{-2}(t)\sum_{i=1}^3\left(-\int_0^1 \tilde{\sigma}^i(t,s)\,\partial_s\eta^i(t,s)\cdot\partial_s\gamma^i(t,s)\,ds\right). \tag{3.16}$$

3.2.2 Evolution of the variance and the normalized flow

We now exploit the gradient flow structure to derive some evolution properties of the length and the "variance" of the θ-network. Remember from Section 1.3 that we can view the evolving string as a stochastic process and so it has a well-defined variance at every time. Throughout this subsection, we remain formal and assume the existence of solutions to the gradient flow (3.14).

The first proposition is about the evolution of the total length functional.

Proposition 3.2. *Let η be solution to* (3.14)*. Then*

$$\partial_t L(\eta) = -\sum_{i=1}^{3} \rho^{-3} \int_0^1 \tilde{\sigma}^i |\partial_{ss}\eta^i|^2 ds.$$

Proof. Using that $|\partial_s \eta^i(s,t)| = \rho(t)$, then

$$\begin{aligned}\partial_t \rho &= \partial_t |\partial_s \eta^i| \\ &= \frac{\partial_s \eta^i(t,s)}{|\partial_s \eta^i(t,s)|} \cdot \partial_{st} \eta^i(t,s) \\ &= \frac{\partial_s \eta^i(t,s)}{\rho} \cdot \partial_s \left(\frac{\tilde{\sigma}^i \partial_{ss} \eta^i}{\rho^2} + \frac{\partial_s \tilde{\sigma}^i \partial_s \eta^i}{\rho^2} \right)\end{aligned}$$

Note the $\partial_t \rho$ does not depend on s. Thus, summing in i, integrating in s from 0 to 1, and then by parts we obtain the desired equality. □

In the next proposition, we show that the variance of the solution decays with constant speed.

Proposition 3.3. *Let η be a solution to the gradient flow* (3.14)*. Let*

$$M(t) := \frac{1}{2} \sum_{i=1}^{3} \int_0^1 |\eta^i(t,s)|^2 ds \tag{3.17}$$

be the variance (up to a multiplicative constant[1]*). Then $\partial_t M(t) = -3$.*

Proof. We let $\gamma^i = \eta^i$ in (3.16) to obtain

$$\partial_t M(t) = \sum_{i=1}^{3} \int_0^1 \partial_t \eta^i \cdot \eta^i ds = -\rho^2 \sum_{i=1}^{3} \int_0^1 \tilde{\sigma}^i |\partial_s \eta^i|^2 ds.$$

Using that $\rho = |\partial_s \eta^i|$ and $\int_0^1 \tilde{\sigma}^i ds = 1$, we obtain $\partial_t M(t) = -3$. □

An immediate consequence of the proposition above is that the flows extinct in finite time.

Corollary 3.4. *$M(t) \to 0$ as $t \to t^*$, where $t^* = \frac{1}{3} M(0) = \frac{1}{6} \sum_{i=1}^{3} \int_0^1 |\eta^i(0,s)|^2 ds$.*

It is also possible to obtain the decay rate of $\rho(t)$ near the extinct time t^*.

Corollary 3.5. *Let $L_0 := L(0) > 0$ and let t^* be the extinction time as in Corollary 3.4. Then for all $t \in [0,t^*)$,*

$$L(t) \leq L_0 \sqrt[6]{\frac{t^* - t}{t^*}}.$$

[1] Remember that the center of mass $\sum_{i=1}^{3} \int_0^1 \eta^i(t,s) ds$ is fixed at the origin, hence the variance is $\frac{1}{3} \sum_{i=1}^{3} \int_0^1 |\eta^i(t,s)|^2 ds$ in the spirit of (1.7).

Proof. By Proposition 3.3, we have
$$M(t) = 3\left(t^* - t\right).$$
For a generic i, by Hölder's inequality the following is true
$$\left(\int_0^1 \partial_t \eta^i \cdot \eta^i ds\right)^2 \leq \int_0^1 |\partial_t \eta^i|^2 ds \int_0^1 |\eta^i|^2 ds.$$
Now, we sum in i
$$\sum_{i=1}^3 \left(\int_0^1 \partial_t \eta^i \cdot \eta^i ds\right)^2 \leq \sum_{i=1}^3 \left(\int_0^1 |\partial_t \eta^i|^2 ds \int_0^1 |\eta^i|^2 ds\right).$$
We can estimate the right-hand side using an algebraic inequality and similarly the left-hand side
$$\begin{aligned}\left(\sum_{i=1}^3 \int_0^1 \partial_t \eta^i \cdot \eta^i ds\right)^2 &\leq 3\sum_{i=1}^3 \left(\int_0^1 \partial_t \eta^i \cdot \eta^i ds\right)^2 \\ &\leq 3\sum_{i=1}^3 \left(\int_0^1 |\partial_t \eta^i|^2 ds \int_0^1 |\eta^i|^2 ds\right) \leq 3\sum_{i=1}^3 \left(\int_0^1 |\partial_t \eta^i|^2 ds\right) \sum_{i=1}^3 \left(\int_0^1 |\eta^i|^2 ds\right).\end{aligned}$$
Using that $\partial_t M(t) = \sum_{i=1}^3 \int_0^1 \partial_t \eta^i \cdot \eta^i ds = -3$ and $M(t) = \frac{1}{2}\sum_{i=1}^3 \int_0^1 |\eta^i|^2 ds$, we obtain
$$9 \leq 3\sum_{i=1}^3 \left(\int_0^1 |\partial_t \eta^i|^2 ds\right) 2M(t).$$
From the last inequality, we immediately have
$$\frac{3}{2M(t)} \leq \sum_{i=1}^3 \int_0^1 |\partial_t \eta^i|^2 ds.$$
We can rewrite the last expression as
$$\frac{1}{2(t^* - t)} \leq \sum_{i=1}^3 \int_0^1 |\partial_t \eta^i|^2 ds.$$
On the other hand, by the gradient flow structure
$$\partial_t L = -\sum_{i=1}^3 \rho \int_0^1 |\partial_t \eta^i|^2 ds \leq \frac{-L}{6(t^* - t)}.$$
Thus, $-\partial_t \ln L(t) \leq \frac{1}{6(t^*-t)}$. We integrate over t from 0 to t and obtain the upper bound
$$L(t) \leq L_0 \sqrt[6]{\frac{t^* - t}{t^*}}$$
for $t \in (0, t^*)$. □

We first work with a slow time variable and start to normalize the flow. More precisely, for $t \in [0, t^*)$ let

$$\tau(t) := -\ln L(t).$$

Note that by Proposition 3.2 and Corollary 3.5, $\tau(t)$ is monotone increasing t and $\tau \to +\infty$ iff $t \to t^*$. Next we consider the normalization

$$\xi^i(\tau, s) := \frac{\eta^i(\tau(t), s)}{\rho(\tau(t))}. \tag{3.18}$$

One advantage of using such renormalization is that the curves $\xi(\tau, s)$ have the unit speed parametrization, i.e.,

$$|\partial_s \xi^i(\tau, s)| = 1, \ (\tau, s) \in [0, \infty) \times [0, 1].$$

We emphasize that we also change the initial value with transformation (3.18) and it is $\beta(s) := \xi^i(0, s) = \frac{\alpha^i(s)}{\rho(\tau(0))}$. A direct computation shows that ξ^i satisfies for each $i = 1, 2, 3$ the following equation

$$\partial_\tau \xi^i = \partial_s \left(\sigma^i \partial_s \xi^i\right) + \xi^i \tag{3.19}$$

Here $\sigma^i(\tau, s)$ can be viewed as a Lagrange multiplier coming from the constraint $|\partial_s \xi^i| = 1$. It satisfies

$$\partial_{ss}\sigma^i - \sigma^i |\partial_{ss}\xi^i|^2 = -1. \tag{3.20}$$

Indeed, from the change of variable $\frac{d\tau}{dt} = -\frac{\partial_t L(t)}{L(t)} = -\frac{\partial_t \rho(t)}{\rho(t)}$. Thus

$$\partial_\tau \xi^i = \frac{\partial_t \eta^i}{\rho}\frac{dt}{d\tau} - \eta^i \frac{\partial_t \rho}{\rho^2}\frac{dt}{d\tau} = -\frac{\partial_t \eta^i}{\partial_t \rho} + \frac{\eta^i}{\rho}.$$

Using the equation of η^i, we have

$$\partial_\tau \xi^i = -\frac{\partial_s\left(\tilde{\sigma}^i \partial_s \eta^i\right)}{\rho^2 \partial_t \rho} + \frac{\eta^i}{\rho}.$$

Letting

$$\sigma^i = \frac{\tilde{\sigma}^i}{-\rho \partial_t \rho} \tag{3.21}$$

and writing the above equation in terms of ξ^i, we arrive at (3.19) together with constraint $|\partial_s \xi^i| \equiv 1$.

With the similar argument as in previous section we can easily see that the normalized flow (3.19) and (3.20) can be viewed as the positive gradient flow of the variance $M(\xi) := \frac{1}{2}\sum_{i=1}^{3}\int_0^1 |\xi^i(s)|^2$ on the manifold of θ-networks with arc-length parametrization

$$\tilde{\mathscr{A}} := \left\{\xi \in \mathscr{A} : |\partial_s \xi^i(s)| = 1 \ \text{ for all } s \in \Omega \text{ and } i = 1, 2, 3\right\},$$

with the Riemannian metric inherited from $\mathscr{A}$, i.e.,

$$\partial_t \xi = +\nabla_{\tilde{\mathscr{A}}} M(\xi(t)). \tag{3.22}$$

Remark 3.3 (Initial data). We fix once and for all Lipschitz initial data $\beta^i \in W^{1,\infty}(\Omega)^d$, $i = 1,2,3$, satisfying the compatibility conditions

$$\beta^1 = \beta^2 = \beta^3 \text{ at points } s = 0,1 \text{ for all times } t, \tag{3.23}$$

$$\sum_{i=1}^{3} \int_0^1 \beta^i ds = 0 \tag{3.24}$$

and

$$|\partial_s \beta^i(s)| = 1 \text{ a.e. in } \Omega. \tag{3.25}$$

Since (3.25) is only required to hold almost everywhere, the arms of the θ-network can have shape of any rectifiable curve at the initial moment. We will moreover assume that the arms of the θ-network are not fully straight at the initial moment. Thus the only case we exclude is when all the arms are coinciding straight lines.

3.3 The approximation problem

In order to analyze system (3.2) that we have just derived we use the approximations and ansatz similar to the ones presented in Section 2.3. We omit the majority of manipulations to avoid redundancies and we write down only the important parts.

We assume that $\sigma^i \geq 0$ for all s and t (this is the ansatz). The approximation problem reads

$$\begin{cases} \partial_t \xi_\varepsilon^i = \partial_s \left(G_\varepsilon \left(\partial_s \xi_\varepsilon^i \right) \right) + \xi_\varepsilon^i \\ \xi_\varepsilon^1 = \xi_\varepsilon^2 = \xi_\varepsilon^3 \text{ at } s = 0,1 \text{ for all } t \\ \sum_{i=1}^3 G_\varepsilon \left(\partial_s \xi_\varepsilon^i \right) = 0 \text{ at } s = 0,1 \text{ for all } t \\ \xi_\varepsilon^i(0,s) = \beta^i(s). \end{cases} \tag{3.26}$$

In comparison with the original equation (3.2) we define

$$\kappa_\varepsilon^i := G_\varepsilon \left(\partial_s \xi_\varepsilon^i \right), \quad \sigma_\varepsilon^i := G_\varepsilon \left(\partial_s \xi_\varepsilon^i \right) \cdot \partial_s \xi_\varepsilon^i \tag{3.27}$$

As discussed in Section 2.3, there exists a bounded smooth positive scalar function γ_ε such that $G_\varepsilon(\tau) = \gamma_\varepsilon \left(|\tau^2| \right) \tau, \tau \in \mathbb{R}^d$. In particular, this implies for θ-networks that

$$\sigma_\varepsilon^i \geq 0. \tag{3.28}$$

We consider the associated total energy to the above approximation system

$$\begin{cases} \mathscr{E}_\varepsilon(\xi) := \sum_{i=1}^3 \left(\int_0^1 Q_\varepsilon \left(\partial_s \xi^i \right) ds - \frac{1}{2} \int_0^1 |\xi^i|^2 ds \right) \\ \text{for } \xi \in AC^2(\Omega; \mathbb{R}^{3d}) \text{ satisfying } \ \xi^1(0) = \xi^2(0) = \xi^3(0), \xi^1(1) = \xi^2(1) = \xi^3(1); \\ +\infty \text{ for any } \xi \in L^2(\Omega; \mathbb{R}^{3d}) \text{ except those above,} \end{cases} \tag{3.29}$$

where as before

$$Q_\varepsilon : \mathbb{R}^d \to \mathbb{R},\ \ Q_\varepsilon(u) = \left(\frac{|G_\varepsilon(u)|^2}{2} - \frac{1}{\sqrt{\varepsilon + |G_\varepsilon(u)|^2}}\right) + \sqrt{\varepsilon}. \tag{3.30}$$

Then (3.26) can be interpreted as a gradient flow with respect to the flat Hilbertian structure inherited from L^2, which is driven by this functional:

$$\frac{d}{dt}\xi_\varepsilon = -\nabla_{L^2}\mathscr{E}_\varepsilon\left(\xi_\varepsilon\right),\ \xi_\varepsilon(0) = \beta.$$

We opt for using the same notation $\mathscr{E}_\varepsilon$ for a different energy in comparison with the other chapters.

3.4 Evolution by pseudomonotone maps and solvability of the approximation problem

Our next goal is to apply theory presented in [59] to show the existence and regularity of solutions to the approximation problem (3.26) using Appendix A. Fix the initial data $\beta^i \in W^{1,\infty}(\Omega)$ satisfying the compatibility conditions as in Remark 3.3. Let us rewrite our approximation system (3.26) with the help of the simple transformation

$$\upsilon^i(t,s) := \xi_\varepsilon^i(t,s) - \beta^i(s),$$

arriving at

$$\begin{cases} \partial_t \upsilon^i - \partial_s\left(G_\varepsilon\left(\partial_s\left(\upsilon^i + \beta^i\right)\right)\right) - \upsilon^i - \beta^i = 0,\ \forall i, \\ \upsilon^1(t,0) = \upsilon^2(t,0) = \upsilon^3(t,0), \\ \upsilon^1(t,1) = \upsilon^2(t,1) = \upsilon^3(t,1), \\ \upsilon^i(0,s) = 0,\ \forall i, \\ \sum_{i=1}^3 G_\varepsilon\left(\partial_s\left(\upsilon^i + \beta^i\right)\right)(0) = 0. \end{cases} \tag{3.31}$$

Let us rewrite this system in the form of the Cauchy problem (A.2). Let $H = L^2\left(\Omega;\mathbb{R}^{3d}\right)$ be the Hilbert space of triples with the natural scalar product. Consider the set

$$V := \{u = \{u^i\} \in AC^2\left(\overline{\Omega};\mathbb{R}^{3d}\right) \text{ such that } u^1(1) = u^2(1) = u^3(1) \text{ and } u^1(0) = u^2(0) = u^3(0)\}.$$

It is a separable reflexive Banach space with the norm inherited from H^1. Define a seminorm on V by $|\{u_i\}|_V := \|\{\partial_s u_i\}\|_H$. The required Poincaré inequality, cf. the Appendix A, obviously holds. Let $\bar{\mathbf{A}} : V \to V^*$ and $\tilde{\mathbf{A}} : V \to V^*$ be the mappings that are defined by duality as follows:

$$\langle \bar{\mathbf{A}}(\upsilon), \zeta\rangle = \sum_{i=1}^{3}\int_0^1 G_\varepsilon\left(\partial_s\left(\upsilon_i + \beta_i\right)\right)\cdot \partial_s\zeta_i\, ds, \tag{3.32}$$

$$\langle \tilde{\mathbf{A}}(\upsilon), \zeta \rangle = -\sum_{i=1}^{3} \int_0^1 \left(\upsilon_i + \beta_i\right) \cdot \zeta_i \, ds. \tag{3.33}$$

Let $\mathbf{A} := \bar{\mathbf{A}} + \tilde{\mathbf{A}}$. Then (3.31) rewrites as

$$\frac{d}{dt}\xi + \mathbf{A}\left(\xi(t)\right) = 0, \; \xi(0) = 0. \tag{3.34}$$

Note that the last equality of (3.31) is hidden in the duality in (3.32).

In order to check that Theorem A.8 is applicable to (3.34) we need to prove several auxiliary statements. Since this material is very similar and the results are the same as Chapter 2.4, for the majority of the statements below we do not give the proofs. The difference with the previous chapter is the external force and, although it destroys the monotonicity, still it is very easy to deal with it as it is an affine term. This is the reason why we do not give extra details. However, when it is needed, we write the proofs. To avoid messy notation, we will omit the subscript ε.

Lemma 3.6. *The mapping* $\bar{\mathbf{A}}$ *satisfies the inequality*

$$\langle \bar{\mathbf{A}}(\upsilon_1) - \bar{\mathbf{A}}(\upsilon_2), \upsilon_1 - \upsilon_2 \rangle \geq c_0 |\upsilon_1 - \upsilon_2|_V^2$$

for some constant $c_0 > 0$ *(depending on* ε*) and any* $\upsilon_1, \upsilon_2 \in V$.

Corollary 3.7. *The mapping* $\mathbf{A}$ *satisfies the inequality*

$$\langle \mathbf{A}(\upsilon_1) - \mathbf{A}(\upsilon_2), \upsilon_1 - \upsilon_2 \rangle \geq c_0 |\upsilon_1 - \upsilon_2|_V^2 - ||\upsilon_1 - \upsilon_2||_H$$

for some constant $c_0 > 0$ *(depending on* ε*) and any* $\upsilon_1, \upsilon_2 \in V$.

Corollary 3.8. *The mapping* $\mathbf{A}$ *is semicoercive.*

Lemma 3.9. *The mapping* $\bar{\mathbf{A}}$ *is bounded.*

Lemma 3.10. *The mapping* $\bar{\mathbf{A}}$ *is radially continuous.*

Lemma 3.11. *The mapping* $\mathbf{A}$ *is pseudomonotone.*

Proof. By Lemma A.4 and the results above $\bar{\mathbf{A}}$ is pseudomonotone. On the other hand, since the Sobolev embeddings $V \subset H$ and hence $V \subset V^*$ are compact, the affine map $\tilde{\mathbf{A}}$ is totally continuous. Now Lemma A.6 yields the result. □

Corollary 3.12. *Given* $\beta \in W^{1,\infty}(\Omega)$*, the system* (3.34) *has a solution* $\upsilon = \{\upsilon_i\} \in W^{1,\infty}(0,T;H) \cap AC^2\left([0,T];V\right)$ *that is understood in the same sense as in Theorem A.8.*

Returning back to the variable ξ and leveraging elementary properties of G_ε and ∇G_ε, we get the existence of approximate solutions.

Corollary 3.13. *Given* β *as in Remark 3.3, there exists a solution* $\xi = \xi_\varepsilon$ *to* (3.26) *in* Ω_T *that belongs to the following regularity class:*

$$\xi^i \in W^{1,\infty}\left(0,T;L^2(\Omega)\right)^d \cap AC^2\left([0,T];AC^2\left(\overline{\Omega}\right)\right)^d,$$

$$\partial_s \xi^i \in AC^2\left([0,T];L^2(\Omega)\right)^d,$$

$$\kappa^i := G_\varepsilon(\partial_s \xi^i) \in L^\infty\left(0,T;L^2(\Omega)\right)^d,$$

$$\nabla G_\varepsilon(\partial_s \xi^i) \in L^\infty\left(0,T;L^\infty(\Omega)\right)^d,$$

$$\partial_t \eta^i \in L^\infty\left(0,T;L^2(\Omega)\right)^d \cap L^2\left(0,T;H^1(\Omega)\right)^d,$$

$$\partial_s \kappa^i = \partial_s\left(G_\varepsilon\left(\partial_s \eta^i\right)\right) \in L^\infty\left(0,T;L^2(\Omega)\right)^d \cap L^2\left(0,T;H^1(\Omega)\right)^d,$$

$$\partial_{ss} \xi^i \in L^\infty\left(0,T;L^2(\Omega)\right)^d.$$

The norms of the functions above in the corresponding spaces may depend on ε. Note that at this stage we cannot infer an L^∞ bound on $\partial_s \xi^i$ (even ε-dependent) because we do not control $\partial_s \xi^i$ on $\partial\Omega$.

It is straightforward to see that $\xi = \xi_\varepsilon$ from Corollary 3.13 coincides with the unique solution of the gradient flow

$$\frac{d}{dt}\xi \in -\partial_{L^2(\Omega;\mathbb{R}^{3d})}\mathscr{E}_\varepsilon(\xi) \tag{3.35}$$

in the sense of Theorem B.2, where the driving functional $\mathscr{E}_\varepsilon$ was defined in (3.29). This in particular implies that $t \mapsto \mathscr{E}_\varepsilon(\xi(t))$ is a continuous and non-increasing function.

3.5 Uniform estimates of the approximation solutions

Now we derive some energy inequalities with uniform (in ε) bounds for the solutions ξ^i_ε in terms of initial datum. We use always C for a constant independent of ε. We keep omitting ε.

Proposition 3.14. *Let $\xi = \{\xi^i\}$ be a solution of the approximation problem* (3.26) *in Ω_T as constructed in Corollary 3.13 Then for any $T \in (0,\infty)$*

1.

$$\sum_{i=1}^{3} \max_{t\in[0,T]} \int_\Omega |\xi^i_\varepsilon|^2 ds + \int_{\Omega_t} |\partial_s \xi^i_\varepsilon|^2 ds dt \le C\left(\sum_{i=1}^{3} e^{2T}\left(\int_\Omega |\beta^i|^2 ds\right) + 1\right), \tag{3.36}$$

2.

$$\sum_{i=1}^{3} \int_{\Omega_T} |\partial_t \xi^i_\varepsilon|^2 + |\nabla G\left(\partial_s \xi^i_\varepsilon\right) \cdot \partial_{ss} \xi^i_\varepsilon|^2 ds dt$$
$$\le C \sum_{i=1}^{3}\left(\mathscr{E}_\varepsilon\left(\beta^i\right) + e^{T/2}\left(||\beta^i||_{L^2(\Omega)}\right) + 1\right) \tag{3.37}$$

Proof. (*Proof of* (3.36)) We start by the following observation: remember the expression of F in (2.34) and $G = F^{-1}$. Consequently one has $\tau = \varepsilon G(\tau) + \left(\varepsilon + |G(\tau)|^2\right)^{-1/2} G(\tau), \tau \in \mathbb{R}^d$. Thus,

$$G(\tau)\cdot\tau = \frac{|\tau|^2}{\varepsilon + \left(\varepsilon + |G(\tau)|^2\right)^{-1/2}}.$$

Using the monotonicity of the scalar function $r \to \tilde{F}(r) := \varepsilon r + \frac{r}{\sqrt{\varepsilon + r^2}}$ in $[0,\infty)$ and $\tilde{F}\left(\frac{1}{\sqrt{\varepsilon}}\right) < 1 + \sqrt{\varepsilon}$ one can conclude that

$$|G(\tau)| \geq \frac{1}{\sqrt{\varepsilon}} \text{ in } \{\tau : |\tau| \geq 1 + \sqrt{\varepsilon}\}. \tag{3.38}$$

Hence by the conclusion above we have

$$G(\tau) \cdot \tau \geq \frac{|\tau|^2}{\varepsilon + \left(\varepsilon + \varepsilon^{-1}\right)^{-1/2}} \geq \frac{1}{\varepsilon + \sqrt{\varepsilon}} |\tau|^2$$

if $|\tau| \geq 1 + \sqrt{\varepsilon}$. This can be rewritten as

$$G\left(\partial_s \xi^i\right) \cdot \partial_s \xi^i \geq \frac{1}{\varepsilon + \sqrt{\varepsilon}} |\partial_s \xi^i|^2 \text{ in } \{(t,s) \in \mathfrak{Q}_\mathrm{t} : |\partial_s \xi^i(t,s)| \geq 1 + \sqrt{\varepsilon}\}. \tag{3.39}$$

We take the inner product of each equation (3.26) with ξ^i and summing all, we have

$$\sum_{i=1}^{3} \partial_t \xi^i \cdot \xi^i = \sum_{i=1}^{3} \partial_s \left(G\left(\partial_s \xi^i\right)\right) \cdot \xi^i + \sum_{i=1}^{3} |\xi^i|^2.$$

Firstly we integrate over $\mathfrak{Q}_\mathrm{t}$ and then apply integration by parts

$$\frac{1}{2} \sum_{i=1}^{3} \int_\Omega |\xi^i|^2 ds = -\sum_{i=1}^{3} \int_{\mathfrak{Q}_\mathrm{t}} G\left(\partial_s \xi^i\right) \cdot \partial_s \xi^i ds dt + \sum_{i=1}^{3} \int_{\mathfrak{Q}_\mathrm{t}} |\xi^i|^2 ds dt + \frac{1}{2} \sum_{i=1}^{3} \int_\Omega |\beta^i(s)|^2 ds, \tag{3.40}$$

we have used the boundary conditions when we applied integration by parts: firstly we used that all arms of the whips have the same value at boundaries $s = 0, 1$ and then the sum of $G\left(\partial_s \xi^i\right)$ are 0 at boundaries. Notice that these calculations could not be done for the triod case because of the different boundary conditions at $s = 1$. Hence

$$\frac{1}{2} \sum_{i=1}^{3} \int_\Omega |\xi^i|^2 ds + \sum_{i=1}^{3} \int_{\mathfrak{Q}_\mathrm{t}} G\left(\partial_s \xi^i\right) \cdot \partial_s \xi^i ds dt$$
$$= \frac{1}{2} \sum_{i=1}^{3} \int_\Omega |\beta^i(s)|^2 ds + \sum_{i=1}^{3} \int_{\mathfrak{Q}_\mathrm{t}} |\xi^i|^2 ds. \tag{3.41}$$

After noticing that $G\left(\partial_s \xi^i\right) \cdot \partial_s \xi^i$ is always positive, by Grönwall type inequality for $t \in [0,T]$ we have

$$\sum_{i=1}^{3} \int_\Omega |\xi^i|^2 ds \leq \sum_{i=1}^{3} e^{2t} \left(\int_\Omega |\beta^i|^2 ds \right) \tag{3.42}$$

Maximizing both sides in t, we obtain the estimate for $\max_{\mathrm{t} \in [0,T]} \|\xi^i(t,\cdot)\|_{L^2(\Omega)}$.

To show the estimate for $\|\partial_s\xi^i\|_{L^2(\mathfrak{Q}_t)}$, we use the estimate (3.39) and apply it to (3.41) to obtain

$$\begin{aligned}\sum_{i=1}^{3}\int_{\mathfrak{Q}_t}|\partial_s\xi^i|^2 dsdt &\le \left(\varepsilon+\sqrt{\varepsilon}\right)\sum_{i=1}^{3}\int_{\mathfrak{Q}_t} G\left(\partial_s\xi^i\right)\cdot\partial_s\xi^i dsdt+\left(1+\sqrt{\varepsilon}\right)^2 3|\mathfrak{Q}_t| \\ &\le \frac{\varepsilon+\sqrt{\varepsilon}}{2}\sum_{i=1}^{3}\int_{\Omega}|\beta^i(s)|^2 ds+\left(\varepsilon+\sqrt{\varepsilon}\right)\sum_{i=1}^{3}\int_{\mathfrak{Q}_t}|\xi^i|^2 dsdt \\ &+3\left(1+\sqrt{\varepsilon}\right)^2|\mathfrak{Q}_t|\end{aligned}$$

After taking the suprema over $\mathfrak{t}$, the last inequality with (3.42) yields

$$\sum_{i=1}^{3}\max_{t\in[0,T]}\int_{\Omega}|\xi^i|^2 ds+\sum_{i=1}^{3}\int_{\mathfrak{Q}_T}|\partial_s\xi^i|^2 dsdt\le C\left(\sum_{i=1}^{3}e^{2T}\left(\int_{\Omega}|\beta^i|^2 ds\right)+1\right).$$

(*Proof of* (3.37)) We take the inner product of each equation with $\partial_t\xi^i$ with related i and integrate it over $\mathfrak{Q}_{t,}$ and we sum them up

$$\sum_{i=1}^{3}\int_{\mathfrak{Q}_t}|\partial_t\xi^i|^2 ds=\sum_{i=1}^{3}\int_{\mathfrak{Q}_{t,}}\partial_s\left(G\left(\partial_s\xi^i\right)\right)\partial_t\xi^i ds+\sum_{i=1}^{3}\int_{\mathfrak{Q}_t}\xi^i\partial_t\xi^i ds$$

After an integration by parts and using the boundary conditions we have the following equality

$$\sum_{i=1}^{3}\int_{\mathfrak{Q}_t}|\partial_t\xi^i|^2 ds=-\sum_{i=1}^{3}\int_{\mathfrak{Q}_t}G\left(\partial_s\xi^i\right)\partial_{st}\xi^i ds+\frac{1}{2}\sum_{i=1}^{3}\int_{\Omega}|\xi^i|^2(\mathfrak{t},\cdot)\,ds-\frac{1}{2}\sum_{i=1}^{3}\int_{\Omega}|\beta^i|^2 ds.$$

Note now that we have $G\left(\partial_s\xi^i\right)\partial_{st}\xi^i=\partial_t Q\left(\partial_s\xi^i\right)$ (cf. (2.51)), using this we obtain

$$\begin{aligned}\sum_{i=1}^{3}\int_{\mathfrak{Q}_t}|\partial_t\xi^i|^2 dsdt+\sum_{i=1}^{3}\int_{\Omega}Q\left(\partial_s\xi^i\right)(\mathfrak{t},\cdot)\,dsdt-\frac{1}{2}\sum_{i=1}^{3}\int_{\Omega}|\xi^i|^2(\mathfrak{t},\cdot)\,ds \\ =\sum_{i=1}^{3}\int_{\Omega}Q\left(\partial_s\beta^i\right)ds-\frac{1}{2}\sum_{i=1}^{3}\int_{\Omega}|\beta^i|^2 ds.\end{aligned}$$

Then we conclude that

$$\sum_{i=1}^{3}\int_{\mathfrak{Q}_t}|\partial_t\xi^i|^2 dsdt+\mathscr{E}\left(\xi(t,\cdot)\right)\le\mathscr{E}\left(\beta\right)$$

This in particular gives the decay of the energy

$$\mathscr{E}\left(\xi(\mathfrak{t},\cdot)\right)\le\mathscr{E}\left(\beta\right)<\infty \text{ for any } \mathfrak{t}\in(0,T]$$

The lower bound of $\mathscr{E}\left(\xi(\mathfrak{t},\cdot)\right)$ follows automatically by inequality (3.42). Indeed, from the definition of $\mathscr{E}\left(\xi(\mathfrak{t},\cdot)\right)$, we see that the first term is always positive using the explicit definition of $Q_\varepsilon(\cdot)$. Thus,

$$-\frac{1}{2}\sum_{i=1}^{3}e^{2t}\left(\int_{\Omega}|\beta^i|^2 ds\right)\le-\frac{1}{2}\sum_{i=1}^{3}\int_{\Omega}|\xi^i|^2 ds\le\mathscr{E}\left(\xi(\mathfrak{t},\cdot)\right)$$

On the other hand, from the equality $\partial_s G\left(\partial_s \xi^i\right) = \partial_t \xi^i - \xi^i$ we deduce that for each $i = 1,2,3$

$$\int_{\Omega_T} |\nabla G \cdot \partial_{ss}\xi^i|^2 dsdt = \int_{\Omega_T} |\partial_s G\left(\partial_s \xi^i\right)|^2 dsdt \leq 2\int_{\Omega_T} |\partial_t \xi^i|^2 dsdt + 2\int_{\Omega_T} |\xi^i|^2 dsdt$$

The last two inequalities together yield (3.37). □

Corollary 3.15. *The norm $\|\sigma^i\|_{L^1\left(0,T:L^1(\Omega)\right)}$ is bounded uniformly in ε.*

Proof. Remember the definition of σ^i in (3.27) and we rewrite (3.40) using the definition

$$\frac{1}{2}\sum_{i=1}^{3}\int_{\Omega}|\xi^i(\mathrm{t},\cdot)|^2 ds = -\sum_{i=1}^{3}\int_{\Omega_\mathrm{t}}\sigma^i dsdt + \sum_{i=1}^{3}\int_{\Omega_\mathrm{t}}|\xi^i|^2 dsdt + \frac{1}{2}\sum_{i=1}^{3}\int_{\Omega}|\beta^i(s)|^2 ds.$$

As in the proof of (3.36) we use the positiveness of $\int_{\Omega_\mathrm{t}} G\left(\partial_s \xi^i\right)\cdot \partial_s \xi^i$ and we can write

$$\frac{1}{2}\sum_{i=1}^{3}\int_{\Omega}|\xi^i(\mathrm{t},\cdot)|^2 ds + \sum_{i=1}^{3}\int_{\Omega_\mathrm{t}}\sigma^i dsdt \leq \sum_{i=1}^{3}\int_{\Omega_\mathrm{t}}|\xi^i|^2 dsdt + \frac{1}{2}\sum_{i=1}^{3}\int_{\Omega}|\beta^i(s)|^2 ds.$$

We conclude the result using (3.42). □

In the rest of this chapter, we do not give the majority of the proofs of the statements as the proofs follow the same ideas as in Section 2.5. However, we provide a proof if there is an essentially different result or proof.

Corollary 3.16. *The energy of the approximation problem $\mathscr{E}\left(\xi(t)\right)$ is bounded from below for all $t \in [0,T]$ uniformly in ε.*

Lemma 3.17. *Given $\delta > 0$, the norm $||\partial_t \xi^i_\varepsilon||_{L^\infty\left(\delta,T;L^2(\Omega)\right)}$ is bounded uniformly in ε. Furthermore, $||\partial_s \kappa^i_\varepsilon||_{L^\infty\left(\delta,T;L^2(\Omega)\right)}$ is also uniformly bounded in ε.*

Lemma 3.18. *For fixed $\delta > 0$, the product $|\kappa^i||\partial_{ss}\xi^i - \varepsilon\partial_s \xi^i|$ is bounded in $L^\infty\left(\delta,T;L^2(\Omega)\right)$ uniformly with respect to ε, $i = 1,2,3$.*

The next lemma follows the same idea as Lemma 2.15. It has some differences in the proof and it is an elaborate proof, so we prefer to write it down.

Proposition 3.19. *Given $\delta > 0$, the norm $\|\kappa^i_\varepsilon\|_{L^\infty\left(\delta,T;L^\infty(\Omega)\right)}$ is bounded uniformly in ε for each $i = 1,2,3$.*

Proof. This proof is done for one of the junction points, and it is valid for the other one too. From now on, we do not omit the subscript ε. However, in this proof we decided to swap the sub and upper-indices for the sake of convenience and readability.

Step 1. We argue by contradiction. Assume that there is a sequence $\varepsilon^n \to 0$ such that

$$||\kappa_1^{\varepsilon^n}||_{L^\infty\left(\delta,T;L^\infty(\Omega)\right)} \to +\infty.$$

Here, without loss of generality, we have chosen the generic i to be equal to 1. By the regularity of $\partial_s \kappa^{\varepsilon^n}$ and $\partial_{ss}\xi^{\varepsilon^n}$ there exists a set $\mathfrak{T}_n$ of full measure in $[\delta, T]$ such that $\kappa_i^{\varepsilon^n}(t)$ and $\xi_i^{\varepsilon^n}(t)$ are C^1-smooth in $\overline{\Omega}$ whereas $\partial_{ss}\xi_i^{\varepsilon^n}(t) \in L^2(\Omega)$ for every i and every $t \in \mathfrak{T}_n$. Furthermore, by Lemma 3.18 without loss of generality we can assume that $\partial_s \kappa_i^{\varepsilon^n}(t)$ and $|\kappa_i^{\varepsilon^n}(t,\cdot)||\partial_{ss}\eta_i^{\varepsilon^n}(t,\cdot) - \varepsilon^n \partial_s \kappa_i^{\varepsilon^n}(t,\cdot)|$ are bounded in $L^2(\Omega)$ uniformly w.r.t. n and $t \in \mathfrak{T}_n$. Let $\mathfrak{T} := \cap_{n\in\mathbb{N}} \mathfrak{T}_n$. Then there is a sequence $(t^n, s^n) \in \mathfrak{T} \times \Omega$ such that $|\kappa_1^{\varepsilon^n}(t^n, s^n)| \to +\infty$ as $n \to \infty$. Thus,

$$\kappa_1^{\varepsilon^n}\left(t^n, s\right) = \underbrace{\kappa_1^{\varepsilon^n}\left(t^n, s^n\right)}_{\to+\infty} + \underbrace{\int_{s^n}^{s} \partial_\xi \kappa_1^{\varepsilon^n}\left(t^n, \xi\right) \partial\xi}_{\leq C}$$

when $n \to \infty$. Accordingly, $|\kappa_1^{\varepsilon^n}(t^n)| \to +\infty$ uniformly in s.

Step 2. By the boundary conditions,

$$\sum_{i=1}^{3} \kappa_i^{\varepsilon^n}(t^n, 0) = 0. \tag{3.43}$$

By the previous step, $|\kappa_1^{\varepsilon^n}(t^n, 0)| \to +\infty$. Hence we have two possible scenarios: The first option is $|\kappa_2^{\varepsilon^n}(t^n, 0)| \to +\infty$ and $|\kappa_3^{\varepsilon^n}(t^n, 0)| \leq C$ as $n \to +\infty$ (up to swapping the second and the third arms). The second one is $|\kappa_2^{\varepsilon^n}(t^n, 0)| \to +\infty$ and $|\kappa_3^{\varepsilon^n}(t^n, 0)| \to +\infty$ as $n \to +\infty$.

Step 3. We start by examining the second scenario. An argument similar to the one of Step 1 shows that $|\kappa_i^{\varepsilon^n}(t^n)| \to +\infty$ uniformly in s, $i = 1, 2, 3$. Since $t^n \in \mathfrak{T}$, we know that

$$|\kappa_i^{\varepsilon^n}\left(t^n, \cdot\right)||\partial_{ss}\xi_i^{\varepsilon^n}\left(t^n, \cdot\right) - \varepsilon \partial_s \kappa_i^{\varepsilon^n}\left(t^n, \cdot\right)|$$

is uniformly bounded in $L^2(\Omega)$. Hence,

$$|\partial_{ss}\xi_i^{\varepsilon^n}\left(t^n, \cdot\right) - \varepsilon^n \partial_s \kappa_i^{\varepsilon^n}\left(t^n, \cdot\right)| \to 0$$

in $L^2(\Omega)$ as $n \to +\infty$. On the other hand, $\partial_s \kappa_i^{\varepsilon^n}(t^n)$ is uniformly bounded in $L^2(\Omega)$, whence

$$|\varepsilon^n \partial_s \kappa_i^{\varepsilon^n}\left(t^n\right)| \to 0$$

in $L^2(\Omega)$. We conclude that $|\partial_{ss}\xi_i^{\varepsilon^n}(t^n, \cdot)| \to 0$ in $L^2(\Omega)$ as $n \to +\infty$. By Remark 2.4, $|\kappa_i^{\varepsilon^n}(t^n, \cdot)| \geq 1$ implies $|\partial_s \xi_i^{\varepsilon^n}(t^n, \cdot)| \geq 1$ (assuming n to be large enough).

Step 4. Let us prove that the points $p_i^n := \xi_i^{\varepsilon^n}(t^n, 0) + \partial_s \xi_i^{\varepsilon^n}(t^n, 0)$ are close to the junction $\xi_i^{\varepsilon^n}(t^n, 1)$ for large n. Indeed

$$|\partial_s \xi_i^{\varepsilon^n}\left(t^n, 0\right) - \partial_s \xi_i^{\varepsilon^n}\left(t^n, \varsigma\right)| = \left| \int_0^{\varsigma} \partial_{ss}\xi_i^{\varepsilon^n}\left(t^n\right) ds \right|$$
$$\leq \int_0^{\varsigma} |\partial_{ss}\xi_i^{\varepsilon^n}\left(t^n\right)| ds \leq \sqrt{\int_0^1 |\partial_{ss}\xi_i^{\varepsilon^n}\left(t^n\right)|^2 ds} \to 0 \text{ uniformly in } \varsigma \text{ as } n \to \infty.$$

Hence,

$$\begin{aligned}|p_i^n - \xi_i^{\varepsilon^n}\left(t^n,1\right)| &= |\xi_i^{\varepsilon^n}\left(t^n,0\right) - \xi_i^{\varepsilon^n}\left(t^n,1\right) + \partial_s \xi_i^{\varepsilon^n}\left(t^n,0\right)| \\ &= \left| \int_0^1 \partial_s \xi_i^{\varepsilon^n}\left(t^n,0\right) - \partial_s \xi_i^{\varepsilon^n}\left(t^n,s\right) ds \right| \to 0 \text{ as } n \to \infty.\end{aligned}$$

Since the radius of the smallest enclosing ball is a continuous function of the points of a set, it follows that the radius of the smallest enclosing ball of the three points p_i^n is small for n sufficiently large. Since the junction point $\eta_i^{\varepsilon^n}(t^n,0)$ does not depend on i, the radius of the smallest enclosing ball of the three points $\tilde{p}_i^n := \partial_s \xi_i^{\varepsilon^n}(t^n,0)$ is the same as the previous one (so again small). By Step 3, $|\tilde{p}_i^n| \geq 1$. Moreover, since $\sum_{i=1}^3 \kappa_i^{\varepsilon^n}(t^n,0) = 0$ and $\tilde{p}_i^n = F_{\varepsilon^n}(\kappa_i^{\varepsilon^n}(t^n,0))$, we conclude that the convex hull of $\{\tilde{p}_i^n\}$ contains the origin. We arrive at a contradiction because by Lemma 2.14 the radius of the smallest enclosing ball of $\{\tilde{p}_i^n\}$ must be greater than or equal to 1.

Step 5. We now study the first scenario. Define p_i^n and $\tilde{p}_i^n$ as in Step 4. The plan is to look at the angle θ_n between the position vectors of $\tilde{p}_1^n$ and $\tilde{p}_2^n$ and to obtain a contradiction from that.

We first show that θ_n must tend to 0. Indeed, mimicking the arguments of Steps 3 and 4, we can prove that for $i = 1,2$ one has $|\partial_s \xi_i^{\varepsilon^n}(t^n,\cdot)| \geq 1$ with n large enough, $|\partial_{ss} \xi_i^{\varepsilon^n}(t^n,\cdot)| \to 0$ in $L^2(\Omega)$ and

$$|p_i^n - \xi_i^{\varepsilon^n}\left(t^n,1\right))| \to 0 \text{ as } n \to \infty.$$

Hence,

$$|\tilde{p}_1^n - \tilde{p}_2^n| = |p_1^n - p_2^n| \to |\xi_1^{\varepsilon^n}\left(t^n,1\right) - \xi_2^{\varepsilon^n}\left(t^n,1\right))| = 0.$$

Since we have $|\tilde{p}_1^n| \geq 1, |\tilde{p}_2^n| \geq 1$, the angle θ_n should converge to 0.

To obtain a contradiction, it remains to observe that θ_n cannot tend to 0. Indeed, taking the scalar product of relation (3.43) with

$$\frac{1}{|\partial_s \xi_1^{\varepsilon^n}(t^n,0)||\kappa_2^{\varepsilon^n}(t^n,0)|} \partial_s \xi_1^{\varepsilon^n}\left(t^n,0\right)$$

we get

$$\frac{\kappa_1^{\varepsilon^n}(t^n,0)}{|\kappa_2^{\varepsilon^n}(t^n,0)|} \cdot \frac{\partial_s \xi_1^{\varepsilon^n}(t^n,0)}{|\partial_s \xi_1^{\varepsilon^n}(t^n,0)|} + \frac{\kappa_2^{\varepsilon^n}(t^n,0)}{|\kappa_2^{\varepsilon^n}(t^n,0)|} \cdot \frac{\partial_s \xi_1^{\varepsilon^n}(t^n,0)}{|\partial_s \xi_1^{\varepsilon^n}(t^n,0)|} + \frac{\kappa_3^{\varepsilon^n}(t^n,0)}{|\kappa_2^{\varepsilon^n}(t^n,0)|} \cdot \frac{\partial_s \xi_1^{\varepsilon^n}(t^n,0)}{|\partial_s \xi_1^{\varepsilon^n}(t^n,0)|} = 0. \quad (3.44)$$

The first term is equal to $\frac{\sigma_1^{\varepsilon^n}}{|\kappa_2^{\varepsilon^n}(t^n,0)||\partial_s \xi_1^{\varepsilon^n}(t^n,0)|} \geq 0$ by (3.27) and (3.28). The third term converges to 0. Accordingly, the second term, which is equal to $\cos\theta_n$, cannot tend to 1. □

Corollary 3.20. *Given $\delta > 0$, the norm $||\xi_\varepsilon^i||_{L^\infty\left(\delta,T;W^{1,\infty}(\Omega)\right)}$ is uniformly bounded with respect to ε.*

Corollary 3.21. *The norm $\|\kappa_\varepsilon^i\|_{L^1\left(0,T;H^1(\Omega)\right)}$ is bounded uniformly in ε.*

Proof. This proof is independent of the choice of i, so we write it as generic i upper-index for all.

Firstly, we show that $|\kappa^i|$ is bounded in $L^1\left(0,T;L^1(\Omega)\right)$ uniformly in ε. Indeed, since $F_\varepsilon\left(\kappa^i\right) = \partial_s \xi^i$ we have $|\kappa^i| \leq 1$ where $|\partial_s \xi^i| \leq 1$. But at the same time we have $|\kappa^i| \leq \frac{|\sigma^i|}{|\partial_s \xi^i|} \leq |\sigma^i|$ for $|\partial_s \xi^i| \geq 1$. These imply that $|\kappa^i|$ is in the same space as $|\sigma^i|$ and by Corollary 3.15 we conclude.

Using a Poincaré type inequality for $\|\kappa^i\|_{L^1\left(0,T;L^1(\Omega)\right)} \leq C$ and that $\|\partial_s \kappa^i\|$ is uniformly bounded in $L^2\left(0,T;L^2(\Omega)\right)$ we conclude that $\|\kappa^i\|_{L^1\left(0,T;H^1(\Omega)\right)}$ is also bounded. □

Lemma 3.22. *The norms* $\|\sigma_\varepsilon^i\|_{L^1\left(0,T;W^{1,1}(\Omega)\right)}$ *and* $\|\sigma_\varepsilon^i\|_{L^\infty\left(\delta,T;H^1(\Omega)\right)}$ *are bounded uniformly in* ε *for each* $i = 1,2,3$.

Proof. This proof is independent of the choice of upper-index. Firstly, we start showing the boundness of $\|\sigma_\varepsilon^i\|_{L^1\left(0,T;W^{1,1}(\Omega)\right)}$. Using the equality $\sigma^i = \partial_s \xi^i \cdot \kappa^i$ with a direct computation we have $\partial_s \sigma^i = \partial_s \kappa^i \cdot \partial_s \xi^i + \kappa^i \cdot \partial_{ss} \xi^i$. We estimate these terms separately. Firstly, by Proposition 3.14 we have the uniform bounds for $\partial_s \kappa^i = \partial_s G\left(\partial_s \xi^i\right)$ and $\partial_s \xi^i$ in $L^2(\mathfrak{Q}_T)$. This gives us the uniform boundedness of $\partial_s \kappa^i \cdot \partial_s \xi^i$ in $L^1(\mathfrak{Q}_T)$.

Now, we start to estimate $\kappa^i \cdot \partial_{ss} \xi^i$. Observe that the explicit expression of λ_ε in (2.36) for $\tau \in \mathbb{R}^d$ gives us

$$\lambda_\varepsilon(\tau) = \frac{\sqrt{\varepsilon + |G_\varepsilon(\tau)|^2}}{\varepsilon\sqrt{\varepsilon + |G_\varepsilon(\tau)|^2} + 1} \geq \frac{|G_\varepsilon(\tau)|}{\varepsilon |G_\varepsilon(\tau)| + 1} \tag{3.45}$$

Thus

$$\begin{aligned} \left|G_\varepsilon\left(\partial_s \xi^i\right)\right| |\partial_{ss} \xi^i| &\leq \left(\varepsilon \left|G_\varepsilon\left(\partial_s \xi^i\right)\right| + 1\right) |\lambda_\varepsilon \partial_{ss} \xi^i| \\ &\leq \left(\varepsilon \left|G_\varepsilon\left(\partial_s \xi^i\right)\right| + 1\right) \left|\nabla G_\varepsilon\left(\partial_s \xi^i\right) \partial_{ss} \xi^i\right| \end{aligned}$$

By the explicit definition of F_ε this gives

$$\left|G_\varepsilon\left(\partial_s \xi^i\right) \cdot \partial_{ss} \xi^i\right| \leq \left(|\partial_s \xi^i| + 1\right) \left|\nabla G_\varepsilon\left(\partial_s \xi^i\right) \partial_{ss} \xi^i\right|. \tag{3.46}$$

By the boundedness of $|\partial_s \xi^i| \in L^2\left(0,T;L^2(\Omega)\right)$ and inequality (3.37) we conclude that $\kappa^i \cdot \partial_{ss} \xi^i = G_\varepsilon\left(\partial_s \xi^i\right) \cdot \partial_{ss} \xi^i$ are uniformly bounded in $L^1(\mathfrak{Q}_T)$, and then using Corollary 3.15 we finally obtain $\sigma^i \in L^1\left(0,T;W^{1,1}(\Omega)\right)$.

The bound for $\|\sigma_\varepsilon^i\|_{L^\infty\left(\delta,T;L^2(\Omega)\right)}$ follows immediately from the equality $\sigma^i = \partial_s \xi^i \cdot \kappa^i$. Indeed, Proposition 3.19 and Proposition 3.14 imply the required result. The H^1 bound follows as in Section 2.5. □

3.6 Existence of generalized solutions

In this section, we relate the results obtained for approximation problem to original gradient flow problem. Now, we are at the position to define generalized solutions to the original problem (3.2) and to prove their existence.

Definition 3.23. *Given initial data* $\beta^i \in W^{1,\infty}(\Omega)^d$ *as in Remark 3.3, we call a pair* $\left(\xi^i, \sigma^i\right)$ *a generalized solution to* (3.2) *in* $\mathfrak{Q}_\infty$ *if*

(i) - $\xi^i \in L^\infty_{loc}\left((0,\infty);W^{1,\infty}(\Omega)\right)^d \cap C_{loc}\left((0,\infty);C\left(\overline{\Omega}\right)\right)^d \cap AC^2_{loc}\left([0,\infty);L^2(\Omega)\right)^d$,

- $\partial_t \xi^i \in L^\infty_{loc}\left((0,\infty);L^2(\Omega)\right)^d \cap L^2_{loc}\left([0,\infty);L^2(\Omega)\right)^d$,
- $\partial_s \xi^i \in L^2_{loc}\left([0,\infty);L^2(\Omega)\right)$
- $\sigma^i \in L^\infty\left((0,\infty);AC^2(\Omega)\right)$, $\partial_s \sigma^i \in \mathscr{M}_{loc}(\mathfrak{Q}_\infty)$
- $\sigma^i \partial_s \xi^i \in L^\infty_{loc}\left((0,\infty);AC^2(\Omega)\right) \cap \mathscr{M}_{loc}\left([0,\infty);AC^2(\Omega)\right)$.

(ii) Each pair $\left(\xi^i,\sigma^i\right)$ *satisfies for a.e.* $(t,s) \in \mathfrak{Q}_\infty$

$$\partial_t \xi^i(t,s) = \partial_s\left(\sigma^i(t,s)\,\partial_s \xi^i(t,s)\right) + \xi^i, \tag{3.47}$$

$$\sigma^i(t,s)\left(|\partial_s \xi^i(t,s)|^2 - 1\right) = 0, \tag{3.48}$$

$$|\partial_s \xi^i(t,s)| \le 1, \tag{3.49}$$

as well as the initial conditions

$$\xi^i(0,s) = \beta^i(s)$$

and the boundary conditions

$$\begin{aligned}
&\xi^1(t,0) = \xi^2(t,0) = \xi^3(t,0),\\
&\xi^1(t,1) = \xi^2(t,1) = \xi^3(t,1),\\
&\sum_{i=1}^{3} \sigma^i(t,0)\,\partial_s \xi^i(t,0) = 0,\\
&\sum_{i=1}^{3} \sigma^i(t,1)\,\partial_s \xi^i(t,1) = 0.
\end{aligned}$$

(iii) The solutions ξ^i *satisfy the energy dissipation inequality*

$$\sum_{i=1}^{3} \int_\Omega |\partial_t \xi^i(t,s)|^2 \, ds \le \sum_{i=1}^{3} \int_\Omega \xi^i \cdot \partial_t \xi^i(t,s) \, ds \tag{3.50}$$

for a.e. $t \in (0,\infty)$.

Remark 3.4. Note that (3.48), (3.49) is a minor relaxation of the non-convex constraint

$$|\partial_s \xi^i(t,s)| = 1. \tag{3.51}$$

However, this is not a banal convexification of the constraint since (3.48) is still not convex. The new constraints (3.48), (3.49) naturally appear from the (ξ,σ,κ)-formulation (2.33). Moreover, if a generalized (in the sense of Definition 3.23) solution (η,σ) is C^2-smooth, then it automatically satisfies the strong constraint (3.51). We claim that any generalized solution $\left(\xi^i,\sigma^i\right)$ with $\xi^i \in C^1\left(\bar{\mathfrak{Q}}_\infty\right) \cap C^2(\mathfrak{Q}_\infty)$ and $|\partial_s \beta^i| = 1$ solves the gradient flow system. The reasoning is a little bit more refined than in Remark 2.5. It is enough to show that the open set $U := \{(t,s) \in \mathfrak{Q}_\infty : |\partial_s \xi^i| < 1\}$ is empty. We argue by contradiction. Suppose it is not empty, which implies $\sigma^i = 0$ a.e. in U due to relaxed constraint (3.48). This gives us immediately that $\partial_t \xi^i = \xi^i$ in U, whence $\partial_t \left(|\partial_s \xi^i|^2\right) = 2|\partial_s \xi^i|^2$. For each $(t_0, s_0) \in U$, let $t_1 = \inf\{t \geq 0 : (t, t_0) \times \{s_0\} \subset U\}$. If $t_1 = 0$ then

$$|\partial_s \xi^i(t_1, s_0)| = 1 \tag{3.52}$$

due to our assumption about the initial data β^i, and if $t_1 > 0$ then (3.52) also holds by continuity of $\partial_s \xi^i$. From $\partial_t \left(|\partial_s \xi^i|^2\right) \geq 0$ in U, we reach to following contradiction

$$|\partial_s \xi^i(t_0, s_0)|^2 \geq |\partial_s \xi^i(t_1, s_0)|^2 = 1$$

and this ends the proof of the claim.

Finally, we emphasize that (3.50) is not direct consequence of (3.47), (3.48) and (3.49).

We remind here again that as in [61], our generalized solutions are, generally speaking, not unique. Yet this has nothing to do with the fact that we slightly relaxed the constraint (3.51). As a matter of fact, non-uniqueness can persist even if the strong constraint (3.51) is imposed, cf. [61, Remark 6.5].

For convenience, we first pass to the limit on finite time intervals. In what follows, we use again the shortcut $\mathfrak{Q}_T^* = (\delta, T) \times \Omega$.

Proposition 3.24. *Fix $T > 0$ and a small $\delta > 0$. Let ξ_ε be a solution to* (3.26) *in $\mathfrak{Q}_T$ as constructed in Section 2.4. Let $\left(\kappa^i, \sigma^i\right)$ be defined as in* (2.39)*. Then (up to selecting a subsequence ε^n) there exists a limit $(\xi^i, \sigma^i, \kappa^i)$ such that as $\varepsilon \to 0$ we have*

$\xi_\varepsilon^i \to \xi^i$ *weakly-* in* $L^\infty\left(\delta, T; W^{1,\infty}(\Omega)\right)^d$, *strongly in* $C\left(\overline{\mathfrak{Q}_T^*}\right)^d$ *and weakly in* $L^2(\mathfrak{Q}_T)^d$,

$\partial_t \xi_\varepsilon^i \to \partial_t \xi^i$ *weakly-* in* $L^\infty\left(\delta, T; L^2(\Omega)\right)^d$ *and weakly in* $L^2(\mathfrak{Q}_T)^d$,

$\partial_s \xi_\varepsilon^i \to \partial_s \xi^i$ *weakly in* $L^2(\mathfrak{Q}_T)^d$

$\sigma_\varepsilon^i \to \sigma^i$ *weakly-* in* $L^\infty\left(\delta, T; H^1(\Omega)\right)$,

$\partial_s \sigma_\varepsilon^i \to \partial_s \sigma^i$ *weakly-* in* $\mathscr{M}(\mathfrak{Q}_T)$,

$\kappa_\varepsilon^i \to \kappa^i$ *weakly-* in* $L^\infty\left(\delta, T; H^1(\Omega)\right)$ *and in* $\mathscr{M}\left([0, T]; H^1(\Omega)\right)$.

The limit satisfies the relation

$$\kappa^i = \sigma^i \partial_s \xi^i \in L^\infty\left(\delta, T; H^1(\Omega)\right)$$

and solves (3.2) *in* $\mathfrak{Q}_T^*$ *in the sense that*

$$\begin{aligned}
&\partial_t \xi^i = \partial_s \left(\sigma^i \partial_s \xi^i\right) + \xi^i \text{ a.e. in } \mathfrak{Q}_T^*,\\
&\sigma^i \left(|\partial_s \xi^i|^2 - 1\right) = 0 \text{ a.e. in } \mathfrak{Q}_T^*,\\
&\xi^1(t,0) = \xi^2(t,0) = \xi^3(t,0),\\
&\xi^1(t,1) = \xi^2(t,1) = \xi^3(t,1),\\
&\sum_{i=1}^{3} \kappa^i = 0 \text{ at } s = 0 \text{ and ats} = 1 \text{ for a.e. } t \in (\delta, T).
\end{aligned}$$

We do not give the proof of the proposition as it is similar to the proof of Proposition 2.19. One of the novelties is the presence of the spaces of measures but they do not create any trouble being dual to separable spaces of continuous functions. The validity of the initial condition $\xi^i(0,s) = \beta^i(s)$ will be discussed in next remark.

Remark 3.5 (Initial conditions). By the Aubin-Lions-Simon theorem, the embedding

$$H^1(0,T;L^2(\Omega)) \subset C([0,T];H^{-1}(\Omega))$$

is compact. Since ξ_ε^i (w.l.o.g.) converge weakly in $H^1(0,T;L^2(\Omega))$ we can pass to the limit in the initial conditions to obtain $\xi^i(0,\cdot) = \beta^i$ in $H^{-1}(\Omega)$. However, since $H^1(0,T;L^2(\Omega)) = AC^2(0,T;L^2(\Omega))$, the initial conditions actually hold in $L^2(\Omega)$.

Proposition 3.25. *Let* $\left(\xi^i, \sigma^i\right)$ *be the limiting solution obtained in Proposition 3.24. Then*

(i) $|\partial_s \xi^i(t,s)| \le 1$ *for a.e.* $(t,s) \in \mathfrak{Q}_T^*$*;*

(ii) $\sigma \ge 0$ *for a.e.* $(t,s) \in \mathfrak{Q}_T^*$*;*

(iii) (3.50) *holds for a.a.* $t \in (\delta, T)$.

We omit the proof since it follows the same lines as the proof of Proposition 2.20.

Theorem 3.26 (Global existence of generalized solutions)**.** *For every initial configuration* $\beta^i \in W^{1,\infty}(\Omega)^d$, $i = 1,2,3$, *meeting the assumptions of Remark 3.3, there exists a generalized solution to the system* (3.2) *in* $\mathfrak{Q}_\infty$. *Moreover, those solutions satisfy* $\sigma^i(t,s) \ge 0$ *for almost every* $(t,s) \in \mathfrak{Q}_\infty$.

Employing a diagonal argument and taking into account Proposition 3.25 and Remark 3.5, we can deduce Theorem 3.26 from Proposition 3.24. This proof follows the same ideas as the proof of Theorem 2.21.

Chapter 4

Inhomogeneous Whips

4.1 Introduction to inhomogeneous inextensible strings

In this chapter we study the equations of motion of a single inextensible *inhomogeneous* string, in other words, we take $n = 1$ in system (1.1) but allow for variable $\rho(s)$. We derive uniform energy estimates for the approximation problem and show that the limiting functions are generalized solutions to the overdamped inhomogeneous whip equation (4.4). At the end of this chapter, we show exponential decay of the energy. The long time behaviour of solutions is the highlight of this chapter. In this chapter, for the global existence of approximate solutions we used a simpler technique in comparison to the previous chapters. The reason is explained in this chapter. Due to the complexity of the previous models we did not have any results on long time behaviour, but in this chapter we manage to derive exponential decay of the energy and long time behaviour results. Vorotnikov and Shi obtained similar results in [61] for homogeneous whips. In this chapter, our results are an extension of their results to inhomogeneous whips. Furthermore, to the best of our knowledge this is the very first work that does mathematical analysis of inextensible strings that are inhomogeneous.

For an unspecified external force $\psi = \psi\left(s, \eta, \partial_s \eta\right)$, and inhomogeneity function $\rho = \rho(s)$ a general system of equations of motion of inextensible strings reads as following:

$$\begin{cases} \rho \partial_{tt} \eta = \partial_s \left(\sigma \partial_s \eta\right) + \psi, \\ |\partial_s \eta| = 1, \\ \eta\left(0, s\right) = \alpha\left(s\right),\ \partial_t \eta\left(0, s\right) = \beta\left(s\right). \end{cases}$$

Let us directly pass to our specific system of this chapter: we study the equation of motion of inhomogeneous whips under the effect of gravitational force. Inhomogeneous whip means that the material that the string is made of is not a constant function, it is a function of s, and it will be denoted by $\rho\left(s\right)$. Physically, $\rho(s)$ is the density of the string which does not change with time because the string is inextensible. Throughout this chapter even if we do not particularly mention that a whip is inextensible or inhomogeneous, it is considered to be inhomogeneous and inextensible.

The full dynamical system of motion of inhomogeneous inextensible string under the force of constant gravity (that will be derived below) reads as

$$\begin{cases} \rho \partial_{tt} \eta = \partial_s (\sigma \partial_s \eta) + \rho g, \\ |\partial_s \eta| = 1, \end{cases} \tag{4.1}$$

here $s \in [0,1]$ is the arc length parameter and $t \in \mathbb{R}_+$ is the time, $\rho = \rho(s)$ comes from the inhomogeneity of the string, $\sigma = \sigma(t,s)$ is the tension, it is a Lagrange multiplier as in the previous chapters, and $\eta = \eta(t,s)$ is the displacement. Due to the inextensibility of the string, we still have $|\partial_s \eta| = 1$. We impose the whip boundary conditions:

$$\eta(t,1) = 0 \text{ and } \sigma(t,0) = 0. \tag{4.2}$$

The potential energy of the string is

$$\mathscr{E}(\eta) := \int_0^1 (-\rho g) \cdot \eta. \tag{4.3}$$

Throughout this chapter, we assume that

$$0 \le \rho(s) \le 1, \ \ \forall s \in [0,1].$$

Physically this makes sense as the density $\rho(s)$ of a material should always be a bounded nonnegative function. Here we allow that ρ can be 0 for some $s \in [0,1]$ or even take one of the two values 0 and 1 everywhere. In particular, ρ is not assumed to be continuous in s. This is inspired by the Muskat problem (imagine a whip made of a porous material).

The system (4.1) is related to the following gradient flow (the calculations that justify it can be found in the next section):

$$\begin{cases} \partial_t \eta = \partial_s (\sigma \partial_s \eta) + \rho g, \\ |\partial_s \eta| = 1, \\ \eta(t,1) = 0 \text{ and } \sigma(t,0) = 0, \\ \eta(0,s) = \alpha(s). \end{cases} \tag{4.4}$$

Physically speaking, when (4.1) is overdamped by a heavily dense environment, we obtain (4.4). Note that we lowered the order of the system and we remain just with one initial datum as explained in Remark 2.1. The new system can be viewed as the gradient flow on $\mathscr{A} := \{\eta \in H^2\left(0,1;\mathbb{R}^d\right) : \eta(1) = 0 \text{ and } |\partial_s \eta| = 1, \forall s \in [0,1]\}$ driven by the potential energy (4.3):

$$\frac{d}{dt}\eta = -\nabla_{\mathscr{A}} \mathscr{E}(\eta), \tag{4.5}$$

where the space of arcs $\mathscr{A}$ is viewed as an infinite-dimensional Riemannian submanifold of $L^2\left(0,1;\mathbb{R}^d\right)$, similarly to related observations in the previous chapters. We assume that $|g| = 1$, we remind that this assumption is only for simplicity.

As we wrote in the first chapter that Muskat equation is a system where you have inhomogeneity of an incompressible fluid [12, 16–18, 64]. Here, the inhomogeneity of the string represents a similar feature. Remember that we discussed the physical meanings of each equation in Muskat system in Chapter 1. In system (4.4), the first equation is reminiscent to the Darcy law in the Muskat equation, where the velocity is the projection of ρg towards admissible velocities and the second equation is like the incompressibility of a fluid.

In comparison with the previous two chapters, in this chapter we use the theory developed by H. Amann in [1] and Ladyzenskaja, Solonikov and Ural'ceva in [34]. The reason is that as we have stressed out before, the presence of uncommon boundary conditions in the previous two systems. Here, the boundary condition (4.2) is more standard. The result in [1] is used for the existence of a unique smooth solution for the approximation system and a technique from [34] is used to prove a weak type maximum principle.

4.2 Derivation of full dynamical system and of the gradient flow

In this section, we show that system (4.1),(4.2) can be viewed as a manifestation of the celebrated physical principle of least action [5, 23]. This section is only intended to show the derivation of main system (4.1),(4.2) and to obtain the gradient flow (4.4). It has the same spirit and ideas as Section 2.2. Our motivation of doing this again for inhomogeneous inextensible strings with whip boundary conditions is that it is not written explicitly anywhere in literature. In this section we are formal and assume that ρ and other functions are sufficiently regular.

We define the *action functional* $S(\eta)$ as the time integral of the difference between the total kinetic energy $K(t) := \int_0^1 \rho \frac{1}{2}|\partial_t \eta|^2 ds$ and the total potential energy $P(t) := \int_0^1 -g\cdot\rho\eta ds$

$$S(\eta) = \int_0^T K(t) - P(t) dt = \int_{\Omega_T} \left(\frac{1}{2}\rho|\partial_t\eta|^2 + g\cdot\rho\eta\right) dsdt. \tag{4.6}$$

Consider the following set of inextensible strings with whip boundary conditions and with fixed initial and final configurations:

$$\mathfrak{W} := \{\eta \in C^1\left(\bar{\Omega};\mathbb{R}^d\right) : |\partial_s\eta|^2 = 1, \eta(T,s) = \eta_T(s), \tag{4.7}$$

$$\eta(t,1) = 0,\ \eta(0,s) = \alpha\}, \tag{4.8}$$

and let us look for minimizers of the functional S within the constraint set $\mathfrak{W}$. We claim that for each local constrained minimizer η there is a scalar function σ such that the pair (η,σ) is a solution to (4.1), (4.2). Indeed, take any local minimizer η. Let ε be a positive small parameter. Let $h = h(\varepsilon)$ be arbitrary element of $C^1\left(\bar{\Omega},\mathbb{R}^d\right)$, satisfying the following conditions

$$h(t,1) = 0,\ h(0,s) = 0, h(T,s) = 0, \tag{4.9}$$

$$2\partial_s h\cdot\partial_s\eta + \varepsilon|\partial_s h|^2 = 0. \tag{4.10}$$

We claim that

$$\eta + \varepsilon h \in \mathfrak{W}. \tag{4.11}$$

By the construction of h, we only show that $|\partial_s(\eta+\varepsilon h)|^2=1$ as it is the only nonobvious condition. Indeed,

$$|\partial_s(\eta+\varepsilon h)|^2=|\partial_s\eta|^2+2\varepsilon\partial_s\eta\cdot\partial_s h+\varepsilon^2|\partial_s h|=1$$

due to (4.10). Since η is a constrained minimizer, we have the following inequality

$$\int_{\Omega_T}\left(\rho\frac{1}{2}|\partial_t\eta+\varepsilon\partial_t h|^2+g\cdot\rho\,(\eta+\varepsilon h)\right)dsdt \geq\int_{\Omega_T}\left(\rho\frac{1}{2}|\partial_t\eta|^2+g\cdot\rho\eta\right)dsdt. \tag{4.12}$$

Dividing by ε, we can recast this in the form

$$\int_{\Omega_T}\left(\rho\partial_t\eta\cdot\partial_t h+\rho\frac{1}{2}\varepsilon|\partial_t h|^2+g\cdot\rho\eta\right)\,dsdt\geq 0. \tag{4.13}$$

Letting $\varepsilon\to 0$ we get

$$\int_{\Omega_T}\rho\left(\partial_t\eta\cdot\partial_t h+g\cdot h\right)dsdt\geq 0. \tag{4.14}$$

Observe that the condition (4.10) as $\varepsilon\to 0$ becomes

$$\partial_s h\cdot\partial_s\eta=0. \tag{4.15}$$

The possibility of replacing h and $-h$ in (4.14) without violating the constraints (4.9) and (4.15) allows us to have the equality in (4.14):

$$\int_{\Omega_T}\rho\left(\partial_t\eta\cdot\partial_t h+g\cdot h\right)dsdt=0. \tag{4.16}$$

Now, we apply integration by parts to the equality (4.16) in order to obtain

$$\int_{\Omega_T}\rho\left(\partial_{tt}\eta-g\right)\cdot h\,dsdt=0 \tag{4.17}$$

for all h satisfying (4.9),(4.15). Denote

$$Z(t,s):=\int_0^s\rho\left(\zeta\right)\left(\partial_{tt}\eta\left(t,\zeta\right)-g\right)d\zeta.$$

We rewrite (4.17)

$$\int_{\Omega_T}\partial_s Z\cdot hdsdt=0$$

and apply integration by parts

$$\int_{\Omega_T}Z\cdot\partial_s h\,dsdt=0 \tag{4.18}$$

for all h satisfying (4.9),(4.15). Finally, by Hilbertian duality argument, it is possible to deduce from (4.18) that there exists a measurable scalar function $\sigma(t,s)$ such that $Z=\partial_s\eta\sigma$. Notice that $Z(t,0)=0$ which automatically allows us to secure the other boundary condition $\sigma(t,0)=0$. Hence, we derived the system (4.1),(4.2).

In the rest of this section, we derive the gradient flow (4.4) from (4.1).

The gradient flow (4.4) is a model of motion of an inextensible string with whip boundary conditions, which is overdamped by a heavily dense environment. The motion of the inextensible string is subject to a frictional force $f_d = c\partial_t\eta$ (here c is the damping coefficient on the string, it is important to notice that the amount of the friction is independent of the material of the string., i.e. of ρ) and a gravitational force f_g. Thus it is governed by the system

$$\begin{cases} \rho(s)\,\partial_{tt}\eta(t,s) = \partial_s\left(\zeta(t,s)\,\partial_s\eta(t,s)\right) + f_g - f_d \\ |\partial_s\eta(t,s)| = 1. \end{cases}$$

Assume that the gravity is of the same order as the damping, that is, $f_g = \rho c g$ for some constant vector g (the gravitational force acts on the mass and not on the volume, so it depends on ρ as in (4.1)). We divide the equations by c, and letting $\sigma = \zeta/c$ and $c \to \infty$ we formally deduce

$$\begin{cases} \partial_t\eta(t,s) = \partial_s\left(\sigma(t,s)\,\partial_s\eta(t,s)\right) + \rho g \\ |\partial_s\eta(t,s)| = 1. \end{cases}$$

We complement the system with the initial/boundary conditions

$$\begin{cases} \sigma(t,0) = 0, \\ \eta(t,1) = 0, \\ \eta(0,s) = \alpha(s). \end{cases}$$

4.3 Approximation problem and energy bounds

In this section, we introduce an approximation to system (4.4) by L^2-gradient flows.

We will use the same functions F_ε and G_ε as in Section 2.3.

Let $\kappa = \sigma\partial_s\eta$, and then the system (4.4) can be written as

$$\begin{cases} \partial_t\eta = \partial_s\kappa + \rho g \\ \kappa = \sigma\partial_s\eta \\ \sigma = \kappa\cdot\partial_s\eta, \end{cases} \tag{4.19}$$

here $(\eta,\sigma,\kappa) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ solves the new system, and initial/boundary conditions are the same as (4.4). The heuristics beyond this system is similar to the one in Section 2.3. Fix $\varepsilon \in \left(0, 1/16\right)$. It is convenient to work with the following approximation problem

$$\partial_t\eta_\varepsilon = \partial_s\left(G_\varepsilon(\partial_s\eta_\varepsilon)\right) + \rho_\varepsilon g \text{ in } (0,\infty)\times(0,1) \tag{4.20}$$

with initial/boundary conditions

$$\begin{aligned}\eta_\varepsilon(t,1) = 0, \quad \partial_s \eta_\varepsilon(t,0) = 0 \\ \eta_\varepsilon(0,s) = \alpha_\varepsilon(s).\end{aligned} \tag{4.21}$$

For technical reasons, the new initial data $\alpha_\varepsilon(s)$ is chosen to approximate the possibly nonsmooth given initial datum α in $C[0,1]$. Here, $\alpha_\varepsilon(s)$ and $\rho_\varepsilon(s)$ are smooth functions in $[0,1]$. Using the fact that ∇G_ε is smooth in its argument and positive definite, the system (4.20),(4.21) is well-posed. Given any initial data $\alpha_\varepsilon(s)$ satisfying the above boundary conditions, the existence of a unique smooth solution $\eta_\varepsilon : C^\infty\left((0,T]\times[0,1];\mathbb{R}^d\right) \cap C\left([0,T]\times[0,1];\mathbb{R}^d\right)$ to the system (4.20),(4.21) follows from Amann's theory [1].

In comparison with the original equation (4.19), we define

$$\begin{aligned}\kappa_\varepsilon &:= G_\varepsilon\left(\partial_s \eta_\varepsilon\right), \\ \sigma_\varepsilon &:= G_\varepsilon\left(\partial_s \eta_\varepsilon\right)\cdot \partial_s \eta_\varepsilon.\end{aligned} \tag{4.22}$$

We consider a new energy which is associated with the approximation system (4.20), (4.21):

$$\begin{cases} \mathscr{E}_\varepsilon(\eta) = \int_0^1 Q_\varepsilon\left(\partial_s \eta_\varepsilon\right) ds + \int_0^1 \left(-\rho_\varepsilon g\right)\cdot \eta\, ds \\ \text{for } \eta \in AC^2(\Omega;\mathbb{R}^{3d}) \text{ satisfying } \ \eta(0) = 0; \\ +\infty \text{ for any } \eta \in L^2(\Omega;\mathbb{R}^d) \text{ except those above,} \end{cases} \tag{4.23}$$

where $Q_\varepsilon : \mathbb{R}^d \to \mathbb{R}$ is given as before

$$Q_\varepsilon(u) = \varepsilon\left(\frac{|G_\varepsilon(u)|^2}{2} - \frac{1}{\sqrt{\varepsilon + |G_\varepsilon(u)|^2}}\right) + \sqrt{\varepsilon}. \tag{4.24}$$

In this way, (4.20) can be interpreted as a gradient flow with respect to the flat Hilbertian structure of L^2, which is driven by this functional

$$\frac{d}{dt}\eta_\varepsilon = -\nabla_{L^2}\mathscr{E}_\varepsilon(\eta_\varepsilon),\ \eta_\varepsilon(0,s) = \alpha_\varepsilon(s).$$

We derive some energy inequalities with uniform (in ε) bounds for the solutions η_ε in terms of the initial datum. In sequel, the generic constant C will be always independent of ε, also we will drop the dependence of ε and only write $\eta = \eta_\varepsilon, G = G_\varepsilon$, etc. We start by stating a remark about initial data and density function ρ.

Remark 4.1. Given any initial datum $\alpha \in W^{1,\infty}(0,1)^d$ with $\alpha(1) = 0$ and $|\partial_s \alpha(s)| \le 1$ to the original problem (4.1)-(4.2), it is possible to find smooth α_ε satisfying $\alpha_\varepsilon(1) = 0$ and $\partial_s \alpha_\varepsilon(0) = 0$ such that $\alpha_\varepsilon \to \alpha$ uniformly and $|\partial_s \alpha_\varepsilon| \le 1$, see [61]. For such approximations the initial energies $\mathscr{E}(\alpha_\varepsilon)$ are uniformly bounded in $\varepsilon \in (0,1/16)$. To see this, we define $\tilde{F}(r) := \varepsilon r + \dfrac{r}{\sqrt{\varepsilon + r^2}}$, and $\tilde{F}\left(\frac{1}{\sqrt{\varepsilon}}\right) > 1$ and using the monotonicity of $\tilde{F}$, we have $|G_\varepsilon(\tau)| \le \frac{1}{\sqrt{\varepsilon}}$ if $|\tau| \le 1$. Then, from (4.23) we see that $\mathscr{E}_\varepsilon(\alpha_\varepsilon) \le 2$ if $|\partial_s \alpha_\varepsilon| \le 1$.

Given any function $\rho \in L^{\infty}\left([0,1]\right)$ with $0 \leq \rho(s) \leq 1$ which is allowed to be a discontinuous function in original problem (4.1), we can find $\rho_{\varepsilon} \to \rho$ weakly* in $L^{\infty}\left([0,1]\right)$ such that $0 \leq \rho_{\varepsilon} \leq 1$ is a sequence of smooth functions.

We start from an energy estimate that is similar to the ones in the previous chapters. We however prefer to present the proof for completeness.

Proposition 4.1. *Given $\alpha_{\varepsilon}, \rho_{\varepsilon} \in C^{\infty}$ as in Remark 4.1, let η_{ε} be the solution to the approximation problem* (4.20)-(4.21) *in $\mathfrak{Q}_{\infty}$. Then for any $T \in (0,\infty)$*

$$\max_{t\in[0,T]} \int_{\Omega} |\eta_{\varepsilon}(t,\cdot)|^2 ds + \int_{\mathfrak{Q}_T} |\partial_s \eta_{\varepsilon}|^2 dsdt \leq C\left(e^T \int_{\Omega} |\alpha_{\varepsilon}|^2 ds + 1\right), \tag{4.25}$$

$$\int_{\mathfrak{Q}_T} |\partial_t \eta_{\varepsilon}| + |\nabla G_{\varepsilon}\left(\partial_s \eta\right) \cdot \partial_{ss}\eta_{\varepsilon}|^2 dsdt \leq C\left(\mathscr{E}_{\varepsilon}(\alpha_{\varepsilon}) + e^T ||\alpha_{\varepsilon}||_{L^2(\Omega)} + 1\right) \tag{4.26}$$

Here $\mathscr{E}_{\varepsilon}$, as defined in (4.23)*, is associated energy for the approximation problem.*

Proof. In this proof, as we agreed above we drop the dependence of ε for readability.

Proof of (4.25). Start by taking the inner product of the equation with η and integrate in space and time. After an integration by parts in space we obtain

$$\int_{\mathfrak{Q}_t} \partial_t \eta \cdot \eta \; dsdt = -\int_{\mathfrak{Q}_t} G\left(\partial_s \eta\right) \cdot \partial_s \eta \; dsdt + \int_{\mathfrak{Q}_t} \rho g \cdot \eta \; dsdt, \quad \forall t \in [0,T].$$

Application of the Young inequality yields

$$\begin{aligned} \frac{1}{2}\int_{\Omega} |\eta(t,s)|^2 ds &+ \int_{\mathfrak{Q}_t} G\left(\partial_s \eta\right) \cdot \partial_s \eta \; dsdt \\ &\leq \frac{1}{2}\int_{\Omega} |\alpha(s)|^2 ds + \frac{1}{2}\int_{\mathfrak{Q}_t} |\eta|^2 dsdt + \frac{1}{2}\int_{\mathfrak{Q}_t} |\rho|^2 |g|^2 dsdt. \end{aligned} \tag{4.27}$$

Using the non-negativity of the second term on left hand side of the inequality (4.27) and the Grönwall's inequality, we get

$$\int_{\Omega} |\eta(t,s)|^2 ds \leq e^t \left(\int_{\Omega} |\alpha|^2 ds + 1\right) \tag{4.28}$$

We use that $\int_0^1 |\rho|^2 ds \leq 1$. Maximizing both sides in t, we obtain the estimates for $\max_{t\in[0,T]} ||\eta(t,\cdot)||_{L^2(\Omega)}$.

Before we give the estimates for $||\partial_s \eta||_{L^2(\mathfrak{Q}_T)}$, we remind the following result

$$G\left(\partial_s \eta\right) \cdot \partial_s \eta \geq \frac{1}{\varepsilon + \sqrt{\varepsilon}} |\partial_s \eta|^2 \text{ in } \{(t,s) \in \mathfrak{Q}_t : |\partial_s \eta(t,s)| \geq 1 + \sqrt{\varepsilon}\} \tag{4.29}$$

from the proof of Proposition 3.14. Now, we will give the estimates for $||\partial_s\eta||_{L^2(\mathfrak{Q}_T)}$: Apply (4.29) to (4.27):

$$\begin{aligned}\int_{\mathfrak{Q}_t}|\partial_s\eta|^2 dsdt &\leq \left(\varepsilon+\sqrt{\varepsilon}\right)\int_{\mathfrak{Q}_t} G\left(\partial_s\eta\right)\cdot\partial_s\eta\ dsdt+\left(1+\sqrt{\varepsilon}\right)^2\int_{\mathfrak{Q}_t} dsdt\\ &\leq \frac{\varepsilon+\sqrt{\varepsilon}}{2}\int_\Omega|\alpha(s)|^2 ds+\frac{\varepsilon+\sqrt{\varepsilon}}{2}\int_{\mathfrak{Q}_t}|\eta|^2 dsdt+\left(\left(1+\sqrt{\varepsilon}\right)^2+\frac{\varepsilon+\sqrt{\varepsilon}}{2}\right)|\mathfrak{Q}_t|\end{aligned}$$

After taking the suprema over t, with (4.28) yields

$$\max_{t\in[0,T]}\int_\Omega|\eta(t,\cdot)|^2+\int_{\mathfrak{Q}_T}|\partial_s\eta|^2 dsdt\leq C\left(e^T\int_\Omega\|\alpha\|^2 ds+1\right). \tag{4.30}$$

Proof of (4.26).We start by taking the inner product of equation with $\partial_t\eta$ and integrate over $\mathfrak{Q}_t$. After integration by parts, we end up with

$$\int_{\mathfrak{Q}_t}|\partial_t\eta|^2 dsdt=-\int_{\mathfrak{Q}_t} G\left(\partial_s\eta\right)\partial_{st}\eta\ dsdt+\int_\Omega\rho g\cdot\eta(t,\cdot)ds-\int_\Omega\rho g\cdot\alpha\ ds.$$

Here we have used the boundary conditions $\partial_t\eta(t,1)=0$ and $\partial_s\eta(t,0)=0$. Now, we will use that

$$G\left(\partial_s\eta\right)\partial_{st}\eta=\partial_t Q\left(\partial_s\eta\right),$$

remember that this follows from $\partial_s\eta_\varepsilon=\varepsilon\kappa+\dfrac{\kappa}{\sqrt{\varepsilon+|\kappa|^2}}$ and the following equality

$$\kappa\cdot\partial_{st}\eta=\varepsilon\frac{d}{dt}\left(\frac{|\kappa|^2}{2}-\frac{1}{\sqrt{\varepsilon+|\kappa|^2}}\right).$$

Then we can write the following

$$\begin{aligned}\int_{\mathfrak{Q}_t}|\partial_t\eta|^2 dsdt+\int_\Omega Q\left(\partial_s\eta\right)(t,\cdot)\ ds+\int_\Omega\left(-\rho g\right)\cdot\eta(t,\cdot)\ ds&\\ =\int_\Omega Q\left(\partial_s\alpha\right)\ ds+\int_\Omega\left(-\rho g\right)\cdot\alpha\ ds,&\end{aligned}$$

or using the definition of $\mathscr{E}$, we can rewrite

$$\int_{\mathfrak{Q}_t}|\partial_t\eta|^2 dsdt+\mathscr{E}\left(\eta(t,\cdot)\right)\leq\mathscr{E}(\alpha).$$

By this, we deduce the decay of the energy

$$\mathscr{E}\left(\eta(t,\cdot)\right)\leq\mathscr{E}(\alpha)<\infty\quad\text{for any } t\in(0,T].$$

Using (4.28) and the definition of $\mathscr{E}$, we immediately have the lower bound, taking in account that initial data is bounded

$$-e^t\left(\int_\Omega|\alpha|^2 ds+1\right)\leq-\frac{1}{2}\int_\Omega|\eta(t,s)|^2 ds-\frac{1}{2}\int_\Omega|\rho|^2|g|^2 ds\leq\mathscr{E}\left(\eta(t,\cdot)\right)$$

On the other hand, from the equation $\partial_s G\left(\partial_s \eta\right) = \partial_t \eta - \rho g$ we deduce

$$
\begin{aligned}
\int_{\mathfrak{Q}_T} |\nabla G \cdot \partial_{ss}\eta|^2 dsdt = \int_{\mathfrak{Q}_T} |\partial_s G\left(\partial_s \eta\right)|^2 dsdt &\le 2\int_{\mathfrak{Q}_T} |\partial_t \eta|^2 + 2\int_{\mathfrak{Q}_T} |\rho g|^2 \\
&\le 2\int_{\mathfrak{Q}_T} |\partial_t \eta|^2 + 2|\mathfrak{Q}_t|
\end{aligned}
$$

From the last two inequalities, we have (4.26). □

Next we estimate $\sup_{\mathfrak{Q}_T} |\partial_s \eta_\varepsilon|$. For this we take the spatial derivative of the equation and let $u_\varepsilon := \partial_s \eta_\varepsilon$. Then u_ε solves the following system

$$
\begin{cases}
\partial_t u_\varepsilon = \partial_s\left(\nabla G\left(u_\varepsilon\right)\partial_s u_\varepsilon\right) + \partial_s \rho_\varepsilon g, \\
u_\varepsilon(t,0) = 0,\ \partial_s G\left(u_\varepsilon\right)(t,1) = -\rho_\varepsilon g, \\
u_\varepsilon(0,s) = \partial_s \alpha_\varepsilon(s) := \beta_\varepsilon(s).
\end{cases}
\tag{4.31}
$$

Proposition 4.2. *Let u_ε and β_ε be as above. Then there exists a positive constant $C = C(T)$ such that*

$$
\sup_{\mathfrak{Q}_T} |u_\varepsilon| \le C \sup_{\Omega} |\beta_\varepsilon|.
$$

Proof. Let k be a constant with $k \ge \max\{\sup_\Omega |\beta|, \left(1+\sqrt{\varepsilon}\right)^2\}$. We use the following sets $A_k(t) := \{s \in \Omega :\ |u|^2(t,s) > k\}$ and $\mathfrak{Q}_k := \{(t,s) \in \mathfrak{Q}_T : |u|^2(t,s) > k\}$. Denote $v^{(k)} := \left(|u|^2 - k\right)_+$ and take the inner product of the equation (4.31) with $uv^{(k)}$, integrate over $\mathfrak{Q}_T$ we obtain

$$
\begin{aligned}
\int_{\mathfrak{Q}_T} \partial_t u \cdot uv^{(k)} dsdt = &\underbrace{\int_{\mathfrak{Q}_T} \partial_s\left(\nabla G(u)\,\partial_s u\right) \cdot uv^{(k)} dsdt}_{:=A} \\
&+ \underbrace{\int_{\mathfrak{Q}_T} \partial_s(\rho g) \cdot \left(uv^{(k)}\right) dsdt}_{:=B}
\end{aligned}
\tag{4.32}
$$

Before we start to show the estimates term by term, notice that

$$
\partial_t u \cdot uv^{(k)} = \frac{1}{4}\partial_t |v^{(k)}|^2
$$

and with the choice of k (such that $v^{(k)}(0,\cdot) = 0$) we deduce

$$
\int_{\mathfrak{Q}_T} \partial_t u \cdot uv^{(k)} dsdt = \frac{1}{4}\int_\Omega |v^{(k)}|^2 (T,\cdot)\, ds.
$$

We start by using integration by parts on the first term of the right hand side of (4.32) and we have

$$
\begin{aligned}
A = &\int_0^T (-\rho g) \cdot \left(uv^{(k)}\right)(t,1)\, dt \\
&- \int_{\mathfrak{Q}_T} \nabla G(u)\,\partial_s u \cdot \partial_s uv^{(k)} - \int_{\mathfrak{Q}_T} \nabla G(u)\,\partial_s u \cdot u \partial_s v^{(k)} dsdt.
\end{aligned}
\tag{4.33}
$$

For the last two terms of the right hand side of the last equality, using $\lambda_\varepsilon(\tau) \geq 1$ if $|\tau|^2 \geq \left(1+\sqrt{\varepsilon}\right)^2$, which follows from $\varepsilon \in \left(0, 1/16\right)$ and from the expression for λ_ε in (2.35) and from the estimate

$$|G(\tau)| \geq \frac{1}{\sqrt{\varepsilon}} \text{ in } \{\tau : |\tau| \geq 1+\sqrt{\varepsilon}\},$$

we get

$$\int_{\Omega_T} \nabla G(u)\, \partial_s u \cdot \partial_s u v^{(k)} dsdt \geq \int_{\Omega_T} |\partial_s u|^2 v^{(k)} dsdt.$$

By very similar arguments, we get

$$\int_{\Omega_T} \nabla G(u)\, \partial_s u \cdot u \partial_s v^{(k)} dsdt \geq \frac{1}{2} \int_{\Omega_T} |\partial_s v^{(k)}|^2 dsdt.$$

For the first term on the right hand side of (4.33), we use the boundary condition $u(t,0) = 0$ and the fundamental theorem of calculus to write

$$\begin{aligned}
\int_0^T (-\rho g) \cdot u v^{(k)}(t,1)\, dt &= \int_{\Omega_T} \partial_s \left((-\rho g) \cdot \left(u v^{(k)} \right) \right) dsdt \\
&= \underbrace{\int_{\Omega_T} (-\partial_s \rho g) \cdot \left(u v^{(k)} \right)}_{:=D} - \int_{\Omega_T} (-\rho g) \cdot \left(\partial_s u v^{(k)} \right) + \int_{\Omega_T} (-\rho g) \cdot \left(u \partial_s v^{(k)} \right)
\end{aligned} \tag{4.34}$$

Let us start by rewriting D, here we use integration by parts and boundary conditions

$$\begin{aligned}
-D &= -\int_{\Omega_T} (\rho g) \cdot \partial_s \left(u v^{(k)} \right) dsdt + \int_0^T (\rho(1) g) \cdot \left(u v^{(k)} \right)(t,1)\, dt \\
&= -\int_{\Omega_T} (\rho g) \cdot \left(\partial_s u v^{(k)} \right) dsdt - \int_{\Omega_T} (\rho g) \cdot \left(u \partial_s v^{(k)} \right) dsdt \\
&\quad + \rho(1) g \int_0^T \left(u v^{(k)} \right)(t,1)\, dt.
\end{aligned} \tag{4.35}$$

Remember that we dropped the dependence of ε for readability. So $\rho = \rho_\varepsilon$ is a smooth function and the value $\rho(1)$ is defined and bounded by 0 and 1.

Now, we give estimates for B, we use integration by parts and boundary conditions

$$\begin{aligned}
B = -\int_{\Omega_T} (\rho g) \cdot \left(\partial_s u v^{(k)} \right) dsdt &- \int_{\Omega_T} (\rho g) \cdot \left(u \partial_s v^{(k)} \right) dsdt \\
&+ \rho(1) g \int_0^T \left(u v^{(k)} \right)(t,1)\, dt
\end{aligned} \tag{4.36}$$

We will use the Young inequality for the first two terms and remember that $|g| = 1$ and ρ is bounded

$$\begin{aligned}
-\int_{\Omega_T} (\rho g)\cdot\left(\partial_s u v^{(k)}\right) dsdt &\leq \int_{\Omega_T} |\rho||\partial_s u v^{(k)}| \\
&\leq \int_{\Omega_T} |\partial_s u||v^{(k)}| dsdt \\
&= \int_{\Omega_T} \left(|\partial_s u||v^{(k)}|^{1/2}\right)|v^{(k)}|^{1/2} dsdt \\
&\leq \frac{1}{4}\int_{\Omega_T} |\partial_s u|^2 v^{(k)} dsdt + \int_{\Omega_T} v^{(k)} dsdt \qquad (4.37)
\end{aligned}$$

Similarly, we estimate the next term

$$\begin{aligned}
-\int_{\Omega_T} (\rho g)\cdot\left(u\partial_s v^{(k)}\right) dsd &\leq \int_{\Omega_T} |u||\partial_s v^{(k)}| \\
&\leq \int_{\Omega_T} |u|^2 + \frac{1}{4}\int_{\Omega_T} |\partial_s v^{(k)}|^2 dsdt \qquad (4.38)
\end{aligned}$$

Notice that in (4.35) and (4.36), the last terms are same, we will estimate them together. We use fundamental theorem of calculus and boundedness of ρ with the Young inequality,

$$\begin{aligned}
\rho(1)\int_0^T g\cdot\left(uv^{(k)}\right)(t,1)\,dt &= \rho(1)\int_{\Omega_T} g\cdot\partial_s\left(uv^{(k)}\right) dsdt \\
&= \rho(1)\left(\int_{\Omega_T} g\cdot\partial_s u v^{(k)} dsdt + \int_{\Omega_T} g\cdot u\partial_s v^{(k)} dsdt\right) \\
&\leq \int_{\Omega_T} |\partial_s u||v^{(k)}| dsdt + \int_{\Omega_T} |u||\partial_s v^{(k)}| dsdt \qquad (4.39)
\end{aligned}$$

The right hand sides can be further estimated in the same manner as above. We combine all above inequalities, we deduce that there exists a universal constant C (independent of ε) such that

$$\begin{aligned}
&\sup_{t\in[0,T]}\int_{\Omega} |v^{(k)}|^2(t,\cdot)\,ds + \int_{\Omega_T} |\partial_s u|^2 v^{(k)} dsdt + \int_{\Omega_T} |\partial_s v^{(k)}|^2 dsdt \qquad (4.40)\\
&\leq C\int_{\Omega_T} v^{(k)} dsdt + C\int_{\Omega_T} |u|^2 dsdt \\
&\leq C\int_{\Omega_T} v^{(k)} dsdt + C\int_{\Omega_T} kdsdt
\end{aligned}$$

where the last inequality follows from $|u|^2 \leq \left(|u|^2 - k\right)_+ + k = v^{(k)} + k$. Using the Young inequality and Hölder inequalities, we have

$$C\int_{\Omega_T} v^{(k)} dsdt + C\int_{\Omega_T} kdsdt \leq \frac{1}{4}\sup_{t\in[0,T]}\int_{\Omega} |v^{(k)}|^2(t,\cdot)\,ds + C\int_0^T |A_k(t)|\,dt + Ck\int_0^T |A_k| dt$$

Note that the constants are generic and can depend on T. Note that the first term can be absorbed by the left hand side of (4.40). Thus,

$$\sup_{t\in[0,T]}\int_{\Omega}|v^{(k)}|^2(t,\cdot)\,ds+\int_{\Omega_T}|\partial_s v^{(k)}|^2 dsdt\le Ck\int_0^T|A_k(t)|dt.$$

Using the Theorem 6.1 in Chapter II of the book [34] of Ladyzenskaja, Solonnikov and Ural'ceva, we get

$$\sup_{\Omega_T}|u|^2(t,s)\le 2(1+C)\sup_{\Omega}|\beta|^2(s)$$

□

4.4 Existence of generalized solutions

In this section we state the main theorem after defining the solution and showing the convergences.

Proposition 4.3. *Given any $T>0$, let η_ε be a solution to* (4.20) *in Ω_T with the initial/boundary conditions* (4.21). *Let $(\kappa_\varepsilon,\sigma_\varepsilon)$ be as in* (4.22). *Assume that $||\alpha_\varepsilon||_{L^2(\Omega)}$ and $\mathscr{E}_\varepsilon(\alpha_\varepsilon)$ are bounded uniformly in ε. Then*

(i) Along a subsequence $\varepsilon\to 0$ one has

- $\eta_\varepsilon\to\eta$ *weakly* in* $L^\infty\left(0,T;W^{1,\infty}(\Omega)\right)^d$ *and strongly in* $L^2(\Omega_T)^d$
- $\partial_t\eta_\varepsilon\to\partial_t\eta$ *weakly in* $L^2(\Omega_T)^d$
- $\sigma_\varepsilon\to\sigma$ *weakly in* $L^2\left(0,T;H^1(\Omega)\right)$

(ii) The limit (η,σ) satisfies

$$\sigma\partial_s\eta\in L^2\left(0,T;H^1(\Omega)\right)^d$$

and solves (4.4),(4.2) *in the sense that*

$$\partial_t\eta=\partial_s(\sigma\partial_s\eta)+\rho g,\ \ \sigma\left(|\partial_s\eta|^2-1\right)=0$$

$$\eta(t,1)=0\ \forall t,\ \eta(0,s)=\alpha(s)\ \forall s,\ \sigma(t,0)=0\ \textit{for a.e. } t.$$

Proof. Proof of (i) The compactness results for η_ε follow immediately from the uniform energy bound in Proposition 4.26 and the L^∞ bound for $\partial_s\eta_\varepsilon$ from the Proposition 4.2. Also for the uniform boundedness of σ_ε we use Proposition 4.2. We do not give more details of this proof as it follows the similar ideas in proofs of Lemma 2.17 and Proposition 2.19.

By a very direct computation $\partial_s\sigma_\varepsilon=\partial_s\kappa_\varepsilon\cdot\partial_s\eta_\varepsilon+\kappa_\varepsilon\cdot\partial_{ss}\eta_\varepsilon$. We will estimate the two terms in the summation separately. First by (4.26) and Poincare's inequality we immediately obtain that $\kappa_\varepsilon=G_\varepsilon(\partial_s\eta_\varepsilon)$ are uniformly bounded in $L^2\left(0,T;H^1(\Omega)\right)$; note that the whip boundary conditions help us in comparison with the previous chapters. This together with Proposition 4.2 gives the uniform boundedness of $\partial_s\kappa_\varepsilon\cdot\partial_s\eta_\varepsilon$ in $L^2(\Omega_T)$. The second term $\kappa_\varepsilon\cdot\partial_{ss}\eta_\varepsilon$ can be estimated as in Proposition 2.19.

Proof of (ii). From the energy uniform bounded showed in (4.26), we see that there exists $\kappa := \lim \kappa_\varepsilon$ in the weak topology of $L^2\left(0,T;H^1(\Omega)\right)$.

The passage to the limit from the approximating system to the original problem is done as in the proof of Proposition 2.19. However, let us point out some differences. The omitted parts of the proof heavily rely on the estimate (4.2). In the proof of Proposition 2.19, we had different estimates but still similar bounds that we could use the same ideas in the proof. Finally since we approximated α and ρ let us see how we proceed with the corresponding terms.

It is easy to see that $\rho_\varepsilon g \to \rho g$ weakly* in $L^\infty(\Omega)$ as g is constant and $0 \le \rho_\varepsilon \le 1$.

We now observe that by the Aubin-Lions-Simon theorem

$$L^\infty\left(0,T;W^{1,\infty}(\Omega)\right) \cap H^1\left(0,T;L^2(\Omega)\right) \subset C\left([0,T];C(\bar{\Omega})\right)$$

and the embedding is compact. Without loss of generality, we may therefore assume that $\eta \to \eta_\varepsilon$ strongly in $C\left([0,T]\times\bar{\Omega}\right)$. Hence, $\alpha = \eta_\varepsilon(0,\cdot) \to \eta(0,\cdot)$ uniformly in s, thus $\eta(0,\cdot) = \alpha$. In a very similar way we obtain the required boundary condition at the fixed end.

To check the validity of the boundary condition for σ, we swap the variables t and s, noting that σ_ε are uniformly bounded and weakly converging in $H^1\left(0,1;L^2(0,T)\right)$. Employing for instance [71, Corollary 2.2.1], we get

$$H^1\left(0,1;L^2(0,T)\right) \subset C\left([0,1];L^2(0,T)\right).$$

Hence, by the Aubin-Lions-Simon theorem, the embedding

$$H^1\left(0,1;L^2(0,T)\right) \subset C\left([0,1];H^{-1}(0,T)\right)$$

is compact, whence we may assume that $\sigma_\varepsilon \to \sigma$ strongly in $C\left([0,1];H^{-1}(0,T)\right)$. Using (4.20) and (4.22), we get $0 = \sigma_\varepsilon(\cdot,0) \to \sigma(\cdot,0)$ in $H^{-1}(0,T)$. Consequently, $\sigma(t,0) = 0$ in $L^2(0,T)$ and for a.e. t. □

Definition 4.4. *Given an initial datum $\alpha \in W^{1,\infty}(\Omega)^d$ with $\alpha(1) = 0$, $|\partial_s\alpha(s)| \le 1$ and $0 \le \rho(s) \le 1$ for almost every $s \in \Omega$, we can call a pair (η,σ) a generalized solution to* (4.4),(4.2) *in $\mathfrak{Q}_\infty := (0,\infty)\times\Omega$ if*

(i) The pair (η,σ) and their derivatives or products belong to given spaces as following

$$\eta \in L^\infty_{loc}\left([0,\infty);W^{1,\infty}(\Omega)\right)^d \cap AC^2_{loc}\left([0,\infty);L^2(\Omega)\right)\Big)^d,$$

$$\sigma \in L^2_{loc}\left([0,\infty);H^2(\Omega)\right)\Big), \; \sigma\partial_s\eta \in L^2_{loc}\left([0,\infty);H^1(\Omega)\right)\Big)^d$$

(ii) The pair (η,σ) *for a.e.* $(t,s)\in\mathfrak{Q}_\infty$,

$$\partial_t\eta(t,s)=\partial_s\left(\sigma(t,s)\,\partial_s\eta(t,s)\right)+\rho g \tag{4.41}$$

$$\sigma(t,s)\left(|\partial_s\eta(t,s)|^2-1\right)=0 \tag{4.42}$$

$$|\partial_s\eta(t,s)|\leq 1 \tag{4.43}$$

and the initial/boundary conditions

$$\eta(t,1)=0\ \textit{for all}\ t,\ \eta(0,s)=\alpha(s)\ \forall s,\ \sigma(t,0)=0\ \textit{for a.e}\ t$$

(iii) The solutions η *satisfy the energy dissipation inequality*

$$\int_\Omega|\partial_t\eta(t,s)|^2ds\leq\int_\Omega g\rho\cdot\partial_t\eta(t,s)\,ds \tag{4.44}$$

for a.e. $t\in(0,\infty)$.

Proposition 4.5. *Let* (η,σ) *be a limiting solution obtained in Proposition 4.3. Then*

(i) $|\partial_s\eta(t,s)|\leq 1$ *for a.e.* $(t,s)\in\mathfrak{Q}_T$

(ii) $\sigma(t,s)\geq 0$ *for a.e.* $(t,s)\in\mathfrak{Q}_T$

(iii) (4.44) *holds for a.a.* $t\in(0,T)$

The proof of the proposition above is very similar to the proof of Proposition 2.20.

We state the theorem that provides the global existence of generalized solutions.

Theorem 4.6 (Global existence of generalized solutions)**.** *For every* $\alpha\in W^{1,\infty}(\Omega)^d$ *with* $\eta(1)=0$, $|\partial_s\alpha|\leq 1$ *for a.e.* $s\in\Omega$ *and given* $0\leq\rho(s)\leq 1$ *in* $L^\infty(\Omega)$ *there exists a generalized solution to* (4.4), (4.2) *in* $\mathfrak{Q}_\infty$. *Moreover, those solutions satisfy* $\sigma(t,s)\geq 0$ *for almost* $(t,s)\in\mathfrak{Q}_\infty$.

We use a diagonal argument for existence of subsequences and take into account Proposition 4.5 and Proposition 4.3. After these, we deduce Theorem 4.6 from Proposition 4.2. This proof follows the same ideas as the proof of Theorem 2.21.

4.5 Exponential decay and long time behaviour of solutions

The main goal of this section is to show that the relative energy decays along the trajectories of the generalized solutions of (4.4), (4.2) exponentially fast. Remember that the potential energy is given in (4.3), and we define the relative energy as

$$\tilde{\mathscr{E}}(t):=\mathscr{E}(t)-\mathscr{E}(\eta_\infty),$$

where η_∞ is the downwards vertical stationary solution:

$$\eta_\infty(s):=(1-s)\,g\ \text{ and }\ \sigma_\infty(s):=\int_0^s\rho.$$

We will show that $\tilde{\mathscr{E}}(t)$ has an upper bound which decays exponentially fast to zero as $t \to \infty$ and this together with non-negativity of the relative energy implies the convergence of $\eta(t,\cdot)$ to η_∞ in $L^2(\Omega)$ with an exponential convergence rate.

We notice the following equality

$$\mathscr{E}(t) = \int_0^1 -\rho g \cdot \eta \, ds = \int_0^1 \sigma_\infty g \cdot \partial_s \eta \, ds$$

because of the boundary conditions (4.2). Observe also that $\mathscr{E}(t)$ is continuous in time because $\eta \in AC^2_{loc}\left([0,\infty); L^2(\Omega)\right)^d$.

Now, an easy computation gives us that $\partial_s \eta_\infty = -g$, so we can write

$$\mathscr{E}(t) = -\int_0^1 \sigma_\infty \partial_s \eta_\infty \cdot \partial_s \eta ds.$$

Thus, by adding and subtracting same terms we deduce the following

$$\begin{aligned}
\tilde{\mathscr{E}}(t) =& \mathscr{E}(t) - \mathscr{E}(\eta_\infty) \\
=& -\int_0^1 \sigma_\infty \partial_s \eta_\infty \cdot \partial_s (\eta - \eta_\infty) ds \\
=& -\int_0^1 \sigma_\infty \partial_s \eta_\infty \cdot \partial_s \eta + \int_0^1 \sigma_\infty |\partial_s \eta_\infty|^2 ds \\
=& -\int_0^1 \sigma_\infty \partial_s \eta_\infty \cdot \partial_s \eta ds + \int_0^1 \sigma_\infty |\partial_s \eta_\infty|^2 ds \\
&+ \int_0^1 \sigma_\infty |\partial_s \eta|^2 ds - \int_0^1 \sigma_\infty |\partial_s \eta|^2 ds + \int_0^1 \sigma_\infty \partial_s \eta \cdot \partial_s \eta_\infty ds - \int_0^1 \sigma_\infty \partial_s \eta_\infty \cdot \partial_s \eta ds \\
=& -\int_0^1 \sigma_\infty \partial_s \eta \cdot \partial_s (\eta - \eta_\infty) ds + \int_0^1 \sigma_\infty |\partial_s (\eta - \eta_\infty)|^2 ds.
\end{aligned}$$

Also, notice the following equality by using the definition of the relative energy

$$\tilde{\mathscr{E}}(t) = \int_0^1 \sigma_\infty \partial_s \eta \cdot \partial_s (\eta - \eta_\infty) ds - \int_0^1 \sigma_\infty \left(|\partial_s \eta|^2 - 1\right) ds.$$

We have used that $\mathscr{E}(\eta_\infty) = -\int_0^1 \sigma_\infty \partial_s \eta_\infty \cdot \partial_s \eta_\infty ds = -\int_0^1 \sigma_\infty ds$. Now, combining both expressions of the relative energy, we obtain

$$\tilde{\mathscr{E}}(t) = \frac{1}{2}\int_0^1 \sigma_\infty |\partial_s (\eta - \eta_\infty)|^2 ds - \frac{1}{2}\int_0^1 \sigma_\infty \left(|\partial_s \eta|^2 - 1\right) ds. \tag{4.45}$$

Remark 4.2. From the equivalent expression of the relative energy $\tilde{\mathscr{E}}(t)$ in (4.45), $|\partial_s \eta| \leq 1$ a.e. and the continuity of $t \mapsto \tilde{\mathscr{E}}(t)$, we immediately obtain that $\tilde{\mathscr{E}}(t) \geq 0$ for all $t \in [0,\infty)$.

Remark 4.3. We will now proceed with proving the exponential decay under the assumptions $\tilde{C}s^{-2} \geq \sigma_\infty^{-1}$ for a positive constant $\tilde{C}$ and $\sigma \geq 0$. The assumption $\sigma \geq 0$ a.e. in Ω_∞ is not restrictive. It is satisfied by the generalized solutions existing by Theorem 4.6 for all Lipschitz initial data with $\alpha(1) = 0$ and $|\partial_s \alpha(s)| \leq 1$ for a.e. $s \in \Omega$. The first assumption is also not very restrictive and just says that the density $\rho(s)$ near $s=0$ should either be bounded away from zero or decay to zero at most linearly.

Lemma 4.7. *Let (η,σ) be a generalized solution in the sense of Definition 4.4. Assume that $\sigma \geq 0$ almost everywhere in $\mathfrak{Q}_\infty$ and $\tilde{C}s^{-2} \geq \sigma_\infty^{-1}$ for a positive constant $\tilde{C}$. Then there exists a universal constant $c_0 > 0$ such that*

$$\tilde{\mathscr{E}}(t) \leq \bar{c}_0 \int_0^1 |\partial_t \eta(t,s)|^2 ds \tag{4.46}$$

for a.a. $t>0$. Here, $\tilde{\mathscr{E}}(t) = \mathscr{E}(t) - \mathscr{E}(\eta_\infty)$ is the relative energy, defined as before.

Proof. Using the equation of η and Hardy's inequality, and remembering that $\kappa = \sigma\partial_s\eta$ (in particular, we can define accordingly $\kappa_\infty := \sigma_\infty\partial_s\eta_\infty = -g\int_0^s \rho$) we obtain for a.e. $t \in (0,\infty)$:

$$\begin{aligned}
\int_0^1 |\partial_t\eta|^2 ds = &\int_0^1 |\partial_s\kappa + \rho g|^2 ds \\
= &\int_0^1 |\partial_s\kappa - \partial_s\kappa_\infty|^2 \\
\geq &\overline{C}\int_0^1 s^{-2}|\kappa - \kappa_\infty|^2 ds \\
\geq &\overline{C}\int_0^1 \sigma_\infty^{-1}|\kappa - \kappa_\infty|^2 ds \\
= &\overline{C}\int_0^1 \sigma_\infty^{-1}|\sigma\partial_s\eta - \sigma_\infty\partial_s\eta_\infty|^2 ds.
\end{aligned}$$

Here $\overline{C}$ is a universal constant which is independent of t and we used the assumption that we have for the density $\tilde{C}s^{-2} \geq \sigma_\infty^{-1}$. This implies for any $\phi = \phi(t) \in C_c^\infty(0,\infty)$, $\phi \geq 0$,

$$\int_{\mathfrak{Q}_\infty} |\partial_t\eta|^2\phi\, ds dt \geq \overline{C}\int_{\mathfrak{Q}_\infty} \sigma_\infty^{-1}|\sigma\partial_s\eta - \sigma_\infty\partial_s\eta_\infty|^2\phi\; ds dt. \tag{4.47}$$

We take the precise representatives of $\partial_s\eta$ and σ and define

$$\begin{aligned}
\Omega_1 :&= \{(t,s) \in \mathfrak{Q}_\infty : |\partial_s\eta(t,s)| = 1\}, \\
\Omega_{2,1} :&= \{(t,s) \in \mathfrak{Q}_\infty : |\partial_s\eta(t,s)| \neq 1,\ \sigma(t,s) = 0\}, \\
\Omega_{2,2} :&= \{(t,s) \in \mathfrak{Q}_\infty : |\partial_s\eta(t,s)| \neq 1,\ \sigma(t,s) \neq 0\}.
\end{aligned} \tag{4.48}$$

Using that $\sigma = 0$ in $\Omega_{2,1}$, (4.47) has the following estimate

$$\int_{\mathfrak{Q}_\infty} |\partial_t\eta|^2\phi\; ds dt \geq \overline{C}\int_{\Omega_{2,1}} \sigma_\infty^{-1}|\sigma_\infty\partial_s\eta_\infty|^2\phi\; ds dt = \overline{C}\int_{\Omega_{2,1}} \sigma_\infty\phi\; ds dt.$$

Combining all we end up with

$$\int_{\mathfrak{Q}_\infty} |\partial_t\eta|^2\phi\, ds dt \geq \frac{\overline{C}}{2}\int_{\Omega_{2,1}} \sigma_\infty\phi\; ds dt + \frac{\overline{C}}{2}\int_{\mathfrak{Q}_\infty} \sigma_\infty^{-1}|\sigma\partial_s\eta - \sigma_\infty\partial_s\eta_\infty|^2\phi\; ds dt. \tag{4.49}$$

Now, we will give an upper bound for the relative energy $\tilde{\mathscr{E}}(t)$ such that we will have same terms that are lower bounds for $\int_0^1 |\partial_t \eta|^2 ds$, this will complete the proof:

$$\begin{aligned}\int_0^\infty \tilde{\mathscr{E}}(t)\,\phi dt =& \frac{1}{2}\int_{\Omega_1\cup\Omega_{2,1}} \sigma_\infty |\partial_s(\eta-\eta_\infty)|^2 \phi \; dsdt + \frac{1}{2}\int_{\Omega_{2,2}} \sigma_\infty |\partial_s(\eta-\eta_\infty)|^2 \phi \; dsdt \\ & -\frac{1}{2}\int_{\Omega_{2,1}} \sigma_\infty \left(|\partial_s\eta|^2-1\right)\phi \; dsdt - \frac{1}{2}\int_{\Omega_{2,2}} \sigma_\infty \left(|\partial_s\eta|^2-1\right)\phi \; dsdt\end{aligned}$$

Note that by (4.42) and (4.48) we have $|\Omega_{2,2}|=0$. This together with $\eta \in L^\infty_{loc}\left([0,\infty);W^{1,\infty}(\Omega)\right)^d$ and $\rho \in L^\infty(\Omega)$ yields that integrals over $\Omega_{2,2}$ are zero:

$$\begin{aligned}\int_{\Omega_{2,2}} \sigma_\infty |\partial_s(\eta-\eta_\infty)|^2 \phi \; dsdt &= 0 \\ \int_{\Omega_{2,2}} \sigma_\infty \left(|\partial_s\eta|^2-1\right)\phi \; dsdt &= 0\end{aligned}$$

To estimate the integrals over $\Omega_{2,1}$, we note that by (4.43) in Definition 4.4 $|\Omega_{2,1} \cap \{(t,s)\in \Omega_T : |\partial_s\eta(t,s)|>1\}|=0$. This allows us to have

$$\begin{aligned}&\frac{1}{2}\int_{\Omega_{2,1}} \sigma_\infty |\partial_s(\eta-\eta_\infty)|^2 \phi \; dsdt - \frac{1}{2}\int_{\Omega_{2,1}} \sigma_\infty \left(|\partial_s\eta|^2-1\right)\phi dsdt \\ &= \int_{\Omega_{2,1}} \sigma_\infty \left(1-\partial_s\eta\cdot\partial_s\eta_\infty\right)\phi \; dsdt \\ &\le \int_{\Omega_{2,1}} 2\sigma_\infty \phi \; dsdt\end{aligned}$$

For the integral over Ω_1, we want to find similar terms in order to relate it with the lower bound of $\int_{\Omega_\infty} |\partial_t\eta|^2 dsdt$:

$$\begin{aligned}\frac{1}{2}\int_{\Omega_1} \sigma_\infty |\partial_s(\eta-\eta_\infty)|^2 \phi \; dsdt &= \frac{1}{2}\int_{\Omega_1} \sigma_\infty^{-1} |\sigma\partial_s\eta - \sigma_\infty\partial_s\eta_\infty + (\sigma_\infty-\sigma)\partial_s\eta|^2 \phi \; dsdt \\ &\le \int_{\Omega_\infty} \sigma_\infty^{-1} |\sigma\partial_s\eta - \sigma_\infty\partial_s\eta_\infty|^2 \phi \; dsdt + \int_{\Omega_1} \sigma_\infty^{-1}|\sigma-\sigma_\infty|^2\phi \; dsdt,\end{aligned}$$

we used that $|\partial_s\eta|=1$ a.e. in Ω_1. Knowing that $\sigma \ge 0$, we observe in Ω_1 the following using the triangle inequality

$$|\sigma_\infty - \sigma| = \Big||\sigma_\infty\partial_s\eta_\infty| - |\sigma\partial_s\eta|\Big| \le |\sigma\partial_s\eta - \sigma_\infty\partial_s\eta_\infty|,$$

thus we have

$$\int_{\Omega_1} \sigma_\infty^{-1}|\sigma_\infty-\sigma|^2\phi \; dsdt \le \int_{\Omega_1} \sigma_\infty^{-1} |\sigma\partial_s\eta - \sigma_\infty\partial_s\eta_\infty|^2 \phi \; dsdt.$$

Therefore, combining all above estimates for all $\phi=\phi(t)\ge 0$ and $\phi \in C_c^\infty(0,\infty)$, we end up with

$$\int_0^\infty \tilde{\mathscr{E}}(t)\phi(t)dt \le 2\int_{\Omega_\infty} \sigma_\infty^{-1} |\sigma\partial_s\eta - \sigma\partial_s\eta|^2 \phi(t)\; dsdt + \int_{\Omega_{2,1}} 2\sigma_\infty\phi(t)\; dsdt.$$

Finally, using (4.49) we conclude that

$$\begin{aligned}&\int_0^\infty \tilde{\mathscr{E}}(t)\phi(t)dt\\ &\leq 2\int_{\mathfrak{Q}_\infty}\sigma_\infty^{-1}|\sigma\partial_s\eta-\sigma_\infty\partial_s\eta_\infty|^2\phi(t)\,dsdt+\int_{\Omega_{2,1}}2\sigma_\infty\phi(t)\,dsdt \\ &\leq \bar{c}_0\int_{\mathfrak{Q}_\infty}|\partial_t\eta|^2\phi(t)\,dsdt.\end{aligned} \tag{4.50}$$

for some positive $\bar{c}_0$. This implies the claim of the lemma because both sides of (4.46) are locally integrable. □

Theorem 4.8. *Let (η,σ) be a generalized solution in the sense of Definition 4.4. Assume that $\sigma \geq 0$ almost everywhere in $\mathfrak{Q}_\infty$ and $\tilde{C}s^{-2} \geq \sigma_\infty^{-1}$ for a positive constant $\tilde{C}$. Then there exists a universal constant $c_0 > 0$ such that*

$$\tilde{\mathscr{E}}(t) \leq e^{-c_0 t}\tilde{\mathscr{E}}(0), \quad t\in[0,\infty). \tag{4.51}$$

Here, $\tilde{\mathscr{E}}(t)$ is the relative energy, defined as before.

Proof. Employing the energy dissipation inequality (4.44), the relation (4.46) and integration by parts we obtain

$$\begin{aligned}\int_0^\infty \tilde{\mathscr{E}}(t)\phi(t)\,dt &\leq \bar{c}_0\int_0^\infty\int_0^1 \rho g\cdot\partial_t\eta(t,s)\,\phi\,dsdt\\ &= \bar{c}_0\int_0^\infty \tilde{\mathscr{E}}(t)\frac{d}{dt}\phi(t)\,dt.\end{aligned}$$

for all $\phi=\phi(t)\geq 0$, $\phi\in C_c^\infty(0,\infty)$. Denoting $c_0^{-1}=\bar{c}_0$, we can rewrite this in the following way

$$\int_0^\infty \tilde{\mathscr{E}}(t)e^{c_0 t}\frac{d}{dt}\left(e^{-c_0 t}\phi(t)\right)\geq 0.$$

This implies that $d\left(e^{c_0 t}\tilde{\mathscr{E}}(t)\right)/dt$ is a non-positive distribution. Since $t\to\mathscr{E}(t)$ and thus $t\to\tilde{\mathscr{E}}(t)$ is continuous, we have

$$\tilde{\mathscr{E}}(t)\leq e^{-c_0 t}\tilde{\mathscr{E}}(0)$$

for all $t\in[0,\infty)$. □

The exponential decay of the relative energy implies the exponential decay of the generalized solution to the stationary solution η_∞ in $L^2(\Omega)$ as $t\to\infty$.

Corollary 4.9. *Let (η,σ) be generalized solution in the sense of Definition 4.4 with $\sigma\geq 0$ and $\tilde{C}s^{-2}\geq\sigma_\infty^{-1}$ for a positive constant $\tilde{C}$. Let $\tilde{\mathscr{E}}(t) =: \mathscr{E}\left(\eta(t,\cdot)\right)-\mathscr{E}(\eta)$ be the relative energy as in Theorem 4.8, where $(\eta_\infty,\sigma_\infty)=\left((1-s)g,\int_0^s\rho\right)$ is the stable stationary solution. Then there exist universal constants $C_0, c_0 > 0$ such that for all $t\in[0,\infty)$,*

$$||\eta(t,\cdot)-\eta_\infty(\cdot)||^2_{L^2(\Omega)}\leq C_0\tilde{\mathscr{E}}(0)e^{-c_0 t}. \tag{4.52}$$

Proof. By (4.51) for nonnegative $\phi = \phi(t) \in C_c^\infty\left((0,\infty)\right)$ we have

$$\int_0^\infty \tilde{\mathscr{E}}(t)\,\phi \le \int_0^\infty e^{-c_0 t}\tilde{\mathscr{E}}(0)\,\phi dt.$$

Using the equivalent expression in (4.45) for $\tilde{\mathscr{E}}(t)$ we obtain

$$\int_{\mathfrak{Q}_\infty} \sigma_\infty |\partial_s(\eta-\eta_\infty)|^2\phi\; dsdt - \int_{\mathfrak{Q}_\infty} \sigma_\infty\left(|\partial_s\eta|^2-1\right)\phi\; dsdt \le 2\int_0^\infty e^{-c_0 t}\tilde{\mathscr{E}}(0)\,\phi\; dt.$$

Since $\partial_s\eta \in L^2_{loc}(\mathfrak{Q}_\infty)$ and $|\partial_s\eta| \le 1$ for almost every $(t,s)\in\mathfrak{Q}_\infty$, we have

$$\int_{\mathfrak{Q}_\infty} \sigma_\infty\left(|\partial_s\eta|^2-1\right)\phi\; dsdt \le 0.$$

Thus we immediately obtain the exponential decay of the Sobolev distance:

$$||(\sigma_\infty)^{1/2}\left(\partial_s\eta-\partial_s\eta_\infty\right)||^2_{L^2(\Omega)} \le 2\tilde{\mathscr{E}}(0)\,e^{-c_0 t} \quad \text{for almost every } \; t\in[0,\infty). \tag{4.53}$$

To derive the decay for the L^2 distance we apply again Hardy's inequality

$$\int_0^1 |\eta(t,s)-\eta_\infty(t,s)|^2\; ds \le C\int_0^1 s^2|\partial_s(\eta-\eta_\infty)|^2\; ds$$

for almost every t. Using $s^2 \le C\sigma_\infty$ we infer from (4.53) that

$$\int_{\mathfrak{Q}_\infty} |(\eta-\eta_\infty)|^2\phi\; dsdt \le C\int_0^\infty e^{-c_0 t}\tilde{\mathscr{E}}(0)\,\phi\; dt.$$

Since this holds for arbitrary nonnegative $\phi \in C_c^\infty\left((0,\infty)\right)$ and since the map $t \mapsto ||\eta(t,\cdot)-\eta_\infty(\cdot)||_{L^2(\Omega)}$ is continuous, we conclude that

$$||\eta(t,\cdot)-\eta_\infty(\cdot)||^2_{L^2} \le C_0\tilde{\mathscr{E}}(0)\,e^{-c_0 t}$$

with some constants c_0, C_0 (actually c_0 is the same as in Theorem 4.8). □

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Appendix A

Evolution by Pseudomonotone Maps

Here we recall some concepts and definitions related to evolution by pseudomonotone maps mainly following the book [59]. Let V be a separable reflexive Banach space, and V^* be the dual space of V. We use the bracket notation for the duality.

Definition A.1. *A mapping $A : V \to V^*$ is called monotone if $\forall u, v \in V$ we have $\langle A(u) - A(v), u - v\rangle \geq 0$.*

Definition A.2. *A mapping $A : V \to V^*$ is called radially continuous if $\forall u, v \in V : t \to \langle A(u + vt), v\rangle$ is continuous.*

Definition A.3. *A mapping $A : V \to V^*$ is called pseudomonotone provided*
(i) A is bounded (i.e., the image of any bounded set is bounded),
(ii) for any sequence $u_k \rightharpoonup u$ weakly with

$$\limsup_{k\to\infty} \langle A(u_k), u_k - u\rangle \leq 0$$

and for every $v \in V$ it is true that

$$\langle A(u), u - v\rangle \leq \liminf_{k\to\infty} \langle A(u_k), u_k - v\rangle.$$

We will need the following useful criterion of pseudomonotonicity from [14].

Lemma A.4. *A bounded, radially continuous and monotone mapping is pseudomonotone.*

Definition A.5. *A mapping $A : V_1 \to V_2$, where V_1, V_2 are Banach spaces is called totally continuous if it maps weakly convergent sequences to strongly convergent ones.*

Lemma A.6. *A perturbation of a pseudomonotone mapping by a totally continuous mapping is pseudomonotone, i.e. if A_1 is pseudomonotone and A_2 is totally continuous then $u \to A_1(u) + A_2(u)$ is pseudomonotone.*

Assume that there is a continuous embedding operator $i : V \to H$, and $i(V)$ is dense in H, where H is a Hilbert space. This generates the *Gelfand triple* $V \subset H \subset V^*$ by the following well-known observation. The adjoint operator $i^* : H^* \to V^*$ is continuous and, since $i(V)$ is dense in H, one-to-one.

Since i is one-to-one, $i^*(H^*)$ is dense in V^*, and one may identify H^* with a dense subspace of V^*. Due to the Riesz representation theorem, one may also identify H with H^*. Moreover, the H-scalar product of $f \in H, u \in V$ coincides with the value of the functional f from V^* on the element $u \in V$:

$$(f,u)_H = \langle f,u \rangle. \tag{A.1}$$

Assume that there is a seminorm $|\cdot|_V$ on V that satisfies the "abstract Poincaré inequality"

$$\|u\|_V \lesssim \|u\|_H + |u|_V, \quad \forall u \in V,$$

where $\|\cdot\|_H$ is the Euclidean norm in H.

Definition A.7. *A mapping $A : V \to V^*$ is called semicoercive if for $u \in V$ we have*

$$\langle A(u), u \rangle \geq c_0 |u|_V^2 - c_1 |u|_V - c_2 \|u\|_H^2,$$

where c_0, c_1 and c_2 are nonnegative constants.

Consider the following abstract initial value problem on the time interval $(0,T)$:

$$\frac{d}{dt}u + A\big(u(t)\big) = f(t),\ u(0) = u_0. \tag{A.2}$$

The following result can be found in [59, Theorem 8.18].

Theorem A.8. *Let $A : V \to V^*$ be a pseudomonotone and semicoercive mapping and*

$f \in AC^2\big([0,T], V^*\big)$,

$u_0 \in V$ is such that $A(u_0) - f(0) \in H$,

$\langle A(u_1) - A(u_2), u_1 - u_2 \rangle \geq c_0 |u_1 - u_2|_V^2 - c_2 \|u_1 - u_2\|_H^2$ for $u_1, u_2 \in V$ with some constants $c_0, c_2 > 0$.

Then there exists $u \in W^{1,\infty}(0,T;H) \cap AC^2\big([0,T];V\big)$ that solves the Cauchy problem (A.2) *(the first equality in* (A.2) *holds in the space V^* for a.a. in $(0,T)$, whereas the second one holds in the space V).*

Appendix B

Solvability of Hilbertian Gradient Flows

For the sake of completeness we present here Theorem 17.2.3 from [7]. This result is about solvability of gradient flows in Hilbert spaces.

Definition B.1. *Let $(V, \|\cdot\|)$ be a normed space and $f : V \to \mathbb{R} \cup \{+\infty\}$ be a closed convex proper function. We say that an element $u^* \in V^*$ belongs to the subdifferential of f at $u \in V$ if*

$$\forall u \in V \quad f(v) \geq f(u) + \langle u^*, v - u \rangle.$$

We then write $u^* \in \partial f(u)$.

Theorem B.2. *Let $\mathscr{H}$ be a Hilbert space, and $\Phi : \mathscr{H} \to \mathbb{R} \cup \{+\infty\}$ be convex, lower semicontinuous, and proper. Suppose that Φ is minorized, i.e., $\inf_{\mathscr{H}} \Phi > -\infty$. Let $u_0 \in \overline{dom\, \Phi}$. Then there exists a unique strong global solution $u : [0, \infty) \to \mathscr{H}$ of the Cauchy problem*

$$\begin{cases} \frac{d}{dt} u(t) + \partial \Phi\left(u(t)\right) \ni 0, \\ u(0) = u_0, \end{cases}$$

which is given by $u(t) = S(t, u_0)$. The Cauchy problem above is satisfied in the following sense:

- *$u \in C\left([0, \infty); \mathscr{H}\right)$;*
- *$u(t) \in dom\, \partial \Phi$ for all $t > 0$;*
- *u is Lipschitz continuous on $[\delta, +\infty)$ for any $\delta > 0$;*
- *the gradient flow equation above is satisfied for almost all $t > 0$;*
- *$\sqrt{t} \frac{d}{dt} u \in L^2(0, T; \mathscr{H})$ for all $T > 0$;*
- *for each t>0,*

$$\|\partial \Phi\left(u(t)\right)^0\| \leq \|\partial \Phi(v)^0\| + \frac{1}{t}\|u_0 - v\| \quad \forall v \in dom\, \partial \Phi$$

- *For each $t > 0$, u has a right derivative, and*

$$\frac{d^+}{dt}(t) = -\partial \phi\left(u(t)\right)^0,$$

where $\partial\Phi\left(u(t)\right)^0$ *is the element of minimal norm of* $\partial\Phi\left(u(t)\right)$.

- $t \mapsto \left\| \frac{d^+}{dt} u(t) \right\|$ *is nonincreasing.*
- $t \mapsto \Phi\left(u(t)\right)$ *is nonincreasing, absolutely continuous on each bounded interval* $[\delta, T]$, $\delta > 0$, *and*

$$\frac{d}{dt}\Phi\left(u(t)\right) = -\left\| \frac{d}{dt} u(t) \right\|^2 \text{ for almost all } t > 0.$$