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Ana Belén Avilez García

A point-free study of z-embeddings, more general classes of localic maps, and uniform continuity

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Abstract

This thesis is concerned with two different aspects of localic maps. First, they are classified and characterized according to their interaction with zero sublocales. Second, uniform continuity is studied within the more restricted setting of localic real-valued maps of preuniform locales.

Localic maps are classified according to the properties that their preimages and images of zero sublocales satisfy. Some of the classes of localic maps defined by their behavior on preimages of zero sublocales extend the notions of C-, C*- and z-embedded sublocales to localic maps. These maps are then used to characterize normality, and weaker forms of normality, in a manner akin to the characterization of normal locales as the locales in which every closed localic embedding is a C-map. On the other hand, localic maps defined by conditions on their behavior on images of zero sublocales are presented, and the relations between them and closed and open localic maps are studied. This leads to the investigation of three other types of localic maps: w-, n- and wz-maps.

A study of uniform continuity of real-valued functions on a preuniform frame is developed. The aim is to characterize uniform continuity of such frame homomorphisms, in terms of a farness relation, and to provide an insertion result for preuniform frames. Separation and extension results for uniform locales are obtained as easy corollaries. As a byproduct, we identify sufficient conditions under which a scale in a frame with a preuniformity generates a real-valued uniform map. The proof of the main theorem relies heavily on (pre)diameters in locales as a substitute for classical pseudometrics. Along the way, several general properties concerning these (pre)diameters are also shown.

Resumo

Esta dissertação consta de duas partes distintas onde duas facetas das funções *locálicas* são abordadas. Na primeira parte, as funções locálicas são classificadas e caracterizadas de acordo com o seu comportamento sobre os *sublocales* de zeros. Na segunda, estuda-se a continuidade uniforme no contexto mais restrito das funções locálicas, em *locales* pré-uniformes, com valores reais.

As funções locálicas são classificadas de acordo com as propriedades que as suas imagens e pré-imagens de sublocales de zeros satisfazem. Algumas das classes de funções locálicas definidas pelo seu comportamento sobre as pré-imagens de sublocales de zeros estendem as noções bem conhecidas de C-, C^*- e z-imersões de sublocales. Estas funções são usadas para caracterizar a propriedade de normalidade (e algumas das suas variantes fracas) do locale em questão, de um modo parecido com a caracterização dos locales normais como os locales nos quais qualquer imersão fechada é uma C-imersão. Por outro lado, as funções locálicas definidas pelo seu comportamento sobre as imagens de sublocales de zeros são também apresentadas e as suas relações com as funções fechadas ou abertas são estudadas. Isto conduz à investigação de três tipos de funções locálicas: as funções w, n e wz.

Um estudo da continuidade uniforme das funções definidas num locale pré-uniforme, com valores reais, é desenvolvido com o objectivo de caracterizar a continuidade uniforme de tais funções em termos de uma relação de afastamento entre elementos (e, mais geralmente, sublocales do locale) e de obter um resultado de inserção para locales pré-uniformes. Resultados de separação e extensão para locales pré-uniformes são depois obtidos como corolários. Identificam-se ainda condições suficientes sob as quais uma escala num locale com uma pré-uniformidade permite gerar uma função uniformemente contínua com valores reais. A prova do teorema principal de inserção baseia-se num estudo prévio de pré-diâmetros em locales onde alguns resultados da literatura são generalizados; diversas propriedades gerais destes diâmetros são apresentadas e provadas ao longo da exposição.

Resumen

Esta tesis se centra principalmente en dos aspectos de las funciones locálicas. En primer lugar, se lleva a cabo una clasificación y caracterización de estas funciones de acuerdo a la interacción que éstas tienen con los sublocales cero. En segundo lugar, se estudia la continuidad uniforme en el entorno de las funciones locálicas reales en locales preuniformes.

Las funciones locálicas se clasifican de acuerdo a las propiedades que las imágenes inversas e imágenes de los sublocales cero satisfacen. Algunas de las clases de las funciones definidas a partir del comportamiento de las imágenes inversas de sublocales cero, generalizan las nociones de sublocales C-, C^*- y z-encajados, a funciones locálicas. Éstas se utilizan para caracterizar normalidad (y variantes débiles de normalidad) de manera similar a como se caracterizan los locales normales, donde todo encaje cerrado es una C-función locálica. Por otro lado, se definen y estudian aquellas funciones que son caracterizadas por el comportamiento de las imágenes de los sublocales cero. Se estudian la relaciones que éstas tienen con otros tipos de funciones, como las funciones locálicas abiertas y cerradas. Este análisis da origen a la investigación de otras tres clases de funciones: w-, n- y wz-funciones locálicas.

Se lleva a cabo un estudio sobre la continuidad uniforme de funciones reales en marcos preuniformes. El próposito es caracterizar la continuidad uniforme de dichos morfismos de marcos a través de la relación de lejanía y proporcionar un teorema de inserción. A partir de este teorema, los resultados de separación y extensión para marcos uniformes se deducen fácilmente como corolarios. Debido a que la demostración del teorema principal utiliza, en gran parte, la noción de prediámetro (que sustituye la idea clásica de pseudométrica), se recordarán y demostrarán algunas propiedades generales de prediámetros. Derivado de esta investigación, se identifican condiciones suficientes para que una escala en un marco preuniforme determine, no sólo una función real continua, si no una uniformemente continua.

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Introduction

Point-free topology is regarded as an algebraic (more specifically, lattice-theoretic) counterpart of classical topology. Locales and frames, the objects of study of point-free topology, are a generalized version of topological spaces where one focuses on their lattices of open sets, leaving aside the points. Replacing the category of topological spaces and continuous maps by the category of locales and their maps has allowed to approach topology from a different perspective, with advantages over the classical setting.

For instance, some classical results with non-constructive content (e.g., that require some version of the axiom of choice) turn out to be provable in the localic setting with constructive arguments, independently of any form of choice. Furthermore, since the dual of the category of locales (the category of frames) is an algebraic category, we have at our disposal such familiar constructions from algebra as free objects and a description of quotients by congruences that provide presentations of frames by generators and relations. For example, instead of working with the frame of open sets of the real-line with their Euclidean topology, Banaschewski in [10] defined the frame of reals through generators and relations.

Therefore, point-free topology is not only a mere translation of classical results to the localic context, but it also offers a new insight into the way we may conceive and work with spaces.

The main motivation of this thesis was to formulate and obtain the localic version of some classical notions and results, taking advantage of all the machinery of point-free topology. First, we were interested on separability characterizations expressed in terms of zero-sets of real functions and related subspaces. More precisely, the goal was to obtain the localic counterpart of topological results regarding normality, z-embeddings, and z-open and z-closed maps. Second, our aim was to expand the theory of insertion theorems on the point-free setting by providing an insertion result for uniform frames.

In order to give a better perspective of the work developed here, we should trace back our steps and take a look at the classical notions and results that inspired this thesis. We are mainly interested in all the theory that has been developed around continuous real-valued functions and the separating conditions they entail.[1] Recall that two subsets Y and Z of a space X are said to be *completely separated* if they can be separated by a continuous real-valued function on X (that is, if $Y \subseteq f^{-1}[\{0\}]$ and $Z \subseteq f^{-1}[\{1\}]$ for some continuous $f\colon X \to \mathbb{R}$). Now, a *zero-set* of a topological space X is a subset that is equal to the preimage of zero under some continuous real-valued function $f\colon X \to \mathbb{R}$

[1] A fundamental reference for this is Gillman and Jerison's book [36].

(that is, a set of the form $f^{-1}[\{0\}]$). On the other hand, a *cozero-set* is the complement of a zero-set, or the preimage of $\mathbb{R} \setminus \{0\}$ under some continuous-real-valued function. Naturally, as a wonderment of how these sets behave for subspaces, comes the defintion of z-embedding. A subspace S of a space X is *z-embedded* if every zero-set of S is equal to the intersection of some zero-set of X with S. There are other types of important and useful embeddings, namely C and C^*-embeddings. A subspace S of a space X is C-*embedded* if any continuous-real valued function on S can be extended to the whole space X. Similarly, a subspace S of a space X is C^*-*embedded* if any bounded continuous-real valued function on S can be extended to X. These three types of embedded sets are related to each other. More precisely, C-embedded implies C^*-embedded which in turn, implies z-embedded. Furthermore, there is the famous Urysohn Extension Theorem ([36, 1.17]) that characterizes C^*-embeddings in terms of complete separation. It states that a subspace S of X is C^*-embedded if and only if every pair of completely separated sets in S is completely separated in the whole space X. As anticipated, these notions play a role characterizing separation axioms such as normality. Let us cite, from [2], the following well-known characterizations of normal spaces.

Theorem A. *The following are equivalent for a space X:*

(i) *X is normal.*

(ii) *Any two disjoint closed sets are completely separated.*

(iii) *Every closed set is* C*-embedded.*

(iv) *Every closed set is* C^**-embedded.*

(v) *Every closed set is z-embedded.*

The equivalence between (i) and (ii) is also known as Urysohn's Separation Lemma, and the one between (i) and (iv) is the famous Tietze's Extension Theorem.

Now, let us shift our attention to the point-free setting. The embedded subsets mentioned before have been extended to point-free topology using cozero elements (which play the role of cozero-sets) and frame quotients (which are the counterpart of subspaces in the category of frames). Ball and Walters-Wayland in [9] do a thorough study of C- and C^*-quotients. They work with these embeddings and related notions such as *coz*-codense and almost *coz*-codense quotients, among others, all of which have a classical counterpart, but approached in a much more algebraic way. There are two main results in [9]. The first one is Theorem 7.1.1, and it is a characterization of C^*-quotients that extends Urysohn's Extension Theorem. The second result is Theorem 7.2.6 which characterizes C-quotients. As an application of all this theory, the authors prove in Corollary 8.3.2 and Theorem 8.3.3 a characterization of normal frames using C- and C^* quotients. That is, the point-free version of equivalences (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (iv) of Theorem A.

Later in [31], Dube and Walters-Wayland do a study of *coz*-onto frame quotients, the counterpart of z-embedded sets. In fact, they give a general definition of *coz*-ontoness for frame homomorphisms, not necessarily frame quotients. They prove a series of characterizations of these frame homomorphisms, Proposition 3.3 being one of the most important ones for this thesis. Finally, in Proposition 4.11 they characterize normal frames in terms of *coz*-onto quotients, which is the extension of the equivalence (i) $\iff$ (v) of Theorem A.

There is another way to extend the notions of C-, C*- and z-embeddings to point-free topology. Classically, these notions are defined for subspaces of a space. In [9] and [31] the authors use frame quotients, but we take a different approach. Instead of working in the category of frames and thinking of quotients as generalized subpaces, we use the category of locales (the dual category of frames) and the subobjects of this category, sublocales. We then define and explore the notions of C-, C* and z-embedded sublocales. This approach has several advantages. First, one is able to differentiate cozero and zero sublocales in much the same way as one studies cozero and zero-sets in classical topology. Second, the notions are more clearly depicted in this language. For instance, complete separation is very easily described with sublocales, but it is quite hard to use and define for quotients. The notions like *coz*-codense and almost *coz*-codense are related to complete separation, but the relation among these concepts seems somewhat obscure in the setting of quotients. Third, some results are formulated and proven in a simpler way. For example, in [31] the proof of the point free version of Urysohn's Extension Theorem requires some background results on the localic Yosida representation, complete separation in Archimedean f-rings and uniformities. The proof that we give uses only basic facts about localic real functions and sublocale lattices. There is one more example of how concise the sublocale language can be. Proposition 4.3 in [31] is stated in terms of frame quotients, but a closer inspection of some of the assertions in this result reveals, when formulated in terms of sublocales, that they express precisely the same fact.

As it often happens in mathematics, looking at the same object from different perspectives, not only helps the intuition, but enriches the theory. In this case, the localic language allowed us to present a different formulation of some known results and extend some notions and results. We extend the notions of C- C*- and z-embeddings to general localic maps, what we call C-, C*- and z-maps. The same is done for *coz*-codense and almost *coz*-codense quotients. By doing this one realizes that all this reduces to the study of the behaviour of zero and cozero sublocales under preimages of localic maps.

Naturally, one then wonders about the localic maps defined by the behaviour of their images on zero sublocales. In spaces, these maps have been studied in Weir's book [79] and Blair and van Douwen's article [23] (where the notions such as z-open and z-closed are discussed). The only place where this has been studied point-freely is in [44], where the authors briefly introduce the notion of z-open and z-closed localic maps and give one result. In this thesis we continue this line of investigation, and extend the study of these maps. Classically, they are related with three other classes of continuous maps, namely W-, WZ- and WN-maps. Therefore, we were led to define the counterpart of these notions for localic maps, using as a guideline Dube's paper [28] (where he extended the notion of WN-maps to frame homomorphisms).

Insertion theorems are of the following nature: given a space X and two, not necessarily continuous, real-valued functions f, g on the space X, $f \geq g$, an insertion result provides necessary and sufficient conditions for the existence of a continuous real-valued function h in between. Usually, f and g are inside a more general class of real-valued functions, and h is required to be continuous, continuous and bounded, etc.

An insertion result usually yields some separation and extension results. Applying the insertion theorem to adequate (characteristic) functions one obtains a (separation) result where certain types of sets are separated by a continuous real-valued function. And then the extension result determines the subspaces of a space where certain classes of real-valued functions on the subspace can be extended to the whole space. Most of the classical insertion theorems have been already extended to point-free topology. For example, in [38, Theorem 8.1] and [66, XIV.7.4.3] the Katětov-Tong Insertion Theorem for normal spaces ([54, 76]) is extended to normal frames. As corollaries of this result, the point-free counterparts of Urysohn's Separation Lemma (which was first proven in [26] and more recent in [9, 10, 66]) and Tietze's Extension Theorem [38, Theorem 8.4] (see also [66, XIV.7.6.1]) are obtained.

Another example is in Gutiérrez García and Kubiak's paper [37] where they prove the point-free extension of the Topological Insertion Theorem due to Blair [21] and Lane [57, 58]. In that same article, [37, Theorem 5.2] is the localic version of Mrówka's Extension Theorem ([63]) for complemented sublocales. This result is quite important for this thesis and will be discussed and generalized in Chapter 3. For more relevant insertion results in point-free topology see e.g. [33, 39, 42].

Looking carefully in the literature to the list of point-free insertion results, one realizes and important gap: an insertion theorem in the uniform setting corresponding to the Preiss-Vilimovský Insertion Theorem for Uniform Spaces:

Topological Insertion Theorem for Uniform Spaces. *Let X be a uniform space and $f,g\colon X \to \mathbb{R}$ two maps with $f \geq g$. Then the following are equivalent:*

(i) *There is a uniformly continuous $h\colon X \to \mathbb{R}$ such that $f \geq h \geq g$.*

(ii) *For ever $\delta > 0$ there is a uniform cover $\mathfrak{U}$ of X such that for all $n \in \mathbb{N}$ the subspaces $f^{-1}(-\infty, r]$ and $g^{-1}[s, +\infty)$ are $\mathsf{St}^n(\mathfrak{U})$-far whenever $s - r > (n+1)\delta$.*

Our main goal was to prove this result in the point-free context, but the existing theory regarding uniform continuity was not enough to even state a result of this nature. Therefore, before proving the desired result, we do a deep study of uniform continuity for general real-valued functions on a locale and introduce the farness relation between sublocales of a locale.

Uniform locales have been studied in point-free topology, and there are several ways to define uniformities. We use the approach via covers introduced by Isbell [47] and Pultr [72, 73]. With this and the study of general real-valued functions in [38] we were able to do two things. First, we defined a relation of farness of sublocales inspired by the classical notion of the proximal relation of farness between sets due to Efremovič and Smirnov [74]. This notion is key for the proof of the insertion theorem; it plays a similar role as the one played by complete separation in the Topological Insertion Theorem in [37] (mentioned previously). Second, we extend the definition of uniform continuity to general, not necessarily continuous, real-valued functions on a frame L. We encounter yet another problem in the process of proving the insertion theorem: the classical proof relied on points and pseudometrics. This led us to revisit prediameters (which are an extension of pseudometrics in point-free topology) and extend some of the results of Pultr in [73]. Finally, as expected, this insertion result also gave rise to an extension and a separation result for uniform frames.

Let us now present a detailed outline of the thesis. In Chapter 1 we recall all the general background on point-free topology needed along the dissertation. Chapter 2 is also part of the preliminaries material. Here we focus on the frame of reals and the corresponding real-valued functions. We survey all the definitions and results regarding this topic.

In Chapter 3 we do a thorough study of the classes of localic maps defined by the behavior of their preimages and images on zero sublocales. In the first four sections of this chapter the notions of C-, C*-, *coz*-onto, *coz*-codense and almost *coz*-codense quotients are not only revisited, but they are also formulated in the more general localic language. Most of the results are an extension to localic maps of results from [9, 31]. One of the most important results in this chapter is Theorem 3.3.6. It is a generalization of Mrówka's Extension Theorem to localic maps, and its Corollary 3.3.7 extends [37, Theorem 5.2] to general sublocales. In the last section of this chapter we do an investigation of z-open, *coz*-open and z-closed localic maps, extending most of the the classical results in [79] to point-free topology. The content of this chapter is mostly based on the author's published papers [6] and [5], the first of which is joint work with Picado.

In Chapter 4 we study normality and variants of this separation axiom in terms of z-embeddings. We first recall the point-free version of the characterization of normality (mentioned above). Then, inspired by the classical result [2, Theorem 7.15], we add some new characterizing conditions to this theorem. There are also other variants of normality, so naturally, one wonders if there are characterizations in terms of z-embeddings for these frames. We identify sufficient conditions under which the characterizations hold for certain variants of normality. As a consequence of this study, we obtain a characterization of mildly normal frames via z-embeddings. Finally, in the last section of this chapter, we collect all the results that characterize certain types of locales in terms of embeddings. Most of the results in this section are not original, but we decided it was worth to make such a survey since these results are scattered around the literature and are rarely formulated in the language of sublocales. The first two sections of this chapter are based on the author's papers [6] (joint with Picado) and [5].

In Chapter 5 we define w-, n- and wz-localic maps. We explore the relation of these notions with some of the classes of localic maps defined in Chapter 3, and provide some examples.

Chapters 6 and 7 are devoted to uniform continuity of real-valued functions on a preuniform frame and the insertion theorem for uniform frames. The structure and organization of these two chapters reflects the way in which our research was conducted. In Chapter 6 we work only with continuous real-valued functions. We define the relation of farness for elements of a frame and characterize uniform continuity of continuous real-valued functions. In this context, the farness relation can be defined through Galois adjunctions, which allows us to present a proof of a separation result for preuniform frames using a purely algebraic (order-theoretic) construction. We also prove a corresponding Tietze-type extension result for dense sublocales in preuniform frames. All this work was previously developed on the author's paper with Jorge Picado [7].

Later on, in Chapter 7 we extend the relation of farness to sublocales and generalize the results of the previous chapter to general real-valued functions. We define uniform continuity of a general real-valued function and characterize it in terms of the farness relation. Finally, we prove the desired insertion theorem and obtain general extension and separation results as corollaries. The content of this chapter is based on the author's paper with Igor Arrieta [3].

Chapter 1

Preliminaries I: Frames and Locales

This chapter is a collection of the relevant (for our purposes) background on frames and locales. The intention is to provide the reader with the basic notions in point-free topology and fix the notation that will be used throughout the thesis. Our main references are [50] and the more recent [66]. For general category theory we refer to [1] and [60].

1.1 Galois Adjunctions

A *Galois adjunction* [32] between posets A and B is a pair (f,g) of maps $f\colon A \to B$ and $g\colon B \to A$ such that

$$f(a) \leq b \iff a \leq g(b) \quad \text{for all } a \in A \text{ and } b \in B, \tag{1.1.1}$$

or, equivalently, a pair of monotone maps $f\colon A \to B$ and $g\colon B \to A$ satisfying

$$a \leq g(f(a)) \text{ for all } a \in A \;\text{ and }\; f(g(b)) \leq b \text{ for all } b \in B.$$

The maps f and g uniquely determine each other. We say f is the *right adjoint* and g the *left adjoint*, and write $f \dashv g$. Sometimes we will also write f^* (resp. f_*) to denote the left (resp. right adjoint) of a map f. If A and B are complete lattices, a monotone map $f\colon A \to B$ is a left (resp. right) adjoint if and only if it preserves all suprema (resp. infima). Furthermore, its right adjoint (resp. left) is given by the formula:

$$f_*(b) = \bigvee \{a \in A \mid f(a) \leq b\} \quad (\text{resp. } f^*(b) = \bigwedge \{a \in A \mid b \leq f(a)\}). \tag{1.1.2}$$

Originally, *Galois connections* were expressed in a contravariant form with maps that reverse order ([18, 64]); these are dual adjunctions between posets A and B, that is, pairs (f,g) of maps $f\colon A \to B$ and $g\colon B \to A$ such that

$$b \leq f(a) \iff a \leq g(b) \quad \text{for all } a \in A \text{ and } b \in B, \tag{1.1.3}$$

or, equivalently, pairs of order-reversing maps $f\colon A \to B$ and $g\colon B \to A$ satisfying

$$a \leq g(f(a)) \text{ for all } a \in A \;\text{ and }\; b \leq f(g(b)) \text{ for all } b \in B.$$

If we take the maps $f'\colon A\to B^{op}$ and $g'\colon B^{op}\to A$ (B^{op} being the dual poset of B) given by f and g respectively, then (f,g) is a Galois connection if and only if $f'\dashv g'$. Similarly, for $f''\colon A^{op}\to B$ and $g''\colon B\to A^{op}$, we have that (f,g) is a Galois connection if and only if $g''\dashv f''$. Thus, if A and B are complete lattices, f is a complete join-homomorphism from A to B^{op}, while g is a complete join-homomorphism $B\to A^{op}$. Clearly, (f,g) is a Galois connection if and only if (g,f) is one. Both composites of the partners of a Galois connection are closure operators, and their ranges are dually isomorphic.

1.2 Frames and Locales

Frames and Locales

A *frame* (or *locale*) L is a complete lattice in which the following distributive law holds

$$a\wedge\bigvee S=\bigvee\{a\wedge s\mid s\in S\} \tag{1.2.1}$$

for all $a\in L$ and $S\subseteq L$. A *frame homomorphism* is a map $h\colon L\to M$ between frames that preserves arbitrary joins (in particular, the bottom element 0) and finite meets (in particular, the top element 1). These objects and homomorphisms constitute the category of frames, that we will denote by Frm.

A generalization of the notion of frame, namely σ-frame (introduced in [24]), will appear naturally when working with cozero elements (see Section 2.3). A *σ-frame* A is a lattice in which each countable subset has a join and the distributive law (1.2.1) holds for any $a\in A$ and countable $S\subseteq A$. A *σ-frame homomorphism* is a map $h\colon A\to B$ between σ-frames that preserves countable joins and finite meets. The category of σ-frames and σ-frame homomorphisms is denoted by $\sigma\mathsf{Frm}$.

The Heyting Operator

In any frame L, for any $a\in L$, the mapping $(x\mapsto(a\wedge x))\colon L\to L$ preserves arbitrary joins (by (1.2.1) holds); hence, it has a right Galois adjoint $(y\mapsto(a\to y))\colon L\to L$, satisfying

$$a\wedge x\le y \iff x\le a\to y \tag{1.2.2}$$

and making L a complete Heyting algebra. The formula in (1.1.2) gives an easy way to compute the Heyting operator:

$$a\to y=\bigvee\{x\in L\mid a\wedge x\le y\}.$$

Being $a\to(-)\colon L\to L$ a right adjoint, for each $x,y\in L$ and $\{b_i\}_{i\in J}\subseteq L$ we have

$$x\le y \implies a\to x\le a\to y \quad\text{and}\quad a\to\Big(\bigwedge_{i\in J}b_i\Big)=\bigwedge_{i\in J}(a\to b_i). \tag{1.2.3}$$

Moreover, equation (1.2.2) yields

$$a\le b\to c \iff b\le a\to c \tag{1.2.4}$$

for any $a,b,c \in L$, showing that the map $(y \mapsto (y \to c))\colon L \to L$ is a self-dual Galois adjoint. That is, $((-) \to c, (-) \to c)$ is a Galois connection for every $c \in L$. Thus, for every $x,y \in L$ and any $\{a_i\}_{i\in J} \subseteq L$, one has

$$\Big(\bigvee_{i\in I} a_i\Big) \to c = \bigwedge_{i\in I}(a_i \to c) \quad \text{and} \quad x \le y \implies y \to c \le x \to c. \tag{1.2.5}$$

Further properties of the Heyting operator:

(H1) $1 \to a = a$,

(H2) $a \le b$ if and only if $a \to b = 1$,

(H3) $a \le b \to a$,

(H4) $a \to b = a \to (a \wedge b)$,

(H5) $a \wedge (a \to b) = a \wedge b$,

(H6) $a \wedge b = a \wedge c$ if and only if $a \to b = a \to c$,

(H7) $(a \wedge b) \to c = a \to (b \to c)$ and $a \to (b \to c) = b \to (a \to c)$,

(H8) $a = (a \vee b) \wedge (b \to a)$,

(H9) $a \le (a \to b) \to b$,

(H10) $((a \to b) \to b) \to b = a \to b$.

The *pseudocomplement* of $a \in L$ is the element

$$a^* := a \to 0 = \bigvee \{x \in L \mid x \wedge a = 0\}.$$

An element a of a frame L is *regular* if $a^{**} = a$, and we denote by L^* the set of all regular elements of L. An $a \in L$ is *complemented* if $a \vee a^* = 1$. Every complemented element is regular, but the converse is not true. For any $a,b \in L$:

(1) $a \wedge a^* = 0$,

(2) $a \le a^{**}$,

(3) $a^{***} = a^*$,

(4) $a \le b$ implies $b^* \le a^*$,

(5) $(\bigvee_{i\in I} a_i)^* = \bigwedge_{i\in I} a_i^*$ for any $\{a_i\}_{i\in J} \subseteq L$.

Localic maps

The category of locales and localic maps, denoted by Loc, is the dual category of Frm; that is, $\mathsf{Loc} = \mathsf{Frm}^{op}$. Localic maps can be seen as more than reversed arrows. Indeed, since frame homomorphisms preserve arbitrary meets, a frame homomorphism $h\colon L \to M$ has a unique right adjoint $h_*\colon M \to L$. Then a map between locales $f\colon M \to L$ is a localic map (that is, the unique right adjoint of a frame homomorphism) if and only if

(1) it preserves arbitrary meets,

(2) $f(h(a \to b)) = a \to f(b)$ for every $a \in M$ and $b \in L$ and

(3) $f(a) = 1 \implies a = 1$ (where h is the left adjoint of f).

Locales and spaces

For any topological space X, the lattice of open sets $\Omega(X)$ is a frame. Moreover, if $f\colon X \to Y$ is a continuous map between topological spaces, the preimage $f^{-1}[-]\colon \Omega(Y) \to \Omega(X)$ is a frame homomorphism, giving us a functor $\Omega\colon \mathsf{Top} \to \mathsf{Loc} = \mathsf{Frm}^{op}$ from the category of topological spaces to the category of locales. This functor has a right adjoint $\Sigma\colon \mathsf{Loc} \to \mathsf{Top}$ defined below.
The *points* of a locale L are the *prime* (or *meet-irreducible*) elements, that is, the $p \in L \setminus \{1\}$ such that $p = a \wedge b$ implies $p = a$ or $p = b$ (equivalently, $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$). The set of all prime elements of L is denoted by $\mathsf{Pt}(L)$. A special kind of points are the *covered prime* elements of L that satisfy the condition

$$p = \bigwedge S \implies p \in S$$

for any $S \subseteq L$ ([17]). For every $a \in L$, set $\Sigma_a = \{p \in \mathsf{Pt}(L) \mid a \nleq p\}$. The family $\{\Sigma_a \mid a \in L\}$ is a topology on $\mathsf{Pt}(L)$. We write

$$\Sigma(L) = (\mathsf{Pt}(L), \{\Sigma_a \mid a \in L\})$$

to denote the respective topological space (referred to as the *spectrum* of L). Since localic maps send prime elements to prime elements, a localic map $f\colon L \to M$ induces a continuous map $\Sigma(f)\colon \Sigma(L) \to \Sigma(M)$ defined by the restriction of f to the points of L. One gets a functor $\Sigma\colon \mathsf{Loc} \to \mathsf{Top}$ and an adjunction

$$\mathsf{Top} \underset{\Sigma}{\overset{\Omega}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{Frm}\ . \tag{1.2.6}$$

The components of the unit η and counit ε of this adjunction are given as follows:

$$\begin{array}{rcl} \eta_X : X & \to & \Sigma(\Omega(X)) \\ x & \mapsto & X - \overline{\{x\}} \end{array} \quad \text{and} \quad \begin{array}{rcl} \varepsilon_L \colon \Omega(\Sigma(L)) & \to & L \\ \Sigma_a & \mapsto & \bigvee\{b \mid \Sigma_b \subseteq \Sigma_a\} \end{array}.$$

A space X is said to be *sober* if η_X is an homeomorphism; equivalently if $X = \Sigma L$ for some frame L. The map η_X is known as the *soberification* of a space X. A frame L is *spatial* if $L \cong \Omega(X)$ for some space X; equivalently, if ε_L is an isomorphism. The right adjoint of ε_L is known as the *spatialization* of a frame L. The adjunction (1.2.6) restricts to an equivalence of categories between the full subcategory Sob of sober topologial spaces and the subcategory SLoc of spatial locales.

The category Frm is algebraic

The category Frm is equationally presentable (i.e., its objects are described by a proper class of operations and equations) and algebraic (that is, the forgetful functor Frm $\to$ Set is monadic). Let us mention some of the consequences of this fact taken from [50, I.3.8] and [68, 4.3]:

(1) Frm is complete and cocomplete. Thus, Loc is also complete and cocomplete.

(2) Quotients are described by congruences.

(3) One has free frames. The free functor from Frm to Set is built by composing the free functor Frm $\to$ Slat and the free functor Slat $\to$ Set , where Slat is the category of meet-semilattices with top (we assume infima of all finite subsets) and their $(\wedge, 1)$-homomorphisms ([50, 66]). Specifically, what we have is that given a set G of generators and a set R of relations (equalities given by combinations of arbitrary joins and finite meets), there is a frame F and a function $\eta_G \colon G \to F$ with the following universal property: For every frame L and every map $f\colon G \to L$ that turns the relations R into identities in L, there is a unique frame homomorphism $\overline{f}\colon F \to L$ such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\eta_G} & F \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \overline{f}} \\ & & L. \end{array} \tag{1.2.7}$$

(4) Monomorphisms in Frm are precisely the one-one frame homomorphisms, and regular epimorphisms (which in this case concide with both extremal and strong epimorphisms) are exactly the onto frame homomorphisms. Dually, epimorphisms in Loc are the onto localic maps, and regular monomorphisms (that concide with both extremal and strong monomorphsims) are precisely the one-one localic maps. A frame homomorphism $h\colon L \to M$ can be naturally decomposed as follows

$$\begin{array}{ccc} L & \xrightarrow{\quad h \quad} & M \\ {\scriptstyle \overline{h}}\searrow & & \nearrow{\scriptstyle k} \\ & h[M] & \end{array} \tag{1.2.8}$$

where $\overline{h}(x) = h(x)$ and k is the inclusion. The frame homomorphism $\overline{h}$ is onto and k is one-one. This yields a factorization system $(\mathscr{E}, \mathscr{M})$ in Frm with $\mathscr{E}$ the class of all strong (extremal, regular) epimorphisms and $\mathscr{M}$ the class of all monomorphisms. Similarly, in Loc we have the factorization system $(\mathscr{E}, \mathscr{M})$ with $\mathscr{E}$ the class of epimorphisms and $\mathscr{M}$ the class of all strong (extremal, regular) monomorphisms. A localic map $f\colon M \to L$ then decomposes into:

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & L \\ {\scriptstyle \overline{f}}\searrow & & \nearrow{\scriptstyle j} \\ & f[M] & \end{array} \tag{1.2.9}$$

where $\overline{f}(x) = f(x)$ and j is the inclusion.

1.3 Subfit, Regular, Completely Regular and Normal Frames

Subfitness

In point-free topology, the role of the classical T_1-axiom is usually taken by the so called subfit axiom (see [69])[1]. One speaks of a *subfit frame* whenever

$$a \nleq b \implies \exists c,\ a \vee c = 1 \neq b \vee c.$$

Regularity

For $a, b \in L$, a is *rather below* b, and one writes $a \prec b$, if $a^* \vee b = 1$ (equivalently, if there is a $c \in L$ such that $a \wedge c = 0$ and $c \vee b = 1$).

Properties 1.3.1. (1) $a \prec b \implies a \leq b$.

(2) $0 \prec a \prec 1$.

(3) $x \leq a \prec b \leq y \implies x \prec y$.

(4) $a \prec b \implies b^* \prec a^*$.

(5) $a \prec b \implies a^{**} \prec b$.

(6) If $a_i \prec b_i$ for $i = 1, 2$, then $a_1 \vee a_2 \prec b_1 \vee b_2$ and $a_1 \wedge a_2 \prec b_1 \wedge b_2$.

(7) If a is complemented, then $a \prec a$.

A frame L is said to be *regular* if $a = \bigvee \{x \in L \mid x \prec a\}$ for every $a \in L$.

Complete Regularity

For $a, b \in L$, a is *completely below* b, and one writes $a \prec\!\!\prec b$, if there is a subset

$$\{a_r \mid r \in \mathbb{Q},\ 0 \leq r \leq 1\} \subseteq L$$

such that $a_0 = a$, $a_1 = b$ and $a_r \prec a_s$ for every $r < s$. The relation $\prec\!\!\prec$ is the largest interpolative relation[2] contained in $\prec$. Thus Properties 1.3.1 yield:

Properties 1.3.2. (1) $a \prec\!\!\prec b \implies a \leq b$.

(2) $0 \prec\!\!\prec a \prec\!\!\prec 1$.

(3) $x \leq a \prec\!\!\prec b \leq y \implies x \prec\!\!\prec y$.

(4) $a \prec\!\!\prec b \implies b^* \prec\!\!\prec a^*$.

(5) $a \prec\!\!\prec b \implies a^{**} \prec\!\!\prec b$.

[1] In spaces, the subfit property is in fact slightly weaker than T_1.
[2] A relation R is interpolative if $aRb \implies \exists c,\ aRcRb$.

(6) If $a_i \prec\!\!\prec b_i$ for $i = 1,2$, then $a_1 \vee a_2 \prec\!\!\prec b_1 \vee b_2$ and $a_1 \wedge a_2 \prec\!\!\prec b_1 \wedge b_2$.

(7) If a is complemented, then $a \prec\!\!\prec a$.

A frame L is *completely regular* if $a = \bigvee\{x \in L \mid x \prec\!\!\prec a\}$ for every $a \in L$. Every completely regular frame is regular, and every regular frame is subfit. Regularity and complete regularity are conservative notions; that is, a topological space X is regular (resp. completely regular) if and only if the frame $\Omega(X)$ is regular (resp. completely regular).

Normality

A frame L is said to be *normal* if $a \vee b = 1$ for $a, b \in L$ implies the existence of $u, v \in L$ such that $u \wedge v = 0$ and $a \vee u = 1 = b \vee v$.

Proposition 1.3.3. [66, V.5.9.1] *In any normal frame, the relation $\prec$ is interpolative. Consequently, in normal frames $\prec\!\!\prec$ coincides with $\prec$, and regularity coincides with complete regularity.*

Proposition 1.3.4. [66, V.5.9.2] *Normality in conjunction with subfitness implies complete regularity.*

Regularity and Normality in σ-Frames

Let A be a σ-frame and $a, b \in A$. One writes $a \prec b$ if there is a $c \in A$ such that $a \wedge c = 0$ and $c \vee b = 1$. A σ-frame A is *regular* if every element a can be written as $\bigvee_{n\in\mathbb{N}} a_n$ for some $a_n \prec a$. Normality and the completely below relation for σ-frames are defined exactly like in general frames. Any regular σ-frame is normal ([13, Corollary 2]), and in particular, the relation $\prec$ interpolates. Hence, in a regular σ-frame A,

$$a = \bigvee_{n\in\mathbb{N}} a_n \quad \text{with } \ a_n \prec\!\!\prec a$$

for any $a \in A$.

1.4 The Coframe $\mathsf{S}(L)$ of Sublocales

Sublocales

A subset S of a locale L is a *sublocale* if it is a locale in the induced order and the embedding map $j\colon S \hookrightarrow L$ is a localic map. Equivalenty, a subset $S \subseteq L$ is a sublocale if it is closed under all meets, and for every $s \in S$ and every $x \in L$, $x \to s \in S$. In particular, $1 \in S$, and the Heyting operation in S coincides with that in L.

There are alternative ways to describe sublocales (see [50] and [66] for these descriptions): congruences, nuclei and frame quotients. A *frame quotient* of a frame L is a frame surjection $f\colon L \twoheadrightarrow M$. Since onto frame homomorphisms are left adjoints of injective localic maps, isomorphism classes of frame surjections with domain L are in bijective correspondence with sublocales of a locale L. If $j\colon S \hookrightarrow L$ is a localic embedding, then the left adjoint $j^*: L \twoheadrightarrow S$, given by $j^*(x) = \bigwedge\{s \in S \mid s \geq x\}$, is a frame quotient. Conversely, if $h\colon L \twoheadrightarrow M$ is an onto frame homomorphism, then $h_*[M]$ is a sublocale of L.

Of course, arbitrary meets and the Heyting operation in a sublocale S of L concide with those in L. In particular, $1_S = 1_L$. In general, the bottom element in S, given by $0_S = \bigwedge S$, differs from the zero in L. Consequently, pseudocomplements are different too:[3]

$$a^{*_L} \leq a^{*_S} = a \to 0_S.$$

Arbitrary joins in S are given by the formula

$$\bigvee^{S} \{a_i \mid i \in J\} = \bigwedge \Big\{ s \in S \mid s \geq \bigvee^{L}_{i \in J} a_i \Big\}$$

for any $\{a_i\}_{i \in J} \subseteq S$.

The Coframe of Sublocales

The system $\mathsf{S}(L)$ of all sublocales of L, partially ordered by inclusion $\subseteq$, is a *coframe*; that is, its dual lattice $\mathsf{S}(L)^{op}$ is a frame. The top element in $\mathsf{S}(L)$ is L and the bottom is the sublocale $\{1\}$, which we will denote by O. Infima and suprema[4] in $\mathsf{S}(L)$ are given by

$$\bigwedge_{i \in J} S_i = \bigcap_{i \in J} S_i \quad \text{and} \quad \bigvee_{i \in J} S_i = \Big\{ \bigwedge M \mid M \subseteq \bigcup_{i \in J} S_i \Big\}. \tag{1.4.1}$$

$\mathsf{S}(L)$, being a coframe, is a co-Heyting algebra. Thus, there is a co-Heyting operator $S \smallsetminus T$ in $\mathsf{S}(L)$ given by the formula $\bigcap \{A \in \mathsf{S}(L) \mid S \subseteq T \vee A\}$ and characterized by the condition

$$S \smallsetminus T \subseteq A \iff S \subseteq T \vee A$$

for any $S, T, A \in \mathsf{S}(L)$. We give a list of properties of this operator ([34]):

(1) $(\bigvee_{i \in J} S_i) \smallsetminus T = \bigvee_{i \in J} (S_i \smallsetminus T)$,

(2) $S \smallsetminus T \subseteq A \iff S \smallsetminus A \subseteq T$,

(3) $S \smallsetminus \bigwedge_{i \in J} T_i \iff \bigvee_{i \in J} (S \smallsetminus T_i)$,

(4) $S \smallsetminus S = \mathsf{O}$ and $S \smallsetminus \mathsf{O} = S$ for all $S \in \mathsf{S}(L)$,

(5) $S \subseteq T \implies S \smallsetminus T = \mathsf{O}$,

(6) $S \smallsetminus T \subseteq S$,

(7) $S \subseteq T \implies S \smallsetminus A \subseteq T \smallsetminus A$,

(8) $S \subseteq T \implies A \smallsetminus T \subseteq A \smallsetminus S$.

[3] In general, to avoid confusion, we will use a subscript (or superscript for joins) to refer to the locale in which the element or operation belongs.

[4] We will always use the inclusion, the intersection and the join as described above to write the operations between sublocales, even when working in the frame $\mathsf{S}(L)^{op}$. We will not introduce a new notation to refer to the order, joins and meets, and top and bottom element in $\mathsf{S}(L)^{op}$.

In particular, the *co-pseudocomplement* of S (also called *supplement* [34]) is the sublocale $S^{\#} := L \smallsetminus S$. Then $S^{\#} \vee S = L$ and

$$S \vee T = L \iff S^{\#} \subseteq T \quad \text{and} \quad S \cap T = \mathsf{O} \implies S \subseteq T^{\#} \tag{1.4.2}$$

for every $S, T \in \mathsf{S}(L)$. For any $S, T \in \mathsf{S}(L)$:

(1) $S \subseteq T \implies T^{\#} \subseteq S^{\#}$,

(2) $S^{\#\#} \subseteq S$ and $S^{\#\#\#} = S^{\#}$,

(3) $\mathsf{O}^{\#} = L$ and $S^{\#} = \mathsf{O} \iff S = L$,

(4) $(\bigcap_{i \in J} S_i)^{\#} = \bigvee_{i \in J} S_i^{\#}$ for any $\{S_i\}_{i \in J} \subseteq \mathsf{S}(L)$.

We refer to [34] for the proof of these properties and for more information about supplements in $\mathsf{S}(L)$.

A sublocale S is *complemented* if $S^{\#} \cap S = \mathsf{O}$. In this case one then gets

$$S \cap T = \mathsf{O} \iff T \subseteq S^{\#} \tag{1.4.3}$$

for any $T \in \mathsf{S}(L)$. The distributive law

$$S \cap \bigvee_{i \in I} T_i = \bigvee_{i \in I} (S \cap T_i) \tag{1.4.4}$$

holds whenever S is complemented ([66, VI.4.4.1]).

Closed and Open Sublocales

For each $a \in L$, the sublocales

$$\mathfrak{c}_L(a) = {\uparrow} a = \{x \in L \mid x \geq a\} \quad \text{and} \quad \mathfrak{o}_L(a) = \{a \to b \mid b \in L\}$$

are the *closed* and *open* sublocales of L, respectively[5]. For each $a \in L$, the sublocales $\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ are complements of each other in $\mathsf{S}(L)$ and satisfy the identities

$$\mathfrak{c}\Big(\bigvee_{i \in I} a_i\Big) = \bigcap_{i \in I} \mathfrak{c}(a_i), \quad \mathfrak{c}(a \wedge b) = \mathfrak{c}(a) \vee \mathfrak{c}(b), \quad \mathfrak{c}(0) = L, \quad \mathfrak{c}(1) = \mathsf{O}, \tag{1.4.5}$$

$$\mathfrak{o}\Big(\bigvee_{i \in I} a_i\Big) = \bigvee_{i \in I} \mathfrak{o}(a_i), \quad \mathfrak{o}(a \wedge b) = \mathfrak{o}(a) \cap \mathfrak{o}(b), \quad \mathfrak{o}(0) = \mathsf{O} \quad \text{and} \quad \mathfrak{o}(1) = L. \tag{1.4.6}$$

Open sublocales have a further distributivity property:

$$S \cap \bigvee_{i \in I} \mathfrak{o}(a_i) = \bigvee_{i \in I} S \cap \mathfrak{o}(a_i) \tag{1.4.7}$$

for every $\{a_i\}_{i \in I} \subseteq L$ and every $S \in \mathsf{S}(L)$ (see [70]).

[5] We shall denote them simply by $\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ when there is no danger of confusion.

For every $a \in L$, $\downarrow a := \{x \in L \mid x \leq a\}$ is a frame, isomorphic in Frm to $\mathfrak{o}(a)$. The onto frame homomorphism $r_a \colon L \twoheadrightarrow \downarrow a$ with $r_a(x) = a \wedge x$ is the left adjoint of the localic embedding $\mathfrak{o}(a) \hookrightarrow L$.

Furthermore, for every sublocale S of L,

$$S = \bigcap_{i \in J} \mathfrak{c}(a_i) \wedge \mathfrak{o}(b_i)$$

for suitable subsets of L, $\{a_i\}_{i \in J}$ and $\{b_i\}_{i \in J}$ ([66, III.6.5, VI.4]), making $\mathsf{S}(L)^{op}$ a *zero-dimensional frame* (i.e., a frame where each element is a join of complemented ones). Therefore, $\mathsf{S}(L)^{op}$ is completely regular; in particular, it is also subfit.

We denote by $\mathfrak{c}(L)$ and $\mathfrak{o}(L)$ the classes of all closed and open sublocales of L, respectively, partially ordered by inclusion. The identities (1.4.5) and (1.4.6) make $\mathfrak{o}(L)$ a subframe of $\mathsf{S}(L)$ and $\mathfrak{c}(L)^{op}$ a subframe of $\mathsf{S}(L)^{op}$; both frames are isomorphic to L. Furthermore, the cannonical map[6]

$$\begin{array}{rcl} \mathfrak{c}_L \colon L & \to & \mathsf{S}(L)^{op} \\ a & \mapsto & \mathfrak{c}(a) \end{array} \tag{1.4.8}$$

is injective, since it is the composition of the isomorphism $\iota \colon L \to \mathfrak{c}(L)^{op}$ and the embedding $\mathfrak{c}(L)^{op} \hookrightarrow \mathsf{S}(L)^{op}$, and it has a universal property. Namely, if $h \colon L \to M$ is a frame homomorphism such that the image of every $a \in L$ is complemented in M, then there is a unique frame homomorphism $\overline{h} \colon \mathsf{S}(L)^{op} \to M$ such that the following diagram commutes:

$$\begin{array}{ccc} L & \xrightarrow{\ \mathfrak{c}_L\ } & \mathsf{S}(L)^{op} \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle \overline{h}} \\ & & M. \end{array}$$

Closure and Interior

The *closure* $\overline{S}$ of a sublocale S of L is the smallest closed sublocale containing S, and the *interior* S° is the largest open sublocale contained in S. That is,

$$S^\circ = \bigvee \{\mathfrak{o}(a) \mid \mathfrak{o}(a) \subseteq S\},$$

and for the closure there is a particularly simple formula:

$$\overline{S} = \mathfrak{c}\Big(\bigwedge S\Big). \tag{1.4.9}$$

Hence,

$$\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*) \quad \text{and} \quad \mathfrak{c}(a)^\circ = \mathfrak{o}(a^*) \tag{1.4.10}$$

[6] We shall denote it simply by $\mathfrak{c}$ when there is no danger of confusion.

from which it follows that

$$a \prec b \iff \mathfrak{c}(b) \subseteq \mathfrak{c}(a)^\circ \iff \overline{\mathfrak{o}(a)} \subseteq \mathfrak{o}(b), \tag{1.4.11}$$

for any $a, b \in L$. More generally,

$$(T^\circ)^\# = \overline{T^\#} \tag{1.4.12}$$

for every sublocale S ([34, Equation 4.2]). A sublocale S of L is *dense* if its closure is equal to L ($\overline{S} = L$); equivalently, by (1.4.9), if $0 \in S$.

Sublocales of a Sublocale

Let S be a sublocale of L. The following properties regarding $\mathsf{S}(S)$ hold:

(1) A sublocale T of S is also a sublocale of L.

(2) $\mathsf{S}(S) = \{T \cap S \mid T \in \mathsf{S}(L)\}$ and the lattice operations in $\mathsf{S}(S)$ are given by those of $\mathsf{S}(L)$. Nevertheless, $\mathsf{S}(S)$ and $\mathsf{S}(L)$ may have different top elements.

(3) For any open (resp. closed) sublocale T of L, $T \cap S$ is an open (resp. closed) sublocale of S. More precisely, if $T = \mathfrak{o}_L(a)$ (resp. $T = \mathfrak{c}_L(a)$) and $j_S \colon S \hookrightarrow L$ is the localic embedding of S in L,

$$T \cap S = \mathfrak{o}_S(j_S^*(a)) \quad (\text{resp. } \; T \cap S = \mathfrak{c}_S(j_S^*(a)). \tag{1.4.13}$$

(4) If U is an open (resp. closed) sublocale of S, then $U = T \cap S$ for some open sublocale $T = \mathfrak{o}_L(a)$ of L (resp. closed sublocale $T = \mathfrak{c}_L(a)$ of L) with $a \in S$. More precisely, if $U = \mathfrak{o}_S(a)$ (resp. $U = \mathfrak{c}_S(a)$) for some $a \in L$, then $U = \mathfrak{o}_S(a) = \mathfrak{o}_L(a) \cap S$ (resp. $U = \mathfrak{c}_S(a) = \mathfrak{c}_L(a) \cap S$).

(5) For any sublocale T of S, its closure in S, denoted by $\overline{T}^S$, is precisely the intersection of its closure in L with S. That is, $\overline{T}^S = \overline{T} \cap S$.

(6) From (5), it is clear that every sublocale is dense in its closure.

(7) If S is complemented, the relative pseudocomplement $T^{\#_S}$, in $\mathsf{S}(S)$, of a sublocale T of S is computed as

$$T^{\#_S} = S \cap T^\#. \tag{1.4.14}$$

Boolean Sublocales

For each $a \in L$, the *boolean sublocale*

$$\mathfrak{b}(a) := \{x \to a \mid x \in L\}$$

is the smallest sublocale containing a. The bottom element of a boolean sublocale $\mathfrak{b}(a)$ is a; hence

$$\overline{\mathfrak{b}(a)} = \mathfrak{c}\Big(\bigwedge \mathfrak{b}(a)\Big) = \mathfrak{c}(a). \tag{1.4.15}$$

More generally, for any sublocale S of L, $\mathfrak{b}(\bigwedge S)$ is the smallest dense sublocale of S. In particular, any frame L has a smallest dense sublocale $\mathfrak{b}(0)$. For any $p \in \mathsf{Pt}(L)$ and $x \in L$,

$$p = (x \vee p) \wedge (x \to p) \tag{1.4.16}$$

and, therefore, $p = x \vee p$ or $p = x \to p$. Hence $\mathfrak{b}(p) = \{1, p\}$ (these are called the *one-point sublocales* [66, III.10.1]).

F_σ- and G_δ-Sublocales

A sublocale F of L is an *F_σ-sublocale* if it is a countable join of closed sublocales. The dual notion is that of *G_δ-sublocale*; i.e., a sublocale that is a countable intersection of open sublocales.

Remark 1.4.1. Unlike classical topology, the supplement of an F_σ-sublocale need not be a G_δ-sublocale. Nevertheless, since $\mathsf{S}(L)$ is a coframe, the supplement of a G_δ-sublocale is an F_σ-sublocale.

Subfitness and Normality in Terms of Sublocales

Later we will need the separating axioms of subfitness and normality phrased in terms of sublocales instead of their first order definitions.

Proposition 1.4.2. [66, V.1.4] *The following conditions are equivalent for a frame L:*

(i) *L is subfit.*

(ii) *If $S \neq L$ for a sublocale $S \subseteq L$, then there is a closed $\mathfrak{c}(x) \neq \mathsf{O}$ such that $S \cap \mathfrak{c}(x) = \mathsf{O}$.*

It follows from the identities (1.4.5) and (1.4.6) that a frame L is normal if and only if for every disjoint pair[7] of closed sublocales $\mathfrak{c}(a)$ and $\mathfrak{c}(b)$ there are disjoint open sublocales $\mathfrak{o}(u)$ and $\mathfrak{o}(v)$ such that $\mathfrak{c}(a) \subseteq \mathfrak{o}(u)$ and $\mathfrak{c}(b) \subseteq \mathfrak{o}(v)$.

1.5 Images and Preimages of Localic Maps

Let $f\colon L \to M$ be a localic map. For any sublocale S of L, its set-theoretic image $f[S]$ is a sublocale of M. On the other hand, the set-theoretic preimage $f^{-1}[T]$ of a sublocale T of M may not be a sublocale of L. But, since f is meet preserving, $f^{-1}[T]$ is closed under meets, and thus, by formula (1.4.1), there exists the largest sublocale of L contained in $f^{-1}[T]$, usually denoted as $f_{-1}[T]$ ([66, III.4]). This is the *localic preimage* of T that provides the image/preimage Galois adjunction

$$\mathsf{S}(L) \underset{f_{-1}[-]}{\overset{f[-]}{\rightleftarrows}} \mathsf{S}(M) \quad (\bot)$$

between coframes $\mathsf{S}(L)$ and $\mathsf{S}(M)$ of sublocales of L and sublocales of M, respectively. This means that

$$f[S] \subseteq T \iff S \subseteq f_{-1}[T],$$

[7] We will say that two sublocales S and T are disjoint whenever $S \cap T = \mathsf{O}$.

and, consequently,

$$S \subseteq f_{-1}[f[S]] \quad \text{and} \quad f[f_{-1}[T]] \subseteq T$$

for every $S \in \mathsf{S}(L)$ and $T \in \mathsf{S}(M)$. The right adjoint $f_{-1}[-]$ is a coframe homomorphism that preserves complements while $f[-]$ is a colocalic map ([66, III.9]).

Being $f_{-1}[-]$ a coframe homomorphism it satisfies the following properties:

(1) $f_{-1}[\mathsf{O}_M] = \mathsf{O}_L$,

(2) $f_{-1}[M] = L$,

(3) $f_{-1}[S \vee T] = f_{-1}[S] \vee f_{-1}[T]$,

(4) $f_{-1}[\bigcap_{i\in J} S_i] = \bigcap_{i\in J} f_{-1}[S_i]$.

Being $f[-]$ a colocalic map the following properties hold:

(1) $f[S] = \mathsf{O}_M \implies S = \mathsf{O}_L$ (in particular, $f[\mathsf{O}_L] = f[\mathsf{O}_M]$),

(2) $f[\bigvee_{i\in J} S_i] = \bigvee_{i\in J} f[S_i]$,

(3) $f[S \smallsetminus f_{-1}[T]] = f[S] \smallsetminus T$.

Remark 1.5.1. Let S be a sublocale of L, and let $j\colon S \hookrightarrow L$ be the localic embedding of S in L. Then the preimage $j_{-1}[-]\colon \mathsf{S}(L) \to \mathsf{S}(S)$ is simply given by the intersection: $j_{-1}[T] = T \cap S$ for every $T \in \mathsf{S}(L)$.

Localic preimages of closed (resp. open) sublocales are closed (resp. open). More specifically, denoting by f^* the frame homomorphism left adjoint to f, we have

$$f_{-1}[\mathfrak{c}_M(a)] = \mathfrak{c}_L(f^*(a)) \quad \text{and} \quad f_{-1}[\mathfrak{o}_M(a)] = \mathfrak{o}_L(f^*(a)) \tag{1.5.1}$$

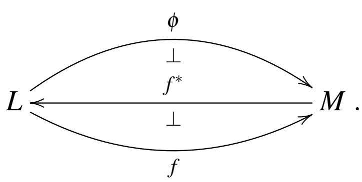
for any $a \in M$.

Closed and Open Localic Maps

A localic map $f\colon L \to M$ is *closed* if the image of every closed sublocale is closed. Since $\mathfrak{c}(a) = \uparrow a$, if f is closed, then $f[\mathfrak{c}_L(a)] = \mathfrak{c}_M(f(a))$ for any $a \in L$. Furthermore, a localic $f\colon L \to M$ is said to be *open* if the image of every open sublocale is open. There is an important characterization of localic maps due to Joyal and Tierney [52]:

Proposition 1.5.2. *A localic map $f\colon L \to M$ is open if and only if f^* is a complete Heyting homomorphism.*

Remark 1.5.3. The fact that f^* is a complete Heyting homomorphism implies the existence of a left adjoint ϕ (since f^* preserves arbitrary meets). Thus, if f is open we have the following situation:



(1.5.2)

(1) If f is open, then $f[\mathfrak{o}(a)] = \mathfrak{o}(\phi(a))$ for every $a \in L$.

(2) A localic map $f\colon L \to M$ is open if and only if f^* admits a left adjoint ϕ that satisfies the identity

$$\phi(a \wedge f^*(b)) = \phi(a) \wedge b \text{ for all } a \in L \text{ and } b \in M. \tag{1.5.3}$$

(3) A localic map $f\colon L \to M$ is open if and only if f^* admits a left adjoint ϕ that satisfies the identity

$$f(a \to f^*(b)) = \phi(a) \to b \text{ for all } a \in L \text{ and } b \in M. \tag{1.5.4}$$

In particular, for $b = 0$, $f(a^*) = \phi(a)^*$.

Other Classes of Localic Maps

A localic map $f\colon L \to M$ is said to be:

(a) *proper* if it is closed and preserves directed joins ([78]).

(b) *dense* if $f[L]$ is a dense sublocale of M (or equivalently, if $f(0) = 0$). The corresponding frame homomorphism $f^*\colon M \to L$ satisfies $f^*(x) = 0 \implies x = 0$ (and this condition is taken as the definition of *dense frame homomorphism*).

(c) *codense* if for every $b \in M$ with $b < 1$, there is $1 \neq a \in L$ such that $b \leq f(a)$. The corresponding frame homomorphism $f^*\colon M \to L$ satisfies $f^*(a) = 1 \implies a = 1$ (and this condition is taken as the definition of *codense frame homomorphism*).

Chapter 2

Preliminaries II: The Frame of Reals and Real-Valued Functions

This chapter is devoted to the frame of reals due to its important role and countless appearances in the main body of the thesis. The intention is to provide the necessary background and notation used later on. Our main references are [10], [66] and [44].

2.1 The Frame of Reals and Continuous Real-Valued Functions

Recall that the *frame of reals* $\mathfrak{L}(\mathbb{R})$ (see [44] or [66]) is the frame presented by generators $(p,—)$ and $(—,p)$ for all rationals $p \in \mathbb{Q}$, and relations

(r1) $(p,—) \wedge (—,q) = 0$ if $q \leq p$,

(r2) $(p,—) \vee (—,q) = 1$ if $p < q$,

(r3) $(p,—) = \bigvee_{r>p}(r,—)$,

(r4) $(—,q) = \bigvee_{s<q}(—s)$,

(r5) $\bigvee_{p\in\mathbb{Q}}(p,—) = 1$,

(r6) $\bigvee_{q\in\mathbb{Q}}(—,q) = 1$.

It follows easily that $(p,—)^* = (—,p)$ and $(—,q)^* = (q,—)$ for every $p,q \in \mathbb{Q}$. Set $(p,q) := (p,—) \wedge (—,q)$ for every pair of rationals with $p < q$. Then $(p,q)^* = (—,p) \vee (q,—)$ whenever $p < q$.

Originally, this frame was defined in [10] as the frame presented by all ordered pairs (p,q) where $p,q \in \mathbb{Q}$, and subject to relations

(R1) $(p,q) \wedge (r,s) = (p \vee r, q \wedge s)$,

(R2) $(p,q) \vee (r,s) = (p,s)$ whenever $p \leq r < q \leq s$,

(R3) $(p,q) = \bigvee\{(r,s) \mid p < r < s < q\}$,

(R4) $\bigvee\{(p,q) \mid p,q \in \mathbb{Q}\}$.

Note that $(p,q) = \bigvee \emptyset = 0$ whenever $p \geq q$. In this presentation $(p,—)$ and $(—,q)$ are recovered as

$$(p,—) = \bigvee \{(p,q) \mid q \in \mathbb{Q}\} \quad \text{and} \quad (—,q) = \bigvee \{(p,q) \mid p \in \mathbb{Q}\}.$$

Both presentations of $\mathfrak{L}(\mathbb{R})$ are equivalent (see [66]). Furthermore, $\mathfrak{L}(\mathbb{R})$ is a completely regular frame ([10, Section 1, Corollary 2]).
Also, denoting by $\mathbb{R}$ the usual space of reals with the Euclidean topology, $\mathfrak{L}(\mathbb{R}) \cong \Omega(\mathbb{R})$ and $\Sigma(\mathfrak{L}(\mathbb{R})) \cong \mathbb{R}$ ([10]). From the adjunction between Top and Frm (1.2.6) one knows that

$$\mathsf{Frm}(L, \Omega(X)) \cong \mathsf{Top}(X, \Sigma(L)).$$

In particular, for $\mathfrak{L}(\mathbb{R})$ we have

$$\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \Omega(X)) \cong \mathsf{Top}(X, \mathbb{R})$$

which motivated Banaschewski [10] to introduce *continuous real-valued functions* on a frame L as frame homomorphisms $f\colon \mathfrak{L}(\mathbb{R}) \to L$. We shall denote the set of all continuous real-valued functions on a frame L by $\mathscr{R}(L)$. It is partially ordered by:

$$f \leq g \;\equiv\; f(p,—) \leq g(p,—) \text{ for every } p \in \mathbb{Q} \iff g(—,q) \leq f(—,q) \text{ for every } q \in \mathbb{Q}. \tag{2.1.1}$$

Remark 2.1.1. Let $f \in \mathscr{R}(L)$. Then

$$\begin{aligned} f(s,—) \leq f(—,s)^* \leq f(s',—) \quad & \text{for any } s' < s, \text{ and} \\ f(—,r) \leq f(r,—)^* \leq f(—,r') \quad & \text{for any } r' > r. \end{aligned}$$

Examples 2.1.2. Let us consider some examples of real-valued functions on a frame L.

(1) For each $p \in \mathbb{Q}$ the *constant function* $\mathbf{p}\colon \mathfrak{L}(\mathbb{R}) \to L$ is defined by

$$\mathbf{p}(r,—) = \begin{cases} 1 & \text{if } r < p \\ 0 & \text{if } r \geq p \end{cases} \quad \text{and} \quad \mathbf{p}(—,r) = \begin{cases} 0 & \text{if } r \leq p \\ 1 & \text{if } r > p. \end{cases}$$

(2) Let a be a complemented element of L. The *characteristic function* $\chi_a \in \mathscr{R}(L)$ is defined for each $r \in \mathbb{Q}$ as follows:

$$\chi_a(r,—) = \begin{cases} 1 & \text{if } r < 0 \\ a^* & \text{if } 0 \leq r < 1 \\ 0 & \text{if } r \geq 1 \end{cases} \quad \text{and} \quad \chi_a(—,r) = \begin{cases} 0 & \text{if } r \leq 0 \\ a & \text{if } 0 \leq r \leq 1 \\ 1 & \text{if } r > 1. \end{cases}$$

A function $f \in \mathscr{R}(L)$ is *bounded* if there are $p,q \in \mathbb{Q}$ $(p \leq q)$ such that and $\mathbf{p} \leq f \leq \mathbf{q}$. Equivalently (use (2.1.1) and Examples 2.1.2(1)), if and only if

$$f(—,p) \vee f(q,—) = 0, \tag{2.1.2}$$

or if and only if

$$f(r,—)\wedge f(—,s)=1 \text{ for every } r,s\in\mathbb{Q} \text{ with } r<p\leq q<s. \tag{2.1.3}$$

If $f(p,q)=1$, then $\mathbf{p}\leq f\leq \mathbf{q}$, but the converse is not true. In [39] the authors introduce the following notation:

$$f<g \;\equiv\; \bigvee_{p\in\mathbb{Q}} f(—,p)\wedge g(p,—)=1.$$

Of course, $f<g$ implies $f\leq g$ as in (2.1.1). Note that the equivalence

$$f\leq g \iff f=g \text{ or } f<g$$

does not hold. For $L=\Omega(X)$, $f<g$ if and only if $\widetilde{f}(x)<\widetilde{g}(x)$ for every $x\in X$, where $\widetilde{f}$ and $\widetilde{g}$ are the associated real-valued functions on the space X ([39, Remark 5.1]). With this notation, it is clear that $\mathbf{p}<f<\mathbf{q}$ if and only if $f(p,q)=1$.
Algebraic operations in $\mathscr{R}(L)$ are defined as follows ([66, XIV]):

(1) **Additive inverse.** For $f\in\mathscr{R}(L)$, the *additive inverse* $-f\in\mathscr{R}(L)$ is given by the formulas

$$-f(r,—)=f(—,-r) \quad\text{and}\quad -f(—,r)=(-r,—)$$

for every $r\in\mathbb{Q}$.

(2) **Product with scalar.** For any positive rational λ and any $f\in\mathscr{R}(L)$, the product $\lambda\cdot f\colon \mathfrak{L}(\mathbb{R})\to L$ is given by

$$(\lambda\cdot f)(r,—)=f(\tfrac{r}{\lambda},—) \quad\text{and}\quad (\lambda\cdot f)(—,r)=(—,\tfrac{r}{\lambda})$$

for every $r\in\mathbb{Q}$.

(3) **Binary join and meet.** Given f and g in $\mathscr{R}(L)$, the *supremum* $f\vee g\in\mathscr{R}(L)$ is defined for each $r\in Q$ by

$$(f\vee g)(r,—)=f(r,—)\vee g(r,—) \quad\text{and}\quad (f\vee g)(—,r)=f(—,r)\wedge g(—,r).$$

The *infimum* $f\wedge g\in\mathscr{R}(L)$ is given by

$$(f\wedge g)(r,—)=f(r,—)\wedge g(r,—) \quad\text{and}\quad (f\wedge g)(—,r)=f(—,r)\vee g(—,r)$$

for every $r\in\mathbb{Q}$.

(4) **Sum.** Let $f,g\in\mathscr{R}(L)$. The *sum* $f+g\in\mathscr{R}(L)$ is defined for every $r\in Q$ as follows:

$$(f+g)(r,—)=\bigvee_{t\in\mathbb{Q}}\left(f(t,—)\wedge g(r-t,—)\right) \quad\text{and}\quad (f+g)(—,r)=\bigvee_{t\in\mathbb{Q}}\left(f(—,t)\wedge g(—,r-t)\right).$$

(5) **Product.** The *product* $f\cdot g$ of two functions $f,g\in\mathscr{R}(L)$ is given by

$$(f\cdot g)(p,—)=\bigvee\{f(r,s)\wedge g(t,u)\mid \langle r,s\rangle\cdot\langle t,u\rangle\subseteq\langle p,—\rangle\}$$

and

$$(f \cdot g)(—,q) = \bigvee \{ f(r,s) \wedge g(t,u) \mid \langle r,s \rangle \cdot \langle t,u \rangle \subseteq \langle —,q \rangle \}$$

where $\langle \cdot, \cdot \rangle$ denotes open intervals in $\mathbb{Q}$, $\langle p, — \rangle$ and $\langle —, q \rangle$ stand for

$$\{x \in \mathbb{Q} \mid p < x\} \quad \text{and} \quad \{x \in \mathbb{Q} \mid x < q\}$$

respectively, and $\langle r,s \rangle \cdot \langle t,u \rangle = \{x \cdot y \mid x \in \langle r,s \rangle, y \in \langle t,u \rangle\}$.

(6) **Absolute value**. Let $f \in \mathscr{R}(L)$, set $f^+ := f \vee \mathbf{0}$ and $f^- := (-f) \vee \mathbf{0}$. Then $f^+, f^- \geq \mathbf{0}$ and $f = f^+ - f^-$. The *absolute value* of f is the sum $|f| := f^+ + f^-$.

Sum, product, infima and suprema can be defined alternatively with the other set of generators ([10, Chapter 4]). For $\diamond = +, \cdot, \wedge, \vee$,

$$(f \diamond g)(p,q) = \bigvee \{ f(r,s) \wedge g(t,u) \mid \langle r,s \rangle \diamond \langle t,u \rangle \subseteq \langle p,q \rangle \} \tag{2.1.4}$$

where $\langle \cdot, \cdot \rangle$ denotes open intervals in $\mathbb{Q}$ and $\langle r,s \rangle \diamond \langle t,u \rangle = \{x \diamond y \mid x \in \langle r,s \rangle, y \in \langle t,u \rangle\}$.

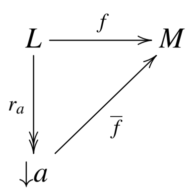
Remark 2.1.3. It may be proved that $\mathscr{R}(L)$ is a bounded strong f-ring. This means that it is an Archimedian commutative lattice-ordered ring with unit (we recall that a lattice-ordered ring A is a ring with a lattice structure such that $(a \wedge b) + c = (a+c) \wedge (b+c)$ for every $a, b, c \in A$), such that $(f \wedge g)h = (fh) \wedge (gh)$, every $f \geq \mathbf{1}$ is invertible, and every f satisfies $f \vee (-f) \leq \mathbf{n}$ for some $n \in \mathbb{N}$ ([10]).

2.2 Frames of Real Intervals

Let us first recall [9, Lemma 2.1.1]:

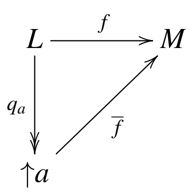
Lemma 2.2.1. *Let $f \colon L \to M$ be a frame homomorphism.*

(1) *If $f(a) = 1$ then there exists a (unique) frame homomorphism $\overline{f} \colon {\downarrow}a \to M$ such that the diagram*



(where $r_a \colon L \to {\downarrow}a$ is the onto frame homomorphism given by $x \mapsto x \wedge a$) commutes.

(2) *If $f(a) = 0$ then there exists a (unique) frame homomorphism $\overline{f} \colon {\uparrow}a \to M$ such that the diagram*



(where $q_a \colon L \to {\uparrow}a$ is the onto frame homomorphism given by $x \mapsto x \vee a$) commutes.

There are other frames of importance related to $\mathfrak{L}(\mathbb{R})$ that one also needs to consider. For each pair of rationals $p < q$, $\mathfrak{L}(p,q)$ denotes the *open interval frame*

$$\downarrow(p,q) = \{a \in \mathfrak{L}(\mathbb{R}) \mid a \leq (p,q)\} \cong \mathfrak{o}_{\mathfrak{L}(\mathbb{R})}\left((p,q)\right).$$

Furthermore, $\mathfrak{L}[p,q]$ is the *closed interval frame* defined by

$$\uparrow((-,p) \vee (q,-)) = \{a \in \mathfrak{L}(\mathbb{R}) \mid a \geq (-,p) \vee (q,-)\} = \mathfrak{c}_{\mathfrak{L}(\mathbb{R})}((p,q)^*) = \overline{\mathfrak{o}_{\mathfrak{L}(\mathbb{R})}\left((p,q)\right)}.$$

The following result is mentioned in [10] without proof. We provide here a sketch of the proof since some of its details and notation will be used in Section 3.3.

Proposition 2.2.2. *Let p and q be rationals such that $p < q$. Then*

$$\mathfrak{L}(p,q) \cong \mathfrak{L}(\mathbb{R}).$$

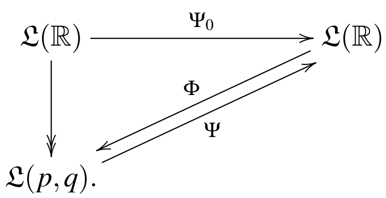
Proof. Let $p < q$ in $\mathbb{Q}$. Consider an order isomorphism ψ from the open rational interval $\langle p,q \rangle$ into $\mathbb{Q}$ and let $\varphi = \psi^{-1}$. The map φ allows to define a frame homomorphism $\Phi\colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(p,q)$ by

$$\Phi(r,-) = (\varphi(r),q) \quad \text{and} \quad \Phi(-,r) = (p,\varphi(r)).$$

With the map ψ one defines a frame homomorphism $\Psi_0\colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(\mathbb{R})$ given by:

$$\Psi_0(r,s) = \begin{cases} 1 & \text{if } r \leq p < q \leq s, \\ (-,\psi(s)) & \text{if } r \leq p < s < q, \\ (\psi(r),\psi(s)) & \text{if } p < r < s < q, \\ (\psi(r),-) & \text{if } p < r < q \leq s \\ 0 & \text{if } s \leq p \text{ or } q \leq r. \end{cases}$$

The restriction of Ψ_0 to $\mathfrak{L}(p,q)$ is, by Lemma 2.2.1 (1), a frame homomorphism $\Psi\colon \mathfrak{L}(p,q) \to \mathfrak{L}(\mathbb{R})$. Furthermore, Ψ and Φ are inverse of each other, and we get the following commutative diagram:



□

Remarks 2.2.3. (1) One can show similarly that both $\mathfrak{L}(p,-) = \downarrow(p,-)$ and $\mathfrak{L}(-,q) = \downarrow(-,q)$ are isomorphic to $\mathfrak{L}(\mathbb{R})$ since $\mathbb{Q}$ has order isomorphisms into $\langle p,+\infty \rangle = \{x \in \mathbb{Q} \mid x > p\}$ and $\langle -\infty,q \rangle = \{x \in \mathbb{Q} \mid x < q\}$.

(2) It can be also proved that $\mathfrak{L}(p,q)$ is isomorphic to the frame K presented by generators

$$(r,q),\ r\in\mathbb{Q},\ p\leq r<q \quad \text{and} \quad (p,s),\ s\in\mathbb{Q},\ p<s\leq q$$

and relations

(r1') $(r,q)\wedge(p,s)=0$ if $s\leq r$,

(r2') $(r,q)\vee(p,s)=1$ if $r<s$,

(r3') $(r',q)=\bigvee_{r>r'}(r,q)$,

(r4') $(p,s')=\bigvee_{s<s'}(p,s)$,

(r5') $\bigvee_{r<q,r>p}(r,q)=1$,

(r6') $\bigvee_{s<q,s>p}(p,s)=1$.

Indeed, the stipulations $(r,q)\mapsto(r,q)$ and $(p,s)\mapsto(p,s)$ turn the defining relations (r1')-(r6') into identities in $\mathfrak{L}(p,q)$, yielding an isomorphism $F\colon K\to\mathfrak{L}(p,q)$.

(3) From (2), it is clear that the frame homomorphism $\Psi\colon\mathfrak{L}(p,q)\to\mathfrak{L}(\mathbb{R})$, defined on generators by

$$\Psi(r,q)=(\psi(r),\text{—}) \quad \text{and} \quad \Psi(p,r)=(\text{—},\psi(r))$$

(for $p<r<q$) is the inverse to the Φ of Proposition 2.2.2 (and provides an alternative proof to Proposition 2.2.2).

Corollary 2.2.4. *Let p and q be rationals such that $p<q$. For each frame L,*

$$\mathscr{R}(L)=\mathsf{Frm}(\mathfrak{L}(\mathbb{R}),L)\cong(\mathfrak{L}(p,q),L).$$

Corollary 2.2.5. *Let p and q be rationals such that $p<q$. For each frame L,*

$$\mathscr{R}(L)\cong\mathsf{Frm}(\mathfrak{L}(p,q),L)\cong\{f\in\mathscr{R}(L)\mid\mathbf{p}<f<\mathbf{q}\}=\{f\in\mathscr{R}(L)\mid f(p,q)=1\}.$$

By dropping (r5) and (r6) in the description of $\mathfrak{L}(\mathbb{R})$, one gets the *extended* variant $\mathfrak{L}(\overline{\mathbb{R}})$ of $\mathfrak{L}(\mathbb{R})$ [16]. The spectrum $\Sigma\mathfrak{L}(\overline{\mathbb{R}})$ is homeomorphic to the extended real line $\overline{\mathbb{R}}$. It may be worth noting that $\mathfrak{L}(\overline{\mathbb{R}})$ is not isomorphic to the frame presented by generators $(p,q)\in\mathbb{Q}\times\mathbb{Q}$ and relations (R1), (R2) and (R3). For more details on $\mathfrak{L}(\overline{\mathbb{R}})$ we refer to [16].

The proof of the following result is similar to that of Proposition 2.2.2. All the details can be found in [16, Remark 2].

Proposition 2.2.6. *Let p and q be rationals such that $p<q$. Then*

$$\mathfrak{L}[p,q]\cong\mathfrak{L}(\overline{\mathbb{R}}).$$

An *extended continuous real-valued function* on a frame L is a frame homomorphism $f\colon\mathfrak{L}(\overline{\mathbb{R}})\to L$. We denote by $\overline{\mathscr{R}}(L)$ the collection of all extended continuous real functions on L.

Corollary 2.2.7. [16, Lemma 3] *Let p and q be rationals such that $p < q$. For each frame L,*

$$\overline{\mathscr{R}}(L) \cong \mathsf{Frm}(\mathfrak{L}[p,q], L) \cong \{f \in \mathscr{R}(L) \mid \mathbf{p} \leq f \leq \mathbf{q}\} = \{f \in \mathscr{R}(L) \mid f(-,p) \vee f(q,-) = 0\}.$$

2.3 Cozero Elements

Cozero elements have been studied to a great extent in point-free topology. They are the point-free counterpart of cozero sets of a topological space. The main references for this topic are [10] and [14]. We refer also to [9], and [66] and [44] where the notation and approach is closer to ours.

A *cozero element* of L, here denoted by $\mathsf{coz}(f)$, is an element of the form

$$f(-,0) \vee f(0,-)$$

for some $f \in \mathscr{R}(L)$. The set of all cozero elements of L is denoted by $\mathsf{Coz}\, L$.

Proposition 2.3.1. [66, XIV.6.1.2][44, 5.3] *The following properties hold for any $f, g \in \mathscr{R}(L)$:*

(1) $\mathsf{coz}(\mathbf{1}) = 1$.

(2) $\mathsf{coz}(f) = 0$ *if and only if* $f = \mathbf{0}$.

(3) $\mathsf{coz}(f) = \mathsf{coz}(|f|)$.

(4) *If* $\mathbf{0} \leq f \leq g$ *then* $\mathsf{coz}(f) \leq \mathsf{coz}(g)$.

(5) $\mathsf{coz}(f+g) \leq \mathsf{coz}(f) \vee \mathsf{coz}(g)$.

(6) $\mathsf{coz}(f \cdot g) = \mathsf{coz}(f) \wedge \mathsf{coz}(g) = \mathsf{coz}(|f| \wedge |g|)$.

(7) $\mathsf{coz}(f) \vee \mathsf{coz}(g) = \mathsf{coz}(f+g)$ *if* $f, g \geq \mathbf{0}$.

(8) $\mathsf{coz}(f - \mathbf{p}) = f(-,p) \vee f(p,-)$ *for every* $p \in \mathbb{Q}$.

Remarks 2.3.2. (1) For any $f \in \mathscr{R}(L)$, $\mathsf{coz}(f) = \mathsf{coz}(|f| \wedge \mathbf{1})$, and $|f| \wedge \mathbf{1}$ is bounded. Thus, bounded continuous real-valued functions yield the same cozero elements as the real valued-functions of $\mathscr{R}(L)$.

(2) For any $g \in \mathscr{R}(L)$, (2.1.2) and (2.1.3) yield

$$g \geq \mathbf{0} \iff \mathsf{coz}(g) = g(0,-) \iff g(r,-)^* = 0 \;\forall r < 0 \iff g(-,0)^* = 1.$$

Obviously, frame homomorphisms preserve cozero elements (indeed, if $f\colon L \to M$ is a frame homomorphism and $a = \mathsf{coz}(g) \in \mathsf{Coz}\, L$, then $f(a) = fg(-,0) \vee fg(0,-) = \mathsf{coz}(fg) \in \mathsf{Coz}\, M$). So we have:

Proposition 2.3.3. *Let $f\colon L \to M$ be a frame homomorphism. Then $f(a) \in \mathsf{Coz}\, M$ for every $a \in \mathsf{Coz}\, L$.*

The following result regarding multiplicative inverses in $\mathscr{R}(L)$ is proven in [9, Proposition 3.3.1] (a weaker version appears in [10]). Since the notation and presentation of the frame of reals in [9] are different from ours, we include here a sketch of the proof using our own notation.

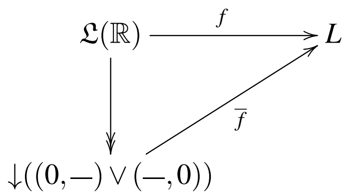
Proposition 2.3.4. *A frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to L$ has a multiplicative inverse if and only if* $\mathsf{coz}(f) = 1$.

Proof. Clearly, if f has a multiplicative inverse f^{-1}, by Proposition 2.3.1 (6) we get

$$1 = \mathsf{coz}(\mathbf{1}) = \mathsf{coz}(f \cdot f^{-1}) = \mathsf{coz}(f) \wedge \mathsf{coz}(f^{-1}).$$

Hence, $\mathsf{coz}(f) = 1$. The idea for the converse mimicks the classical proof that constructs the multiplicative inverse of a function $f\colon X \to \mathbb{R}$ by composing it with $g\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ $(x \mapsto \frac{1}{x})$ provided the image of f is contained in $\mathbb{R} \setminus \{0\}$.

Indeed, if $\mathsf{coz}(f) = 1$ there is, by Lemma 2.2.1(1), a frame homomorphism $\overline{f}$ such that the diagram



commutes. We can compose $\overline{f}$ with

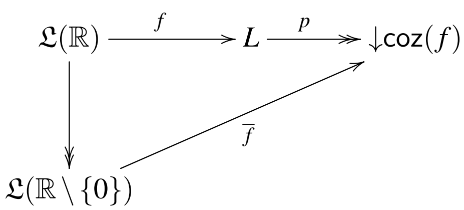
$$g\colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(\mathbb{R} \setminus \{0\}) := {\downarrow}((0,\text{—}) \vee (\text{—},0)),$$

the point-free version of the mapping $x \mapsto \frac{1}{x}$ above, given by

$$g(p,\text{—}) = \begin{cases} (0,\frac{1}{p}) & \text{if } p > 0 \\ (0,\text{—}) & \text{if } p = 0 \\ (\text{—},\frac{1}{p}) \vee (0,\text{—}) & \text{if } p < 0 \end{cases} \quad \text{and} \quad g(\text{—},q) = \begin{cases} (\frac{1}{q},0) & \text{if } q < 0 \\ (\text{—},0) & \text{if } q = 0 \\ (\text{—},0) \vee (\frac{1}{q},\text{—}) & \text{if } q > 0. \end{cases}$$

It is easy to check that g turns relations (r1)–(r6) into identities in $\mathfrak{L}(\mathbb{R} \setminus \{0\})$, making g a frame homomorphism. Furthermore, the composite $\overline{f}g$ is the multiplicative inverse of f, as can be readily verified. □

Remarks 2.3.5. (1) Classically, when a function does not have a multiplicative inverse, one restricts it to its cozero set in order to compose it with $x \mapsto \frac{1}{x}$. Similarly, if $f\colon \mathfrak{L}(\mathbb{R}) \to L$ is a frame homomorphism, by 2.2.1 (1) there exists $\overline{f}$ such that the diagram



commutes. Then $\mathfrak{L}(\mathbb{R}) \xrightarrow{g} \mathfrak{L}(\mathbb{R} \setminus \{0\}) \xrightarrow{\overline{f}} L$ is the multiplicative inverse of pf.

(2) If $f \geq \mathbf{1}$ then $f(0,\text{—}) \vee f(\text{—},0) = 1$. Thus, f has a multiplicative inverse.

The following result is a well-known characterization of cozero elements ([14, Proposition 1]).

Proposition 2.3.6. *The following are equivalent for an $a \in L$:*

(i) $a \in \mathsf{Coz}\, L$.

(ii) $a = \bigvee_{n\in\mathbb{N}} a_n$ *where* $a_n \prec\!\!\prec a$ *for all* $n \in \mathbb{N}$.

(iii) $a = \bigvee_{n\in\mathbb{N}} a_n$ *where* $a_n \prec\!\!\prec a_{n+1}$ *for all* $n \in \mathbb{N}$.

Remark 2.3.7. We list some consequences of the previous result that appear in [14] and [10]:

(1) Clearly, $0, 1 \in \mathsf{Coz}\, L$. Furthermore, $\mathsf{Coz}\, L$ is a sub-σ-frame of L.

(2) For any $a, b \in L$, $a \prec\!\!\prec b$ if and only if there is some $c \in \mathsf{Coz}\, L$ such that $a \prec\!\!\prec c \prec\!\!\prec b$.

(3) In particular, one gets that for every $a \in \mathsf{Coz}\, L$, $a = \bigvee_{n\in\mathbb{N}} a_n$ with $a_n \prec a$ and $a_n \in \mathsf{Coz}\, L$, meaning that $\mathsf{Coz}\, L$ is a regular σ-frame. Thus, it is normal and $\prec = \prec\!\!\prec$ in $\mathsf{Coz}\, L$.

(4) The correspondence $L \mapsto \mathsf{Coz}\, L$ is functorial. There is a functor from Frm to the category of regular σ-frames $\mathsf{Coz}\colon \mathsf{Frm} \to \mathsf{R\sigma Frm}$ ([10, 61]).

Proposition 2.3.6 yields the following characterization of completely regular frames in terms of cozero elements (for the proof see [10, Section 5,Corollary 2] or [66, XIV.6.2.5]).

Corollary 2.3.8. *A frame L is completely regular if and only if every element of L is a join of cozero elements.*

Proposition 2.3.9. [44, 5.4.3] *The following are equivalent for any elements a and b of a frame L:*

(i) $b \prec\!\!\prec a$.

(ii) *There is an* $f \in \mathscr{R}(L)$ *with* $\mathbf{0} \leq f \leq \mathbf{1}$ *such that* $f(0,\text{—}) \wedge b = 0$ *and* $f(\text{—},1) \leq a$.

(iii) *There are* $c, d \in \mathsf{Coz}\, L$ *such that* $c^* \vee a = 1$, $c \vee d = 1$ *and* $b \wedge d = 0$.

Corollary 2.3.10. [9, Proposition.5.1.2] *Let* $a, b \in \mathsf{Coz}\, L$ *such that* $a \vee b = 1$. *There is an* $f \in \mathscr{R}(L)$ *with* $\mathbf{0} \leq f \leq \mathbf{1}$ *such that* $a = f(0,\text{—})$ *and* $b = f(\text{—},1)$.

2.4 Scales in Frames

A useful way to construct continuous real-valued functions is through scales (also called trails in [10]). Here we follow the notation used in [38]. A *descending scale* in a frame L is a family $(a_p)_{p\in\mathbb{Q}} \subseteq L$ such that the following conditions hold:

(s1) $a_p \prec a_q$ for every $q < p$.

(s2) $\bigvee a_p = 1 = \bigvee a_p^*$.

Dually, we say $(a_p)_{p\in\mathbb{Q}} \subseteq L$ is an *ascending scale* in L if it satisfies (s2) and (s1') $a_p \prec a_q$ for every $p < q$.

Remark 2.4.1. Consider the following weaker version of (s1) and (s1'):

(ws1) $a_p \leq a_q$ for every $q < p$,

(ws1') $a_p \leq a_q$ for every $p < q$.

If a_p is complemented for every $p \in \mathbb{Q}$, then $(a_p)_{p\in\mathbb{Q}} \subseteq L$ is a descending scale (resp. ascending scale) if and only if (ws1) (resp. (ws1')) and (s2) hold.

Proposition 2.4.2. [38, Lemma 4.3]

(1) *Let $(a_p)_{p\in\mathbb{Q}}$ be a descending scale in L. Then the formulas*

$$f(p,\text{—}) = \bigvee_{r>p} a_r \quad \text{and} \quad f(\text{—},q) = \bigvee_{s<q} a_s^*$$

define a frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to L$ (and one says that f is generated by the descending scale $(a_p)_{p\in\mathbb{Q}}$).

(2) *Let $(a_p)_{p\in\mathbb{Q}}$ be an ascending scale in L. Then the formulas*

$$g(p,\text{—}) = \bigvee_{r>p} a_r^* \quad \text{and} \quad g(\text{—},q) = \bigvee_{s<q} a_s$$

define a frame homomorphism $g\colon \mathfrak{L}(\mathbb{R}) \to L$ (and one says that f is generated by the ascending scale $(a_p)_{p\in\mathbb{Q}}$).

Conversely, we have:

Proposition 2.4.3. *Let L be a frame and $f \in \mathscr{R}(L)$. Then:*

(1) *The family $(f(p,\text{—}))_{p\in\mathbb{Q}}$ is a descending scale.*

(2) *The family $(f(\text{—},q))_{q\in\mathbb{Q}}$ is an ascending scale.*

2.5 General Real-Valued Functions

A *general real-valued function* on a frame L is a frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(L)^{op}$ ([38]). The collection of all general real-valued functions on a frame L is denoted by $\mathsf{F}(L)$.[1]

$\mathsf{F}(L)$ is partially order by:

$$f \leq g \;\equiv\; f(\text{—},r) \subseteq g(\text{—},r) \;\text{ for every } r \in \mathbb{Q} \iff f(r,\text{—}) \supseteq g(r,\text{—}) \;\text{ for every } r \in \mathbb{Q}. \tag{2.5.1}$$

Remark 2.5.1. For any $f \in \mathsf{F}(L)$, we have

$$\begin{aligned} f(s',\text{—}) \subseteq f(\text{—},s)^{\#} \subseteq f(s,\text{—}) &\quad \text{for any } s' < s, \text{ and} \\ f(\text{—},r') \subseteq f(r,\text{—})^{\#} \subseteq f(\text{—},r) &\quad \text{for any } r' > r. \end{aligned}$$

[1] Note that $\mathsf{F}(L) = \mathscr{R}(\mathsf{S}(L)^{op})$.

Examples 2.5.2. (1) For each $p \in \mathbb{Q}$ the *constant function* $\mathbf{p} \in \mathsf{F}(L)$ is given by

$$\mathbf{p}(r,\text{—}) = \begin{cases} \mathsf{O} & \text{if } r < p \\ L & \text{if } r \geq p \end{cases} \quad \text{and} \quad \mathbf{p}(\text{—},r) = \begin{cases} L & \text{if } r \leq p \\ \mathsf{O} & \text{if } r > p. \end{cases}$$

(2) For each complemented sublocale S of L, the *characteristic function* $\chi_S \in \mathsf{F}(L)$ of S is defined by

$$\chi_S(r,\text{—}) = \begin{cases} \mathsf{O} & \text{if } r < 0 \\ S^{\#} & \text{if } 0 \leq r < 1 \\ L & \text{if } r \geq 1 \end{cases} \quad \text{and} \quad \chi_S(\text{—},r) = \begin{cases} L & \text{if } r \leq 0 \\ S & \text{if } 0 < r \leq 1 \\ \mathsf{O} & \text{if } r > 1. \end{cases}$$

Note that $\mathbf{0} \leq \chi_S \leq \mathbf{1}$.

An $f \in \mathsf{F}(L)$ is *bounded* if there are $p,q \in \mathbb{Q}$ with $p \leq q$ such that $\mathbf{p} \leq f \leq \mathbf{q}$. By (2.5.1), $\mathbf{p} \leq f \leq \mathbf{q}$ holds if and only if

$$f(q,\text{—}) = L \quad \text{and} \quad f(\text{—},p) = L. \tag{2.5.2}$$

Or equivalently, if and only if for every $r,s \in \mathbb{Q}$ with $r < p$ and $s > q$,

$$f(r,\text{—}) = \mathsf{O} \quad \text{and} \quad f(\text{—},s) = \mathsf{O}. \tag{2.5.3}$$

If $f(p,q) = \mathsf{O}$, then f is bounded and $\mathbf{p} \leq f \leq \mathbf{q}$. The converse is not true (recall Section 2.1).

A general real-valued function $f \in \mathsf{F}(L)$ is:

(1) *lower semicontinuous* if $f(r,\text{—})$ is closed for every $r \in \mathbb{Q}$,

(2) *upper semicontinuous* if $f(\text{—},r)$ is closed for every $r \in \mathbb{Q}$,

(3) *continuous* if $f(\text{—},r)$ and $f(s,\text{—})$ are closed for every $r,s \in \mathbb{Q}$.

The collection of all continuous $f \in \mathsf{F}(L)$ is denoted by $\mathsf{C}(L)$, and the collection of all continuous and bounded $f \in \mathsf{F}(L)$ is denoted by $\mathsf{C}^*(L)$.

Remark 2.5.3. As expected, the notion of continuity of a real-valued function just defined is equivalent to the one defined in Section 2.1. Indeed, for every $f \in \mathscr{R}(L)$, the composite $\mathfrak{c}f\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(L)^{op}$ is continuous. Conversely, if $f \in \mathsf{C}(L)$ then take $g\colon \mathfrak{L}(\mathbb{R}) \to L$ given by $g(r,\text{—})) = \imath - 1 f(r,\text{—})$ and $g(\text{—},s) = \imath^{-1} f(\text{—},s)$, where $\imath$ is the frame isomorphism form L to $\mathfrak{c}(L)^{op}$ (recall Section 1.4). Then $\mathfrak{c}g = f$. Consequently, $\mathscr{R}(L)$ and $\mathsf{C}(L)$ are isomorphic rings.

2.6 Scales in $\mathsf{S}(L)$

In this section we present the application of the definitions and results of Section 2.4 to the frame $\mathsf{S}(L)^{op}$. Since scales in $\mathsf{S}(L)^{op}$ give general real-valued functions in $\mathsf{F}(L)$ we recall the conditions that make a scale in $\mathsf{S}(L)^{op}$ give a continuous, and not just general, real valued-function in L. The content in this section is based on [38] and [42].

A family $(S_p)_{p\in\mathbb{Q}}$ of sublocales of L is a *descending scale* in $\mathsf{S}(L)^{op}$ if

(S1) $S_p^\# \cap S_q = \mathsf{O}$ for every $q < p$ and

(S2) $\bigcap_{p\in\mathbb{Q}} S_p = \mathsf{O} = \bigcap_{p\in\mathbb{Q}} S_p^\#$

hold. On the other hand, $(S_p)_{p\in\mathbb{Q}}$ is an *ascending scale* in $\mathsf{S}(L)^{op}$ if it satisfies (S2) and

(S1') $S_p^\# \cap S_q = \mathsf{O}$ for every $p < q$.

Remark 2.6.1. Similarly, as in Remark 2.4.1, consider the following conditions:

(wS1) $S_q \subseteq S_p$ for every $q < p$ and

(wS1') $S_q \subseteq S_p$ for every $p < q$.

Clearly, (S1) (resp. (S1')) implies (wS1) (resp. (wS1')). Moreover, if S_p is complemented for every $p \in \mathbb{Q}$, the family $(S_p)_{p\in\mathbb{Q}} \subseteq \mathsf{S}(L)$ is a descending scale (resp. ascending scale) in $\mathsf{S}(L)^{op}$ if and only if (wS1) (resp. (wS1')) and (S2) hold.

An application of Propositions 2.4.2 and 2.4.3 to the frame $\mathsf{S}(L)^{op}$ give the following two results:

Proposition 2.6.2. (1) *Let $(S_p)_{p\in\mathbb{Q}}$ be a descending scale in $\mathsf{S}(L)^{op}$. Then the formulas*

$$f(p,\text{—}) = \bigcap_{r>p} S_r \quad \textit{and} \quad f(\text{—},q) = \bigcap_{s<q} S_s^\#$$

define a frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(L)^{op}$ (and one says that f is generated by the descending scale $(S_p)_{p\in\mathbb{Q}}$).

(2) *Let $(S_p)_{p\in\mathbb{Q}}$ be an ascending scale in $\mathsf{S}(L)^{op}$. Then the formulas*

$$g(p,\text{—}) = \bigcap_{r>p} S_r^\# \quad \textit{and} \quad g(\text{—},q) = \bigcap_{s<q} S_s$$

define a frame homomorphism $g\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(L)^{op}$ (and one says that f is generated by the ascending scale $(S_p)_{p\in\mathbb{Q}}$).

Proposition 2.6.3. *Let L be a frame and $f \in \mathsf{F}(L)$. Then:*

(1) *The family $(f(p,\text{—}))_{p\in\mathbb{Q}}$ is a descending scale in $\mathsf{S}(L)^{op}$.*

(2) *The family $(f(\text{—},q))_{q\in\mathbb{Q}}$ is an ascending scale in $\mathsf{S}(L)^{op}$.*

Proposition 2.6.4. [38, Lemma 4.4] *Let $f_1, f_2 \in \mathsf{F}(L)$ be generated by descending scales $(S_r)_{r\in\mathbb{Q}}$ and $(T_r)_{r\in\mathbb{Q}}$ respectively. Then:*

(1) *$f_1(\text{—},r)^\# \subseteq S_r \subseteq f_1(r,\text{—})$ for every $r \in \mathbb{Q}$.*

(2) *$f_2 \leq f_1$ if and only if $S_r \subseteq T_s$ for every $r < s$.*

Scales in $\mathsf{S}(L)^{op}$ establish general real-valued functions on a frame L, but they may not be continuous. To understand when do scales in $\mathsf{S}(L)^{op}$ generate continuous real-valued functions on L, consider the following conditions on a family $(S_p)_{p\in\mathbb{Q}}$:

(IC) $\overline{S_p} \subseteq S_q^\circ$ for every $p < q$,

(IC') $\overline{S_p} \subseteq S_q^\circ$ for every $q < p$.

In [42, Lemma 2.2] the authors present a proof of a weaker version of the next result. We have not found in the literature a proof of this stronger version so we include it here.

Proposition 2.6.5. *Let $(S_p)_{p\in Q}$ be a descending (resp. ascending) scale in $\mathsf{S}(L)^{op}$. The formulas*

$$f(p,\text{—}) = \bigcap_{r>p} S_r \quad and \quad f(\text{—},q) = \bigcap_{s<q} S_s^\#$$

$$(resp.\ f(p,\text{—}) = \bigcap_{r>p} S_r^\# \quad and \quad f(\text{—},q) = \bigcap_{s<q} S_s)$$

define a function $f \in \mathsf{C}(L)$ if and only if (IC) (resp. (IC')) holds.

Proof. Let $(S_p)_{p\in Q}$ be a descending scale in $\mathsf{S}(L)^{op}$, and let $f \in \mathsf{F}(L)$ be the function it defines (Proposition 2.6.2). Then

$$f(p,\text{—}) = \bigcap_{r>p} S_r \quad \text{and} \quad f(\text{—},q) = \bigcap_{s<q} S_s^\#. \tag{2.6.1}$$

Assume that $f\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(L)^{op}$ is continuous. Then there is a frame homomorphism $\overline{f}\colon \mathfrak{L}(\mathbb{R}) \to L$ such that $\mathfrak{c}\overline{f} = f$. Let $p < q$, and take $r,s \in \mathbb{Q}$ such that $p < r < s < q$. The fact that $\overline{f} \in \mathscr{R}(L)$ and (2.1.1) imply

$$\overline{f}(\text{—},r)^* \vee \overline{f}(s,\text{—})^* \geq \overline{f}(r,\text{—}) \vee \overline{f}(\text{—},s) = 1.$$

Taking the respective open sublocales and using (1.4.10) we obtain

$$L = \mathfrak{o}(\overline{f}(\text{—},r)^*) \vee \mathfrak{o}(\overline{f}(s,\text{—})^*) = \mathfrak{c}(\overline{f}(\text{—},r))^\circ \vee \mathfrak{c}(\overline{f}(s,\text{—}))^\circ = f(\text{—},r)^\circ \vee f(s,\text{—})^\circ.$$

Equivalently, by (1.4.2) and (1.4.12), $\overline{f(\text{—},r)^\#} = (f(\text{—},r)^\circ)^\# \subseteq f(s,\text{—})^\circ$. Take $p' \in \mathbb{Q}$ with $p < p' < r$ so that $p < p' < r < s < q$. Using (2.6.1) we get

$$\overline{S_p} \subseteq \overline{S_{p'}^{\#\#}} \subseteq \overline{f(\text{—},r)^\#} = (f(\text{—},r)^\circ)^\# \subseteq f(s,\text{—})^\circ \subseteq S_q^\circ$$

where the first inclusion holds because $(S_p)_{p\in\mathbb{Q}}$ is a descending scale (it satisfies (S1)). Hence, $S_p \subseteq S_{p'}^{\#\#}$ for every $p < p'$.

For the converse suppose (IC) holds. To show that f is continuous we need to prove that $f(p,\text{—})$ and $f(\text{—},q)$ are closed subocales for every $p,q \in \mathbb{Q}$. Note that (IC) implies that

$$f(p,\text{—}) = \bigcap_{r>p} S_r \subseteq \bigcap_{r>p} \overline{S_r} \subseteq \bigcap_{q>p} S_q^\circ \subseteq \bigcap_{q>p} S_q = f(p,\text{—})$$

and

$$f(-,q) = \bigcap_{s<q} S_s^\# \subseteq \bigcap_{s<q} (S_s^\circ)^\# \subseteq \bigcap_{r<q} \overline{S_r}^\# \subseteq \bigcap_{r<q} S_r^\# = f(-,q).$$

Hence, $f(p,-)$ and $f(-,q)$ are closed; that is, $f \in \mathsf{C}(L)$.
The dual statement for ascending scales is proven in a similar way. □

Condition (IC) (resp. (IC')) implies (S1) (resp. (S1')). Indeed, let $p<q$ (resp. $q<p$). Then $\overline{S_p} \subseteq S_q^\circ$. Equivalently, since S_q° is open (and in particular complemented), $\overline{S_p} \cap (S_q^\circ)^\# = \mathsf{O}$. Thus, $S_p \cap S_q^\# \subseteq \overline{S_p} \cap (S_q^\circ)^\# = \mathsf{O}$ for rationals $p<q$ (resp. $q<p$). It then follows from Propositions 2.6.2 and 2.6.5 that:

Corollary 2.6.6. *Let $(S_p)_{p\in\mathbb{Q}} \subseteq \mathsf{S}(L)^{op}$ satisfy conditions (IC) (resp. (IC')) and (S2). The formulas*

$$f(p,-) = \bigcap_{r>p} S_r \quad \textit{and} \quad f(-,q) = \bigcap_{s<q} S_s^\#$$

$$(\textit{resp. } f(p,-) = \bigcap_{r>p} S_r^\# \quad \textit{and} \quad f(-,q) = \bigcap_{s<q} S_s)$$

define a continuous $f\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(L)^{op}$.

Remark 2.6.7. Recall from (1.4.11) and (1.4.10) that $a \prec b$ if and only if $\mathfrak{c}(b) \subseteq \mathfrak{c}(a)^\circ = \mathfrak{o}(a^*)$, if and only if $\mathfrak{c}(a)^* = \overline{\mathfrak{o}(a)} \subseteq \mathfrak{o}(b)$. We have:

(1) The family $(a_r)_{r\in\mathbb{Q}}$ is a descending (resp. ascending) scale in L if and only if $(\mathfrak{c}(a_r))_{r\in\mathbb{Q}}$ is a descending (resp. ascending) scale in $\mathsf{S}(L)^{op}$ that satisfies (IC) (resp. (IC')). Furthermore, the function $\overline{f}\colon \mathfrak{L}(\mathbb{R}) \to L$ defined by $(a_r)_{r\in\mathbb{Q}}$ as in Proposition 2.4.2 and the function $f\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(L)^{op}$ defined by $(\mathfrak{c}(a_r))_{r\in\mathbb{Q}}$ as in Proposition 2.6.2 are related by the identity $\mathfrak{c}\overline{f} = f$.

(2) The family $(a_r)_{r\in\mathbb{Q}}$ is a descending (resp. ascending) scale in L if and only if $(\mathfrak{o}(a_r))_{r\in\mathbb{Q}}$ is an ascending (resp. descending) scale in $\mathsf{S}(L)^{op}$ that satisfies (IC') (resp. (IC)). Furthermore, the function $\overline{f}\colon \mathfrak{L}(\mathbb{R}) \to L$ defined by $(a_r)_{r\in\mathbb{Q}}$ as in Proposition 2.4.2 and the function $f\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(L)^{op}$ defined by $(\mathfrak{c}(a_r))_{r\in\mathbb{Q}}$ as in Proposition 2.6.2 are related by the identity $\mathfrak{c}\overline{f} = f$.

In general, a scale $(\mathfrak{c}(a_r))_{r\in\mathbb{Q}}$ (resp. $(\mathfrak{o}(a_r))_{r\in\mathbb{Q}}$) in $\mathsf{S}(L)^{op}$ does not give a continuous real-valued function on L (it may not satisfy (IC) or (IC')). In fact, it will only define a lower (resp. upper) semicontinuous function. For a thorough study of this case see [38].

2.7 Cozero Sublocales and Complete Separation

A *zero* (resp. *cozero*) *sublocale* of L is a sublocale $S \subseteq L$ of the form $\mathfrak{c}_L(a)$ (resp. $\mathfrak{o}_L(a)$) with $a \in \mathsf{Coz}\, L$. We denote by

$$\mathsf{CoZS}(L) \quad \text{and} \quad \mathsf{ZS}(L)$$

the classes of cozero and zero sublocales respectively.

Remarks 2.7.1. (1) It follows from the isomorphism $\mathscr{R}(L) \cong \mathsf{C}(L)$ that a sublocale S of L is a zero sublocale if and only if $S = f(0,-) \cap f(-,0)$ for some $f \in \mathsf{C}(L)$. In fact, S is a zero sublocale if and only if $S = f(0,\text{—})$ for some $f \in \mathsf{C}(L)$ with $\mathbf{0} \leq f \leq \mathbf{1}$ (recall Remark 2.3.2). Similarly, S is a cozero sublocale if and only if $S = f(0,\text{—})^{\#} \vee f(\text{—},0)^{\#}$ for some $f \in \mathsf{C}(L)$.

(2) Since Coz L is a regular sub-σ-frame of L, $\mathsf{ZS}(L)^{op}$ (resp. $\mathsf{CoSZ}(L)$) is a regular sub-σ-frame of $\mathfrak{c}(L)^{op}$ (resp. $\mathfrak{o}(L)$), and Coz $L \cong \mathsf{ZS}(L)^{op} \cong \mathsf{CoSZ}(L)$.

Since $\mathscr{R}(L)$ and $\mathsf{C}(L)$ are isomorphic (recall Remark 2.5.3) and $\mathsf{ZS}(L) = \mathfrak{c}(\text{Coz } L)$, one can take all the results from Section 2.3 and formulate them with zero sublocales and continuous functions in $\mathsf{C}(L)$. We will need the following two formulations:

Proposition 2.7.2. *An $f \in \mathsf{C}(L)$ has a multiplicative inverse if and only if $f(\text{—},0) \cap f(0,\text{—}) = \mathsf{O}$.*

Proposition 2.7.3. *Let Z_1 and Z_2 be disjoint zero sublocales of a frame L. Then*

$$Z_1 = f(0,\text{—}) \quad \textit{and} \quad Z_2 = f(\text{—},1)$$

for some $f \in \mathsf{C}(L)$ satisfying $\mathbf{0} \leq f \leq \mathbf{1}$.

Two sublocales S and T of L are *completely separated*[2] in L if there is an $f \in \mathsf{C}(L)$ with $\mathbf{0} \leq f \leq \mathbf{1}$ such that

$$S \subseteq f(0,\text{—}) \quad \text{and} \quad T \subseteq f(\text{—},1).$$

Remarks 2.7.4. (1) Sublocales $f(p,\text{—})$ and $f(\text{—},q)$ are zero sublocales for every $p,q \in \mathbb{Q}$ and $f \in \mathsf{C}(L)$.

(2) By Proposition 2.7.3 and the previous (1), two sublocales are completely separated in L if and only if they are contained in disjoint zero sublocales; i.e., if there are $Z_1, Z_2 \in \mathsf{ZS}(L)$ such that

$$S \subseteq Z_1, \quad T \subseteq Z_2 \quad \text{and} \quad Z_1 \cap Z_2 = \mathsf{O}.$$

(3) Any two disjoint zero sublocales of L are completely separated in L.

(4) S and T are completely separated if and only if their closures $\overline{S}$ and $\overline{T}$ are completely separated.

(5) If S and T are completely separated in L, then there exist $Z \in \mathsf{ZS}(L)$ and $C \in \mathsf{CoZS}(L)$ such that $S \subseteq Z \subseteq C \subseteq T^{\#}$. The converse does not hold in general.

Proposition 2.7.5. [44, Lemma.5.4.2.] *The following are equivalent for $a, b \in L$:*

(i) *$b \prec\!\!\prec a$.*

(ii) *$\mathfrak{o}(b)$ and $\mathfrak{c}(a)$ are completely separated in L.*

(iii) *There exist $Z \in \mathsf{ZS}(L)$ and $C \in \mathsf{CoZS}(L)$ such that $\mathfrak{c}(a) \subseteq Z^{\circ} \subseteq Z \subseteq C \subseteq \mathfrak{c}(b)$.*

(iv) *There exist $Z \in \mathsf{ZS}(L)$ and $C \in \mathsf{CoZS}(L)$ such that $\mathfrak{o}(b) \subseteq Z \subseteq C \subseteq \overline{C} \subseteq \mathfrak{o}(a)$.*

[2] When there is no risk of confusion we will avoid specifying the frame in which two sublocales completely separated. Note that if two sublocales of a sublocale S of L are completely separated in L, then they are completely separated in S. However, the converse does not hold: two sublocales of S completely separated sublocales in S may not be completely separated in L.

Remark 2.7.6. If S and T are completely separated in L there exist $a,b \in \mathsf{Coz}\, L$ such that

$$S \subseteq \mathfrak{c}(a), \quad T \subseteq \mathfrak{c}(b) \quad \text{and} \quad \mathfrak{c}(a) \cap \mathfrak{c}(b) = \mathsf{O}.$$

Then, since $\mathsf{Coz}\, L$ is normal (Remark 2.3.7 (3)) and $a \vee b = 1$, there exist $u,v \in \mathsf{Coz}\, L$ such that $u \wedge v = 0$, $v \prec\!\!\prec a$ and $u \prec\!\!\prec b$. Hence,

$$S \subseteq \mathfrak{c}(a) \subseteq \mathfrak{o}(u), \quad T \subseteq \mathfrak{c}(b) \subseteq \mathfrak{o}(u) \quad \text{and} \quad \mathfrak{o}(u) \cap \mathfrak{o}(v) = \mathsf{O}.$$

Moreover, by Proposition 2.7.5, $\mathfrak{o}(u)$ and $\mathfrak{c}(b)$ are completely separated, and so are $\mathfrak{o}(v)$ and $\mathfrak{c}(a)$.

Proposition 2.3.8, combined with Proposition 2.7.5, yields immediately the following corollary:

Corollary 2.7.7. [44, 5.5.] *The following are equivalent for a frame L:*

(i) *L is completely regular.*

(ii) *$\mathfrak{c}(a) = \bigcap \{Z \in \mathsf{ZS}(L) \mid \mathfrak{c}(a) \subseteq Z\}$ for every $a \in L$.*

(iii) *$\mathfrak{o}(a) = \bigvee \{C \in \mathsf{CoZS}(L) \mid C \subseteq \mathfrak{o}(a)\}$ for every $a \in L$.*

Proposition 2.7.8. [44, Corollary 5.6.1] *Let S be a sublocale of L.*

(1) *S is a zero sublocale of L if and only if $S = \bigcap_{n\in\mathbb{N}} \mathfrak{c}(a_n)$ for some $a_n \in L$, and there exist $Z_n \in \mathsf{ZS}(L)$ and $C_n \in \mathsf{CoZS}(L)$ such that $S \subseteq Z_n^\circ \subseteq Z_n \subseteq C_n \subseteq \mathfrak{c}(a_n)$ for every $n \in \mathbb{N}$.*

(2) *S is a cozero sublocale of L if and only if $S = \bigvee_{n\in\mathbb{N}} \mathfrak{o}(b_n)$ for some $b_n \in L$, and there exist $Z_n \in \mathsf{ZS}(L)$ and $C_n \in \mathsf{CoZS}(L)$ such that $\mathfrak{o}(b_n) \subseteq Z_n \subseteq C_n \subseteq \overline{C_n} \subseteq S$ for every $n \in \mathbb{N}$.*

(3) *S is both a zero and a cozero sublocale of L if and only if it is both closed and open.*

Since every zero sublocale is closed, then it is clearly F_σ. Furthermore, by Proposition 2.7.8 every zero sublocale is also G_δ. Similiarly, every cozero sublocale is a G_δ- and F_σ-sublocale.

Remark 2.7.9. In (1) of Proposition 2.7.8, the elements a_n may be taken in $\mathsf{Coz}\, L$ and such that $a_n \leq a_{n+1}$ for every $n \in \mathbb{N}$. Similarly, in (2) of Proposition 2.7.8 the b_n's may be taken in $\mathsf{Coz}\, L$ and such that $b_n \leq b_{n+1}$ for every $n \in \mathbb{N}$.

Proposition 2.7.10. [44, Corollary.5.6.2] *Every cozero sublocale of a cozero sublocale of L is a cozero sublocale of L.*

The relation of complete separation between sublocales has been used to prove insertion theorems. Article [37] gives a point-free extension of the Topological Insertion Theorem due to Blair [21] and Lane [57, 58]:

Theorem 2.7.11. *Let L be a frame and $f_1, f_2 \in \mathsf{F}(L)$. The following statements are equivalent.*

(i) *There exists $h \in \mathsf{C}(L)$ such that $f_2 \leq h \leq f_1$.*

(ii) *The sublocales $f_2(-,s)$ and $f_1(r,-)$ are completely separated in L for every $r < s$ in $\mathbb{Q}$.*

Theorem 2.7.11 implies the following insertion theorem for normal frames ([38, Theorem 8.1], [66, XIV.7.4.3]):

Theorem 2.7.12. *A frame L is normal if and only if for each upper semicontinuous $f \in \mathsf{F}(L)$ and lower semicontinuous $g \in \mathsf{F}(L)$ with $f \leq g$, there is an $h \in \mathsf{C}(L)$ such that $f \leq h \leq g$.*

An insertion theorem usually provides a separation result and an extension result as byproducts. In the case of Theorem 2.7.12, the separation result is the well-known point-free version of Urysohn's Separation Lemma [26] (and more recent [9, 10, 66]). The extension result is the point-free counterpart of Tietze's Extension Theorem [38, Theorem 8.4] (see also [66, XIV.7.6.1]). We state below these two results which will be later revisited in detail in Section 4.1.

Proposition 2.7.13. *A frame L is normal if and only if every pair of disjoint closed sublocales is completely separated in L.*

Proposition 2.7.14. *A frame L is normal if and only if for each closed sublocale S of L and each $h \in \mathsf{C}^*(S)$, there exists a continuous extension $\widetilde{h} \in \mathsf{C}^*(L)$ (i.e., there is an $\widetilde{h} \in \mathsf{C}^*(L)$ such that $j_{-1}[-]\widetilde{h} = h$ where $j\colon S \hookrightarrow L$).*

Chapter 3

Zero Sublocales and Localic Maps

In this chapter we explore the interaction between zero sublocales and localic maps. That is, we study the classes of localic maps defined by the behaviour of their images and preimages on zero sublocales. Recall Section 1.5 and the image/preimage Galois adjunction

$$\mathsf{S}(L) \underset{f_{-1}[-]}{\overset{f[-]}{\underset{\perp}{\rightleftarrows}}} \mathsf{S}(M)$$

for a localic map $f: L \to M$. A relevant case throughout this chapter is the localic embedding of a sublocale S of L, that is $j: S \hookrightarrow L$ where the preimage $j_{-1}: \mathsf{S}(L) \to \mathsf{S}(S)$ is given by the intersection: $j_{-1}[T] = S \cap T$. Most of the definitions and results presented in this chapter applied to this embedding give the notions and results of [6] and [9]. In this sense the work here generalizes the articles [6, 9]. Our approach focuses on general localic maps.

3.1 Classes of Localic Maps Defined by the Behaviour of Their Preimages on Zero Sublocales

Since frame homomorphisms preserve cozero elements (Proposition 2.3.3), the preimage map $f_{-1}[-]: \mathsf{S}(M) \to \mathsf{S}(L)$ restricts to maps

$$f^{z}_{-1}[-]: \mathsf{ZS}(M) \to \mathsf{ZS}(L) \quad \text{and} \quad f^{coz}_{-1}[-]: \mathsf{CoZS}(M) \to \mathsf{CoZS}(L).$$ [1]

The former is a σ-coframe homomorphism and the latter is a σ-frame homomorphism. Whenever $f^{z}_{-1}[-]$ is surjective, we say that f is a *z-map*. These maps are the right adjoints of the *coz*-onto frame homomorphisms of [31]. Note that when L is completely regular, a z-map is always injective because a completely regular frame is join-generated by its cozero σ-frame (recall Proposition 2.3.8).

In the particular case where the embedding $j: S \hookrightarrow L$ of a sublocale S of L is a z-map, one refers to S as *z-embedded* in L ([6]) and we have immediately the following result:

[1] Usually, when applying $f^{z}_{-1}[-]$ or $f^{coz}_{-1}[-]$ we only write $f_{-1}[-]$.

Proposition 3.1.1. [44, Proposition 6.2.1] *A sublocale S of L is z-embedded if and only if for each zero sublocale Z of S there is a zero sublocale T of L such that $T \cap S = Z$. Equivalently, for every cozero sublocale C of S there is a cozero sublocale T of L such that $T \cap S = C$.*

Remarks 3.1.2. (1) If $f\colon L \to M$ is a z-map, then $f[L]$ is a z-embedded sublocale of M. Indeed, consider the standard factorization in Loc ([66, IV.1.4] and (1.2.9))

$$L \xrightarrow{\ \phi\ } f[L] \xhookrightarrow{\ j\ } M, \qquad f = j\circ\phi$$

with ϕ onto and j injective localic maps. Since the correspondence $L \mapsto \mathsf{ZS}(L)$ is functorial (recall Remark 2.3.7 (4) and Remark 2.7.1 (2)), we get the following commutative diagram in Frm:

$$\mathsf{ZS}(L) \xleftarrow{\ \phi^z_{-1}[-]\ } \mathsf{ZS}(f[L]) \xleftarrow{\ j^z_{-1}[-]\ } \mathsf{ZS}(M), \qquad f^z_{-1}[-] = \phi^z_{-1}[-]\circ j^z_{-1}[-]\colon \mathsf{ZS}(M)\to\mathsf{ZS}(L).$$

Then, for any $\mathfrak{c}(a) \in \mathsf{ZS}(f[L])$, $\phi^z_{-1}[\mathfrak{c}(a)] \in \mathsf{ZS}(L)$. Since f is a z-map,

$$\phi^z_{-1}[\mathfrak{c}(a)] = f^z_{-1}[\mathfrak{c}(b)] = \phi^z_{-1}[j^z_{-1}[\mathfrak{c}(b)]]$$

for some $\mathfrak{c}(b) \in \mathsf{ZS}(M)$. This means that $\mathfrak{c}(\phi^* j^*(b)) = \mathfrak{c}(\phi^*(a))$. Finally, since ϕ^* is injective (being the left adjoint of an onto localic map), $j^*(b) = a$ and thus $\mathfrak{c}(b) \cap f[L] = j_{-1}[\mathfrak{c}(b)] = \mathfrak{c}(j^*(b)) = \mathfrak{c}(a)$.

(2) By Proposition 3.1.1, it clearly follows that if T is a z-embedded sublocale of S and S is z-embedded in L, then T is z-embedded in L.

(3) If S is a sublocale of L and T is sublocale of S such that T is z-embedded in L, then T is z-embedded in S. Indeed, let $Z \in \mathsf{ZS}(T)$. By assumption and Proposition 3.1.1, $Z = T \cap Z'$ with $Z' \in \mathsf{ZS}(L)$. Then $Z = T \cap Z' = T \cap Z' \cap S$ and $Z' \cap S \in \mathsf{ZS}(L)$. So by Proposition 3.1.1, T is z-embedded in S.

We say that a localic map $f\colon L \to M$ is *z-dense* if

$$f^z_{-1}[Z] = \mathsf{O} \implies Z = \mathsf{O}.$$

Remarks 3.1.3. (1) Equivalently, f is z-dense if the cozero map $f^{coz}_{-1}[-]$ is a codense σ-frame homomorphism, that is,

$$f^{coz}_{-1}[C] = L \implies C = M.$$

The z-dense localic maps are the right adjoints of the *coz*-codense frame homomorphisms of [9].

(2) Any codense localic map is z-dense. Indeed, if $f\colon L \to M$ is a codense localic map and $f_{-1}[\mathfrak{c}_M(a)] = \mathsf{O}$ for $a \in \mathrm{Coz}\, M$, then $f_{-1}[\mathfrak{c}_M(a)] = \mathfrak{c}_L(f^*(a)) = \mathfrak{c}_L(1)$. Since f is codense (i.e., f^* is a codense frame homomorphism) we get $a = 1$ which means $\mathfrak{c}_M(a) = \mathsf{O}$.

(3) For a sublocale S of L, the embedding $j\colon S \hookrightarrow L$ is z-dense if and only if $Z \cap S = \mathsf{O}$ implies $Z = \mathsf{O}$ for every $Z \in \mathsf{ZS}(L)$.

Furthermore, we say that an f is *almost z-dense* if for every $Z \in \mathsf{ZS}(M)$ such that $f_{-1}^{z}[Z] = \mathsf{O}$, there exists a $Z' \in \mathsf{ZS}(M)$ such that $f_{-1}^{z}[Z'] = L$ and $Z \cap Z' = \mathsf{O}$.

Remarks 3.1.4. (1) Almost z-dense maps are the right adjoints of the almost *coz*-codense frame homomorphisms of [9, 31]. Clearly, any z-dense localic map is almost z-dense.

(2) If f is dense and almost z-dense, then it is z-dense. Indeed, let $Z \in \mathsf{ZS}(L)$ such that $f_{-1}^{z}[Z] = \mathsf{O}$. Then there is $Z' \in \mathsf{ZS}(L)$ such that $f_{-1}^{z}[Z'] = L$ and $Z \cap Z' = \mathsf{O}$. Now, since f is dense (i.e., f^* is a dense frame homomorphism) and $f_{-1}^{z}[Z'] = f_{-1}[\mathfrak{c}_M(a)] = \mathfrak{c}_L(f^*(a)) = L = \mathfrak{c}_L(0)$, we have $a = 0$. Thus, $Z' = L$ which implies $Z = \mathsf{O}$.

(3) For each sublocale S of L, the embedding $j\colon S \hookrightarrow L$ is almost z-dense if and only if for every $Z \in \mathsf{ZS}(L)$ such that $Z \cap S = \mathsf{O}$, there exists a $Z' \in \mathsf{ZS}(L)$ such that $S \subseteq Z'$ and $Z \cap Z' = \mathsf{O}$ (i.e., S is completely separated from every zero sublocale disjoint from it).

The following result characterizes almost z-dense maps. It was proved in [9, 7.2.1] for almost *coz*-codense frame homomorphisms.

Proposition 3.1.5. *A localic map $f\colon L \to M$ is almost z-dense if and only if for every $Z \in \mathsf{ZS}(M)$ such that $f_{-1}[Z] = \mathsf{O}$, there exists a bounded $g \in \mathsf{C}(M)$ such that $Z \subseteq g(0,\text{—})$ and $f_{-1}[g(\text{—},1)] = L$.*

Proof. The implication '$\Longrightarrow$' follows from Proposition 2.7.3. The converse is clear since $g(0,\text{—})$ and $g(\text{—},1)$ are disjoint zero sublocales for every bounded $g \in \mathsf{C}(M)$. □

Let $f\colon L \to M$ be a localic map. We say that two sublocales S and T of M are *f-separated* whenever there exist $Z_1, Z_2 \in \mathsf{ZS}(M)$ such that

$$S \subseteq Z_1, \quad T \subseteq Z_2, \quad \text{and} \quad f_{-1}^{z}[Z_1] \cap f_{-1}^{z}[Z_2] = \mathsf{O}.$$

In particular, for a localic embedding $j\colon S \hookrightarrow L$, the sublocales R and T are j-separated if and only if there exist $Z_1, Z_2 \in \mathsf{ZS}(L)$ such that $R \subseteq Z_1$, $T \subseteq Z_2$ and $Z_1 \cap Z_2 \cap S = \mathsf{O}$. In this case we say that R and T are *S-separated*. Note that Definition 3.4 of [31] is the equivalent notion of S-separated sublocales for frame quotients.

Remarks 3.1.6. (1) If S and T are f-separated sublocales of M and f is z-dense, then S and T are completely separated in M.

(2) Any two completely separated sublocales of M are f-separated for any localic $f\colon L \to M$.

3.2 More on z-Maps

This section presents a couple of useful characterizations of z-maps which immediately yield as corollaries characterizations for z-embedded sublocales.

The next result is a characterization of z-maps that appears in [31, Prop. 3.3] phrased in terms of *coz*-onto frame homomorphisms.

Proposition 3.2.1. *The following are equivalent for any localic map $f\colon L \to M$:*

(i) *f is a z-map.*

(ii) *For any* $C \in \mathsf{CoZS}(L)$ *and* $Z \in \mathsf{ZS}(L)$ *such that* $C \subseteq Z$*, there exist* $C' \in \mathsf{CoZS}(M)$ *and* $Z' \in \mathsf{ZS}(M)$ *with* $C' \subseteq Z'$ *such that* $f_{-1}[C'] = C$ *and* $f_{-1}[Z'] = Z$*.*

(iii) *For any* $C \in \mathsf{CoZS}(L)$ *and* $Z \in \mathsf{ZS}(L)$ *such that* $C \subseteq Z$*, there exist* $C' \in \mathsf{CoZS}(M)$ *and* $Z' \in \mathsf{ZS}(M)$ *with* $C' \subseteq Z'$ *such that* $C \subseteq f_{-1}[C'] \subseteq f_{-1}[Z'] \subseteq Z$*.*

Proof. (i) $\Longrightarrow$ (ii): Let f be a z-map. Consider $Z \in \mathsf{ZS}(L)$ and $C \in \mathsf{CoZS}(L)$ such that $C \subseteq Z$. Since f is a z-map we have

$$f_{-1}[\mathfrak{c}(a)] = Z \quad \text{and} \quad f_{-1}[\mathfrak{o}(b)] = C$$

for some $a, b \in \mathsf{Coz}\, M$. Since $\mathfrak{c}(a)$ is a zero sublocale of M, we get from Proposition 2.7.8 and Remark 2.7.9 that $\mathfrak{c}(a) = \bigcap_{n\in\mathbb{N}} \mathfrak{c}(a_n)$ for some $a_n \in \mathsf{Coz}\, M$ such that, for every $n \in \mathbb{N}$,

$$\mathfrak{c}(a) \subseteq \mathfrak{o}(x_n) \subseteq \mathfrak{c}(a_n) \quad \text{and} \quad \mathfrak{c}(a_{n+1}) \subseteq \mathfrak{c}(a_n) \tag{3.2.1}$$

for some $x_n \in \mathsf{Coz}\, M$. Analogously, $\mathfrak{o}(b) = \bigvee_{n\in\mathbb{N}} \mathfrak{o}(b_n)$ for some $b_n \in \mathsf{Coz}\, M$ and

$$\mathfrak{o}(b_n) \subseteq \mathfrak{c}(y_n) \subseteq \mathfrak{o}(b) \quad \text{and} \quad \mathfrak{o}(b_n) \subseteq \mathfrak{o}(b_{n+1}) \tag{3.2.2}$$

for some $y_n \in \mathsf{Coz}\, M$. Now, let

$$Z' = \bigcap_{n\in\mathbb{N}} (\mathfrak{c}(a_n) \vee \mathfrak{c}(y_n)) \quad \text{and} \quad C' = \bigvee_{n\in\mathbb{N}} (\mathfrak{o}(b_n) \cap \mathfrak{o}(x_n)).$$

Clearly, $Z' \in \mathsf{ZS}(M)$ and $C' \in \mathsf{CoZS}\, M$. Fix an $m \in \mathbb{N}$. From (3.2.1) and (3.2.2) we know that

$$\begin{aligned}
&\mathfrak{o}(b_m) \cap \mathfrak{o}(x_m) \subseteq \mathfrak{o}(b_m) \subseteq \mathfrak{c}(y_n) \subseteq \mathfrak{c}(y_n) \vee \mathfrak{c}(a_n) \quad \forall n \geq m \quad \text{and} \\
&\mathfrak{o}(b_m) \cap \mathfrak{o}(x_m) \subseteq \mathfrak{o}(x_m) \subseteq \mathfrak{c}(a_n) \subseteq \mathfrak{c}(y_n) \vee \mathfrak{c}(a_n) \quad \forall n \leq m.
\end{aligned}$$

Hence, $\mathfrak{o}(b_m) \vee \mathfrak{o}(x_m) \subseteq Z'$ for every $m \in \mathbb{N}$, and $C' \subseteq Z'$. Moreover,

$$\mathfrak{c}(a) = \bigcap_{n\in\mathbb{N}} \mathfrak{c}(a_n) \subseteq Z' \subseteq \bigcap_{n\in\mathbb{N}} \mathfrak{c}(a_n) \vee \mathfrak{o}(b) = \mathfrak{c}(a) \vee \mathfrak{o}(b)$$

and $Z = f_{-1}[\mathfrak{c}(a)] \subseteq f_{-1}[Z'] \subseteq f_{-1}[\mathfrak{c}(a)] \vee f_{-1}[\mathfrak{o}(b)] = Z \vee C = Z$. Similarly,

$$\mathfrak{o}(b) \cap \mathfrak{c}(a) = \bigvee_{n\in\mathbb{N}} \mathfrak{o}(b_n) \cap \mathfrak{c}(a) \subseteq C' \subseteq \bigvee_{n\in\mathbb{N}} \mathfrak{o}(b_n) = \mathfrak{o}(b).$$

Finally, $C = C \cap Z = f_{-1}[\mathfrak{o}(b)] \cap f_{-1}[\mathfrak{c}(a)] \subseteq f_{-1}[C'] \subseteq f_{-1}[\mathfrak{o}(b)] = C$ as required.

(ii) $\Longrightarrow$ (iii) is trivial.

(iii) $\Longrightarrow$ (i): Let $\mathfrak{c}(a) = \bigcap_{n\in\mathbb{N}} \mathfrak{c}(a_n)$ be a zero sublocale with $a_n \in \mathsf{Coz}\, L$ such that for each natural n there is a cozero sublocale $\mathfrak{o}(x_n)$ satisfying $\mathfrak{c}(a) \subseteq \mathfrak{o}(x_n) \subseteq \mathfrak{c}(a_n)$ (recall Proposition 2.7.8 and Remark 2.7.9). By hypothesis, there exist zero and cozero sublocales $\mathfrak{c}(b_n)$ and $\mathfrak{o}(d_n)$ in M, such that

$$\mathfrak{o}(d_n) \subseteq \mathfrak{c}(b_n) \quad \text{and} \quad \mathfrak{o}(x_n) \subseteq f_{-1}[\mathfrak{o}(d_n)] \subseteq f_{-1}[\mathfrak{c}(b_n)] \subseteq \mathfrak{c}(a_n)$$

for every n. We claim that $\mathfrak{c}(a) = f_{-1}[\bigcap_{n\in\mathbb{N}} \mathfrak{c}(b_n)]$. Indeed,

$$\mathfrak{c}(a) \subseteq \bigcap_{n\in\mathbb{N}} \mathfrak{o}(x_n) \subseteq \bigcap_{n\in\mathbb{N}} f_{-1}[\mathfrak{o}(d_n)] \subseteq \bigcap_{n\in\mathbb{N}} f_{-1}[\mathfrak{c}(b_n)] \subseteq \bigcap_{n\in\mathbb{N}} \mathfrak{c}(a_n) = \mathfrak{c}(a)$$

and $\bigcap_{n\in\mathbb{N}} f_{-1}[\mathfrak{c}(b_n)] = f_{-1}[\bigcap_{n\in\mathbb{N}} \mathfrak{c}(b_n)]$, where $\bigcap_{n\in\mathbb{N}} \mathfrak{c}(b_n)$ is clearly a zero sublocale of M. □

Remark 3.2.2. Conditions (ii) and (iii) above can be equivalently written as follows:

(ii)' For any disjoint pair of cozero sublocales $C_1, C_2 \in \mathsf{CoZS}(L)$, there exist disjoint $C_1', C_2' \in \mathsf{CoZS}(M)$ such that $f_{-1}[C_1'] = C_1$ and $f_{-1}[C_2'] = C_2$.

(ii)" For any pair of zero sublocales $Z_1, Z_2 \in \mathsf{ZS}(L)$ such that $Z_1 \vee Z_2 = L$, there exist $Z_1', Z_2' \in \mathsf{ZS}(M)$ such that $Z_1 \vee Z_2 = M$, $f_{-1}[Z_1'] = Z_1$ and $f_{-1}[Z_2'] = Z_2$.

(iii)' For any disjoint pair of cozero sublocales $C_1, C_2 \in \mathsf{CoZS}(L)$, there exist disjoint $C_1', C_2' \in \mathsf{CoZS}(M)$ such that $C_1 \subseteq f_{-1}[C_1']$ and $C_2 \subseteq f_{-1}[C_2']$.

(iii)" For any pair of zero sublocales $Z_1, Z_2 \in \mathsf{ZS}(L)$ such that $Z_1 \vee Z_2 = L$, there exist $Z_1', Z_2' \in \mathsf{ZS}(M)$ such that $Z_1' \vee Z_2' = M$, $f_{-1}[Z_1'] \subseteq Z_1$ and $f_{-1}[Z_2'] \subseteq Z_2$.

Proposition 3.2.1 applied to the case of a sublocale embedding $S \hookrightarrow L$ yields immediately the following corollary:

Corollary 3.2.3. *The following are equivalent for any sublocale S of L:*

(i) *S is z-embedded.*

(ii) *For any $C \in \mathsf{CoZS}(S)$ and $Z \in \mathsf{ZS}(S)$ such that $C \subseteq Z$, there exist $C' \in \mathsf{CoZS}(L)$ and $Z' \in \mathsf{ZS}(L)$ with $C' \subseteq Z'$ such that $S \cap C' = C$ and $S \cap Z' = Z$.*

(iii) *For any $C \in \mathsf{CoZS}(S)$ and $Z \in \mathsf{ZS}(S)$ such that $C \subseteq Z$, there exist $C' \in \mathsf{CoZS}(L)$ and $Z' \in \mathsf{ZS}(L)$ with $C' \subseteq Z'$ such that $C \subseteq S \cap C' \subseteq S \cap Z' \subseteq Z$.*

(iv) *For any pair of disjoint cozero sublocales $C_1, C_2 \in \mathsf{CoZS}(S)$, there exist disjoint $C_1', C_2' \in \mathsf{CoZS}(L)$ such that $S \cap C_1' = C_1$ and $S \cap C_2' = C_2$.*

(v) *For any pair of zero sublocales $Z_1, Z_2 \in \mathsf{ZS}(S)$ such that $Z_1 \vee Z_2 = S$, there exist $Z_1', Z_2' \in \mathsf{ZS}(L)$ such that $Z_1' \vee Z_2' = L$, $S \cap Z_1' = Z_1$ and $S \cap Z_2' = Z_2$.*

We have a further characterization of z-maps in terms of complete separation and f-separation:

Proposition 3.2.4. *The following are equivalent for any localic map $f\colon L \to M$:*

(i) *f is a z-map.*

(ii) *If S and T are completely separated sublocales of L, then there exists $Z \in \mathsf{ZS}(M)$ such that $S \subseteq f_{-1}[Z]$ and $T \subseteq f_{-1}[Z^\#] = f_{-1}[Z]^\#$ (equivalently, $f[S] \subseteq Z$ and $f[T] \subseteq Z^\#$).*

(iii) *If S and T are completely separated sublocales of L, then $f[S]$ and $f[T]$ are f-separated.*

Proof. (i) $\implies$ (ii): Let S and T be completely separated sublocales of L. There exists a $Z \in \mathsf{ZS}(L)$ such that $S \subseteq Z$ and $T \subseteq Z^{\#}$. Then, since f is a z-map, $Z = f_{-1}[Z']$ where $Z' \in \mathsf{ZS}(M)$.

(ii) $\implies$ (iii): It suffices to show condition (iii) for disjoint zero sublocales (recall Remark 2.7.4(2)). Consider $Z_1, Z_2 \in \mathsf{ZS}(L)$ such that $Z_1 \cap Z_2 = \mathsf{O}$. By assumption, there exists $Z \in \mathsf{ZS}(M)$ such that $Z_1 \subseteq f_{-1}[Z]$ and $Z_2 \subseteq f_{-1}[Z^{\#}]$. Then

$$Z_2 \cap f_{-1}[Z] \subseteq f_{-1}[Z^{\#}] \cap f_{-1}[Z] = f_{-1}[Z \cap Z^{\#}] = f_{-1}[\mathsf{O}_M] = \mathsf{O}_L,$$

and thus Z_2 and $f_{-1}[Z]$ are completely separated in L. We apply once again (ii) to obtain $Z_2 \subseteq f_{-1}[Z']$ and $f_{-1}[Z] \subseteq f_{-1}[Z'^{\#}]$ for some $Z' \in \mathsf{ZS}(M)$. Then

$$f_{-1}[Z'] \cap f_{-1}[Z] \subseteq f_{-1}[Z'] \cap f_{-1}[Z'^{\#}] = f_{-1}[Z' \cap Z'^{\#}] = f_{-1}[\mathsf{O}_M] = \mathsf{O}_L.$$

Finally, by the image/preimage adjunction, we get $f[Z_1] \subseteq Z$ and $f[Z_2] \subseteq Z'$. Hence, $f[Z_1]$ and $f[Z_2]$ are f-separated.

(iii) $\implies$ (i): In order to show that f is a z-map, let $Z \in \mathsf{ZS}(L)$. By Proposition 2.7.8, $Z = \bigcap_{n\in\mathbb{N}} \mathfrak{c}(a_n)$ where for each n there exist zero and cozero sublocales Z_n and C_n such that $Z \subseteq Z_n \subseteq C_n \subseteq \mathfrak{c}(a_n)$. In particular, Z and $\mathfrak{o}(a_n)$ are completely separated sublocales in L. Then, by assumption, there are $F_n', F_n \in \mathsf{ZS}(M)$ such that

$$f[Z] \subseteq F_n, \quad f[\mathfrak{o}(a_n)] \subseteq F_n' \quad \text{and} \quad f_{-1}[F_n] \cap f_{-1}[F_n'] = \mathsf{O}.$$

Clearly, $\bigcap_{n\in\mathbb{N}} F_n \in \mathsf{ZS}(M)$ and $Z \subseteq f_{-1}\left[\bigcap_{n\in\mathbb{N}} F_n\right]$. For the other inclusion, since $\mathfrak{o}(a_n) \subseteq f_{-1}[F_n']$, we have

$$f_{-1}\Big[\bigcap_{n\in\mathbb{N}} F_n\Big] = \bigcap_{n\in\mathbb{N}} f_{-1}[F_n] \subseteq \bigcap f_{-1}[F_n']^{\#} \subseteq \bigcap_{n\in\mathbb{N}} \mathfrak{c}(a_n) = Z. \qquad \square$$

The following corollary (which is Proposition 7.3 of [6]) is the application of Proposition 3.2.4 to the case of a sublocale embedding $S \hookrightarrow L$. This result can be found in [31, Proposition 3.5] formulated in terms of frame quotients. We would like to point out that the formulation in terms of sublocales is advantageous because it resembles the corresponding result in classical topology in [19] (see also [2, Theorem 7.2]).

Corollary 3.2.5. *The following are equivalent for a sublocale S of L:*

(i) *S is z-embedded in L.*

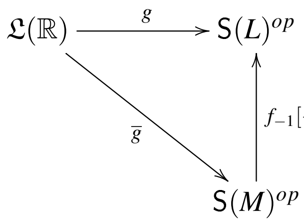
(ii) *If T and R are completely separated sublocales of S, then there exists $Z \in \mathsf{ZS}(L)$ such that $T \subseteq Z$ and $R \subseteq Z^{\#}$.*

(iii) *If T and R are completely separated sublocales of S, then they are S-separated.*

3.3 C- and C*-Maps

Similar to the extension of z-embedded sublocales to z-maps, we can generalize the notions of C- and C^*-embedded sublocales of [6, 9] to localic maps. We say that a localic map $f\colon L \to M$

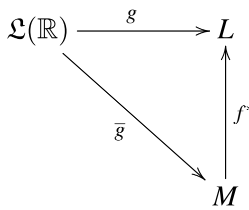
is a C-*map* (resp. C*-*map*) if for every continuous (resp. bounded and continuous) real-valued function $g\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(L)^{op}$, there exists a continuous (resp. bounded and continuous) function $\overline{g}\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(M)^{op}$ such that the diagram



(3.3.1)

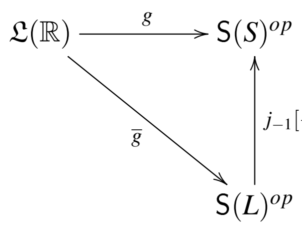
commutes. That is, f is a C-map precisely when g factors through $f_{-1}[-]$ for every $g \in \mathsf{C}(L)$. Note that if $\mathbf{p} \leq g \leq \mathbf{q}$ (i.e., $g \in \mathsf{C}^*(L)$) and $\overline{g} \in \mathsf{C}(L)$ makes the diagram (3.3.1) commute, then so does $(\overline{g} \vee \mathbf{q}) \wedge \mathbf{p}$. Thus, f is a C*-map if and only if g factors through $f_{-1}[-]$ for every $g \in \mathsf{C}^*(L)$.

Remarks 3.3.1. (1) By the isomorphism $\mathsf{C}(L) \cong \mathscr{R}(L)$ (Remark 2.5.3) and property (1.5.1), it follows that $f\colon L \to M$ is a C-*map* (resp. C*-map) if and only if for every $g \in \mathscr{R}(L)$ (resp. bounded $g \in \mathscr{R}(L)$) there exists $\overline{g} \in \mathscr{R}(L)$ (resp. bounded $\overline{g} \in \mathscr{R}(L)$) such that the diagram



commutes.

(2) Recall from [6] that a sublocale S is C-embedded (resp. C*-embedded) if for every continuous (resp. bounded and continuous) real function $g\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(S)^{op}$ there exists a continuous (resp. bounded and continuous) function such that the diagram



commutes, where j is the localic embedding of S in L. Because of how preimages of localic embeddings are computed, the diagram above just means that $\overline{g}(a) \cap S = g(a)$ for every $a \in \mathfrak{L}(\mathbb{R})$. In this case, we say that $\overline{g}$ is a *continuous extension* (resp. *continuous and bounded extension*) of g. Hence, a sublocale S of L is C-*embedded* (resp. C*-*embedded*) if and only if the embedding $j\colon S \hookrightarrow L$ is a C-map (resp. C*-map). These localic embeddings are precisely the right adjoints of the C-quotients (resp. C*-quotients) of [9].

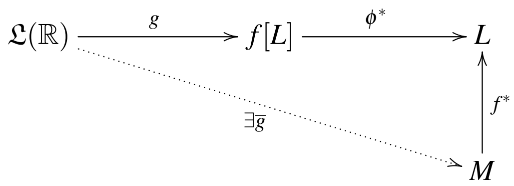
(3) Every C-map is a C*-map, and every C*-map is a z-map. Indeed, if $f\colon L \to M$ is a C-map and $g \in \mathsf{C}(L)$ with $\mathbf{p} \leq g \leq \mathbf{q}$, there exists $\overline{g} \in \mathsf{C}(M)$ such that $f_{-1}[-]\,\overline{g} = g$. Nevertheless, $\overline{g}$ might not be

bounded so consider $h := (\mathbf{p} \vee \overline{g}) \wedge \mathbf{q}$. Then h is bounded and factors g through $f_{-1}[-]$. Furthermore, to show that a C^*-map is a z-map, let $f\colon L \to M$ be a localic C^*-map and take $Z \in \mathsf{ZS}(L)$. By Proposition 2.7.3, there is $g \in \mathsf{C}^*(L)$ such that $Z = g(0,\text{—})$. Since f is a C^*-map, there is $\overline{g} \in \mathsf{C}^*(M)$ such that $f_{-1}[-]\,\overline{g} = g$. Thus, $f_{-1}[\overline{g}(0,\text{—})] = g(0,\text{—}) = Z$ and $\overline{g}(0,\text{—}) \in \mathsf{ZS}(M)$.

(4) If $f\colon L \to M$ is a C-map (resp. C^*-map), then $f[L]$ is a C-embedded (resp. C^*-embedded) sublocale of L. Indeed, by (1.2.9), the localic map f can be factorized in Loc as

$$L \xrightarrow{\ \phi\ } f[L] \xhookrightarrow{\ j\ } M\,.$$

So if f is a C-map, by (1) above, for every $g \in \mathscr{R}(f[L])$ (resp. bounded $g \in \mathscr{R}(f[L])$) there exists $\overline{g} \in \mathscr{R}(M)$ (resp. bounded $g \in \mathscr{R}(M)$) such that

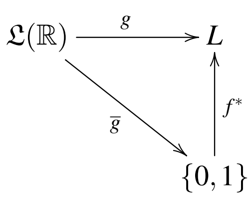


commutes. Then

$$\phi^* g = f^* \overline{g} = \phi^* j^* \overline{g},$$

and, since ϕ^* is injective, $g = j^*\overline{g}$, showing that j is a C-map (resp. C^*-map). Equivalently, $f[L]$ is C-embedded (resp. C^*-embedded).

(5) For the case when f is an injective C-map, $L \cong f[L]$. By (4) we get the notion of a C-embedded sublocale. Let us provide an example of a C-map that is not injective. Consider the frame L given by the Sierpińksi topology, that is, $L = \{0 < a < 1\}$. We claim that the unique localic map $f\colon L \to \{0,1\}$ is a C-map. Indeed, the only frame homomorphisms $g\colon \mathfrak{L}(\mathbb{R}) \to L$ are the constant functions. Hence there is always a $\overline{g}\colon \mathfrak{L}(\mathbb{R}) \to \{0,1\}$ such that



commutes.

We now present characterizations of C- and C^*-maps that generalize those that appear in [6, 9] for C- and C^*-embedded sublocales (or quotients), which will naturally come as immediate corollaries. The fact that these results hold for general localic maps, and not only embeddings, is rather surprising, since the notions of C and C^*-maps are stronger. Note also that a direct proof of the corollaries in this section can be produced by mimicking the idea of the proofs of the corresponding general results.

Proposition 3.3.2. *Every* C*-map is almost* z*-dense.*

Proof. Let $f\colon L\to M$ be a localic C-map and $Z\in \mathsf{ZS}(M)$ such that $f_{-1}[Z]=\mathsf{O}$. Consider $g\in \mathsf{C}^*(M)$, with $\mathbf{0}\le g\le \mathbf{1}$, such that $g(0,—)=Z$. The composite $f_{-1}[-]\,g$ is a bounded continuous function and its zero sublocale is given by

$$(f_{-1}[-]\,g)(0,—)=f_{-1}[g(0,—)]=\mathsf{O}.$$

Thus, by Proposition 2.7.2, $f_{-1}[-]\,g$ has a multiplicative inverse function, say $h\colon \mathfrak{L}(\mathbb{R})\to \mathsf{S}(L)^{op}$. Since f is a C-map there exists a continuous $\overline{h}\colon \mathfrak{L}(\mathbb{R})\to \mathsf{S}(M)^{op}$ such that $h=f_{-1}[-]\,\overline{h}$. We claim that $(\overline{h}\cdot g)(—,1)$ is the zero sublocale of M we are looking for. Indeed

$$f_{-1}[(\overline{h}\cdot g)(—,1)]=\bigcap_{s>0} f_{-1}[(\overline{h}(—,s)]\vee f_{-1}[g(—,\tfrac{1}{s})]=\bigcap_{s>0} h(—,s)\vee f_{-1}[g(—,\tfrac{1}{s})]=\mathbf{1}(—,1)=L.$$

Moreover,

$$(\overline{h}\cdot g)(0,—)=\bigcap_{s>0}\overline{h}(s,-)\vee g(0,-)=\overline{h}(0,-)\vee Z\supseteq Z.$$

Finally, this means that $(\overline{h}\cdot g)(—,1)\cap Z=(\overline{h}\cdot g)(—,1)\cap(\overline{h}\cdot g)(0,—)=\mathsf{O}$. □

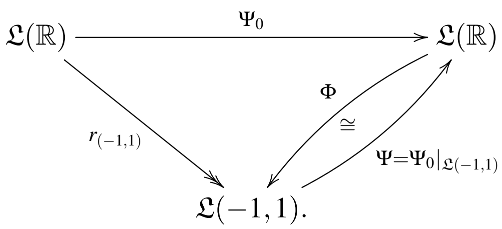
Recall Remarks 3.3.1 (2) and 3.1.4 (3). The application of Proposition 3.3.2 to the localic embedding $j\colon S\hookrightarrow L$ yields immediately:

Corollary 3.3.3. *Let S be a sublocale of L. If S is* C*-embedded then S is completely separated from every zero sublocale disjoint from it.*

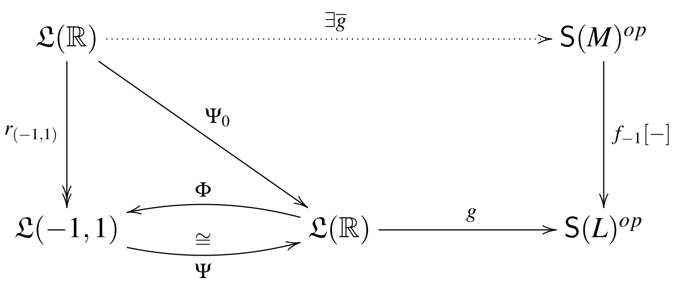
In classical topology, it is well known that a subspace is C-embedded if and only if it is C*-embedded and it is completely separated from every zero sublocale disjoint from it [36, 1.18]. Next, not only do we give the equivalent point-free result for sublocales (Corollary 3.3.5), instead of using quotients like in [9, 7.2.2], but we also generalize the result for localic maps.

Proposition 3.3.4. *A localic map $f\colon L\to M$ is a* C*-map if and only if it is an almost z-dense* C**-map.*

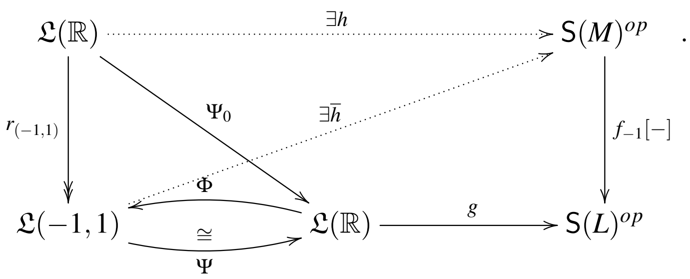
Proof. If $f\colon L\to M$ is a C-map it is clearly a C*-map; furthermore, by Proposition 3.3.2, it is almost z-dense.
Conversely, let $f\colon L\to M$ be an almost z-dense C*-map. In order to show that f is a C-map, let $g\in \mathsf{C}(L)$. Now recall Proposition 2.2.2 and consider an order isomorphism ψ from the rational interval $\langle -1,1\rangle$ into $\mathbb{Q}$. Using the notation from Proposition 2.2.2, we have the commutative diagram



The composite $g\Psi_0$ is a bounded frame homomorphism (since $\Psi_0(-1,1)=1$). Hence, since f is C*-map, there is a $\overline{g}\in \mathsf{C}^*(M)$ (with $\mathbf{p}<\overline{g}<\mathbf{q}$) such that the diagram



commutes. Note that $\overline{g}$ may not be an extension of g. Nevertheless, to show that f is a C-map it suffices to find an $h\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(M)^{op}$ such that $h(-1,1) = \mathsf{O}$ and $f_{-1}[-]h = g\Psi_0$, because then, by Lemma 2.2.1, there exists $\overline{h}\colon \mathfrak{L}(-1,1) \to \mathsf{S}(M)^{op}$ such that $h = \overline{h} f_{-1}[-]$.
We have that the following commutative diagram



.

Further, the composite $\overline{h}\Phi$ is a continuous extension of g. Indeed,

$$g\Psi r_{(-1,1)} = g\Psi_0 = f_{-1}[-]h = f_{-1}[-]\overline{h}r_{(-1,1)} \implies g\Psi = f_{-1}[-]\overline{h} \iff g = f_{-1}[-]\overline{h}\Phi.$$

We conclude the proof by showing how to get such map h.
Let $\overline{g}(-1,1) = \mathfrak{c}_M(a) \in \mathsf{ZS}(M)$. Then $f_{-1}[\mathfrak{c}_M(a)] = g\Psi_0(-1,1) = g(1) = \mathsf{O}$. Since f is almost z-dense, there exists $D \in \mathsf{ZS}(M)$ such that $f_{-1}[D] = L$ and $\mathfrak{c}_M(a) \cap D = \mathsf{O}$. Any two disjoint zero sublocales are always completely separated (Remark 2.7.4 (3)), so there exists $t\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(M)^{op}$ ($\mathbf{0} \le t \le \mathbf{1}$) such that $\overline{g}(-1,1) = \mathfrak{c}_M(a) \subseteq t(0,\text{—})$ and $D \subseteq t(\text{—},1)$. We claim that $t \cdot \overline{g}$ is the function h we are searching for. We only need to check that $(t\cdot\overline{g})(-1,1) = \mathsf{O}$ and $f_{-1}[-](t\cdot\overline{g}) = g\Psi_0$. Using (2.1.4) we get

$$\begin{aligned}(t\cdot\overline{g})(-1,1) &= \bigcap\{t(r,s)\vee\overline{g}(t,u) \mid \langle r,s\rangle\cdot\langle t,u\rangle \subseteq \langle -1,1\rangle\} \subseteq \bigcap\{t(-y,y)\vee\overline{g}(\tfrac{-1}{y},\tfrac{1}{y}) \mid 1<y\} \\ &\overset{(*)}{=} \bigcap\{\overline{g}(\tfrac{-1}{y},\tfrac{1}{y}) \mid 1<y\} = \overline{g}(-1,1)\end{aligned}$$

(the equality $(*)$ follows from the fact that $\mathbf{0} \le t \le \mathbf{1}$). Consequently,

$$\begin{aligned}(t\cdot\overline{g})(-1,1) &= (t\cdot\overline{g})(-1,1)\cap\overline{g}(-1,1) = \bigcap\{t(r,s)\vee\overline{g}(u,v) \mid \langle r,s\rangle\cdot\langle u,v\rangle \subseteq \langle -1,1\rangle\}\cap\overline{g}(-1,1) \\ &\subseteq \bigcap\{t(\tfrac{-1}{y},\tfrac{1}{y})\vee\overline{g}(-y,y) \mid y>\max\{|p|,|q|,1\}\}\cap\overline{g}(-1,1) \\ &\overset{(*)}{=} \bigcap\{t(\tfrac{-1}{y},\tfrac{1}{y}) \mid y>\max\{|p|,|q|,1\}\}\cap\overline{g}(-1,1) \\ &\subseteq \bigcap\{t(\tfrac{-1}{y},\tfrac{1}{y}) \mid y>\max\{|p|,|q|,1\}\}\cap t(0,\text{—})\end{aligned}$$

$$
\begin{aligned}
&= \bigcap\{t(\tfrac{-1}{y},\tfrac{1}{y}) \cap t(0,—) \mid y > \max\{|p|,|q|,1\}\} \\
&= \bigcap\{t(\tfrac{-1}{y},—) \mid y > \max\{|p|,|q|,1\}\} \stackrel{(**)}{=} \mathsf{O}
\end{aligned}
$$

(where $(*)$ follows from $\mathbf{p} < \overline{g} < \mathbf{q}$ and $(**)$ from $t \geq \mathbf{0}$).

Finally, in order to show that $f_{-1}[-](t\cdot\overline{g}) = g\Psi_0$ note first that, for any $(u,v) \in \mathfrak{L}(\mathbb{R})$, if $1 \notin \langle u,v\rangle$, then

$$
f_{-1}[-]t(u,v) \supseteq f_{-1}[-]t\,((1,—)\vee(—,1)) = f_{-1}[(t(1,—)\cap t(—,1))] \supseteq f_{-1}[M\cap D] = f_{-1}[D] = L
$$

otherwise,

$$
\begin{aligned}
\mathsf{O} = f_{-1}[t(1)] &= f_{-1}[t\,((-,1)\vee(1,-)\vee(u,v))] \supseteq f_{-1}[D\cap M\cap t(u,v)] = f_{-1}[D]\cap f_{-1}[t(u,v)] \\
&= L \cap f_{-1}[t(u,v)] = f_{-1}[t(u,v)].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
f_{-1}[-](t\cdot\overline{g})(r,s) &= \bigcap\{f_{-1}[t(u,v)] \vee f_{-1}[\overline{g}(z,w)] \mid \langle u,v\rangle\cdot\langle z,w\rangle \subseteq \langle r,s\rangle\} \\
&= \bigcap\{f_{-1}[t(u,v)] \vee g\Psi_0(z,w) \mid 1 \in \langle u,v\rangle, \langle u,v\rangle\cdot\langle z,w\rangle \subseteq \langle r,s\rangle\} \\
&= \bigcap\{g\Psi_0(z,w) \mid 1 \in \langle u,v\rangle, \langle u,v\rangle\cdot\langle z,w\rangle \subseteq \langle r,s\rangle\} \\
&= \bigcap\{g\Psi_0(z,w) \mid r < z < w < s\} = g\Psi_0(r,s)
\end{aligned}
$$

as required. ☐

Again we can apply Proposition 3.3.4 to a localic embedding $j\colon S \hookrightarrow L$ and obtain:

Corollary 3.3.5. *A sublocale S of L is* C*-embedded if and only if it is* C*-embedded and it is completely separated from every zero sublocale disjoint from it.*

The following theorem gives a criteria for when can we factorize a real-valued function through a localic map.

Theorem 3.3.6. *Let $f\colon L \to M$ be a localic map. The following statements about a $g \in \mathsf{C}^*(L)$ are equivalent:*

(i) *There exists $\overline{g} \in \mathsf{C}^*(M)$ such that $f_{-1}[-]\overline{g} = g$.*

(ii) *$f[g(r,—)]$ and $f[g(—,s)]$ are completely separated in M for every $r < s$ in $\mathbb{Q}$.*

(iii) *For every $r < s$ in $\mathbb{Q}$ there are disjoint $Z_1, Z_2 \in \mathsf{ZS}(M)$ such that $g(r,—) \subseteq f_{-1}[Z_1]$ and $g(—,s) \subseteq f_{-1}[Z_2]$.*

Proof. (ii) $\iff$ (iii): $f[g(r,—)]$ and $f[g(—,s)]$ are completely separated in M if and only if there are disjoint zero sublocales Z_1 and Z_2 such that $f[g(r,—)] \subseteq Z_1$ and $f[g(—,s)] \subseteq Z_2$. From the image/preimage Galois adjunction, $f[g(r,—)] \subseteq Z_1$ and $f[g(—,s)] \subseteq Z_2$ is equivalent to $g(r,—) \subseteq f_{-1}[Z_1]$ and $g(—,s) \subseteq f_{-1}[Z_2]$.

(i) $\implies$ (iii): By assumption there is $\overline{g} \in \mathsf{C}^*(M)$ such that $f_{-1}[-]\overline{g} = g$. Then, for every $r < s$, we have

$$g(r,—) = f_{-1}[\overline{g}(r,-)] \quad \text{and} \quad Z_2 \subseteq g(-,s) = f_{-1}[\overline{g}(-,s)]$$

and $\overline{g}(r,—)$ and $\overline{g}(—,s)$ are disjoint zero sublocales of M.

(iii) $\implies$ (i): Without loss of generality we may assume that $\mathbf{0} \leq g \leq \mathbf{1}$. For each $r \in \mathbb{Q}$ set

$$S_r = \begin{cases} \mathsf{O} & \text{if } r < 0 \\ \bigcap\{Z \in \mathsf{ZS}(M) \mid g(r,—) \subseteq f_{-1}[Z]\} & \text{if } 0 \leq r < 1 \\ M & \text{if } r \geq 1 \end{cases}$$

and

$$T_r = \begin{cases} \mathsf{O} & \text{if } r \leq 0 \\ \bigvee\{C \in \mathsf{CoZS}(M) \mid g(—,r) \subseteq f_{-1}[C]^{\#}\} & \text{if } 0 < r \leq 1 \\ M & \text{if } r > 1. \end{cases}$$

Each S_r is a closed sublocale of M while each T_r is open. For any $r < s$, we have $S_r \subseteq S_s$, since $g(r,—) \subseteq g(s,—)$, and $T_r \subseteq T_s$, since $g(—,s) \subseteq g(—,r)$. Note that

$$T_r^{\#} = \bigcap\{Z \in \mathsf{ZS}(M) \mid g(—,r) \subseteq f_{-1}[Z]\}$$

for any $0 < r \leq 1$.

Further, $\bigcap_{r\in\mathbb{Q}} S_r = \mathsf{O} = \bigcap_{r\in\mathbb{Q}} T_r$. Hence $(S_r)_{r\in\mathbb{Q}}$ and $(T_r)_{r\in\mathbb{Q}}$ are (descending) scales, with corresponding functions $f_1, f_2 \in \mathsf{F}(M)$ defined by (recall Proposition 2.4.2)

$$\begin{array}{ll} f_1\colon \mathfrak{L}(\mathbb{R}) \to S(M)^{op} & f_2\colon \mathfrak{L}(\mathbb{R}) \to S(M)^{op} \\ f_1(r,—) = \bigcap_{p>r} S_p & f_2(r,—) = \bigcap_{p>r} T_p \\ f_1(—,s) = \bigcap_{q<s} S_q^{\#} & f_2(—,s) = \bigcap_{q<s} T_q^{\#}. \end{array}$$

Claim 1: $f_2 \leq f_1$.

We will show this using Proposition 2.6.4, by proving that $S_r \subseteq T_s$ for every $r < s$. If $r < 0$ or $1 < s$ we clearly have $S_r \subseteq T_s$. If $0 \leq r < s \leq 1$, then $g(r,—)$ and $g(—,s)$ are disjoint zero sublocales of L thus, by assumption, there exist disjoint $Z_1, Z_2 \in \mathsf{ZS}(M)$ such that $g(r,—) \subseteq f_{-1}[Z_1]$ and $g(—,s) \subseteq f_{-1}[Z_2]$. Hence $S_r \subseteq Z_1$ and $T_s^{\#} \subseteq Z_2$. Consequently,

$$S_r \subseteq Z_1 \subseteq Z_2^{\#} \subseteq T_s^{\#\#} = T_s.$$

Claim 2: *There exists $h \in \mathsf{C}^*(M)$ such that $f_2 \leq h \leq f_1$.*

By Theorem 2.7.11 it suffices to show that $f_1(r,—)$ and $f_2(—,s)$ are completely separated for any $r < s$. Again, the cases $r < 0$ and $s > 1$ are trivial. If $0 \leq r < s \leq 1$ consider $p, q \in \mathbb{Q}$ such that $0 \leq r < p < q < s \leq 1$. By the assumption, there are disjoint $Z_1, Z_2 \in \mathsf{ZS}(M)$ such that $g(p,-) \subseteq f_{-1}[Z_1]$ and $g(-,q) \subseteq f_{-1}[Z_2]$. Then

$$f_1(r,—) = \bigcap_{p>r} S_p \subseteq S_p \subseteq Z_1$$

and

$$f_2(—,s) = \bigcap_{q<s} T_q^{\#} \subseteq T_q^{\#} \subseteq Z_2$$

Hence $f_1(r,—)$ and $f_2(—,s)$ are completely separated.

Claim 3: *h is a continuous bounded real-valued function that factors g through $f_{-1}[-]$.*
We need to show that $f_{-1}[h(r,—)] = g(r,—)$ for every $r \in \mathbb{Q}$. We have the following three cases:

(Case 1): If $r < 0$ we have $h(r,—) \subseteq f_2(r,—) = \bigcap_{p>r} T_p = \mathsf{O}_M$ and $g(r,—) = \mathsf{O}_L$ (because $\mathbf{0} \leq f$). Hence $f_{-1}[h(r,—)] = \mathsf{O}_L = g(r,—)$.

(Case 2): For $r \geq 1$ we get $M = f_1(r,—) \subseteq h(r,—)$ and $g(r,—) = L$ since $g \leq \mathbf{1}$. Hence $f_{-1}[h(r,—)] = L = g(r,—)$.

(Case 3): When $0 \leq r < 1$ we have that for every $p > r$, $g(r,—) \subseteq g(p,—) \subseteq f_{-1}[S_p]$. Hence $g(r,—) \subseteq \bigcap_{p>r} f_{-1}[S_p] = f_{-1}[f_1(r,—)] \subseteq f_{-1}[h(r,—)]$. On the other hand, since $f_2 \leq h$, then

$$f_{-1}[h(r,—)] \subseteq f_{-1}[f_2(r,—)] = f_{-1}\Big[\bigcap_{p>r} T_p\Big] = \bigcap_{p>r} f_{-1}[T_p].$$

Moreover,

$$\begin{aligned} g(-,p) &\subseteq \bigcap \{f_{-1}[Z] \mid Z \in ZS(M), g(—,p) \subseteq f_{-1}[Z]\} \\ &= f_{-1}\Big[\bigcap \{Z \in ZS(M) \mid g(—,p) \subseteq f_{-1}[Z]\}\Big] \\ &= f_{-1}[T_p^{\#}] = f_{-1}[T_p]^{\#} \end{aligned}$$

which means $f_{-1}[T_p] \subseteq g(-,p)^{\#} \subseteq g(p,-)$ for every $p > r$ (even for $p > 1$ because $g(p,-) = L$). Thus,

$$f_{-1}[h(r,—)] \subseteq \bigcap_{p>r} f_{-1}[T_p] \subseteq \bigcap_{p>r} g(p,—) = g(r,—).$$

In conclusion, h is a continuous real-valued function that factors g through $f_{-1}[-]$. It is bounded because $f_2(—,0) = \bigcap_{q<0} T_q^{\#} = M$ and $f_1(1,—) = \bigcap_{p>1} S_p = M$. Hence,

$$M = f_2(—,0) \cap f_1(1,—) \subseteq h(—,0) \cap h(1,—)$$

meaning $\mathbf{0} \leq h \leq \mathbf{1}$. □

An immediate consequence of this result (applied to a localic embedding) is the point-free counterpart of the Mrówka's Extension Theorem ([63]). A direct proof of the following corollary can be found in [6, Theorem 4.2]. It generalizes [37, Theorem 5.2] where the authors only show it for complemented sublocales.

Corollary 3.3.7. *Let S be a sublocale of L. The following statements about a $g \in \mathsf{C}^*(S)$ are equivalent:*

(i) *There exists a continuous bounded extension of g to L.*

(ii) *The sublocales $g(r,—)$ and $g(—,s)$ are completely separated in L for every $r < s$ in $\mathbb{Q}$.*

We will use Theorem 3.3.6 to prove the following result. Nevertheless, notice that one can give a direct proof mimicking the arguments and using the same function as in Theorem 3.3.6.

Theorem 3.3.8. *Let $f\colon L \to M$ be a localic map. Then the following are equivalent:*

(i) *f is a C^*-map.*

(ii) *For every pair of disjoint zero sublocales $Z_1, Z_2 \in \mathsf{ZS}(L)$ there are disjoint sublocales $Z_1', Z_2' \in \mathsf{ZS}(M)$ such that $Z_1 \subseteq f_{-1}[Z_1']$ and $Z_2 \subseteq f_{-1}[Z_2']$ (equiv. $f[Z_1] \subseteq Z_1'$ and $f[Z_2] \subseteq Z_2'$).*

(iii) *If S and T are completely separated sublocales of L, then $f[S]$ and $f[T]$ are completely separated in M.*

Proof. (i) $\Longrightarrow$ (ii): Let Z_1 and Z_2 be disjoint zero sublocales of L. By Proposition 2.7.3, there is a $g \in \mathsf{C}^*(L)$ such that $g(0,\text{—}) = Z_1$ and $g(\text{—},1) = Z_2$. Since f is a C^*-map there is $\overline{g} \in \mathsf{C}^*(M)$ such that $f_{-1}[-]\overline{g} = g$. Hence,

$$Z_1 = g(0,\text{—}) = f_{-1}[\overline{g}(0,\text{—})] \quad \text{and} \quad Z_2 = g(\text{—},1) = f_{-1}[\overline{g}(\text{—},1)]$$

where $\overline{g}(0,\text{—})$ and $\overline{g}(\text{—},1)$ are disjoint zero sublocales of M.
(ii) $\Longrightarrow$ (i): Let $g \in \mathsf{C}^*(L)$. We have to show that there is $\overline{g} \in \mathsf{C}^*(M)$ such that $f_{-1}[-]\overline{g} = g$. For this, we will use Theorem 3.3.6. Let $r < s$ in $\mathbb{Q}$. Since $g(r,\text{—})$ and $g(\text{—},s)$ are disjoint zero sublocales, by assumption (ii), there are $Z_1, Z_2 \in \mathsf{ZS}(M)$ such that $g(r,\text{—}) \subseteq f_{-1}[Z_1]$ and $g(\text{—},s) \subseteq f_{-1}[Z_2]$. Thus we have shown that Theorem 3.3.6 (iii) holds, so there is an extension $\overline{g} \in \mathsf{C}^*(M)$ of g that factors through f, as required.
(ii) $\Longleftrightarrow$ (iii): It follows trivially from (2) and the image/preimage Galois adjunction. □

Notice that condition (iii) characterizes localic C^*-maps as those that preserve complete separation under taking images.

Finally, Theorem 3.3.8 applied to a localic embedding, yields the point-free version of Urysohn's Extension Theorem [6, Theorem 6.1] (see for example [36, 1.17] or [2, Theorem 6.6] for the result in classical topology). The following corollary, was proved in terms of frame quotients in [9, Theorem 7.11], but the proof there uses more complex arguments than those used in our formulation. One should also point out that the complete separation relation is simpler and easier to work with when using sublocales rather than frame quotients.

Corollary 3.3.9. *Let S be a sublocale of a locale L. Then the following are equivalent:*

(i) *S is C^*-embedded.*

(ii) *If R and T are completely separated sublocales of S, then R and T are also completely separated in L.*

3.4 Relations Among C-, C^*- and z-Maps

We know that any C-map is a C^*-map hence a z-map too (recall Remark 3.3.1 (3)). In general the converse does not hold. In this section we investigate conditions that may give the converse implications. Inspired by the classical topological results in [19, 22] (see also [2]) we obtain, not only their counterparts in point-free topology, but also their generalizations to localic maps.

Proposition 3.4.1. *The following are equivalent for a localic map $f\colon L \to M$:*

(i) *f is a C^*-map.*

(ii) *f is a z-map and for each $T \in \mathsf{S}(L)$ and each $Z \in \mathsf{ZS}(M)$, if $f[T]$ and $f[f_{-1}[Z]]$ are f-separated then they are completely separated in M.*

Proof. (i) $\implies$ (ii): If f is a C^*-map then it is a z-map (Remark 3.3.1 (3)). Moreover, let $T \in \mathsf{S}(L)$ and $Z \in \mathsf{ZS}(M)$ such that $f[T]$ and $f[f_{-1}[Z]]$ are f-separated. That is, there are $Z_1, Z_2 \in \mathsf{ZS}(M)$ such that

$$f[T] \subseteq Z_1, \quad f[f_{-1}[Z]] \subseteq Z_2 \quad \text{and} \quad f_{-1}[Z_1] \cap f_{-1}[Z_2] = \mathsf{O}.$$

By the image/preimage Galois adjunction we have $T \subseteq f_{-1}[Z_1]$ and $f_{-1}[Z] \subseteq f_{-1}[Z_2]$. Since $f_{-1}[Z_1]$ and $f_{-1}[Z_2]$ are disjoint zero sublocales of L, T and $f_{-1}[Z]$ are completely separated in L. By Theorem 3.3.8, $f[T]$ and $f[f_{-1}[Z]]$ are completely separated in M.
(ii) $\implies$ (i): We will use Proposition 3.3.8 (ii) to show that f is a C^*-map. Let Z_1 and Z_2 be disjoint zero sublocales of L. Since f is a z-map, there are $D_1, D_2 \in \mathsf{ZS}(M)$ such that $f_{-1}[D_1] = Z_1$ and $f_{-1}[D_2] = Z_2$. Hence,

$$f[Z_1] = f[f_{-1}[D_1]] \subseteq D_1 \quad f[Z_1] = f[f_{-1}[D_1]] \subseteq D_1 \quad \text{and} \quad f_{-1}[D_1] \cap f_{-1}[D_2] = \mathsf{O}.$$

That is, $f[Z_1]$ and $f[Z_2]$ are f-separated. By assumption, $f[Z_1]$ and $f[Z_2]$ are completely separated in M so there are disjoint sublocales $Z_1', Z_2' \in \mathsf{ZS}(M)$ such that $f[Z_1] \subseteq Z_1'$ and $f[Z_2] \subseteq Z_2'$. □

In order to obtain the next corollary we apply Proposition 3.4.1 to the localic embeding $j\colon S \hookrightarrow L$. This result is another good example of the advantages of sublocale language in terms of conciseness and clarity. Indeed, the result is stated in [31, Proposition 4.3] in terms of frame quotients, and a closer inspection to assertions (2) and (3) there reveals, when formulated in terms of sublocales, that they express precisely the same fact.

Corollary 3.4.2. *The following are equivalent for a sublocale S of L:*

(i) *S is C^*-embedded.*

(ii) *S is z-embedded and for each $T \in \mathsf{S}(S)$ and each $Z \in \mathsf{ZS}(L)$, if T and $Z \cap S$ are S-separated then they are completely separated in L.*

Proposition 3.4.3. *A localic map $f\colon L \to M$ is a C-map if and only if it is an almost z-dense z-map.*

Proof. If f is a C-map it is a z-map, and it is almost z-dense by Proposition 3.3.2. Conversely, assume f is an almost z-dense z-map. To show that it is a C-map it suffices, by Proposition 3.3.4, to check that f is a C^*-map. We will do that using Theorem 3.3.8. Consider a pair of disjoint sublocales $Z_1, Z_2 \in \mathsf{ZS}(L)$. Since f is a z-map, $f[Z_1]$ and $f[Z_2]$ are f-separated (by Proposition 3.2.4), that is, there exist $Z_1', Z_2' \in \mathsf{ZS}(M)$ such that $f[Z_1] \subseteq Z_1'$, $f[Z_2] \subseteq Z_2'$ and $f_{-1}[Z_1'] \cap f_{-1}[Z_2'] = \mathsf{O}$. Then by almost z-density, there exists an $F \in \mathsf{ZS}(M)$ such that $f_{-1}[F] = L$ and $Z_1' \cap Z_2' \cap F = \mathsf{O}$. Thus, Z_1' and $Z_2' \cap F$ are disjoint zero sublocales of M such that $Z_1 \subseteq f_{-1}[Z_1']$ and $Z_2 \subseteq f_{-1}[Z_2' \cap F]$. □

The following result was proven in [9, 7.2.3] in terms of C-, *coz*-onto and almost *coz*-codense quotients. Our formulation via sublocales and localic maps comes closer to that in classical topology.

Corollary 3.4.4. *A sublocale S of L is* C*-embedded if and only if it is z-embedded and it is completely separated from every zero sublocale disjoint from it.*

Remark 3.3.1 (3) and Proposition 3.4.3 yield immediately:

Corollary 3.4.5. *Let f be an almost z-dense localic map. Then the following are equivalent:*

(i) *f is a* C*-map.*

(ii) *f is a* C^**-map.*

(iii) *f is a z-map.*

If Z is a zero sublocale of a locale L the embedding $j\colon Z \hookrightarrow L$ is always an almost z-dense sublocale, because every two disjoint zero sublocales are completely separated (Remarks 2.7.4 (2) and 3.1.4 (3)). Hence, we have:

Corollary 3.4.6. *Let L be a locale. The following are equivalent for any zero sublocale Z of L:*

(i) *Z is* C*-embedded in L.*

(ii) *Z is* C^**-embedded in L.*

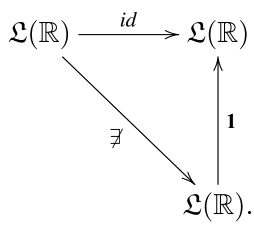
(iii) *Z is z-embedded in L.*

The next result gives a class of sublocales where z-embedded implies C-embedded. First, recall that a sublocale S of L is *G_δ-dense* if $T \neq \mathsf{O}$ implies $T \cap S \neq \mathsf{O}$ for every G_δ-sublocale T of L.

Proposition 3.4.7. *Let S be a G_δ-dense sublocale of L. If S is z-embedded, then it is* C*-embedded.*

Proof. Let S be a z-embedded G_δ-sublocale of L. We use Corollary 3.4.4 to show that S is C-embedded. If $S \cap Z = \mathsf{O}$, then $Z = \mathsf{O}$ because Z is G_δ (Proposition 2.7.8 (1)) and S is G_δ-dense. Then S and Z are clearly completely separated in L. ☐

The following theorem generalizes [9, Theorem 8.3.3] and [31, Proposition 4.11], but when working with general localic maps one looses the equivalent condition of M being a normal frame. Trivilally, the equivalent conditions (i),(ii) and (iii) of Theorem 3.4.8 imply that M is normal, but not the other way around. For example, $\mathfrak{L}(\mathbb{R})$ is a normal frame, the localic map $\mathbf{1}_*\colon \mathfrak{L}(\mathbb{R}) \to \mathfrak{L}(\mathbb{R})$ is closed, but it is clearly not a C- and C^*-map.



Theorem 3.4.8. *The following assertions are equivalent for a locale M:*

(i) *Every closed localic map with codomain M is a z-map.*

(ii) *Every closed localic map with codomain M is a C^*-map.*

(iii) *Every closed localic map with codomain M is a C-map.*

Proof. (iii) $\implies$ (ii) $\implies$ (i) is trivial because

$$C\text{-map} \implies \mathsf{C}^*\text{-map} \implies z\text{-map}.$$

(i) $\implies$ (iii): Consider a closed localic map $f\colon L \to M$. By assumption it is a z-map; we will use Proposition 3.4.3 in order to prove that it is a C-map. It suffices to show that f is almost z-dense so consider $Z \in \mathsf{ZS}(M)$ such that $f_{-1}[Z] = \mathsf{O}$. Since $f[L]$ is a closed sublocale of M we have

$$f[L] \cap Z^{\#} \stackrel{(*)}{=} f[L] \smallsetminus Z \stackrel{(**)}{=} f[L \smallsetminus f_{-1}[Z]] = f[L \smallsetminus \mathsf{O}] = f[L]$$

($(*)$ holds because Z and $f[L]$ are closed, hence complemented; in $(**)$ we use the fact that $f[-]$ is a colocalic map).The equality above shows that $f[L] \subseteq Z^{\#}$, hence $f[L] \cap Z = \mathsf{O}$. Consider the closed sublocale $T = f[L] \vee Z$ of M. The localic embedding $j\colon T \hookrightarrow M$ is closed and, by hypothesis, it is a z-map. Therefore T is z-embedded in M. Note that $T \cap f[L] = (f[L] \vee Z) \cap f[L] = f[L]$ and

$$T \cap Z^{\#} = (f[L] \vee Z) \cap Z^{\#} = f[L] \cap Z^{\#} = f[L].$$

This means that $f[L]$ is both closed and open in T. Consequently, by Proposition 2.7.8 (3), $f[L]$ is both a zero and a cozero sublocale of T. Since T is z-embedded in M, there exists $Z' \in \mathsf{ZS}(M)$ such that $T \cap Z' = f[L]$. Then $f[L] \subseteq Z'$, that is, $L = f_{-1}[Z']$. Moreover,

$$Z' \cap Z = Z' \cap (Z \cap T) = (Z' \cap T) \cap Z = f[L] \cap Z = \mathsf{O},$$

which shows that f is almost z-dense. □

Corollary 3.4.9. *The following assertions are equivalent for a locale L:*

(i) *Every closed sublocale of L is z-embedded in L.*

(ii) *Every closed sublocale of L is C^*-embedded in L.*

(iii) *Every closed sublocale of L is C-embedded in L.*

3.5 Classes of Localic Maps Defined by the Behaviour of Their Images on Zero Sublocales

So far we have discussed classes of localic maps defined by conditions on the behavior of their preimages on zero and cozero sublocales. In this section, inspired by [79], we introduce similar classes of localic maps defined by conditions on the behavior of their images on zero and cozero sublocales.

Definition 3.5.1. Let $f\colon L\to M$ be a localic map. We say that f is

(a) *z-closed* if $f[Z]$ is a closed sublocale of M for every $Z\in \mathsf{ZS}(L)$;

(b) *coz-open* if $f[C]$ is an open sublocale of M for every $C\in \mathsf{CoZS}(L)$;

(c) *z-open* if $\overline{f[Z]}\subseteq f[C]^{\circ}$ for every $Z\in \mathsf{ZS}(L)$ and every $C\in \mathsf{CoZS}(L)$ such that $Z\subseteq C$;

(d) *z-preserving* (resp. *coz-preserving*) if the image of every zero (resp. cozero) sublocale is a zero (resp. cozero) sublocale.

Remarks 3.5.2. (1) The σ-coframe homomorphism $f^{z}_{-1}[-]\colon \mathsf{ZS}(M)\to \mathsf{ZS}(L)$ has a left adjoint if and only if f is z-preserving, and the σ-frame homomorphism $f^{coz}_{-1}[-]\colon \mathsf{CoZS}(M)\to \mathsf{CoZS}(L)$ has a right adjoint if and only if f is coz-preserving.

(2) Clearly, any open or any coz-preserving map is coz-open. Similarly, any closed or z-preserving localic map is z-closed.

(3) If f is z-closed and coz-open, then it is z-open.

(4) Recall (1.4.12). We have

$$Z\subseteq C \iff Z\cap C^{\#}=\mathsf{O} \quad\text{and}\quad \overline{f[Z]}\subseteq f[C]^{\circ} \iff \overline{f[Z]}\cap (f[C]^{\circ})^{\#}=\overline{f[Z]}\cap \overline{f[C]^{\#}}=\mathsf{O}$$

for any $Z\in \mathsf{ZS}(L)$ and $C\in \mathsf{CoZS}(L)$. Therefore, f is z-open if and only if for any disjoint $Z_1,Z_2\in \mathsf{ZS}(L)$, the sublocales $\overline{f[Z_1]}$ and $\overline{f[Z_2^{\#}]^{\#}}$ are also disjoint.

The following result was proved in [44, 6.3.2].

Proposition 3.5.3. *Let $f\colon L\to M$ be a localic map. If L is completely regular and f is coz-open, then f is open.*

We can prove a similar result for z-open maps:

Proposition 3.5.4. *Let $f\colon L\to M$ be a localic map. If L is completely regular and f is z-open, then f is open.*

Proof. Let $\mathfrak{o}(a)$ be an open sublocale of L. By complete regularity, $\mathfrak{o}(a)=\bigvee\{\mathfrak{o}(b)\mid b\prec\!\prec a\}$. Moreover, by [44, 5.4.2], the sublocales $\mathfrak{c}(a)$ and $\mathfrak{o}(b)$ are completely separated, that is, there exist $Z_b\in \mathsf{ZS}(L)$ and $C_b\in \mathsf{CoZS}(L)$ such that $\mathfrak{o}(b)\subseteq Z_b\subseteq C_b\subseteq \mathfrak{o}(a)$. Hence, by the z-openness of f,

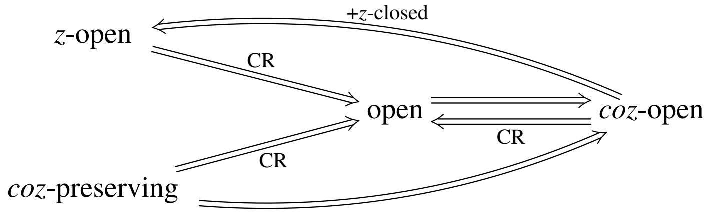
$$f[\mathfrak{o}(b)]\subseteq f[Z_b]\subseteq \overline{f[Z_b]}\subseteq f[C_b]^{\circ}\subseteq f[C_b]\subseteq f[\mathfrak{o}(a)].$$

Finally, taking joins we obtain

$$f[\mathfrak{o}(a)]=f\left[\bigvee\{\mathfrak{o}(b)\mid b\prec\!\prec a\}\right]=\bigvee\{f[\mathfrak{o}(b)]\mid b\prec\!\prec a\}\subseteq\bigvee\{f[C_b]^{\circ}\mid b\prec\!\prec a\}\subseteq f[\mathfrak{o}(a)],$$

which shows that $f[\mathfrak{o}(a)]$ is a join of open sublocales of M, hence open. □

Summing up, we have the following diagram depicting the relations among the mentioned classes of maps



(3.5.1)

(where CR indicates that we need to assume complete regularity in the domain of the localic map).

Proposition 3.5.5. *Let $f\colon L \to M$ be a localic map. If for any completely separated sublocales S and T of L, $f[S]$ and $f[T^\#]^\#$ are completely separated in M, then f is z-open.*

Proof. Let Z_1 and Z_2 be disjoint zero sublocales of L. By assumption, $f[Z_1]$ and $f[Z_2^\#]^\#$ are completely separated in M. In particular, $\overline{f[Z_1]}$ and $\overline{f[Z_2^\#]^\#}$ are disjoint sublocales hence f is z-open by Remark 3.5.2 (4). □

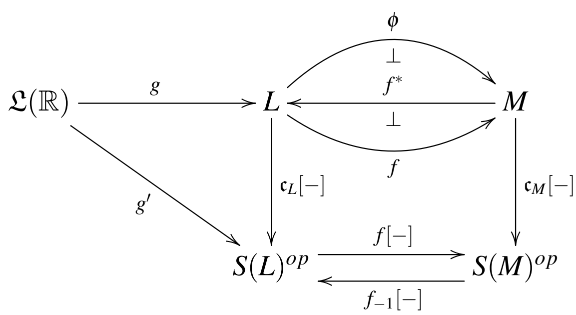
The converse holds under complete regularity:

Theorem 3.5.6. *The following are equivalent for a localic map $f\colon L \to M$ with completely regular domain:*

(i) *f is z-open.*

(ii) *If S and T are completely separated sublocales of L, then $f[S]$ and $f[T^\#]^\#$ are completely separated in M.*

Proof. Let f be a z-open map. If S and T are completely separated sublocales of L, they are contained in disjoint zero sublocales Z_1 and Z_2. Clearly, in order to show that $f[S]$ and $f[T^\#]^\#$ are completely separated, it suffices to show that so are $f[Z_1]$ and $f[Z_2^\#]^\#$.
By Proposition 2.7.3, there exists a continuous $g'\colon \mathfrak{L}(\mathbb{R}) \to S(L)^{op}$, with $\mathbf{0} \le g' \le \mathbf{1}$ such that $Z_1 = g'(0,—)$ and $Z_2 = g'(—,1)$. From Proposition 3.5.4, we know that f is open. Hence, f^* has a left adjoint ϕ such that $f[\mathfrak{o}(a)] = \mathfrak{o}(\phi(a))$ for any $a \in L$. We have the following diagram



where the left triangle and the square $f_{-1}[-]\mathfrak{c}_M[-] = \mathfrak{c}_L[-]f^*$ commute.
For each $r \in \mathbb{Q}$ let

$$C_r = g'(-,r)^\# = \mathfrak{o}_L(g(-,r)) \quad \text{and} \quad F_r = g'(r,-) = \mathfrak{c}_L(g(r,—)).$$

Clearly, $C_r \in \mathsf{CoZS}(L)$ and $F_r \in \mathsf{ZS}(L)$. If $r < s$ then $g'(r,—) \vee g'(—,r) = L$ and $g'(r,—) \cap g'(—,s) = \mathsf{O}$. Hence, by (1.4.2), $C_r \subseteq F_r \subseteq C_s$, and, since f is z-open,

$$\overline{f[C_r]} \subseteq \overline{f[F_r]} \subseteq f[C_s]^\circ = f[C_s] \tag{3.5.2}$$

for every $r < s$. Consider further the following family of open sublocales of M ($r \in \mathbb{Q}$):

$$U_r = \begin{cases} \mathsf{O} & \text{if } r < 0 \\ f[C_r] & \text{if } 0 \leq r \leq 1 \\ M & \text{if } r > 1. \end{cases}$$

Since every U_r is complemented and $U_r \subseteq U_s$ for every $r < s$, it is easy to see that $(U_r)_{r\in\mathbb{Q}}$ is a descending scale in $S(M)^{op}$. By Proposition 2.6.5, it generates a frame homomorphism $h\colon \mathfrak{L}(\mathbb{R}) \to S(M)^{op}$ given by

$$h(p,-) = \bigcap_{r>p} f[C_r] \quad \text{and} \quad h(-,q) = \bigcap_{s<q} f[C_s]^{\#}.$$

Clearly, $h(—,q)$ is closed for every $q \in \mathbb{Q}$ and, by (3.5.2)

$$h(p,—) = \bigcap_{r'>p} f[C_{r'}] \subseteq \bigcap_{r'>p} \overline{f[C_{r'}]} \subseteq \bigcap_{r>p} f[C_r] = h(p,—).$$

So $h(p,—)$ is closed for every $p \in \mathbb{Q}$ and h is continuous. Note that h is also bounded, since $h(—,0) \cap h(1,—) = M$. We claim that h completely separates $f[Z_1]$ and $f[Z_2^{\#}]^{\#}$ in M. Indeed:

$$\begin{aligned} h(—,1) &= \bigcap_{s<1} f[C_s]^{\#} = \bigcap_{s<1} f[\mathfrak{o}_L(g(—,s))]^{\#} = \bigcap_{s<1} \mathfrak{o}_M(\phi(g(—,s)))^{\#} \\ &= \bigcap_{s<1} \mathfrak{c}_M(\phi(g(—,s))) = \mathfrak{c}_M\Big(\phi\Big(g\Big(\bigvee_{s<1}(—,s)\Big)\Big)\Big) = \mathfrak{c}_M(\phi(g(—,1))) \\ &= \mathfrak{o}_M(\phi(g(—,1)))^{\#} = f[\mathfrak{o}_L(g(—,1))]^{\#} = f[g'(—,1)^{\#}]^{\#} = f[Z_2^{\#}]^{\#}. \end{aligned}$$

Notice that we are using the fact that ϕ and g, being left adjoints, preserve arbitrary joins. Moreover,

$$\begin{aligned} h(0,—) &= \bigcap_{0<r} f[C_r] \overset{(*)}{\supseteq} \bigcap_{0<s} \overline{f[C_s]} = \bigcap_{0<s} \overline{\mathfrak{o}_M(\phi(g(—,s)))} = \bigcap_{0<s} \mathfrak{c}_M(\phi(g(—,s))^*) \\ &\overset{(**)}{=} \bigcap_{0<s} \mathfrak{c}_M(f(g(—,s)^*)) = \mathfrak{c}_M\Big(f\Big(\bigvee_{0<s} g(—,s)^*\Big)\Big) \overset{(***)}{\supseteq} \mathfrak{c}_M(f(g(0,—))) \\ &\supseteq f[\mathfrak{c}_L(g(0,—))] = f[g'(0,—)] = f[Z_1] \end{aligned}$$

where $(*)$ follows from (3.5.2), $(**)$ from (1.5.4), and $(***)$ holds since $g(—,s)^* \leq g(0,—)$ (because $g(0,—) \vee g(—,s) = 1$) for every $s > 0$. □

We get a similar result by replacing the condition on the domain with normality on the codomain. For proving it, we need to recall that in normal locales, disjoint closed sublocales are always completely separated (Proposition 2.7.13).

Proposition 3.5.7. *The following are equivalent for a localic map $f\colon L\to M$ with normal M:*

(i) *f is z-open.*

(ii) *If S and T are completely separated sublocales of L, then $f[S]$ and $f[T^\#]^\#$ are completely separated in M.*

Proof. Assume that f is z-open. As in Theorem 3.5.6, it suffices to show (ii) for Z_1 and Z_2 disjoint zero sublocales of L. Since $Z_1\subseteq Z_2^\#$, we have, by hypothesis, $\overline{f[Z_1]}\subseteq f[Z_2^\#]^\circ$. Hence $\overline{f[Z_1]}$ and $(f[Z_2^\#]^\circ)^\# = \overline{f[Z_2^\#]^\#}$ are disjoint closed sublocales and since M is normal, they are completely separated. In particular, $f[Z_1]$ and $f[Z_2^\#]^\#$ are completely separated. □

Proposition 3.5.8. *Let $f\colon L\to M$ be a localic map between subfit locales, with L normal. Then f is z-open if and only if it is open and closed.*

Proof. By (3.5.1), if f is open and closed then it is z-open. Conversely, let f be z-open. Since L is completely regular, we know by Proposition 3.5.4 that f is open. To prove that it is also closed, let $\mathfrak{c}_L(a)\subseteq L$ and let $\mathfrak{c}_M(b)=\overline{f[\mathfrak{c}_L(a)]}$. It suffices to show that $\mathfrak{c}_M(b)\subseteq f[\mathfrak{c}_L(a)]$. We will proceed by contradiction.

If $\mathfrak{c}_M(b)\nsubseteq f[\mathfrak{c}_L(a)]$, then, by (1.4.2), $f[\mathfrak{c}_L(a)]\vee\mathfrak{o}_M(b)\neq M$, and since M is subfit there would exist some $\mathfrak{c}_M(d)\neq \mathsf{O}_M$ in M such that $(f[\mathfrak{c}_L(a)]\vee\mathfrak{o}_M(b))\cap\mathfrak{c}_M(d)=\mathsf{O}_M$ (see Proposition 1.4.2). Then, $f[\mathfrak{c}_L(a)]\subseteq f[\mathfrak{c}_L(a)]\vee\mathfrak{o}_M(b)\subseteq\mathfrak{o}_M(d)$ and, consequently, $\mathfrak{c}_L(a)\subseteq f_{-1}[\mathfrak{o}_M(d)]=\mathfrak{o}_L(f^*(d))$. This would mean that $\mathfrak{c}_L(a)$ and $\mathfrak{c}_L(f^*(d))$ are disjoint closed sublocales, hence completely separated (by the normality of L). It then would follow, by Theorem 3.5.6, the existence of $Z_1,Z_2\in\mathsf{ZS}(M)$ such that

$$f[\mathfrak{c}_L(a)]\subseteq Z_1,\quad f[\mathfrak{o}_L(f^*(d))]^\#\subseteq Z_2,\quad\text{and}\quad Z_1\cap Z_2=\mathsf{O}_M. \tag{3.5.3}$$

Indeed, $\mathfrak{c}_M(d)\subseteq f[\mathfrak{o}_L(f^*(d))]^\#\subseteq Z_2$ since

$$\mathfrak{c}_M(d)\cap f[\mathfrak{o}_L(f^*(d))]=\mathfrak{c}_M(d)\cap\mathfrak{o}_M(\phi(f^*(d)))\subseteq\mathfrak{c}_M(d)\cap\mathfrak{o}_M(d)=\mathsf{O}_M$$

(where ϕ denotes the left adjoint of f^* provided by the openness of f). Moreover, by (3.5.3),

$$f[\mathfrak{c}_L(a)]\subseteq\mathfrak{c}_M(b)\subseteq Z_1\subseteq Z_2^\#\subseteq\mathfrak{o}_M(d)$$

Then we would get $M=\mathfrak{o}_M(b)\vee\mathfrak{c}_M(b)\subseteq\mathfrak{o}_M(d)$, contradicting the fact that $\mathfrak{c}_M(d)$ is nonempty. □

Chapter 4

Forms of Normality and z-Embeddings

Asking for certain classes of sublocales of a locale to be z-embedded gives rise to characterizations of certain types of locales. For example, a locale L is normal if every closed sublocale is z-embedded. In Section 4.1 we will add other characterizing conditions of normality in terms of z-embeddings. Next, in Section 4.2 we analyze for which classes of closed sublocales would a characterization of this type still hold, and what conditions should be imposed. In particular, we obtain a general result that implies characterizations for normal and mildly normal locales. Finally, we make a recap of all the frames that can be characterized via classes of z-embedded sublocales (Section 4.3).

4.1 Normality via z-Embeddings

There are several well-known characterizations of normal locales in terms of complete separation, C-, C^*- and z-embedded sublocales. These results have been studied as a consequence of the insertion theorem for normal frames (Theorem 2.7.12 and Propositions 2.7.13 and 2.7.14), or in the context of frame quotients ([9, Theorem 8.3.3] and [31, Proposition 4.11]). For the sake of completeness, we gather all of these characterizing conditions of normality into Theorem 4.1.1 and present a proof which uses only the tools we have discussed so far. Later in this section, we add new equivalent conditions inspired by those that appear in classical topology (see [2, Theorem 7.15]). This section is based on the author's published article with Jorge Picado [6].

Theorem 4.1.1. *The following statements are equivalent for a locale L*

(i) *L is normal.*

(ii) *Every pair of disjoint closed sublocales is completely separated in L.*

(iii) *Every closed sublocale of L is C-embedded.*

(iv) *Every closed sublocale of L is C^*-embedded.*

(v) *Every closed sublocale of L is z-embedded.*

Proof. (iii) $\iff$ (iv) $\iff$ (v) is precisely Corollary 3.4.9.
(i) $\implies$ (ii): Let $\mathfrak{c}(a)$ and $\mathfrak{c}(b)$ be disjoint sublocales of L. Then $a \vee b = 1$. By assumption, there are

$u,v\in L$ such that $a\vee u=1=b\vee v$ and $u\wedge v=0$. Hence, $u\leq v^*$ and $v\leq u^*$, meaning $a\vee v^*=1=b\vee u^*$. Thus, $v\prec a$ and $u\prec b$. In fact, since L is normal, we have $v \prec\!\!\prec a$ and $u\prec\!\!\prec b$ (Proposition 1.3.3). Recall that from Remark 2.3.7 (2) there is $c\in \mathsf{Coz}\, L$ such that $v\prec\!\!\prec c\prec\!\!\prec a$. By Proposition 2.7.5, we have that $\mathfrak{o}(c)$ is completely separated from $\mathfrak{c}(a)$. Now, $\mathfrak{o}(v)\subseteq \mathfrak{o}(c)$ and $\mathfrak{c}(b)\subseteq \mathfrak{o}(v)$ because $b\vee v=1$. Thus, $\mathfrak{c}(b)$ is completely separated from $\mathfrak{c}(a)$ in L, as required.
(ii) $\Longrightarrow$ (i): Let $a\vee b=1$. Then $\mathfrak{c}(a)$ and $\mathfrak{c}(b)$ are disjoint sublocales of L. By assumption, there are $Z_1,Z_2\in \mathsf{ZS}(L)$ such that $Z_1\cap Z_2=\mathsf{O}$, $\mathfrak{c}(a)\subseteq Z_1$ and $\mathfrak{c}(b)\subseteq Z_2$. Since $\mathsf{Coz}\, L$ is a normal σ-frame, there are $u,v\in \mathsf{Coz}\, L$ such that $u\wedge v=0$, $Z_1\subseteq \mathfrak{o}(u)$ and $Z_2\subseteq \mathfrak{o}(v)$. Consequently, $a\vee u=1=b\vee v$ as required.
(ii) $\Longrightarrow$ (iv): Let $\mathfrak{c}_L(a)$ be a closed sublocale of L. To show that it is C^*-embedded we will use Corollary 3.3.9. Let S and T be sublocales of $\mathfrak{c}_L(a)$ completely separated in $\mathfrak{c}_L(a)$. Then there are $Z_1,Z_2\in \mathsf{ZS}(\mathfrak{c}_L(a))$ such that $Z_1\cap Z_2=\mathsf{O}$, $S\subseteq Z_1$ and $T\subseteq Z_2$. Since Z_1 and Z_2 are closed sublocales of a closed sublocale $\mathfrak{c}_L(a)$, then Z_1 and Z_2 are disjoint closed sublocales of L. By assumption there are $D_1,D_2\in \mathsf{ZS}(L)$ such that $D_1\cap D_2=\mathsf{O}$, $Z_1\subseteq D_1$ and $Z_2\subseteq D_2$. In particular, $S\subseteq D_1$ and $T\subseteq D_2$, which prove that S and T are completely separated in L. With a similar argument one could prove the implication (ii) $\Longrightarrow$ (v).
(iv) $\Longrightarrow$ (ii): Let $a,b\in L$ such that $\mathfrak{c}_L(a)\cap \mathfrak{c}_L(b)=\mathsf{O}$. Consider the closed sublocale $M:=\mathfrak{c}_L(a)\vee \mathfrak{c}_L(b)=\mathfrak{c}_L(a\wedge b)$. Note that $\mathfrak{c}_L(a)$ and $\mathfrak{c}_L(b)$ are zero sublocales of $\mathfrak{c}_L(a\wedge b)$. Indeed, since $\mathfrak{c}_L(a)\cap \mathfrak{c}_L(b)=\mathsf{O}$,

$$\mathfrak{c}_L(a)=\mathfrak{o}_L(b)\cap\mathfrak{c}_L(a)=\mathfrak{o}_L(b)\cap(\mathfrak{c}_L(a)\vee\mathfrak{c}_L(b))=\mathfrak{o}_L(b)\cap M, \quad \text{and}$$

$$\mathfrak{c}_L(b)=\mathfrak{o}_L(a)\cap\mathfrak{c}_L(b)=\mathfrak{o}_L(a)\cap(\mathfrak{c}_L(b)\vee\mathfrak{c}_L(a))=\mathfrak{o}_L(a)\cap M.$$

Consequently, $\mathfrak{c}_L(a)$ and $\mathfrak{c}_L(b)$ are clopen in $\mathfrak{c}_L(a\wedge b)$. By Proposition 2.7.8 (3), they are disjoint zero sublocales of $\mathfrak{c}_L(a\wedge b)$. By assumption, $\mathfrak{c}_L(a\wedge b)$ is C^*-embedded so from Corollary 3.3.9 we can conclude that $\mathfrak{c}_L(a)$ and $\mathfrak{c}_L(b)$ are completely separated in L, as required. □

Other direct proofs of some of the equivalences in the theorem above can be found in the literature. For example, (i) $\Longleftrightarrow$ (ii) is a corollary of Urysohn's Separation Lemma ([66, XIV.7.5.1]), and (i) $\Longleftrightarrow$ (iv) is a consequence of Tietze's Extension Theorem ([66, XIV.7.6.1]). Instead of the proof given above, one could show the equivalence between conditions (i) and (ii) and (v) avoiding conditions (iii) and (iv). This work will be presented in detail in Section 4.2.

Now, we would like to enlarge the list of characterizations of normal frames in Theorem 4.1.1, by adding conditions in terms of F_σ-sublocales and F_σ-generalized sublocales. We say that a sublocale $S\in\mathsf{S}(L)$ is an *F_σ-generalized sublocale* if whenever $S\subseteq \mathfrak{o}_L(a)$ for some $a\in L$, there is an F_σ-sublocale F such that $S\subseteq F\subseteq \mathfrak{o}_L(a)$. Clearly, every closed sublocale is F_σ, and every F_σ-sublocale is F_σ-generalized.

Lemma 4.1.2. *In a normal locale, every F_σ-sublocale is z-embedded.*

Proof. Let S be an F_σ-sublocale of L, say $S=\bigvee_{n=1}^{\infty}\mathfrak{c}_L(a_n)$. Consider a zero sublocale $Z=g(0,\text{—})$ in S for some $g\in\mathsf{C}^*(S)$ with $\mathbf{0}\leq g\leq \mathbf{1}$. Let $\mathfrak{c}_L(b)$ be the closure of Z in L. Note that, since $\mathfrak{c}_L(b)$ is the closure in L of $g(0,—)$, $\mathfrak{c}_L(b)\cap S = g(0,—)$. Furthermore, consider for each $n = 1,2,\dots$

$$T_n = \mathfrak{c}_L(a_n)\vee\mathfrak{c}_L(b) = \mathfrak{c}_L(a_n\wedge b)$$

and $g_n\colon \mathfrak{L}(\mathbb{R})\to \mathsf{S}(T_n)^{op}$ defined by

$$g_n(p,—) = \begin{cases} \mathsf{O} & \text{if } p<0\\ \mathfrak{c}_L(b)\vee(g(p,—)\cap\mathfrak{c}_L(a_n)) & \text{if } 0\le p<1\\ \mathfrak{c}_L(b)\vee\mathfrak{c}_L(a_n) & \text{if } p\ge 1\end{cases}$$

and

$$g_n(—,q) = \begin{cases} \mathfrak{c}_L(b)\vee\mathfrak{c}_L(a_n) & \text{if } q\le 0\\ g(—,q)\cap\mathfrak{c}_L(a_n) & \text{if } 0<q\le 1\\ \mathsf{O} & \text{if } q>1.\end{cases}$$

Let us confirm that this defines a frame homomorphism, by checking that it turns relations (r1)–(r6) into identities in the frame $\mathsf{S}(T_n)^{op}$:

(r1): $g_n(p,—)\vee g_n(—,q) = \mathfrak{c}_L(b)\vee\mathfrak{c}_L(a_n)$ whenever $p\ge q$.
The only nontrivial case is when $0\le p<1$ and $0<q\le 1$ where we have

$$\begin{aligned} g_n(p,—)\vee g_n(—,q) &= (\mathfrak{c}_L(b)\vee(g(p,—)\cap\mathfrak{c}_L(a_n)))\vee(g(—,q)\cap\mathfrak{c}_L(a_n))\\ &= \mathfrak{c}_L(b)\vee(\mathfrak{c}_L(a_n)\cap(g(p,—)\vee g(—,q)) = \mathfrak{c}_L(b)\vee(\mathfrak{c}_L(a_n)\cap S)\\ &= \mathfrak{c}_L(b)\vee\mathfrak{c}_L(a_n).\end{aligned}$$

(r2): $g_n(p,—)\cap g_n(—,q) = \mathsf{O}$ whenever $p<q$.
The only nontrivial case is when $0\le p<1$ and $0<q\le 1$ and we have

$$\begin{aligned} g_n(p,—)\cap g_n(—,q) &= (\mathfrak{c}_L(b)\vee(g(p,—)\cap\mathfrak{c}_L(a_n)))\cap(g(—,q)\cap\mathfrak{c}_L(a_n))\\ &= (\mathfrak{c}_L(b)\cap g(—,q)\cap\mathfrak{c}_L(a_n))\vee(g(p,—)\cap\mathfrak{c}_L(a_n)\cap g(—,q))\\ &= (\mathfrak{c}_L(b)\cap g(—,q)\cap\mathfrak{c}_L(a_n)\cap S)\vee\mathsf{O}\\ &= g(0,—)\cap g(—,q)\cap\mathfrak{c}_L(a_n) = \mathsf{O}.\end{aligned}$$

(r3): $\bigcap_{r>p} g_n(r,—) = g_n(p,—)$.
For the only nontrivial case, when $0\le p<1$, we have

$$\begin{aligned} \bigcap_{r>p} g_n(r,—) &= \mathfrak{c}_L(b)\vee\bigcap_{1>r>p}(g(r,—)\cap\mathfrak{c}_L(a_n))\\ &= \mathfrak{c}(b)\vee(g(p,—)\cap\mathfrak{c}_L(a_n)) = g_n(p,—).\end{aligned}$$

Note that the second equality holds since g is a frame homomorphism and $g\le\mathbf{1}$.

(r4): $\bigcap_{s<q} g_n(-,s) = g_n(-,q)$.
For the only nontrivial case $0 < q \leq 1$ we have

$$\bigcap_{s<q} g_n(-,s) = \bigcap_{0<s<q} (g(-,s) \cap \mathfrak{c}_L(a_n)) = g(-,q) \cap \mathfrak{c}_L(a_n) = g_n(-,q)$$

(the second equality holds since g is a frame homomorphism and $\mathbf{0} \leq g$).

(r5): $\bigcap_{p\in\mathbb{Q}} g_n(p,-) = \mathsf{O}$ is clear.

(r6): $\bigcap_{q\in\mathbb{Q}} g_n(-,q) = \mathsf{O}$ is also obvious.

In order to see that g_n is continuous for every n it suffices to check that $g(p,-) \cap \mathfrak{c}_L(a_n)$ and $g(-,q) \cap \mathfrak{c}_L(a_n)$ are closed sublocales in T_n for every $0 \leq p < 1$ and $0 < q \leq 1$. Regarding the former, since g is continuous, $g(p,-)$ is closed in S and thus there is a $d \in L$ such that $\mathfrak{c}_L(d) \cap S = g(p,-)$. Hence

$$\begin{aligned}(\mathfrak{c}_L(d) \cap \mathfrak{c}_L(a_n)) \cap (\mathfrak{c}_L(a_n) \vee \mathfrak{c}_L(b)) &= (\mathfrak{c}_L(d) \cap \mathfrak{c}_L(a_n)) \vee (\mathfrak{c}_L(d) \cap \mathfrak{c}_L(a_n) \cap \mathfrak{c}_L(b)) \\ &= (g(p,-) \cap \mathfrak{c}_L(a_n)) \vee (g(p,-) \cap \mathfrak{c}_L(a_n) \cap g(0,-)) \\ &= (g(p,-) \cap \mathfrak{c}_L(a_n)) \vee (\mathfrak{c}_L(a_n) \cap g(0,-)) \\ &= \mathfrak{c}_L(a_n) \cap (g(p,-) \vee g(0,-)) = g(p,-) \cap \mathfrak{c}_L(a_n).\end{aligned}$$

Similarly, if $\mathfrak{c}_L(d) \cap S = g(-,q)$ we have

$$\begin{aligned}(\mathfrak{c}_L(d) \cap \mathfrak{c}_L(a_n)) \cap (\mathfrak{c}_L(a_n) \vee \mathfrak{c}_L(b)) &= (\mathfrak{c}_L(d) \cap \mathfrak{c}_L(a_n)) \vee (\mathfrak{c}_L(d) \cap \mathfrak{c}_L(a_n) \cap \mathfrak{c}_L(b)) \\ &= (g(-,q) \cap \mathfrak{c}_L(a_n)) \vee (g(-,q) \cap \mathfrak{c}_L(a_n) \cap g(0,-)) \\ &= (g(-,q) \cap \mathfrak{c}_L(a_n)) \vee \mathsf{O} = g(-,q) \cap \mathfrak{c}_L(a_n).\end{aligned}$$

By Theorem 4.1.1 we know that T_n is C-embedded in L. Consequently, there are $f_n \in \mathsf{C}(L)$ $(n = 1,2,\ldots)$ such that $(j_n)_{-1}[-] f_n = g_n$ where j_n is the localic embedding of T_n in L. Take

$$F = \bigcap_{n=1}^{\infty} (f_n(0,-) \cap f_n(-,0)) \in \mathsf{ZS}(L).$$

We claim that $F \cap S = Z$. First note that

$$\begin{aligned}g_n(0,-) \cap g_n(-,0) &= (\mathfrak{c}_L(b) \vee (g(0,-) \cap \mathfrak{c}_L(a_n))) \cap (\mathfrak{c}_L(b) \vee \mathfrak{c}_L(a_n)) \\ &= \mathfrak{c}_L(b) \vee ((\mathfrak{c}_L(b) \vee (g(0,-) \cap \mathfrak{c}_L(a_n))) \cap \mathfrak{c}_L(a_n)) \\ &= \mathfrak{c}_L(b) \vee (\mathfrak{c}_L(b) \cap \mathfrak{c}_L(a_n)) \vee (\mathfrak{c}_L(a_n) \cap g(0,-)) \\ &= \mathfrak{c}_L(b) \vee (\mathfrak{c}_L(a_n) \wedge g(0,-)) = (\mathfrak{c}_L(b) \vee \mathfrak{c}_L(a_n)) \cap (\mathfrak{c}_L(b) \vee g(0,-)) \\ &= (\mathfrak{c}_L(b) \vee \mathfrak{c}_L(a_n)) \cap \mathfrak{c}_L(b) = \mathfrak{c}_L(b).\end{aligned}$$

Hence

$$g(0,-) \subseteq \mathfrak{c}_L(b) = g_n(0,-) \cap g_n(-,0) \subseteq f_n(0,-) \cap f_n(-,0)$$

for every n. Therefore, $g(0,-) \subseteq F \cap S$. For the converse inclusion we have

$$\begin{aligned} F \cap S &= \bigvee_{n\in\mathbb{N}} F \cap \mathfrak{c}_L(a_n) \subseteq \bigvee_{n\in\mathbb{N}} f_n(0,-) \cap f_n(-,0) \cap \mathfrak{c}_L(a_n) \\ &= \bigvee_{n\in\mathbb{N}} f_n(0,-) \cap f_n(-,0) \cap \mathfrak{c}_L(a_n) \cap T_n = \bigvee_{n\in\mathbb{N}} g_n(0,-) \cap g_n(-,0) \cap \mathfrak{c}_L(a_n) \\ &= \bigvee_{n\in\mathbb{N}} \mathfrak{c}_L(b) \cap \mathfrak{c}_L(a_n) = \mathfrak{c}_L(b) \cap S = g(0,-) \end{aligned}$$

where the first and the last equalities hold by (1.4.4), since F and $\mathfrak{c}_L(b)$ are closed sublocales (hence, complemented). We have shown that an arbitrary zero sublocale of S is the intersection of S with a zero sublocale in L. In conclusion, S is z-embedded in L. □

Lemma 4.1.3. *If S is a sublocale of L with the property that whenever $S \subseteq \mathfrak{o}(a)$ there is a normal (resp. normal and z-embedded) sublocale F such that $S \subseteq F \subseteq \mathfrak{o}(a)$, then S is normal (resp. normal and z-embedded).*

Proof. First we show the statement that does not involve z-embedded sublocales. To prove that S is normal it suffices to show, by Theorem 4.1.1, that every closed sublocale of S is z-embedded. So let $F := \mathfrak{c}_S(a)$ be a closed sublocale of S, and $A := \mathfrak{c}_F(b) \in \mathsf{ZS}(F)$. Since every zero sublocale is G_δ (recall Proposition 2.7.8(1)), $\mathfrak{c}_F(b) = \bigcap_{n\in\mathbb{N}} \mathfrak{o}_F(b_n)$ with $b_n \in F$ for $n \in \mathbb{N}$. We have that

$$S \cap \mathfrak{c}_L(a \vee b \vee b_n) = F \cap \mathfrak{c}_L(b) \cap \mathfrak{c}_L(b_n) = A \cap \mathfrak{c}_F(b_n) = \mathsf{O}$$

where the last equality holds because $A \subseteq \mathfrak{o}_F(b_n)$. Thus, $S \subseteq \mathfrak{o}_L(a \vee b \vee b_n)$. By assumption, for every $n \in \mathbb{N}$, there is a normal sublocale T_n of L such that $S \subseteq T_n \subseteq \mathfrak{o}_L(b \vee b_n \vee a)$. Furthermore, $\mathfrak{c}_L(b) \cap T_n$ and $\mathfrak{c}_L(a \vee b_n) \cap T_n$ are disjoint closed sublocales of T_n; indeed,

$$\mathfrak{c}_L(b) \cap T_n \cap \mathfrak{c}_L(a \vee b_n) \subseteq \mathfrak{c}_L(b \vee a \vee b_n) \cap \mathfrak{o}_L(b \vee a \vee b_n) = \mathsf{O}.$$

Since T_n is normal (Theorem 4.1.1 (ii)), $\mathfrak{c}_L(b) \cap T_n$ and $\mathfrak{c}_L(a \vee b_n) \cap T_n$ are completely separated in T_n, so for every $n \in \mathbb{N}$ there is $Z_n \in \mathsf{ZS}(T_n)$ such that

$$\mathfrak{c}_L(b) \cap T_n \subseteq Z_n \quad \text{and} \quad T_n \cap \mathfrak{c}_L(a \vee b_n) \cap Z_n = \mathsf{O}. \tag{4.1.1}$$

For every $n \in \mathbb{N}$, $Z_n \cap S$ is a zero sublocale in F because S is a sublocale of T_n. Consider the zero sublocale $Z = S \cap \bigcap_{n\in\mathbb{N}} Z_n = \bigcap_{n\in\mathbb{N}} (S \cap Z_n)$ of S. We claim that $A = Z \cap S$, which shows that F is z-embedded in S. Indeed, since

$$A \subseteq F \subseteq S \quad \text{and} \quad A = \mathfrak{c}_L(b) \cap F \subseteq \mathfrak{c}_L(b) \cap S \subseteq \mathfrak{c}_L(b) \cap T_n \subseteq Z_n$$

for every $n \in \mathbb{N}$, we have $A \subseteq Z \cap S$. For the other inclusion note that, from (4.1.1), we have that $T_n \cap \mathfrak{c}_L(a) \cap Z_n \subseteq \mathfrak{o}_L(b_n)$, then

$$Z_n \cap F = Z_n \cap T_n \cap \mathfrak{c}_L(a) \cap F \subseteq \mathfrak{o}_L(b_n) \cap F = \mathfrak{o}_F(b_n)$$

where the first equality holds since $F \subseteq \mathfrak{c}_L(a)$ and $F \subseteq T_n$. Thus,

$$Z \cap F = F \cap \bigcap_{n\in\mathbb{N}} S \cap Z_n = \bigcap_{n\in\mathbb{N}} F \cap Z_n \subseteq \bigcap_{n\in\mathbb{N}} \mathfrak{o}_F(b_n) = A.$$

For the assertion involving z-embedded sublocales (the one inside parenthesis) there is only left to show that S is z-embedded. Let $A = \mathfrak{c}_S(b)$ be a zero sublocale of S. Then A is a G_δ-sublocale of S, that is, $A = \bigcap_{n\in\mathbb{N}} \mathfrak{o}_S(b_n)$ for some $b_n \in S$. Consider the open sublocales $\mathfrak{o}_L(b \vee b_n)$ for $n \in \mathbb{N}$. Since $\mathfrak{c}_S(b) \subseteq \mathfrak{o}_S(b_n)$ we have

$$S \cap \mathfrak{c}_L(b \vee b_n) = S \cap \mathfrak{c}_L(b) \cap \mathfrak{c}_L(b_n) = \mathfrak{c}_S(b) \cap \mathfrak{c}_S(b_n) = \mathsf{O}.$$

Hence $S \subseteq \mathfrak{o}_L(b \vee b_n)$. By assumption, for each $n \in \mathbb{N}$ there is a normal and z-embedded sublocale T_n such that $S \subseteq T_n \subseteq \mathfrak{o}_L(b \vee b_n)$. Note that $\mathfrak{c}_L(b) \cap T_n$ and $\mathfrak{c}_L(b_n) \cap T_n$ are disjoint; indeed

$$\mathfrak{c}_L(b) \cap T_n \cap \mathfrak{c}_L(b_n) \subseteq \mathfrak{c}_L(b \vee b_n) \cap \mathfrak{o}_L(b \vee b_n) = \mathsf{O}.$$

By the normality of T_n (Theorem 4.1.1 (ii)), $\mathfrak{c}_L(b) \cap T_n$ and $\mathfrak{c}_L(b_n) \cap T_n$ are then completely separated in T_n. Consequently, there is a $Z_n \in \mathsf{ZS}(T_n)$ such that

$$T_n \cap \mathfrak{c}_L(b) \subseteq C_n \quad \text{and} \quad T_n \cap \mathfrak{c}_L(b_n) \cap C_n = \mathsf{O}.$$

On the other hand, by z-embeddedness of T_n, there is a $Z'_n \in \mathsf{ZS}(L)$ such that $T_n \cap Z'_n = Z_n$. Finally, consider the zero sublocale $\bigcap_{n\in\mathbb{N}} Z'_n$. We claim that $A = S \cap \bigcap_{n\in\mathbb{N}} Z'_n$. The inclusion '$\subseteq$' is clear because

$$A \subseteq S \quad \text{and} \quad A \subseteq \mathfrak{c}_L(b) \cap T_n \subseteq C_n \subseteq C'_n$$

for every n. Conversely,

$$S \cap \bigcap_{n\in\mathbb{N}} Z'_n = S \cap \bigcap_{n\in\mathbb{N}} (Z'_n \cap T_n) = S \cap \bigcap_{n\in\mathbb{N}} Z_n \overset{(*)}{\subseteq} S \cap \bigcap_{n\in\mathbb{N}} \mathfrak{o}_L(b_n) = \bigcap_{n\in\mathbb{N}} \mathfrak{o}_S(b_n) = A.$$

where $(*)$ holds because $T_n \cap \mathfrak{c}_L(b_n) \cap Z_n = \mathsf{O}$. Hence $Z_n = T_n \cap Z_n \subseteq \mathfrak{o}_L(b_n)$. ☐

Proposition 4.1.4. *Let L be a normal frame. If S is an F_σ-generalized sublocale of L, then it is normal and z-embedded.*

Proof. We will use Lemma 4.1.3 to show that S is normal and z-embedded. Let $S \subseteq \mathfrak{o}_L(a)$. Then there is an F_σ-sublocale $F = \bigvee_{n\in\mathbb{N}} \mathfrak{c}_L(a_n)$ such that $S \subseteq F \subseteq \mathfrak{o}_L(a)$. By Lemma 4.1.2, F is z-embedded. In order to show that F is normal consider T a closed sublocale of F. Then $T = \mathfrak{c}_L(b) \cap F$ for some $b \in F$. Moreover,

$$T = \mathfrak{c}_L(b) \cap F = \mathfrak{c}_L(b) \cap \bigvee_{n\in\mathbb{N}} \mathfrak{c}_L(a_n) = \bigvee_{n\in\mathbb{N}} \mathfrak{c}_L(b \vee a_n)$$

where the third equality holds by (1.4.4). This shows that T is an F_σ-sublocale of L so, by Lemma 4.1.2, it is z-embedded. In particular, T is z-embedded in F. Hence, by Theorem 4.1.1 (v), F is normal. ☐

This proposition implies immediately Proposition 6.4 of [41]:

Corollary 4.1.5. *An F_σ-sublocale of a normal locale is normal.*

Theorem 4.1.6. *The following statements about a locale L are equivalent:*

(i) *L is normal.*

(ii) *Every F_σ-generalized sublocale of L is z-embedded in L.*

(iii) *Every F_σ-sublocale of L is z-embedded in L.*

(iv) *For any closed sublocale F of L and any zero sublocale Z of L, $F \vee Z$ is z-embedded in L.*

(v) *For any closed sublocale F of L and any zero sublocale Z of L such that $F \cap Z = \mathsf{O}$, $F \vee Z$ is z-embedded in L.*

(vi) *Every closed sublocale of L is z-embedded in L.*

Proof. (i) $\Longrightarrow$ (ii) follows from Proposition 4.1.4.
(ii) $\Longrightarrow$ (iii) is trivial.
(iii) $\Longrightarrow$ (iv): It is clear since $F \vee Z$, being a closed sublocale, is an F_σ-sublocale.
(iv) $\Longrightarrow$ (v) is obvious.
(v) $\Longrightarrow$ (i): It is simply Theorem 4.1.1 (v). □

4.2 Variants of Normality and z-Embeddings

As discussed in Section 4.1, normality can be characterized in terms of C-,C^*- and z-embeddings. Similar results hold for some weaker forms of normality as e.g. mild normality. We will present now some general results that cover and unify all such characterizations under a single proof. Besides, the setting will allow us to identify general conditions under which this kind of characterizations may hold.

Definition 4.2.1. A *selection function on sublocales* $\mathfrak{S}$ is a function which assigns to each locale L a subset $\mathfrak{S}L$ of $\mathsf{S}(L)$.

In this Section we will only work with *closed selection functions on sublocales*; that is, a selection function $\mathfrak{S}$ given by

$$\mathfrak{S}L = \{\mathfrak{c}(a) \mid a \in \mathfrak{s}L\}$$

where $\mathfrak{s}$ is a function which assigns to each locale L a subset $\mathfrak{s}L$ of L. We say that the sublocales in $\mathfrak{S}L$ are the *$\mathfrak{S}$-closed* sublocales of L. Accordingly, we say that L is *completely separated $\mathfrak{S}$-normal* (briefly, *c. s. $\mathfrak{S}$-normal*) if every two disjoint $\mathfrak{S}$-closed sublocales of L are completely separated in L.

The standard examples for $\mathfrak{S}$ are given by selecting respectively all elements, regular elements, cozero elements, δ-elements and δ-regular elements.[1] In the sequel, these classes will be denoted as

$$\mathfrak{S}_1,\ \mathfrak{S}_{\mathrm{reg}},\ \mathfrak{S}_{\mathsf{coz}},\ \mathfrak{S}_\delta,\ \mathfrak{S}_{\delta\mathrm{reg}} \tag{4.2.1}$$

[1] An $a \in L$ is a *δ-element* [62] if $a = \bigvee\{x \in L \mid x \text{ is regular}, x \le a\}$; it is a *$\delta$-regular element* if $a = \bigvee_{n=1}^{\infty} a_n$ for some $a_n \prec a$ (we may assume that each a_n is regular, since $a_n \prec a$ implies $a_n^{**} \prec a$, hence any δ-regular element is a δ-element); see [43] for more information.

respectively.

Given a closed selection $\mathfrak{S}$, a locale L is called *$\mathfrak{S}$-normal* [43] whenever $a \vee b = 1$ for $a, b \in \mathfrak{s}L$ implies the existence of $u, v \in \mathfrak{s}L$ such that $u \wedge v = 0$ and $a \vee u = 1 = v \vee b$. It will be useful to introduce the following (formally) weaker variant of this notion: we say that a locale L is *weakly $\mathfrak{S}$-normal* whenever $a \vee b = 1$ for $a, b \in \mathfrak{s}L$ implies the existence of $u, v \in L$ such that $u \wedge v = 0$ and $a \vee u = 1 = v \vee b$. Of course, translated to sublocales, a frame is $\mathfrak{S}$-normal (resp. weakly $\mathfrak{S}$-normal) if any pair of disjoint $\mathfrak{S}$-closed sublocales can be separated by disjoint $\mathfrak{S}$-open[2] (resp. open) sublocales. Clearly, since $\mathsf{Coz}\, L$ is a normal σ-frame, any c. s. $\mathfrak{S}$-normal locale is weakly $\mathfrak{S}$-normal.

Examples 4.2.2. Notice that $\mathfrak{S}_1$-normality and weak $\mathfrak{S}_1$-normality are just standard normality, while weak $\mathfrak{S}_{\mathrm{reg}}$-normality is precisely the definition of *mild normality* ([62]), and it coincides with $\mathfrak{S}_{\mathrm{reg}}$-normality. In fact, when $\mathfrak{s}L$ contains all regular elements, $\mathfrak{S}$-normality is equivalent to weak $\mathfrak{S}$-normality because in any frame (more generally, any distributive pseudocomplemented algebra [43, Proposition 1.4]), $u \wedge v = 0$ if and only if $u^{**} \wedge v^{**} = 0$. This is also the case of $\mathfrak{S}_\delta$.
Moreover, since $\mathsf{Coz}\, L$ is a normal σ-frame, $\mathfrak{S}_{\mathsf{coz}}$-normality is a property satisfied by any locale.

The fact that $\prec$ interpolates in normal locales, and thus $\prec \;=\; \prec\!\!\prec$, plays an important role in the proof that a locale is normal if and only if every pair of disjoint closed sublocales is completely separated. Certainly, the following conditions on a locale L might also play some role if we want to obtain similar results for other variants of normality:

(I) For every $a, b \in \mathfrak{s}L$, if $a \prec b$ then there is a $c \in \mathfrak{s}L$ such that $a \prec c \prec b$.

(II) For every $a \in L$ and $b \in \mathfrak{s}L$, if $a \prec b$ then there is a $c \in \mathfrak{s}L$ such that $a \prec c \prec b$.

(wI) For every $a, b \in \mathfrak{s}L$, if $a \prec b$ then $a \prec\!\!\prec b$.

(wII) For every $a \in L$ and $b \in \mathfrak{s}L$, if $a \prec b$ then $a \prec\!\!\prec b$.

(wI') For every $a, b \in \mathfrak{s}L$, if $a \prec b$ then there is a $c \in \mathsf{Coz}\, L$ such that $a \prec c \prec b$.

(wII') For every $a \in L$ and $b \in \mathfrak{s}L$, if $a \prec b$ then there is a $c \in \mathsf{Coz}\, L$ such that $a \prec c \prec b$.

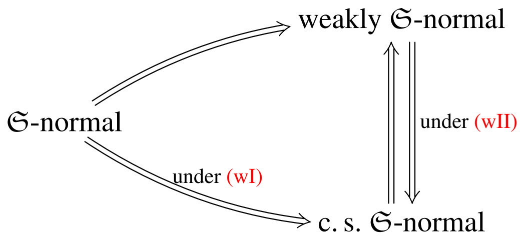
Clearly, we have:

$$\begin{array}{ccccc} \text{(II)} & \Longrightarrow & \text{(wII)} & \Longrightarrow & \text{(wII')} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{(I)} & \Longrightarrow & \text{(wI)} & \Longrightarrow & \text{(wI')} \end{array}$$

Remark 4.2.3. If L is a $\mathfrak{S}$-normal (resp. weakly $\mathfrak{S}$-normal) locale and satisfies (wI) (resp. (wII)) then L is c. s. $\mathfrak{S}$-normal. Indeed, if $a, b \in \mathfrak{s}L$ are such that $a \vee b = 1$ then there are $u, v \in \mathfrak{s}L$ (resp. in L) such that $u \wedge v = 0$ and $a \vee u = 1 = v \vee b$. This implies $a \vee v^* = 1$ meaning $v \prec a$. By (wI) (resp. (wII)), $v \prec\!\!\prec a$. Thus, from Proposition 2.7.5, $\mathfrak{c}(a)$ is completely separated from $\mathfrak{o}(v)$. Because $\mathfrak{c}(b) \subseteq \mathfrak{o}(v)$, then $\mathfrak{c}(a)$ and $\mathfrak{c}(b)$ are also completely separated in L.

[2] A sublocale S of L is $\mathfrak{S}$-open if $S = \mathfrak{o}(a)$ for some $a \in \mathfrak{s}L$.

Summing up, we have



Consider now the following further conditions on a selection $\mathfrak{S}$:

($\mathfrak{s}$1) If $a, b \in \mathfrak{s}L$ then $a \wedge b \in \mathfrak{s}L$.

($\mathfrak{s}$2) If $a \in \mathfrak{s}L$ and $b \in \mathfrak{s}\mathfrak{c}_L(a)$, then $b \in \mathfrak{s}L$.

($\mathfrak{s}$3) If $a, b \in \mathsf{Coz}\, L$ are such that $a \vee b = 1$, then there are $u, v \in \mathfrak{s}L$ such that $v \leq a$, $u \leq b$ and $u \vee v = 1$.

When a selection function $\mathfrak{S}$ satisfies all of them, we say that $\mathfrak{S}$ is an *adequate selection*. E.g. $\mathfrak{S}_1$ and $\mathfrak{S}_{\text{reg}}$ are examples of adequate selections.

Proposition 4.2.4. *Let $\mathfrak{S}$ be a selection with property ($\mathfrak{s}$2). If L is a weakly $\mathfrak{S}$-normal locale, then $\mathfrak{c}_L(a)$ is weakly $\mathfrak{S}$-normal for every $a \in \mathfrak{s}L$.*

Proof. Let S and T be disjoint $\mathfrak{S}$-closed sublocales of $\mathfrak{c}_L(a)$ for some $a \in \mathfrak{s}L$. By ($\mathfrak{s}$2), S and T are $\mathfrak{S}$-closed sublocales of L. By assumption, there are open sublocales $\mathfrak{o}_L(x)$ and $\mathfrak{o}_L(y)$ of L such that $S \subseteq \mathfrak{o}_L(x)$ and $T \subseteq \mathfrak{o}_L(y)$. Thus, $S \subseteq \mathfrak{o}_L(x) \cap \mathfrak{c}_L(a)$ and $T \subseteq \mathfrak{o}_L(y) \cap \mathfrak{c}_L(a)$ where $\mathfrak{o}(x)_L \cap \mathfrak{c}_L(a)$ and $\mathfrak{o}(y)_L \cap \mathfrak{c}_L(a)$ are open sublocales of $\mathfrak{c}_L(a)$. ☐

Proposition 4.2.5. *Let $\mathfrak{S}$ be a selection with properties ($\mathfrak{s}$2) and ($\mathfrak{s}$3). If L is completely separated $\mathfrak{S}$-normal, then every $\mathfrak{S}$-closed sublocale of L is C^*-embedded in L.*

Proof. Let $\mathfrak{c}_L(a)$ be a $\mathfrak{S}$-closed sublocale of L. We will use Corollary 3.3.9 to show that $\mathfrak{c}_L(a)$ is C^*-embedded. Let Z_1 and Z_2 be disjoint zero sublocales of $\mathfrak{c}_L(a)$. By ($\mathfrak{s}$3) there are disjoint $\mathfrak{S}$-closed sublocales D_1 and D_2 of $\mathfrak{c}_L(a)$ such that $Z_1 \subseteq D_1$ and $Z_2 \subseteq D_2$. Since ($\mathfrak{s}$2) holds, D_1 and D_2 are $\mathfrak{S}$-closed in L. Because L is c. s. $\mathfrak{S}$-normal, D_1 and D_2 are completely separated in L, and so are Z_1 and Z_2. ☐

Proposition 4.2.6. *Let $\mathfrak{S}$ be a selection with properties ($\mathfrak{s}$2) and ($\mathfrak{s}$3). Consider the following statements for a locale L:*

(1) *For every pair of disjoint $\mathfrak{S}$-closed sublocales $\mathfrak{c}(a)$ and $\mathfrak{c}(b)$ of L there is a zero sublocale Z such that $\mathfrak{c}(a) \subseteq Z$ and $\mathfrak{c}(b) \subseteq Z^{\#}$.*

(2) *Every $\mathfrak{S}$-closed sublocale of L is z-embedded in L.*

Then (1) $\Longrightarrow$ (2).

Proof. Let $\mathfrak{c}_L(a)$ be a $\mathfrak{S}$-closed sublocale. We will use Corollary 3.2.5 (iii) to prove that $\mathfrak{c}_L(a)$ is z-embedded. It suffices to take disjoint zero sublocales instead of general completely separated sublocales (recall Remark 2.7.4(2)). Let Z_1 and Z_2 be disjoint zero sublocales of $\mathfrak{c}_L(a)$. By ($\mathfrak{s}3$) there are disjoint $\mathfrak{S}$-closed sublocales D_1 and D_2 of $\mathfrak{c}(a)$ such that $Z_1 \subseteq D_1$ and $Z_2 \subseteq D_2$. Since ($\mathfrak{s}2$) holds, D_1 and D_2 are $\mathfrak{S}$-closed in L. Finally, by assumption, there is a zero sublocale Z of L such that $Z_1 \subseteq Z$ and $Z_2 \subseteq Z^{\#}$. □

Corollary 4.2.7. *Let $\mathfrak{S}$ be a selection satisfying ($\mathfrak{s}2$) and ($\mathfrak{s}3$). If L is completely separated $\mathfrak{S}$-normal, then every $\mathfrak{S}$-closed sublocale of L is z-embedded in L.*

The following proposition gives a sufficient condition for weak $\mathfrak{S}$-normality that only requires property ($\mathfrak{s}1$); hence it covers also the selections $\mathfrak{S}_\delta$ and $\mathfrak{S}_{\delta\mathrm{reg}}$.

Proposition 4.2.8. *Let $\mathfrak{S}$ be a selection with property ($\mathfrak{s}1$). If L is a locale in which every $\mathfrak{S}$-closed sublocale is z-embedded, then L is weakly $\mathfrak{S}$-normal.*

Proof. Let $\mathfrak{c}_L(a)$ and $\mathfrak{c}_L(b)$ be disjoint $\mathfrak{S}$-closed sublocales of L. Consider the sublocale $M = \mathfrak{c}_L(a) \vee \mathfrak{c}_L(b) = \mathfrak{c}_L(a \wedge b)$. By ($\mathfrak{s}1$), M is $\mathfrak{S}$-closed in L. Then the sublocales $\mathfrak{c}_L(a)$ and $\mathfrak{c}_L(b)$ are clopen in M; indeed $\mathfrak{c}_L(a) = \mathfrak{c}_L(a) \cap M$, $\mathfrak{c}_L(b) = \mathfrak{c}_L(b) \cap M$ and, since $\mathfrak{c}_L(a) \cap \mathfrak{c}_L(b) = \mathsf{O}$,

$$\mathfrak{c}_L(a) = \mathfrak{o}_L(b) \cap \mathfrak{c}_L(a) = \mathfrak{o}_L(b) \cap (\mathfrak{c}_L(a) \vee \mathfrak{c}_L(b)) = \mathfrak{o}_L(b) \cap M,$$

$$\mathfrak{c}_L(b) = \mathfrak{o}_L(a) \cap \mathfrak{c}_L(b) = \mathfrak{o}_L(a) \cap (\mathfrak{c}_L(b) \vee \mathfrak{c}_L(a)) = \mathfrak{o}_L(a) \cap M.$$

Consequently (recall Proposition 2.7.8 (3)), $\mathfrak{c}_L(a)$ and $\mathfrak{c}_L(b)$ are disjoint cozero sublocales of M. By assumption, M is z-embedded so from Corollary 3.2.3 (iv) we know that there are disjoint cozero sublocales $\mathfrak{o}_L(v)$ and $\mathfrak{o}_L(u)$ in L such that

$$\mathfrak{o}_L(u) \cap \mathfrak{o}_L(v) = \mathsf{O}, \quad \mathfrak{c}_L(a) \subseteq \mathfrak{o}_L(u) \quad \text{and} \quad \mathfrak{c}_L(b) \subseteq \mathfrak{o}_L(v),$$

as required. □

Proposition 4.2.9. *Let $\mathfrak{S}$ be a selection with properties ($\mathfrak{s}2$) and ($\mathfrak{s}3$). If L is a weakly $\mathfrak{S}$-normal locale and (wII') holds, then every $\mathfrak{S}$-closed sublocale of L is z-embedded.*

Proof. To prove that every $\mathfrak{S}$-closed sublocale is z-embedded we will show that condition (1) of Proposition 4.2.6 holds. Let $\mathfrak{c}(a)$ and $\mathfrak{c}(b)$ be disjoint $\mathfrak{S}$-closed sublocales. Then $a \vee b = 1$. Since L is weakly $\mathfrak{S}$-normal, there are $u, v \in L$ such that $u \wedge v = 0$ and $a \vee u = 1 = b \vee v$. This implies $v \prec a$. By (wII'), there is a $c \in \operatorname{Coz} L$ such that $v \prec c \prec a$. In particular, $c \leq a$, which means $\mathfrak{c}(a) \subseteq \mathfrak{c}(c)$. Furthermore, $v^* \vee c = 1$ so $v \leq v^{**} \leq c$. Hence, $1 = v \vee b \leq c \vee b$, that is, $\mathfrak{c}(b) \subseteq \mathfrak{o}(c) = \mathfrak{c}(c)^{\#}$ as required. □

Corollary 4.2.10. *Let $\mathfrak{S}$ be an adequate selection. If (wII') holds on a locale L, then L is weakly $\mathfrak{S}$-normal if and only if every $\mathfrak{S}$-closed sublocale is z-embedded.*

Mimicking the proof of Proposition 4.2.9, we can show a similar result for $\mathfrak{S}$-normality:

Proposition 4.2.11. *Let $\mathfrak{S}$ be a selection with properties (𝔰2) and (𝔰3). If L is an $\mathfrak{S}$-normal locale and (wI') holds, then every $\mathfrak{S}$-closed sublocale is z-embedded.*

Putting together all the results above we obtain the following theorems:

Theorem 4.2.12. *Let $\mathfrak{S}$ be an adequate selection. Consider the following statements for a locale L:*

(1) *Any pair of disjoint $\mathfrak{S}$-closed sublocales of L are completely separated in L (i.e. L is completely separated $\mathfrak{S}$-normal).*

(2) *Every $\mathfrak{S}$-closed sublocale of L is C^*-embedded.*

(3) *Every $\mathfrak{S}$-closed sublocale of L is z-embedded.*

(4) *L is weakly $\mathfrak{S}$-normal.*

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4).

Theorem 4.2.13. *Let $\mathfrak{S}$ be an adequate selection. The following statements are equivalent for any locale L with property* (wII)*:*

(i) *Any pair of disjoint $\mathfrak{S}$-closed sublocales of L are completely separated in L (i.e. L is completely separated $\mathfrak{S}$-normal).*

(ii) *Every $\mathfrak{S}$-closed sublocale of L is C^*-embedded.*

(iii) *Every $\mathfrak{S}$-closed sublocale of L is z-embedded.*

(iv) *L is weakly $\mathfrak{S}$-normal.*

Theorem 4.2.14. *Let $\mathfrak{S}$ be an adequate selection. Consider the following statements for a locale L with property* (wI)*:*

(1) *L is $\mathfrak{S}$-normal.*

(2) *Any pair of disjoint $\mathfrak{S}$-closed sublocales of L are completely separated in L (i.e. L is completely separated $\mathfrak{S}$-normal).*

(3) *Every $\mathfrak{S}$-closed sublocale of L is C^*-embedded.*

(4) *Every $\mathfrak{S}$-closed sublocale of L is z-embedded.*

(5) *L is weakly $\mathfrak{S}$-normal.*

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5).

In the standard example $\mathfrak{S} = \mathfrak{S}_1$, an important fact is that $\mathsf{Coz}\, L \subseteq \mathfrak{s}L$. For a general $\mathfrak{S}$, we do not have necessarily $\mathsf{Coz}\, L \subseteq \mathfrak{s}L$, but we need cozero elements to behave "normally" with respect to $\mathfrak{s}L$ in order to get the converses. For this we consider the following condition on L:

(D) For every $a, b \in \mathsf{Coz}\, L$ such that $a \vee b = 1$ there are $u, v \in \mathfrak{s}L$ such that $u \wedge v = 0$ and $a \vee u = 1 = b \vee v$.

Note that if (D) holds and L is c. s. $\mathfrak{S}$-normal, then L is $\mathfrak{S}$-normal. Hence:

Theorem 4.2.15. *Let $\mathfrak{S}$ be an adequate selection. The following statements are equivalent for any locale L with properties (wII) and (D):*

(i) *L is $\mathfrak{S}$-normal.*

(ii) *Any pair of disjoint $\mathfrak{S}$-closed sublocales of L is completely separated in L (i.e., L is completely separated $\mathfrak{S}$-normal).*

(iii) *Every $\mathfrak{S}$-closed sublocale of L is C^*-embedded.*

(iv) *Every $\mathfrak{S}$-closed sublocale of L is z-embedded.*

(v) *L is weakly $\mathfrak{S}$-normal.*

Remarks 4.2.16. (1) If $\mathsf{Coz}\, L \subseteq \mathfrak{s}L$, then clearly (D) holds. Furthermore, in this case we can add one more equivalent statement to Theorem 4.2.15, namely:

(iv) *Every $\mathfrak{S}$-closed sublocale of L is C-embedded.*

Indeed, let $\mathfrak{c}(a)$ be a $\mathfrak{S}$-closed sublocale. If L is c. s. $\mathfrak{S}$-normal then $\mathfrak{c}(a)$ is C^*-embedded. From [6, 6.2] it suffices to show that $\mathfrak{c}(a)$ is completely separated from every zero sublocale disjoint form it, but this is immediate since $\mathsf{Coz}\, L \subseteq \mathfrak{s}L$ and L is c. s. $\mathfrak{S}$-normal.
This assertion can also be added to Theorems 4.2.12 (between statements (1) and (2)) and 4.2.13 whenever $\mathsf{Coz}\, L \subseteq \mathfrak{s}L$.

(2) On the other hand, the property that $\mathfrak{s}L$ contains the set L^* of regular elements is equivalent to the following condition (by the property that $u \wedge v = 0$ if and only if $u^{**} \wedge v^{**} = 0$):

(DC) For every $a, b \in L$ such that $a \wedge b = 0$ there are $u, v \in \mathfrak{s}L$ such that $u \wedge v = 0$, $a \leq u$ and $b \leq v$.

(DC) is stronger than (D): Indeed, let $a, b \in \mathsf{Coz}\, L$ with $a \vee b = 1$. Since $\mathsf{Coz}\, L$ is a normal σ-frame, there are $a', b' \in \mathsf{Coz}\, L$ such that $a' \wedge b' = 0$ and $a' \vee a = 1 = b' \vee b$. By (DC), there are $u, v \in \mathfrak{s}L$ such that $u \wedge v = 0$, $a' \leq u$ and $b' \leq v$. Hence $a \vee u = 1 = a \vee v$, as required.
This means that if L is c. s. $\mathfrak{S}$-normal then it is $\mathfrak{S}$-normal. Furthermore, if L is $\mathfrak{S}$-normal then (I) holds. Indeed, if $a, b \in \mathfrak{s}L$ are such that $a \prec b$ then $a^* \vee b = 1$, and since $a^* \in L^* \subseteq \mathfrak{s}L$, there are $u, v \in \mathfrak{s}L$ such that $u \wedge v = 0$ and $a^* \vee u = 1 = b \vee v$; thus, $a \prec u$ and $u \prec b$, as required. Hence, L is c. s. $\mathfrak{S}$-normal.
Together with Examples 4.2.2 this shows that whenever $\mathfrak{s}L$ contains all regular elements, the notions of c. s. $\mathfrak{S}$-normality, weak $\mathfrak{S}$-normality and $\mathfrak{S}$-normality are equivalent.

4.3 Frames in Which Special Sublocales are z-Embedded

As its title suggests, this section was inspired by Blair's paper [20], where the author characterizes some classes topological spaces in terms of z-embedded sets. In the previous two sections we characterized types of frames where closed (or subclasses of closed) sublocales are z-embedded. Here we will discuss frames where other type of sublocales are z-embedded. Many of the results presented in [20] were already studied point-freely, but they are scattered around in the literature and rarely formulated in terms of sublocales. Thus, we provide here a survey that gathers most of the results that characterize frames where certain type of sublocales are z-, C-, or C*-embedded. Proofs will only be included when they are missing in the literature.

Following [40], we say that a frame a L is *perfect* if every open sublocale is an F_σ-sublocale.

Remarks 4.3.1. (1) A space X is perfect if and only if $\Omega(X)$ is a perfect frame ([40, Proposition 3.4]).

(2) Notice that, in general, perfectness is not equivalent to the condition that every closed sublocale is G_δ (Remark 1.4.1). In fact, in [40] the authors discuss in depth these two notions. They call a frame F_σ-perfect (resp. G_δ-perfect) if every open sublocale is F_σ (resp. if every closed sublocale is G_δ). They show that G_δ-perfectness is not a conservative extension of topological perfectness, making F_σ-perfectness the only convenient way to define perfectness in the point-free setting. Nevertheless, under normality the two concepts coincide ([40, Proposition 3.7]).

(3) Every perfect locale is subfit (see [40]).

A frame L is *perfectly normal* if for each $a \in L$ there is a subset $\{b_n\}_{n\in\mathbb{N}} \subseteq L$ such that $a = \bigvee_{n\in\mathbb{N}} b_n$ and $b_n \prec a$ for every $n \in \mathbb{N}$. An element a of this form is called *regular F_σ-element*. This definition is taken from [39] where more details on perfectly normal locales can be found.

Probably more useful then the definition itself is the following characterization of perfect normality [39, Proposition 4.2]:

Proposition 4.3.2. *The following are equivalent for a frame L:*

(i) *L is perfectly normal.*

(ii) *L is normal and each closed sublocale is G_δ.*

(iii) *L is normal and perfect.*

(iv) $\mathsf{Coz}\, L = L$.

Remarks 4.3.3. (1) Every perfectly normal frame is completely regular. This is obvious from Proposition 4.3.2 (iv) and Corollary 2.3.8.

(2) Note that condition (iv) of Proposition 4.3.2 in terms of sublocales asserts that every closed sublocale of L is a zero sublocale. Equivalently, every open sublocale of L is a cozero sublocale.

(3) The frame $\mathfrak{L}(\mathbb{R})$ is perfectly normal. This can be easily checked by noticing that every open set in the real line is a countable union of open intervals. Thus, every open set is a cozero set. This will provide, by Proposition 4.3.6 below, examples of sublocales of $\mathfrak{L}(\mathbb{R})$ that are z-embedded but not C*-embedded.

Proposition 4.3.4. [39, Proposition 4.3] *Any sublocale of a perfectly normal frame is perfectly normal.*

Now we recall that a frame L is *completely normal* ([33] or [48]) if for every pair of sublocales S and T of L such that $S\cap\overline{T}=\mathsf{O}=\overline{S}\cap T$, there are open sublocales U and V of L such that $S\subseteq U$, $T\subseteq V$ and $U\cap V=\mathsf{O}$.

The following characterization [33, Theorem 3.7] will then allow us to compare some of these notions.

Proposition 4.3.5. *The following are equivalent for a frame L:*

(i) *L is completely normal.*

(ii) *Every sublocale of L is normal (that is, L is hereditary normal).*

(iii) *Every open sublocale of L is normal.*

One can show that perfect normality implies complete normality (Proposition 4.3.6), but before showing this let us define the weaker notion of weak perfect normality. By mimicking the classical notion in [20], we say that a frame L is *weakly perfectly normal* if every sublocale of L is z-embedded.

Proposition 4.3.6. *Every perfectly normal frame is weakly perfectly normal. Moreover, every weakly perfectly normal frame is completely normal.*

Proof. Let L be a perfectly normal frame. Consider S a sublocale of L and $Z\in\mathsf{ZS}(S)$. Since Z is closed in S we have that $Z=S\cap\mathfrak{c}_L(a)$ for some $a\in L$. From Proposition 4.3.2 (iv), $\mathfrak{c}_L(a)$ is a zero sublocale of L. Thus, S is z-embedded in L.
Now, let L be a weakly perfectly normal frame and S be a sublocale of L. We will show that S is normal using Theorem 4.1.1 (v). So let and $\mathfrak{c}_S(a)$ be sublocale of S. Since L is weakly perfectly normal, $\mathfrak{c}_S(a)$ and S are z-embedded in L. By Remark 3.1.2(3), $\mathfrak{c}_S(a)$ is also z-embedded in S, as required. □

Remark 4.3.7. Regarding the converse of the first statement in Proposition 4.3.6, even in classical topology, the situation is not clear. In [20], Blair shows that under the hypothesis of the existence of measurable cardinals, there is a weakly perfectly normal space (of measurable power) that is not perfectly normal. As far as we know, without this assumption the question of whether weak perfect normality is weaker than perfect normality is still open.

Recall that a sublocale S of L is *locally closed* if it is the intersection of a closed sublocale with an open sublocale. The following result is the point-free counterpart of [20, Proposition 4.11].

Proposition 4.3.8. *The following are equivalent for a frame L:*

(i) *L is weakly perfectly normal.*

(ii) *Every locally closed sublocale of L is z-embedded in L.*

(iii) *Every open sublocale of L is normal and z-embedded in L.*

(iv) *L is completely normal and every open sublocale of L is z-embedded in L.*

Proof. Clearly (i) implies (ii). For (ii) $\Longrightarrow$ (iii) let $\mathfrak{o}(a)$ be a sublocale of L. Trivially every open sublocale is locally closed, so $\mathfrak{o}(a)$ is z-embedded. To show it is also normal let F be a closed sublocale of $\mathfrak{o}_L(a)$. It is locally closed in L, so F is z-embedded in L. By Remark 3.1.2 (3), it is also z-embedded in $\mathfrak{o}_L(a)$. Hence, $\mathfrak{o}_L(a)$ is normal (recall Theorem 4.1.1 (v)). Implication (iii) $\Longrightarrow$ (iv) follows immediately from Proposition 4.3.5 (iii). Finally, from Lemma 4.1.3, one can easily deduce (iv) $\Longrightarrow$ (i). □

Oz-frames were introduced in [15] (see also [12, 29]). The definition translated to our own sublocale language says that a frame L is an *Oz-frame* if every open sublocale is z-embedded in L. The following result gathers Propositions 2.2 and 2.3 of [12]. Here, we need to recall that a *regular closed* (resp. *regular open*) sublocale (what we called $\mathfrak{S}_{\mathrm{reg}}$-closed in Section 4.2) is a sublocale of L of the form $\mathfrak{c}_L(a)$ (resp. $\mathfrak{o}_L(a)$) with $a \in L^*$. Equivalently, by (1.4.10), a sublocale S is regular closed (resp. regular open) if and only if $\overline{S^\circ} = S$ (resp. $\overline{S}^\circ = S$).

Proposition 4.3.9. *The following are equivalent for a frame L:*

(i) *L is an Oz-frame.*

(ii) *Every dense open sublocale of L is z-embedded in L.*

(iii) *Every regular closed sublocale is a zero sublocale.*

(iv) *Every regular open sublocale is a cozero sublocale.*

(v) *Every dense sublocale of L is z-embedded in L.*

(vi) *For every regular element $a \in L$, $a \vee a^*$ is a cozero element.*

(vii) *For all $a, b \in L$ with $a \wedge b = 0$, there are $c, d \in \mathsf{Coz}\, L$ such that $a \leq c$, $b \leq d$ and $c \wedge d = 0$.*

(viii) *The sub σ-frame of L generated by L^* is regular.*

Remarks 4.3.10. (1) If L is *Oz* then every open, dense and regular closed sublocale of L is also *Oz*. Indeed, one only needs to notice that an open (resp. dense) sublocale of an open (resp. dense) sublocale of L is open (resp. dense) in L and use the definition of *Oz*-frame (resp. use Proposition 4.3.9 (v)). For the regular closed sublocale one uses Proposition 4.3.9 (iii) and the fact that a regular closed sublocale of a regular closed sublocale is regular closed (recall that in Section 4.2 we mentioned that $\mathfrak{S}_{\mathrm{reg}}$ satisfies ($\mathfrak{s}$2)).

(2) Clearly, every weakly perfectly normal frame is *Oz*. Moreover, every *Oz*-frame is mildly normal (see [62, Proposition 3.3.7]).

(3) In [40] the authors introduce the notion of a *perfectly mildly normal* (or *pm-normal* for short) frame; that is, a frame where every regular element is regular-F_σ. Most important is the fact that pm-normal frames are precisely *Oz*-frames ([40, Proposition 4.4]).

Note that conditions (iii) and (iv) of Proposition 4.3.9 are just saying that every regular element of L is a cozero element (i.e., $L^* \subseteq \mathsf{Coz}\, L$). A weaker version of this is given in [12]: a frame L is *weak Oz* if $a^* \in \mathsf{Coz}\, L$ for each $a \in \mathsf{Coz}\, L$ (i.e., $(\mathsf{Coz}\, L)^* \subseteq \mathsf{Coz}\, L$). In the language of sublocales, this means that $\overline{\mathfrak{o}(a)} \in \mathsf{ZS}(L)$ for every $\mathfrak{o}(a) \in \mathsf{CoZS}(L)$. Equivalently, $\mathfrak{c}(a)^{\circ} \in \mathsf{CoZS}(L)$ for every $\mathfrak{c}(a) \in \mathsf{ZS}(L)$. The next proposition [12, Proposition 5.2] identifies when a weak Oz-frame is actually Oz.

Proposition 4.3.11. *A weak Oz-frame L is Oz if and only if every regular element of L is equal to c^* for some $c \in \mathsf{Coz}\, L$ (i.e., $L^* = (\mathsf{Coz}\, L)^*$).*

Proposition 4.3.12. *The following are equivalent for a frame L:*

(i) *L is a weak Oz-frame.*

(ii) *For any disjoint $a \in L$ and $b \in \mathsf{Coz}\, L$, there exists $c \in \mathsf{Coz}\, L$ such that $a \leq c$ and $b \wedge c = 0$.*

(iii) *For any open sublocale $\mathfrak{o}(a)$ of L, if $\mathfrak{o}(a) \subseteq Z$ for some $Z \in \mathsf{ZS}(L)$, then there is $C \in \mathsf{CoZS}(L)$ such that $\mathfrak{o}_L(a) \subseteq C \subseteq Z$.*

(iv) *For any open sublocale $\mathfrak{o}(a)$ of L, if $\mathfrak{o}(a) \cap C = \mathsf{O}$ for some $C \in \mathsf{CoZS}(L)$, then there is $Z \in \mathsf{ZS}(L)$ such that $C \subseteq Z$ and $\mathfrak{o}(a) \cap Z = \mathsf{O}$.*

The previous characterization of weak Oz-frames is Proposition 5.3 in [12], but we added two more conditions (iii) and (iv) which are only a formulation in terms of sublocales of (ii).

Inspired by a Remark in [20, p. 685], we say that a frame L is *regularly normal* if it is normal and every regular closed sublocale is a G_δ-sublocale. We have:

Proposition 4.3.13. *A frame L is regularly normal if and only if its open sublocales and its closed sublocales are z-embedded in L.*

Proof. Let L be regularly normal. In particular it is normal, so every closed sublocale is z-embedded (Theorem 4.1.1 (v)). To show that every open sublocale is z-embedded we use Proposition 4.3.9 (iii). Let S be a regular closed sublocale of L, that is, $S = \mathfrak{c}(a^*)$ for some $a \in L$. By assumption, $\mathfrak{c}(a^*) = \bigcap_{n\in\mathbb{N}} \mathfrak{o}(a_n)$. Then for every $n \in \mathbb{N}$ we have that $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*) \subseteq \mathfrak{o}(a_n)$. Thus, $\mathfrak{c}(a^*)$ and $\mathfrak{c}(a_n)$ are disjoint closed sublocales. Since L is normal (Theorem 4.1.1 (ii)), there are $Z_1^n, Z_2^n \in \mathsf{ZS}(L)$ such that

$$\mathfrak{c}(a^*) \subseteq Z_1^n, \quad \mathfrak{c}(a_n) \subseteq Z_2^n \quad \text{and} \quad Z_1^n \cap Z_2^n = \mathsf{O}.$$

In particular, $\mathfrak{c}(a^*) \subseteq Z_1^n \subseteq (Z_2^n)^\# \subseteq \mathfrak{o}(a_n)$ for every natural number n. Hence,

$$\mathfrak{c}(a^*) \subseteq \bigcap_{n\in\mathbb{N}} Z_1^n \subseteq \bigcap_{n\in\mathbb{N}} \mathfrak{o}(a_n) = \mathfrak{c}(a^*),$$

showing that $\mathfrak{c}(a^*)$ is a zero sublocale of L.
The converse is immediate from Theorem 4.1.1 (v) and Proposition 4.3.9, since every zero sublocale is a G_δ-sublocale. □

Remark 4.3.14. Clearly, by Proposition 4.3.13 every weakly perfectly normal frame is regularly normal, and every regularly normal frame is Oz.

A locale L is *extremally disconnected* if $a^* \vee a^{**} = 1$ for every $a \in L$; i.e., every regular element of L is complemented (see [50, 65]). We put together some of the well-known characterizations of extremal disconnectedness (see, for example, [69] and [9]) into the following proposition.

Proposition 4.3.15. *The following are equivalent for a frame L:*

(i) *L is extremally disconnected.*

(ii) *The interior of a closed sublocale of L is clopen.*

(iii) *The closure of an open sublocale of L is clopen.*

(iv) *Every open sublocale is C^*-embedded.*

(v) *Every dense sublocale is C^*-embedded.*

(vi) *Every open and dense sublocale is C^*-embedded.*

(vii) *Every pair of disjoint open sublocales of L is completely separated in L.*

Remarks 4.3.16. (1) Note that extremal disconnectedness is a kind of dual notion of normality (more details can be found in [43]).

(2) From condition (iv) it is clear why every extremally disconnected frame is Oz.

(3) In fact, we have that a frame is extremally disconnected if and only if it is Oz and for every sublocale S that is a finite union of regular open sublocales, the localic embedding $S \hookrightarrow L$ is almost z-dense ([12, Proposition 4.2]).

Furthermore, when a frame is extremally disconnected and completely normal we have:

Proposition 4.3.17. [33, Corollary 6.7] *The following are equivalent for a frame L:*

(i) *L is completely normal and extremally disconnected.*

(ii) *Every sublocale of L is C^*-embedded.*

A frame L is *basically disconnected* if $a^* \vee a^{**} = 1$ for every $a \in \mathsf{Coz}\, L$ (every regular cozero element is complemented) [9]. Every extremally disconnected frame is basically disconnected.

We have the following result ([9, 8.4.3] and [44, 7.1.1]):

Proposition 4.3.18. *The following conditions are equivalent for a frame L:*

(i) *L is basically disconnected.*

(ii) *For all $a \in \mathsf{Coz}\, L$ and $b \in L$, $a \wedge b = 0$ implies $a^* \vee b^* = L$.*

(iii) *For every open sublocale $\mathfrak{o}(a)$ of L and $C \in \mathsf{CoZS}(L)$, if $\mathfrak{o}(a) \cap C = \mathsf{O}$ then they are completely separated in L.*

(iv) *The closure of every cozero sublocale of L is open.*

(v) *The interior of every zero sublocale of L is closed.*

The first appearance of almost P-spaces (referred to as P'-spaces) was in [77] (see also [59]). In the point-free setting, recall from [9] that a *P-frame* is a frame L where $c \vee c^* = 1$ for every $c \in \mathsf{Coz}\, L$; that is, every cozero element is complemented. Further, L is an *almost P-frame* if $c = c^{**}$ for every $c \in \mathsf{Coz}\, L$; i.e., every cozero element is regular ($\mathsf{Coz}\, L \subseteq L^*$). Clearly, every P-frame is an almost P-frame. The following proposition gathers the characterization for P-frames given in [9, Proposition 8.4.7], [44, Proposition 7.1.2] and [31, Proposition 4.9].

Proposition 4.3.19. *The following are equivalent for a frame L:*

(i) *L is a P-frame.*

(ii) *L is a basically disconnected almost P-frame.*

(iii) *Every cozero sublocale is closed.*

(iv) *Every zero sublocale is open.*

(v) *Every cozero sublocale is C-embedded.*

(vi) *Every z-embedded sublocale is C-embedded.*

(vii) *For every sublocale S of L, S is completely separated form every zero sublocale disjoint form it (i.e., the localic map $S \hookrightarrow L$ is almost z-dense for every $S \in \mathsf{S}(L)$).*

Remark 4.3.20. Notice that conditions (iii) and (iv) are actually saying that in a P-frame the classes of zero sublocales and cozero sublocales do concide. Furthermore, from Proposition 2.7.8 (3), we have that:

$$\mathsf{CoZS}(L) = \mathsf{ZS}(L) = \{S \in \mathsf{S}(L) \mid S \text{ is a clopen sublocale }\}.$$

For almost P-frames we have the following result ([44, Proposition 7.1.2] and [27, Proposition 3.3]):

Proposition 4.3.21. *The following are equivalent for a frame L:*

(i) *L is an almost P-frame.*

(ii) *For every $C \in \mathsf{CoZS}(L)$, $C = \overline{C}^{\circ}$, (i.e. every cozero sublocale is regular open).*

(iii) *For every $Z \in \mathsf{ZS}(L)$, $Z = \overline{Z^{\circ}}$ (i.e., every zero sublocale is regular closed).*

(iv) *For every dense sublocale S of L, the localic embedding $S \hookrightarrow L$ is z-dense.*

(v) *Every dense z-embedded sublocale of L is C-embedded.*

(vi) *Every cozero dense sublocale of L is C-embedded.*

Remark 4.3.22. By definition, a frame L is almost P and Oz if and only if $L^* = \mathsf{Coz}\, L$.

In classical topology the notion of F-spaces and F'-spaces first appeared in [35] (see also [36]), and quasi F-spaces were introduced in [25]. In [9] the authors studied the respective notions in point-free topology: L is an *F-frame* (a *quasi F-frame*) if every (dense) cozero sublocale is C^*-embedded. Further, a frame L is an *F'-frame* if $a \wedge b = 0$ for $a, b \in \mathsf{Coz}\, L$ implies $a^* \vee b^* = 1$.

Proposition 4.3.23. [9, Proposition 8.4.10] *The following are equivalent for a frame L:*

(i) *L is an F-frame.*

(ii) *Disjoint cozero sublocales are completely separated in L.*

(iii) *For all $a, b \in \mathsf{Coz}\, L$ such that $a \wedge b = 0$ there exist $c, d \in \mathsf{Coz}\, L$ such that $a \wedge c = 0 = b \wedge d$ and $c \vee d = 1$.*

Remarks 4.3.24. (1) It is not hard to see that F'-frames are precisely those frames where the closure of two disjoint cozero sublocales is also disjoint (use (1.4.10)).

(2) Clearly, every F-frame is a quasi F-frame and an F'-frame.

(3) In [62] it is proved that every almost P-frame and every F'-frame is a quasi F-frame.

(4) Any normal F'-frame is an F-frame (this is clear using (1) and Theorem 4.1.1 (ii)). In fact, every mildly normal F'-frame is an F-frame [62, Proposition 3.3.8].

There are some more interesting facts regarding quasi F-, F- and F'-frames in [9, Proposition 8.4.12] and [31, Lemma 4.4]. We formulate them here in terms of sublocales.

Proposition 4.3.25. [9, Proposition 8.4.12] *Let L be an F-frame. Any $C \in \mathsf{CoZS}(L)$ is also an F-frame.*

Proposition 4.3.26. [31, Lemma 4.4] *Let L be an F'-frame, and let S a sublocale of L. If S is z-embedded in L, then S is also an F'-frame.*

The following proposition puts together the characterizations [31, Proposition 4.8], [31, Proposition 4.6] and [62, Corollary 4.2.22], so that we can see the connection between these types of frames.

Proposition 4.3.27. *Let L be a completely regular frame.*

(1) *L is an F-frame if and only if every z-embedded sublocale of L is C^*-embedded in L.*

(2) *L is an F'-frame if and only if every z-embedded sublocale of L is C^*-embedded in its closure.*

(3) *L is a quasi F-frame if and only if every z-embedded dense sublocale of L is C^*-embedded in L.*

The next two results (Propositions 4.1 and 5.4 in [12]) relate Oz- and weak Oz-frames with quasi F-, F- and F'-frames.

Proposition 4.3.28. *The following are equivalent for an Oz-frame L:*

(i) *L is extremally disconnected.*

(ii) *L is an F-frame.*

(iii) *L is an F'-frame.*

(iv) *L is a quasi F-frame.*

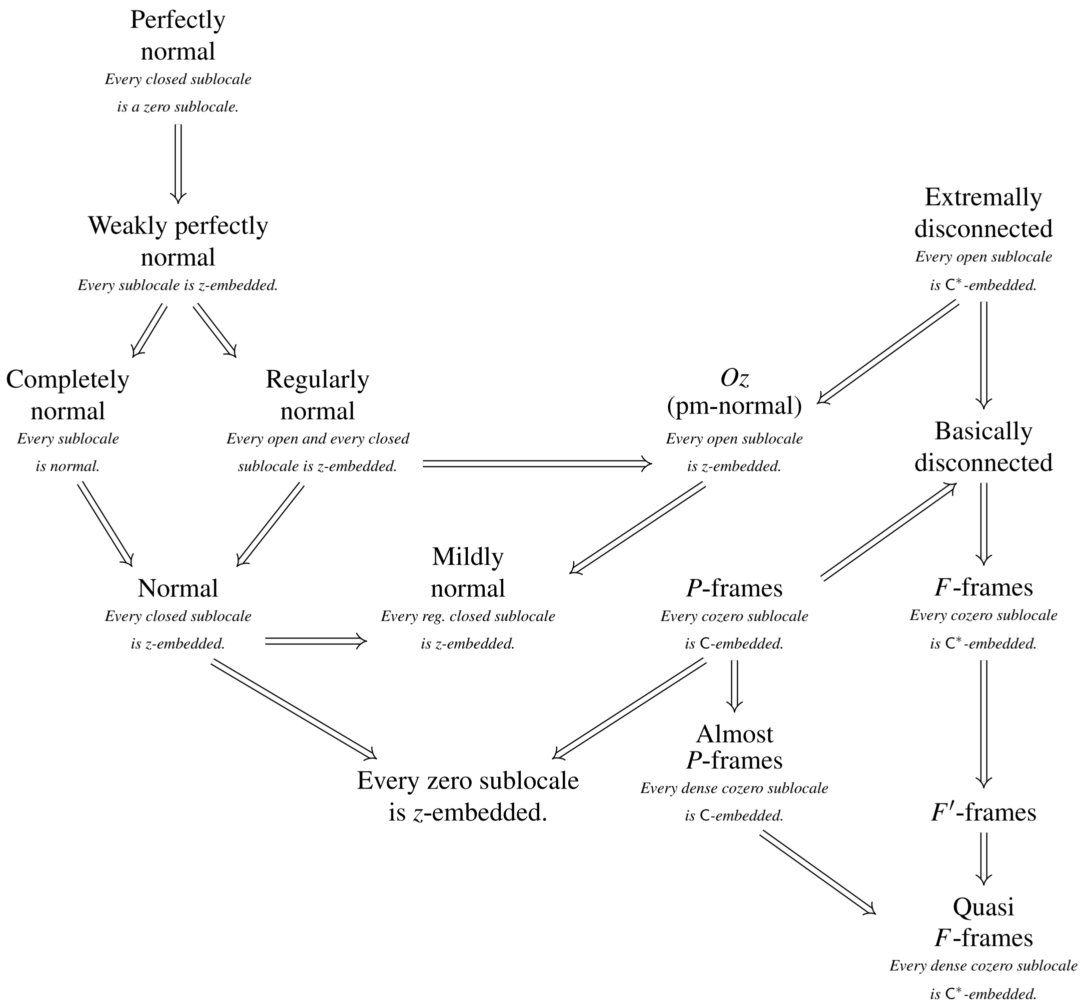
Proposition 4.3.29. *The following are equivalent for a weak Oz-frame L:*

(i) *L is basically disconnected.*

(ii) *L is an F-frame.*

(iii) *L is an F'-frame.*

We conclude this chapter with a diagram depicting all the implications among the several types of frames mentioned along the chapter.



Lastly, we should remark that the locales where every zero sublocale is z-embedded have not yet been characterized (see [4, Problem 10.1]), but this condition is definitely weaker than normality (see [36, 8.20 and 8J]).

Chapter 5

Localic w-, n- and wz- Maps

Because of their relation with the localic maps studied in Section 3.5, in this chapter we study the point-free counterpart of what is classically known as WZ-, WN- and N-maps. Continuous WZ-maps were introduced by Isiwata in [49] (see also [79]). Later on, Woods extended this notion and defined WN- and N-maps in [80]. Dube in [28] already translated the notion of a WN-map to frame homomorphisms (he calls them W-maps, and we keep this name and write it in lowercase for the respective localic maps) and characterized them in a very useful way ([28, Theorem 4.3]). Here we will use this characterization to extend the classical notions of WZ-, WN- and N-maps to localic maps that will be called wz-, w- and n-maps respectively. In fact, we will give a general definition which encompasses all three notions, allowing us to obtain more general results than those found in the literature. We should point out that classically the maps mentioned before are defined in terms of the Čech-Stone compactification. We do not take this approach, and it is yet to be studied in the point-free setting the connection of these maps to compactifications. The content of this Chapter is based on the author's published article [5].

5.1 Localic n- and w-Maps

In this section, we present a study for the localic counterpart of continuous N-maps and WN-maps similar to the one of Woods in [80]. Both notions can be mimicked in the category of locales. The WN-maps were already studied by Dube [28] in terms of frame homomorphisms, referred to as W-maps; here we will call them w-maps.

Definition 5.1.1. A localic map $f\colon L \to M$ is

(a) a *w-map* if whenever an open sublocale $\mathfrak{o}_L(a)$ is completely separated from $f_{-1}[Z]$, for some $Z \in \mathsf{ZS}(M)$, then there exists an open sublocale $\mathfrak{o}_M(b)$ such that $\mathfrak{o}_L(a) \subseteq f_{-1}[\mathfrak{o}_M(b)]$, and $\mathfrak{o}_M(b)$ is completely separated from Z.

(b) an *n-map* if whenever an open sublocale $\mathfrak{o}_L(a)$ is completely separated from $f_{-1}[\mathfrak{c}_M(b)]$, for some $b \in M$, then there exists an open sublocale $\mathfrak{o}_M(d)$ such that $\mathfrak{o}_L(a) \subseteq f_{-1}[\mathfrak{o}_M(d)]$, and $\mathfrak{o}_M(d)$ is completely separated from $\mathfrak{c}_M(b)$.

Clearly, every n-map is a w-map.

We may unify both notions by defining the concept in terms of a selection function $\mathfrak{S}$ on sublocales (recall Definition 4.2.1). We will keep in mind the selections introduced in Section 4.2 (cf. (4.2.1)); they will be our guiding examples for the results and definitions below. But we will not assume our selection to be closed; we will work with arbitrary selections on sublocales.

Definition 5.1.2. We say that a locale L is $\mathfrak{S}$-*normally separated* when every $S \in \mathfrak{S}L$ is completely separated from every closed sublocale of L disjoint from it.
Accordingly, a localic map $f\colon L \to M$ is an $\mathfrak{S}$-*map* if whenever an open sublocale $\mathfrak{o}_L(a)$ is completely separated from $f_{-1}[T]$ for some $T \in \mathfrak{S}M$, there exists an open sublocale $\mathfrak{o}_M(d)$ such that $\mathfrak{o}_L(a) \subseteq f_{-1}[\mathfrak{o}_M(d)]$, and $\mathfrak{o}_M(d)$ is completely separated from T.

Of course, $\mathfrak{S}_1$-normally separated locales are just normal locales, and $\mathfrak{S}_1$-maps are precisely the n-maps.

The definition of a δ-*normally separated* frame was introduced in [30, 3.13]. Rephrasing it in terms of sublocales and localic maps we have that a locale L is δ-*normally separated* if every zero sublocale is completely separated from every closed sublocale disjoint from it (that is, if the embedding $\mathfrak{c}_L(a) \hookrightarrow L$ is almost z-dense for every $a \in L$). Hence, $\mathfrak{S}_{\mathsf{coz}}$-normally separated locales are precisely δ-normally separated locales, while $\mathfrak{S}_{\mathsf{coz}}$-maps are just the w-maps.

Remark 5.1.3. Note that every P-frame is δ-normally separated (recall Proposition 4.3.19 (vii)).

Our first result shows that in $\mathfrak{S}$-normally separated locales M (and only in them), any closed, z-closed, or proper map (Section 1.5) with codomain M is an $\mathfrak{S}$-map.

Theorem 5.1.4. *Let $\mathfrak{S}$ be a selection function on sublocales, and let M be a locale such that every $T \in \mathfrak{S}M$ is complemented. The following are equivalent:*

(i) *M is $\mathfrak{S}$-normally separated.*

(ii) *Every z-closed localic map $f\colon L \to M$ is an $\mathfrak{S}$-map.*

(iii) *Every closed localic map $f\colon L \to M$ is an $\mathfrak{S}$-map.*

(iv) *Every proper localic map $f\colon L \to M$ is an $\mathfrak{S}$-map.*

Proof. (i) $\Longrightarrow$ (ii): Let $f\colon L \to M$ be a z-closed localic map and take $\mathfrak{o}_L(a)$ and $f_{-1}[S]$ with $S \in \mathfrak{S}M$ such that they are completely separated in L. Then there exists a zero sublocale $\mathfrak{c}_L(d)$ such that $\mathfrak{o}_L(a) \subseteq \mathfrak{c}_L(d)$ and $f_{-1}[S] \cap \mathfrak{c}_L(d) = \mathsf{O}$ (which implies that $\mathfrak{c}_L(d) \subseteq f_{-1}[S]^{\#}$). Taking images we obtain

$$f[\mathfrak{o}_L(a)] \subseteq f[\mathfrak{c}_L(d)] \subseteq f[f_{-1}[S]^{\#}] = f[f_{-1}[S^{\#}]] \subseteq S^{\#}.$$

Note that the equality above holds because preimages preserve complements, and that the last inclusion holds due to the adjunction between image and preimage. Hence, since S is complemented, $f[\mathfrak{c}_L(d)] \cap S = \mathsf{O}$ and $f[\mathfrak{c}_L(d)]$ is closed (because f is z-closed). Then, since M is $\mathfrak{S}$-normally separated, $f[\mathfrak{c}_L(d)]$ and S are completely separated in M and, using the fact that $\mathsf{Coz}\, M$ is a normal σ-frame, there is a cozero sublocale $\mathfrak{o}_M(x)$, completely separated from S, such that $f[\mathfrak{c}_L(d)] \subseteq \mathfrak{o}_M(x)$ (recall Remark 2.7.6). In particular, $f[\mathfrak{o}_L(a)] \subseteq \mathfrak{o}_M(x)$. Hence $\mathfrak{o}_L(a) \subseteq f_{-1}[\mathfrak{o}_M(x)]$, as required.
(ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) are trivial since every closed map is z-closed and every proper map is closed.

(iv) $\Longrightarrow$ (i): Let $\mathfrak{c}_M(a)$ and $S \in \mathfrak{S}M$ be disjoint sublocales of M. To prove that M is $\mathfrak{S}$-normally separated we will show that $\mathfrak{c}_M(a)$ and S are completely separated in M. Consider the embedding $j\colon \mathfrak{c}_M(a) \hookrightarrow M$. By assumption, since j is a proper map, it is an $\mathfrak{S}$-map. Consider $\mathfrak{c}_M(a)$ (which is open in $\mathfrak{c}_M(a)$) and $j_{-1}[S]$; they are completely separated in $\mathfrak{c}_M(a)$ because $j_{-1}[S] = S \cap \mathfrak{c}_M(a) = \mathsf{O}$. Thus, there exists $\mathfrak{o}_M(d)$ such that $\mathfrak{c}_M(a) \subseteq j_{-1}[\mathfrak{o}_M(d)]$ and $\mathfrak{o}_M(d)$ is completely separated from S in M. In particular, $\mathfrak{c}_M(a)$ is completely separated from S in M, since $\mathfrak{c}_M(a) = j[\mathfrak{c}_M(a)] \subseteq \mathfrak{o}_M(d)$. $\square$

Hence, we may conclude that any closed, z-closed or proper map is an n-map precisely when the codomain is normal.

Corollary 5.1.5. *The following are equivalent for a locale M:*

(i) *M is normal.*

(ii) *Every z-closed localic map $f\colon L \to M$ is an n-map.*

(iii) *Every closed localic map $f\colon L \to M$ is an n-map.*

(iv) *Every proper localic map $f\colon L \to M$ is an n-map.*

On the other hand, for the selection $\mathfrak{S} = \mathfrak{S}_{\mathsf{coz}}$ we get:

Corollary 5.1.6. *The following are equivalent for a locale M:*

(i) *M is δ-normally separated.*

(ii) *Every z-closed localic map $f\colon L \to M$ is a w-map.*

(iii) *Every closed localic map $f\colon L \to M$ is a w-map.*

(iv) *Every proper localic map $f\colon L \to M$ is a w-map.*

We present one more unifying result that shows that under some assumptions on L and M, n-maps $f\colon L \to M$ are *$\mathfrak{S}$-closed*, that is, $f[S]$ is closed for every $S \in \mathfrak{S}L$. Of course, for $\mathfrak{S} = \mathfrak{S}_1$, $\mathfrak{S}$-closed maps are the closed localic maps, and for $\mathfrak{S} = \mathfrak{S}_{\mathsf{coz}}$, $\mathfrak{S}$-closed maps are the z-closed maps.
This result constitutes a further interesting example of the important role of subfitness in point-free topology (cf. [69]).

Theorem 5.1.7. *Let $f\colon L \to M$ be a localic n-map. If L is $\mathfrak{S}$-normally separated and M is subfit, then f is $\mathfrak{S}$-closed.*

Proof. Let $S \in \mathfrak{S}L$ and consider $\mathfrak{c}_M(b) = \overline{f[S]}$. Clearly, $f[S] \subseteq \mathfrak{c}_M(b)$. To prove that f is $\mathfrak{S}$-closed it suffices to show that $\mathfrak{c}_M(b) \subseteq f[S]$. Suppose $\mathfrak{c}_M(b) \nsubseteq f[S]$. Then $\mathfrak{o}_M(b) \vee f[\mathfrak{c}_L(a)] \neq M$, by (1.4.2), and, by a well-known characterization of subfit locales (Proposition 1.4.2), there is a closed sublocale $\mathfrak{c}_M(d) \neq \mathsf{O}$ such that

$$(\mathfrak{o}_M(b) \vee f[S]) \cap \mathfrak{c}_M(d) = \mathsf{O}. \tag{5.1.1}$$

Then $f[S] \subseteq \mathfrak{o}_M(b) \vee f[S] \subseteq \mathfrak{o}_M(d)$. Taking preimages we obtain

$$S \subseteq f_{-1}[f[S]] \subseteq f_{-1}[\mathfrak{o}_M(d)] = \mathfrak{o}_L(f^*(d)),$$

from which it follows that $S \cap \mathfrak{c}_L(f^*(d)) = \mathsf{O}$. Since L is $\mathfrak{S}$-normally separated, S and $\mathfrak{c}_L(f^*(d))$ are completely separated in L. Therefore there are $Z_1, Z_2 \in \mathsf{ZS}(L)$ such that

$$S \subseteq Z_1, \quad \mathfrak{c}_L(f^*(d)) \subseteq Z_2 \quad \text{and} \quad Z_1 \cap Z_2 = \mathsf{O}.$$

In fact, since Coz L is a normal σ-frame, there is a cozero sublocale $\mathfrak{o}_L(y)$ such that $S \subseteq Z_1 \subseteq \mathfrak{o}_L(y)$, and $\mathfrak{o}_L(y)$ is completely separated from Z_2 (recall Remark 2.7.6). In particular, $\mathfrak{o}_L(y)$ is completely separated from $\mathfrak{c}_L(f^*(d)) = f_{-1}[\mathfrak{c}_M(d)]$. Since f is an n-map, there is $\mathfrak{o}_M(z)$ such that

$$S \subseteq Z_1 \subseteq \mathfrak{o}_L(y) \subseteq f_{-1}[\mathfrak{o}_M(z)] \tag{5.1.2}$$

and $\mathfrak{o}_M(z)$ is completely separated from $\mathfrak{c}_M(d)$ in M. Taking images in (5.1.2)) we deduce that $f[S] \subseteq f[f_{-1}[\mathfrak{o}_M(z)]] \subseteq \mathfrak{o}_M(z)$. So, in fact, $f[S]$ is completely separated from $\mathfrak{c}_M(d)$ in M. In particular, $\overline{f[S]}$ is completely separated from $\mathfrak{c}_M(d)$. Thus $\mathfrak{c}_M(b) = \overline{f[S]} \subseteq \mathfrak{o}_M(d)$, and it follows from (5.1.1) that $\mathfrak{o}_M(b) \subseteq \mathfrak{o}_M(b) \vee f[S] \subseteq \mathfrak{o}_M(d)$. Consequently, $M = \mathfrak{c}_M(b) \vee \mathfrak{o}_M(b) \subseteq \mathfrak{o}_M(d)$, which contradicts the fact that $\mathfrak{c}_M(d) \neq \mathsf{O}$. Hence, $\overline{f[S]} = \mathfrak{c}_M(b) \subseteq f[S]$, as required. □

In particular, we have:

Corollary 5.1.8. *Let $f: L \to M$ be a localic n-map with M a subfit locale.*

(a) *If L is normal, then f is closed.*

(b) *If L is $\mathcal{S}$-normally separated, then f is z-closed.*

5.2 Localic wz-Maps

In this section we study wz-maps which are the point-free counterparts of continuous WZ-maps ([81]). This notion in spaces is very much dependent on the points of a space. We extend it to locales by selecting one-point sublocales in Definition 5.1.2. We will obtain results similar to those that appear classically, and we will study how this selection interacts with separation axioms such as normality and complete regularity.

Let $\mathfrak{S}_p$ denote the sublocale selection given by

$$\mathfrak{S}_p L = \{\mathfrak{b}(p) \mid p \in \mathsf{Pt}(L)\}.$$

$\mathfrak{S}_p$-maps will be called *wz-maps*.

We show now that any completely regular locale is $\mathfrak{S}_p$-normally separated. To simplify terminology, we say that a point p is *completely separated* from a sublocale T whenever the sublocales $\mathfrak{b}(p)$ and T are completely separated.

Proposition 5.2.1. *In a locale L, every point is completely separated from every zero sublocale disjoint from it.*

Proof. Let $\mathfrak{c}(a) \in \mathsf{ZS}(L)$ such that $\mathfrak{b}(p) \cap \mathfrak{c}(a) = \mathsf{O}$. Then $\mathfrak{b}(p) \subseteq \mathfrak{o}(a)$. By Proposition 2.7.8 (2) and Remark 2.7.9,

$$\mathfrak{o}(a) = \bigvee_{n\in\mathbb{N}} \mathfrak{o}(b_n) = \bigvee_{n\in\mathbb{N}} \mathfrak{c}(c_n)$$

where $\mathfrak{o}(b_n) \subseteq \mathfrak{c}(c_n)$ and $c_n \in \mathsf{Coz}\, L$ for every $n \in \mathbb{N}$. Since $\mathfrak{b}(p) \subseteq \mathfrak{o}\left(\bigvee_{n\in\mathbb{N}} b_n\right)$, we have

$$p = \Big(\bigvee_{n\in\mathbb{N}} b_n\Big) \to p = \bigwedge_{n\in\mathbb{N}} (b_n \to p)\,.$$

Since p is prime, by (1.4.16), $b_n \to p = 1$ or $b_n \to p = p$, so there is a $k \in \mathbb{N}$ such that $b_k \to p = p$. Therefore, $\mathfrak{b}(p) \subseteq \mathfrak{o}(b_k) \subseteq \mathfrak{c}(c_k) \subseteq \mathfrak{o}(a)$, as required. □

In other words,

the localic embedding $\mathfrak{b}(p) \hookrightarrow L$ is almost z-dense for every $p \in \mathsf{Pt}(L)$.

Corollary 5.2.2. *In a completely regular locale L, every point is completely separated from every closed sublocale disjoint from it. That is, any completely regular locale is $\mathfrak{S}_p$-normally separated.*[1]

Proof. Let $\mathfrak{c}(a)$ be a sublocale of L such that $\mathfrak{b}(p) \cap \mathfrak{c}(a) = \mathsf{O}$. Since L is completely regular, $\mathfrak{c}(a) = \bigcap\{Z \in \mathsf{ZS}(L) \mid \mathfrak{c}(a) \subseteq Z\}$ (Corollary 2.7.7 (ii)). Consequently, since $p \notin \mathfrak{c}(a)$, there is a $Z \in \mathsf{ZS}(L)$ such that $\mathfrak{c}(a) \subseteq Z$ and $Z \cap \mathfrak{b}(p) = \mathsf{O}$. By Proposition 5.2.1, $\mathfrak{c}(a)$ and $\mathfrak{b}(p)$ are completely separated. □

Remark 5.2.3. Hence, in any completely regular locale L, for each $p \in \mathsf{Pt}(L)$ and every $a \in L$ such that $a \nleq p$, there is a continuous real-valued function $f\colon \mathfrak{L}(\mathbb{R}) \to L$ such that $\mathbf{0} \le f \le \mathbf{1}$, $\mathfrak{b}(p) \subseteq f(0,\text{—})$ and $\mathfrak{c}(a) \subseteq f(\text{—},1)$.

Recall covered prime elements from Section 1.2. A one-point sublocale $\mathfrak{b}(p)$ is complemented if and only if p is a covered prime ([34, Proposition 10.2]). Moreover, in regular locales every prime is covered ([34, Proposition 10.3]). Hence, $\mathfrak{S}_p$ satisfies the assumptions of Theorem 5.1.4 whenever the codomain M is regular and we have:

Corollary 5.2.4. *The following assertions are equivalent for a regular locale M:*

(i) *M is $\mathfrak{S}_p$-normally separated.*

(ii) *Every z-closed localic map $f\colon L \to M$ is a wz-map.*

(iii) *Every closed localic map $f\colon L \to M$ is a wz-map.*

(iv) *Every proper localic map $f\colon L \to M$ is a wz-map.*

Moreover, applying Theorem 5.1.7 to the selection $\mathfrak{S}_p$ we obtain:

Corollary 5.2.5. *Let $f\colon L \to M$ be a localic n-map with M a subfit locale. If L is $\mathfrak{S}_p$-normally separated, then f is a wz-map.*

Let us investigate also the more general (boolean) selection $\mathfrak{S}_\mathfrak{b}$ defined by

$$\mathfrak{S}_\mathfrak{b} L = \{\mathfrak{b}(x) \mid x \in L\}.$$

Proposition 5.2.6. *Each $\mathfrak{S}_\mathfrak{b}$-normally separated locale is normal and subfit.*

[1] The converse does not hold since there are pointless locales that are not completely regular.

Proof. Let L be a $\mathfrak{S}_{\mathfrak{b}}$-normally separated locale. If $\mathfrak{c}(a)\cap\mathfrak{c}(b)=\mathsf{O}$ then $\mathfrak{b}(a)\cap\mathfrak{c}(b)\subseteq\mathfrak{c}(a)\cap\mathfrak{c}(b)=\mathsf{O}$. By assumption, $\mathfrak{b}(a)$ and $\mathfrak{c}(b)$ are completely separated. By (1.4.15),

$$\overline{\mathfrak{b}(a)}=\mathfrak{c}\Big(\bigwedge\mathfrak{b}(a)\Big)=\mathfrak{c}(a).$$

Hence $\mathfrak{c}(a)$ and $\mathfrak{c}(b)$ are also completely separated (recall Remarks 2.7.4 (4)). This means that every pair of disjoint closed sublocales are completely separated, which characterizes normality.

Regarding subfitness, consider $a,b\in L$ such that $a\not\leq b$ (equivalently, $a\to b\neq 1$). Given $d=a\to b\geq b$, consider $\mathfrak{b}(d)$ and $\mathfrak{c}(a)$. Notice that if $x\in\mathfrak{b}(d)\cap\mathfrak{c}(a)$, then $x\geq a$ and $x=(x\to d)\to d$. Hence

$$\begin{aligned} a\leq(x\to d)\to d &\iff (x\to d)\wedge a\leq d\\ &\iff x\to d\leq a\to d=a\to(a\to b)=a\to b=d\\ &\iff 1\leq(x\to d)\to d=x.\end{aligned}$$

Thus, $\mathfrak{b}(d)\cap\mathfrak{c}(a)=\mathsf{O}$. Then, since L is $\mathfrak{S}_{\mathfrak{b}}$-normally separated, $\mathfrak{b}(d)$ and $\mathfrak{c}(a)$ are completely separated: there exist $x,y\in\mathsf{Coz}\,L$ such that

$$\mathfrak{b}(d)\subseteq\mathfrak{c}(x),\quad \mathfrak{c}(a)\subseteq\mathfrak{c}(y)\quad\text{and}\quad \mathfrak{c}(x)\cap\mathfrak{c}(y)=\mathsf{O}.$$

This means that $\mathfrak{c}(a)\subseteq\mathfrak{o}(x)$, that is, $a\vee x=1$. Moreover, $x\leq d$ and $b\leq d$, hence $x\vee b\leq d\neq 1$ and L is subfit. □

Summing up, since each subfit normal locale is completely regular (Proposition 1.3.4) we have:

$$\mathfrak{S}_{\mathfrak{b}}\text{-norm. sep.}\Longrightarrow\text{normal + subfit}\Longrightarrow\text{c. regular}\Longrightarrow\mathfrak{S}_{p}\text{-norm. sep.}$$

We end this section with an example of z-embedded sublocales that are also C-embedded.

Proposition 5.2.7. *Let $f\colon L\to M$ be a z-closed localic map with L and M completely regular locales. For any $p\in\mathsf{Pt}(M)$, if $f_{-1}[\mathfrak{b}(p)]$ is z-embedded, then it is C-embedded.*

Proof. Applying Corollary 3.4.4 to the localic embedding $f_{-1}[\mathfrak{b}(p)]\hookrightarrow L$, it suffices to show that $f_{-1}[\mathfrak{b}(p)]$ is completely separated from every zero sublocale disjoint from it. So consider $\mathfrak{c}_L(a)\in\mathsf{ZS}(L)$ such that $f_{-1}[\mathfrak{b}(p)]\cap\mathfrak{c}_L(a)=\mathsf{O}$. By the regularity of M, $\mathfrak{b}(p)$ is complemented, hence $f_{-1}[\mathfrak{b}(p)]$ is also complemented (since preimages preserve complements). Then

$$f_{-1}[\mathfrak{b}(p)]\cap\mathfrak{c}_L(a)=\mathfrak{c}_L(a)\smallsetminus f_{-1}[\mathfrak{b}(p)]^{\#}=\mathfrak{c}_L(a)\smallsetminus f_{-1}[\mathfrak{b}(p)^{\#}]$$

because $f_{-1}[-]$ is a coframe homomorphism. Furthermore, $f[-]$ is a colocalic map hence

$$\begin{aligned}\mathsf{O}=f[f_{-1}[\mathfrak{b}(p)]\cap\mathfrak{c}_L(a)]&=f[\mathfrak{c}_L(a)\smallsetminus f_{-1}[\mathfrak{b}(p)^{\#}]]=f[\mathfrak{c}_L(a)]\smallsetminus\mathfrak{b}(p)^{\#}\\ &=f[\mathfrak{c}_L(a)]\cap\mathfrak{b}(p)=\mathfrak{c}_M(f(a))\cap\mathfrak{b}(p)\end{aligned}$$

(where the last equality follows from f being z-closed). By Corollary 5.2.2, $\mathfrak{b}(p)$ and $\mathfrak{c}_M(f(a))$ are completely separated in M. Then $f_{-1}[\mathfrak{b}(p)]$ and $f_{-1}[\mathfrak{c}_M(f(a))]$ are completely separated in L. Since $\mathfrak{c}_L(a)\subseteq f_{-1}[\mathfrak{c}_M(f(a))]$, this completes the proof. □

5.3 Examples

In this final section we will present a class of examples of w- and n-maps, inspired by an example in classical topology from [81, Section 2].

We will first build a certain frame P_a. Let L be a frame and $a \in L$. Consider the onto frame homomorphism $p_a \colon L \to \mathfrak{c}(a)$ given by $x \mapsto x \vee a$. Let $\mathbf{2}$ denote the two-element frame $\{0,1\}$. There is a unique frame homomorphism $\iota \colon \mathbf{2} \to \mathfrak{c}(a)$. The frame P_a is given by the pullback

$$\begin{array}{ccc} P_a & \xrightarrow{k} & \mathbf{2} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \iota} \\ L & \xrightarrow{p_a} & \mathfrak{c}(a) \end{array}$$

of morphisms ι and p_a in the category of frames. Since the pullback is the equalizer of

$$L \times \mathbf{2} \xrightarrow{p_L} L \xrightarrow{p_a} \mathfrak{c}(a) \quad \text{and} \quad L \times \mathbf{2} \xrightarrow{p_2} \mathbf{2} \xrightarrow{\iota} \mathfrak{c}(a)$$

(where p_L and p_2 are the product projections), P_a is explicitly the subframe of $L \times \mathbf{2}$ given by

$$P_a = \{(x,0) \in L \times \mathbf{2} \mid x \leq a\} \cup \{(x,1) \in L \times \mathbf{2} \mid x \vee a = 1\},$$

and $h = p_L i$ and $k = p_2 i$ (i being the subframe inclusion $P_a \subseteq L \times \mathbf{2}$).

Proposition 5.3.1. *The cozero elements of P_a are precisely the $(x,0) \in P_a$ with $x \in \mathrm{Coz}\, L$, and the $(w,1) \in P_a$ with $w \in \mathrm{Coz}\, L$ such that $\mathfrak{c}(w)$ is completely separated from $\mathfrak{c}(a)$ in L.*

Proof. Let $(x,y) \in \mathrm{Coz}\, P_a$, then there exists a frame homomorphism $f \colon \mathfrak{L}(\mathbb{R}) \to P_a$ with $\mathbf{0} \leq f \leq \mathbf{1}$ such that $f(0,\text{—}) = (x,y)$. Consider the composite $hf \colon \mathfrak{L}(\mathbb{R}) \to L$, then $hf(0,\text{—}) = h(x,y) = x$. Thus, $x \in \mathrm{Coz}\, L$. For the case when $y = 0$, we are done. Now, if $y = 1$ we have that $x \vee a = 1$. We claim that $hf(0,\text{—}) \leq x$ and $hf(\text{—},p) \leq a$ for some $p > 0$ (which implies that $\mathfrak{c}(a)$ and $\mathfrak{c}(x)$ are completely separated in L). For $a = 1$ this trivially holds, so assume $a \neq 1$ and suppose $hf(p,\text{—})^* \nleq a$ for every $p > 0$. Then $p_a hf(p,\text{—}) = hf(p,\text{—}) \vee a \neq 1$. This implies that $p_a hf(p,\text{—}) = hf(p,\text{—}) = a$, since $p_a h = \iota k$. Then

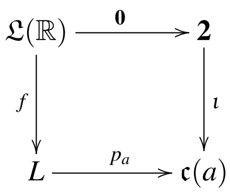
$$x = hf(0,\text{—}) = \bigvee_{p>0} hf(p,\text{—}) = a,$$

which contradicts the fact that $x \vee a = 1$. Thus, there is some $p > 0$ such that $hf(p,\text{—})^* \leq a$. Take a rational q such that $p > q > 0$. Then $hf(\text{—},q) \leq hf(p,\text{—})^* \leq a$.

For the converse we have two cases:

(Case 1): $(x,0) \in P_a$ and $x \in \mathrm{Coz}\, L$.
Then $x \leq a$, and there exists a frame homomorphism $f \colon \mathfrak{L}(\mathbb{R}) \to L$ such that $f(0,\text{—}) \vee f(\text{—},0) = x$.

Consider the constant function $\mathbf{0}\colon \mathfrak{L}(\mathbb{R}) \to \mathbf{2}$ (Example 2.1.2(1)). The following diagram commutes:



Indeed, if $p < 0 < q$ we have $\mathbf{0}(p,q) = 1$, so $\iota\mathbf{0}(p,q) = \iota(1) = 1$. Further,

$$1 = f(p,q) \vee f(0,\text{—}) \vee f(\text{—},0) = f(p,q) \vee x$$

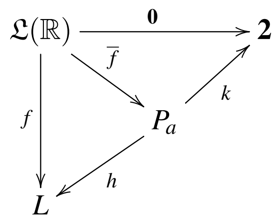
which implies

$$1 = p_a(1) = p_a(f(p,q) \vee x) = p_a(f(p,q)) \vee p_a(x) = p_a(f(p,q)) \vee a = p_a(f(p,q)).$$

If $p < q \le 0$ or $0 \le p < q$, then $\iota(\mathbf{0}(p,q)) = \iota(0) = a$. Moreover, $(p,q) \le (0,\text{—}) \vee (\text{—},0)$ so

$$p_a f(p,q) \le p_a(f(0,\text{—}) \vee f(\text{—},0)) = p_a(x) = a.$$

Thus, by the universal property of the pullback, there exists a frame homomorphism $\overline{f}\colon \mathfrak{L}(\mathbb{R}) \to P_a$ such that the diagram

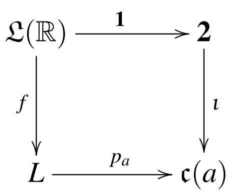


commutes. Consequently,

$$h(\overline{f}(0,\text{—}) \vee \overline{f}(\text{—},0)) = f(0,\text{—}) \vee f(\text{—},0) = x \quad \text{and} \quad k(\overline{f}(0,\text{—}) \vee \overline{f}(\text{—},0)) = \mathbf{0}((0,\text{—}) \vee (\text{—},0)) = 0.$$

Thus, $\overline{f}(0,\text{—}) \vee \overline{f}(\text{—},0) = (x,0)$ meaning $(x,0) \in \mathsf{Coz}\, P_a$.

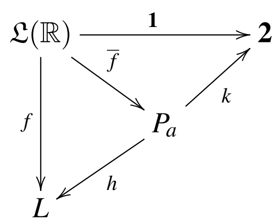
(Case 2): $(x,1) \in P_a$ with $x \in \mathsf{Coz}\, L$ such that $\mathfrak{c}(x)$ is completely separated from $\mathfrak{c}(a)$ in L.
In particular we have $a \vee x = 1$. As a consequence of the complete separation there is a frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to L$ with $\mathbf{0} \le f \le \mathbf{1}$ such that $f(0,\text{—}) = x$ and $f(\text{—},1) \le a$. Consider the diagram



where $\mathbf{1}\colon \mathfrak{L}(\mathbb{R}) \to \mathbf{2}$ is the constant function (Example 2.1.2(1)). For $p \le 1$ we have $f(\text{—},p) \le f(\text{—},1) \le a$ so $p_a f(\text{—},p) = a$, and $\iota\mathbf{1}(\text{—},p) = \iota(0) = a$. For any $p > 1$, $f(\text{—},p) = 1$ (because $f \le \mathbf{1}$ and $\mathbf{1}(\text{—},p) = 1$). Thus, $p_a f(\text{—},p) = 1 = \iota\mathbf{1}(\text{—},p)$. For any $p \ge 1$, $\iota\mathbf{1}(p,\text{—}) = \iota(0) = a$. Further, since $f \le \mathbf{1}$, $f(p,\text{—}) \le \mathbf{1}(p,\text{—}) = 0$. Thus, $p_a f(p,\text{—}) = a$. Now, for $p < 1$, $\iota\mathbf{1}(p,\text{—}) = \iota(1) = 1$. If $p \le 0$, then $x = f(0,—) \le f(p,—)$, meaning $1 = x \vee a \le p_a f(p,—)$. If $0 < p < 1$, consider $k \in \mathbb{Q}$ such that $p < k < 1$, then $f(p,—) \vee f(—,k) = 1$. Consequently,

$$1 = p_a f(p,—) \vee p_a f(—,k) \le p_a f(p,—) \vee p_a(a) = p_a f(p,—) \vee a = p_a f(p,—)$$

where the inequality holds because $f(—,k) \le f(—,1) \le a$. Thus, $p_a f(—,p) = \iota\mathbf{1}(—,p)$ and $p_a f(p,—) = \iota\mathbf{1}(p,—)$ for every $p \in \mathbb{Q}$, which shows that the diagram above is commutative. Hence, by the universal property of the pullback, there exists a frame homomorphism $\overline{f} \colon \mathfrak{L}(\mathbb{R}) \to P_a$ such that



commutes. We claim $\overline{f}(0,—) \vee \overline{f}(—,0) = (x,1)$. Indeed,

$$k\overline{f}((0,—) \vee (—,0)) = \mathbf{1}(0,—) \vee \mathbf{1}(—,0) = 1 \vee 0 = 1$$

$$h\overline{f}((0,—) \vee (—,0)) = f(0,—) \vee f(—,0) = x \vee 0 = x$$

where $f(—,0) = 0$, since $f \ge \mathbf{0}$. In conclusion, $(x,1) \in \mathrm{Coz}\, P_a$, as required. □

Clearly, from Proposition 5.3.1 we know that if $\mathfrak{c}(a)$ is completely separated from every zero sublocale disjoint from it, then

$$\mathrm{Coz}\, P_a = \{(x,y) \in P_a \mid x \in \mathrm{Coz}\, L\}.$$

Consequently, we have:

Corollary 5.3.2. *A frame L is δ-normally separated if and only if* $\mathrm{Coz}\, P_a = \{(x,y) \in P_a \mid x \in \mathrm{Coz}\, L\}$ *for every $a \in L$.*

Regarding separation properties inherited from L to P_a we have the following results.

Proposition 5.3.3. *Let $a \in L$.*

(1) *If L is subfit then P_a is subfit.*

(2) *If L is normal then P_a is normal.*

Proof. (1): Let $(x,y),(w,z) \in P_a$ with $(x,y) \not\le (w,z)$.

(Case 1): $y = 1$, $x \vee a = 1$, $z = 0$ and $w \le a$.
Take $(a,0) \in P_a$. Clearly $(a,0) \vee (x,y) = (1,1)$ but $(a,0) \vee (w,z) = (a,0) \ne (1,1)$.

(Case 2): $y = z = 1$ and $x \vee a = 1 = w \vee a$.
Since $x \not\le w$, there exists $v \in L$ such that $x \vee v = 1 \ne w \vee v$. Take $(v \wedge a, 0) \in P_a$. Clearly, $(x,y) \vee (v \wedge a,0) = (1,1)$, but $(w,z) \vee (v \wedge a,0) \le (w \vee v,1) \ne (1,1)$.

(Case 3): $y = 0$, $x \leq a$, $z = 1$ and $a \vee w = 1$.
Since $x \not\leq w$, there exists $v \in L$ such that $x \vee v = 1 \neq w \vee v$. Then $(v,1) \in P_a$ since $v \vee a \geq v \vee x = 1$. Clearly, $(x,y) \vee (v,1) = (1,1)$, but $(w,z) \vee (v,1) = (w \vee v, 1) \neq (1,1)$.

(Case 4): $y = 0$, $x \leq a$, $z = 0$ and $w \leq a$.
Since $x \not\leq w$, there exists $v \in L$ such that $x \vee v = 1 \neq w \vee v$. Then $(v,1) \in P_a$ since $v \vee a \geq v \vee x = 1$. Clearly, $(x,y) \vee (v,1) = (1,1)$, but $(w,z) \vee (v,1) = (w \vee v, 1) \neq (1,1)$.

(2): Let $(x,y),(w,z) \in P_a$ with $(x,y) \vee (w,z) = (1,1)$.

(Case 1): $y = 0$, $x \leq a$, $z = 1$ and $a \vee w = 1$.
By hypothesis, there exist $u, v \in L$ such that $x \vee u = 1 = w \vee v$ and $u \wedge v = 0$. Then

$$(u,1) \in P_a \text{ and } (x,0) \vee (u,1) = (1,1)$$

(since $u \vee a \geq u \vee x = 1$), and

$$(v \wedge a, 0) \in P_a \text{ and } (w,1) \vee (v \wedge a, 0) = (1,1)$$

(since $w \vee (v \wedge a) = (w \vee v) \wedge (w \vee a) = 1$). Moreover, $(u,1) \wedge (v \wedge a, 0) = (0,0)$ since $u \wedge v = 0$.

(Case 2): $y = z = 1$ and $x \vee a = 1 = w \vee a$.
By hypothesis, there exist $u, v \in L$ such that $x \vee u = 1 = w \vee v$ and $u \wedge v = 0$. Then

$$(u \wedge a, 0) \in P_a \text{ and } (x,1) \vee (u \wedge a, 0) = (1,1)$$

(since $x \vee (u \wedge a) = (x \vee u) \wedge (x \vee a) = 1$), and

$$(v \wedge a, 0) \in P_a \text{ and } (w,1) \vee (v \wedge a, 0) = (1,1)$$

(since $w \vee (v \wedge a) = (w \vee v) \wedge (w \vee a) = 1$). Moreover, $(u \wedge a, 0) \wedge (v \wedge a, 0) = (0,0)$ since $u \wedge v = 0$. ☐

Since subfitness together with normality yield complete regularity (Proposition 1.3.4), we get:

Corollary 5.3.4. *If L is normal and subfit, then P_a is completely regular for every $a \in L$.*

Proposition 5.3.5. *If P_a is completely regular then $\mathfrak{c}(a)$ is completely separated in L from every closed sublocale disjoint from it.*

Proof. If P_a is completely regular then, for every $(x,y) \in P_a$

$$(x,y) = \bigvee \{(u,v) \in \mathsf{Coz}\, P_a \mid (u,v) \leq (x,y)\}.$$

If $y = 1$ then $x \vee a = 1$. In this case notice that in the join above there must be a cozero element of the form $(z,1)$. Otherwise,

$$(x,y) = \bigvee \{(u,0) \in \mathsf{Coz}\, P_a \mid u \leq x \text{ and } u \leq a\},$$

and so $(x,1) \leq (a,0)$, a contradiction. This means that there is a cozero element in P_a, say $(z,1)$, such that $(z,1) \leq (x,1)$; in particular, $z \leq x$. Furthermore, since $(z,1) \in \mathsf{Coz}\, P_a$ we know that $\mathfrak{c}(z)$ and $\mathfrak{c}(a)$ are completely separated in L. Consequently, $\mathfrak{c}(x)$ is completely separated from $\mathfrak{c}(a)$ in L. □

Proposition 5.3.6. *If L is completely regular and $\mathfrak{c}(a)$ is completely separated from every zero sublocale of L disjoint from it, then P_a is completely regular.*

Proof. Let $\mathfrak{c}(a)$ be completely separated from every zero sublocale disjoint from it. By Proposition 5.3.1, we know that $\mathsf{Coz}\, P_a = \{(x,y) \in P_a \mid x \in \mathsf{Coz}\, L\}$. We will show that P_a is completely regular by proving that it is join-generated by its cozero elements. Let $(x,0) \in P_a$. Then $x \leq a$ and since L is completely regular

$$x = \bigvee \{c \in \mathsf{Coz}\, L \mid c \leq x\}.$$

In particular, $c \leq a$ for every cozero element in the join above. This means that $(c,0) \in P_a$. Hence,

$$(x,0) = \bigvee \{(c,0) \in \mathsf{Coz}\, P_a \mid (c,0) \leq (x,0)\}.$$

Now, consider $(x,1) \in P_a$, that is, $x \vee a = 1$. Since L is completely regular

$$x = \bigvee \{c \in \mathsf{Coz}\, L \mid c \leq x\}.$$

In fact, we have the following

$$x = \bigvee \{c \in \mathsf{Coz}\, L \mid c \leq x\} \leq \bigvee \{c \in \mathsf{Coz}\, L \mid c \leq x \text{ and } (c \leq a \text{ or } c \vee a = 1)\} \leq x$$

where the first inequality holds because if $\mathfrak{c}(x)$ and $\mathfrak{c}(a)$ are completely separated. Indeed, by Remark 2.7.4 (2), there is $w, v \in \mathsf{Coz}\, L$ such that $\mathfrak{c}(a) \subseteq \mathfrak{c}(w)$, $\mathfrak{c}(x) \subseteq \mathfrak{c}(v)$ and $\mathfrak{c}(w) \cap \mathfrak{c}(v) = \mathsf{O}$. So if $c \in \mathsf{Coz}\, L$ such that $c \leq x$, $c \vee a \neq 1$ and $c \nleq a$, we take $c \vee v \in \mathsf{Coz}\, L$ and we have $c \vee v \leq x$, $c \leq c \vee v$, and $c \vee v \vee a = 1$. Now, since $\mathsf{Coz}\, P_a = \{(x,y) \in P_a \mid x \in \mathsf{Coz}\, L\}$ we get

$$(x,1) = \bigvee \{(c,d) \in \mathsf{Coz}\, P_a \mid (c,d) \leq (x,1)\}.$$ □

Corollary 5.3.7. *A completely regular frame L is normal if and only if P_a is completely regular for every $a \in L$.*

Proof. Let L be a completely regular frame. If P_a is completely regular for every $a \in L$, then by Proposition 5.3.5, any pair of disjoint closed sublocales is completely separated in L. Thus, L is normal. On the other hand, if L is normal, then every pair of disjoint closed sublocales is completely separated (Theorem 4.1.1 (ii)). By Proposition 5.3.6, P_a is completely regular for every $a \in L$. □

Finally, consider the localic map $h_* \colon L \to P_a$, right adjoint to h. To simplifiy notation we denote h_* by f.

Proposition 5.3.8. *$f \colon L \to P_a$ is always a w-map.*

Proof. Let $u \in L$ and $(x,y) \in \mathsf{Coz}\, P_a$ such that $\mathfrak{o}_L(u)$ and $f_{-1}[\mathfrak{c}_{P_a}(x,y)]$ are completely separated in L. Then there exists $v \in \mathsf{Coz}\, L$ such that

$$\mathfrak{o}_L(u) \subseteq \mathfrak{c}_L(v) \quad \text{and} \quad \mathfrak{c}_L(v) \cap f_{-1}[\mathfrak{c}_{P_a}(x,y)] = \mathsf{O}. \tag{5.3.1}$$

Note that $f_{-1}[\mathfrak{c}_{P_a}(x,y)] = \mathfrak{c}_L(h(x,y)) = \mathfrak{c}_L(x)$, and by Proposition 5.3.1, $x \in \mathsf{Coz}\, L$. By construction of P_a we have two cases:

(Case 1): $y = 0$. Thus, $x = h(x,y) \leq a$.
Furthermore, from (5.3.1), $\mathfrak{c}_L(v)$ and $\mathfrak{c}_L(x)$ are disjoint zero sublocales; thus, they are completely separated in L. Since $\mathfrak{c}_L(a) \subseteq \mathfrak{c}_L(x)$, $\mathfrak{c}_L(a)$ and $\mathfrak{c}_L(v)$ are completely separated in L. By Proposition 5.3.1, $(v,1) \in \mathsf{Coz}\, P_a$. Moreover, $(x,y) \vee (v,1) = (x \vee v, 0 \vee 1) = (1,1)$ meaning $\mathfrak{c}_{P_a}(x,y)$ and $\mathfrak{c}_{P_a}(v,1)$ are completely separated in P_a. By Remark 2.7.6, there is a cozero sublocale $\mathfrak{o}_{P_a}(x',y')$ completely separated from $\mathfrak{c}_{P_a}(x,y)$ in P_a such that $\mathfrak{c}_{P_a}(v,1) \subseteq \mathfrak{o}_{P_a}(x',y')$. Hence,

$$\mathfrak{o}_L(u) \subseteq \mathfrak{c}_L(v) = \mathfrak{c}_L(h(v,1)) = f_{-1}[\mathfrak{c}_{P_a}(v,1)] \subseteq f_{-1}[\mathfrak{o}_{P_a}(x',y')]$$

as required.

(Case 2): $y = 1$. Thus, $x \vee a = h(x,y) \vee a = 1$.
From Proposition 5.3.1, we know that $\mathfrak{c}_L(x)$ and $\mathfrak{c}_L(a)$ are completely separated in L. Hence, there is $z \in \mathsf{Coz}\, L$ such that

$$\mathfrak{c}_L(a) \subseteq \mathfrak{c}_L(z) \quad \text{and} \quad \mathfrak{c}_L(z) \cap \mathfrak{c}_L(x) = \mathsf{O}. \tag{5.3.2}$$

Consider the zero sublocale $\mathfrak{c}_L(z \wedge v)$ of L. By (5.3.2), $z \wedge v \leq a$ so $(z \wedge v, 0) \in \mathsf{Coz}\, L$ (by Proposition 5.3.1). Moreover,

$$(z \wedge v, 0) \vee (x,y) = ((z \wedge v) \vee x, 0 \vee 1) = ((z \vee x) \wedge (v \vee x), 1) = (1,1)$$

where the last equality holds from (5.3.2) and (5.3.1). This means that $\mathfrak{c}_{P_a}(z \wedge v, 0)$ and $\mathfrak{c}_{P_a}(x,y)$ are disjoint zero sublocales of P_a. By Remark 2.7.6, there is a cozero sublocale $\mathfrak{o}_{P_a}(x',y')$ of P_a completely separated from $\mathfrak{c}_{P_a}(x,y)$ such that $\mathfrak{c}_{P_a}(z \wedge v, 0) \subseteq \mathfrak{o}_{P_a}(x',y')$. Finally, taking preimages we get

$$\mathfrak{o}_L(u) \subseteq \mathfrak{c}_L(v) \subseteq \mathfrak{c}_L(z \wedge v) = \mathfrak{c}_L(h(z \wedge v, 0)) = f_{-1}[\mathfrak{c}_{P_a}(z \wedge v, 0)] \subseteq f_{-1}[\mathfrak{o}_{P_a}(x',y'),]$$

as required. □

Remark 5.3.9. Note that $(a,0)$ is a prime element of P_a. Indeed, $(a,0) \neq (1,1)$ and if $(x,y) \wedge (u,v) \leq (a,0)$, then $y = 0$ or $v = 0$. This means that $x \leq a$ or $u \leq a$. Thus, $(x,y) \leq (a,0)$ or $(u,v) \leq (a,0)$.

Proposition 5.3.10. *If P_a is completely regular then $f \colon L \to P_a$ is an n-map.*

Proof. Let $u \in L$ and $(x,y) \in P_a$ such that $\mathfrak{o}_L(u)$ and $f_{-1}[\mathfrak{c}_{P_a}(x,y)]$ are completely separated in L. Then there exists $v, w \in \mathsf{Coz}\, L$ such that

$$\mathfrak{o}_L(u) \subseteq \mathfrak{c}_L(v), \quad f_{-1}[\mathfrak{c}_{P_a}(x,y)] \subseteq \mathfrak{c}_L(w) \quad \text{and} \quad \mathfrak{c}_L(v) \cap \mathfrak{c}_L(w) = \mathsf{O}. \tag{5.3.3}$$

Note that $f_{-1}[\mathfrak{c}_{P_a}(x,y)] = \mathfrak{c}_L(h(x,y)) = \mathfrak{c}_L(x)$. By definition of P_a we have two cases:

(Case 1): $y=0$. Thus, $x=h(x,y)\leq a$.
Furthermore, we have that $w\leq x\leq a$ so $(w,0)\in \mathsf{Coz}\, P_a$. From (5.3.3) we know that $\mathfrak{o}_L(u)$ and $\mathfrak{c}_L(w)=\mathfrak{c}_L(h(w,0))=f_{-1}[\mathfrak{c}_{P_a}(w,0)]$ are completely separated in L. Since f is a w-map (Proposition 5.3.8) there is an open sublocale $\mathfrak{o}_{P_a}(x',y')$ completely separated from $\mathfrak{c}_{P_a}(w,0)$ in P_a such that $\mathfrak{o}_L(u)\subseteq f_{-1}[\mathfrak{o}_{P_a}(x',y')]$. In particular, since $(w,0)\leq(x,y)$, $\mathfrak{c}_{P_a}(x,y)$ is completely separated from $\mathfrak{o}_{P_a}(x',y')$ in P_a.

(Case 2): $y=1$. Thus, $x\vee a=x\vee h(x,y)=1$. Since $(a,0)$ is a prime element of P_a and $(x,y)=(x,1)\nleq(a,0)$, the sublocales $\mathfrak{b}_{P_a}(a,0)$ and $\mathfrak{c}_{P_a}(x,y)$ are disjoint. By Corollary 5.2.2, they are completely separated; thus, there are $(c,d),(c',d')\in \mathsf{Coz}\, P_a$ such that

$$\mathfrak{c}_{P_a}(a,0)\subseteq\mathfrak{c}_{P_a}(c,d),\quad \mathfrak{c}_{P_a}(x,y)\subseteq\mathfrak{c}_{P_a}(c',d')\quad\text{and}\quad \mathfrak{c}_{P_a}(c,d)\cap\mathfrak{c}_{P_a}(c',d')=\mathsf{O} \tag{5.3.4}$$

(we are using here Remark 2.7.4 (4) and the fact that $\overline{\mathfrak{b}_{P_a}(a,0)}=\mathfrak{c}_{P_a}(a,0)$). This implies that $d=0$ and $c\leq a$. Therefore $d'=1$, since $\mathfrak{c}_{P_a}(c,d)$ and $\mathfrak{c}_{P_a}(c',d')$ are disjoint. By Proposition 5.3.1, $\mathfrak{c}_L(c')$ and $\mathfrak{c}_L(a)$ are completely separated in L. Consider $v\wedge c, w\vee c'\in L$, and note that they are both cozero elements of L. Since $\mathfrak{c}_L(w\vee c')\subseteq\mathfrak{c}_L(w)$, the sublocale $\mathfrak{c}_L(w\vee c')$ is also completely separated from $\mathfrak{c}_L(a)$ in L. Then, by Proposition 5.3.1, $(w\vee c',1)\in \mathsf{Coz}\, P_a$. Further, $v\wedge c\leq c\leq a$ so $(v\wedge c,0)\in \mathsf{Coz}\, P_a$. Note that $\mathfrak{c}_{P_a}(w\vee c',1)$ and $\mathfrak{c}_{P_a}(v\wedge c,0)$ are disjoint zero sublocales of P_a; indeed

$$(v\wedge c,0)\vee(w\vee c',1)=((v\wedge c)\vee w\vee c',1)=((v\vee w\vee c')\wedge(c\vee w\vee c')),1)=(1,1)$$

because $v\vee w=1$ and $c\vee c'=1$. Using Remark 2.7.6 we obtain $(x',y')\in P_a$ such that $\mathfrak{o}_{P_a}(x',y')$ is completely separated from $\mathfrak{c}_{P_a}(w\vee c',1)$in P_a and $\mathfrak{c}_{P_a}(v\wedge c,0)\subseteq\mathfrak{o}_{P_a}(x',y')$. By (5.3.3) and (5.3.4)), we get that $(w\vee c',1)\leq(x,y)$; thus, $\mathfrak{o}_{P_a}(x',y')$ is also completely separated from $\mathfrak{c}_{P_a}(x,y)$. Finally, we have

$$\mathfrak{o}_L(u)\subseteq\mathfrak{c}_L(v)\subseteq\mathfrak{c}_L(v\wedge c)=\mathfrak{c}_L(h(v\wedge c,0))=f_{-1}[\mathfrak{c}_{P_a}(v\wedge c,0)]\subseteq f_{-1}[\mathfrak{o}_{P_a}(x',y'),]$$

which concludes the proof. □

Finally, putting together Corollary 5.3.4 and Proposition 5.3.10 we have:

Corollary 5.3.11. *If L is normal and subfit, then $f\colon L\to P_a$ is an n-map for all $a\in L$.*

Chapter 6

Covering Farness and Uniform Continuity

In this chapter we study uniform continuity of real-valued functions on a preuniform frame. Our aim is to characterize uniform continuity of such frame homomorphisms in terms of a farness relation between elements in a frame, and then to derive from it a separation and an extension theorem for real-valued uniform maps on uniform frames. The approach, purely order-theoretic, uses properties of the Galois maps associated with the farness relation. As a byproduct, we identify sufficient conditions under which a (continuous) scale in a frame with a preuniformity generates a real-valued uniform map.

The content of this chapter is based on the author's paper with Jorge Picado [7].

6.1 Background: Covers and Uniform Frames

In this section we present the general background needed to work with covering (pre)uniformities. Our main references are [72] and [66].

Covers

A *cover* of a frame L is a subset $U \subseteq L$ such that $\bigvee U = 1$. A cover U *refines* (or is a *refinement* of) a cover V, and we write $U \leq V$, if for every $u \in U$ there is some $v \in V$ such that $u \leq v$. For covers U, V we have the largest common refinement $U \wedge V = \{u \wedge v \mid u \in U, v \in V\}$.

For any $U \subseteq L$ and any $x \in L$ the *star* of x in U is the element of L

$$U \cdot x = \bigvee \{u \in U \mid u \wedge x \neq 0\}. \tag{6.1.1}$$

For any $U, V \subseteq L$, set

$$U \cdot V = \{U \cdot v \mid v \in V\}.$$

If U and V are covers, then $U \cdot V$ is also a cover. One usually writes Ux and UV instead of $U \cdot x$ and $U \cdot V$. Since this operator is neither commutative nor associative, we will use parenthesis when needed.

The following proposition lists some of the basic properties of these operators (see [66] or [72]).

Proposition 6.1.1. *For any covers $U,V \subseteq L$ and any frame homomorphism $h\colon L \to M$, we have:*

(1) *UV is a cover of L,*

(2) *$x \leq Ux$,*

(3) *$Ux \leq y$ implies $x \prec y$,*

(4) *$U \leq UU$,*

(5) *$U \leq V$ and $x \leq y$ imply $Ux \leq Vy$,*

(6) *$U(Vx) \leq (UV)x = U(V(Ux))$,*

(7) *$U\left(\bigvee_{i\in I} x_i\right) = \bigvee_{i\in I} Ux_i$,*

(8) *$h[U]\,h(x) \leq h(Ux)$.*

For a cover U, define a cover U^n for $n \geq 1$ inductively by setting

$$U^1 = U \quad \text{and} \quad U^{n+1} = UU^n. \tag{6.1.2}$$

Hence

$$U^{n+1} = \{Ux \mid x \in U^n\}, \quad n = 1,2,\ldots.$$

Clearly, from Proposition 6.1.1 (5), for any $n \geq 1$, $U \leq V$ implies $U^n \leq V^n$.

We shall need certain strengthenings of the notion of refinement of covers (see [51]). For covers U,V we say that

(a) U is a *star refinement* of V, denoted by $U \leq^* V$, if $U^2 \leq V$.

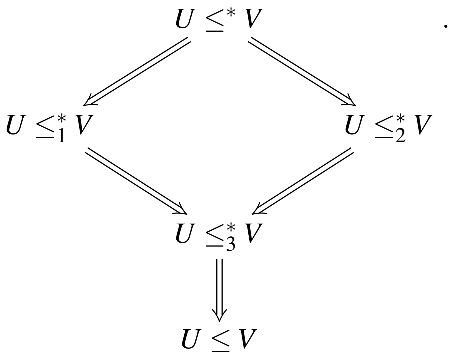
(b) U is a *barycentric refinement* of V, denoted by $U \leq^*_1 V$, if there is a cover W of L with $UW \leq V$.

(c) U is a *connected refinement* of V, denoted by $U \leq^*_2 V$, if for all $S \subseteq U$ such that $a \wedge b \neq 0$ for all $a,b \in S$, there is a $v \in V$ with $\bigvee S \leq v$.

(d) U is a *regular refinement* of V, denoted by $U \leq^*_3 V$, if for all $a,b \in U$ with $a \wedge b \neq 0$, there is a $v \in V$ with $a \vee b \leq v$.

Clearly, (a) $\Longrightarrow$ (b) and (c) $\Longrightarrow$ (d). Moreover, (a) $\Longrightarrow$ (c). Indeed, if $U \leq^* V$ and $\emptyset \neq S \subseteq U$ such that $a \wedge b \neq 0$ for all $a,b \in S$, then there is $v \in V$ such that $\bigvee S \leq \bigvee\{u \in U \mid u \wedge s \neq 0\} = Us \leq v$ for some $s \in S$. If S is empty the condition trivially holds; thus $U \leq^*_2 V$.
Furthermore, (b) $\Longrightarrow$ (d). If $UW \leq V$ and $a,b \in U$ with $a \wedge b \neq 0$, since W is a cover of L there is $w \in W$ such that $a \wedge b \wedge w \neq 0$. In particular we have $a \wedge w \neq 0$ and $b \wedge w \neq 0$. Hence, $a \vee b \leq Uw \leq v$ for some $v \in V$. Consequently, $U \leq^*_3 V$.
In conclusion, the star refinement is the strongest relation, and the regular refinement is the weakest and it trivially implies ordinary refinement. Further, conditions (b) and (c) are generally unrelated,

even classically, as displayed in the following diagram:



Uniform Frames

Let $\mathscr{U}$ be a system of covers of a frame L and $a, b \in L$. The element a is *uniformly below* b, and we write $a \lhd_{\mathscr{U}} b$, if

$$\exists U \in \mathscr{U}: \quad Ua \leq b.$$

The following proposition lists some basic properties of the uniformly below relation (we refer to [72] and [66] for the proofs).

Proposition 6.1.2. *Let $\mathscr{U}$ (resp. $\mathscr{V}$) be a system of covers on L (resp. M). The following statements hold:*

(1) *If $h\colon L \to M$ is a frame homomorphism with the property that for every $U \in \mathscr{U}$ there is a $V \in \mathscr{V}$ such that $V \leq h[U]$, then*

$$a \lhd_{\mathscr{U}} b \implies h(a) \lhd_{\mathscr{V}} h(b).$$

(2) *If any pair $U_1, U_2 \in \mathscr{U}$ has a common refinement $V \in \mathscr{U}$, then*

$$a \lhd_{\mathscr{U}} b_1, b_2 \implies a \lhd_{\mathscr{U}} b_1 \wedge b_2 \quad \text{and}$$

$$a_1, a_2 \lhd_{\mathscr{U}} b_2 \implies a_1 \vee a_2 \lhd_{\mathscr{U}} b.$$

(3) $a' \leq a \lhd_{\mathscr{U}} b \leq b' \implies a' \lhd_{\mathscr{U}} b'$.

(4) $a \lhd_{\mathscr{U}} b \implies a \prec b$.

(5) $a \lhd_{\mathscr{U}} b \implies a^{**} \lhd_{\mathscr{U}} b$.

A (covering) *uniformity* on a frame L is a nonempty system $\mathscr{U}$ of covers of L such that

(U1) $U \in \mathscr{U}$ and $U \leq V$ implies $V \in \mathscr{U}$,

(U2) $U, V \in \mathscr{U}$ implies $U \wedge V \in \mathscr{U}$,

(U3) for every $U \in \mathscr{U}$ there is a $V \in \mathscr{U}$ such that $VV \leq U$,

(U4) for every $a \in L$, $a = \bigvee\{b \mid b \lhd_{\mathscr{U}} a\}$.

Without (U4) one speaks of a *preuniformity*; without (U1) one speaks of a *basis of a (pre)uniformity* (in the latter case one obtains a (pre)uniformity by adding all the V with $V \geq U \in \mathscr{U}$).

A *uniform frame* (resp. *preuniform frame*) is a pair $(L, \mathscr{U})$ where $\mathscr{U}$ is a uniformity (resp. preuniformity) on L. A frame homomorphism $h\colon L \to M$ is a *uniform homomorphism* $(L, \mathscr{U}) \to (M, \mathscr{V})$ if $h[U] \in \mathscr{V}$ for every $U \in \mathscr{U}$ (if $\mathscr{U}, \mathscr{V}$ are bases of (pre)uniformities this condition is replaced by the existence of some $V \in \mathscr{V}$ such that $h[U] \geq V$).

Remark 6.1.3. (1) If (U3) holds for a system of covers $\mathscr{U}$ of a frame L then the uniformly below relation $\triangleleft_{\mathscr{U}}$ is interpolative:

$$x \triangleleft_{\mathscr{U}} y \implies \exists z\, (x \triangleleft_{\mathscr{U}} z \triangleleft_{\mathscr{U}} y).$$

Then, together with Proposition 6.1.2 (4), we conclude that

$$x \triangleleft_{\mathscr{U}} y \implies x \prec\!\!\prec y$$

for every $x, y \in L$.

(2) Hence, any frame that admits a uniformity is completely regular. The converse is also true, as it is well-known ([72],[66]).

The Metric Uniformity of $\mathfrak{L}(\mathbb{R})$

The frame of reals $\mathfrak{L}(\mathbb{R})$ carries a natural uniformity, its *metric uniformity* ([10]), generated by covers

$$C_n = \left\{(p,q) \in \mathfrak{L}(\mathbb{R}) \mid 0 < q - p < \tfrac{1}{n}\right\}, \quad n = 1, 2, \ldots$$

Alternatively, we may consider the covers

$$D_n = \left\{(r,s) \in \mathfrak{L}(\mathbb{R}) \mid s - r = \tfrac{1}{n}\right\}, \quad n = 1, 2, \ldots.$$

Clearly, for each $n \leq m$, $D_m \leq D_n$ and $C_m \subseteq C_n$. Moreover, for every $n \in \mathbb{N}$, $C_n \leq D_n$ and $D_{n+1} \subseteq C_n$. Hence these covers also constitute a basis for the metric uniformity on $\mathfrak{L}(\mathbb{R})$.

We will consider, more generally, the covers

$$D_\delta = \left\{(p,q) \in \mathfrak{L}(\mathbb{R}) \mid\ q - p = \tfrac{1}{\delta}\right\}, \quad \delta \in \mathbb{Q}^+$$

(where $\mathbb{Q}^+$ denotes the set of positive rational numbers).

From now on we will be interested in *uniformly continuous real-valued functions* on a preuniform frame $(L, \mathscr{U})$ (with $\mathfrak{L}(\mathbb{R})$ equipped with its metric uniformity). That is, uniform homomorphisms from a preuniform frame $(L, \mathscr{U})$ to $\mathfrak{L}(\mathbb{R})$ with its metric uniformity. An $f\colon \mathfrak{L}(\mathbb{R}) \to (L, \mathscr{U})$ is uniformly continuous if

$$\forall n \in \mathbb{N} \quad \exists U \in \mathscr{U} \text{ such that } U \leq f[D_n].$$

Equivalently, if

$$\forall \delta \in \mathbb{Q}^+ \quad \exists U \in \mathscr{U} \text{ such that } U \leq f[D_\delta.] \tag{6.1.3}$$

Proposition 6.1.4. *For any $\gamma,\delta \in \mathbb{Q}^+$, $D_\gamma \cdot D_\delta = D_{\frac{\gamma\delta}{\gamma+2\delta}}$.*

Proof. By definition $D_\gamma \cdot D_\delta = \{D_\gamma \cdot (p-\frac{1}{2\delta}, p+\frac{1}{2\delta}) \mid p \in \mathbb{Q}\}$ and

$$D_\gamma \cdot (p-\tfrac{1}{2\delta}, p+\tfrac{1}{2\delta}) = \bigvee\{(r,s) \mid s-r=\tfrac{1}{\gamma}, (r,s)\wedge(p-\tfrac{1}{2\delta}, p+\tfrac{1}{2\delta}) \neq 0\}.$$

From $(r,s)\wedge(p-\frac{1}{2\delta}, p+\frac{1}{2\delta}) \neq 0$, it follows that

$$(r,s) \leq \left(p-\tfrac{1}{2\delta}-\tfrac{1}{\gamma}, p+\tfrac{1}{2\delta}+\tfrac{1}{\gamma}\right) = \left(p-\tfrac{\gamma+2\delta}{2\delta\gamma}, p+\tfrac{\gamma+2\delta}{2\delta\gamma}\right).$$

Hence

$$D_\gamma \cdot (p-\tfrac{1}{2\delta}, p+\tfrac{1}{2\delta}) \leq (p-\tfrac{\gamma+2\delta}{2\delta\gamma}, p+\tfrac{\gamma+2\delta}{2\delta\gamma}).$$

Now,

$$D_\gamma \cdot (p-\tfrac{1}{2\delta}, p+\tfrac{1}{2\delta}) \geq \bigvee_{\alpha<\frac{\gamma+2\delta}{2\delta\gamma}} (p-\alpha, p+\alpha) = (p-\tfrac{\gamma+2\delta}{2\delta\gamma}, p+\tfrac{\gamma+2\delta}{2\delta\gamma}).$$

Hence $D_\gamma \cdot (p-\frac{1}{2\delta}, p+\frac{1}{2\delta}) = (p-\frac{\gamma+2\delta}{2\delta\gamma}, p+\frac{\gamma+2\delta}{2\delta\gamma})$. Therefore

$$D_\gamma \cdot D_\delta = \{D_\gamma \cdot (p-\tfrac{1}{2\delta}, p+\tfrac{1}{2\delta}) \mid p \in \mathbb{Q}\} = \left\{(p-\tfrac{\gamma+2\delta}{2\delta\gamma}, p+\tfrac{\gamma+2\delta}{2\delta\gamma}) \mid p \in \mathbb{Q}\right\} = D_{\frac{\gamma\delta}{2\delta+\gamma}}. \qquad \square$$

Proposition 6.1.5. *For every $n \in \mathbb{N}$ and $\delta \in \mathbb{Q}^+$, $D_\delta^n = D_{\frac{\delta}{2n-1}}$.*

Proof. Let $\delta \in \mathbb{Q}^+$. We show the result by induction over n. The case $n=1$ is trivial. Assume it holds for n. Using Proposition 6.1.4, we get

$$D_\delta^{n+1} = D_\delta \cdot D_\delta^n = D_\delta \cdot D_{\frac{\delta}{2n-1}} = D_{\frac{\delta^2}{2\delta+\delta(2n-1)}} = D_{\frac{\delta}{2n+1}} = D_{\frac{\delta}{2(n+1)-1}}. \qquad \square$$

6.2 Covering Farness

In this section we introduce a key concept, the definition of farness, which should be considered as the point-free extension of the proximal relation of farness betweeen sets due to Efremovič and Smirnov [74, 75]. This relation is defined for elements of a frame. Later on, in Section 7.2, we will generalize it for sublocales. There is an underlying Galois adjunction in the notion of farness between elements. Such adjunction will be helpful in Section 6.4 to prove the separation theorem.

Farness relation

If U is a cover of a frame L we say that elements $a,b \in L$ are U*-far* if

$$\forall u \in U \quad u \wedge a \neq 0 \implies u \wedge b = 0.$$

Remark 6.2.1. Note that if a and b are U-far and $V \leq U$, then a and b are V-far. Further, if a and b are U-far and $a' \leq a$ and $b' \leq b$ then a' and b' are U-far.

The farness relation can be defined in different ways as the following result shows:

Proposition 6.2.2. *Let L be a frame, U a cover of L and $a,b \in L$. The following conditions are equivalent:*

(i) *a and b are U-far.*

(ii) $Ua \wedge b = 0$.

(iii) $Ub \wedge a = 0$.

(iv) $Ua \leq b^*$ *(equivalently, $b \leq (Ua)^*$).*

(v) $Ub \leq a^*$ *(equivalently, $a \leq (Ub)^*$).*

(vi) $U \leq \{a^*, b^*\}$.

(vii) *a^{**} and b^{**} are U-far.*

Proof. (i) $\iff$ (ii) holds immediately by the definitions of farness and the star operator: $Ua \wedge b = 0$ if and only if $(\bigvee\{u \in U \mid u \wedge a \neq 0\}) \wedge b = 0$ if and only if $u \wedge a \neq 0$ implies $u \wedge b = 0$ for every $u \in U$. Since the farness relation is symmetric, the argument for (i) $\iff$ (iii) is similar. The equivalences (ii) $\iff$ (iv) and (iii) $\iff$ (v) follow immediately from (1.2.2). For (vi) $\iff$ (i) note that $U \leq \{a^*, b^*\}$ if and only if for every $u \in U$ we have $u \leq a^*$ or $u \leq b^*$. By (1.2.2) this is equivalent to a and b being U-far. One uses this same argument and the fact that $x^* = x^{***}$ for any $x \in L$, to show the equivalence between (vii) and (vi). □

Remark 6.2.3. There is an obvious link between the farness relation and the uniformly below relation. Let $\mathscr{U}$ be a system of covers for a frame L, then a and b are U-far for some $U \in \mathscr{U}$ if and only if $a \triangleleft_{\mathscr{U}} b^*$ (recall Proposition 6.2.2 (ii)). Since $a \triangleleft_{\mathscr{U}} b$ implies $a \prec b$ (Proposition 6.1.2 (4)). Then $a^* \vee b^* = 1$ (compare this with Proposition 6.2.2 (vii)).
Furthermore, if $a \triangleleft_{\mathscr{U}} b$ then $a \triangleleft_{\mathscr{U}} b \leq b^{**}$. Hence, a and b^* are U-far for some $U \in \mathscr{U}$.

Farness and Galois Connections

Given a cover U of L we define the map

$$\begin{aligned} \mathsf{S}_U : L &\to L \\ a &\mapsto \mathsf{S}_U(a) = Ua = \bigvee\{u \in U \mid u \wedge a \neq 0\}. \end{aligned}$$

From Proposition 6.1.1 (7) we know that S_U preserves arbitrary joins. Hence, S_U has a right adjoint:

$$\begin{aligned} \widetilde{\mathsf{S}_U} : L &\to L \\ b &\mapsto \widetilde{\mathsf{S}_U}(b) = \bigvee\{x \in L \mid Ux \leq b\}. \end{aligned}$$

Moreover, we denote by S_U^n the result of composing S_U with itself n times. In general, we have $\mathsf{S}_{U^n} \neq \mathsf{S}_U^n$, but as noted in [53, Fact 2.4], the operators S_U^n and S_{U^n} are closely related. We have the following properties:

Proposition 6.2.4. *Let U be a cover of a frame L and $n,m \in \mathbb{N}$. Then:*

(1) $a \in U^{n+1}$ *if and only if there is $u \in U$ such that $a = \mathsf{S}_U^n(u)$,*

(2) $U^{n+1} = \{\mathsf{S}_U^n(x) \mid x \in U\} = \mathsf{S}_U^n[U]$,

(3) $\mathsf{S}_{U^n} = \mathsf{S}_U^{2n-1}$,

(4) $U^n U = U^{2n}$,

(5) $U^{nm} \leq (U^n)^m$.

Proof. (1): By definition of the star operator, this statement is clear for $n = 1$. We proceed by induction on n. Assume it holds for some $n > 1$. Recall (6.1.2). Then $a \in U^{n+1} = UU^n$ if and only if $a = \mathsf{S}_U(y)$ for some $y \in U^n$. By inductive hypothesis, $y \in U^n$ if and only if there is $u \in U$ such that $y = \mathsf{S}_U^{n-1}(u)$. Hence, $a = \mathsf{S}_U(y) = \mathsf{S}_U(\mathsf{S}_U^{n-1}(u)) = \mathsf{S}_U^n(u)$.
(2): Immediate from (1).
(3): We proceed by induction on n. Clearly, the equality holds for $n = 1$ so assume it holds for some $n > 1$. Let $a \in L$, then

$$\begin{aligned}\mathsf{S}_{U^{n+1}}(a) &= U^{n+1}a = (UU^n)(a) = (U(U^n(Ua))) = \mathsf{S}_U(\mathsf{S}_{U^n}(\mathsf{S}_U(a))) \\ &= \mathsf{S}_U(\mathsf{S}_U^{2n-1}\mathsf{S}_U(a)) = \mathsf{S}_U^{2n+1}(a) = \mathsf{S}_U^{2(n+1)-1}(a)\end{aligned}$$

as required (we are using Proposition 6.1.1 (6) and the inductive hypothesis in the identities above).
(4): By definition, $a \in U^n U$ if and only if there is a $u \in U$ with $a = U^n U = \mathsf{S}_{U^n}(u) = \mathsf{S}_U^{2n-1}(u)$, and by (1) this is equivalent to $a \in U^{2n}$.

(5): If $n = 1$ or $m = 1$ the desired equality trivially holds. We assume $n,m > 1$. By an application of (1) one has $a \in U^{nm}$ if and only if $a = \mathsf{S}_U^{nm-1}(u)$ for some $u \in U$. Further, by (1), $b \in (U^n)^m$ if and only if $b = \mathsf{S}_{U^n}^{m-1}(v)$ for some $v \in U^n$. By another application of (1), the latter is equivalent to the existence of a $w \in U$ such that

$$b = \mathsf{S}_{U^n}^{m-1}(\mathsf{S}_U^{n-1}(w)) = \mathsf{S}_U^{(m-1)(2n-1)}(\mathsf{S}_U^{n-1}(w)) = \mathsf{S}_U^{(m-1)(2n-1)+n-1}(w).$$

The result thus follows from the obvious fact that $nm \leq (m-1)(2n-1)+n$: we have that for every $a \in U^{nm}$ there is some $u \in U$ such that

$$a = \mathsf{S}_U^{nm-1}(u) \leq \mathsf{S}_U^{(m-1)(2n-1)+n-1}(u) \in (U^n)^m.$$ □

Remark 6.2.5. For a better geometric understanding of the operator S_U it is important to notice the following:

(1) Let $n \in \mathbb{N}$ and $x,a \in L$. We have that $\mathsf{S}_U^n(x) \wedge a \neq 0$ if and only if there are $u_1,u_2,\ldots,u_n \in U$ such that $x \wedge u_1 \neq 0$, $u_{i-1} \wedge u_i \neq 0$ for $i = 2,\ldots,n$ and $u_n \wedge a \neq 0$. Indeed, for $n = 1$ we know that

$$\mathsf{S}_U(x) \wedge a = Ux \wedge a = \left(\bigvee\{u \in U \mid u \wedge x \neq 0\}\right) \wedge a = \bigvee\{u \wedge a \mid u \in U, u \wedge x \neq 0\}$$

so $\mathsf{S}_U(x) \wedge a \neq 0$ if and only if there is $u \in U$ such that $x \wedge u \neq 0$ and $u \wedge a \neq 0$. Assuming the statement holds for n, we will prove it for $n+1$. If $\mathsf{S}_U^n(\mathsf{S}_U(x)) \wedge a = \mathsf{S}_U^{n+1}(x) \wedge a \neq 0$, then by inductive hypothesis, there are $u_1, \dots, u_n \in U$ such that $\mathsf{S}_U(x) \wedge u_1 \neq 0$, $u_{i-1} \wedge u_i \neq 0$ for $i = 2, \dots, n$ and $u_n \wedge a \neq 0$. Furthermore, since $\mathsf{S}_U(x) \wedge u_1 \neq 0$, there is $u_0 \in U$ such that $x \wedge u_0 \neq 0$ and $u_0 \wedge u_1 \neq 0$.

(2) Consequently, (1) above yields immediately

$$\begin{aligned} \mathsf{S}_U^n(x) &= \bigvee \{u \in U \mid \mathsf{S}_U^{n-1}(x) \wedge u \neq 0\} \\ &= \bigvee \{u \in U \mid \exists u_1, \dots, u_{n-1} \in U, u \wedge u_1 \neq 0, u_{i-1} \wedge u_i \neq 0 \text{ for } i = 2, \dots, n-1 \text{ and } u_n \wedge x \neq 0\}. \end{aligned}$$

Using "chains" of connected elements ([8]) gives a better intuition of how the star operator and the relation between the several types of refinements defined in Section 6.1 work. In Section 7.1, particularly in Proposition 7.1.7, one can see how this approach is advantageous when working with prediameters.

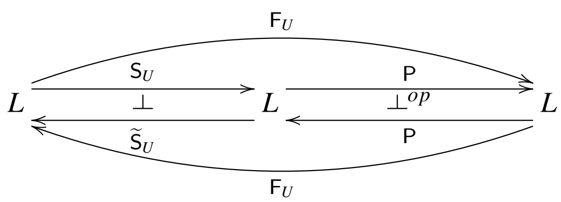
Now, we define the *pseudocomplement operator* $\mathsf{P}\colon L \to L$ given by pseudocomplements, that is, $\mathsf{P}(a) = a^*$. This map is a self-dual Galois adjoint (that is, the pair (P, P) is a Galois connection); indeed,

$$a \leq \mathsf{P}(b) = b^* \iff b \leq \mathsf{P}(a) = a^*.$$

Then, the composition $\mathsf{S}_U\mathsf{P}$, that we denote by F_U, must be a dual Galois adjoint. Note that from Proposition 6.2.2 we have that

$$b \leq \mathsf{F}_U(a) \iff \mathsf{S}_U(a) \leq \mathsf{P}(b) \iff \mathsf{S}_U(b) \leq \mathsf{P}(a) \iff a \leq \mathsf{F}_U(b) \tag{6.2.1}$$

meaning F_U is a self-dual Galois Adjoint, and by uniqueness of adjoints $\mathsf{F}_U = \widetilde{\mathsf{S}}_U\mathsf{P}$. Summarizing, we have the following diagram:



The pair $(\mathsf{F}_U, \mathsf{F}_U)$ being a Galois connection yields immediately the following properties:

(F1) $\mathsf{F}_U(\bigvee_{i \in I} a_i) = \bigwedge_{i \in I} \mathsf{F}_U(a_i)$ (in particular, $\mathsf{F}_U(0) = 1$). In this case, we have also $\mathsf{F}_U(1) = 0$.

(F2) $\mathsf{F}_U\mathsf{F}_U = \mathsf{F}_U^2 \geq \mathrm{id}_L$.

(F3) $\mathsf{F}_U\mathsf{F}_U\mathsf{F}_U = \mathsf{F}_U^3 = \mathsf{F}_U$.

Now, we can use this in association with the farness relation. First, elements a, b in L are *U-far* if and only if $\mathsf{S}_U(a) \leq \mathsf{P}(b)$. Hence, by Proposition 6.2.2 and (6.2.1),

$$a \text{ and } b \text{ are } U\text{-far} \iff a \leq \mathsf{F}_U(b) \iff b \leq \mathsf{F}_U(a), \tag{6.2.2}$$

and it follows from property (F2) that $\mathsf{F}_U(a)$ is the largest element in L that is U-far from a. Indeed, since $a \leq \mathsf{F}_U\mathsf{F}_U(a)$ we know a and $\mathsf{F}_U(a)$ are U-far. Further, $\mathsf{F}_U(a)$ is the largest element because if a and b are U-far, then $b \leq \mathsf{F}_U(a)$ (by (6.2.2)). These are the general properties of the farness relation and its associated Galois connection $(\mathsf{F}_U, \mathsf{F}_U)$.

Proposition 6.2.6. *Elements a and b are U^n-far if and only if $\mathsf{S}_U^k(a)$ and $\mathsf{S}_U^k(b)$ are U^{n-k}-far for every $1 \leq k < n$.*

Proof. $\mathsf{S}_U^k(a)$ and $\mathsf{S}_U^k(b)$ are U^{n-k}-far if and only if

$$\mathsf{S}_U^k(b) \leq \mathsf{F}_{U^{n-k}}(\mathsf{S}_U^k(a)) = \mathsf{PS}_{U^{n-k}}\mathsf{S}_U^k(a).$$

By Proposition 6.2.4 (3),

$$\mathsf{PS}_{U^{n-k}}\mathsf{S}_U^k(a) = \mathsf{PS}_U^{2n-2k-1}\mathsf{S}_U^k(a) = \mathsf{PS}_U^{2n-k-1}(a).$$

Hence, by (6.2.1) and using Proposition 6.2.4 (3) again, we may conclude that $\mathsf{S}_U^k(a)$ and $\mathsf{S}_U^k(b)$ are U^{n-k}-far if and only if $b \leq \mathsf{PS}_U^{2n-1}(a) = \mathsf{PS}_{U^n}(a) = \mathsf{F}_{U^n}(a)$. □

In particular, a and b are U^n-far if and only if $\mathsf{S}_U(a)$ and $\mathsf{S}_U(b)$ are U^{n-1}-far.

Corollary 6.2.7. *If a and b are U^n-far then $(\mathsf{S}_U^j(a))^* \vee (\mathsf{S}_U^k(b))^* = 1$ for every $1 \leq j \leq k < n$.*

Proof. Clearly, $(\mathsf{S}_U^j(a))^* \vee (\mathsf{S}_U^k(b))^* \geq (\mathsf{S}_U^k(a))^* \vee (\mathsf{S}_U^k(b))^*$. By Proposition 6.2.6, $\mathsf{S}_U^k(a)$ and $\mathsf{S}_U^k(b)$ are V-far for some V. Hence, by Remark 6.2.3,

$$(\mathsf{S}_U^k(a))^* \vee (\mathsf{S}_U^k(b))^* = 1.$$ □

It may be worth pointing out that, by Proposition 6.2.4 (3), $(\mathsf{S}_U^n(x))^*$ is given by

$$\begin{cases} \mathsf{PS}_U^{2k-1}(x) = \mathsf{PS}_{U^k}(x) = \mathsf{F}_{U^k}(x) & \text{if } n = 2k-1 \\ \mathsf{PS}_U^{2k}(x) = \mathsf{PS}_U^{2k-1}\mathsf{S}_U(x) = \mathsf{PS}_{U^k}\mathsf{S}_U(x) = \mathsf{F}_{U^k}(\mathsf{S}_U(x)) & \text{if } n = 2k. \end{cases}$$

6.3 Uniform Continuity and Scales in Uniform Frames

The purpose of this section is to describe uniformly continuous real-valued functions in terms of the farness relation. The new characterizing conditions of uniform continuity that we get allow us to impose conditions on scales in order to generate uniform frame homomorphisms

Uniform Continuity via Covering Farness

Let $f, g \in \mathscr{R}(L)$ such that $f \geq g$. For each $\delta \in \mathbb{Q}^+$,

$$D_\delta^{f,g} := \{f(r,-) \wedge g(-,s) \mid (r,s) \in D_\delta\}$$

is a cover of L. Indeed, since D_δ is a cover of $\mathfrak{L}(\mathbb{R})$,

$$\bigvee_{(r,s)\in D_\delta} (f(r,\text{—})\wedge g(\text{—},s)) \geq \bigvee_{(r,s)\in D_\delta} (g(r,\text{—})\wedge g(\text{—},s))$$
$$= g\Big(\bigvee_{(r,s)\in D_\delta} (r,s)\Big) = g(1) = 1.$$

Note that $D_\delta^{f,g}$ is a refinement of both covers $\{f(r,\text{—}) \mid r\in\mathbb{Q}\}$ and $\{g(\text{—},r) \mid r\in\mathbb{Q}\}$. Clearly, $D_\delta^{f,f} = f[D_\delta]$, and we denote this cover by D_δ^f.

Also note that by Remark 2.1.1 one immediately gets the following result:

Lemma 6.3.1. *Let U be a cover of L, $f,g\in\mathscr{R}(L)$ and $\delta\in\mathbb{Q}^+$. The following are equivalent:*

(i) *$f(\text{—},r)$ and $g(s,\text{—})$ are U-far for every $r,s\in\mathbb{Q}$ such that $s-r>\frac{1}{\delta}$.*

(ii) *$f(\text{—},r)$ and $g(\text{—},s)^*$ are U-far for every $r,s\in\mathbb{Q}$ such that $s-r>\frac{1}{\delta}$.*

(iii) *$f(r,\text{—})^*$ and $g(s,\text{—})$ are U-far for every $r,s\in\mathbb{Q}$ such that $s-r>\frac{1}{\delta}$.*

(iv) *$f(r,\text{—})^*$ and $g(\text{—},s)^*$ are U-far for every $r,s\in\mathbb{Q}$ such that $s-r>\frac{1}{\delta}$.*

Proposition 6.3.2. *Let $(a_p)_{p\in\mathbb{Q}},(b_q)_{q\in\mathbb{Q}}\subseteq L$ such that*

$$a_q\leq a_p \ \textit{ and } \ b_p\leq b_q \ \textit{ for every } \ p\leq q.$$

If $U_\delta=\{a_r\wedge b_s \mid (r,s)\in D_\delta\}$ is a cover for $\delta\in\mathbb{Q}^+$, then a_r^* and b_s^* are U_δ-far for every $s-r>\frac{1}{\delta}$.

Proof. Let $s-r>\frac{1}{\delta}$. By Proposition 6.2.2 (vi), we need to show that $U_\delta\leq\{a_r^{**},b_s^{**}\}$. Let $(p,q)\in D_\delta$. Then $s>q$ or $r<p$. In the former case we have $b_s\geq b_q$ and thus $a_p\wedge b_q\leq b_s\leq b_s^{**}$; otherwise, in the latter case, $a_r\geq a_p$ hence $a_p\wedge b_q\leq a_r\leq a_r^{**}$. □

Corollary 6.3.3. *Let $f,g\in\mathscr{R}(L)$ such that $f\geq g$. For each $\delta\in\mathbb{Q}^+$ and every $r,s\in\mathbb{Q}$ such that $s-r>\frac{1}{\delta}$, the elements $f(\text{—},r)$ and $g(s,\text{—})$ are $D_\delta^{f,g}$-far.*

Proof. As noted above, $D_\delta^{f,g}$ is a cover for every $\delta\in\mathbb{Q}^+$ whenever $f\geq g$. From Proposition 6.3.2, taking $a_r=f(r,\text{—})$ and $b_s=g(\text{—},s)$ we have that $f(r,\text{—})^*$ and $g(\text{—},s)^*$ are $D_\delta^{f,g}$-far. Then, by Remark 6.3.1, $f(\text{—},r)$ and $g(s,\text{—})$ are also $D_\delta^{f,g}$-far. □

Proposition 6.3.4. *Let $(a_r)_{r\in\mathbb{Q}},(b_s)_{s\in\mathbb{Q}}\subseteq L$ satisfy the following conditions:*

(1) *$a_r^{**}\leq a_p$ for $p<r$,*

(2) *$b_s^{**}\leq b_q$ for $s<q$*

(3) *$\bigvee_{r\in\mathbb{Q}} a_r^*=1$,*

(4) *$\bigvee_{s\in\mathbb{Q}} b_s^*=1$.*

If U is a cover of L such that a_r^* and b_s^* are U-far for all $r,s \in \mathbb{Q}$ such that $s-r > \frac{1}{\delta}$, then

$$U \leq \{a_r \wedge b_s \mid (r,s) \in D_\gamma\} \quad \textit{for every } \gamma < \delta \textit{ in } \mathbb{Q}^+.$$

In particular, $\{a_r \wedge b_s \mid (r,s) \in D_\gamma\}$ is a cover of L.

Proof. For each $r,s \in \mathbb{Q}$ such that $s-r > \frac{1}{\delta}$, $U \leq \{a_r^{**}, b_s^{**}\}$. Therefore, for every $u \in U$,

$$u \wedge a_r^* \neq 0 \implies u \wedge b_s^* = 0. \tag{6.3.1}$$

Let $u \neq 0$ in U. Since $\{b_s^* \mid s \in \mathbb{Q}\}$ is a cover of L, there exists $s_0 \in \mathbb{Q}$ such that $u \wedge b_{s_0}^* \neq 0$. By (6.3.1), $u \wedge a^*_{s_0 - \frac{2}{\delta}} = 0$. Thus, $u \leq a^{**}_{s_0-\frac{2}{\delta}} \leq a_{s_0 - \frac{3}{\delta}}$ and the set $\{r \in \mathbb{Q} \mid u \leq a_r\}$ is nonempty. It should be also noted that $u \not\leq a_r$ for some $r \in \mathbb{Q}$ (and therefore $u \not\leq a_{r'}$ for any $r' \geq r$), otherwise

$$u \leq \bigwedge_{r\in\mathbb{Q}} a_r \leq \bigwedge_{r\in\mathbb{Q}} a_r^{**} = \Big(\bigvee_{r\in\mathbb{Q}} a_r^*\Big)^* = 0,$$

a contradiction. Hence,

$$r_1 = \sup\{r \in \mathbb{Q} \mid u \leq a_r\} \in \mathbb{R}.$$

Now, let $\gamma \in \mathbb{Q}^+$ with $\gamma < \delta$. Set $\varepsilon = \frac{\delta-\gamma}{\delta\gamma} > 0$. Take $r \in \mathbb{Q}$ such that $0 < r_1 - r < \frac{\varepsilon}{5}$ and $p \in \mathbb{Q}$ such that $0 < p - r_1 < \frac{\varepsilon}{5}$. Then $u \leq a_r$ and $u \not\leq a_p$. Since $a_p \geq a^{**}_{p+\frac{\varepsilon}{5}}$, we have further that $u \not\leq a^{**}_{p+\frac{\varepsilon}{5}}$, that is, $u \wedge a^*_{p+\frac{\varepsilon}{5}} \neq 0$ and, by (6.3.1), $u \wedge b^*_{p+\frac{2\varepsilon}{5}+\frac{1}{\delta}} = 0$, that is,

$$u \leq b^{**}_{p+\frac{2\varepsilon}{5}+\frac{1}{\delta}} \leq b_{p+\frac{3\varepsilon}{5}+\frac{1}{\delta}}.$$

In conclusion,

$$u \leq a_r \wedge b_{p+\frac{3\varepsilon}{5}+\frac{1}{\delta}} \leq a_r \wedge b_{r+\frac{1}{\delta}+\varepsilon} \in \{a_r \wedge b_s \mid (r,s) \in D_\gamma\},$$

since

$$p + \tfrac{3\varepsilon}{5} + \tfrac{1}{\delta} - r < \tfrac{\varepsilon}{5} + r_1 + \tfrac{3\varepsilon}{5} + \tfrac{1}{\delta} + \tfrac{\varepsilon}{5} - r_1 = \tfrac{1}{\delta} + \varepsilon$$

and

$$\tfrac{1}{\delta} + \varepsilon = \tfrac{1}{\delta} + \tfrac{\delta-\gamma}{\gamma\delta} = \tfrac{1}{\gamma}.$$

□

Corollary 6.3.5. *Let $f,g \in \mathscr{R}(L)$ such that $f \geq g$ and $\delta \in \mathbb{Q}^+$. If U is a cover of L such that $f(—,r)$ and $g(s,—)$ are U-far for all $r,s \in \mathbb{Q}$ with $s-r > \frac{1}{\delta}$, then*

$$U \leq D_\gamma^{f,g} \quad \textit{for every } \gamma < \delta \textit{ in } \mathbb{Q}^+.$$

Proof. We will use Proposition 6.3.4. Let $a_r = f(r,—)$ and $b_s = g(—,s)$. Since $f,g \in \mathscr{R}(L)$, conditions (1)–(4) hold. Furthermore, by assumption and Lemma 6.3.1, a_r^* and b_s^* are U-far for every $s-r > \frac{1}{\delta}$. Thus,

$$U \leq \{a_r \wedge b_s \mid (r,s) \in D_\gamma\} = D_\gamma^{f,g}.$$

□

Corollary 6.3.6. *The following are equivalent for any $f,g \in \mathscr{R}(L)$ such that $f \geq g$:*

(i) *For every $\delta \in \mathbb{Q}^+$, there exists a cover U of L such that $U \leq D_\delta^{f,g}$.*

(ii) *For every $\delta \in \mathbb{Q}^+$, there exists a cover U of L such that $f(—,r)$ and $g(s,—)$ are U-far for any $s-r > \frac{1}{\delta}$.*

Proof. (i) $\Longrightarrow$ (ii): Let $\delta \in \mathbb{Q}^+$ and U such that $U \leq D_\delta^{f,g}$. By Corollary 6.3.3 and Remark 6.2.1, $f(—,r)$ and $g(s,—)$ are U-far for every $s-r > \frac{1}{\delta}$.
(ii) $\Longrightarrow$ (i): Let $\delta \in \mathbb{Q}^+$. By assumption, there is a cover U such that $f(—,r)$ and $g(s,—)$ are U-far for any $s-r > \frac{1}{\delta+1}$. Then, by Corollary 6.3.5, $U \leq D_\delta^{f,g}$. □

More generally, we have:

Proposition 6.3.7. *The following are equivalent for any $f,g \in \mathscr{R}(L)$ such that $f \geq g$:*

(i) *For every $\delta \in \mathbb{Q}^+$, there exists a cover U of L such that $U^n \leq D^{f,g}_{\frac{\delta}{2n-1}}$ for every $n \in \mathbb{N}$.*

(ii) *For every $\delta \in \mathbb{Q}^+$, there exists a cover U of L such that $f(—,r)$ and $g(s,—)$ are U^n-far for every $s-r > \frac{n}{\delta}$ and $n \in \mathbb{N}$.*

Proof. (i) $\Longrightarrow$ (ii): Let $\delta \in \mathbb{Q}^+$ and consider $\varepsilon = 2\delta$. By assumption, there is some U such that $U^n \leq D^{f,g}_{\frac{\varepsilon}{2n-1}}$ for every $n \in \mathbb{N}$. Let $s-r > \frac{n}{\delta} = \frac{2n}{\varepsilon} > \frac{2n-1}{\varepsilon}$. By Corollary 6.3.3, $f(—,r)$ and $g(s,—)$ are $D^{f,g}_{\frac{\varepsilon}{2n-1}}$-far. In particular, they are U^n-far, as required.
(ii) $\Longrightarrow$ (i): Let $\delta \in \mathbb{Q}^+$. By assumption, there is some U such that $f(—,r)$ and $g(s,—)$ are U^n-far for every $s-r > \frac{n}{\delta+1}$ and $n \in \mathbb{N}$. Then, by Corollary 6.3.5, since $\frac{\delta}{2n-1} < \frac{\delta+1}{n}$, we have $U^n \leq D^{f,g}_{\frac{\delta}{2n-1}}$ as required. □

Remark 6.3.8. Note that the assumption $f \geq g$ is crucial here. For instance, the condition $f \leq g$ does not even imply that $\{f(r,—) \wedge g(—,s) \mid (r,s) \in D_\delta\}$ is a cover of L. Moreover, if $f \leq g$ and, for every $\delta \in \mathbb{Q}^+$, there is some U such that $f(—,r)$ and $g(s,—)$ are U-far for any $s-r > \frac{1}{\delta}$, then $f = g$. Indeed, for every pair $r < s$, $f(—,r) \wedge g(s,—) = 0$ (consequence of the farness), thus, $g(s,—) \leq \bigwedge_{r<s} f(—,r)^* = f(—,s)^*$. This means that, for every $s \in \mathbb{Q}$, $g(s,—) \leq f(—,s)^*$ and then, for any rational q,

$$g(q,—) = \bigvee_{s>q} g(s,—) \leq \bigvee_{s>q} f(—,s)^* \leq \bigvee_{s'>q} f(s',—) = f(q,—),$$

which shows that $f = g$.

Theorem 6.3.9. *Let $(L,\mathscr{U})$ be a preuniform frame. The following are equivalent for any $f \in \mathscr{R}(L)$:*

(i) *f is uniformly continuous.*

(ii) *For every $\delta \in \mathbb{Q}^+$, there is some $U \in \mathscr{U}$ such that $f(—,r)$ and $f(s,—)$ are U-far for all $r,s \in \mathbb{Q}$ such that $s-r > \frac{1}{\delta}$.*

(iii) *For every $\delta \in \mathbb{Q}^+$, there is some $U \in \mathscr{U}$ such that $f(—,r)$ and $f(s,—)$ are U^n-far for every natural n and every $r,s \in \mathbb{Q}$ such that $s-r > \frac{n}{\delta}$.*

(iv) *For every $\delta \in \mathbb{Q}^+$, there is some $U \in \mathscr{U}$ such that $U^n \leq D^f_{\frac{\delta}{2n-1}}$ for every $n \in \mathbb{N}$.*

Proof. (i) $\iff$ (ii): Recall that uniform continuity is defined by (6.1.3) and apply Corollary 6.3.6 to $f = g$.
(i) $\implies$ (iii): Let $\delta \in \mathbb{Q}^+$ and consider a natural m such that $\frac{1}{m} \leq \frac{1}{\delta}$. By assumption, there is a uniform cover $U \in \mathscr{U}$ such that $U \leq f[D_{2m}] = D^f_{2m}$. We claim this is the cover we are looking for. Let $n \in \mathbb{N}$ and $r,s \in \mathbb{Q}$ such that $s - r > \frac{n}{\delta}$. If $n = 1$ then, $s - r > \frac{1}{m} > \frac{1}{2m}$. By Corollary 6.3.3, $f(\text{—}, r)$ and $f(s, \text{—})$ are D^f_{2m}-far, and since $U \leq D^f_{2m}$, they are U-far. For $n \geq 2$, suppose $f(\text{—}, r)$ and $f(s, \text{—})$ are not U^n-far. Since $U \leq D^f_{2m}$, using Proposition 6.1.1 (7) and Proposition 6.1.5 we obtain

$$U^n \leq (D^f_{2m})^n = f[D_{2m}]^n \leq f[(D_{2m})^n] = f[D_{\frac{2m}{2n-1}}] = D^f_{\frac{2m}{2n-1}}.$$

Thus, $f(\text{—}, r)$ and $f(s, \text{—})$ are not $D^f_{\frac{2m}{2n-1}}$-far. This means that there is some pair $(p,q) \in D_{\frac{2m}{2n-1}}$ such that

$$f(\text{—}, r) \wedge f(p,q) \neq 0 \quad \text{ and } \quad f(s, \text{—}) \wedge f(p,q) \neq 0.$$

It then follows that $p < r$ and $s < q$ and therefore that

$$\tfrac{n}{\delta} < s - r < q - p = \tfrac{2n-1}{2m} < \tfrac{n}{m} \leq \tfrac{n}{\delta},$$

a contradiction. Hence $f(\text{—}, r)$ and $f(s, \text{—})$ have to be U^n-far.
(iii) $\implies$ (ii) is obvious.
(iv) $\iff$ (iii): By Proposition 6.3.7. □

Recall Section 2.4. In the present situation, we have:

Corollary 6.3.10. *Let $(L, \mathscr{U})$ be a preuniform frame and let $f \in \mathscr{R}(L)$ be given by a descending (resp. ascending) scale $(a_p)_{p \in \mathbb{Q}}$. Then f is uniformly continuous if and only if for every $\delta \in \mathbb{Q}^+$ there is a $U \in \mathscr{U}$ such that a^*_r and a_s (resp. a_r and a^*_s) are U-far for any $s - r > \frac{1}{\delta}$.*

Proof. Assume f is uniformly continuous and let $\delta \in \mathbb{Q}^+$. By Theorem 6.3.9, there exists $U \in \mathscr{U}$ such that $f(\text{—}, r) = \bigvee_{p<r} a^*_p$ and $f(s, \text{—}) = \bigvee_{q>s} a_q$ are U-far for any $s - r > \frac{1}{\delta}$. Take $q - p > \frac{1}{\delta}$ and $s, r \in \mathbb{Q}$ such that $p < r < s < q$ and $s - r > \frac{1}{\delta}$. Then $a_q \leq f(s, \text{—})$ and $a^*_p \leq f(\text{—}, r)$. Hence, a_q and a^*_p are also U-far.

Conversely, let $\delta \in \mathbb{Q}^+$ and $s - r > \frac{1}{\delta}$. Take the U provided by the hypothesis and consider $u \in U$ such that $u \wedge f(s, \text{—}) \neq 0$. Then there exists $q > s$ such that $u \wedge a_q \neq 0$. By the farness hypothesis, $a^*_p \wedge u = 0$ for every $p < r$. Hence, $u \wedge f(\text{—}, r) = 0$. Therefore, $f(s, \text{—})$ and $f(\text{—}, r)$ are U-far and we may use Theorem 6.3.9 to conclude that f is uniformly continuous. □

Scales for Uniform Frames

We now identify sufficient conditions on a scale on a preuniform frame $(L, \mathscr{U})$ under which it generates a uniformly continuous real function on L.
Let $(L, \mathscr{U})$ be a preuniform frame. Consider the following conditions on a family $(a_r)_{r \in \mathbb{Q}}$ of elements of L:

(far) For every $\delta \in \mathbb{Q}^+$ there is a $V \in \mathscr{U}$ such that a^*_r and a_s are V-far for any $s - r > \frac{1}{\delta}$.

(far') For every $\delta \in \mathbb{Q}^+$ there is a $V \in \mathscr{U}$ such that a_r and a_s^* are V-far for any $s - r > \frac{1}{\delta}$.

We know already from Corollary 6.3.10 that if $(a_r)_{r\in\mathbb{Q}}$ satisfies (s1) (resp. (s1')), (s2) and (far) (resp. (far')), it induces a uniformly continuous real-valued function on L. Instead of taking (s1) and (s1') one can consider weaker conditions, namely:

(**) $a_p^{**} \leq a_q$ for every $q < p$.

(**') $a_q^{**} \leq a_p$ for every $q < p$.

Remark 6.3.11. Condition (**) together with (far) imply (s1): if $r < s$ then there is $q \in \mathbb{Q}$ such that $r < q < s$ and, by (far), there is $U \in \mathscr{U}$ such that a_q^* and a_s are U-far. In particular, this means that $a_q^{**} \vee a_s^* = 1$. Hence, using (**), we get $1 = a_q^{**} \vee a_q^* \leq a_q \vee a_s^*$, which means that $a_s \prec a_r$. Similarly, (**') together with (far') imply (s1').

Hence, by Corollary 6.3.10 and Remark 6.3.11 we have:

Proposition 6.3.12. *Let $(L,\mathscr{U})$ be a preuniform frame. If a family $(a_r)_{r\in\mathbb{Q}} \subseteq L$ satisfies (far) (resp. (far')), (**) (resp. (**')) and (s2), then the formulas*

$$f(p,\text{—}) = \bigvee_{r>p} a_r \quad \textit{and} \quad f(\text{—},q) = \bigvee_{s<q} a_s^*$$

$$(\textit{resp.} \quad f(p,\text{—}) = \bigvee_{r>p} a_r^* \quad \textit{and} \quad f(\text{—},q) = \bigvee_{s<q} a_s).$$

define a uniform homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to L$.

Inspired by condition (iv) of Theorem 6.3.9 we can define uniform scales in a different way. For that, consider the following properties, clearly stronger than (s2):

(c) For each $\delta \in \mathbb{Q}^+$ there is a $V \in \mathscr{U}$ such that

$$V \leq \{a_p \wedge a_q^* \mid (p,q) \in D_\delta\}.$$

(c') For each $\delta \in \mathbb{Q}^+$ there is a $V \in \mathscr{U}$ such that

$$V \leq \{a_p^* \wedge a_q \mid (p,q) \in D_\delta\}.$$

Remarks 6.3.13. Let $(L,\mathscr{U})$ be a preuniform frame.

(1) (c) + (ws1) $\Longrightarrow$ (s1): Let $p < q$ and consider $\delta \in \mathbb{Q}^+$ such that $q - p > \frac{1}{\delta}$. For each $r \in \mathbb{Q}$, $r \leq q$ or $r - \frac{1}{\delta} \geq p$. In the former case,

$$a_{r-\frac{1}{\delta}} \wedge a_r^* \leq a_r^* \leq a_q^* \leq a_p \vee a_q^*.$$

Otherwise, if $r - \frac{1}{\delta} \geq p$ then $a_{r-\frac{1}{\delta}} \wedge a_r^* \leq a_{r-\frac{1}{\delta}} \leq a_p \leq a_p \vee a_q^*$. Thus, $1 = \bigvee\{a_r \wedge a_s^* \mid (r,s) \in D_\delta\} \leq a_p \vee a_q^*$, that is, $a_q \prec a_p$.

(2) (far) + (**) + (s2) $\implies$ (c): Let $\delta \in \mathbb{Q}^+$, $d_r = a_r$ and $e_s = a_s^*$ for $r,s \in \mathbb{Q}$. By (**), $d_r^{**} = a_r^{**} \leq a_p^{**} = d_p$ for $p < r$, and $e_s^{**} = a_s^* \leq a_q^* = e_q$ for $s < q$. From (s2) we have

$$\bigvee_{r\in\mathbb{Q}} d_r^* = \bigvee_{r\in\mathbb{Q}} a_r^* = 1 \quad \text{and} \quad \bigvee_{s\in\mathbb{Q}} e_s = \bigvee_{s\in\mathbb{Q}} a_s^{**} \geq \bigvee_{s\in\mathbb{Q}} a_s = 1.$$

Note that by Proposition 6.2.2 (vii) the condition (far) implies that for every $\delta \in \mathbb{Q}^+$ there is a $V \in \mathscr{U}$ such that a_r^* and a_s^{**} are V-far for any $s - r > \frac{1}{\delta}$. Then for $\delta + 1$ there is a cover $U \in \mathscr{U}$ such that d_r^* and e_s^* are U-far for every $s - r > \frac{1}{\delta+1}$. Finally, from Proposition 6.3.4 we get

$$U \leq \{d_r \wedge e_s \mid (r,s) \in D_\delta\} = \{a_r \wedge a_s^* \mid (r,s) \in D_\delta\}.$$

(3) (ws1) + (c) $\implies$ (far): Let $\delta \in \mathbb{Q}^+$. By assumption there is a $V \in \mathscr{U}$ such that $V \leq \{a_r \wedge a_s^* \mid (r,s) \in D_\delta\}$. In particular, $\{a_r \wedge a_s^* \mid (r,s) \in D_\delta\}$ is a cover in $\mathscr{U}$. Furthermore, $(a_r)_{r\in\mathbb{Q}}$ is descending while $(a_r^*)_{r\in\mathbb{Q}}$ is ascending. Thus, by Proposition 6.3.2, a_r^* and a_s^{**} are $\{a_r \wedge a_s^* \mid (r,s) \in D_\delta\}$-far for every $s - r > \frac{1}{\delta}$. In particular, a_r^* and a_s are also V-far.

It follows immediately from Proposition 6.3.12 and Remarks 6.3.13 that

Proposition 6.3.14. *Let $(L,\mathscr{U})$ be a preuniform frame. If $(a_p)_{p\in\mathbb{Q}} \subseteq L$ satisfies (ws1) (resp. (ws1')) and (c) (resp. (c')), then it is an ascending uniform scale.*

It seems natural that a condition on a scale defined in terms of the uniform strong relation $\lhd_{\mathscr{U}}$ may induce a uniformly continuous function. Thus, consider the following properties:

(u) For every $\delta \in \mathbb{Q}^+$ there is some $U \in \mathscr{U}$ such that $Ua_q \leq a_p$ for every $q - p > \frac{1}{\delta}$.

(u') For every $\delta \in \mathbb{Q}^+$ there is some $U \in \mathscr{U}$ such that $Ua_p \leq a_q$ for every $q - p > \frac{1}{\delta}$.

Remarks 6.3.15. Let $(L,\mathscr{U})$ be a preuniform frame.

(1) (u) $\implies$ (s1) is obvious since (u) implies that $a_s \lhd_{\mathscr{U}} a_r$ for every $r < s$.

(2) (far) + (**) $\iff$ (u): Indeed, assume (u) and let $\delta \in \mathbb{Q}^+$. There is a cover $U \in \mathscr{U}$ such that $Ua_s \leq a_r$ whenever $s - r > \frac{1}{\delta}$. Hence, $Ua_s \wedge a_r^* = 0$, that is, a_r^* and a_s are U-far. Conversely, assume (far) and (**) and let $\delta \in \mathbb{Q}^+$ and $r,s \in \mathbb{Q}$ such that $s - r > \frac{1}{\delta}$. Consider the cover $U \in \mathscr{U}$ given by (far) and take $r' \in \mathbb{Q}$ such that $s - r' > \frac{1}{\delta}$ and $r < r'$. Then $a_{r'}^*$ and a_s are U-far. Thus, by farness and (**), $Ua_s \leq a_{r'}^{**} \leq a_r$, showing that (u) holds.

Now, it follows immediately from Proposition 6.3.12 and Remark 6.3.15 that

Proposition 6.3.16. *Let $(L,\mathscr{U})$ be a preuniform frame. If a family $(a_r)_{r\in\mathbb{Q}} \subseteq L$ satisfies (u) (resp. (u')) and (s2), then the formulas*

$$f(p,\text{—}) = \bigvee_{r>p} a_r \quad \textit{and} \quad f(\text{—},q) = \bigvee_{s<q} a_s^*$$

$$(\textit{resp.} \quad f(p,\text{—}) = \bigvee_{r>p} a_r^* \quad \textit{and} \quad f(\text{—},q) = \bigvee_{s<q} a_s).$$

define a uniform homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to L$.

Corollary 6.3.10 and Remark 6.3.15 yield:

Proposition 6.3.17. *Let $(L,\mathscr{U})$ be a preuniform frame and $f \in \mathscr{R}(L)$ be defined by a descending (resp. ascending) scale $(a_r)_{r\in\mathbb{Q}}$. Then the following conditions are equivalent:*

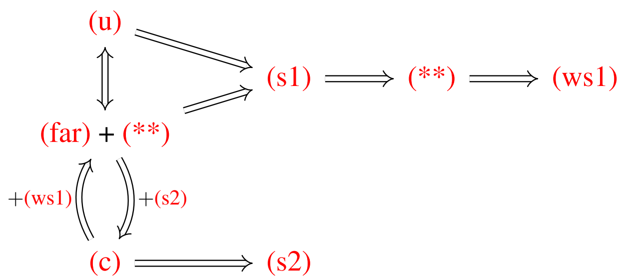
(i) *f is uniformly continuous.*

(ii) *$(a_r)_{r\in\mathbb{Q}}$ satisfies (far) (resp. (far')).*

(iii) *$(a_r)_{r\in\mathbb{Q}}$ satisfies (u) (resp. (u')).*

(iv) *$(a_r)_{r\in\mathbb{Q}}$ satisfies (c) (resp. (c'))*

The following diagram shows the implications that hold among all the conditions discussed in this section:



(6.3.2)

6.4 A Separation Result for Uniform Frames

We now prove the point-free counterpart to Smirnov functional separation result [46, p. 292]. This result provides a way to construct a uniformly continuous real-valued function that separates far elements in a preuniform frame. The proof consists of a purely algebraic (order-theoretic) construction using the Galois Adjunction $\mathsf{F}_U \dashv \mathsf{F}_U$ introduced in Section 6.2.

Denote by $\mathbb{D}$ the set of dyadic rationals in the closed unit interval $[0,1] \subseteq \mathbb{R}$:

$$\mathbb{D} = \left\{ \tfrac{m}{2^n} \mid n \in \mathbb{N},\ m = 0,1,\dots,2^n \right\} = \{0,1\} \,\dot\cup \bigcup_{n\in\mathbb{N}} \left\{ \tfrac{2k-1}{2^n} \mid k = 1,2,\dots,2^{n-1} \right\}.$$

Given a preuniformity $\mathscr{U}$ in L, let $a,b \in L$ be U-far for some $U \in \mathscr{U}$ and consider a chain of uniform covers

$$\cdots \leq U_3 \leq U_2 \leq U_1 \leq U_0 = U$$

such that $U_{n+1}^2 \leq U_n$ for every n.

By Corollary 6.2.7, if x and y are U_m-far then

$$\mathsf{F}_{U_n}(x) \vee \mathsf{F}_{U_n}(y) = 1 \tag{6.4.1}$$

for every $n > m$.

Now, define, recursively, two families $(a_d)_{d\in\mathbb{D}}$ and $(b_d)_{d\in\mathbb{D}}$, in the following way:

Definition 6.4.1. For $n = 0$,

$$a_0 = a, \quad a_1 = 1 \quad \text{and} \quad b_0 = 1, \quad b_1 = b.$$

For each $n \geq 1$,

$$a_{\frac{2k-1}{2^n}} = \mathsf{F}_{U_n}(b_{\frac{k}{2^{n-1}}}) \quad \text{and} \quad b_{\frac{2k-1}{2^n}} = \mathsf{F}_{U_n}(a_{\frac{k-1}{2^{n-1}}})$$

(cf. Table 6.1).

Lemma 6.4.2. *$a_{\frac{m-1}{2^n}}$ and $b_{\frac{m}{2^n}}$ are U_n-far for every $n \in \mathbb{N}$ and $m = 1, 2, \ldots, 2^n$.*

Proof. We proceed by induction. The fact that $a_0 = a$ and $b_1 = b$ are U_0-far is obvious. Assuming that the fact holds for $1, 2, \ldots, n$ we need to show that it also holds for $n+1$, that is, that $a_{\frac{m-1}{2^{n+1}}}$ and $b_{\frac{m}{2^{n+1}}}$ are U_{n+1}-far for $m = 1, 2, \ldots 2^{n+1}$. There are two cases:

(Case 1): $m = 2k$ for $1 \leq k \leq 2^n$.
Then

$$a_{\frac{m-1}{2^{n+1}}} = a_{\frac{2k-1}{2^{n+1}}} = \mathsf{F}_{U_{n+1}}(b_{\frac{k}{2^n}}) \quad \text{and} \quad b_{\frac{m}{2^{n+1}}} = b_{\frac{k}{2^n}},$$

that is,

$$a_{\frac{m-1}{2^{n+1}}} = \mathsf{F}_{U_{n+1}}(b_{\frac{m}{2^{n+1}}}),$$

which implies that $a_{\frac{m-1}{2^{n+1}}}$ and $b_{\frac{m}{2^{n+1}}}$ are U_{n+1}-far.

(Case 2): $m = 2k-1$ for $1 \leq k \leq 2^n$.
In this case, $a_{\frac{m-1}{2^{n+1}}} = a_{\frac{2k-2}{2^{n+1}}} = a_{\frac{k-1}{2^n}}$, and thus

$$b_{\frac{m}{2^{n+1}}} = b_{\frac{2k-1}{2^{n+1}}} = \mathsf{F}_{U_{n+1}}(a_{\frac{k-1}{2^n}}) = \mathsf{F}_{U_{n+1}}(a_{\frac{m-1}{2^{n+1}}}).$$ □

Lemma 6.4.3. *$a_{\frac{m}{2^n}} \vee b_{\frac{m}{2^n}} = 1$ for every $n \in \mathbb{N}$ and $m = 0, 1, \ldots, 2^n$.*

Proof. We proceed by induction. For $n = 0$ we clearly have

$$a_0 \vee b_0 = a \vee 1 = 1 \quad \text{and} \quad a_1 \vee b_1 = 1 \vee b = 1.$$

Assume it holds for n, and consider $a_{\frac{m}{2^{n+1}}}$ and $b_{\frac{m}{2^{n+1}}}$. Again, if $m = 2k$ for some $0 \leq k \leq 2^n$, then by the inductive hypothesis we have

$$a_{\frac{m}{2^{n+1}}} \vee b_{\frac{m}{2^{n+1}}} = a_{\frac{k}{2^n}} \vee b_{\frac{k}{2^n}} = 1.$$

Otherwise, if $m = 2k-1$ for some $1 \leq k \leq 2^n$, then

$$a_{\frac{m}{2^{n+1}}} \vee b_{\frac{m}{2^{n+1}}} = \mathsf{F}_{U_{n+1}}(b_{\frac{k}{2^n}}) \vee \mathsf{F}_{U_{n+1}}(a_{\frac{k-1}{2^n}}) = 1$$

where the last equality follows from (6.4.1) and the fact that, by 6.4.2, $a_{\frac{k-1}{2^n}}$ and $b_{\frac{k}{2^n}}$ are U_n-far. □

Lemma 6.4.4. *$(a_d)_{d \in \mathbb{D}}$ is an ascending family while $(b_d)_{d \in \mathbb{D}}$ is a descending family.*

Proof. It suffices to show that $a_0 \leq a_1$ (which is obvious) and that

$$a_{\frac{k-1}{2^n}} \leq a_{\frac{2k-1}{2^{n+1}}} \leq a_{\frac{k}{2^n}} \quad \text{for} \;\; n \in \mathbb{N},\; k = 1, 2, \ldots, 2^n.$$

0	1	2	3	4
$a_1 = 1$				
				$a_{\frac{15}{16}} = \mathsf{F}_{U_4}(b)$
			$a_{\frac{7}{8}} = \mathsf{F}_{U_3}(b)$	
				$a_{\frac{13}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_3}\mathsf{F}_{U_2}(b)$
		$a_{\frac{3}{4}} = \mathsf{F}_{U_2}(b)$		
				$a_{\frac{11}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_2}\mathsf{F}_{U_1}(b)$
			$a_{\frac{5}{8}} = \mathsf{F}_{U_3}\mathsf{F}_{U_2}\mathsf{F}_{U_1}(b)$	
				$a_{\frac{9}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_3}\mathsf{F}_{U_1}(b)$
	$a_{\frac{1}{2}} = \mathsf{F}_{U_1}(b)$			
				$a_{\frac{7}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_1}(a)$
			$a_{\frac{3}{8}} = \mathsf{F}_{U_3}\mathsf{F}_{U_1}(a)$	
				$a_{\frac{5}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_3}\mathsf{F}_{U_2}\mathsf{F}_{U_1}(a)$
		$a_{\frac{1}{4}} = \mathsf{F}_{U_2}\mathsf{F}_{U_1}(a)$		
				$a_{\frac{3}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_2}(a)$
			$a_{\frac{1}{8}} = \mathsf{F}_{U_3}\mathsf{F}_{U_2}(a)$	
				$a_{\frac{1}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_3}(a)$
$a_0 = a$				
$b_1 = b$				
				$b_{\frac{15}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_3}(b)$
			$b_{\frac{7}{8}} = \mathsf{F}_{U_3}\mathsf{F}_{U_2}(b)$	
				$b_{\frac{13}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_2}(b)$
		$b_{\frac{3}{4}} = \mathsf{F}_{U_2}\mathsf{F}_{U_1}(b)$		
				$b_{\frac{11}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_3}\mathsf{F}_{U_2}\mathsf{F}_{U_1}(b)$
			$b_{\frac{5}{8}} = \mathsf{F}_{U_3}\mathsf{F}_{U_1}(b)$	
				$b_{\frac{9}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_1}(b)$
	$b_{\frac{1}{2}} = \mathsf{F}_{U_1}(a)$			
				$b_{\frac{7}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_3}\mathsf{F}_{U_1}(a)$
			$b_{\frac{3}{8}} = \mathsf{F}_{U_3}\mathsf{F}_{U_2}\mathsf{F}_{U_1}(a)$	
				$b_{\frac{5}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_2}\mathsf{F}_{U_1}(a)$
		$b_{\frac{1}{4}} = \mathsf{F}_{U_2}(a)$		
				$b_{\frac{3}{16}} = \mathsf{F}_{U_4}\mathsf{F}_{U_3}\mathsf{F}_{U_2}(a)$
			$b_{\frac{1}{8}} = \mathsf{F}_{U_3}(a)$	
				$b_{\frac{1}{16}} = \mathsf{F}_{U_4}(a)$
$b_0 = 1$				

Table 6.1 Definition of $(a_d)_{d\in\mathbb{D}}$ and $(b_d)_{d\in\mathbb{D}}$ for $n = 0,1,2,3,4$.

By Lemma 6.4.2, $a_{\frac{k-1}{2^n}}$ and $b_{\frac{k}{2^n}}$ are U_n-far. Since $U_{n+1} \leq U_n$, they are also U_{n+1}- far, hence

$$a_{\frac{k-1}{2^n}} \leq \mathsf{F}_{U_{n+1}}(b_{\frac{k}{2^n}}) = a_{\frac{2k-1}{2^{n+1}}}.$$

Furthermore, by Lemma 6.4.3,

$$1 = b_{\frac{k}{2^n}} \vee a_{\frac{k}{2^n}} \leq \mathsf{S}_{U_{n+1}}(b_{\frac{k}{2^n}}) \vee a_{\frac{k}{2^n}},$$

which implies

$$a_{\frac{2k-1}{2^{n+1}}} = \mathsf{F}_{U_{n+1}}(b_{\frac{k}{2^n}}) \leq a_{\frac{k}{2^n}}.$$

The proof for $(b_d)_{d\in\mathbb{D}}$ is similar. □

Theorem 6.4.5. *Let $\mathscr{U}$ be a preuniformity on a frame L. If a and b are U-far, for some $U \in \mathscr{U}$, then there is a uniformly continuous $f \in \mathscr{R}(L)$ such that $\mathbf{0} \leq f \leq \mathbf{1}$, $f(0,—) \leq a^*$ and $f(—,1) \leq b^*$.*

Proof. Let $(a_d)_{d\in\mathbb{D}}$ and $(b_d)_{d\in\mathbb{D}}$ be the families defined in Definition 6.4.1. We extend $(a_d)_{d\in\mathbb{D}}$ to $\mathbb{Q}$ using the procedure of Banaschewski in the proof of the point-free Urysohn's Lemma [10, Proposition. 5]: for every $r,s \in \mathbb{Q}$ let

$$c_r = \begin{cases} 0 & \text{if } r < 0 \\ \bigvee\{a_{\frac{m}{2^n}} \mid \frac{m}{2^n} \leq r\} & \text{if } 0 \leq r \leq 1 \\ 1 & \text{if } 1 < r \end{cases}$$

We will show that $(c_r)_{r\in\mathbb{Q}}$ satisfies (s2) and (u'). Observe that $(c_r)_{r\in\mathbb{Q}}$ is ascending and, trivially, (s2) holds:

$$\bigvee_{r\in\mathbb{Q}} c_r = 1 = \bigvee_{r\in\mathbb{Q}} c_r^*.$$

Now, we claim that property (u') also holds. Indeed, let $1 \leq \delta \in \mathbb{Q}$ (because of how we defined the c_r's notice that the case $\delta < 1$ is trivial). Take $n \in \mathbb{N}$ such that $\delta \leq 2^n$. We will show U_{n+1} is the cover we are looking for. Let $s,r \in \mathbb{Q}$ such that $s - r > \frac{1}{\delta}$. Clearly, for $r < 0$ or $1 < s$, we have $U_{n+1}c_r \leq c_s$. Consider $0 \leq r < s \leq 1$, since $\mathbb{D}$ is dense and $s - r > \frac{1}{\delta} \geq \frac{1}{2^n}$, there is $0 \leq m \leq 2^n$ such that $r \leq \frac{m}{2^n} \leq s$. Let

$$m_0 = \max\{m \mid 0 \leq m \leq 2^n \text{ and } r \leq \tfrac{m}{2^n} \leq s\}.$$

Then $r < \frac{m_0}{2^n} \leq s$ and $r \leq \frac{2m_0-1}{2^{n+1}} < \frac{2m_0}{2^{n+1}} \leq s$. By Lemma 6.4.2, $a_{\frac{2m_0-1}{2^{n+1}}}$ and $b_{\frac{2m_0}{2^{n+1}}}$ are U_{n+1}-far. Thus, $U_{n+1} \cdot a_{\frac{2m_0-1}{2^{n+1}}} \leq b^*_{\frac{2m_0}{2^{n+1}}}$. Since $(a_r)_{r\in\mathbb{D}}$ is ascending (Lemma 6.4.4),

$$U_{n+1} \cdot c_r = U_{n+1} \cdot \Big(\bigvee_{\frac{m}{2^n} \leq r} a_{\frac{m}{2^n}}\Big) \leq U_{n+1} \cdot a_{\frac{2m_0-1}{2^{n+1}}} \leq b^*_{\frac{2m_0-1}{2^{n+1}}}.$$

By Lemma 6.4.3, $b^*_{\frac{2m_0-1}{2^{n+1}}} \leq a_{\frac{2m_0-1}{2^{n+1}}} \leq c_s$. Hence, $U_{n+1} \cdot c_r \leq c_s$, as required. By Proposition 6.3.16, $(c_p)_{p\in\mathbb{Q}}$ defines a uniformly continuous $f \in \mathscr{R}(L)$ given by the formulas

$$f(—,r) = \bigvee_{p<r} c_p \quad \text{and} \quad f(s,—) = \bigvee_{q>s} c_q^*.$$

Notice that $a \leq f(-,r)$ for every $r > 0$. Indeed, $f(-,r) = \bigvee_{q<r} c_q \geq c_0 = a_0 = a$. Moreover, $b \leq f(s,-)$ for every $s < 1$. Since $s < 1$, there is some $n \in \mathbb{N}$ such $s < \frac{2^n-1}{2^n} < 1$, and $a_{\frac{2^n-1}{2^n}}$ and b_1 are U_n-far (by Lemma 6.4.2). Hence,

$$f(s,-) = \bigvee_{q>s} c_q^* \geq c^*_{\frac{2^n-1}{2^n}} = a^*_{\frac{2^n-1}{2^n}} \geq b_1 = b.$$

Moreover, $f(0,-) \leq a^*$:

$$a \wedge f(0,-) = a_0 \wedge \bigvee_{q>0} c_q^* \leq a_0 \wedge \bigvee_{n\in\mathbb{N}} a^*_{\frac{1}{2^n}} = \bigvee_{n\in\mathbb{N}} a_0 \wedge a^*_{\frac{1}{2^n}} \leq \bigvee_{n\in\mathbb{N}} a_0 \wedge b_{\frac{1}{2^n}} = 0$$

(where the last inequality follows from Lemma 6.4.3 and the last equality from Lemma 6.4.2). Similarly, $f(-,1) \leq b^*$:

$$b \wedge f(-,1) = b_1 \wedge \bigvee_{q<1} c_q \leq b_1 \wedge \bigvee_{n\in\mathbb{N}} a_{\frac{2^n-1}{2^n}} = \bigvee_{n\in\mathbb{N}} b_1 \wedge a_{\frac{2^n-1}{2^n}} = 0$$

(where the last equality holds by Lemma 6.4.2).

Finally, it is obvious from the definition of f that $f(-,0) \vee f(1,-) = 0$. Hence, $\mathbf{0} \leq f \leq \mathbf{1}$ and f is bounded. □

Corollary 6.4.6. *For each preuniformity $\mathscr{U}$ on a frame L, elements $a,b \in L$ are U-far for some $U \in \mathscr{U}$ if and only if there is a uniformly continuous $f \in \mathscr{R}(L)$ such that $\mathbf{0} \leq f \leq \mathbf{1}$, $f(0,-) \leq a^*$ and $f(-,1) \leq b^*$.*

Proof. If $f(0,-) \leq a^*$ and $f(-,1) \leq b^*$ for some uniformly continuous $f \in \mathscr{R}(L)$, then $a \leq f(0,-)^*$ and $b \leq f(-,1)^*$ and thus, by Theorem 6.3.9 (recall also Lemma 6.3.1 and Remark 6.2.1), a and b are U-far for some $U \in \mathscr{U}$. □

6.5 An Extension Result for Uniform Frames

In this final section we prove a Tietze-type extension theorem for uniform homomorphisms. This result provides a uniform extension of any uniformly continuous real function on a dense sublocale S of L to the whole of L and is based on a well-known general result for lattices (Lemma 6.5.1 below, known as the Katětov Lemma) that extends the original basic lemma of Katětov, formulated for power sets in his 1951 celebrated paper (corrected in 1953 [54]).

Recall that a binary relation $\Subset$ on a lattice L is a *Katětov relation* if it satisfies the following conditions for all $a,b,a',b' \in L$:

(K1) $a \Subset b \implies a \leq b$;

(K2) $a' \leq a \Subset b \leq b' \implies a' \Subset b'$;

(K3) $a \Subset b$ and $a' \Subset b \implies (a \vee a') \Subset b$;

(K4) $a \Subset b$ and $a \Subset b' \implies a \Subset (b \wedge b')$;

(K5) $a \Subset b \implies \exists c \in L\colon a \Subset c \Subset b$.

The following result ([55, 56]) extends the original idea of Katětov from power sets to complete lattices.

Lemma 6.5.1 (Katětov's Lemma)**.** *Let L be a complete lattice, $\Subset$ a Katětov relation on L and $\lhd$ a transitive and irreflexive relation on a countable set D. Further, let $(a_d)_{d\in D}$ and $(b_d)_{d\in D}$ be two families of elements of L such that*

$$d_1 \lhd d_2 \quad \textit{implies} \quad a_{d_2} \leq a_{d_1}, \quad b_{d_2} \leq b_{d_1} \quad \textit{and} \quad a_{d_2} \Subset b_{d_1}.$$

Then there exists a family $(c_d)_{d\in D} \subseteq L$ such that

$$d_1 \lhd d_2 \quad \textit{implies} \quad c_{d_2} \Subset c_{d_1}, \quad a_{d_2} \Subset c_{d_1} \quad \textit{and} \quad c_{d_2} \Subset b_{d_1}.$$

Example 6.5.2. Let $(L,\mathscr{U})$ be a preuniform frame. From Proposition 6.1.2 and Remark 6.1.3 (1) we know that the uniformly below relation $\lhd_{\mathscr{U}}$ given by the preuniformity $\mathscr{U}$ is a Katětov relation.

Let $(L,\mathscr{U})$ be a (pre)uniform frame and S a sublocale of L with $j_S\colon S \hookrightarrow L$ the localic embedding of S in L. It is shown in [11, Lemma 2.2] that the system

$$\mathscr{U}_S^L := \{j_S^*[U] \mid U \in \mathscr{U}\}$$

is a (pre)uniformity in S. [1]

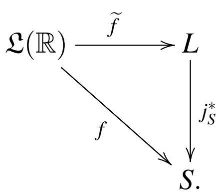
Remarks 6.5.3. (1) $a \lhd_{\mathscr{U}} b \implies j_S^*(a) \lhd_{\mathscr{U}_S} j_S^*(b)$ for any $a,b \in L$.

(2) $\mathscr{U}_S \subseteq \mathscr{U}$ (since $U \leq j_S^*[U]$ for every $U \in \mathscr{U}$).

(3) In case S is dense, since meets in S are computed as in L and $0_S = 0_L$, then, for any $a,b \in S$, if a and b are U-far in S for some $U \in \mathscr{U}_S$, they are also U-far in L.

(4) Let S be a sublocale of L, and let T be a sublocale of S. It is then easy to see that $\mathscr{U}_T^L = \left(\mathscr{U}_S^L\right)_T^S$.

Let $f\colon \mathfrak{L}(\mathbb{R}) \to (S,\mathscr{U}_S^L)$ be a uniform homomorphism, we say that $\widetilde{f}\colon \mathfrak{L}(\mathbb{R}) \to (L,\mathscr{U})$ is a *uniform extension* of f if the following diagram commutes:



(6.5.1)

Now, we present an extension result for dense sublocales ([7, Theorem 7.3]):

Lemma 6.5.4. *Let $(L,\mathscr{U})$ be a preuniform frame and S be a dense sublocale of L. Any uniform homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to (S,\mathscr{U}_S)$ has a uniform extension $\tilde{f}\colon \mathfrak{L}(\mathbb{R}) \to (L,\mathscr{U})$. Furthermore, if f is bounded, then so is $\widetilde{f}$.*

[1] We shall simply denote $\mathscr{U}_S^L$ by $\mathscr{U}_S$ when there is no danger of confusion.

Proof. Let $f \in \mathscr{R}(S)$ be a uniform homomorphism. By Theorem 6.3.9 (ii) and Lemma 6.3.1 we know that for each $\delta \in \mathbb{Q}^+$ there is some $U_\delta \in \mathscr{U}_S$ such that $f(p,—)^{*s}$ and $f(q,—)$ are U_δ-far in S for every $p,q \in \mathbb{Q}$ with $q - p > \frac{1}{\delta}$. By Remark 6.5.3 (3) we have

$$\forall \delta \in \mathbb{Q}^+ \ \exists U_\delta \in \mathscr{U} \text{ such that } f(p,—)^{*s} \text{ and } f(q,—) \text{ are } U_\delta\text{-far in } L \text{ whenever } q - p > \tfrac{1}{\delta}. \quad (6.5.2)$$

Equivalently, by Proposition 6.2.2 (vii),

$$\forall \delta \in \mathbb{Q}^+ \ \exists U_\delta \in \mathscr{U} \text{ such that } f(p,—)^{*s**} \text{ and } f(q,—)^{**} \text{ are } U_\delta\text{-far in } L \text{ whenever } q - p > \tfrac{1}{\delta}. \quad (6.5.3)$$

By Remark 6.2.3, condition (6.5.2) implies

$$f(q,—) \lhd_{\mathscr{U}} f(p,—)^{*s*}$$

for every $p < q$. Denoting $f(q,—)$ by a_q and $f(p,—)^{*s*}$ by b_p and taking $D = \mathbb{Q}$, $\lhd = <$ and $\Subset = \lhd_{\mathscr{U}}$ (Example 6.5.2) the assumptions of Katětov's Lemma hold; thus, there exists a family $(c_p)_{p\in\mathbb{Q}}$ of elements of L such that

$$c_q \lhd_{\mathscr{U}} c_p, \quad a_q \lhd_{\mathscr{U}} c_p \quad \text{and} \quad c_q \lhd_{\mathscr{U}} b_p \qquad (6.5.4)$$

for every rationals $p < q$. Note that $a_p^{*s*} = b_p$.

Claim 1 *$(c_p)_{p\in\mathbb{Q}}$ defines a uniform homomorphism $\widetilde{f} \in \mathscr{R}(L)$.*

To prove the claim, we will use Proposition 6.3.12 and show that $(c_p)_{p\in\mathbb{Q}}$ satisfies (**), (s2) and (far).

(**): Let $p < q$. Then $c_q \lhd_{\mathscr{U}} c_p$. In particular, by Proposition 6.1.2 (4), $c_q \prec c_p$, which implies $c_q^{**} \le c_p$.

(s2): Let $\delta \in \mathbb{Q}^+$. Since f is a uniform homomorphism there is a $U \in \mathscr{U}$ such that

$$j_S^*[U] \le f[D_\delta] = \{f(r,s) \mid (r,s) \in D_\delta\} \le \{a_r \mid r \in \mathbb{Q}\}.$$

From (6.5.4) and Remark 6.5.3 (2), we have $U \le j_S^*[U] \le \{c_p \mid p \in \mathbb{Q}\}$. Then $1 = \bigvee U \le \bigvee_{p\in\mathbb{Q}} c_p$. On the other hand,

$$j_S^*[U] \le \{f(r,s) \mid (r,s) \in D_\delta\} \le \{f(—,s) \mid s \in \mathbb{Q}\}$$

and therefore, by (2.1.1),

$$j_S^*[U] \le \{f(p,—)^{*s} \mid p \in \mathbb{Q}\} \le \{f(p,—)^{*s**} \mid p \in \mathbb{Q}\} = \{b_p^* \mid p \in \mathbb{Q}\}.$$

It then follows from (6.5.4) that $b_p^* \le c_q^*$ for every $p < q$. Hence

$$\{b_p^* \mid p \in \mathbb{Q}\} \le \{c_p^* \mid p \in \mathbb{Q}\}$$

and from Remark 6.5.3 (2), $U \le j_S^*[U] \le \{c_p^* \mid p \in \mathbb{Q}\}$ so $1 = \bigvee U \le \bigvee_{p\in\mathbb{Q}} c_p^*$.

(far): Let $\delta \in \mathbb{Q}^+$. We will show that there is a $U \in \mathscr{U}$ such that c_p^* and c_q are U-far whenever $q - p > \frac{1}{\delta}$. We claim that the cover U_δ given by (6.5.3) satisfies this property. Let $p,q \in \mathbb{Q}$ such that $q - p > \frac{1}{\delta}$. Then there exist $r,s \in \mathbb{Q}$ such that $p < r < s < q$ and $s - r > \frac{1}{\delta}$. Since $p < r$, by (6.5.4), we have

$$a_r \lhd_{\mathscr{U}} c_p \implies a_r \le c_p \implies c_p^* \le a_r^*$$

and $a_r^* \le a_r^{*s} \le a_r^{*s**} = b_r^*$. Hence, $c_p^* \le b_r^*$. Again, by (6.5.4) we have

$$c_q \lhd_{\mathscr{U}} b_s \implies c_q \le b_s \implies b_s^* \le c_q^*$$

and $a_s^* \le a_s^{*s} \le a_s^{*s**} = b_s^*$. Thus, $c_q \le c_q^{**} \le b_s^{**} \le a_s^{**}$. By (6.5.3), b_r^* and a_s^{**} are U_δ-far, and since $c_p^* \le b_r^*$ and $c_q \le a_s^{**}$, so are c_p^* and c_q.

From **Claim 1** it follows, using Proposition 6.3.12, that the formulas

$$\widetilde{f}(p,\text{—}) = \bigvee_{r>p} c_r \quad \text{and} \quad \widetilde{f}(\text{—},q) = \bigvee_{s<q} c_s^*$$

define a uniformly continuous $\widetilde{f} \in \mathscr{R}(L)$.

Claim 2 *$\widetilde{f}$ extends f, that is, $j_S^*\widetilde{f} = f$.*
By (6.5.4), we know that $\bigsqcup_{r>p} j_S^*(c_r) \ge \bigsqcup_{r>p} j_S^*(a_r)$.[2] Hence,

$$j_S^*\widetilde{f}(p,\text{—}) = \bigsqcup_{r>p} j_S^*(c_r) \ge \bigsqcup_{r>p} j_S^*(a_r) = \bigsqcup_{r>p} a_r = \bigsqcup_{r>p} f(r,\text{—}) = f(p,\text{—}).$$

For the other inequality notice that, from (6.5.4), we have that $\bigsqcup_{r>p} j_S^*(c_r) \le \bigsqcup_{r>p} j_S^*(b_r)$. Then,

$$\begin{aligned} j_S^*\widetilde{f}(p,\text{—}) &= \bigsqcup_{r>p} j_S^*(c_r) \le \bigsqcup_{r>p} j_S^*(b_r) = \bigsqcup_{r>p} j_S^*(f(r,\text{—})^{*s*}) \\ &\le \bigsqcup_{r>p} j_S^*(f(r,\text{—})^{*s*s}) = \bigsqcup_{r>p} (f(r,\text{—})^{*s*s}). \end{aligned}$$

Finally, since $f(r,\text{—}) \le f(r,\text{—})^{*s*s} \le f(t,\text{—})$ for every $t < r$, we obtain $j_S^*\widetilde{f}(p,\text{—}) \le \bigsqcup_{t>p} f(t,\text{—}) = f(p,\text{—})$ for every $p \in \mathbb{Q}$.

Furthermore, if f is bounded, say $\mathbf{p} \le f \le \mathbf{q}$ for rationals $p \le q$, then (recall (2.1.3)):

$$f(r,\text{—}) \wedge f(\text{—},s) = 1 \text{ for every } r,s \in \mathbb{Q} \text{ with } r < p \le q < s.$$

Thus, for any $r < p$ we have that

$$\widetilde{f}(r,\text{—}) = \bigvee_{t>r} c_t \ge c_{t'} \ge a_{r'} = f(r',\text{—}) = 1$$

for some $t', r' \in \mathbb{Q}$ with $r < t' < r' < p$. And for any $s > q$ we get that

$$\widetilde{f}(\text{—},s) = \bigvee_{t<s} c_t^* \ge c_{t'}^* \ge b_{s'}^* = f(s',\text{—})^{*s**} \ge f(s',\text{—})^{*s} \ge f(\text{—},s')$$

for some $t', s' \in \mathbb{Q}$ with $q < s' < t' < s$. Consequently, $\mathbf{p} \le \widetilde{f} \le \mathbf{q}$. □

[2] In this proof, to simplify notation, we denote by $\bigsqcup$ the joins in S.

Remark 6.5.5. There is an alternative proof for Lemma 6.5.4. One can replace **Claim 1** by:

$(c_p)_{p\in\mathbb{Q}}$ *defines a frame homomorphism* $\overline{f} \in \mathscr{R}(L)$.

Then one applies the following general principle:

Uniform Extension Principle. *In the commutative diagram* (6.5.1)*, if S is dense and the given f is uniform then $\widetilde{f}$ is also uniform.*

Proof. Let $\delta \in \mathbb{Q}^+$. By Theorem 6.3.9 (ii), there is a $U \in \mathscr{U}$ such that $f(\text{—},r)$ and $f(s,\text{—})$ are $j_S^*[U]$-far in S for every $r,s \in \mathbb{Q}$ such that $s-r > \frac{1}{\delta}$. By Remark 6.5.3 (3), $f(\text{—},r)$ and $f(s,\text{—})$ are also $j_S^*[U]$-far in L. Furthermore, since

$$U \le j_S^*[U], \quad \widetilde{f}(\text{—},r) \le j_S^*\widetilde{f}(\text{—},r) = f(\text{—},r) \quad \text{and} \quad \widetilde{f}(s,\text{—}) \le j_S^*\widetilde{f}(s,\text{—}) = f(s,\text{—}),$$

$\widetilde{f}(\text{—},r)$ and $\widetilde{f}(s,\text{—})$ are U-far in L for every $r,s \in \mathbb{Q}$ such that $s-r > \frac{1}{\delta}$ (by Remark 6.2.1). Hence, by Proposition 7.3.14 (iii), the extension $\widetilde{f}$ is also uniform. □

Chapter 7

An Insertion Theorem for Uniform Frames

In this chapter we prove the point-free version of Preiss and Vilimovský's insertion result for uniform spaces [71]. For this purpose, in Section 7.1 we recall and extend some results for prediameters from [73]. Further, to state (and prove) our insertion theorem we need to extend the farness relation from Section 6.2 to sublocales and to define uniform continuity for general real-valued functions (Section 7.2). In Sections 7.3 and 7.4 we present a study of uniform continuity similar to the one in the previous chapter, but in a more general setting. Finally, Section 7.5 is devoted to the proof of the insertion theorem, and Section 7.6 presents two important corollaries. This chapter should be thought of as a generalization of the theory developed in Chapter 6.2, and it is based on the author's paper with Igor Arrieta [3].

7.1 Prediameters

Let us recall that a *prediameter* on a frame L is a function $d\colon L \to [0,+\infty]$ with the following properties ([73, 1.2] or [66, XI.3.1]):

(PD1) $d(0) = 0$.

(PD2) $a \leq b \implies d(a) \leq d(b)$ for all $a, b \in L$.

(PD3) For all $\varepsilon > 0$, the set $\{\, a \in L \mid d(a) < \varepsilon \,\}$ is a cover of L.

Consider now the following two properties:

(PD4) If $a, b \in L$ are such that $a \wedge b \neq 0$, then $d(a \vee b) \leq d(a) + d(b)$.

(PD5) If $a, b \in L$ are such that $a \wedge b \neq 0$, then $d(a \vee b) \leq 2\max\{\, d(a), d(b) \,\}$ (and so, in particular, $d(a \vee b) \leq 2d(a) + 2d(b)$).

Clearly, (PD4) implies (PD5). A prediameter satisfying (PD4) is called a *diameter*. Moreover, a prediameter satisfying (PD5) is a *weak diameter* ([73]). The latter should not be confused with the notion of *strong prediameter* ([66]); i.e., a prediameter which additionally satisfies

(PD6) If $S \subseteq L$ is such that $a \wedge b \neq 0$ for all $a, b \in S$, then $d(\bigvee S) \leq 2\sup\{\, d(s) \mid s \in S \,\}$.

Clearly, every strong prediameter is a weak diameter. For our purposes we shall be interested only in weak diameters, but in passing we shall also present an application to strong prediameters. The following lemma about weak diameters will be crucial in the proof of our uniform insertion theorem.

Lemma 7.1.1. *Let $d\colon L \to [0,+\infty]$ be a weak diameter on a frame L. Let $a_1,\dots,a_k \in L$ with $a_{i-1} \wedge a_i \neq 0$ for all $i = 2,\dots,k$. Then,*

$$d\Big(\bigvee_{j=1}^{k} a_j\Big) \leq 2d(a_1 \vee a_2) + 4\sum_{i=3}^{k-1} d(a_{i-1} \vee a_i) + 2d(a_{k-1} \vee a_k).$$

Proof. Obviously we can assume that every summand in the right hand side is finite (and in that case, by (PD5), the left hand side is also readily seen to be finite, and so each $d(\bigvee_{j=1}^{i} a_j)$ is also finite). We proceed by induction over k. If $k = 1$ or $k = 2$ there is nothing to prove. If $k = 3$, we have $d(a_1 \vee a_2 \vee a_3) \leq 2d(a_1 \vee a_2) + 2d(a_2 \vee a_3)$ by (PD5). Assume now it holds for all sequences of length $< k$ and let $a_1,\dots,a_k \in L$ with $a_{i-1} \wedge a_i \neq 0$ for all $i = 2,\dots,k$. Let

$$A := \Big\{ i \in \{1,\dots,k\} \mid d\Big(\bigvee_{j=1}^{k} a_j\Big) \leq 2d\Big(\bigvee_{\ell=1}^{i} a_\ell\Big)\Big\}.$$

One has trivially $k \in A$, so $A \neq \varnothing$, hence there is a well-defined $m = \min A$. If $m = 1$ or $m = 2$, the formula in the statement holds trivially so assume $m > 2$. By minimality (and because $m > 1$) $m - 1 \notin A$; i.e., $2d(\bigvee_{\ell=1}^{m-1} a_\ell) < d(\bigvee_{j=1}^{k} a_j)$.

Now, by way of contradiction suppose $2d(\bigvee_{\ell=m-1}^{k} a_\ell) < d(\bigvee_{j=1}^{k} a_j)$. Then

$$2\max\Big\{d\Big(\bigvee_{\ell=1}^{m-1} a_\ell\Big), d\Big(\bigvee_{\ell=m-1}^{k} a_\ell\Big)\Big\} < d\Big(\bigvee_{j=1}^{k} a_j\Big). \tag{7.1.1}$$

But

$$d\Big(\bigvee_{j=1}^{k} a_j\Big) = d\Big(\bigvee_{\ell=1}^{m-1} a_\ell \vee \bigvee_{\ell=m-1}^{k} a_\ell\Big)$$

and $(\bigvee_{\ell=1}^{m-1} a_\ell) \wedge (\bigvee_{\ell=m-1}^{k} a_\ell) \geq a_{m-1} \neq 0$, so by (PD5) it follows that

$$d\Big(\bigvee_{j=1}^{k} a_j\Big) \leq 2\max\Big\{d\Big(\bigvee_{\ell=1}^{m-1} a_\ell\Big), d\Big(\bigvee_{\ell=m-1}^{k} a_\ell\Big)\Big\}. \tag{7.1.2}$$

Combining (7.1.1) and (7.1.2) we reach a contradiction. Hence, we have

$$d\Big(\bigvee_{j=1}^{k} a_j\Big) \leq 2d\Big(\bigvee_{\ell=m-1}^{k} a_\ell\Big). \tag{7.1.3}$$

Now, if $m = k$, from (7.1.3) we see that the desired formula holds, so we may as well assume $m < k$. Now, we have

$$d\Big(\bigvee_{j=1}^{k} a_j\Big) = \tfrac{1}{2}d\Big(\bigvee_{j=1}^{k} a_j\Big) + \tfrac{1}{2}d\Big(\bigvee_{j=1}^{k} a_j\Big) \leq d\Big(\bigvee_{\ell=1}^{m} a_\ell\Big) + d\Big(\bigvee_{\ell=m-1}^{k} a_\ell\Big) \tag{7.1.4}$$

(by (7.1.3) and the fact that $m \in A$). We use induction twice:

$$d\Big(\bigvee_{\ell=1}^{m} a_\ell\Big) \leq 2d(a_1 \vee a_2) + 4\sum_{i=3}^{m-1} d(a_{i-1} \vee a_i) + 2d(a_{m-1} \vee a_m)$$

and

$$d\Big(\bigvee_{\ell=m-1}^{k} a_\ell\Big) \leq 2d(a_{m-1} \vee a_m) + 4\sum_{i=m+1}^{k-1} d(a_{i-1} \vee a_i) + 2d(a_{k-1} \vee a_k).$$

This together with (7.1.4) gives the desired inequality. □

The combination of the previous lemma with (PD5) yields the following:

Corollary 7.1.2. *Let $d\colon L \to [0,+\infty]$ be a weak diameter on a frame L. Let $a_1,\dots,a_k \in L$ with $a_i \wedge a_{i-1} \neq 0$ for all $i = 2,\dots,k$. Then*

$$d\Big(\bigvee_{j=1}^{k} a_j\Big) \leq 4d(a_1) + 12d(a_2) + 16\sum_{i=3}^{k-2} d(a_i) + 12d(a_{k-1}) + 4d(a_k).$$

Remark 7.1.3. The last corollary is, in a certain sense, an improvement of [73, Lemma 3.9] (cf. also [66, Lemma XI.3.2.4]), which shows a similar inequality whenever d satisfies a property stronger than (PD5) (too strong for our purposes), namely:

(3W) If $a,b,c \in L$ are such that $a \wedge b \neq 0 \neq b \wedge c$, then $d(a \vee b \vee c) \leq 2\max\{d(a), d(b), d(c)\}$.

Of course, the price one has to pay for considering (PD5) instead of (3W) is that the inequality in Corollary 7.1.2 is not as sharp as that in [73, Lemma 3.9].

The following result is an application of Lemma 7.1.1 to strong prediameters. For that, we first recall the definition of a star-additive diameter ([66, XI.1.2]). It is an important notion, since any such diameter immediately induces a uniformity on L, and it can be satisfactorily approximated by a *metric diameter* ([66, XI.1.3]). A diameter d is said to be *star-additive* whenever the following condition holds:

(DS) If $a \in L$ and $S \subseteq L$ are such that $a \wedge b \neq 0$ for all $b \in S$, then

$$d\Big(a \vee \bigvee S\Big) \leq d(a) + \sup\{d(b) + d(c) \mid b,c \in S\}.$$

We then have the following (compare with [66, Proposition XI.3.2.5]):

Proposition 7.1.4. *Let L be a frame and d a strong prediameter on L. Then there is a star-additive diameter d' on L such that*

$$\tfrac{1}{32}d \leq d' \leq d.$$

We omit the details of the proof, as it is very similar to [66, Proposition XI.3.2.5] (instead of [66, Lemma XI.3.2.4] and property (3W), one uses Lemma 7.1.1).

In the following lemma we define a weak diameter using a family of covers. The application of Corollary 7.1.2 to this weak diameter will give rise to Proposition 7.1.7, which will be crucial for the proof of the insertion theorem in Section 7.5.

Lemma 7.1.5. *Let L be a frame, $\{V_n\}_{n\in\mathbb{Z}}$ a sequence of covers with $V_{n-1} \leq_3^* V_n$ for all $n \in \mathbb{Z}$ and set $d\colon L \to [0,+\infty]$ by*

$$d(a) = \inf\{\, 2^n \mid \exists v \in V_n \text{ with } a \leq v \,\}.$$

Then d is a weak diameter on L.

Proof. Note that if $d(a) \leq 2^n$, then there is a $v \in V_n$ such that $a \leq v$. Properties (PD1) and (PD2) are obvious, and (PD3) follows from the fact that each V_n is a cover and $V_n \subseteq \{\, a \in L \mid d(a) < 2^{n+1} \,\}$. Let us show (PD5) holds. Let $a, b \in L$ such that $a \wedge b \neq 0$. If $d(a) = +\infty$ or $d(b) = +\infty$, then there is nothing to prove. Now assume without loss of generality that $d(a) \leq d(b) < +\infty$. If $d(b) = 0$, then $d(a) = 0$; i.e., for all $n \in \mathbb{Z}$ there are $u_n, v_n \in V_n$ with $a \leq u_n$ and $b \leq v_n$. Now, let $n \in \mathbb{Z}$. Then $u_{n-1} \wedge v_{n-1} \geq a \wedge b \neq 0$, and since $V_{n-1} \leq_3^* V_n$, there is a $v \in V_n$ with $u_{n-1} \vee v_{n-1} \leq v$. Consequently $a \vee b \leq v$ and so $d(a \vee b) = 0$. Assume $d(b) = 2^n$. Then there are $u, v \in V_n$ with $a \leq u$ and $b \leq v$. Since $u \wedge v \neq 0$ and $V_n \leq_3^* V_{n+1}$, there is a $w \in V_{n+1}$ such that $u \vee v \leq w$. Hence, $a \vee b \leq w$ and so $d(a \vee b) \leq 2^{n+1} = 2 \cdot 2^n$, as required. □

Remarks 7.1.6. (1) It is easy to check that the previous lemma also holds when one replaces the relation $\leq_3^*$ by $\leq_2^*$ and the words "weak diameter" by "strong prediameter".

(2) The lemma above can be easily adapted to a sequence $\{V_n\}_{n\in\mathbb{N}}$ with $V_{n+1} \leq_3^* V_n$; it suffices to define $d\colon L \to [0,+\infty]$ by $d(a) = \inf\{\, 2^{-n} \mid \exists u \in V_n \text{ with } a \leq u \,\}$.

We also state the following for future reference:

Proposition 7.1.7. *Let L be a frame, $\{V_n\}_{n\in\mathbb{Z}}$ a sequence of covers with $V_{n-1} \leq_3^* V_n$ for all $n \in \mathbb{Z}$. Let $a_1, \ldots, a_k \in L$ with $a_{i-1} \wedge a_i \neq 0$ for all $i = 2, \ldots, k$, and suppose that $a_i \in V_{n_i}$ for all $i = 1, \ldots, k$. Suppose also that*

$$\sum_{i=1}^{k} 2^{n_i+4} < 2^n.$$

Then there is a $v \in V_{n-1}$ such that $a_1 \vee a_k \leq v$.

Proof. Let d denote the weak diameter given by Lemma 7.1.5. By the definition of d, we have $d(a_i) \leq 2^{n_i}$ for all $i = 1, \ldots, k$. In particular,

$$4d(a_1) + 12d(a_2) + 16\sum_{i=3}^{k-2} d(a_i) + 12d(a_{k-1}) + 4d(a_k) \leq 16\sum_{i=1}^{k} d(a_i) \leq \sum_{i=1}^{k} 2^{n_i+4} < 2^n.$$

Then, by Corollary 7.1.2, $d(a_1 \vee a_k) < 2^n$, so it follows by the definition of d that there is a $v \in V_{n-1}$ such that $a_1 \vee a_k \leq v$. □

7.2 Covering Farness in $\mathsf{S}(L)$

We say that a subset $\mathfrak{U} \subseteq \mathsf{S}(L)$ is a *cover* of $\mathsf{S}(L)$ if $\bigvee \mathfrak{U} = L$. This definition is not to be confused with the notion of cover in the frame $\mathsf{S}(L)^{op}$. In this context, we say that a cover $\mathfrak{V}$ *refines* (or is a *refinement* of) a cover $\mathfrak{U}$ if for every $S \in \mathfrak{V}$ there is some $T \in \mathfrak{U}$ such that $S \subseteq T$, and we write $\mathfrak{V} \leq \mathfrak{U}$.

In particular, we will be interested in *open covers* of $\mathsf{S}(L)$, that is, covers of the form

$$\mathfrak{o}[U] := \{\mathfrak{o}(u) \mid u \in U\}$$

for a cover U of L. Notice from (1.4.6) that U is a cover of L if and only if $\mathfrak{o}[U]$ is a cover of $\mathsf{S}(L)$, and that for covers U and V of L, $\mathfrak{o}[U] \leq \mathfrak{o}[V]$ if and only if $U \leq V$ in the sense of Section 6.1.

Let $\mathfrak{U}$ be a cover of $\mathsf{S}(L)$. We say that sublocales S and T of L are $\mathfrak{U}$-*far*[1] in the coframe $\mathsf{S}(L)$ if

$$\forall D \in \mathfrak{U} \quad D \cap S \neq \mathsf{O} \implies D \cap T = \mathsf{O}.$$

The following observations are trivial:

Remark 7.2.1. Let $\mathfrak{U}$ be a cover of $\mathsf{S}(L)$, and let S, T be sublocales of L. Then:

(1) If S and T are $\mathfrak{U}$-far and $S' \subseteq S$ and $T' \subseteq T$, then S' and T' are also $\mathfrak{U}$-far;

(2) If $\mathfrak{U} \leq \mathfrak{V}$ and S and T are $\mathfrak{V}$-far, then S and T are also $\mathfrak{U}$-far;

(3) If S and T are $\mathfrak{U}$-far, then $S^{\#} \vee T^{\#} = L$. Indeed, we have

$$\begin{aligned} L = \bigvee \mathfrak{U} &= \bigvee \{D \in \mathfrak{U} \mid D \cap S \neq \mathsf{O}\} \vee \bigvee \{D \in \mathfrak{U} \mid D \cap S = \mathsf{O}\} \\ &\subseteq \bigvee \{D \in \mathfrak{U} \mid D \cap T = \mathsf{O}\} \vee \bigvee \{D \in \mathfrak{U} \mid D \cap S = \mathsf{O}\} \subseteq S^{\#} \vee T^{\#} \end{aligned}$$

With only a couple of exceptions, we shall be interested in the case where the cover $\mathfrak{U}$ is open, say $\mathfrak{U} = \mathfrak{o}[U]$ for a cover U of L. In that case, we shall simply say that S and T are U-*far* when they are $\mathfrak{o}[U]$-far. This notion coincides with that of Section 6.2 above, in the sense that elements a and b of L are U-far if and only if $\mathfrak{o}(a)$ and $\mathfrak{o}(b)$ are U-far.

Given a cover U of L and a sublocale $S \subseteq L$, we set

$$U * S := \bigvee \{\, \mathfrak{o}(u) \mid u \in U,\ \mathfrak{o}(u) \cap S \neq \mathsf{O} \,\}$$

and we say that the sublocale $U * S$ is the *star* of the sublocale S with respect to U. One can define this star operation for a general cover $\mathfrak{U}$ [51] (see also [45, 67] for this concept in the more general context of nearness structures), but for our purposes this is not necessary.

Clearly, $U * S$ is an open sublocale of L. From (1.4.7) and the fact that U is a cover of L, one can also very easily deduce that $S \subseteq U * S$. Note also that for every $a \in L$ one has $U * \mathfrak{o}(a) = \mathfrak{o}(Ua)$ (recall (1.4.6)). Moreover, if $S \subseteq T$, then $U * S \subseteq U * T$.

In the case of open covers, we can give a few more characterizations of farness:

Proposition 7.2.2. *Let L be a frame and U a cover of L. For sublocales S and T of L, the following conditions are equivalent:*

(i) *S and T are U-far.*

(ii) $(U * S) \cap T = \mathsf{O}$.

[1] A word of caution: this definition of farness is not equivalent to the one given in Section 6.2 applied to the frame $\mathsf{S}(L)^{op}$.

(iii) $(U * T) \cap S = \mathsf{O}$.

(iv) $T \subseteq (U * S)^{\#}$.

(v) $S \subseteq (U * T)^{\#}$.

(vi) $\overline{S}$ *and* $\overline{T}$ *are* U*-far.*

Moreover, if S *and* T *are* U*-far, then* $\overline{S} \cap \overline{T} = \mathsf{O}$.

Proof. (i) $\Longleftrightarrow$ (ii): Since families of open covers are distributive (recall (1.4.7)),

$$(U * S) \cap T = \bigvee \{ \mathfrak{o}(u) \cap T \mid u \in U,\ \mathfrak{o}(u) \cap S \neq \mathsf{O} \}.$$

Then, $(U * S) \cap T = \mathsf{O}$ if and only if for each $u \in U$, $\mathfrak{o}(u) \cap S \neq \mathsf{O}$ implies $\mathfrak{o}(u) \cap T = \mathsf{O}$. That is, if and only if S and T are U-far.
(i) $\Longleftrightarrow$ (iii): By definition, U-farness is a symmetric relation, so the proof of this equivalence is analogous to (i) $\Longleftrightarrow$ (ii).
(ii) $\Longleftrightarrow$ (iv): This equivalence follows because $U * S$ is open and hence complemented (recall (1.4.3)).
(iii) $\Longleftrightarrow$ (v): Since $U * T$ is complemented, by (1.4.3) we have $(U * T) \cap S = \mathsf{O} \iff S \subseteq (U * T)^{\#}$.
(i) $\Longleftrightarrow$ (vi): Assume that S and T are U-far; equivalently one has $T \subseteq (U * S)^{\#}$ and since $(U * S)^{\#}$ is closed, it follows that $\overline{T} \subseteq (U * S)^{\#}$. The latter is in turn equivalent to $\overline{T}$ and S being U-far. Now, (vi) follows repeating the argument with S and $\overline{T}$. The reverse implication is trivial by Remark 7.2.1 (1).

For the last assertion, if S and T are U-far, then so are $\overline{S}$ and $\overline{T}$ and by (ii) it follows that $\overline{S} \cap \overline{T} \subseteq (U * \overline{S}) \cap \overline{T} = \mathsf{O}$. □

Remark 7.2.3. For each cover U of L and every $a \in L$, $U * \mathfrak{o}(a) = \mathfrak{o}(Ua)$. Thus,

$$U * \mathfrak{o}(a) \subseteq \mathfrak{o}(b) \iff Ua \leq b$$

for any $a, b \in L$. Then, by Proposition 7.2.2, we have

$$\begin{aligned} U * \mathfrak{c}(a) \subseteq \mathfrak{c}(b) &\iff (U * \mathfrak{c}(a)) \cap \mathfrak{o}(b) = \mathsf{O} \iff (U * \mathfrak{o}(b)) \cap \mathfrak{c}(a) = \mathsf{O} \iff U * \mathfrak{o}(b) \subseteq \mathfrak{o}(a) \\ &\iff Ub \leq a \end{aligned}$$

for any $a, b \in L$.

We also have the following:

Corollary 7.2.4. *Let* L *be a frame and* U *be cover of* L*. For sublocales* S *and* T *of* L*, the following conditions are equivalent*:

(i) $U * S \subseteq T$.

(ii) $U * \overline{S} \subseteq T^{\circ}$.

(iii) S *and* $T^{\#}$ *are* U*-far.*

(iv) $\overline{S}$ *and* $(T^{\circ})^{\#}$ *are* U*-far.*

Proof. The equivalence between (i) and (ii) follows since

$$\begin{aligned} U * S \subseteq T &\iff U * S \subseteq T^\circ && \text{(because } U * S \text{ is open),} \\ &\iff (U * S) \cap (T^\circ)^\# = 0 && \text{(because } T^\circ \text{ is complemented),} \\ &\iff (U * \overline{S}) \cap (T^\circ)^\# = 0 && \text{(by Proposition 7.2.2),} \\ &\iff U * \overline{S} \subseteq T^\circ && \text{(because } T^\circ \text{ is complemented).} \end{aligned}$$

Now, $U * \overline{S} \subseteq T^\circ$ if and only if $\overline{S}$ and $(T^\circ)^\# = \overline{T^\#}$ are U-far (recall (1.4.12) for the equality), which by Proposition 7.2.2 holds if and only if S and $T^\#$ are U-far. Thus the equivalence between (ii) and (iii) follows. Finally, (iii) $\iff$ (iv) follows from Proposition 7.2.2 (vi) and 1.4.12. □

7.3 Uniform Continuity for General Real-Valued Functions

We now fix some special kind of covers induced by the basis of the metric uniformity in $\mathfrak{L}(\mathbb{R})$ and by some general real-valued functions on a frame L. These covers generalize those defined in Section 6.3.

Let $f, g \in \mathsf{F}(L)$ and $f \geq g$. For every $\delta \in \mathbb{Q}^+$ let

$$\begin{aligned} \mathfrak{D}_\delta^{f,g} &:= \{(f(r,\text{—}) \vee g(\text{—},s))^\# \mid (r,s) \in D_\delta\}, \\ \mathfrak{C}_\delta^{f,g} &:= \{f(r,\text{—})^\# \cap g(\text{—},s)^\# \mid (r,s) \in D_\delta\} \quad \text{and} \\ \mathfrak{F}_\delta^{f,g} &:= \{f(\text{—},r) \cap g(s,\text{—}) \mid (r,s) \in D_\delta\}. \end{aligned}$$

When $f = g$ we simply write $\mathfrak{D}_\delta^f := \mathfrak{D}_\delta^{f,f}$, $\mathfrak{C}_\delta^f := \mathfrak{C}_\delta^{f,f}$, and $\mathfrak{F}_\delta^f := \mathfrak{F}_\delta^{f,f}$. First note that, since $f \geq g$ and D_δ is a cover of $\mathfrak{L}(\mathbb{R})$, we have:

$$\begin{aligned} \bigvee \mathfrak{D}_\delta^{f,g} &= \bigvee\{\, (f(r,\text{—}) \vee g(\text{—},s))^\# \mid (r,s) \in D_\delta \,\} \supseteq \bigvee\{\, (g(r,\text{—}) \vee g(\text{—},s))^\# \mid (r,s) \in D_\delta \,\} \\ &= \bigvee\{\, g(r,s)^\# \mid (r,s) \in D_\delta \,\} = \Big(\bigcap\{\, g(r,s) \mid (r,s) \in D_\delta \,\}\Big)^\# = L. \end{aligned}$$

Thus, $\mathfrak{D}_\delta^{f,g}$ is a cover in $\mathsf{S}(L)$ for every $\delta \in \mathbb{Q}^+$. Moreover, for every $(r,s) \in D_\delta$ we have

$$\begin{aligned} f(r,\text{—}) \vee g(\text{—},s) \vee f(r,\text{—})^\# &\supseteq f(r,\text{—}) \vee f(r,\text{—})^\# = L \quad \text{and} \\ f(r,\text{—}) \vee g(\text{—},s) \vee g(\text{—},s)^\# &\supseteq g(\text{—},s) \vee g(\text{—},s)^\# = L \end{aligned}$$

which means, by (1.4.2), that

$$(f(r,\text{—}) \vee g(\text{—},s))^\# \subseteq f(r,\text{—})^\# \quad \text{and} \quad (f(r,\text{—}) \vee g(\text{—},s))^\# \subseteq g(\text{—},s)^\#.$$

Therefore, $(f(r,\text{—}) \vee g(\text{—},s))^\# \subseteq f(r,\text{—})^\# \cap g(\text{—},s)^\#$, and since f is a frame homomorphism,

$$f(r,\text{—})^\# \cap g(\text{—},s)^\# \subseteq f((r,\text{—})^*) \cap g((\text{—},s)^*) = f(\text{—},r) \cap g(s,\text{—}).$$

Thus, $\mathfrak{C}_\delta^{f,g}$ and $\mathfrak{F}_\delta^{f,g}$ are also covers and

$$\mathfrak{D}_\delta^{f,g} \leq \mathfrak{C}_\delta^{f,g} \leq \mathfrak{F}_\delta^{f,g}. \tag{7.3.1}$$

Remark 2.5.1 immediately gives:

Lemma 7.3.1. *Let U be a cover of L, $f,g \in \mathsf{F}(L)$ and $\delta \in \mathbb{Q}^+$. The following are equivalent:*

(i) *The sublocales $f(r,\text{—})$ and $g(\text{—},s)$ are U-far for all $s-r>\frac{1}{\delta}$.*

(ii) *The sublocales $f(r,\text{—})$ and $g(s,\text{—})^\#$ are U-far for all $s-r>\frac{1}{\delta}$.*

(iii) *The sublocales $f(\text{—},r)^\#$ and $g(\text{—},s)$ are U-far for all $s-r>\frac{1}{\delta}$.*

(iv) *The sublocales $f(\text{—},r)^\#$ and $g(s,\text{—})^\#$ are U-far for all $s-r>\frac{1}{\delta}$.*

Proposition 7.3.2. *Let U and V be covers of a frame L. If $\mathfrak{o}[U] \leq \mathfrak{F}^f_\gamma$ and $\mathfrak{o}[V] \leq \mathfrak{F}^f_\delta$ for some $\gamma,\delta \in \mathbb{Q}^+$ and $f \in \mathsf{F}(L)$, then $\mathfrak{o}[UV] \leq \mathfrak{F}^f_{\frac{\gamma\delta}{\gamma+2\delta}}$.*

Proof. Let $Uv \in UV$. Since $\mathfrak{o}[V] \leq \mathfrak{F}^f_\delta$, there is $(r,s) \in D_\delta$ such that $\mathfrak{o}(v) \subseteq f(\text{—},r) \cap f(s,\text{—})$. Take $u \in U$ such that $u \wedge v \neq 0$. Since $\mathfrak{o}[U] \leq \mathfrak{F}^f_\gamma$, there is $(p,q) \in D_\gamma$ such that $\mathfrak{o}(u) \subseteq f(\text{—},p) \cap f(q,\text{—})$. Then

$$\mathsf{O} \neq \mathfrak{o}(u) \cap \mathfrak{o}(v) \subseteq f(\text{—},p) \cap f(q,\text{—}) \cap f(\text{—},r) \cap f(s,\text{—}) = f((\text{—},r \vee p) \vee (s \wedge q,\text{—}))$$

which means that $r \leq q$ and $p \leq s$ (otherwise $(\text{—},r \vee p) \vee (s \wedge q,\text{—}) = 1$). Hence,

$$r - \tfrac{1}{\gamma} = r-(q-p) \leq r-r+p = p \quad \text{and} \quad s+\tfrac{1}{\gamma} = s+(q-p) \geq s+q-s = q$$

so

$$\mathfrak{o}(u) \subseteq f(\text{—},p) \cap f(q,\text{—}) \subseteq f(\text{—},r-\tfrac{1}{\gamma}) \cap f(s+\tfrac{1}{\gamma},\text{—}).$$

Consequently, $\mathfrak{o}(Uv) \subseteq f(\text{—},r-\frac{1}{\gamma}) \cap f(s+\frac{1}{\gamma},\text{—})$ and $s+\frac{1}{\gamma}-(r-\frac{1}{\gamma}) = \frac{1}{\delta}+\frac{2}{\gamma} = \frac{\gamma+2\delta}{\delta\gamma}$, showing that $\mathfrak{o}[UV] \leq \mathfrak{F}^f_{\frac{\gamma\delta}{\gamma+2\delta}}$. ☐

Proposition 7.3.3. *Let U be a cover of a frame L. If $\mathfrak{o}[U] \leq \mathfrak{F}^f_\delta$ for some $\delta \in \mathbb{Q}^+$ and $f \in \mathsf{F}(L)$, then $\mathfrak{o}[U^n] \leq \mathfrak{F}^f_{\frac{\delta}{2n-1}}$.*

Proof. We proceed by induction on n. For $n=1$ is trivial. Assume $\mathfrak{o}[U^n] \leq \mathfrak{F}^f_{\frac{\delta}{2n-1}}$. Then, by Proposition 7.3.2,

$$\mathfrak{o}[U^{n+1}] = \mathfrak{o}[UU^n] = \mathfrak{F}^f_{\frac{\delta^2}{2\delta+\delta(2n-1)}} = \mathfrak{F}^f_{\frac{\delta}{2n+1}} = \mathfrak{F}^f_{\frac{\delta}{2(n+1)-1}}.$$ ☐

Next results will help us prove Theorem 7.3.10 below. They generalize the results in Section 6.3.

Lemma 7.3.4. *Let L be a frame and $f,g \in \mathsf{F}(L)$ with $f \geq g$. For each $\delta \in \mathbb{Q}^+$ and every $r,s \in \mathbb{Q}$ such that $s-r>\frac{1}{\delta}$, the sublocales $f(r,\text{—})$ and $g(\text{—},s)$ are $\mathfrak{F}^{f,g}_\delta$-far. In particular, they are also $\mathfrak{D}^{f,g}_\delta$-far and $\mathfrak{C}^{f,g}_\delta$-far.*

Proof. Let $\delta \in \mathbb{Q}^+$. We proceed by contradiction; suppose there are $s,r \in \mathbb{Q}$ with $s-r>\frac{1}{\delta}$ such that $f(r,\text{—})$ and $g(\text{—},s)$ are not $\mathfrak{F}^{f,g}_\delta$-far. This means there must exist $(r',s') \in D_\delta$ such that

$$f(r,\text{—}) \cap f(\text{—},r') \cap g(s',\text{—}) \neq \mathsf{O} \quad \text{and} \quad g(\text{—},s) \cap f(\text{—},r') \cap g(s',\text{—}) \neq \mathsf{O}.$$

In particular, we would have

$$f(r,\text{—})\cap f(\text{—},r')\neq \mathsf{O} \quad \text{and} \quad g(\text{—},s)\cap g(s',\text{—})\neq \mathsf{O}.$$

Thus, $r'\leq r$ and $s\leq s'$. Hence, $\frac{1}{\delta}=s'-r'\geq s-r>\frac{1}{\delta}$, a contradiction. Consequently, for every $s,r\in\mathbb{Q}$ with $s-r>\frac{1}{\delta}$ the sublocales $f(r,\text{—})$ and $g(\text{—},s)$ are $\mathfrak{F}_\delta^{f,g}$-far. From Remark 7.2.1 (2) and (7.3.1), they are also $\mathfrak{D}_\delta^{f,g}$-far and $\mathfrak{C}_\delta^{f,g}$-far. □

Lemma 7.3.5. *Let L be a frame, $f,g\in\mathsf{F}(L)$ with $f\geq g$, and $\delta\in\mathbb{Q}^+$. If $\mathfrak{U}$ is a cover of $\mathsf{S}(L)$ such that for every $s,r\in\mathbb{Q}$ with $s-r>\frac{1}{\delta}$ the sublocales $f(r,\text{—})$ and $g(\text{—},s)$ are $\mathfrak{U}$-far, then*

$$\mathfrak{U}\leq\mathfrak{D}_\gamma^{f,g} \text{ for every } \gamma\in\mathbb{Q}^+ \text{ with } \gamma<\delta.$$

Proof. Let $S\in\mathfrak{U}$ with $S\neq\mathsf{O}$. We claim there is $s_0\in\mathbb{Q}$ such that $S\cap g(\text{—},s_0)\neq\mathsf{O}$. Suppose not, that is, $S\cap g(\text{—},s)=\mathsf{O}$ for every $s\in\mathbb{Q}$. Then, since $g\in\mathsf{F}(L)$ (recall (1.4.2)), we would have $S\subseteq\bigcap_{s\in\mathbb{Q}}g(\text{—},s)^\#\subseteq\bigcap_{s\in\mathbb{Q}}g(s,\text{—})=\mathsf{O}$, a contradiction (because $S\neq\mathsf{O}$). Then, by assumption, $S\cap f(s_0-\frac{2}{\delta},\text{—})=\mathsf{O}$. We may consider the non-empty set

$$A:=\{r\in\mathbb{Q}\mid S\cap f(r,\text{—})=\mathsf{O}\}$$

which is strictly contained in $\mathbb{Q}$ (if this was not the case, then $S\subseteq\bigcap_{s\in\mathbb{Q}}f(r,\text{—})^\#\subseteq\bigcap_{r\in\mathbb{Q}}f(\text{—},r)=\mathsf{O}$ contradicting the fact that $S\neq\mathsf{O}$). Moreover, if $S\cap f(r,\text{—})\neq\mathsf{O}$, then $S\cap f(r',\text{—})\neq\mathsf{O}$ for every $r'>r$. Thus, A is upper bounded, and $\alpha:=\sup A\in\mathbb{R}$. Let $\gamma\in\mathbb{Q}^+$ with $\gamma<\delta$, and take $\varepsilon=\frac{\delta-\gamma}{\delta\gamma}$. Clearly, $\varepsilon>0$. Now, take $r,s\in\mathbb{Q}$ such that $0<\alpha-r<\frac{\varepsilon}{3}$ and $0<s-\alpha<\frac{\varepsilon}{3}$. Then, $S\cap f(r,\text{—})=\mathsf{O}$ and $S\cap f(s,\text{—})\neq\mathsf{O}$. By assumption, we have $S\cap g(\text{—},s+\frac{\varepsilon}{3}+\frac{1}{\delta})=\mathsf{O}$. Since

$$s+\tfrac{\varepsilon}{3}+\tfrac{1}{\delta}-r<\alpha+\tfrac{\varepsilon}{3}+\tfrac{\varepsilon}{3}+\tfrac{1}{\delta}+\tfrac{\varepsilon}{3}-\alpha=\varepsilon+\tfrac{1}{\delta}=\tfrac{1}{\gamma},$$

we can pick $(r',s')\in D_\gamma$ such that $r'<r<s+\frac{\varepsilon}{3}+\frac{1}{\delta}<s'$. Consequently, we have

$$S\cap f(r',\text{—})\subseteq S\cap f(r,\text{—})=\mathsf{O} \quad \text{and} \quad S\cap g(\text{—},s')\subseteq S\cap g(\text{—},s+\tfrac{\varepsilon}{3}+\tfrac{1}{\delta})=\mathsf{O}.$$

That is, $S\cap(f(r',\text{—})\vee g(\text{—},s'))=\mathsf{O}$ which implies that $S\subseteq(f(r',\text{—})\vee g(\text{—},s'))^\#$, showing that $\mathfrak{U}\leq\mathfrak{D}_\gamma^{f,g}$. □

Proposition 7.3.6. *Let L be a frame and $f,g\in\mathsf{F}(L)$ with $f\geq g$. Then the following are equivalent:*

(i) *For every $\delta\in\mathbb{Q}^+$ there is a cover $\mathfrak{U}$ of $\mathsf{S}(L)$ such that $\mathfrak{U}\leq\mathfrak{D}_\delta^{f,g}$.*

(ii) *For every $\delta\in\mathbb{Q}^+$ there is a cover $\mathfrak{U}$ of $\mathsf{S}(L)$ such that the sublocales $f(r,\text{—})$ and $g(\text{—},s)$ are $\mathfrak{U}$-far for any $s-r>\frac{1}{\delta}$.*

Proof. (i) $\Longrightarrow$ (ii): For $\delta\in\mathbb{Q}^+$, let $\mathfrak{U}$ be the cover of $\mathsf{S}(L)$ such that $\mathfrak{U}\leq\mathfrak{D}_\delta^{f,g}$. By Lemma 7.3.4, the sublocales $f(r,\text{—})$ and $g(\text{—},s)$ are $\mathfrak{D}_\delta^{f,g}$-far for any $s-r>\frac{1}{\delta}$. By Remark 7.2.1 (2), $f(r,\text{—})$ and $g(\text{—},s)$ are $\mathfrak{U}$-far for any $s-r>\frac{1}{\delta}$ (since $\mathfrak{U}\leq\mathfrak{D}_\delta^{f,g}$).
(ii) $\Longrightarrow$ (i): Let $\delta\in\mathbb{Q}^+$. By assumption, there is a cover $\mathfrak{U}$ of $\mathsf{S}(L)$ such that the sublocales $f(r,\text{—})$ and $g(\text{—},s)$ are $\mathfrak{U}$-far for any $s-r>\frac{1}{\delta+1}$. Then, by Lemma 7.3.5, $\mathfrak{U}\leq\mathfrak{D}_\delta^{f,g}$. □

Proposition 7.3.7. *Let L be a frame and $f,g \in \mathsf{F}(L)$ with $f \geq g$. Then the following are equivalent:*

(i) *For every $\delta \in \mathbb{Q}^+$ there is a cover U of L such that $\mathfrak{o}[U^n] \leq \mathfrak{D}^{f,g}_{\frac{\delta}{2n-1}}$ for every $n \in \mathbb{N}$.*

(ii) *For every $\delta \in \mathbb{Q}^+$ there is a cover U of L such that for every $n \in \mathbb{N}$ the sublocales $f(r,—)$ and $g(—,s)$ are U^n-far whenever $s-r > \frac{n}{\delta}$.*

Proof. (i) $\Longrightarrow$ (ii): Let $\delta \in \mathbb{Q}^+$ and consider $\varepsilon = 2\delta$. By assumption there is a cover U of L such that $\mathfrak{o}[U^n] \leq \mathfrak{D}^{f,g}_{\frac{\varepsilon}{2n-1}}$ for every $n \in \mathbb{N}$. Let $s,r \in \mathbb{Q}$ with $s-r > \frac{n}{\delta}$. Since $\frac{n}{\delta} = \frac{2n}{\varepsilon} > \frac{2n-1}{\varepsilon}$, by Lemma 7.3.4, the sublocales $f(r,—)$ and $g(—,s)$ are $\mathfrak{D}^{f,g}_{\frac{\varepsilon}{2n-1}}$-far. In particular, they are U^n-far (recall Remark 7.2.1 (2)).
(ii) $\Longrightarrow$ (i): Let $\delta \in \mathbb{Q}^+$. By assumption there is some cover U of L such that for every $n \in \mathbb{N}$ the sublocales $f(r,—)$ and $g(—,s)$ are U^n-far whenever $s-r > \frac{n}{\delta+1}$. Then, by Lemma 7.3.5, since $\frac{\delta}{2n-1} < \frac{\delta+1}{n}$, we have $\mathfrak{o}[U^n] \leq \mathfrak{D}^{f,g}_{\frac{\delta}{2n-1}}$. □

Definition 7.3.8. Let $(L,\mathscr{U})$ be a (pre)uniform frame. An $f \in \mathsf{F}(L)$ is *uniformly continuous* if for every $n \in \mathbb{N}$ there is a $U \in \mathscr{U}$ such that

$$\mathfrak{o}[U] \leq \mathfrak{D}^f_n = \left\{ f(r,s)^\# \mid (r,s) \in D_n \right\}.$$

Equivalently, if for every $\delta \in \mathbb{Q}^+$ there is a $U \in \mathscr{U}$ such that $\mathfrak{o}[U] \leq \mathfrak{D}^f_\delta = \left\{ f(r,s)^\# \mid (r,s) \in D_\delta \right\}$.

Remarks 7.3.9. (1) If $f \in \mathsf{C}(L)$ (i.e., $f\colon \mathfrak{L}(\mathbb{R}) \to \mathsf{S}(L)^{op}$ is of the form $f = \mathfrak{c}\, g$ for a frame homomorphism $g\colon \mathfrak{L}(\mathbb{R}) \to L$), it is clear that f is uniformly continuous (in the sense just defined) if and only if g is a uniform homomorphism. Indeed, we have:

$$\begin{aligned} \mathfrak{o}[U] \leq \mathfrak{D}^f_n &\iff \forall u \in U \quad \exists (r,s) \in D_n: \quad \mathfrak{o}(u) \leq f(r,s)^\# = \mathfrak{c}(g(r,s))^\# = \mathfrak{o}(g(r,s)) \\ &\iff \forall u \in U \quad \exists (r,s) \in D_n: \quad u \leq g(r,s) \\ &\iff U \leq g[D_n] = D^g_n. \end{aligned}$$

(2) Actually, it is not necessary to require f to be continuous in order to recover the usual notion of uniform continuity. Indeed, we shall show below, in Corollary 7.3.11, that uniform continuity (in the sense of Definition 7.3.8) implies continuity. Hence, by virtue of the previous remark, uniformly continuous maps in $\mathsf{F}(L)$ correspond precisely to uniform homomorphisms $\mathfrak{L}(\mathbb{R}) \to (L,\mathscr{U})$, thus ensuring that this is the right notion of uniform continuity for maps in $\mathsf{F}(L)$.

Theorem 7.3.10. *Let $(L,\mathscr{U})$ be a preuniform frame and $f \in \mathsf{F}(L)$. Then the following statements are equivalent:*

(i) *f is uniformly continuous.*

(ii) *for every $\delta \in \mathbb{Q}^+$ there is $U \in \mathscr{U}$ such that the sublocales $f(r,—)$ and $f(—,s)$ are U-far whenever $s-r > \frac{1}{\delta}$.*

(iii) *for every $\delta \in \mathbb{Q}^+$ there is $U \in \mathscr{U}$ such that $\mathfrak{o}[U^n] \leq \mathfrak{D}^f_{\frac{\delta}{2n-1}}$.*

(iv) *for every $\delta \in \mathbb{Q}^+$ there is $U \in \mathscr{U}$ such that for every $n \in \mathbb{N}$ the sublocales $f(r,—)$ and $f(—,s)$ are U^n-far whenever $s-r > \frac{n}{\delta}$.*

Proof. (i) $\Longrightarrow$ (ii): Let $\delta \in \mathbb{Q}^+$. Consider the cover U of L given by the uniform continuity of f, so that $\mathfrak{o}[U] \leq \mathfrak{D}^f_\delta$. By Lemma 7.3.4, the sublocales $f(r,—)$ and $f(—,s)$ are $\mathfrak{D}^f_\delta$-far for any $s-r > \frac{1}{\delta}$. By Remark 7.2.1 (2), since $\mathfrak{o}[U] \leq \mathfrak{D}^f_\delta$, $f(r,—)$ and $f(—,s)$ are U-far for any $s-r > \frac{1}{\delta}$.
(ii) $\Longrightarrow$ (i): Let $\delta \in \mathbb{Q}^+$. By assumption there is a cover U of L such that the sublocales $f(r,—)$ and $f(—,s)$ are U-far for any $s-r > \frac{1}{\delta+1}$. Then, by Lemma 7.3.5, $\mathfrak{o}[U] \leq \mathfrak{D}^f_\delta$.
(iv) $\Longrightarrow$ (ii): Trivial.
(i) $\Longrightarrow$ (iv): Let $\delta \in \mathbb{Q}^+$ and consider a natural number m such that $\frac{1}{m} \leq \frac{1}{\delta}$. By assumption, there is a uniform cover $U \in \mathscr{U}$ such that $\mathfrak{o}[U] \leq \mathfrak{D}^f_{2m}$. We claim U is the cover we are looking for. Let $n \in \mathbb{N}$ and $s-r > \frac{n}{\delta}$. If $n=1$, then $s-r > \frac{1}{m} > \frac{1}{2m}$. By Lemma 7.3.4, sublocales $f(r,—)$ and $g(—,s)$ are $\mathfrak{D}^f_{2m}$-far. Since $\mathfrak{o}[U] \leq \mathfrak{D}^f_{2m}$, they are also U-far. For $n \geq 2$, suppose $f(r,—)$ and $f(—,s)$ are not U^n-far. Since $\mathfrak{o}[U] \leq \mathfrak{D}^f_{2m} \leq \mathfrak{F}^f_{2m}$, by Proposition 7.3.3,

$$\mathfrak{o}[U^n] \leq \mathfrak{F}^f_{\frac{2m}{2n-1}}.$$

It then follows (recall Remark 7.2.1 (2)) that $f(r,—)$ and $f(—,s)$ cannot be $\mathfrak{F}^f_{\frac{2m}{2n-1}}$-far, since we assumed they are not U^n-far. This means that there is some $(p,q) \in D_{\frac{2m}{2n-1}}$ such that

$$f(r,—) \cap f(—,p) \cap f(q,—,) \neq \mathsf{O} \quad \text{and} \quad f(—,s) \cap f(—,p) \cap f(q,—,) \neq \mathsf{O}.$$

In particular,

$$f(r,—) \cap f(—,p) \neq \mathsf{O} \quad \text{and} \quad f(—,s) \cap f(q,—,) \neq \mathsf{O}.$$

meaning that $p \leq r$ and $s \leq q$. Therefore,

$$\tfrac{n}{\delta} < s-r < q-p = \tfrac{2n-1}{2m} < \tfrac{n}{m} \leq \tfrac{n}{\delta},$$

a contradiction. In conclusion, $f(r,—)$ and $f(—,s)$ are U^n-far.
(iii) $\Longleftrightarrow$ (iv): Follows from Proposition 7.3.7. □

Corollary 7.3.11. *Let $(L,\mathscr{U})$ be a preuniform frame and $f \in \mathsf{F}(L)$ be uniformly continuous. Then f is continuous (i.e., $f \in \mathsf{C}(L)$).*

Proof. By Theorem 7.3.10 (ii), for each $\delta \in \mathbb{Q}^+$ there is a uniform cover U_δ such that $f(r,—)$ and $f(—,s)$ are U_δ-far whenever $s-r > \frac{1}{\delta}$. To show that f is continuous we have to prove that for every $r,s \in \mathbb{Q}$, the sublocales $f(r,—)$ and $f(—,s)$ are closed. For each $r \in \mathbb{Q}$, by Proposition 7.2.2 we have

$$f(r,—) \subseteq \left(U_\delta * f(—,r+\tfrac{2}{\delta})\right)^{\#}$$

for every $\delta \in \mathbb{Q}^+$. Thus,

$$f(r,—) \subseteq \bigcap_{\delta \in \mathbb{Q}^+} \left(U_\delta * f(—,r+\tfrac{2}{\delta})\right)^{\#}.$$

From Remark 2.5.1 and the fact that $S \subseteq U * S$ for any sublocale $S \in \mathsf{S}(L)$, we obtain

$$\bigcap_{r<t} f(—,t)^{\#} \subseteq \bigcap_{r<t} f(t,—) = f(r,—) \subseteq \bigcap_{\delta \in \mathbb{Q}^+} \left(U_\delta * f(—,r+\tfrac{2}{\delta})\right)^{\#} \subseteq \bigcap_{\delta \in \mathbb{Q}^+} f(—,r+\tfrac{2}{\delta})^{\#} = \bigcap_{t>r} f(—,t)^{\#}.$$

Since $\bigcap_{\delta\in\mathbb{Q}^+}\left(U_\delta * f(\text{—}, r+\frac{2}{\delta})\right)^{\#}$ is closed, $f(r,\text{—})$ is a closed sublocale for every $r\in\mathbb{Q}$. Similarly, we can conclude that $f(\text{—},s)$ is closed for every $s\in\mathbb{Q}$. □

Remark 7.3.12. Let $f,g\in\mathscr{R}(L)$. Then

$$\mathfrak{D}_\delta^{\mathfrak{c}f,\mathfrak{c}g} = \left\{(\mathfrak{c}(f(r,\text{—}))\vee\mathfrak{c}(g(\text{—},s)))^{\#} \mid (r,s)\in D_\delta\right\} = \{\mathfrak{o}(f(r,\text{—})\wedge g(\text{—},s)) \mid (r,s)\in D_\delta\} = \mathfrak{o}[D_\delta^{f,g}],$$

$$\mathfrak{C}_\delta^{\mathfrak{c}f,\mathfrak{c}g} = \{\mathfrak{o}(f(r,\text{—}))\cap\mathfrak{o}(g(\text{—},s)) \mid (r,s)\in D_\delta\} = \{\mathfrak{o}(f(r,\text{—})\wedge g(\text{—},s)) \mid (r,s)\in D_\delta\} = \mathfrak{o}[D_\delta^{f,g}]$$

and

$$\begin{aligned}\mathfrak{F}_\delta^{\mathfrak{c}f,\mathfrak{c}g} &= \{\mathfrak{c}(f(\text{—},r))\cap\mathfrak{c}(g(s,\text{—})) \mid (r,s)\in D_\delta\} \supseteq \{\mathfrak{c}(f(r,\text{—})^*)\cap\mathfrak{c}(g(\text{—},s)^*) \mid (r,s)\in D_\delta\}\\ &= \left\{\overline{\mathfrak{o}(f(r,\text{—}))}\cap\overline{\mathfrak{o}(g(\text{—},s))} \mid (r,s)\in D_\delta\right\}.\end{aligned}$$

If $f=g$ we have

$$\mathfrak{C}_\delta^{\mathfrak{c}f} = \mathfrak{D}_\delta^{\mathfrak{c}f} = \{\mathfrak{o}(f(r,s)) \mid (r,s)\in D_\delta\} = \mathfrak{o}[D_\delta^f] = \mathfrak{o}[f[D_\delta]] \quad \text{and}$$

$$\mathfrak{F}_\delta^{\mathfrak{c}f} = \{\mathfrak{c}(f(\text{—},r)\vee f(\text{—},s)) \mid (r,s)\in D_\delta\} \supseteq \left\{\overline{\mathfrak{o}(f(r,s))} \mid (r,s)\in D_\delta\right\}.$$

Lemma 7.3.13. *Let L be a frame and U a cover of L. If $\delta\in\mathbb{Q}^+$ and $f,g\in\mathscr{R}(L)$ are such that $f\geq g$, then the following are equivalent*:

(i) *the elements $f(\text{—},r)$ and $g(s,\text{—})$ are U-far whenever $s-r>\frac{1}{\delta}$;*

(ii) *the sublocales $\mathfrak{c}f(r,\text{—})$ and $\mathfrak{c}g(\text{—},s)$ are U-far whenever $s-r>\frac{1}{\delta}$.*

Proof. For any $r,s\in\mathbb{Q}$ we have the following equivalences:

$$\begin{aligned}\mathfrak{c}f(r,\text{—}) \text{ and } \mathfrak{c}g(\text{—},s) \text{ are } U\text{-far} &\iff \forall u\in U \quad \mathfrak{c}f(r,\text{—})\cap\mathfrak{o}(u)=\mathsf{O}\\ &\qquad\qquad \text{or } \mathfrak{c}g(\text{—},s)\cap\mathfrak{o}(u)=\mathsf{O}\\ &\iff \forall u\in U \quad \mathfrak{c}f(r,\text{—})\subseteq\mathfrak{c}(u) \text{ or } \mathfrak{c}g(\text{—},s)\subseteq\mathfrak{c}(u)\\ &\iff \forall u\in U \quad u\leq f(r,\text{—}) \text{ or } u\leq g(\text{—},s).\end{aligned}$$

Now, assume (i), and let $r,s\in\mathbb{Q}$ with $s-r>\frac{1}{\delta}$. Select $p,q\in\mathbb{Q}$ such that $r<p<q<s$ with $q-p>\frac{1}{\delta}$. Then, $f(\text{—},p)$ and $g(q,\text{—})$ are U-far which means that for all $u\in U$ one has $u\leq f(\text{—},p)^*$ or $u\leq g(q,\text{—})^*$ (recall Proposition 6.2.2 (vi)). By Remark 2.1.1, it follows that for all $u\in U$ either $u\leq f(r,\text{—})$ or $u\leq g(\text{—},s)$. By the equivalences above, $\mathfrak{c}f(r,\text{—})$ and $\mathfrak{c}g(\text{—},s)$ are U-far. The converse follows at once from the equivalences above and Remark 2.1.1. □

Proposition 7.3.14. *Let $(L,\mathscr{U})$ be a preuniform frame. Then the following are equivalent for $f\in\mathscr{R}(L)$:*

(i) *$\mathfrak{c}f$ is uniformly continuous.*

(ii) *f is a uniform homomorphism.*

(iii) *For every* $\delta \in \mathbb{Q}^+$ *there is a* $U \in \mathscr{U}$ *such that the elements* $f(-,r)$ *and* $f(s,-)$ *are* U*-far whenever* $s - r > \frac{1}{\delta}$.

(iv) *For every* $\delta \in \mathbb{Q}^+$ *there is a* $U \in \mathscr{U}$ *such that for every* $n \in \mathbb{N}$ *the elements* $f(-,r)$ *and* $f(s,-)$ *are* U^n*-far whenever* $s - r > \frac{n}{\delta}$.

(v) *For every* $\delta \in \mathbb{Q}^+$ *there is* $U \in \mathscr{U}$ *such that for every* $n \in \mathbb{N}$, $U^n \leq D^f_{\frac{\delta}{2n-1}}$.

(vi) *For every* $\delta \in \mathbb{Q}^+$ *there is a* $U \in \mathscr{U}$ *such that the sublocales* $\mathfrak{c}f(r,-)$ *and* $\mathfrak{c}f(-,s)$ *are* U*-far whenever* $s - r > \frac{1}{\delta}$.

(vii) *For every* $\delta \in \mathbb{Q}^+$ *there is a* $U \in \mathscr{U}$ *such that for every* $n \in \mathbb{N}$ *the sublocales* $\mathfrak{c}f(r,-)$ *and* $\mathfrak{c}f(-,s)$ *are* U^n*-far whenever* $s - r > \frac{n}{\delta}$.

(viii) *For every* $\delta \in \mathbb{Q}^+$ *there is* $U \in \mathscr{U}$ *such that for every* $n \in \mathbb{N}$, $\mathfrak{o}[U^n] \leq \mathfrak{D}^{\mathfrak{c}f}_{\frac{\delta}{2n-1}}$.

Proof. The equivalences (i) $\iff$ (vi) $\iff$ (vii) $\iff$ (viii) are proved in Theorem 7.3.10. On the other hand, (ii) $\iff$ (iii) $\iff$ (iv) $\iff$ (v) follow from Theorem 6.3.9. Then by Remark 7.3.9 one knows that (i) and (ii) are equivalent. One can also use Lemma 7.3.13 to show that (iii) $\iff$ (vi) and (iv) $\iff$ (vii), or use Remark 7.3.12 to see that (iv) is equivalent to (viii). □

7.4 Scales in $\mathsf{S}(L)$ for Uniform Frame Homomorphisms

In Section 2.6 we discussed scales for $\mathsf{S}(L)^{op}$ and how they generate a function $f \in \mathsf{F}(L)$. We also proved that under condition (IC), these scales define, not only a general real-valued function, but an $f \in \mathsf{C}(L)$. In this section we will see whether a scale in $\mathsf{S}(L)^{op}$ generates a uniformly continuous real-valued function and, more generally, when does a family of sublocales $(S_r)_{r\in\mathbb{Q}}$ defines a uniformly continuous function in L.

Inspired by the scales studied in Section 6.3 let us first consider the following conditions on a family $(S_r)_{r\in\mathbb{Q}}$ of sublocales of a preuniform frame $(L,\mathscr{U})$:

(U) For every $\delta \in \mathbb{Q}^+$ there is a $U \in \mathscr{U}$ such that $U * S_r \subseteq S_s$ whenever $s - r > \frac{1}{\delta}$.

(U') For every $\delta \in \mathbb{Q}^+$ there is a $U \in \mathscr{U}$ such that $U * S_s \subseteq S_r$ whenever $s - r > \frac{1}{\delta}$.

Remarks 7.4.1. (1) (U) $\Longrightarrow$ (S1): Indeed, let $r < s$ and $\delta \in \mathbb{Q}^+$ such that $s - r > \delta$. By (U), there is $U \in \mathscr{U}$ such that $U * S_r \subseteq S_s$. By Corollary 7.2.4 (iii), S_r and $S_s^\#$ are U-far. In particular $S_r \cap S_s^\# = \mathsf{O}$ (Proposition 7.2.2).

(2) (U) $\Longrightarrow$ (IC): Let $r < s$ and $\delta \in \mathbb{Q}^+$ such that $s - r > \delta$. By (U), there is $U \in \mathscr{U}$ such that $U * S_r \subseteq S_s$. From Corollary 7.2.4 (ii) it follows that $\overline{S_r} \subseteq U * \overline{S_r} \subseteq S_s^\circ$, as required.

(3) Similarly, one can show that (U') $\Longrightarrow$ (S1') and (U') $\Longrightarrow$ (IC').

Lemma 7.4.2. *Let $(L,\mathscr{U})$ be a preuniform frame. If a family $(S_r)_{r\in\mathbb{Q}} \subseteq \mathsf{S}(L)$ satisfies (U) (resp. (U')) and (S2), then the formulas*

$$h(p,\text{—}) = \bigcap_{r>p} S_r \quad \textit{and} \quad h(\text{—},q) = \bigcap_{s<q} S_s^{\#}$$

$$(\textit{resp.} \quad h(p,\text{—}) = \bigcap_{r>p} S_r^{\#} \quad \textit{and} \quad h(\text{—},q) = \bigcap_{s<q} S_s)$$

define a uniformly continuous $h \in \mathsf{F}(L)$.

Proof. Let $(S_r)_{r\in\mathbb{Q}} \subseteq \mathsf{S}(L)$ be a family of sublocales satisfying (U) and (S2). By Remark 7.4.1 (1), $(S_r)_{r\in\mathbb{Q}}$ is an ascending scale in $\mathsf{S}(L)^{op}$. Hence, the formulas

$$h(p,\text{—}) = \bigcap_{r>p} S_r \quad \text{and} \quad h(\text{—},q) = \bigcap_{s<q} S_s^{\#}$$

determine an $h \in \mathsf{F}(L)$. Moreover, by Remark 7.4.1 (2), we have $h \in \mathsf{C}(L)$. Now, to show that h is uniformly continuous let $\delta \in \mathbb{Q}^+$ and take the $U \in \mathscr{U}$ given by (U). Let $p,q \in \mathbb{Q}$ such that $q-p > \frac{1}{\delta}$ and select $r',s' \in \mathbb{Q}$ such that $p < r' < s' < q$ and $s'-r' > \frac{1}{\delta}$. Then $U * S_{r'} \subseteq S_{s'}$. By Corollary 7.2.4, $\overline{S_{r'}}$ and $(S_{s'}^{\circ})^{\#}$ are U-far. Now,

$$h(p,\text{—}) = \bigcap_{r>p} S_r \subseteq S_{r'} \subseteq \overline{S_{r'}} \quad \text{and} \quad h(\text{—},q) = \bigcap_{s<q} S_s^{\#} \subseteq S_{s'}^{\#} \subseteq (S_{s'}^{\circ})^{\#}$$

so, by Remark 7.2.1 (1), $h(p,\text{—})$ and $h(\text{—},q)$ are U-far. Thus h is uniformly continuous by Theorem 7.3.10. The statement inside parenthesis can be proved in a similar way. □

Condition (ii) in Theorem 7.3.10 points that farness is related to uniform continuity. So, instead of working with the star operator as in the preceding lemma, we may consider the following conditions for a preuniform frame $(L,\mathscr{U})$ and a family $(S_r)_{r\in\mathbb{Q}} \subseteq \mathsf{S}(L)$:

(FAR) For every $\delta \in \mathbb{Q}^+$ there is a cover $U \in \mathscr{U}$ such that S_r and $S_s^{\#}$ are U-far whenever $s-r > \frac{1}{\delta}$.

(FAR') For every $\delta \in \mathbb{Q}^+$ there is a cover $U \in \mathscr{U}$ such that $S_r^{\#}$ and S_s are U-far whenever $s-r > \frac{1}{\delta}$.

Remark 7.4.3. It is clear from Corollary 7.2.4 (iii) that (U) $\Longleftrightarrow$ (FAR) and (U') $\Longleftrightarrow$ (FAR').

Lemma 7.4.2 and Remarks 7.4.3 immediately yield:

Lemma 7.4.4. *Let $(L,\mathscr{U})$ be a preuniform frame. If a family $(S_r)_{r\in\mathbb{Q}} \subseteq \mathsf{S}(L)$ satisfies (FAR) (resp. (FAR')) and (S2), then the formulas*

$$h(p,\text{—}) = \bigcap_{r>p} S_r \quad \textit{and} \quad h(\text{—},q) = \bigcap_{s<q} S_s^{\#}$$

$$(\textit{resp.} \quad h(p,\text{—}) = \bigcap_{r>p} S_r^{\#} \quad \textit{and} \quad h(\text{—},q) = \bigcap_{s<q} S_s)$$

define a uniformly continuous $h \in \mathsf{F}(L)$.

We consider two last conditions on a familiy $(S_r)_{r\in\mathbb{Q}} \subseteq \mathsf{S}(L)$:

(C) For every $\delta \in \mathbb{Q}^+$ there is a cover $U \in \mathscr{U}$ such that $\mathfrak{o}[U] \leq \left\{\overline{S_r}^{\#} \cap S_s^\circ \mid (r,s) \in D_\delta\right\}$.

(C') For every $\delta \in \mathbb{Q}^+$ there is a cover $U \in \mathscr{U}$ such that $\mathfrak{o}[U] \leq \left\{\overline{S_s}^{\#} \cap S_r^\circ \mid (r,s) \in D_\delta\right\}$.

Remarks 7.4.5. (1) (C) $\Longrightarrow$ (S2): For any $\delta \in \mathbb{Q}^+$ there is a $U \in \mathscr{U}$ such that

$$\mathfrak{o}[U] \leq \left\{\overline{S_r}^{\#} \cap S_s^\circ \mid (r,s) \in D_\delta\right\} \leq \{\overline{S_r}^{\#}\}_{r\in\mathbb{Q}} \quad \text{and} \quad \mathfrak{o}[U] \leq \left\{\overline{S_r}^{\#} \cap S_s^\circ \mid (r,s) \in D_\delta\right\} \leq \{S_s^\circ\}_{r\in\mathbb{Q}}$$

so $\bigvee_{r\in\mathbb{Q}} \overline{S_r}^{\#} = L$ and $\bigvee_{s\in\mathbb{Q}} S_s^\circ = L$. Let us write $\overline{S}_r = \mathfrak{c}(a_r)$ for some $a_r \in L$, then

$$\mathsf{O} = \mathfrak{c}\Big(\bigvee_{r\in\mathbb{Q}} a_r\Big) = \bigcap_{r\in\mathbb{Q}} \mathfrak{c}(a_r) = \bigcap_{r\in\mathbb{Q}} \overline{S_r} \supseteq \bigcap_{r\in\mathbb{Q}} S_r,$$

since $\mathfrak{o}\left(\bigvee_{r\in\mathbb{Q}} a_r\right) = \bigvee_{r\in\mathbb{Q}} \mathfrak{o}(a_r) = \bigvee_{r\in\mathbb{Q}} \overline{S_r}^{\#} = L$. Writing $S_s^\circ = \mathfrak{o}(b_s)$ for some $b_s \in L$, we have

$$\mathsf{O} = \mathfrak{c}\Big(\bigvee_{s\in\mathbb{Q}} b_s\Big) = \bigcap_{s\in\mathbb{Q}} \mathfrak{c}(b_s) = \bigcap_{s\in\mathbb{Q}} (S_s^\circ)^{\#} \supseteq \bigcap_{s\in\mathbb{Q}} S_s^{\#},$$

since $\mathfrak{o}\left(\bigvee_{s\in\mathbb{Q}} b_s\right) = \bigvee_{s\in\mathbb{Q}} \mathfrak{o}(b_s) = \bigvee_{s\in\mathbb{Q}} S_s^\circ = L$.

(2) (C) + (wS1) $\Longrightarrow$ (FAR): Let $\delta \in \mathbb{Q}^+$. By (C) there is $U \in \mathscr{U}$ such that

$$\mathfrak{o}[U] \leq \left\{\overline{S_r}^{\#} \cap S_s^\circ \mid (r,s) \in D_\delta\right\}.$$

We claim that U is the cover that satisies (FAR); that is, S_r and $S_s^{\#}$ are U-far whenever $s - r < \frac{1}{\delta}$. Suppose this is not true. Then there are $s', r' \in \mathbb{Q}$ with $s' - r' > \frac{1}{\delta}$ such that $S_{r'}$ and $S_{s'}^{\#}$ are not U-far. In particular, $S_{r'}$ and $S_{s'}^{\#}$ are not $\left\{\overline{S_r}^{\#} \cap S_s^\circ \mid (r,s) \in D_\delta\right\}$-far (recall Remark 7.2.1 (2)). Thus, there is $(p,q) \in D_\delta$ such that

$$S_{r'} \cap \overline{S_p}^{\#} \supseteq S_{r'} \cap \overline{S_p}^{\#} \cap S_q^\circ \neq \mathsf{O} \quad \text{and} \quad S_{s'}^{\#} \cap S_q^\circ \supseteq S_{s'}^{\#} \cap \overline{S_p}^{\#} \cap S_q^\circ \neq \mathsf{O}.$$

By (wS1), this implies $r' \geq p$ and $s' \leq q$. Then we get a contradiction because

$$\tfrac{1}{\delta} < s' - r' \leq p - q = \tfrac{1}{\delta}.$$

Thus, S_r and $S_s^{\#}$ are U-far whenever $s - r > \frac{1}{\delta}$.

(3) (FAR) + (S2) $\Longrightarrow$ (C): Let $\delta \in \mathbb{Q}^+$. By assumption there is a cover $U \in \mathscr{U}$ such that S_r and $S_s^{\#}$ are U-far for every $s, r \in \mathbb{Q}$ with $s - r > \frac{1}{\delta+1}$. By Remark 6.2.1 and Proposition 7.2.4 (iv),

$$(\overline{S_r})^\circ \text{ and } (S_s^{\#})^\circ \text{ are } U\text{-far whenever } s - r > \tfrac{1}{\delta+1}. \tag{7.4.1}$$

We set $\overline{S_r} = \mathfrak{c}(a_r)$ and $S_s^\circ = \mathfrak{o}(b_r)$ for every $r, s \in \mathbb{Q}$. Since (FAR) implies (IC) we obtain

$$\mathfrak{c}(a_p) = \overline{S_p} \subseteq S_r^\circ \subseteq (\overline{S_r})^\circ \subseteq (\overline{(\overline{S_r})^\circ}) = \overline{\mathfrak{o}(a_r^*)} = \mathfrak{c}(a_r^{**}) \quad \text{and}$$

$$\mathfrak{o}(b_p^{**}) = (\overline{S_p^\circ})^\circ \subseteq \overline{S_p^\circ} \subseteq \overline{S_r} \subseteq S_r^\circ = \mathfrak{o}(b_r).$$

for every $p < r$. Thus, $a_r^{**} \leq a_p$ and $b_p^{**} \leq a_r$. Again by (IC) we have

$$\mathfrak{c}(b_r^*) = \overline{\mathfrak{o}(b_r)} = \overline{S_r^\circ} \subseteq \overline{S_r} \subseteq S_s^\circ \text{ for every } r < s$$

Thus, by (S2), $\bigcap_{r\in\mathbb{Q}} \mathfrak{c}(b_r^*) \subseteq \bigcap_{s\in\mathbb{Q}} S_s = \mathsf{O}$. Equivalently, $\bigvee_{r\in\mathbb{Q}} b_r^* = 1$. Similarly, for every $s < r$,

$$S_s \subseteq \overline{S_s} \subseteq S_r^\circ \subseteq \overline{S_r}^\circ = \mathfrak{c}(a_r)^\circ = \mathfrak{o}(a_r^*).$$

Then, by (S2),

$$L = \Big(\bigcap_{s\in\mathbb{Q}} S_s^\#\Big)^\# = \bigvee_{s\in\mathbb{Q}} S_s^{\#\#} \subseteq \bigvee_{s\in\mathbb{Q}} S_s \subseteq \bigvee_{r\in\mathbb{Q}} \mathfrak{o}(a_r^*)$$

meaning $\bigvee_{r\in\mathbb{Q}} a_r^* = 1$. By (7.4.1), sublocales $\mathfrak{o}(a_r^*)$ and $\mathfrak{o}(b_s^*)$ are U-far whenever $s - r > \frac{1}{\delta+1}$. Hence, the elements a_r^* and b_s^* are U-far whenever $s - r > \frac{1}{\delta+1}$. Thus, by Proposition 6.3.4, $U \leq \{a_r \wedge b_s \mid (r,s) \in D_\delta\}$. Equivalently,

$$\mathfrak{o}[U] \leq \{\mathfrak{o}(a_r) \wedge \mathfrak{o}(b_s) \mid (r,s) \in D_\delta\} = \Big\{\overline{S_r}^\# \cap S_s^\circ \mid (r,s) \in D_\delta\Big\}.$$

(4) Similarly, one can check that (C') $\Longrightarrow$ (S2), (C') + (ws1') $\Longrightarrow$ (FAR'), and (FAR') + (S2) $\Longrightarrow$ (C').

Thus Remark 7.4.5 and Lemma 7.4.4 yield:

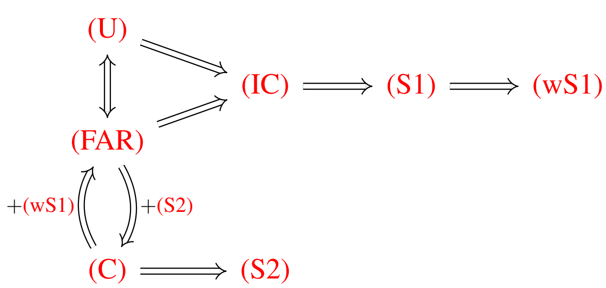
Lemma 7.4.6. *Let $(L, \mathscr{U})$ be a preuniform frame. If a family $(S_r)_{r\in\mathbb{Q}} \subseteq \mathsf{S}(L)$ satisfies (C) and (wS1) (resp. (C') and (wS1')), then the formulas*

$$h(p,\text{—}) = \bigcap_{r>p} S_r \quad \textit{and} \quad h(\text{—},q) = \bigcap_{s<q} S_s^\#$$

$$(\textit{resp.} \quad h(p,\text{—}) = \bigcap_{r>p} S_r^\# \quad \textit{and} \quad h(\text{—},q) = \bigcap_{s<q} S_s)$$

define a uniformly continuous $h \in \mathsf{F}(L)$.

Let us summarize the implications of all the conditions in this section and Section 2.6 in the following diagram:



(7.4.2)

Comparing this diagram with (6.3.2) one can see the importance of Corollary 7.2.4. Indeed, for sublocales one has

$$S \text{ and } T^\# \text{ are } U\text{-far} \iff U * S \subseteq T,$$

but for elements

$$a \text{ and } b^* \text{ are } U\text{-far} \not\Leftrightarrow Ua \leq b.$$

This is the reason why (FAR) is equivalent to (U), but (far) is not equivalent to (u).

We also condense in Table 7.1 the results presented in this section. The table shows what combinations of conditions on a family of sublocales allows us to define a general, continuous or uniformly continuous real-valued function.

	(U) $\Longleftrightarrow$ (FAR)	(IC)	(S1)	(wS1)
(C)	Unif. continuous	Unif. continuous	Unif. continuous	Unif. continuous
(S2)	Unif. continuous	Continuous	General	——

Table 7.1 Type of real-valued function a family of sublocales defines according to the conditions it satisfies.

Note that conditions (FAR), (U) and (C) are equivalent under (S1) and (S2). In fact, we have:

Proposition 7.4.7. *Let $(L, \mathscr{U})$ be a preuniform frame and let $f \in \mathsf{F}(L)$ be induced by a descending (resp. ascending) scale $(S_r)_{r\in\mathbb{Q}}$ in $\mathsf{S}(L)^{op}$. Then the following are equivalent:*

(i) *f is uniformly continuous*

(ii) *$(S_r)_{r\in\mathbb{Q}}$ satisfies (U) (resp. (U'))*.

(iii) *$(S_r)_{r\in\mathbb{Q}}$ satisfies (FAR) (resp. (FAR'))*.

(iv) *$(S_r)_{r\in\mathbb{Q}}$ satisfies (C) (resp. (C'))*.

Proof. Notice that $(S_r)_{r\in\mathbb{Q}}$ is a descending (resp. ascending) scale in $\mathsf{S}(L)^{op}$. From (7.4.2), it suffices to show only one of the statements. We will prove (iii).
Let $f\colon \mathfrak{L}(\mathbb{R}) \to L$ be given by a descending (resp. ascending) $(S_r)_{r\in\mathbb{Q}}$ scale; that is,

$$f(p,\text{—}) = \bigcap_{r>p} S_r \quad \text{and} \quad f(\text{—},q) = \bigcap_{s<q} S_s^{\#} \tag{7.4.3}$$

$$(\text{resp.} \quad f(p,\text{—}) = \bigcap_{r>p} S_r^{\#} \quad \text{and} \quad f(\text{—},q) = \bigcap_{s<q} S_s). \tag{7.4.4}$$

Suppose f is uniformly continuous. We will show that (FAR) holds for the family $(S_r)_{r\in\mathbb{Q}}$. Let $\delta \in \mathbb{Q}^+$ and $s, r \in \mathbb{Q}$ such that $s - r > \frac{1}{\delta}$. Now, since f is uniformly continuous, by Theorem 7.3.10 (ii) and Lemma 7.3.1, there is $U \in \mathscr{U}$ such that for all $p, q \in \mathbb{Q}$ with $q - p > \frac{1}{\delta}$ the sublocales $f(\text{—},p)^{\#}$ and $f(q,\text{—})^{\#}$ are U-far. Take $p', q', r' \in \mathbb{Q}$ such that $r < r' < p' < q' < s$ and $q' - p' > \delta$. From (7.4.3) and (S1) we get:

$$S_r \subseteq S_{r'}^{\#\#} \subseteq f(\text{—},p)^{\#} \quad \text{and} \quad S_s^{\#} \subseteq f(q,\text{—})^{\#}.$$

Then, by Remark 7.2.1 (1), S_r and $S_s^{\#}$ are U-far, as required. The parallel statement (inside parenthesis) is proven in a similar way.

For the converse, assume $(S_r)_{r\in\mathbb{Q}}$ satisfies (FAR) (resp. (FAR')). Because the family $(S_r)_{r\in\mathbb{Q}}$ is a descending (resp. ascending) scale, (S1) (resp. (S1')) and (S2) also hold. By Lemma 7.4.4, the formulas in (7.4.3) (resp. (7.4.4)) define a uniformly continuous function. □

Finally, let us show how the conditions in Section 6.3 relate with the ones discussed here.

Remarks 7.4.8. (1) By Remark 7.2.3, a family $(a_r)_{r\in\mathbb{Q}} \subseteq L$ satisfies (u) (resp. (u')) if and only if (U) (resp. (U')) holds for $(\mathfrak{c}(a_r))_{r\in\mathbb{Q}}$ if and only if (U') (resp. (U)) holds for $(\mathfrak{o}(a_r))_{r\in\mathbb{Q}}$.

(2) A family $(a_r)_{r\in\mathbb{Q}} \subseteq L$ satisfies (far) (resp. (far')) and (**) (resp. (**')) if and only if (FAR) (resp. (FAR')) holds for $(\mathfrak{c}(a_r))_{r\in\mathbb{Q}}$ if and only if (FAR') (resp. (FAR)) holds for $(\mathfrak{o}(a_r))_{r\in\mathbb{Q}}$.

(3) A family $(a_r)_{r\in\mathbb{Q}} \subseteq L$ satisfies (c) (resp. (c')) if and only if (C) (resp. (C')) holds for $(\mathfrak{c}(a_r))_{r\in\mathbb{Q}}$ if and only if (C') (resp. (C)) holds for $(\mathfrak{o}(a_r))_{r\in\mathbb{Q}}$.

7.5 An Insertion Theorem for Uniform Frames

Lemma 7.5.1. *Let $(L,\mathscr{U})$ be a preuniform frame and let $f,g \in \mathsf{F}(L)$ with $f \geq g$. Assume that for every $\delta \in \mathbb{Q}^+$ there is a $U \in \mathscr{U}$ such that for every $n \in \mathbb{N}$, the sublocales $f(r,—)$ and $g(—,s)$ are U^n-far whenever $s-r > \frac{n}{\delta}$. Then, there is a sequence $\{V_n\}_{n\in\mathbb{Z}} \subseteq \mathscr{U}$ such that for every $n \in \mathbb{Z}$ the following properties are satisfied*:

(1) $V_n \leq_1^* V_{n+1}$.

(2) *For every $r,s \in \mathbb{Q}$ such that $s-r > 2^n$, the sublocales $f(r,—)$ and $g(—,s)$ are V_n-far.*

Proof. Let V_0 be the cover given by the assumption for $\delta = 1$. Furthermore, for $n \geq 1$, set

$$V_n := (V_0)^{2^n}.$$

Clearly, property (2) is satisfied when $n \geq 0$. Property (1) is also satisfied for $n \geq 0$. Indeed, by Proposition 6.2.4 (4),

$$V_nV_0 = (V_0)^{2^n}V_0 = (V_0)^{2^{n+1}} = V_{n+1},$$

hence $V_n \leq_1^* V_{n+1}$. Now we define recursively V_n for $n < 0$. First, for each $n < 0$, let U_n denote the cover given by the assumption for $\delta = \frac{1}{2^n}$. For $n = -1$, pick $V_{-1} \in \mathscr{U}$ such that $V_{-1}^2 \leq V_0 \wedge U_{-1}$ (axiom (U3)). Clearly, properties (1) and (2) are satisfied (the refinement $V_{-1} \leq V_0$ is a star-refinement; in particular, it is barycentric). Suppose now that for an $n < 0$ we have constructed $V_n, V_{n+1}, \ldots, V_{-1}$ satisfying (1) and (2). Then we choose $V_{n-1} \in \mathscr{U}$ such that $V_{n-1}^2 \leq V_n \wedge U_{n-1}$. The sequence $\{V_n\}_{n\in\mathbb{Z}}$ clearly satisfies the required conditions. □

Theorem 7.5.2 (Uniform Insertion Theorem)**.** *Let $(L,\mathscr{U})$ be a preuniform frame. The following are equivalent for any $f,g \in \mathsf{F}(L)$ with $f \geq g$.*

(i) *There exists a uniformly continuous $h \in \mathsf{F}(L)$ such that $f \geq h \geq g$.*

(ii) *For every $\delta \in \mathbb{Q}^+$ there is a $U \in \mathscr{U}$ such that for every $n \in \mathbb{N}$ the sublocales $f(r,—)$ and $g(—,s)$ are U^n-far whenever $s-r > \frac{n}{\delta}$.*

Proof. (i) $\Longrightarrow$ (ii): This implication follows from Proposition 7.3.14 (vi), the definition (2.5.1) of the partial order in $\mathsf{F}(L)$ and Remark 7.2.1 (1).
(ii) $\Longrightarrow$ (i): Let $\{V_n\}_{n\in\mathbb{Z}} \subseteq \mathscr{U}$ be the sequence of uniform covers given by Lemma 7.5.1 and define a family $(b_r)_{r\in\mathbb{Q}} \subseteq L$ by

$$b_r := \bigvee_{n\in\mathbb{Z}} \bigvee A_r^n,$$

where

$$\begin{aligned} A_r^n := \Big\{ a \in V_n \mid \exists k \in \mathbb{N},\ \exists n_1, \ldots, n_k \in \mathbb{Z}, \\ \exists a_i \in V_{n_i} \text{ for all } i = 1, \ldots, k \text{ such that } a_1 = a,\ n_1 = n, \\ a_{i-1} \wedge a_i \neq 0\ (i = 2, \ldots, k), \text{ and } \mathfrak{o}(a_k) \cap f\Big(r - \sum_{i=1}^{k} 2^{n_i+5}, -\Big) \neq \mathsf{O} \Big\}. \end{aligned}$$

Set also $B_r := \mathfrak{o}(b_r)$ for every $r \in \mathbb{Q}$. Clearly, $B_r = \bigvee\{\, \mathfrak{o}(a) \mid \exists n \in \mathbb{Z} \text{ with } a \in A_r^n \,\}$.

First we will show that

$$f(-,r)^{\#} \subseteq B_r \subseteq g(-,r)^{\#} \tag{7.5.1}$$

for every $r \in \mathbb{Q}$. For the first inclusion, note that for each $n \in \mathbb{Z}$ one has

$$V_n * f(r - 2^{n+5}, -) = \bigvee\{\, \mathfrak{o}(a) \mid a \in V_n,\ \mathfrak{o}(a) \cap f(r - 2^{n+5}, -) \neq \mathsf{O} \,\} \subseteq B_r.$$

Consequently,

$$\begin{aligned} B_r &\supseteq \bigvee_{n\in\mathbb{Z}} V_n * f(r - 2^{n+5}, -) \supseteq \bigvee_{n\in\mathbb{Z}} f(r - 2^{n+5}, -) = \bigvee_{s<r} f(s, -) \supseteq \bigvee_{s<r} f(-,s)^{\#} \\ &= \Big(\bigcap_{s<r} f(-,s)\Big)^{\#} = f(-,r)^{\#}. \end{aligned}$$

Let us now show the inclusion $B_r \subseteq g(-,r)^{\#}$. Let $a \in A_r^n$; our goal is to show that $\mathfrak{o}(a) \subseteq g(-,r)^{\#}$. Since $a \in A_r^n$, there is a $k \in \mathbb{N}$ and there are $n_i \in \mathbb{Z}$ and $a_i \in V_{n_i}$ for all $i = 1, \ldots, k$ satisfying $n_1 = n$, $a_1 = a$, $a_{i-1} \wedge a_i \neq 0$ for every $i = 2, \ldots, k$, and

$$\mathfrak{o}(a_k) \cap f\Big(r - \sum_{i=1}^{k} 2^{n_i+5}, -\Big) \neq \mathsf{O}.$$

Take an $m \in \mathbb{Z}$ such that

$$2^{m-1} \le \sum_{i=1}^{k} 2^{n_i+4} < 2^m.$$

By Proposition 7.1.7 (recall that barycentric refinement implies regular refinement), there is a $v \in V_{m-1}$ such that $a_1 \vee a_k \le v$. We have that

$$r - \Big(r - \sum_{i=1}^{k} 2^{n_i+5}\Big) > \sum_{i=1}^{k} 2^{n_i+4} \ge 2^{m-1}$$

so by Lemma 7.5.1 (2), $f(r-\sum_{i=1}^{k}2^{n_i+5},—)$ and $g(—,r)$ are V_{m-1}-far. Consequently,

$$\mathfrak{o}(a)=\mathfrak{o}(a_1)\subseteq\mathfrak{o}(v)\subseteq V_{m-1}*f\Big(r-\sum_{i=1}^{k}2^{n_i+5},—\Big)\subseteq g(—,r)^{\#}$$

where the second inclusion holds because $v\in V_{m-1}$ and

$$\mathsf{O}\neq\mathfrak{o}(a_k)\cap f\Big(r-\sum_{i=1}^{k}2^{n_i+5},—\Big)\subseteq\mathfrak{o}(v)\cap f\Big(r-\sum_{i=1}^{k}2^{n_i+5},—\Big).$$

Hence, (7.5.1) holds. Now, we will show that the conditions of Lemma 7.4.2 hold for the family $\{B_r\}_{r\in\mathbb{Q}}$. First, notice that by (7.5.1) one has

$$\bigcap_{r\in\mathbb{Q}}B_r\subseteq\bigcap_{r\in\mathbb{Q}}g(—,r)^{\#}\subseteq\bigcap_{r\in\mathbb{Q}}g(r,—)=\mathsf{O},$$

and similarly,

$$\bigcap_{r\in\mathbb{Q}}B_r^{\#}\subseteq\bigcap_{r\in\mathbb{Q}}f(—,r)^{\#\#}\subseteq\bigcap_{r\in\mathbb{Q}}f(—,r)=\mathsf{O}.$$

Let $\delta\in\mathbb{Q}^+$ and select an $n\in\mathbb{Z}$ such that $\frac{1}{\delta}>2^{n+5}$. Let $s-r>\frac{1}{\delta}$; we will show that

$$V_nb_r\leq b_s, \tag{7.5.2}$$

which is clearly equivalent to $V_n*B_r\subseteq B_s$. Now, since $b_r=\bigvee_{m\in\mathbb{Z}}\bigvee A_r^m$, by virtue of Proposition 6.1.1 (7), to prove (7.5.2) is equivalent to show that if $a\in A_r^m$ and $v\in V_n$ are such that $v\wedge a\neq 0$, then $v\leq b_s$. If $a\in A_r^m$, there is a $k\in\mathbb{N}$ such that for every $i=1,\ldots,k$ there is an $a_i\in V_{n_i}$ satisfying $a_1=a$, $n_1=m$, $a_{i-1}\wedge a_i\neq 0$ for every $i=2,\ldots,k$ and $\mathfrak{o}(a_k)\cap f(r-\sum_{i=1}^{k}2^{n_i+5},—)\neq\mathsf{O}$. Since $s-2^{n+5}>r$, it follows that

$$f\Big(r-\sum_{i=1}^{k}2^{n_i+5},—\Big)\subseteq f\Big(s-2^{n+5}-\sum_{i=1}^{k}2^{n_i+5},—\Big),$$

and so

$$\mathfrak{o}(a_k)\cap f\Big(s-2^{n+5}-\sum_{i=1}^{k}2^{n_i+5},—\Big)\neq\mathsf{O}.$$

Hence, if $v\in V_n$ is such that $v\wedge a\neq 0$, it follows that $v\in A_s^n$, which yields $v\leq b_s$, as required.

By Lemma 7.4.2, the function $h\in\mathsf{F}(L)$ defined by

$$h(p,—)=\bigcap_{r>p}B_r\quad\text{and}\quad h(—,q)=\bigcap_{s<q}B_s^{\#}$$

is uniformly continuous. Finally, $f\geq h\geq g$ because, from (7.5.1) and Remark 2.5.1, we have

$$h(p,—)=\bigcap_{r>p}B_r\subseteq\bigcap_{r>p}g(—,r)^{\#}\subseteq\bigcap_{r>p}g(r,—)=g(p,—)$$

and

$$h(-,q) = \bigcap_{s<q} B_s^{\#} \subseteq \bigcap_{s<q} f(-,s)^{\#\#} \subseteq \bigcap_{s<q} f(-,s) = f(-,q)$$

for every $p,q \in \mathbb{Q}$ (see (2.5.1)). □

Specializing Theorem 7.5.2 one can easily obtain the Uniform Insertion Theorem for bounded functions (Theorem 7.5.4 below). However, we present an alternative (and easier) proof of this special case by using a different technique similar to the one used in Lemma 6.5.4; we will use again Katětov's Lemma (Lemma 6.5.1).

Example 7.5.3. Let $(L,\mathscr{U})$ be a preuniform frame. We know that the uniformly below relation $\lhd_{\mathscr{U}}$ given by the preuniformity $\mathscr{U}$ is a Katětov relation (Example 6.5.2). Moreover, we can extend this relation to sublocales:

$$S \lhd_{\mathscr{U}} T \equiv \text{ there is a } U \in \mathscr{U} \text{ such that } U * S \subseteq T.$$

It can be easily checked that this is a Katětov relation on $\mathsf{S}(L)$.

Theorem 7.5.4 (Uniform Insertion Theorem for Bounded Functions)**.** *Let $(L,\mathscr{U})$ be a preuniform frame and let $f,g \in \mathsf{F}(L)$ be bounded functions with $f \geq g$. Then the following are equivalent*:

(i) *There exists a uniformly continuous $h \in \mathsf{F}(L)$ such that $f \geq h \geq g$.*

(ii) *For every $\delta \in \mathbb{Q}^+$ there is a $U \in \mathscr{U}$ such that the sublocales $f(r,-)$ and $g(-,s)$ are U-far whenever $s-r > \frac{1}{\delta}$.*

Proof. (i) $\Longrightarrow$ (ii): This implication follows from Proposition 7.3.14 (v), the definition of the partial order in $\mathsf{F}(L)$ (2.5.1) and Remark 7.2.1 (1).
(ii) $\Longrightarrow$ (i): Since f and g are bounded, by (2.5.2) take $\alpha,\beta \in \mathbb{Q}$ with $\alpha < \beta$ such that[2]

$$f(\beta,-) = L, \quad f(-,\alpha) = L, \quad g(\beta,-) = L \quad \text{and} \quad g(-,\alpha) = L. \tag{7.5.3}$$

By assumption, one has in particular that $g(-,s) \lhd_{\mathscr{U}} f(r,-)^{\#}$ for every $s > r$. Since $\lhd_{\mathscr{U}}$ is a Katětov relation (Example 6.5.2), by Lemma 6.5.1 there is a family $(C_p)_{p\in\mathbb{Q}} \subseteq \mathsf{S}(L)$ such that

$$g(-,s) \lhd_{\mathscr{U}} C_q \lhd_{\mathscr{U}} C_p \lhd_{\mathscr{U}} f(r,-)^{\#} \tag{7.5.4}$$

whenever $r < p < q < s$. We will use Lemma 7.4.2 to show that $(C_p)_{p\in\mathbb{Q}}$ determines a uniformly continuous function. First, from (7.5.4) it is easy to see that $\bigcap_{p\in\mathbb{Q}} C_p = \mathsf{O} = \bigcap_{p\in\mathbb{Q}} C_p^{\#}$. So we only have to show that

$$\forall \delta \in \mathbb{Q}^+ \text{ there is some } U \in \mathscr{U} \text{ such that } U * C_s \subseteq C_r \text{ for every } s-r > \tfrac{1}{\delta}. \tag{7.5.5}$$

Let $\delta \in \mathbb{Q}^+$. Notice that if $\beta < s$ or $\alpha > r$, from (7.5.3) and (7.5.4) one obtains $C_s \subseteq f(\beta,-)^{\#} = \mathsf{O}$ or $L = g(-,\alpha) \subseteq C_r$ which clearly yields $U * C_s \subseteq C_r$ for any $U \in \mathscr{U}$. Thus, it suffices to show (7.5.5)

[2] Departing from our usual convention of using p,q,r,s to denote rationals, in this proof we also use α and β. We do this in order to simplify notation.

for every $s-r>\frac{1}{\delta}$ with $\alpha \leq r<s\leq \beta$. Select an $n\in\mathbb{N}$ and $t_0,t_1,\dots,t_{n+1}\in\mathbb{Q}$ such that

$$t_0=\alpha<t_1<t_2<\cdots<t_n<\beta=t_{n+1}$$

and $t_{k+1}-t_k<\frac{1}{2\delta}$ for all $k=0,\dots,n$. Set $U:=U_0\wedge U_1\cdots\wedge U_n$, where U_k is the cover that witnesses the relation $C_{t_{k+1}}\lhd_{\mathscr{U}} C_{t_k}$ for $k=0,\dots,n$. Thus $U*C_{t_{k+1}}\subseteq C_{t_k}$ for every $k=0,\dots,n$. Let $s-r>\frac{1}{\delta}$ with $\alpha\leq r<s\leq\beta$ and pick a $k\in\{0,\dots,n\}$ such that $r\leq t_k<t_{k+1}\leq s$. Then

$$U*C_s\subseteq U*C_{t_{k+1}}\subseteq C_{t_k}\subseteq C_r,$$

as required. In conclusion, $(C_p)_{p\in\mathbb{Q}}$ determines a uniformly continuous $h\in\mathsf{F}(L)$ given by

$$h(r,\text{—})=\bigcap_{r<p}C_p^{\#}\quad\text{and}\quad h(\text{—},s)=\bigcap_{q<s}C_q.$$

Furthermore, by (7.5.4) one may easily check that $g\leq h\leq f$. □

Condition (ii) in Theorem 7.5.2 is formally stronger than condition (ii) in Theorem 7.5.4. The following proposition and the remark afterwards explain the reason behind this discrepancy:

Proposition 7.5.5. *Let $(L,\mathscr{U})$ be a preuniform frame and $f,g\in\mathsf{F}(L)$ with $f\geq g$. Fix a $\delta_0\in\mathbb{Q}^+$. Then the following are equivalent:*

(i) *For every $\delta\in\mathbb{Q}^+$ there is a $U\in\mathscr{U}$ such that for every $n\in\mathbb{N}$ the sublocales $f(r,\text{—})$ and $g(\text{—},s)$ are U^n-far whenever $s-r>\frac{n}{\delta}$.*

(ii) *The following two conditions hold:*

   (a) *There is a $U_0\in\mathscr{U}$ such that for every $n\in\mathbb{N}$ the sublocales $f(r,\text{—})$ and $g(\text{—},s)$ are U_0^n-far whenever $s-r>\frac{n}{\delta_0}$.*

   (b) *For every $\delta\in\mathbb{Q}^+$ there is a $U\in\mathscr{U}$ such that the sublocales $f(r,\text{—})$ and $g(\text{—},s)$ are U-far whenever $s-r>\frac{1}{\delta}$.*

Proof. (i) $\Longrightarrow$ (ii) is trivial.

(ii) $\Longrightarrow$ (i): Let $\delta\in\mathbb{Q}^+$ and select an $m\in\mathbb{N}$ such that $\delta<\delta_0 2^m$. For each $n\in\{1,\dots,2^m-1\}$ let U_n be the cover given by (b) for the rational $\frac{\delta_0 2^m}{n}\in\mathbb{Q}^+$. Then

$$f(r,\text{—})\text{ and }g(\text{—},s)\text{ are }U_n\text{-far whenever }s-r>\tfrac{n}{\delta_0 2^m}\tag{7.5.6}$$

for each $n\in\{1,\dots,2^m-1\}$. Now use (U3) and choose a cover W such that

$$W^{2^{m+1}}\leq U_0\wedge\bigwedge_{n=1}^{2^m-1}U_n.$$

We claim that, for any $n\in\mathbb{N}$, the sublocales

$$f(r,\text{—})\text{ and }g(\text{—},s)\text{ are }W^n\text{-far whenever }s-r>\tfrac{n}{\delta_0 2^m}.\tag{7.5.7}$$

Indeed, let $n \in \mathbb{N}$ and $s - r > \frac{n}{\delta_0 2^m}$. We distinguish two cases:

(Case 1): If $n \in \{1, \ldots, 2^m\}$, then $f(r,—)$ and $g(—,s)$ are U_n-far if $n < 2^m$ (by (7.5.6)) and U_0-far if $n = 2^m$ (by (a)). In either case they are $W^{2^{m+1}}$-far by Remark 7.2.1 (2). But $n \leq 2^m \leq 2^{m+1}$ and so $W^n \leq W^{2^{m+1}}$, hence they are also W^n-far.

(Case 2): If $n > 2^m$. Since $n2^{-m} > 1$, select an $\ell \in \mathbb{N}$ such that $\ell < n2^{-m} \leq \ell + 1$. We may write $n = \ell 2^m + j$ for a suitable $j \in \{1, \ldots, 2^m\}$, namely $j = n - \ell 2^m$. Since

$$s - r > \frac{n}{\delta_0 2^m} = \frac{\ell 2^m + j}{\delta_0 2^m} > \frac{\ell}{\delta_0},$$

it follows from (a) that $f(r,—)$ and $g(—,s)$ are U_0^ℓ-far. By Lemma 6.2.4 (5) we conclude then that

$$W^n = W^{\ell 2^m + j} \leq W^{\ell 2^m + 2^m} \leq W^{\ell 2^{m+1}} \leq (W^{2^{m+1}})^\ell \leq U_0^\ell,$$

thus $f(r,—)$ and $g(—,s)$ are W^n-far, as required.

Hence, (7.5.7) is proved. Finally, if $s - r > \frac{n}{\delta}$ we have $s - r > \frac{n}{\delta_0 2^m}$ so $f(r,—)$ and $g(—,s)$ are W^n-far. □

Remark 7.5.6. Let $\mathbf{p} \leq g \leq f \leq \mathbf{q}$ be bounded. Then by taking $\delta_0 = \frac{1}{q-p}$, condition (a) in Proposition 7.5.5 is trivially satisfied. Indeed, if $s - r > n(q-p)$, then $s - r > q - p$ and so either $r < p$ or $s > q$. By (2.5.3), one has $f(r,—) = \mathsf{O}$ or $g(—,s) = \mathsf{O}$, thus $f(r,—)$ and $g(—,s)$ are U-far for any cover U. This explains why condition (ii) in Theorem 7.5.4 is precisely condition (b) in Proposition 7.5.5.

7.6 Separation and Extension Theorems for Uniform Frames

Usually, a Katětov-type insertion theorem yields an Urysohn-type separation result and a Tietze-type extension result as straightforward corollaries. In this final section, we prove the uniform versions of these theorems.

A General Separation Theorem for Uniform Frames

Theorem 7.6.1 (Uniform Separation Theorem). *Let $(L, \mathscr{U})$ be a preuniform frame, and let S and T be sublocales of L. Then the following are equivalent*:

(i) *S and T are U-far for some $U \in \mathscr{U}$.*

(ii) *There is a uniformly continuous $h \in \mathsf{F}(L)$ with $\mathbf{0} \leq h \leq \mathbf{1}$ such that $T \subseteq h(0,—)$ and $S \subseteq h(—,1)$.*

Proof. (i) ⟹ (ii): Assume that S and T are U-far for some $U \in \mathscr{U}$. By Proposition 7.2.2, $\overline{S}$ and $\overline{T}$ are U-far. Consider the characteristic functions of $\overline{S}$ and $\overline{T}^{\#}$ from Example 2.5.2 (2), namely the maps $\chi_{\overline{S}}, \chi_{\overline{T}^{\#}} \in \mathsf{F}(L)$ given by

$$\chi_{\overline{S}}(p,—) = \begin{cases} \mathsf{O} & \text{if } p < 0 \\ \overline{S}^{\#} & \text{if } 0 \leq p < 1 \\ L & \text{if } p \geq 1 \end{cases} \qquad \chi_{\overline{S}}(—,q) = \begin{cases} L & \text{if } q \leq 0 \\ \overline{S} & \text{if } 0 < q \leq 1 \\ \mathsf{O} & \text{if } q > 1 \end{cases}$$

and

$$\chi_{\overline{T}^\#}(p,\text{—}) = \begin{cases} \mathsf{O} & \text{if } p < 0 \\ \overline{T} & \text{if } 0 \le p < 1 \\ L & \text{if } p \ge 1 \end{cases} \qquad \chi_{\overline{T}^\#}(\text{—},q) = \begin{cases} L & \text{if } q \le 0 \\ \overline{T}^\# & \text{if } 0 < q \le 1 \\ \mathsf{O} & \text{if } q > 1. \end{cases}$$

Since $\overline{S}$ and $\overline{T}$ are U-far, one has $\overline{S} \subseteq \overline{T}^\#$, and therefore it follows that $\mathbf{0} \le \chi_{\overline{S}} \le \chi_{\overline{T}^\#} \le \mathbf{1}$. Furthermore, we claim that for every $\delta \in \mathbb{Q}^+$ the sublocales $\chi_{\overline{T}^\#}(r,\text{—})$ and $\chi_{\overline{S}}(\text{—},s)$ are U-far whenever $s - r > \frac{1}{\delta}$. Indeed, if $r < 0$ or $1 < s$, one clearly has that $\chi_{\overline{T}^\#}(r,\text{—})$ and $\chi_{\overline{S}}(\text{—},s)$ are U-far. If $0 \le r < s \le 1$, then $\chi_{\overline{T}^\#}(r,\text{—}) = \overline{T}$ and $\chi_{\overline{S}}(\text{—},s) = \overline{S}$, which by assumption are U-far. Consequently, by Theorem 7.5.4, there is a uniformly continuous $h \in \mathsf{F}(L)$ such that $\mathbf{0} \le \chi_{\overline{S}} \le h \le \chi_{\overline{T}^\#} \le \mathbf{1}$. Moreover, (recall (2.5.1)), we have

$$S \subseteq \overline{S} = \chi_{\overline{S}}(\text{—},1) \subseteq h(\text{—},1) \quad \text{and} \quad T \subseteq \overline{T} = \chi_{\overline{T}^\#}(0,\text{—}) \subseteq h(0,\text{—})$$

as required.
(ii) $\Longrightarrow$ (i): Since h is uniformly continuous, by Theorem 7.3.10 there is a $U \in \mathscr{U}$ such that $h(0,\text{—})$ and $h(\text{—},1)$ are U-far. In particular, S and T are U-far. □

Since uniformly continuous real-valued functions are continuous and $\mathsf{C}(L) \cong \mathscr{R}(L)$, applying Theorem 7.6.1 to open sublocales immediately yields our previous Theorem 6.4.5.

A General Extension Theorem for Uniform Frames

Let $(L,\mathscr{U})$ be a preuniform frame and S a sublocale of L with $j_s \colon S \hookrightarrow L$ the localic embedding in S. Recall from Section 6.5 the preuniformity in S,

$$\mathscr{U}_S^L = \{ j_S^*[U] \mid U \in \mathscr{U} \},$$

induced by $\mathscr{U}$. Notice that in this more general setting where we work with open covers and farness between sublocales, we have that:

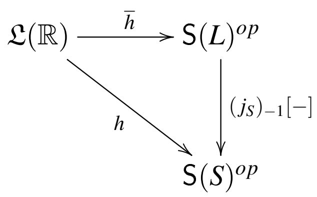
(1) Remark 1.5.1 and (1.5.1) give us a description of $\mathfrak{o}_S[j_s^*[U]]$:

$$\mathfrak{o}_S[j_s^*[U]] = \{\mathfrak{o}_S(j_S^*(u)) \mid u \in U\} = \{(j_S)_{-1}[\mathfrak{o}_L(u)] \mid u \in U\} = \{\mathfrak{o}_L(u) \cap S \mid u \in U\} \qquad (7.6.1)$$

for every $U \in \mathscr{U}$. A word of caution: one should not confuse $\mathfrak{o}_S[j_s^*[U]]$ with $\mathfrak{o}_L[j_s^*[U]]$.

(2) Using (7.6.1) one can easily check that if two sublocales are $j_S^*[U]$-far (in $\mathsf{S}(S)$) for some $U \in \mathscr{U}$, then they are U-far (in $\mathsf{S}(L)$).

For general real-valued functions, let $h \in \mathsf{F}(S)$ be uniformly continuous with respect to $\mathscr{U}_S^L$. We say that an $\overline{h} \in \mathsf{F}(L)$ is a *uniformly continuous extension* of h if it is uniformly continuous with respect to $\mathscr{U}$ and the diagram



(7.6.2)

commutes, where $(j_S)_{-1}[T] = T \cap S$ for each $T \in \mathsf{S}(L)$. For $f \in \mathsf{C}(L)$ we have $f = \mathfrak{c}g$ with $g \in \mathscr{R}(L)$. Note that f has a uniformly continuous extension if and only if g does. Furthermore, the respective diagram (7.6.2) for f commutes if and only if (6.5.1) applied to g does (recall (1.5.1)). We also know, from Corollary 7.3.11, that uniform continuity implies continuity for general real-valued functions. Then, by Proposition 7.3.14, Lemma 6.5.4 yields immediately the following:

Corollary 7.6.2. *Let $(L,\mathscr{U})$ be a preuniform frame and let S be a dense sublocale of L. Every uniformly continuous $h \in \mathsf{F}(S)$ (with respect to $\mathscr{U}_S^L$) has a uniformly continuous extension $\overline{h} \in \mathsf{F}(L)$ (with respect to $\mathscr{U}$). Moreover, if h is bounded, then so is $\overline{h}$.*

Theorem 7.6.3 (Uniform Extension Theorem)**.** *Let $(L,\mathscr{U})$ be a preuniform frame and let S be a sublocale of L. Every bounded uniformly continuous $h \in \mathsf{F}(S)$ (with respect to $\mathscr{U}_S^L$) has a bounded uniformly continuous extension $\overline{h} \in \mathsf{F}(L)$ (with respect to $\mathscr{U}$).*

Proof. Since every sublocale is dense in its closure, by Corollary 7.6.2 and Remark 6.5.3 (4), it suffices to show the result for closed sublocales. More generally, we shall show it for complemented sublocales.

Let S be a complemented sublocale of L and denote by $j_S\colon S \hookrightarrow L$ its localic embedding. Let $h \in \mathsf{F}(S)$ be bounded and uniformly continuous with respect to $\mathscr{U}_S^L$. Select $\alpha,\beta \in \mathbb{Q}$ such that $\boldsymbol{\alpha} \le h \le \boldsymbol{\beta}$ and for each $r \in \mathbb{Q}$ set [3]

$$S_r := \begin{cases} \mathsf{O} & \text{if } r < \alpha \\ h(r,\text{—}) & \text{if } \alpha \le r < \beta \\ L & \text{if } r \ge \beta \end{cases} \quad \text{and} \quad T_r := \begin{cases} L & \text{if } r \le \alpha \\ h(\text{—},r) & \text{if } \alpha < r \le \beta \\ \mathsf{O} & \text{if } r > \beta. \end{cases}$$

For each $r < s$ one has $S_s^\# \cap S_r = \mathsf{O}$. Indeed, if $r < \alpha$ or $s \ge \beta$ it is trivial because either $S_s^\# = \mathsf{O}$ or $S_r = \mathsf{O}$. If $\alpha \le r < s < \beta$ then

$$S_s^\# \cap S_r = h(s,\text{—})^\# \cap h(r,\text{—}) = h(s,\text{—})^\# \cap S \cap h(r,\text{—}) = h(s,\text{—})^{\#_S} \cap h(r,\text{—}) \subseteq h(\text{—},s) \cap h(r,\text{—}) = \mathsf{O}$$

by (1.4.14) and (2.5.1). Hence $(S_r)_{r\in\mathbb{Q}}$ satisfies (S1), and similarly $(T_r)_{r\in\mathbb{Q}}$ satisfies (S1'). Let $f,g \in \mathsf{F}(L)$ be the functions they define (Proposition 2.6.2). From the equalities

$$f(\text{—},r) = \bigcap_{p<r} S_p^\# \quad \text{and} \quad g(\text{—},r) = \bigcap_{q<r} T_q,$$

it follows that for each $r \in \mathbb{Q}$ one has $g(\text{—},r) \subseteq f(\text{—},r)$, that is $f \ge g$. Indeed, let $r \in \mathbb{Q}$ and $p < r$. We have to check that $\bigcap_{q<r} T_q \subseteq S_p^\#$. If $p < \alpha$ or $r > \beta$ one has either $S_p^\# = L$ or $\bigcap_{q<r} T_q = \mathsf{O}$, so the inclusion follows. Suppose now that $\alpha \le p < r \le \beta$ and pick a $q' \in \mathbb{Q}$ with $p < q' < r$. Then

$$\bigcap_{q<r} T_q \subseteq T_{q'} = h(\text{—},q') \subseteq h(p,\text{—})^\# = S_p^\#,$$

as desired.

[3] As in the proof of Theorem 7.5.4, in order to simplify the notation, in this proof we will also use α and β to denote rationals.

Furthermore, the maps f and g satisfy condition (ii) in Theorem 7.5.4. Indeed, let $\delta \in \mathbb{Q}^+$. Since h is uniformly continuous there is a $U \in \mathscr{U}$ such that $h(r,—)$ and $h(—,s)$ are $j_S^*[U]$-far (as sublocales of S) whenever $s - r > \frac{1}{\delta}$. By Remark 6.5.3 (2), $h(r,—)$ and $h(—,s)$ are U-far (as sublocales of L). We claim that $f(r,—)$ and $g(—,s)$ are U-far whenever $s - r > \frac{1}{\delta}$. Clearly it suffices to show the case where $\alpha \leq r < s \leq \beta$ (as otherwise $f(r,—) = \mathsf{O}$ or $g(—,s) = \mathsf{O}$). Pick $r', s' \in \mathbb{Q}$ with $r < r' < s' < s$ and $s' - r' > \frac{1}{\delta}$. Then

$$f(r,—) = \bigcap_{r<p} S_p \subseteq S_{r'} = h(r',—) \quad \text{and} \quad g(—,s) \subseteq T_{s'} = h(—,s').$$

The claim follows from Remark 7.2.1 (1).

Moreover, f and g are bounded by (2.5.3). By Theorem 7.5.4 there is a uniformly continuous $\overline{h} \in \mathsf{F}(L)$ with $f \geq \overline{h} \geq g$. Now it follows trivially from (2.5.3) and (2.5.2) that $S_r \cap S = h(r,—)$ and $T_r \cap S = h(—,r)$ for each $r \in \mathbb{Q}$. Hence $(j_S)_{-1}[-]\, f = h = (j_s)_{-1}[-]\, g$, and so $h \geq (j_S)_{-1}[-]\, \overline{h} \geq h$. Thus, $\overline{h}$ is the desired extension of h. $\square$

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List of Symbols

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