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Market Based Probability of Default Models

Dissertação no âmbito do Mestrado em Métodos Quantitativos em Finanças, orientada pela Professora Doutora Ana Margarida Monteiro e pelo Professor Doutor António Alberto Santos e apresentada ao Departamento de Matemática da Faculdade de Ciências e Tecnologia e à Faculdade de Economia.

março de 2023

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UNIVERSIDADE D
COIMBRA

Master in Quantitative Methods in Finance

Mestrado em Métodos Quantitativos em Finanças

MSc Dissertation | Dissertação de Mestrado

março 2023

Acknowledgements

This work would not be possible without the support of some important people. In one way or another, everyone has been there and made their contribution to the success of this project.

The work presented in this paper was partially carried out in the scope of the RiskBigData project: Complex Risk Management in the Big Data Regime (PTDC/MAT-APL/1286/2021), financed the Portuguese Foundation for Science and Technology (FCT).

To my advisors Professor Ana Margarida Monteiro and Professor António Alberto Santos, I would like to thank all of their availability over these last few months. Your help and encouragement were crucial for the accomplishment of this work and I will carry, through the next challenges of my life, the values of excellence that you passed on to me. It was a true privilege to be able to work with you.

To my family for always being there for me. Mainly to my parents, my sisters and my four wonderful grandparents because without you this achievement would not be possible, your support, patience and affection were essential to be able to trace this path.

To the friends that Coimbra brought me. What a privilege it was to get to know you all! Thank you Catarina, Beatriz, Mariana, Inês, Taís, Constança, Marta, Nuno, Frota, Rúben, Francisco and Miguel. Without you, I can say that this road would not have had the same joy, and such a good road the one we had.

A special thanks to Pollux and to all the friends that this Junior Initiative brought me. Pollux is a place to grow and it was there where I learned a lot, beyond Mathematics. Thank you for all the values that you gave me and on which I will base my professional life.

To my great friend Carolina Marques, without whom these last 5 years at University would not have been so wonderful, thank you for always being there and for helping me becoming a better person.

To a very special person, Tiago Novo, for all the endless support he has given me over the last few years. Sharing life with you motivates me to do more and better every day.

Finally, a huge thank you to this city that welcomed me five years ago. These years have been better than I ever asked for, and I am sure that the Alice from 2018 would not have changed a thing. Thank you, Coimbra.

Cofinanciado por:

Abstract

This dissertation presents the estimation of the probability of default based on default models associated with the prices of financial options. First, the Black-Scholes-Merton option pricing model is presented, as well as some extensions. The first extension presented is based on the mixture of two lognormal distributions which, although it does not involve a default parameter, is relevant in the sense that it allows a better understanding of the following extensions. On the other hand, two extensions to the Black-Scholes-Merton model are addressed in which a parameter that represents the probability of default is introduced. The first one is a lognormal distribution augmented with a probability of default and the second one is a mixture of two lognormal distributions with a probability of default. The goal of both extensions is the estimation of the risk-neutral density function that incorporates this probability. An empirical analysis is carried out in a simulation environment and with market data. In a simulation environment, we intend to verify that, in fact, the models that we are working with are well calibrated, that is, we obtain estimated values considerably close to the theoretical ones. In the analysis with market data, we intend to estimate the risk-neutral density functions, as well as the probability of default for two companies, Intel Corp. and Beyond Meat, Inc., using option price data, in order to compare the probability of default of both companies, associated with their financial stability over the past few years.

Keywords: Options contracts, Probability of default, Risk-neutral density function.

Resumo

A presente dissertação apresenta a estimação da probabilidade de *default* baseada em modelos de *default* associados a preços de opções financeiras. Numa primeira fase, é apresentado o modelo de Black-Scholes-Merton para atribuição de preços de opções financeiras, bem como algumas generalizações do mesmo. A primeira generalização apresentada baseia-se na mistura de duas distribuições lognormais que, apesar de não envolver um parâmetro de *default*, é relevante no sentido em que permite uma melhor compreensão das generalizações seguintes. Por outro lado, são estudadas duas generalizações ao modelo de Black-Scholes-Merton nas quais já é introduzido um parâmetro que representa a probabilidade de *default*. A primeira diz respeito a uma distribuição lognormal aumentada com probabilidade de *default* e a segunda uma mistura de duas distribuições lognormais com probabilidade de *default*. O objetivo de ambas é a estimação da função densidade neutra ao risco que incorpora esta probabilidade. Numa segunda fase, procede-se a uma análise empírica em ambiente de simulação e com dados de mercado. Em ambiente de simulação, pretendemos verificar que, de facto, os modelos com os quais estamos a trabalhar estão bem calibrados, isto é, obtemos valores estimados consideravelmente próximos dos valores teóricos. Na análise com dados de mercado, pretendemos estimar as funções densidade neutras ao risco, assim como a probabilidade de *default* para duas empresas, Intel Corp. e Beyond Meat, Inc., através da utilização de dados de preços de opções financeiras, de forma a comparar a probabilidade de *default* de ambas associada com a sua estabilidade financeira ao longo dos últimos anos.

Palavras-chave: Contratos de opções, Probabilidade de *default*, Função densidade neutra face ao risco.

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Chapter 1

Introduction

As financial markets become more complex, the risk to which companies are subject can assume more relevance, in this sense, it becomes important to study the financial instruments associated with risk management.

Financial risk is associated with changes in financial markets that can have significant impacts on investors' profits and losses, as well as on a country's economy. Financial derivatives appear as risk management instruments, which are agreements established for deferred delivery in time of one or more assets denominated by underlying assets. In other words, it is assumed that the agreements entered in this context are settled on one, or several, pre-established future dates. One of the most essential classes of financial derivatives is options contracts. The most distinctive feature is the possibility of the owner to exercise or not the contract, that is, buying or selling the underlying asset. Options are traded on various financial markets and may reveal market expectations about the underlying asset. Thus options are important in hedging and managing financial risk.

One important risk indicator in financial markets is the probability of default. In the present dissertation, we intend to study market models associated with the probability of default. This dissertation will analyse two companies, namely Intel Corp. and Beyond Meat, Inc., the first one is a financially stable company, while the second one presents itself as one of the most shorted companies in the market. To estimate the probability of default, we based our study on the implementation of three different models: a mixture of two lognormal distributions, a lognormal distribution augmented with a probability of default presented by Câmara et al. [3] and a mixture of two lognormal distributions with a probability of default presented by Taylor et al. [14].

The method presented by Câmara et al. [3] is a modification of the Black-Scholes-Merton option pricing model, where we have a lognormal distribution augmented with a probability of default. On the other hand, the method presented by Taylor et al. [14] consists of a mixture of two lognormal densities with a probability of default. Both of these models intend to adapt the Black-Scholes-Merton model to the evidence of the markets by estimating the probability of default from stock and option market prices.

In the second chapter, we present some basic concepts of financial literacy, which are useful for understanding this dissertation and we make a brief literature review where some fundamental issues used by several authors are exposed, as well as their contributions to the study of this subject. In the third chapter, we present the best-known option pricing model: the Black-Scholes-Merton model. It is also presented the extensions of the Black-Scholes-Merton model, namely the methods of Câmara et al.[3] and Taylor et al. [14] and three optimization problems that are the starting point for the next chapter. In chapter four, an empirical analysis is presented regarding the models under study, using the MatLab software. Finally, in chapter five, we present the conclusions of the work.

Chapter 2

Basic Concepts and Literature Review

Financial derivatives are defined as instruments whose price depends on the value of one or more assets, which we refer to as underlying assets (J. Hull [6]). The main types of derivatives include forward, future and options contracts.

Forward and futures contracts are both derivatives arrangements that involve two parts who agree to buy or sell a specific asset by a set price at a certain date in the future. However, it is important to notice that they might differ in some aspects: forward contracts are traded in decentralized and informal markets and have a high level of flexibility, which means that the contract conditions are better suited to the interests of the two parties involved; nonetheless, futures contracts are standardized agreements and are traded in formal and centralized markets, which results in increased liquidity and reduced transaction costs.

Options contracts involve two parts: a buyer and a seller. In the case of an investor buying an option, we say that he is facing a long position, on the other hand, if he sells the option, we say that he is facing a short position (J.Hull [6]). An option gives the buyer the right, but not the obligation, to buy (call option) or sell (put option) a particular underlying asset, at a future date or during a certain period of time, at a pre-established price. Besides that, we can also distinguish the options in two styles: European and American. An European option can only be exercised on a certain future date, while an American option can be exercised at any time up to (and including) its maturity. The pre-established price is called the strike price, K, and the last day on which an option can be exercised is called maturity, T. Let S_t be the price of the underlying asset at the present time t. We denote by $\tau = T - t$ the time to maturity (which is the time before an option expires).

We can classify options trading at time t as *at-the-money*, *in-the-money* and *out-of-the-money*. For an European call (put) option the ratio $\frac{S_t}{K}$ is called the moneyness; if this ratio is larger (smaller) than 1, the option is said to be *in-the-money*, if it is smaller (larger) than 1, the option is said to be *out-of-the-money* and if the ratio is approximately 1, the option is said to be *at-the-money*.

To estimate a company's probability of default, it is necessary to be familiar with some mathematical and economic concepts. In this dissertation, we adopt the same definition of default as the one pointed out by Jensen and Meckling [7]: in general, default occurs when the firm cannot meet a current payment or a debt obligation.

Arbitrage is an investment strategy in which an investor simultaneously buys and sells an asset in different markets, to take advantage of a price difference and generate a profit. While price differences are typically small and remain during a small interval of time, the returns can be impressive when multiplied by a large volume of transactions.

Consider an European call option, where the option holder has the right to purchase the underlying asset at the strike price at maturity, S_T. If $S_T < K$, the owner of the option can purchase the underlying asset for less than K, which means that it is not optimal to exercise the option, considering that he can buy the underlying asset on the market for less than the strike price. As a consequence of that, the option on maturity T will have the value 0. Furthermore, if $S_T > K$, it will be rational to exercise the option and its value will be $S_T - K$. Considering an European put option, the owner has the right to sell the underlying asset for the strike price at maturity T. It will be reasonable to exercise the put option only when $S_T < K$. Therefore, the payoffs of the call and put options at maturity are, respectively,

$$C(S_T,T,K) = \max(S_T - K, 0) = (S_T - K)^+, \tag{2.1}$$

$$P(S_T,T,K) = \max(0, K - S_T) = (K - S_T)^+ \tag{2.2}$$

.

So far, we have considerable information about European options. American options embed the right to early exercise. As we said before, this means that they can be exercised between the purchased instant and the maturity. Given that American options confer to the buyer an additional right, it may be expected that the option price will be higher than the one of the European counterpart (with the same strike price and time to maturity).

Before concluding this chapter, it is relevant to understand that for European options, the call and put prices denoted by C and P, respectively, with the same strike price and time to maturity and where the underlying does not pay dividends, are related by the so-called *put-call parity*

$$C - P = S_t - Ke^{-r\tau}, \tag{2.3}$$

where r is the risk-free rate.

The proof of the *put-call parity* follows by considering the payoffs of both sides. The positions $C - P$ corresponds to a purchase of a call and a sale of a put: its payoff is

$$\max(S_T - K, 0) - \max(0, K - S_T) = S_T - K,$$

Contemplating the right-hand side of equality (2.3), it is seen that S_t corresponds to the purchase price of the underlying asset that yields a value of S_T at T; also, $Ke^{-r\tau}$ is the discounted value of K, so this right-hand side represents the purchase of the underlying asset and a credit. At time T, this portfolio will worth $S_T - K$. Given that the payoff on both sides is equal, it follows that the value of the assets yielding the payoffs is also equal. Otherwise, would exist an arbitrage opportunity.

Another concept that is taken into account during the study of the Black-Scholes-Merton (BSM) model is the Itô lemma. Itô lemma provides a framework to differentiate the functions of a stochastic process. Itô lemma allows us to derive the stochastic differential equation for the price of derivatives. The derivation of the BSM formula, presented in the next chapter, is one example of this procedure.

The model developed by Black, Scholes [13] and Merton [9], called the BSM model, presents a mathematical formulation for option pricing, that became popular in quantitative finance.

Black and Scholes [13] and Merton [9] presented a formula that allows calculating the price of an option on a date before its expiration. On the other hand, Cox and Ross [5] in the absence of arbitrage opportunities, presented a formula for the option price given by the discounted expected value of its future payoffs, under a risk-neutral measure; if we consider the log-normal distribution, we obtain the formulas for the price of the call and put options, under the BSM model.

Regarding the BSM model, the risk-neutral density function (RNDF) is a log-normal distribution. In this same model, volatility is considered constant. However, this assumption is a limitation, because empirical evidence shows that volatility is not constant. As BSM model plays a central role in option pricing, it is important to analyse alternative methods.

There are several methods for estimating the RNDF: structural and non-structural methods. Structural methods propose a full description of the stock price dynamics and, in some cases, of the volatility process. On the other hand, non-structural methods produce an estimate for the RNDF without describing the price dynamics. This last type of methods can be subdivided into parametric, semiparametric and nonparametric methods. The parametric methods propose a direct expression for the RNDF using a family of distributions, for example, a mixture of lognormal distributions, as proposed by Bahra [2] and Melick and Thomas [8]. On the other hand, non-parametric methods propose approaches without the need to assign an explicit form to the RNDF, such as the maximum entropy method, Kernel methods [11] and spline methods [1]. Finally, the estimation by semiparametric methods considers some particularities of both previous methods, as examples we have the edgeworth expansion [4] and the hermite polynomials approaches. However, there are some authors who classify the hermite polynomials as semi-nonparametric methods.

The mixture of lognormal distributions was initially proposed by Ritchey [12] and later applied by Bahra [2] and Melick and Thomas [8]. Melick and Thomas [8] estimated the RNDF considering American option prices during the Persian Gulf crisis. Bahra [2] considers a mixture of two lognormal distributions using options on commodities traded on LIFFE (London International Financial Futures and Options Exchange) and options on currency traded on the Philadelphia Stock Exchange (PHLX).

Merton [10] presented an extension of the Black-Scholes-Merton model in order to relate the company's credit risk to its capital. He considers that the value of a company's equity is the value of an European call option over the value of its assets, with maturity T. Thus, it is possible to estimate the probability of a company going into default. In this context, a company is said to enter default when the value of its assets falls to a certain level below that of its debt. It is clear that an increase in the value of a company's debt may reduce the market value of its assets and increase its probability of default.

The probability of default can be considered as the probability that the shares will have zero value and, consequently, the shareholders will lose all their rights over the company in question, as considered by Câmara et al. [3]. The authors present a modification of the BSM model for estimating the probability of default and furthermore, they assume that the RNDF is a lognormal density augmented with a probability of default.

Taylor et al. [14] developed a model for the RNDF, which is based in a mixture of two lognormal densities with a probability of default. The authors showed that a model with a mixture of two lognormal distributions with a probability of default is more effective than a model with a single lognormal or a model that considers the mixture of lognormal distributions.

Chapter 3

Black-Scholes-Merton Model

In this chapter we intend to make a brief description of the Black-Scholes-Merton model, explaining the context in which it is applied and some details about the mathematical aspects involved.

The approach proposed by Black, Scholes [13] and Merton [9] considers that the value of the underlying asset is modelled by a Geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{3.1}$$

where μ is the expected annual return of the asset and it is constant, σ is the volatility and W_t is a Wiener process. Both μ and σ are considered constant.

In order to define the price dynamics of the derivative asset, let $f(S_t,t)$ be the price of a derivative asset. Applying Itô's Lemma [6], the process followed by $f(S_t,t)$ is given by

$$df = \left(\frac{\partial f}{\partial S_t}\mu S_t + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}\sigma^2 S_t^2\right)dt + \frac{\partial f}{\partial S_t}\sigma S_t dW_t. \tag{3.2}$$

Creating a portfolio consisting of one unit of the derivative asset and a short position of a certain number of units in the underlying asset, the value of the portfolio is given by

$$B_t = f - \frac{\partial f}{\partial S_t} S_t, \tag{3.3}$$

to which the following stochastic differential equation is associated

$$dB_t = df - \frac{\partial f}{\partial S_t} dS_t \tag{3.4}$$

Replacing (3.1) and (3.2) into (3.4), we obtain

$$\begin{aligned} dB_t &= \left(\frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2} + \mu S_T \frac{\partial f}{\partial S_t} - \mu S_t \frac{\partial f}{\partial S_t}\right)dt + \sigma S_t \frac{\partial f}{\partial S_t} dW_t - \sigma S_t \frac{\partial f}{\partial S_t} dW_t \\ &= \left(\frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2} + \frac{\partial f}{\partial t}\right)dt + 0 dW_t \end{aligned} \tag{3.5}$$

From (3.5) it is possible to conclude that the dynamic of this portfolio is without risks since the coefficient of the dW_t term is null.

To avoid arbitrage, the instantaneous return of portfolio B_t must be the same as the risk-free rate of interest, represented by r. The corresponding differential equation takes the form

$$dB_t = rB_t dt = r\left(f - \frac{\partial f}{\partial S_t} S_t\right) dt. \tag{3.6}$$

Based on equations (3.5) and (3.6), the Black-Scholes-Merton differential equation that relates the price of an option, f, as a function of time t and the value of the underlying asset S_t , is given by

$$\frac{\partial f}{\partial t}(S_t,t) + rS_t \frac{\partial f}{\partial S_t}(S_t,t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2}(S_t,t) = rf(S_t,t). \tag{3.7}$$

It should be noted that the equation described above does not depend on the parameter μ. Hence, in a risk-neutral environment, the expected return is equal to the risk-free interest rate.

This equation has many solutions that correspond to the different derivatives that can be defined, considering S_t as the underlying asset. In order to have a single solution, regarding the price of a call or put option, we must consider initial and boundary conditions. At maturity T, equations (2.1) and (2.2) represent the value of a call and put option, respectively, which constitute the initial conditions. On the other hand, boundary conditions are imposed as S_t approaches 0 and $+\infty$. In the case of a call option, when S_t approaches $+\infty$ the option will be exercised, since the underlying asset, S_t, is much higher than the strike price, K. Hence, the value of the call option is given by:

$$C(K;S_t,r,t) \approx S_t - Ke^{-rt}, \qquad t \geq 0.$$

Considering the exercise price, K, discounted at the risk-free interest rate, r, the first boundary condition will be

$$\lim_{S_t \to +\infty} (S_t - C(K;S_t,r,t)) = Ke^{-r\tau}, \qquad t \geq 0.$$

Additionally, the first boundary condition for a put option when S_t approaches $+\infty$ is given by

$$\lim_{S_t \to +\infty} P(K;S_t,r,t) = 0, \qquad t \geq 0,$$

in fact, the put option will not be exercised when the value of the underlying asset, S_t, is equal to or greater than the value of the exercise price, K.

When S_t approaches 0, the second boundary condition for the call option is as follows

$$\lim_{S_t \to 0} C(K;S_t,r,t) = 0, \qquad t \geq 0,$$

in the case of a call option, the price of the option is null since its underlying asset value will also be null.

Finally, in the case of a put option, when S_t approaches 0, the price must be the same as the strike price, K, discounted at the interest rate, r. Therefore, the boundary condition for this option is given by

$$\lim_{S_t \to 0} (S_t + P(K; S_t, r, t)) = Ke^{-r\tau}, \qquad t \geq 0.$$

The call and put BSM option price formulas can be obtained from the conditions previously described, such that:

$$C(K, \sigma; S_t, r, t) = S_t N(d_1) - e^{-r\tau} K N(d_2), \qquad t \in [0, T], \tag{3.8}$$

$$P(K, \sigma; S_t, r, t) = e^{-r\tau} K N(-d_2) - S_t N(-d1), \qquad t \in [0, T], \tag{3.9}$$

where

$$d_1 = \frac{ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \qquad \text{and} \qquad d_2 = d_1 - \sigma\sqrt{\tau}$$

and $N(.)$ represents the distribution function of a standardized normal variable.
Note that, particularly, if we apply Itô's Lemma [6] to the function $f(S_t, t) = ln(S_t)$, we have

$$d\ln(S_t) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW_t,$$

so

$$\ln(S_T)|S_t \sim N\left(\ln(S_t) + (\mu - \frac{\sigma^2}{2})(T - t), \sigma^2(T - t)\right).$$

However, the Black-Scholes-Merton model has some limitations, namely with regard to the volatility parameter σ .

An approach to pricing call and put options is to use the risk-neutral density function (RNDF) for the price of the underlying asset at maturity. Cox and Ross [5], assuming a market without arbitrage, consider that the prices of European call and put options, at time t, can be determined through the expected value of its future payoffs discounted by the risk-free interest rate, obtained through the RNDF, as we can see in (3.10) and (3.11). For simplifying the notation we will aggregate the variables S_t, r and t as θ.

$$C(K, \sigma, \theta) = e^{-r\tau} \int_K^{+\infty} q(S_T)(S_T - K)\, dS_T, \tag{3.10}$$

$$P(K, \sigma, \theta) = e^{-r\tau} \int_0^K q(S_T)(K - S_T)\, dS_T. \tag{3.11}$$

If q is represented by the lognormal density given by

$$q(S_T) = \frac{1}{S_T \sigma \sqrt{2\pi\tau}} e^{\left(-\frac{ln(S_T) - \zeta}{2\sigma\sqrt{\tau}}\right)^2} \tag{3.12}$$

$$\zeta = ln(S_t) + \left(r - \frac{\sigma^2}{2}\right)\tau.$$

and replacing (3.12) in (3.10) and (3.11) equations, it is possible to obtain the (3.8) and (3.9) BSM formulas. Therefore, for the price of a call and put option we obtain, respectively,

$$C(K, \sigma, \theta) = e^{-r\tau} \int_K^{+\infty} \frac{1}{S_T \sigma \sqrt{2\pi\tau}} e^{\left(-\frac{ln(S_T) - \zeta}{2\sigma\sqrt{\tau}}\right)^2} (S_T - K)\, dS_T$$

and

$$P(K,\sigma,\theta) = e^{-r\tau}\int_0^K \frac{1}{S_T\sigma\sqrt{2\pi\tau}} e^{\left(-\frac{ln(S_T)-\zeta}{2\sigma\sqrt{\tau}}\right)^2}(K-S_T)\,dS_T.$$

To calculate the integrals above, we may consider the following variables

$$I_1 = e^{-r\tau}\int_K^{+\infty} \frac{S_T}{S_T\sigma\sqrt{2\pi\tau}} e^{\left(-\frac{ln(S_T)-\zeta}{2\sigma\sqrt{\tau}}\right)^2}(S_T-K)\,dS_T,$$

$$I_2 = -Ke^{-r\tau}\int_K^{+\infty} \frac{1}{S_T\sigma\sqrt{2\pi\tau}} e^{\left(-\frac{ln(S_T)-\zeta}{2\sigma\sqrt{\tau}}\right)^2}(K-S_T)\,dS_T.$$

By changing the variable, $y = ln(S_T)$:

$$I_1 = e^{-r\tau}\int_{ln(K)}^{+\infty} \frac{e^y}{\sigma\sqrt{2\pi\tau}} e^{\left(-\frac{(y-\zeta)^2}{2\sigma^2\tau}\right)^2} dS_T = e^{-r\tau}e^{\zeta+\frac{\sigma^2\tau}{2}}N(d_1),$$

where it can be verified that $e^{\zeta+\frac{\sigma^2\tau}{2}} = S_t e^{r\tau}$.

Hence, it is obtained

$$I_1 = S_t N(d_1) \qquad \text{and} \qquad I_2 = -Ke^{-r\tau}N(d_2), \tag{3.13}$$

where I_2 is obtained through a similar process.

It follows that for the price of a call option we obtain formula (3.8) and for the price of a put option we obtain formula (3.9).

3.1 Mixture of lognormal distributions

As previously mentioned, the Black-Scholes-Merton model follows a lognormal distribution. However, this model has some limitations, particularly the assumption that volatility is constant, which lead some authors, namely Melick and Thomas [8] and Bahra [2], to describe a RNDF in the context of option pricing as a mixture of lognormal densities, in order to obtain a better approximation for option prices, managing to overcome some of the limitations of the model initially proposed.

By considering the risk-free rate of interest, r, each lognormal density is defined by

$$l(S_T,\sigma) = \frac{1}{S_T\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2}\left(\frac{\log(S_T)-r}{\sigma}\right)^2\right).$$

A mixture of such densities yields

$$q(S_T;\alpha,\sigma) = \sum_{i=1}^{M}\alpha_i l(S_T,\sigma_i), \tag{3.14}$$

where $\alpha = (\alpha_1,\alpha_2,...,\alpha_M)$, $\sigma = (\sigma_1,\sigma_2,...,\sigma_M)$ and M denotes the number of mixtures describing the RNDF.

Substituting (3.14) into (3.10) and (3.11), we obtain

$$C(K,\alpha,\sigma) = e^{-r\tau}\sum_{i=1}^{M}\alpha_i \int_K^{+\infty} l(S_T;\sigma_i)(S_T-K)\,dS_T, \tag{3.15}$$

$$P(K,\alpha,\sigma) = e^{-r\tau}\sum_{i=1}^{M}\alpha_i \int_0^{K} l(S_T;\sigma_i)(K-S_T)\,dS_T. \tag{3.16}$$

In this context, it is possible to directly use the formulas of the Black-Scholes-Merton model to calculate the prices of call and put options:

$$C(K,\alpha,\sigma) = \sum_{i=1}^{M}\alpha_i\left[S_tN(d_{1,i}) - Ke^{-r\tau}N(d_{2,i})\right] \tag{3.17}$$

$$P(K,\alpha,\sigma) = \sum_{i=1}^{M}\alpha_i\left[e^{-r\tau}KN(-d_{2,i}) - S_tN(-d_{1,i})\right], \tag{3.18}$$

where

$$d_{1,i} = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r+\frac{\sigma_i^2}{2}\right)\tau}{\sigma_i\sqrt{\tau}} \qquad \text{and} \qquad d_{2,i} = d_{1,i} - \sigma_i\sqrt{\tau}.$$

The authors Melick and Thomas [8] and Bahra [2] used the formulas (3.17) and (3.18) to define the theoretical option price values.

Starting from the previous generalization, we consider the particular case of mixing two lognormal distributions described by three parameters: σ_1, σ_2 and α; where σ_1 and σ_2 refer to the volatility parameters and α defines the weight of the distribution.

Therefore, and according to the equation (3.14), we have the following expression for the density function:

$$q(S_T;\alpha,\sigma_1,\sigma_2) = \alpha l(S_T,\sigma_1) + (1-\alpha)l(S_T,\sigma_2).$$

Starting from (3.15) and (3.16), we get the following equations for the prices of call and put options, respectively:

$$C(K,\alpha,\sigma_1,\sigma_2) = e^{-r\tau}\int_K^{+\infty}\left[\alpha l(S_T;\sigma_1) + (1-\alpha)l(S_T;\sigma_2)\right](S_T-K)\,dS_T,$$

$$P(K,\alpha,\sigma_1,\sigma_2) = e^{-r\tau}\int_0^{K}\left[\alpha l(S_T;\sigma_1) + (1-\alpha)l(S_T;\sigma_2)\right](K-S_T)\,dS_T.$$

Hence, it is easy to derive formulas for estimating the prices of call and put options for the case of a mixture of two lognormal distributions. In fact, from (3.17) and (3.18) we have:

$$C(K,\alpha,\sigma_1,\sigma_2) = \alpha\left(S_tN(d_{1,1}) - Ke^{-r\tau}N(d_{2,1})\right) + (1-\alpha)\left(S_tN(d_{1,2}) - Ke^{-r\tau}N(d_{2,2})\right), \tag{3.19}$$

$$P(K,\alpha,\sigma_1,\sigma_2) = \alpha\left(Ke^{-r\tau}N(d_{2,1}) - S_tN(-d_{1,1})\right) + (1-\alpha)\left(Ke^{-r\tau}N(d_{2,2}) - S_tN(-d_{1,2})\right), \tag{3.20}$$

where

$$d_{1,1} = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r+\frac{\sigma_1^2}{2}\right)\tau}{\sigma_1\sqrt{\tau}} \qquad , \qquad d_{2,1} = d_{1,1} - \sigma_1\sqrt{\tau}$$

and

$$d_{1,2} = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma_2^2}{2}\right)\tau}{\sigma_2\sqrt{\tau}} \qquad , \qquad d_{2,2} = d_{1,2} - \sigma_2\sqrt{\tau}.$$

Once more, the authors Melick and Thomas [8] and Bahra [2] use the equations (3.19) and (3.20) to estimate the parameters σ_1, σ_2 and α.

3.2 Extensions of the Black-Scholes-Merton model with probability of default

Although the method of the mixture of two lognormal distributions already overcome some of the limitations of the BSM model, some authors considered that this approach was still not close enough to the reality of financial markets because it does not include a parameter that represents the probability of default. Therefore, Câmara et al. [3] and Taylor et al. [14] presented extension models of the Black-Scholes-Merton model that included the probability of default.

Considering δ the probability of default, Câmara et al. [3] propose that the underlying asset can take the value zero with a probability of default δ or follow a Geometric Brownian Motion with probability $1-\delta$. In this way, the prices of call and put options, considering the probability of default, are given by

$$C(K,\sigma,\delta) = S_t N(d_1) - (1-\delta)^\tau e^{-r\tau} K N(d_2) \tag{3.21}$$

$$P(K,\sigma,\delta) = [1-(1-\delta)^\tau]\, e^{-r\tau} K + (1-\delta)^\tau e^{-r\tau} K N(-d_2) - S_t N(-d_1), \tag{3.22}$$

with

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + (r + \frac{1}{2}\sigma^2)\tau - \ln(1-\delta)^{\frac{\tau}{2}}}{\sqrt{(\sigma^2 + \ln(1-\delta))\tau}} \qquad \text{and} \qquad d_2 = d_1 - \sqrt{(\sigma^2 + ln(1-\delta))\tau}.$$

Formulas (3.21) and (3.22) mainly depend on the parameters δ, probability of default, and σ, volatility, and should verify

$$0 < \delta < 1 \qquad \text{and} \qquad \sigma^2 + \ln(1-\delta) \geq 0,$$

that is, $\delta + e^{-\sigma^2} - 1 \leq 0$.

In this case, S_T follows a risk-neutral lognormal distribution augmented with a probability of default implicit in the option price formulas, (3.21) and (3.22), given by

$$S_T \sim \Delta\left[1-(1-\delta)^\tau, \ln(S_t) + r\tau - \frac{\sigma^2}{2}\tau - \ln(1-\delta)^{\frac{3}{2}\tau}, \sigma^2\tau + \ln(1-\delta)^\tau\right].$$

In formula (3.22), where we calculate the price of a put option according to this approach by Câmara et al. [3], there is an extra term that does not exist in the formula for the price of a call option (3.21), namely $[1-(1-\delta)^\tau]e^{-r\tau}K$. This happens since, in an event of default, a call option has zero worth because it makes no sense to buy something at a certain price when has a null value, but a put option is worth K because we have the right to sell it.

Considering $\delta = 0$ in (3.21) and (3.22), we then obtain the formulas to determine the price of call and put options according to the BSM model (3.8) and (3.9).

Next, we will study the approach proposed by Taylor et al. [14]. This extension of the BSM model considers a model for estimating the probability of default defined by a mixture of two log-normal distributions with a probability of default. The authors consider two distribution weights, α_1 and α_2, each corresponding to a lognormal distribution and the remaining weight, $1-\alpha_1-\alpha_2$, corresponding to the respective default probability. The respective risk-neutral distribution function for S_t, $Q(S_T)$, is given by

$$Q(S_T;\alpha_1,\alpha_2,\sigma_1,\sigma_2) = 1-\alpha_1-\alpha_2+\alpha_1 Q_1(S_T;\sigma_1)+\alpha_2 Q_2(S_T;\sigma_2). \tag{3.23}$$

It should be noted that the distribution function Q is defined by the parameters $\alpha_1,\alpha_2,\sigma_1,\sigma_2$ and by two lognormal distributions, Q_1 and Q_2, such that:

$$Q_i(S_T;\sigma_i) = N\left(\frac{\ln(\frac{S_T}{K})+\frac{1}{2}\sigma_i^2\tau}{\sigma_i\sqrt{\tau}}\right), \qquad i = 1,2,$$

with $\sigma_1,\sigma_2 > 0$, $\alpha_1,\alpha_2 \geq 0$ and $\alpha_1+\alpha_2 \leq 1$.

In this dissertation, we used the approach of Taylor et al. [14] but in a different way, since the initial approach involves the price of future contracts and in this dissertation, we do not consider these contracts.

Through (3.23) and considering $\alpha_1 \leq 1$ and $\alpha_2 = 0$, it is possible to obtain the lognormal distribution augmented with a probability of default proposed by Câmara et al. [3].

According to Taylor et al. [14] approach, it is possible to calculate the prices of call and put options as follows:

$$C(K;\alpha_1,\alpha_2,\sigma_1,\sigma_2) = \alpha_1 C(K,\sigma_1;S_t,r,t)+\alpha_2 C(K,\sigma_2;S_t,r,t) \tag{3.24}$$

$$P(K;\alpha_1,\alpha_2,\sigma_1,\sigma_2) = e^{-r\tau}(1-\alpha_1-\alpha_2)K+\alpha_1 P(K,\sigma_1;S_t,r,t)+\alpha_2 P(K,\sigma_2;S_t,r,t), \tag{3.25}$$

where the prices $C(K,\sigma_i;S_t,r,t)$ and $P(K,\sigma_i;S_t,r,t)$, $i=1,2$, are obtained through the formulas of the Black-Scholes-Merton model (3.8) and (3.9). Note that, in the event of default, call options have no value, but put options are worth K, as presented before.

Taylor et al. [14] recognize that a mixture of two lognormal distributions with a probability of default better describes option prices than, for example, the lognormal distribution augmented with a probability of default presented by Câmara et al. [3]. In fact, Taylor et al. [14] considers more parameters than Câmara et al. [3], making this model more flexible, which originates better approximation of market prices.

3.3 Estimation of the optimization problem

In this section are presented three optimization problems, to estimate the parameters according to the approaches mentioned in Sections 3.1 and 3.2.

The models to be used for the estimation are a mixture of two lognormal distributions, a lognormal distribution augmented with a probability of default and a mixture of two lognormal distributions with a probability of default.

As previously mentioned, in the BSM model the RNDF is the lognormal density. Note that this function depends on the volatility parameter σ; however, this parameter is not directly observable, so its estimation is necessary.

To estimate the parameters it is necessary to use formulas (3.19) and (3.20), related to the mixture of two lognormal distributions, to determine the prices of the call and put options. The optimization problem of this approach is given by

$$\begin{aligned} \underset{\sigma_1,\sigma_2,\alpha}{\text{minimize}} \quad & \sum_{i=1}^{NC} \frac{|C(K_i,\alpha,\sigma_1,\sigma_2)-C_i^*|}{C_i^*} + \sum_{j=1}^{NP} \frac{|P(K_j,\alpha,\sigma_1,\sigma_2)-P_j^*|}{P_j^*} \\ \text{subject to} \quad & 0 \le \alpha \le 1, \\ & \sigma_1 - \sigma_2 < 0. \end{aligned} \tag{3.26}$$

where C_i^* and P_j^* represent, respectively, the observed prices of call and put options and $C(K_i,\alpha,\sigma_1,\sigma_2)$ and $P(K_j,\alpha,\sigma_1,\sigma_2)$ represent, respectively, the theoretical prices of call and put options, NC and NP represent, respectively, the number of call and put options over the range of observed strike prices.

According to the results discussed in Section 3.2, that is, the lognormal distribution augmented with a probability of default, the theoretical prices are calculated using the formulas (3.21) and (3.22). In this case, we estimate the volatility σ and the probability of default δ, using the following optimization problem:

$$\begin{aligned} \underset{\sigma,\delta}{\text{minimize}} \quad & \sum_{i=1}^{NC} \frac{|C(K_i,\sigma,\delta)-C_i^*|}{C_i^*} + \sum_{j=1}^{NP} \frac{|P(K_j,\sigma,\delta)-P_j^*|}{P_j^*} \\ \text{subject to} \quad & 0 \le \delta \le 1 \\ & \sigma > 0 \\ & \delta + e^{-\sigma^2} - 1 \le 0 \end{aligned} \tag{3.27}$$

where $C(K_i,\sigma,\delta)$ and $P(K_j,\sigma,\delta)$ represent, respectively, the theoretical prices of call and put options.

For the estimation of these parameters, it is essential to introduce a nonlinear restriction in the optimization problem (3.27). The use of the restriction $\delta + e^{-\sigma^2} - 1 \le 0$ is imperative to guarantee the positivity of the square root in the d_2 formula presented in (3.21) and (3.22).

To calculate the theoretical prices of call and put options according to a mixture of two lognormal distributions with a probability of default, the formulas (3.24) and (3.25) are used. Thus, we intend to estimate σ_1 and σ_2 for each lognormal distribution, as well as the respective weights α_1 and α_2. Therefore, the following optimization problem is used:

$$\begin{aligned} \underset{\sigma_1,\sigma_2,\alpha_1,\alpha_2}{\text{minimize}} \quad & \sum_{i=1}^{NC} \frac{|C(K_i,\sigma_1,\sigma_2,\alpha_1,\alpha_2)-C_i^*|}{C_i^*} + \sum_{j=1}^{NP} \frac{|P(K_j,\sigma_1,\sigma_2,\alpha_1,\alpha_2)-P_j^*|}{P_j^*} \\ \text{subject to} \quad & \alpha_1 + \alpha_2 < 1, \\ & \sigma_1 - \sigma_2 < 0. \end{aligned} \tag{3.28}$$

where $C(K_i, \sigma_1, \sigma_2, \alpha_1, \alpha_2)$ and $P(K_j, \sigma_1, \sigma_2, \alpha_1, \alpha_2)$ represent, respectively, the theoretical prices of call and put options.

The second and third approaches, although both include the probability of default, also differ in the sense that the approach by Câmara et al.[3] is a theoretically grounded approach, while the approach by Taylor et al. [14] is an eminently empirical approach.

Chapter 4

Empirical Analysis

In this chapter an empirical analysis is carried out considering the previous methods. We intend to estimate risk-neutral density functions using a mixture of two lognormal distributions, a lognormal distribution augmented with probability of default and a mixture of two lognormal distributions with probability of default.

A comparative analysis of the previous methods will be done, for the estimation of the probability of default, according to the methods of Câmara et al. [3] and Taylor et al. [14], using simulated and market call and put option prices.

First, we use simulated prices to analyse the estimation approaches and the respective results. Since there is no uncertainty about the model, it is possible to compare the estimates of the variables with the theoretical values.

Second, we will consider market data and essentially focus our study on two companies: one with a very stable financial situation, Intel Corp., and another being one of the most shorted companies in the last months, Beyond Meat, Inc., so that we can compare its probability of default.

4.1 Simulations

This section presents simulated option prices, considering the following values: $S_t = 30$, $r = 0.02$ and $\tau = T - t = 0.25$. However, for a better approach to the fluctuations observed in the market, a perturbation of the theoretical prices is made through the formulas

$$C_i^* = C_i + 0.05 C_i \varepsilon_i \quad and \quad P_j^* = P_j + 0.05 P_j \delta_j,$$

where C_i^*, P_j^* correspond to the perturbed call and put option prices, with $i = 1, .., NC$ and $j = 1, ..., NP$, C_i, P_j represent the theoretical prices generated by the methods and ε_i, $\eta_j \sim$ N(0,1). The price perturbation has the main purpose of introducing heteroscedasticity to the methods under study.

In the first estimation, we intend to generate perturbed theoretical call and put option prices according to a mixture of two lognormal distributions, taking into account the formulas (3.19) and (3.20), representing the call and put option prices of a mixture of two lognormal distributions. To estimate the parameters σ_1, σ_2 and α, we consider the optimization problem given by (3.26). The theoretical parameters are $\sigma_1 = 0.25$, $\sigma_2 = 0.15$, $\alpha = 0.30$ and the exercise price, K, takes values between 10 and 50. The estimated values are as follows: $\hat{\sigma}_1 = 0.2499$, $\hat{\sigma}_2 = 0.1489$ and $\hat{\alpha} = 0.3015$. Thus, we found that the results obtained are good approximations for the theoretical parameter values, as represented in Table 4.1.

Parameters	Theoretical Values	Estimated Values
σ_1	0.25	0.2499
σ_2	0.15	0.1489
α	0.3	0.3015

Table 4.1 Comparison between theoretical and estimated parameters, through a mixture of two lognormal distributions

In Figure 4.1 we have, on the left side, the perturbed theoretical call and put option prices and, on the right, the estimated and the theoretical RNDF, through a mixture of two lognormal distributions approach.

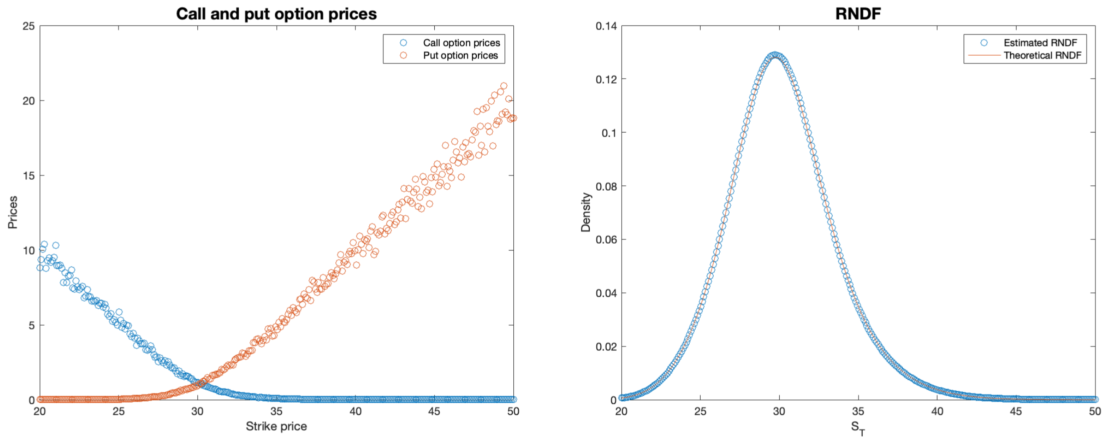

Fig. 4.1 On the left: Perturbed theoretical call and put option prices through a mixture of two lognormal distributions approach; On the right: Estimated and theoretical RNDF through a mixture of two lognormal distributions approach

In the next estimation, we intend to determine an estimative for the probability of default δ, according to the formulas (3.21) and (3.22) by the Câmara et al. [3] method.

To estimate the parameters σ and δ it is necessary to solve the optimization problem (3.27). We consider the following values for the theoretical parameters: $\sigma = 0.25$, $\delta = 0.03$ and, for the exercise prices, K, values between 10 and 50. Thus, the estimated values are the following: $\hat{\sigma} = 0.2501$ and $\hat{\delta} = 0.0301$. Once more, we can conclude that the results obtained are good approximations for the parameters initially defined, as represented in Table 4.2.

Parameters	Theoretical Values	Estimated Values
σ	0.25	0.2501
δ	0.03	0.0301

Table 4.2 Comparison between theoretical and estimated parameters, through a lognormal distribution augmented with a probability of default

In Figure 4.2 we have, on the left side, the perturbed theoretical call and put option prices and, on the right, the estimated and the theoretical RNDF, through a lognormal distribution augmented with a probability of default approach.

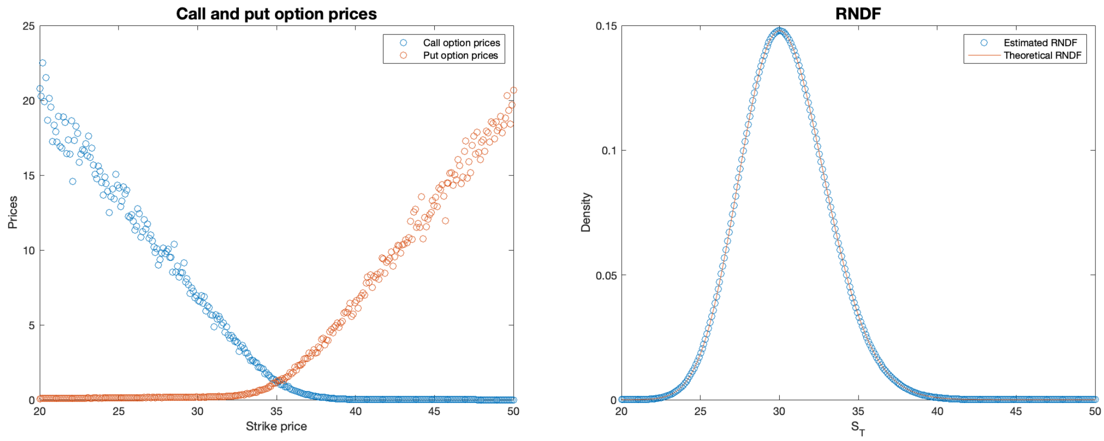

Fig. 4.2 On the left: Perturbed theoretical call and put option prices through a lognormal distribution augmented with a probability of default approach; On the right: Risk Neutral Density Function through a lognormal distribution augmented with a probability of default approach

Finally, we consider a mixture of two lognormal distributions framework augmented with a probability of default. Option prices are calculated based on the formulas (3.24) and (3.25), to estimate the parameters σ_1, σ_2, α_1 and α_2, we solve the problem (3.28). In this way, we consider the following initial theoretical values: $\sigma_1 = 0.25$, $\sigma_2 = 0.75$, $\alpha_1 = 0.70$, $\alpha_2 = 0.25$ and, since $\delta = 1 - \alpha_1 - \alpha_2$, we have $\delta = 0.05$. After the estimation we obtain the following values: $\hat{\sigma}_1 = 0.2481$, $\hat{\sigma}_2 = 0.7508$, $\hat{\alpha}_1 = 0.7006$ and $\hat{\alpha}_2 = 0.2494$.

As shown in Table 4.3, we obtained good approximations. According to the values previously estimated and represented in Table 4.3, it is possible to extract the estimated value for the probability of default, $\hat{\hat{\delta}}$. Thus, since $\hat{\hat{\delta}} = 1 - \hat{\alpha}_1 - \hat{\alpha}_2$, we have the estimated value of $\hat{\hat{\delta}} = 0.051$, which is quite similar to the theoretical value.

In Figure 4.3 we have, on the left side, the perturbed theoretical call and put option prices and, on the right, the estimated and theoretical RNDF, all through a mixture of two lognormal distributions with a probability of default approach.

Parameters	Theoretical Values	Estimated Values
σ_1	0.25	0.2481
σ_2	0.75	0.7508
α_1	0.70	0.7006
α_2	0.25	0.2494

Table 4.3 Comparison between theoretical and estimated parameters, through a mixture of two lognormal distributions with a probability of default

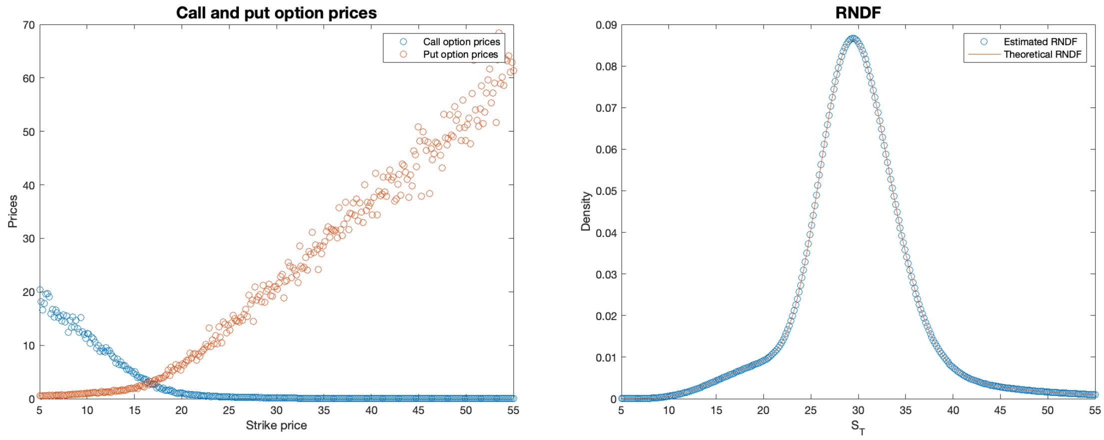

Fig. 4.3 On the left: Perturbed theoretical call and put option prices through a mixture of two lognormal distributions with a probability of default approach; On the right: Estimated and Theoretical RNDF through a mixture of two lognormal distributions with a probability of default approach

4.2 Market Data

In this Section, we present the RNDF and default probability estimates from market option prices, from Intel Corp. and Beyond Meat, Inc.. Intel Corp. is an American multinational technology company and Beyond Meat, Inc. is a Los Angeles–based producer of plant-based meat substitutes, founded in 2009.

It is important to note that option market data are American style, and theoretically we should work with European style options. Thus, we use Out-of-The-Money (OTM) options that in the actual moment, t, will not be exercised, in this sense they can be seen as an approximation to the setup given by European options.

In Figure 4.4 we have a representation of the stock prices for Intel Corp. and Beyond Meat, Inc. from 02/05/2019 until 27/01/2023. As we can verify, there is a colossal difference between the two companies stock prices: while Intel Corp. has remained more stable over the last four years, Beyond Meat, Inc. presents a big range of prices from 2019 until January of 2023, as their prices were once around 200 and are now around 20. Once again, this justifies the reason why we choose these two companies, that is, Intel Corp. is more financially stable than Beyond Meat, Inc., because this last has been suffering from short selling.

For Intel Corp., we have defined six maturities: 17th of March of 2023, 21st of April of 2023, 16th of June of 2023, 21st of July of 2023, 15th of September of 2023 and 19th of January of 2024. For each of the six maturities, there is a different number of strikes and for each strike, there is a different number of observations, not equally spaced. The representation of call and put option prices for Intel Corp. for the first (represented in blue) and last (represented in red) maturities can be seen in Figure 4.5.

On the other hand, for Beyond Meat Inc. we have defined five maturities: 17th of February of 2023, 17th of March of 2023, 19th of May of 2023, 16th of June of 2023 and 19th of January of 2024. For each of the five maturities, there is a different number of strikes and for each strike, there is a different number of observations, not equally spaced. The representation of call and put option prices

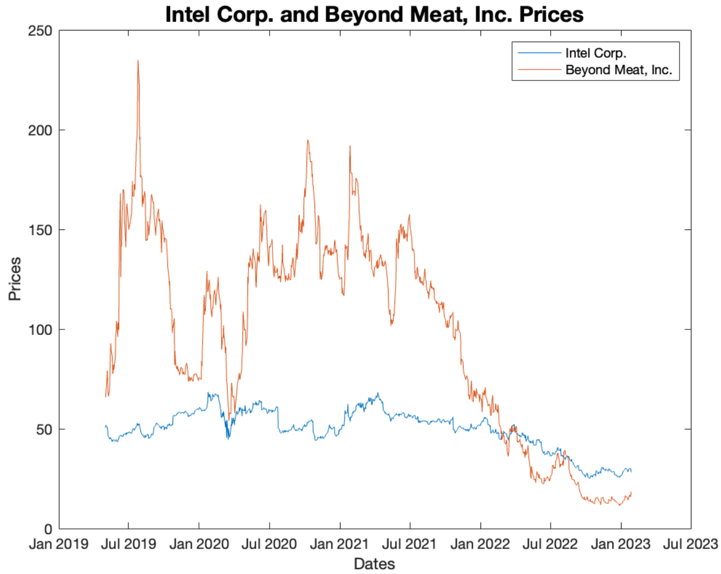

Fig. 4.4 Intel Corp. and Beyond Meat, Inc. stock prices since 02/05/2019 until 27/01/2023

for Beyond Meat, Inc. for the first (represented in blue) and last (represented in red) maturities can be seen in Figure 4.6.

In both figures 4.5 and 4.6, we can verify a preponderant variability of the option prices referring to the last maturity. This results from the uncertainty that exists relative to financial options with such a long maturity, with the existing information not being reliable for the 19th of January of 2024.

Our goal is to estimate the probability of default for both companies following the methods from Câmara et al. [3] and Taylor et al. [14]. However, a measure of uncertainty regarding the estimates is needed. There is no simple way due to the complexity of the problem. One way to overcome this problem is through simulation techniques, including bootstrapping. In each table presented below, a value for the estimation of each parameter will be presented and, underneath, a value associated with the standard error of that estimation.

First, we will consider Intel Corp. option data. Following the approach by Câmara et al. [3], we estimate the parameters σ and δ. As shown in Table 4.4, the company shows a low volatility value, around 35.51% per year, and a probability of default of approximately 2.45%.

Next, following the approach by Taylor et al. [14], we estimated the parameters and the respective probability of default for Intel Corp.. As it is seen in Table 4.5, the company presents a volatility of around 29.35% and 74.56% per year and a probability of default of around 2.43%, which is quite similar to the value found by Câmara et al. [3] method.

In the second place, we will analyse Beyond Meat, Inc. data. Following the approach by Câmara et al. [3], we estimate the parameters σ and δ. As shown in Table 4.6, the company shows high volatility of around 67.2% per year and a probability of default of 11.62%.

Next, following the approach by Taylor et al. [14], we estimated the parameters and the respective probability of default for Beyond Meat, Inc. As seen in Table 4.7, the company presents a volatility of around 59.68% and 262% per year and a probability of default of around 10%.

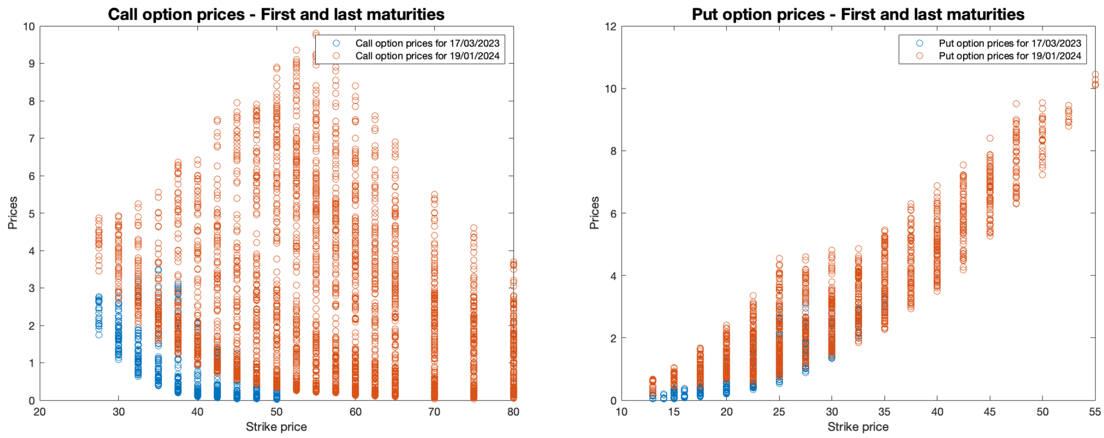

Fig. 4.5 On the left: Call option prices for Intel Corp. with the first (17/03/2023) and last (19/01/2024) maturities; On the right: Put option prices for Intel Corp. with the first (17/03/2023) and last (19/01/2024) maturities

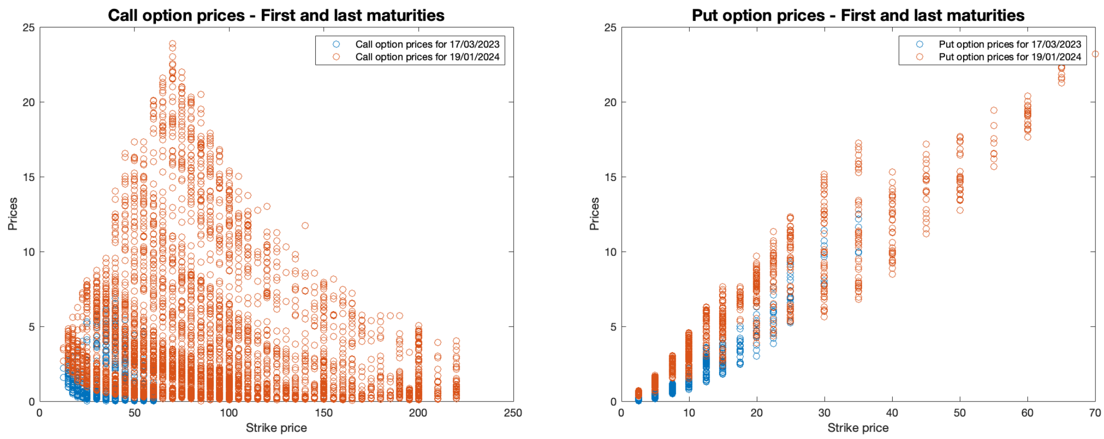

Fig. 4.6 On the left: Call option prices for Beyond Meat, Inc. with the first (17/02/2023) and last (19/01/2024) maturities; On the right: Put option prices for Beyond Meat, Inc. with the first (17/02/2023) and last (19/01/2024) maturities

Parameters	Estimated Values
σ	0.3551
	(0.0006)
δ	0.0245
	(0.0003)

Table 4.4 Parameter estimates for Intel Corp. through a lognormal distribution augmented with a probability of default approach

As we can see in the tables, Beyond Meat, Inc. has a much higher probability of default, of around 10% when compared with the probability of default of 2.5% for Intel Corp.. These results are in line with our expectations since Intel Corp. is more financially stable than Beyond Meat, Inc..

Parameters	Estimated Values
σ_1	0.2935
	(0.0013)
σ_2	0.7456
	(0.0164)
α_1	0.9412
	(0.0027)
α_2	0.0345
	(0.0027)
δ	0.0243
	(0.0004)

Table 4.5 Parameter estimates for Intel Corp. through a mixture of two lognormal distributions with a probability of default approach

Parameters	Estimated Values
σ	0.6720
	(0.0036)
δ	0.1162
	(0.0029)

Table 4.6 Parameter estimates for Beyond Meat, Inc. through a lognormal distribution augmented with a probability of default approach

Parameters	Estimated Values
σ_1	0.5968
	(0.0060)
σ_2	2.6253
	(0.2363)
α_1	0.8776
	(0.0052)
α_2	0.0277
	(0.0036)
δ	0.0948
	(0.0043)

Table 4.7 Parameter estimates for Beyond Meat, Inc. through a mixture of two lognormal distributions with a probability of default approach

In Figure 4.7 we can verify the behaviour of the objective function that we are working with. It should be noted that this representation can only be obtained for the Câmara et al. [3] method, since it is the only one in which we are estimating two variables and, therefore, it is allowed to make a graphical representation.

In Figure 4.8 we have a representation of the RNDF for Intel Corp. on the left, and for Beyond Meat, Inc. on the right, following the estimated values presented in the Tables 4.5 and 4.7, respectively, for $\tau = 0.25$, $\tau = 0.5$ e $\tau = 1.0$. Beyond Meat's density is quite different from Intel. This difference symbolizes the existence of a significant value for Beyond Meat's probability of default, of around

10% as we have calculated before. On the other hand, this probability is quite close to zero for Intel Corp. as it is notable in the graph on the left.

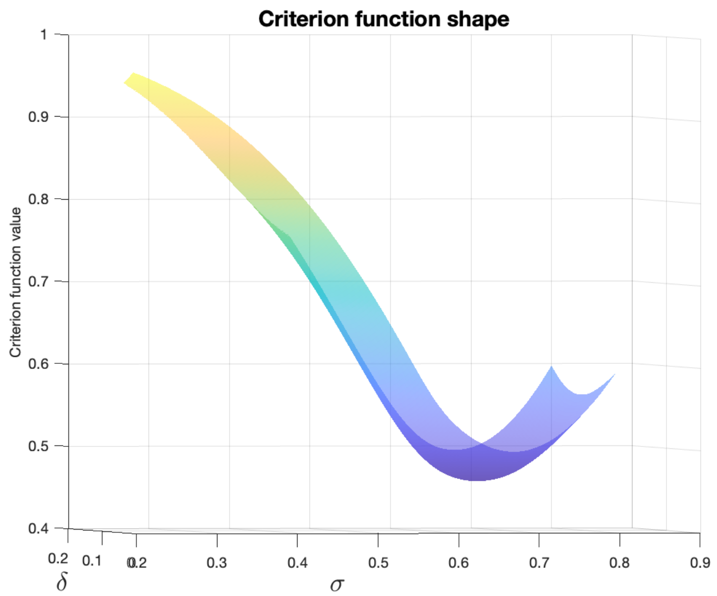

Fig. 4.7 Objective function behaviour associated with Beyond Meat, Inc. through Câmara et al. [3] approach.

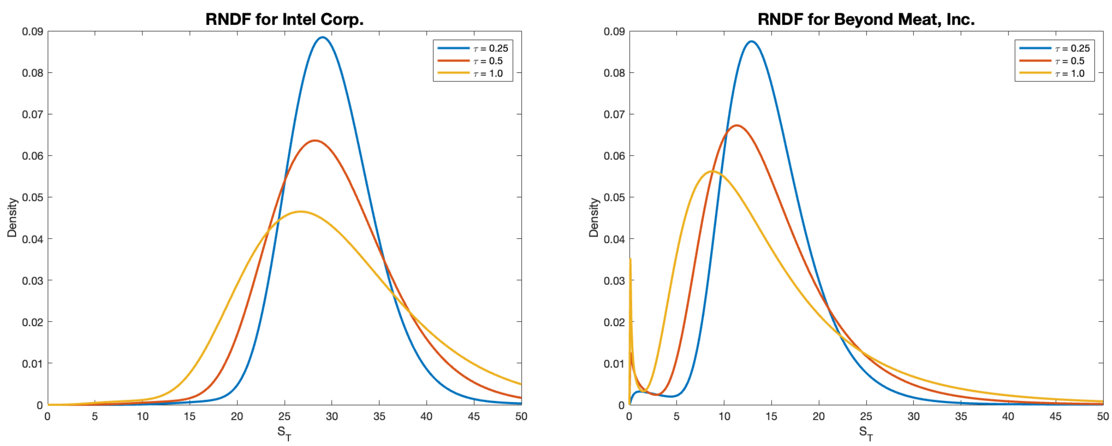

Fig. 4.8 On the left: RNDF for Intel Corp. through the values obtained in Table 4.5 for $\tau = 0.25$, $\tau = 0.5$ e $\tau = 1.0$; On the right: RNDF for Beyond Meat, Inc. through the values obtained in Table 4.7 for $\tau = 0.25$, $\tau = 0.5$ e $\tau = 1.0$.

In a simulation environment, there is no uncertainty regarding the methods in study, therefore the estimated parameters are very close to the theoretical values, which demonstrates the reliability of the methods. It is important to notice that the probability of default for each company is very close in the two methods. This shows that both methods have similar results, even with the simplification we made in Taylor et al. [14] approach.

Chapter 5

Conclusion

The prices of financial options can reveal essential information about investors' future expectations regarding the evolution of the underlying asset price. Through option prices, it is also possible to analyze the financial risk of an investment. Besides that, the estimation of a RNDF may provide relevant information about financial markets.

In this dissertation, we considered different methods for estimating the RNDF and the probability of default, using call and put option prices: a mixture of lognormal distributions, a lognormal distribution augmented with a probability of default (Câmara et al. [3]), and a mixture of two lognormal distributions with a probability of default (Taylor et al. [14]). The aim is to estimate and compare the parameter associated with the probability of default in the last two approaches. It is important to bear in mind that in this dissertation we used the theory of Câmara et al. [3] in full, while for the approach by Taylor et al. [14] there was an adaptation in order to try to simplify the study.

First, we consider an analysis within a simulation environment, for ensuring that the models are capable of estimating the theoretical parameters. Since there is no uncertainty regarding the models, it is possible to compare the estimates with the real values used in the simulation, obtaining credible results. Second, we analyse market data using call and put option prices for Intel Corp. and Beyond Meat, Inc.. These companies were chosen because of their distinct financial stability, which allows us to evaluate the methods used. In this context, we focused our empirical analysis in a lognormal distribution augmented with a probability of default (Câmara et al. [3]) and a mixture of two lognormal distributions with a probability of default (Taylor et al. [14]) and it was found that the default probability for Intel Corp. in both models is very similar as well as the probability of default for Beyond Meat, Inc. presented by the two models. Besides that, we can verify that Intel Corp. probability of default is considerably low when compared to the probability of default of Beyond Meat, Inc. and these results strengthen the initial idea that one company is more financially stable than the other. Besides that, the fact that we obtain similar values for the default probability in both models proves that the modification we made on Taylor et al. [14] approach does not change the intuition and is a more simple way to implement this method.

In a future analysis, it would be interesting to apply these methods to a diversified set of companies and, in addition, analyse the probability of default through a model that involves jumps, so that we can apply approaches that could be closer to what actually happens in financial markets.

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