A NOTE ON 3-QUASI-SASAKIAN GEOMETRY

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Abstract: 3-quasi-Sasakian manifolds were recently studied by the authors as a suitable setting unifying 3-Sasakian and 3-cosymplectic geometries. In this paper some geometric properties of this class of almost 3-contact metric manifolds are briefly reviewed, with an emphasis on those more related to physical applications.

Keywords: Almost contact metric 3-structures, 3-Sasakian manifolds, 3-cosymplectic manifolds.


1. Introduction

The class of 3-quasi-Sasakian manifolds is the analogue in the setting of 3-structures of the class of quasi-Sasakian manifolds, introduced by Blair [3] and later studied among others by Tanno [13], Kanemaki [11], Olszak [12]. More recent are the examples of applications of quasi-Sasakian manifolds to string theory found by Friedrich and his collaborators [2, 9]. Just like quasi-Sasakian manifolds include Sasakian and cosymplectic manifolds, so 3-quasi-Sasakian manifolds unify 3-Sasakian and 3-cosymplectic geometry. A 3-quasi-Sasakian manifold can arise, for example, as the product of a 3-Sasakian manifold and a hyper-Kähler manifold (see Sect. 3 or [7]). The setting of 3-structures has been recently the object of a wider interest from both mathematicians and physicists due to the important role acquired by the 3-Sasakian and the related quaternionic structures in supergravity and superstring theory, where they appear in the so called hypermultiplet solutions (see e. g. [1, 2, 6, 15]). This note contains a concise review of the main properties of 3-quasi-Sasakian manifolds, recently studied by the authors in [7], together with some relevant properties of the two important subclasses of 3-Sasakian and 3-cosymplectic manifolds which were compared in [8].

2. 3-quasi-Sasakian geometry

An almost contact metric manifold is a $(2n+1)$-dimensional manifold $M$ endowed with a field $\phi$ of endomorphisms of the tangent spaces, a vector field $\xi$,

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called Reeb vector field, a 1-form \( \eta \) satisfying \( \phi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1 \) (where \( I: TM \to TM \) is the identity mapping) and a compatible Riemannian metric \( g \) such that \( g(\phi X, \phi Y) = g(X, Y) - \eta(X) \eta(Y) \) for all \( X, Y \in \Gamma(TM) \). The manifold is said to be normal if the tensor field \( N^{(1)} = [\phi, \phi] + 2d\eta \otimes \xi \) vanishes identically. The 2-form \( \Phi \) on \( M \) defined by \( \Phi(X, Y) = g(X, \phi Y) \) is called the fundamental 2-form of the almost contact metric manifold \((M, \phi, \xi, \eta, g)\). Normal almost contact metric manifolds such that both \( \eta \) and \( \Phi \) are closed are called cosymplectic manifolds and those such that \( d\eta = \Phi \) are called Sasakian manifolds. The notion of quasi-Sasakian structure unifies those of Sasakian and cosymplectic structures. A quasi-Sasakian manifold is defined as a normal almost contact metric manifold whose fundamental 2-form is closed. A quasi-Sasakian manifold \( M \) is said to be of rank \( 2p \) (for some \( p \leq n \)) if \((d\eta)^p \neq 0 \) and \( \eta \wedge (d\eta)^p = 0 \) on \( M \), and to be of rank \( 2p + 1 \) if \( \eta \wedge (d\eta)^p \neq 0 \) and \((d\eta)^{p+1} = 0 \) on \( M \) (cf. [3, 13]). Blair proved that there are no quasi-Sasakian manifolds of even rank. Just like Blair and Tanno did, we will only consider quasi-Sasakian manifolds of constant (odd) rank. If the rank of \( M \) is \( 2p + 1 \), then the module \( \Gamma(TM) \) of vector fields over \( M \) splits into two submodules as follows: \( \Gamma(TM) = \mathcal{E}^{2p+1} \oplus \mathcal{E}^{2q}, \) \( p + q = n \), where \( \mathcal{E}^{2q} = \{ X \in \Gamma(TM) \mid i_X d\eta = 0 \text{ and } i_X \eta = 0 \} \) and \( \mathcal{E}^{2p+1} = \mathcal{E}^{2p} \oplus \langle \xi \rangle, \mathcal{E}^{2p} \) being the orthogonal complement of \( \mathcal{E}^{2q} \oplus \langle \xi \rangle \) in \( \Gamma(TM) \). These modules satisfy \( \phi \mathcal{E}^{2p} = \mathcal{E}^{2p} \) and \( \phi \mathcal{E}^{2q} = \mathcal{E}^{2q} \) (cf. [13]).

We now come to the main topic of our paper, i.e. 3-quasi-Sasakian geometry, which is framed into the more general setting of almost 3-contact geometry. An almost 3-contact metric manifold is a \((4n + 3)\)-dimensional smooth manifold \( M \) endowed with three almost contact structures \((\phi_1, \xi_1, \eta_1), (\phi_2, \xi_2, \eta_2), (\phi_3, \xi_3, \eta_3)\) satisfying the following relations, for any even permutation \((\alpha, \beta, \gamma)\) of \( \{1, 2, 3\} \),

\[
\begin{align*}
\phi_\gamma &= \phi_\alpha \phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta \phi_\alpha + \eta_\alpha \otimes \xi_\beta, \\
\xi_\gamma &= \phi_\alpha \xi_\beta = -\phi_\beta \xi_\alpha, \quad \eta_\gamma = \eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha,
\end{align*}
\]

and a Riemannian metric \( g \) compatible with each of them. It is well known that in any almost 3-contact metric manifold the Reeb vector fields \( \xi_1, \xi_2, \xi_3 \) are orthonormal with respect to the compatible metric \( g \) and that the structural group of the tangent bundle is reducible to \( Sp(n) \times I_3 \). Moreover, by putting \( \mathcal{H} = \bigcap_{\alpha=1}^{3} \ker (\eta_\alpha) \) one obtains a \( 4n \)-dimensional horizontal distribution on \( M \) and the tangent bundle splits as the orthogonal sum \( TM = \mathcal{H} \oplus \mathcal{V}, \) where \( \mathcal{V} = \langle \xi_1, \xi_2, \xi_3 \rangle \) is the vertical distribution.
Definition 2.1. A 3-quasi-Sasakian manifold is an almost 3-contact metric manifold \((M, \phi_{\alpha}, \xi_{\alpha}, \eta_{\alpha}, g)\) such that each almost contact structure is quasi-Sasakian.

The class of 3-quasi-Sasakian manifolds includes as special cases the well-known 3-Sasakian and 3-cosymplectic manifolds.

The following theorem combines the results obtained in Theorems 3.4 and 4.2 of [7].

Theorem 2.2. Let \((M, \phi_{\alpha}, \xi_{\alpha}, \eta_{\alpha}, g)\) be a 3-quasi-Sasakian manifold. Then the 3-dimensional distribution \(\mathcal{V}\) generated by \(\xi_1, \xi_2, \xi_3\) is integrable. Moreover, \(\mathcal{V}\) defines a totally geodesic and Riemannian foliation of \(M\) and for any even permutation \((\alpha, \beta, \gamma)\) of \(\{1, 2, 3\}\) and for some \(c \in \mathbb{R}\)

\[\left[\xi_{\alpha}, \xi_{\beta}\right] = c\xi_{\gamma}.\]

Using Theorem 2.2 we may divide 3-quasi-Sasakian manifolds in two classes according to the behaviour of the leaves of the foliation \(\mathcal{V}\): those 3-quasi-Sasakian manifolds for which each leaf of \(\mathcal{V}\) is locally \(SO(3)\) (or \(SU(2)\)) (which corresponds to take in Theorem 2.2 the constant \(c \neq 0\)), and those for which each leaf of \(\mathcal{V}\) is locally an abelian group (this corresponds to the case \(c = 0\)).

The preceding theorem also allows to define a canonical metric connection on any 3-quasi-Sasakian manifold. Indeed, let \(\nabla^B\) be the Bott connection associated to \(\mathcal{V}\), that is the partial connection on the normal bundle \(TM/\mathcal{V} \cong \mathcal{H}\) of \(\mathcal{V}\) defined by \(\nabla^B_V Z := [V, Z]_\mathcal{H}\) for all \(V \in \Gamma(\mathcal{V})\) and \(Z \in \Gamma(\mathcal{H})\). Following [14] we may construct an adapted connection on \(\mathcal{H}\) putting

\[\tilde{\nabla}_{XY} := \begin{cases} \nabla^B_X Y, & \text{if } X \in \Gamma(\mathcal{V}); \\ (\nabla_X Y)_H, & \text{if } X \in \Gamma(\mathcal{H}). \end{cases}\]

This connection can be also extended to a connection on all \(TM\) by requiring that \(\tilde{\nabla}\xi_{\alpha} = 0\) for each \(\alpha \in \{1, 2, 3\}\). Some properties of this global connection have been considered in [8] for any almost 3-contact metric manifold. Now combining Theorem 2.2 with [8, Theorem 3.6] we have:

Theorem 2.3. Let \((M, \phi_{\alpha}, \xi_{\alpha}, \eta_{\alpha}, g)\) be a 3-quasi-Sasakian manifold. Then there exists a unique metric connection \(\tilde{\nabla}\) on \(M\) satisfying the following properties:

(i) \(\tilde{\nabla}\eta_{\alpha} = 0, \tilde{\nabla}\xi_{\alpha} = 0, \text{ for each } \alpha \in \{1, 2, 3\}\),

(ii) \(\tilde{T}(X, Y) = 2 \sum_{\alpha=1}^{3} d\eta_{\alpha}(X, Y)\xi_{\alpha}, \text{ for all } X, Y \in \Gamma(TM)\).
3. The rank of a 3-quasi-Sasakian manifold

For a 3-quasi-Sasakian manifold one can consider the ranks of the three structures \((\phi_\alpha, \xi_\alpha, \eta_\alpha, g)\). The following theorem assures that these three ranks coincide.

**Theorem 3.1** ([7]). Let \((M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)\) be a 3-quasi-Sasakian manifold of dimension \(4n+3\). Then the 1-forms \(\eta_1, \eta_2, \text{ and } \eta_3\) have the same rank \(4l+3\) or \(4l+1\), for some \(l \leq n\), according to \([\xi_\alpha, \xi_\beta] = c\xi_{\gamma}\) with \(c \neq 0\), or \([\xi_\alpha, \xi_\beta] = 0\), respectively.

According to Theorem 3.1, we say that a 3-quasi-Sasakian manifold \((M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)\) has rank \(4l+3\) or \(4l+1\) if any quasi-Sasakian structure has such rank. We may thus classify 3-quasi-Sasakian manifolds of dimension \(4n+3\), according to their rank. For any \(l \in \{0, \ldots, n\}\) we have one class of manifolds such that \([\xi_\alpha, \xi_\beta] = c\xi_{\gamma}\) with \(c \neq 0\), and one class of manifolds with \([\xi_\alpha, \xi_\beta] = 0\). The total number of classes amounts then to \(2n+2\). In the following we will use the notation \(E^{4m} := \{X \in \Gamma(H) \mid i_X d\eta_\alpha = 0\}\), while \(E^{4l} := \Gamma(H)\), \(E^{4l+3} := E^{4l} \oplus \Gamma(V)\), and \(E^{4m+3} := E^{4m} \oplus \Gamma(V)\).

We now consider the class of 3-quasi-Sasakian manifolds such that \([\xi_\alpha, \xi_\beta] = c\xi_{\gamma}\) with \(c \neq 0\) and let \(4l+3\) be the rank. In this case, according to [3], we define for each structure \((\phi_\alpha, \xi_\alpha, \eta_\alpha, g)\) two \((1, 1)\)-tensor fields \(\psi_\alpha\) and \(\theta_\alpha\) by putting

\[
\psi_\alpha X = \begin{cases} 
\phi_\alpha X, & \text{if } X \in E^{4l+3}; \\
0, & \text{if } X \in E^{4m}; 
\end{cases}
\quad \theta_\alpha X = \begin{cases} 
0, & \text{if } X \in E^{4l+3}; \\
\phi_\alpha X, & \text{if } X \in E^{4m}; 
\end{cases}
\]

Note that, for each \(\alpha \in \{1, 2, 3\}\) we have \(\phi_\alpha = \psi_\alpha + \theta_\alpha\). Next, we define a new (pseudo-Riemannian, in general) metric \(\tilde{g}\) on \(M\) setting

\[
\tilde{g}(X, Y) = \begin{cases} 
-d\eta_\alpha(X, \phi_\alpha Y), & \text{for } X, Y \in E^{4l}; \\
g(X, Y), & \text{elsewhere.}
\end{cases}
\]

This definition is well posed by virtue of normality and of [7, Lemma 5.3]. \((M, \phi_\alpha, \xi_\alpha, \eta_\alpha, \tilde{g})\) is in fact a hyper-normal almost 3-contact metric manifold, in general non-3-quasi-Sasakian. We are now able to formulate the following decomposition theorem, proven in [7].

**Theorem 3.2.** Let \((M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, \tilde{g})\) be a 3-quasi-Sasakian manifold of rank \(4l+3\) with \([\xi_\alpha, \xi_\beta] = 2\xi_\gamma\). Assume \([\theta_\alpha, \theta_\alpha] = 0\) for some \(\alpha \in \{1, 2, 3\}\) and
\(\bar{g}\) positive definite on \(E^{4l}\). Then \(M^{4n+3}\) is locally the product of a 3-Sasakian manifold \(M^{4l+3}\) and a hyper-Kählerian manifold \(M^m\) with \(m = n - l\).

We now consider the class of 3-quasi-Sasakian manifolds such that \([\xi, \xi] = 0\) and let \(4l + 1\) be the rank. In this case we define for each structure \((\phi, \xi, \eta, g)\) two \((1,1)\)-tensor fields \(\psi\) and \(\theta\) by putting

\[
\psi X = \begin{cases} 
\phi X, & \text{if } X \in E^{4l}, \\
0, & \text{if } X \in E^{4m+3}, 
\end{cases}
\quad \theta X = \begin{cases} 
0, & \text{if } X \in E^{4l}, \\
\phi X, & \text{if } X \in E^{4m+3}.
\end{cases}
\]

Note that for each \(\alpha\) the maps \(-\psi^2\) and \(-\theta^2 + \eta \otimes \xi\) define an almost product structure which is integrable if and only if \([-\psi^2, -\psi^2] = 0\) or, equivalently, \([\psi, \psi] = 0\). Under this assumption the structure turns out to be 3-cosymplectic:

**Theorem 3.3** ([7]). Let \((M, \phi, \xi, \eta, g)\) be a 3-quasi-Sasakian manifold of rank \(4l + 1\) such that \([\xi, \xi] = 0\) for any \(\alpha, \beta \in \{1, 2, 3\}\) and \([\psi, \psi] = 0\) for some \(\alpha \in \{1, 2, 3\}\). Then \(M\) is a 3-cosymplectic manifold.

As we have remarked before, 3-Sasakian and 3-cosymplectic manifolds belong to the class of 3-quasi-Sasakian manifolds, having respectively rank \(4n + 3 = \dim(M)\) and rank 1. We now briefly collect some additional properties of these two important subclasses. We have seen that the vertical distribution \(V\) is integrable already in any 3-quasi-Sasakian manifold. Ishihara ([10]) has shown that if the foliation defined by \(V\) is regular then the space of leaves is a quaternionic-Kählerian manifold. Boyer, Galicki and Mann have proved the following more general result.

**Theorem 3.4** ([5]). Let \((M^{4n+3}, \phi, \xi, \eta, g)\) be a 3-Sasakian manifold such that the Killing vector fields \(\xi_1, \xi_2, \xi_3\) are complete. Then

(i): \(M^{4n+3}\) is an Einstein manifold of positive scalar curvature equal to \(2(2n + 1)(4n + 3)\).

(ii): Each leaf of the foliation \(V\) is a 3-dimensional homogeneous spherical space form.

(iii): The space of leaves \(M^{4n+3}/V\) is a quaternionic-Kählerian orbifold of dimension \(4n\) with positive scalar curvature equal to \(16n(n + 2)\).

We consider now the horizontal distribution: on the one hand, in the 3-Sasakian subclass \(H\) is never integrable. On the other hand, in any 3-cosymplectic manifold \(H\) is integrable since each \(\eta\) is closed. Furthermore,
the projectability with respect to $\mathcal{V}$ is always granted, as the following theorem shows.

**Theorem 3.5 ([8]).** Every regular 3-cosymplectic manifold projects onto a hyper-Kählerian manifold.

As a corollary, it follows that every 3-cosymplectic manifold is Ricci-flat.

In [8] the horizontal flatness of such structures has been studied. In particular it has been proven to be equivalent to the existence of Darboux-like coordinates, that is local coordinates $\{x_1, \ldots, x_{4n}, z_1, z_2, z_3\}$ with respect to which, for each $\alpha \in \{1, 2, 3\}$, the fundamental 2-forms $\Phi_\alpha = d\eta_\alpha$ have constant components and $\xi_\alpha = a^1_\alpha \frac{\partial}{\partial z_1} + a^2_\alpha \frac{\partial}{\partial z_2} + a^3_\alpha \frac{\partial}{\partial z_3}$, $a^\beta_\alpha$ being functions depending only on the coordinates $z_1, z_2, z_3$. Consequently, in view of Theorem 3.4 and Theorem 3.5 we have the following result.

**Theorem 3.6 ([8]).** A 3-Sasakian manifold does not admit any Darboux-like coordinate system. On the other hand, a 3-cosymplectic manifold admits a Darboux-like coordinate system around each of its points if and only if it is flat.

### 4. Final Remarks

A number of natural questions arose during the development of our work on 3-quasi-Sasakian manifolds. We have seen that 3-Sasakian manifolds do not admit any Darboux coordinate system, while on 3-cosymplectic manifolds such coordinate exist if and only if the manifold is flat, so it is natural to wonder whether these coordinates do not exist on any 3-quasi-Sasakian manifold of rank greater than one. Another important topic would be to study the projectability of 3-quasi-Sasakian manifolds for understanding the general relation between this class and the quaternionic structures, since the 3-Sasakian manifolds project over quaternionic-Kähler structures while the structure of the leaf space turns out to be globally hyper-Kählerian in the 3-cosymplectic case. Finally, as both 3-Sasakian and 3-cosymplectic manifolds are Einstein manifolds a natural question would be to ask whether all 3-quasi-Sasakian manifolds are Einstein. However, since we have already found an example of an $\eta$-Einstein, non-Einstein 3-quasi-Sasakian manifold in [7], the natural problem now becomes to establish if there is any 3-quasi-Sasakian manifolds which is not $\eta$-Einstein. We will try to address some of these questions in the next future.
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