A PALEY-WIENER THEOREM FOR THE ASKEY-WILSON FUNCTION TRANSFORM

LUISS DANIEL ABREU† AND FETHI BOUZEFFOUR

ABSTRACT: We define an analogue of the Paley-Wiener space in the context of the Askey-Wilson function transform, compute explicitly its reproducing kernel and prove that the growth of functions in this space of entire functions is of order two and type $\ln q^{-1}$, providing a Paley-Wiener Theorem for the Askey-Wilson transform. Up to a change of scale, this growth is related to the refined concepts of exponential order and growth proposed by J. P. Ramis. The Paley-Wiener theorem is proved by combining a sampling theorem with a result on interpolation of entire functions due to M. E. H. Ismail and D. Stanton.


1. Introduction

Let $M(r; f) = \sup\{|f(z)| : |z| \leq r\}$ and consider the space $A$, constituted by the analytic continuation to the whole complex plane of the functions $f \in L^2(\mathbb{R})$ satisfying

$$M(r; f) = O(e^{\pi r}). \quad (1)$$

Consider also the space $PW$ constituted by the analytic continuation to the whole complex plane of the functions $f \in L^2(\mathbb{R})$ such that, for some $u \in L^2(-\pi, \pi),$ $f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{izt} u(t) dt. \quad (2)$

A celebrated classical theorem of Paley and Wiener says that

$$A = PW.$$

The growth condition (1) means that $f : \mathbb{C} \rightarrow \mathbb{C}$ has order one and type $\pi$ and the space $PW$ is called the Paley-Wiener space of band-limited functions; it is the reproducing kernel Hilbert space mapped via the Fourier transform into $L^2$ functions supported on the interval $[-\pi, \pi]$. See [24] for more details.

Received July 7, 2009.

†Partial financial assistance by CMUC/FCT and FCT post-doctoral grant SFRH/BPD/26078/2005, POCI 2010 and FSE.
Another famous result, the Whittaker-Shannon-Kolmogorov sampling theorem, asserts that every function in the space \(PW\) admits the following representation

\[
f(x) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi (x - n)}{\pi (x - n)}.
\]  

As a result, research concerning extensions of the sampling theorem has been historically associated with the corresponding extensions of the Paley-Wiener theorem.

The sampling theorem is known to hold for more general transforms, including the Hankel and Jacobi functions transforms [13], [25] and the Paley-Wiener theorem is known to extend to such special functions transforms [9].

Many sampling theorems have been recently considered in the \(q\)-case [1], [2], [5], [16]. When thinking about these extensions, one should keep in mind that many of the classical \(q\)-functions are special cases of a very general basic hypergeometric function known as the Askey-Wilson function. This fact is known as ”the Askey-Wilson transform scheme” [8].

Recently, one of us has found a sampling theorem for the Askey-Wilson function transform [6]. Thus, it is natural to ask for the associated Paley-Wiener theorem. It is the purpose of this paper to address this question, providing a Paley-Wiener theorem for the Askey-Wilson function transform. This will be done after rephrasing the results in [6] in the convenient reproducing kernel Hilbert space setting.

Recent research concerning \(q\)-difference equations [19], interpolation of entire functions [17] and moment problems [4], strongly suggests that in order to deal with basic hypergeometric functions one should use the following concepts. A function \(f\) has logarithmic order \(\rho\) if

\[
\lim_{r \to +\infty} \sup \frac{\ln \ln M(r; f)}{\ln \ln r} = \rho
\]

and \(f\) with logarithmic order \(\rho\) has logarithmic type \(c\) if

\[
\lim_{r \to +\infty} \sup \frac{\ln M(r; f)}{(\ln r)^{\rho}} = c.
\]

This is because basic hypergeometric functions are of order zero and therefore require a refined concept of order to define their growth. However, we will approach the topic in a slightly different manner in this paper: Instead of considering a function in \(\mu\), we will considerer a function in \(z = q^\mu\). Looking
at objects from this point of view, our Askey-Wilson Paley-Wiener space turns out to be constituted by functions of order two with type $\ln(1/q)$. This is equivalent to say that, in the variable $z = q^\mu$, they have logarithmic order two and logarithmic type $\ln(1/q)$.

We have organized the paper in the following way. The next section reviews the definitions of the Askey-Wilson polynomials and functions and provides a short outline of the $L^2$ theory of the Askey-Wilson transform. Then, in the third section, we present a detailed study of the reproducing kernel Hilbert space which is naturally associated to the Askey-Wilson functions transform (in much the same way $PW$ is associated to the Fourier transform). We compute a basis for this space as well as the explicit formula for the reproducing kernel and recover by this method the sampling theorem of [6]. Finally, in the last section we prove a Paley-Wiener theorem, by describing the growth of functions in the reproducing kernel Hilbert space in terms of their order and type.

2. The Askey-Wilson function transform

2.1. The Askey-Wilson polynomials. Choose a number $q$ such that $0 < q < 1$. The notational conventions from [11]

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=1}^{n} (1 - aq^{k-1}),$$

$$(a; q)_{\infty} = \lim_{n \to \infty} (a; q)_n, \quad (a_1, \ldots, a_m; q)_n = \prod_{t=1}^{m} (a_t; q)_n, \quad |q| < 1,$$

where $n = 1, 2, \ldots$, will be used. The symbol $_{r+1}\phi_r$ stands for the function

$$_{r+1}\phi_r \left( \begin{array}{c} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{array} \bigg| q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_{r+1}; q)_n}{(q, b_1, \ldots, b_r; q)_n} z^n.$$

The Askey-Wilson polynomials $p_n(x; a, b, c, d)$, with $x = \frac{z + z^{-1}}{2}$, are defined by

$$p_n \left( \frac{z + z^{-1}}{2}; a, b, c, d \right) = \frac{(ab, ac, ad; q)_n}{a^n} _4\phi_3 \left( \begin{array}{c} q^{-n}, q^{n-1}abcd, az, a/z \\ ab, ac, ad \end{array} \bigg| q; q \right).$$

(4)

If $a, b, c, d \in \mathbb{C}$ are four reals or two reals and one pair of conjugates, or two pairs of conjugates such that $|ab|, |ac|, |ad|, |bc|, |cd| < 1$, then the
Askey-Wilson polynomials are real valued and their orthogonality can be written as an integral over \( x = \frac{z^2 - 1}{2} \in [-1, 1] \) plus a finite sum over a discrete set with mass points outside \([-1, 1]\). This finite sum does not occur if \(|a|, |b|, |c|, |d| < 1\). When \( \max(|a|, |b|, |c|, |d|) < 1\), the Askey-Wilson polynomials satisfy the orthogonality relation
\[
\int_{-1}^{1} p_n(x; a, b, c, d)p_m(x; a, b, c, d)w(x)dx = h_n \delta_{m,n},
\]
where
\[
w(x) = \frac{(x^2, 1/x^2; q)_\infty \sin \theta}{(ax, a/x, bx, b/x, cx, c/x, dx, d/x; q)_\infty},
\]
and
\[
h_n = \frac{2\pi (abcdq^{2n}; q)_\infty (abcdq^{n-1}; q)_n}{(q^{n+1}, abq^n, acq^n, adq^n, bcq^n, bdq^n, cdq^n; q)_\infty}.
\]

The Askey-Wilson function is defined as
\[
\phi_\gamma(z) = \frac{1}{(bc, q/ad; q)_\infty} \frac{4\phi_3(\tilde{a}/\gamma, \tilde{b} \gamma, a z, a/z; q)}{\phi_3(\tilde{a}/\gamma, \tilde{a} \gamma, qb/d, qc/d, az, a/z; q)_\infty} \frac{(q \gamma/d, q/\gamma d, ab, ac, ad/q, qz/d, q/zd; q)_\infty}{(q \gamma/d, q/\gamma d, ab, ac, ad/q, qz/d, q/zd; q)_\infty} \frac{4\phi_3(q \gamma/d, q/\gamma d, q z/d, q/zd; q)}{4\phi_3(q \gamma/d, q/\gamma d, q z/d, q/zd; q)_\infty},
\]
where
\[
\tilde{a} = \sqrt{q^{-1}abcd},
\]
\[
\tilde{b} = ab/\tilde{a} = q\tilde{a}/cd,
\]
\[
\tilde{c} = ac/\tilde{a} = q\tilde{a}/bd,
\]
\[
\tilde{d} = ad/\tilde{a} = q\tilde{a}/bc.
\]

The function \( \phi_\gamma \) is introduced in [15] and it can also be defined as a single \(8\phi_7\) with a very-well poised \(8W_7\) structure [23]. The function \( \phi_\gamma \) is meromorphic in \( \gamma \). Moreover, its poles are simple and can be removed multiplying it by the factor \((q \gamma/d, q/\gamma d; q)_\infty\).

Now we will define the Askey-Wilson function transform, following the construction in [7]. A new weight function is defined as
\[
W(x) = \Delta(x) \Theta(x)
\]
where, using the notation $\theta (x) = (x, q/x; q)_\infty$ for the renormalized Jacobi theta function, the function $\Theta$ is defined as

$$
\Theta (x) = \frac{\theta (dx, d/x)}{\theta (dtx, dt/x)}.
$$

For generic parameters $a, b, c, d$ such that the weight function $W$ has simple poles we define a measure $\nu$, depending on these parameters, by

$$
\int f (x) \, d\nu (x) = \frac{K}{4i\pi} \int_T f(x) \phi_\gamma (x) W (x) \frac{dx}{x} 
$$

$$
+ \frac{K}{2} \sum_{x \in D} \left( f (x) + f (x^{-1}) \right) \text{Res}_{y=x} \left( \frac{W (y)}{y} \right),
$$

where $K$ is a constant (the exact value will not be required), $S = S_- \cup S_+$ is the infinite, discrete set given by

$$
S_- = \{ dtq^k; \ k \in \mathbb{Z}, \ dtq^k < -1 \},
$$

$$
S_+ = \{ aq^k; \ k \in \mathbb{Z}, \ aq^k > 1 \}.
$$

In the next sections we will often refer to the measure defined above as being of the form $\nu = \nu_c + \nu_d$, where $\nu_c$ is the continuous measure

$$
d\nu_c (x) = \Theta (x) \Delta (x) \frac{dx}{x}
$$

continuous and $\nu_d$ is the discrete part, supported in the set $S$.

Now, let $L^2_+ (\nu)$ be the Hilbert space with respect to the measure $\nu$ constituted by functions $f$ satisfying $f (x) = f (x^{-1})$, $\nu$-almost everywhere. The Askey-Wilson function transform is defined by

$$
(F f) (\gamma) = \int f (x) \phi_\gamma (x) \, d\nu (x)
$$

for compactly supported functions $f \in L^2_+ (\nu)$. Let $L^2_+ (\tilde{\nu})$ be the same space with respect to the same measure, but replacing the parameters $a, b, c, d$ by the dual parameters $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$. The main result in [7] states that $F$ extends to an isometric isomorphism

$$
F : L^2_+ (\nu) \rightarrow L^2_+ (\tilde{\nu}) .
$$
3. The Askey-Wilson Paley-Wiener space

3.1. Reproducing kernel Hilbert spaces. We will now introduce some concepts concerning reproducing kernel Hilbert spaces. This exposition is taken from [13], [10] and [20].

Let $H_{rep}$ be a class of complex valued functions, defined in a set $X \subset \mathbb{C}$, such that $H_{rep}$ is a Hilbert space. We say that $k(\gamma, x)$ is a reproducing kernel of $H_{rep}$ if $k(\gamma, x) \in H_{rep}$ for every $\gamma \in X$ and every $f \in H_{rep}$ satisfies the reproducing equation

$$f(\gamma) = \langle f(.), k(., \gamma) \rangle_{H_{rep}}.$$

Now we will use the language in Saitoh [20], and we proceed to give a brief account of the required results.

Consider a second Hilbert space, $H$. For each $t$ belonging to a domain $X$, let $K(., t)$ belong to $H$. Then,

$$k(\gamma, x) = \langle K(., \gamma), K(., x) \rangle_H$$

is defined on $X \times X$. Suppose that we have an isometric transformation

$$(Fg)(\gamma) = \langle g, K(, \gamma) \rangle_H$$

and denote the set of images by $F(H)$. The following theorem can be found in [20]:

**Theorem A** If $F$ is a one to one isometric transformation, the kernel $k(\gamma, x)$ determines uniquely a reproducing kernel Hilbert space for which it is the reproducing kernel. This reproducing kernel Hilbert space is precisely $F(H)$ and it can have no other reproducing kernel. If $\{S_n\}$ is a basis of $F(H)$, then

$$k(\gamma, x) = \sum_n S_n(\gamma)S_n(x).$$

There is a general formulation of the sampling theorem in reproducing kernel Hilbert spaces [14]. We will use the following ”orthogonal basis case”.

**Theorem B** With the notations established earlier, we have: If there exists $\{t_n\}_{n \in \mathbb{I} \subset \mathbb{Z}}$ such that $\{K(., t_n)\}_{n \in \mathbb{I}}$ is an orthogonal basis, we then have the sampling expansion

$$f(t) = \sum_{n \in \mathbb{I}} f(t_n) \frac{k(t, t_n)}{k(t_n, t_n)}.$$
in $F(H)$, pointwise over $I$, and uniformly over any compact subset of $X$ for which $\|K_t\|$ is bounded.

The chief example of a reproducing kernel Hilbert space is $PW$. In this situation the reproducing kernel is the function $\sin \pi (x - \gamma) / \pi (x - \gamma)$, the sampling points are $t_n = n$ and the uniformly convergent expansion is the Whittaker-Shannon-Kolmogorov sampling formula.

### 3.2. The Askey-Wilson function reproducing kernel.

Let us look at the reproducing kernel Hilbert space associated to the Askey-Wilson function transform.

The first task is to consider a proper analogue of bandlimited functions. This is done by defining a finite continuous Askey-Wilson function in much the same way it was done in [6].

We start by removing the poles of the function $\phi_{\tilde{a}q^\mu}$: Consider a function $u_\mu$, analytic in the variable $\mu$, defined as

$$u_\mu (x, a, b, c, d | q) = (\tilde{a}q^\mu; \tilde{a}q^{-\mu}; q)_\infty \phi_{\tilde{a}q^\mu} (e^{i\theta}), \ x = \cos \theta.$$

Then we consider what is going to be the analogue of the transform (2): if $\max(|a|, |b|, |c|, |d|) < 1$, the finite continuous Askey-Wilson transform $J$ is defined by

$$J(f)(\mu) = \int_{-1}^{1} f(x)u_\mu (x; a, b, c, d | q) w(x, a, b, c, d | q) dx. \tag{5}$$

The continuous Askey-Wilson relates to the Askey-Wilson transform as follows: If $\tilde{f}$ is the analytic function such that $f (\cos \theta) = \tilde{f} (e^{i\theta})$, then

$$J(f)(\mu) = \frac{4i\pi}{K}(\tilde{a}q^\mu; \tilde{a}q^{-\mu}; q)_\infty \mathcal{F} \left( \frac{\tilde{f}}{\Theta} \right)(\tilde{a}q^\mu).$$

**Definition 1.** The Askey-Wilson Paley-Wiener space, $PW_{AW}$, is the space constituted by the analytic extension to the complex plane of the functions $f \in L^2_+ (\nu)$ such that, for some $u \in L^2 (w(x, a, b, c, d | q), dx)$,

$$f = J(u).$$

Let us look at this particular setting from the point of view of Theorem A.
Theorem 1. If \( \max(|a|, |b|, |c|, |d|) < 1 \), then the set \( PW_{AW} \) is a Hilbert space of entire functions with reproducing kernel \( k(\gamma, \lambda) \). The functions
\[
S_n^{(\tilde{a})}(\mu; q) = \frac{(-1)^n q^{n(n+1)/2} (1 - \tilde{a}^2 q^{2n}) (\tilde{a}q^{\mu}, \tilde{a}q^{-\mu}; q)_\infty}{(q; q)_n (a, \tilde{a}^2 q^n; q)_\infty (1 - \tilde{a}q^{n+\mu}) (1 - \tilde{a}q^{-n-\mu})}.
\]
constitute an orthogonal basis of \( PW_{AW} \) and the reproducing kernel is given explicitly by
\[
k(\gamma, \lambda) = \sum_{n=0}^{\infty} S_n^{(\tilde{a})}(\gamma; q) S_n^{(\tilde{a})}(\lambda; q).
\]

Proof: To fulfill the conditions in Theorem A, we need to show that the finite continuous Askey-Wilson is a one to one isomorphism between \( A_{AW} \) and \( PW_{AW} \). To see that it is one-to-one, observe that, since, if
\[
\int_{-1}^{1} f(x) u_\mu(x; a, b, c, d | q) w(x, a, b, c, d | q) dx = 0, \quad \text{for all } \mu \in \mathbb{C},
\]
then we have, in particular, that
\[
\int_{-1}^{1} f(x) u_n(x; a, b, c, d | q) w(x, a, b, c, d | q) dx, \quad \text{for } n = 0, 1, \ldots
\]
Since for integer values of \( \mu \), \( u_\mu \) is a multiple of the Askey-Wilson polynomials,
\[
u_n(x; a, b, c, d) = \frac{(-1)^n q^{-n(n-1)/2}}{(ab, ac, bc; q)_n} d^{-n} p_n(x; a, b, c, d), \quad (6)
\]
we can use the completeness of the system of the Askey-Wilson polynomials to get \( f = 0 \). Consequently, \( J(f) \) is one to one. From the definition,
\[
PW_{AW} = J \left[ L^2(w(x, a, b, c, d | q) dx) \right].
\]
Therefore, endowing \( PW_{AW} \) with the inner product
\[
\langle J(f), J(g) \rangle_{PW_{AW}} = \int_{-1}^{1} f(x) g(x) w(x, a, b, c, d | q) dx,
\]
the finite Askey-Wilson transform \( J \) becomes a Hilbert space isometry between \( L^2(w(x, a, b, c, d | q) dx) \) and \( PW_{AW} \).

It remains to show that the functions \( S_n^{(\tilde{a})}(\mu; q) \) provide an orthogonal basis for \( PW_{AW} \). By the definition (5) and (6),
\[
J(u_n)(\mu) = \int_{-1}^{1} u_n(x) u_\mu(x) w(x, a, b, c, d | q) dx
\]
\[ = \frac{(-1)^n q^{-n(n-1)/2}}{d^n(ab, ac, bc; q)_n} \int_{-1}^{1} p_n(x) u_\mu(x) w(x, a, b, c, d \mid q) \, dx. \]

Now we can use Proposition 6 of [6] to conclude that
\[ J(u_n)(\mu) = S_n^{(\tilde{\alpha})} (\mu; q). \]

By (6), \( \{u_n\} \) is an orthogonal basis of \( L^2(w(x, a, b, c, d \mid q) \, dx) \). Since \( J \) is isometric onto \( PW_{AW} \), it follows that \( S_n^{(\tilde{\alpha})} (\mu; q) \) is a basis of \( PW_{AW} \).

**Remark 1.** The functions \( S_n^{(\tilde{\alpha})} (\gamma; q) \) play the same role in our setting as do the functions \( \sin \pi (x - n) / \pi (x - n) \) in the Paley-Wiener space.

Now, Theorem 1 and Theorem B give the following sampling theorem. This has been proved in [6], but the approach with reproducing kernels provides the uniform convergence that will be used in the next section.

**Theorem 2.** For \( f \in PW_{AW} \) we have
\[ f(\mu) = \sum_{n=0}^{\infty} f(n) S_n^{(\tilde{\alpha})} (\mu; q), \quad (8) \]

where \( S_n^{(\tilde{\alpha})} (\mu; q) \) is given by
\[ S_n^{(\tilde{\alpha})} (\mu; q) = \frac{(-1)^n q^{-n(n+1)/2} (1 - \tilde{\alpha}^2 q^{2n}) (\tilde{\alpha} q^\mu, \tilde{\alpha} q^{-\mu}; q)_\infty (1 - \tilde{\alpha} q^{n+\mu}) (1 - \tilde{\alpha} q^{n-\mu})}{(q; q)_n (a, \tilde{\alpha}^2 q^n; q)_\infty (1 - \tilde{\alpha} q^{n+\mu}) (1 - \tilde{\alpha} q^{n-\mu})}. \]

The convergence is uniform on every compact subset of the real line.

**Proof:** Observe that, from
\[ S_n^{(\tilde{\alpha})} (m; q) = \delta_{n,m}, \]
we obtain:
\[ g(\mu, m) = \sum_{n=0}^{\infty} S_n^{(\tilde{\alpha})} (\mu; q) S_n^{(\tilde{\alpha})} (m; q) = S_n^{(\tilde{\alpha})} (\mu; q). \]

Moreover,
\[ g(m, m) = S_n^{(\tilde{\alpha})} (m; q) = 1, \]
and the result follows from Theorem 1 and Theorem B.
4. The Askey-Wilson-Paley-Wiener theorem

Recall that the entire function $f$ is of order $\rho$ if
$$\lim_{r \to \infty} \frac{\ln \ln(M(r; f))}{\ln r} = \rho.$$  
A constant has order zero, by convention.

The entire function $f$ of positive order $\rho$ is of type $\tau$ if
$$\lim \frac{\ln(M(r; f))}{r^\rho} = \tau.$$  

Definition 2. The space $\mathcal{A}_{AW}$, which will be the analogue of $\mathcal{A}$ in the Askey-Wilson setting, is the space constituted by the analytic continuation of the functions from $L^2(w(x, a, b, c, d \mid q), dx)$ such that
$$M(r; f) = O(e^{\ln(1/q)r^2}),$$
that is, of order 2 and type $\ln(1/q)$.

It is easy to see that the functions in $\mathcal{A}_{AW}$ satisfy the conditions in [17, Theorem 3.1]. We rewrite this statement as:

**Theorem C** Every $f \in \mathcal{A}_{AW}$ admits the expansion
$$f(\mu) = \sum_{n=0}^{\infty} f(n) S_n^{(a)}(\mu; q).$$  

The next result is the Paley-Wiener theorem for the Askey-Wilson functions transform. The cornerstone of its proof is the fact that the entire function expansion (9) and the sampling expansion (8) are exactly the same.

**Theorem 3.** If $\max(|a|, |b|, |c|, |d|) < 1$, then $\mathcal{A}_{AW} = PW_{AW}$.

**Proof:** Take $f \in PW_{AW}$. By definition we have, for some $u \in L^2(w(x, a, b, c, d \mid q), dx)$,
$$f(\mu) = \mathcal{J}(u)(\mu) = \int_{-1}^{1} u(x) u_\mu(x) w(x, a, b, c, d \mid q) dx.$$  
We need to study the growth of
$$M(r; u_\mu).$$
From formula (5.4) in [23] and for $0 \leq \theta \leq \pi$, we have
$$u_r(x) = \frac{(ae^{i\theta}, be^{i\theta}, ce^{i\theta}, qe^{i\theta} / d ; q)_\infty}{(ab, ac, bc, e^{2i\theta} ; q)_\infty} (q^{1-r} / e^{i\theta} d ; q)_\infty [1 + o(1)],$$ as $r \to \infty$.  

Let $-1 < \delta < 0$, then

$$M(n + \delta; u_\mu) = O\left((q^{1-\delta-n}/d; q)_n\right).$$

This implies

$$M(n + \delta; u_\mu) = O\left((q/d)^n q^{-n(n+1+2\delta)/2}\right).$$

Therefore,

$$\lim_{r \to \infty} \sup \ln \ln(M(r; u_\mu)) / \ln r = 2,$$

and

$$\lim_{r \to \infty} \frac{\ln(M(r; u_\mu))}{r^2} = \ln(1/q).$$

This condition implies that $u_\mu$ is of order 2 and type at most $\ln(1/q)$. Therefore,

$$u_\mu(x) \in A_{AW}.$$

This shows that $f \in A_{AW}$. Conversely, let $f \in A_{AW}$. By Theorem C,

$$f(\mu) = \sum_{n=0}^{\infty} f(n) S_n^{(\tilde{\alpha})}(\mu; q),$$

In the end of the proof of Theorem 1 we have seen that

$$S_n^{(\tilde{\alpha})}(\mu; q) = \mathcal{J}(u_n)(\mu).$$

Then, the sampling formula of Theorem 2 can be written as

$$f(\mu) = \sum_{n=0}^{\infty} f(n) \mathcal{J}(u_n)(\mu)$$

$$= \sum_{n=0}^{\infty} f(n) \int_{-1}^{1} u_n(x) u_\mu(x) \omega(x, a, b, c, d | q) dx.$$

The uniform convergence of the sampling series allows to interchange the integral with the sum in such a way that

$$f(\mu) = \int_{-1}^{1} \left(\sum_{n=0}^{\infty} f(n) u_n(x)\right) u_\mu(x) \omega(x, a, b, c, d | q) dx.$$

Then we have written $f$ in the form

$$f(\mu) = \mathcal{J}(u)(\mu),$$
with
\[ u(x) = \left( \sum_{n=0}^{\infty} f(n) u_n(x) \right) \in L^2(w(x, a, b, c, d \mid q), dx). \]

As a result, \( f \in PW_{AW} \).

**References**


Luis Daniel Abreu†
CMUC, Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal
E-mail address: daniel@mat.uc.pt

Fethi Bouzeffour
Faculté des Sciences de Bizerte, Tunisie.
E-mail address: Fethi.Bouzeffour@ipeib.rnu.tn