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Implicit regularization in a QCD decay of the Higgs boson into gluons

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Implicit Regularization in a QCD decay of the Higgs boson

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Abstract

Perturbative Quantum Chromodynamics (QCD) involves the appearance of divergences in the amplitudes of a process. However, physical observables must be finite, therefore all the divergences that emerge must be cancelled. The Kinoshita–Lee–Nauenberg (KLN) theorem states that the infrared divergences that appear in a QCD decay rate or cross section must cancel when putting together the contributions from the virtual and real parts that contribute at the same order in perturbation theory. In this work, the main goal is to calculate the decay rate of the Higgs boson into gluons modelled by an effective Lagrangian in the limit of infinite top quark mass and verify the KLN theorem, using the Implicit Regularization (IReg). We derive the Feynman rules of the effective Lagrangian to describe the interaction between gluons and the Higgs boson, and use them to construct the amplitudes of the process' virtual and real diagrams. We then use IReg, which is a regularization scheme that works in the physical dimension of the theory and allows for the separation of the ultraviolet and infrared divergences of an amplitude. The ultraviolet divergent integrals are written as basic divergent integrals and the ultraviolet finite/ infrared divergent integrals are evaluated using the software *Mathematica*. After renormalization of the effective theory, we show that the virtual decay rate of the process can be written as a correction to the tree-level decay rate. We introduce the spinor-helicity formalism to compute the real amplitude. We then study the explicit computation of the phase space of the real decay and integrate the squared real amplitude over the phase space to obtain the real decay. At last, we add the contributions from both virtual and real decay rates to obtain the final result which is finite as expected, reproducing known results in the literature.

Resumo

O regime perturbativo de Cromodinâmica Quântica (QCD) envolve o aparecimento de divergências nas amplitudes de um processo. No entanto, as observáveis físicas devem ser finitas e, portanto, todas as divergências que surgem devem ser canceladas. De acordo com o teorema Kinoshita–Lee–Nauenberg (KLN), as divergências infrevermelhas que aparecem numa taxa de decaimento ou secção eficaz em QCD devem cancelar-se ao juntar as contribuições das partes virtual e real que contribuem para a mesma ordem em teoria de perturbações. Neste trabalho, o objetivo principal é calcular a taxa de decaimento do bosão de Higgs em gluões modelado por um Lagrangiano efetivo no limite da massa do quark top infinita, e verificar o cancelamento das divergências usando regularização implícita (IReg). Para tal, derivamos as regras de Feynman do Lagrangiano efetivo para descrever a interação entre os gluões e o bosão de Higgs e estas são usadas para construir as amplitudes dos diagramas virtuais e reais do processo. Em seguida, usamos IReg, que é um esquema de regularização que trabalha na dimensão física da teoria e permite a separação das divergências de ultravioleta e infravermelhas de uma amplitude. Os integrais divergentes de ultravioleta são escritos como integrais divergentes básicos e os integrais finitos no ultravioleva/ divergentes no infravermelho são avaliados usando o software *Mathematica*. Depois de se fazer renormalização da teoria efetiva, mostramos que a taxa de decaimento virtual do processo pode ser escrita como uma correção à taxa de decaimento a nível árvore. Introduzimos o formalismo de spin-helicidade para calcular a amplitude real. Em seguida, estudamos o cálculo explícito do espaço de fase do decaimento real e integramos a amplitude real ao longo das variáveis de integração do espaço fase para obter o decaimento real. Por fim, adicionamos as contribuições das taxas de decaimento virtual e real para obter o resultado final, que reproduz resultados conhecidos da literatura.

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Abbreviations

QFT - Quantum Field Theory

QCD - Quantum Chromodynamics

KLN - Kinoshita-Lee-Nauenberg

IR - Infrared

UV - Ultraviolet

IReg - Implicit regularization

BDI - Basic Divergent Integrals

ST - Surface Terms

YM - Yang-Mills

Chapter 1

Introduction

The purpose of this thesis is to further investigate if Implicit Regularization (IReg), a non-dimensional regularization scheme satisfies the Kinoshita–Lee–Nauenberg (KLN) theorem. For that we do the computation of a QCD decay rate of the Higgs boson into gluons and verify that infrared (IR) divergences are cancelled in this scheme and the final decay rate is finite.

Regularization and renormalization in QCD

Physical observables are the primary objects of study in physics. Performing precise measurements of the interaction of known particles is very important to corroborate or refute new models in physics, [2]. We cannot compute the exact Green's functions analytically when we have a QCD Lagrangian, therefore we use perturbation theory and expand the terms to arbitrary order. To do so, we first derive the Lagrangian Feynman rules, which we then use as building block to construct the amplitudes of the Feynman diagrams that correspond to the selected order of expansion. However, while doing higher-order computations, ultraviolet (UV) and IR divergences are present, causing Feynman amplitudes to diverge. To allow comparison with experimental data from particle accelerators, precise calculations of physical observables beyond leading order in perturbation theory are required. As a result, one of the most important problems in physics is the regularization and renormalization of these divergent amplitudes in intermediate steps in order to preserve the predictive power of the underlying theory.

Renormalization is a systematic way of subtracting the divergences of a theory by doing a redefinition of the parameters of the bare (unrenormalized) Lagrangian. A renormalizable theory is a theory that has a finite number of superficially divergent diagrams whose divergences can be cancelled order by order by a finite number of redefinitions of the bare parameters. But before we perform renormalization we need to completely separate the UV divergent parts from the UV finite parts of the amplitudes. So we need to regularize the theory.

The choice of a regularization scheme is unphysical but it is otherwise very important in the sense that we must respect several physical requirements such as unitarity, causality and symmetries of the system. Several regularization schemes have been developed over the past years such as conventional dimensional regularization (CDR), 't Hooft-Veltman scheme (HV), four-dimensional helicity (FDH), dimensional reduction (DRED) and six-dimensional formalism (SDF) which are traditional dimensional schemes. Some non-dimensional schemes developed are implicit regularization (IReg), four-dimension regularization (FDR), four-dimensional unsubtraction (FDU) and loop regularization (LORE) [3], [4].

Regularization of chiral, topological and supersymmetric gauge theories with dimensional methods can give rise to inconsistencies in higher loop order or to spurious anomalies. On the other hand, IReg, being a non-dimensional regularization scheme, is expected to preserve the symmetries of the model and we won't need to modify the Lagrangian of the underlying theory, [5]. IReg acts directly on the dimension of the theory and is implemented to all orders in perturbation theory. We assume an implicit regulator that allows us to separate the UV divergent amplitudes that are independent of external momenta from the UV finite ones, which can be IR divergent or IR finite. UV divergent integrals are classified as basic divergent integrals and UV finite integrals are analytically evaluated using *Package-X* of software *Mathematica*, [6].

Spinor-helicity

The traditional method to calculate unpolarized cross sections or decay rates in QCD involves squaring the amplitude and then sum over the external states and the spins. When the number of Feynman diagrams increases, the computation becomes much more complex. The spinor-helicity formalism is an alternative approach to construct amplitudes using only physical on-shell external states. The idea behind this formalism is that instead of using spinor fields that transform under unitary, irreducible infinite-dimensional representations of the Poincaré group, we use helicity spinors, which transform in finite-dimensional representations of the Lorentz group. The choice of this basis automatically eliminates a large number of amplitude terms and the procedure is highly simplified, [7].

Motivation and goals

It is of great relevance to implement a fully mathematical consistent regularization scheme that prevents the emergence of symmetry breaking terms or anomalies and that is valid to arbitrary higher order. As we mentioned, IReg is an invariant regularization scheme that has proven to be a promising candidate that fulfills all the previous requirements. It is also stated by the KLN theorem that perturbative quantum theories must be IR finite. As a result it is of theoretical interest to test the applicability of IReg in a practical calculation involving IR divergences that only cancel at the level of cross sections or decay rates and verify the KLN theorem.

The decay $H \longrightarrow gg$ which is described by an effective model in which quarks are neglected and therefore only gluons are considered, provides a simple yet reliable model to test this regularization scheme, as suggested in [2] and has already been used in [8] to test FDR. The main goal of this work will be the computation of the total decay rate of this process. We will compute the one-loop virtual diagrams that arise from this effective model and use IReg to extract the UV divergences which are absorbed in the process of renormalization, and use the remaining UV finite amplitudes to obtain the regularized virtual decay rate. Then we apply the spinor-helicity formalism to compute the real diagrams of the process and obtain the real decay rate. It is expected to have a cancellation of the IR divergences when combining these decay rates and the final result must be finite.

Chapter 2

Implicit regularization and renormalization in perturbative QCD

In this chapter, we will see that divergences appear in perturbative QCD. As a result, regularization and renormalization techniques must be developed to deal with these divergences in order to obtain finite quantities.

2.1 The Kinoshita–Lee–Nauenberg theorem or KLN theorem

The KLN theorem states that although IR divergences may occur in the expansion of the action when doing perturbative calculations, the IR divergences coming from loop integrals are cancelled by divergences coming from phase space integrals. According to this theorem infrared divergences appear because some of the states are degenerate in energy. Therefore for a suitably defined physical observable IR divergences will always cancel at all orders. The importance of this theorem is that it assures that all quantum field theories are IR finite (free of IR divergences) in the limit of massless particles and this holds in any order in the perturbation theory, [9], [10].

2.2 Perturbative QCD

Quantum Field Theory (QFT) allows the computation of decay rates and cross sections that can be compared with experimental results and give information about the accuracy of the theoretical models. In order to do this, we start by determining what are the relevant degrees of freedom (fields) in the energy scale that we aim to describe and then we write down the Lorenz invariant Lagrangian that incorporates those fields obeying the symmetries of the system.

The theory is defined by its generating functional that allows us to write the Green's functions by differentiation with respect to the sources. Ideally, the computation of correlation functions would provide us the exact answer to the time ordered expectation value of the product of n fields defined in different time-space points. But most of the times, the Lagrangian will contain interaction terms and it is not possible to analytically compute the exact correlation functions, which leads us to the idea of perturbative QFT. Essentially what we do is consider that the interaction parts of a Lagrangian can be written as a series expansion around a small coupling constant and compute as many terms as necessary to achieve the requested precision. These terms in the expansion can be alternatively built from the Feynman diagrams, which work as the building blocks of the theory. The Feynman rules can be derived directly from the Lagrangian as we will do further in this work. We write down all of the Feynman diagrams until we get the order of perturbation we want to compute and then we use the Feynman rules to compute their amplitudes.

Before entering the framework of perturbative QCD it has to be assured that the parameter that we are expanding on is small enough so that higher order terms can be neglected and we still have an accurate result. In the case of QCD only at high energies the strong coupling constant is small enough so that we can apply perturbation theory.

It can happen that the process that we want to describe has some dependence on a long-distance parameter which in the case of QCD means that the coupling constant is larger and we cannot expand around it. In this case we factorize the process in short and long distances. The perturbative part depends on the high energy part and can be computed from the Lagrangian while the non-perturbative cannot but we can measure it experimentally and the result will be universal.

2.2.1 Renormalization and Regularization

When working with interactive theories, as we have seen before, we need to approach the problem of computing amplitudes using perturbation theory. By expanding the S-matrix, the terms of interaction can be represented using Feynman diagrams. We notice that when we compute an amplitude of a Feynman diagram with a loop with some internal momentum k that is integrated to infinity we may get divergent integrals. To understand this, we start by defining the superficial degree of divergence of an integral to be $D =$ Power of momentum in numerator $-$ Powers of momentum in denominator. If $D > 0$, the integrals diverge, if $D = 0$ the integrals diverge logarithmically and if $D < 0$, they do not diverge, [11]. These type of integrals that appear when the internal momentum in a Feynman diagram goes to infinity $k \longrightarrow \infty$ are UV divergent integrals. Another type of divergent integrals are IR divergent integrals relate to divergences that may occur when the internal momentum in a loop goes to zero, $k \longrightarrow 0$. In this case, for $D > 0$ the integrals converge, for $D = 0$ they are logarithmically divergent and $D < 0$ they are divergent.

Before proceeding in the calculation, we first need to regularize all infinities, making the amplitudes finite. The UV divergent integrals depend only on the internal momentum we are integrating, but it is common that the amplitudes have a dependence on the external momenta. Therefore, it is useful to have a framework that allows us to extract the UV divergent integrals completely from the finite ones, so that we can renormalize the theory and compute finite physical observables. Several regularization schemes have been developed over the years. One of those schemes is IReg that we present in the next section.

Our solution to get free of the divergences is by doing renormalization of the theory. The bare parameters of a Lagrangian are the parameters of a unrenormalized theory, which will gives us infinities when computing the amplitudes of the Feynman diagrams. Renormalizing the theory means that we will make a shift from the bare parameters to some physical parameters. As a result, we get the counterterms, which are terms that are added to the bare Lagrangian, that will exactly cancel the divergences we had from the bare parameters. In fact, these counterterms are the by-product of the shift of variables that will give us the physical parameters, which are actually the ones we would observe in nature.

2.3 Renormalization in QCD

The complete QCD Lagrangian is given by

$$L_{QCD} = \overline{\psi}_{q,0}(i\gamma_\mu D^\mu - m_{q,0})\psi_{q,0} - \frac{1}{4}G^a_{\mu\nu,0}G_0^{\mu\nu,a} + \overline{c}^a_0(-\partial^\mu D^{ab}_\mu)c^b_0. \tag{2.1}$$

where D^μ is the covariant derivative, ψ_q represents the Dirac spinor for the quark field, m_q is the quark mass, $G_{\mu\nu}$ is the field-strength tensor of the gluon fields and c^a represent the fields of the Faddeev–Popov ghost. The Lagrangian is not renormalized and the subscript $_0$ stands for the bare fields.

In order to renormalize the theory, we start by making shift in the bare parameters, introducing the field renormalization constants

$$\psi_{q,0} = Z_2^{1/2}\psi_q, \qquad A_{\mu,0} = Z_3^{1/2}A^a_\mu, \qquad c^a_0 = Z_{2c}^{1/2}c^a. \tag{2.2}$$

Doing these substitutions in the Lagrangian we can find a relation between the renormalization constants and the coupling constant of QCD g,

$$Z_2 Z_3^{1/2} g_{s,0} = \mu^{(4-d)/2} Z_1 g_s. \tag{2.3}$$

The renormalization constants are chosen so that the Green's functions have a unit residue. Extensive calculation of these factors is done in [12].

For the case of the effective theory in this work, only interactions between gluons and the Higgs boson are considered. Quarks and ghosts can be neglected and the only renormalization constants we need to compute are the ones for the vacuum polarization and the coupling constant renormalization. There are given by Z_3 and Z_g and are computed in [12]. Their correspondence to IReg in the limit of zero quarks is given by

$$Z_3 = 1 + \alpha_s \frac{1}{(4\pi)} \frac{1}{b} I_{log}(\mu^2) \Big(\frac{13}{6} - \frac{\zeta}{2} \Big) C_A + O(\alpha_s^2) \tag{2.4}$$

for the gluon-field renormalization constant and

$$Z_g = 1 - \alpha_s \frac{1}{(4\pi)} \frac{1}{b} I_{log}(\lambda^2) \frac{11}{6} C_A + O(\alpha_s^2) \tag{2.5}$$

for the renormalization of the coupling constant.

Here ζ is the Gauge parameter, and we use $\zeta = 1$. Also as we are in $SU(3)$, we use $C_A = 3$.

2.4 Implicit regularization

IReg is a regularization method that operates on the momentum space and was shown to respect unitarity, locality and Lorentz invariance. This procedure operates on the specific physical dimension of the theory, therefore we do not need to extend the space-time dimensions. IReg also does not require any changes to the Lagrangian and can be applicable to arbitrary n-loop calculations, making it an alternative to dimensional schemes. In IReg, we use an algebraic identity recursively until the UV behavior is only present in irreducible loop integrals that depend on internal momentum. The UV finite content of the amplitude isolates the dependence on physical parameters (external momenta and masses). The idea behind this scheme is to assume an implicit regulator μ that we rely on to isolate the basic divergent loop integrals from the UV finite parts. For a better illustration, consider the following integral in 4 dimensions,

$$\int_k \frac{1}{k^2(k-p)^2}. \tag{2.6}$$

We will then exemplify the procedure as the following steps. Some extensive discussion is made in the following references, [3], [5], [13], [14],[15],[16].

2.4.1 Separation of divergences

We start by introducing a regulator μ in the denominator like

$$\int_k \frac{1}{(k^2-\mu^2)((k-p)^2-\mu^2)}. \tag{2.7}$$

In the case of IR safe integrals, the regulator μ is needed to avoid spurious IR divergences in the course of the evaluation. It will cancel in the end result. In the case of IR divergent integrals, the μ will survive and parameterize the IR divergences.

By power counting, we notice that as $k \longrightarrow \infty$ this integral diverges, but there is a dependence both on internal and external momenta. We want to isolate the UV divergent content in an integral that is solely dependent on the internal momentum k on which we are integrating. We notice that it is possible to rewrite the portion of the integrand that depends on external momentum as

$$\frac{1}{(k-p)^2-\mu^2} = \frac{1}{k^2-\mu^2} + \frac{2k\cdot p - p^2}{(k^2-\mu^2)((k-p)^2-\mu^2)}, \tag{2.8}$$

where in the second term we diminish one order of divergence and in the first one we have an integral depending only on the internal momentum as we wanted. As a result, this identity can be used to manipulate the expression and isolate the UV divergent content in integrals depending only on the internal momentum.

The procedure exemplified above work in general, where equation 2.8 can be generalised to

$$\frac{1}{(k-p_i)^2-\mu^2} = \sum_{j=0}^{n-1} \frac{(-1)^j (p_i^2 - 2p_i\cdot k)^j}{(k^2-\mu^2)^{j+1}} + \frac{(-1)^n (p_i^2 - 2p_i\cdot k)^n}{(k^2-\mu^2)^n [(k-p_i)^2]-\mu^2}. \tag{2.9}$$

The value of n is chosen so that the UV behaviour, which is regularization dependent is completely separated from the finite part, which is regularization independent. Notice that this identity in no way will change the integrand, therefore no alterations are made in the amplitude.

2.4.2 UV divergent integrals as Basic Divergent Integrals (BDI's)

Following the separation of the divergences of the amplitude, the UV divergent content of the amplitude can be expressed as integrals whose denominator is only dependent on the internal momentum k. These integrals are classified as BDI's and they can take either logarithmic or quadratic forms which are respectively

$$I_{log}^{\nu_1\ldots\nu_{2r}}(\mu^2) = \int_k \frac{k^{\nu_1}\ldots k^{2r}}{(k^2-\mu^2)^{r+2}} \tag{2.10}$$

and

$$I_{quad}^{\nu_1\ldots\nu_{2r}}(\mu^2) = \int_k \frac{k^{\nu_1}\ldots k^{2r}}{(k^2-\mu^2)^{r+1}}. \tag{2.11}$$

Any BDI with odd power of k in the numerator is automatically zero once the integral goes over the entire space-time and all the denominators have even powers of k. These BDI's are written in terms of Lorenz indices and can be rewritten as scalar integrals, multiplying metric tensors, after setting ST's to zero, (see section below). Scalar logarithmic and quadratic divergent integrals are given as

$$I_{log}(\mu^2) = \int_k \frac{1}{(k^2-\mu^2)^2} \tag{2.12}$$

and

$$I_{quad}(\mu^2) = \int_k \frac{1}{(k^2-\mu^2)}. \tag{2.13}$$

2.4.3 Surface terms (ST's)

Previous work has shown that IReg preserves the symmetries of the system, such as Lorenz invariance, non-Abelian gauge invariance and supersymmetry. Any symmetry breaking term can be expressed as a well defined difference between divergent integrals with the same superficial degree of freedom. These are called surface terms and they are not originally fixed, which indicates that they are related to momentum routing invariance in Feynman diagrams meaning that we could make a shift in the integration variables. As their value is associated with symmetry breaking terms, they play a critical role in IReg for the preservation of the symmetries of the system and we must carefully choose a value that allows the symmetries of the underlying theory to be preserved. Nonetheless, in a constrained version of IReg, it has been proven that these regularization dependent surface terms may be set to zero, complying with gauge invariance, [15]. This will actually allow us to reduce BDI's with Lorenz indices $\nu_1 \ldots \nu_{2r}$ to linear combinations of scalar products with the same degree of divergence plus this well defined surface terms (ST's). Generally in the four dimensional Minkowskian space-time a surface term of order j can be written as

$$\Gamma_i^{\nu_1 \ldots \nu_j} = \int_k \frac{\partial}{\partial k_{\nu_1}} \frac{k^{\nu_2} \ldots k^{\nu_j}}{(k^2-\mu^2)^{(2+j-1)/2}}, \tag{2.14}$$

with k being the internal momentum and μ an implicit regulator. The general formula allows for the computation of any order surface terms. As examples one has

$$\Gamma_0^{\mu\nu} = \int_k \frac{\partial}{\partial_\mu} \frac{k^\nu}{(k^2-\mu^2)^2} = 4\Big(\frac{g_{\mu\nu}}{4} I_{log}(\mu^2) - I_{log}^{\mu\nu}(\mu^2)\Big), \tag{2.15}$$

$$\begin{aligned}\Gamma_0^{\mu\nu\alpha\beta} &= \int_k \frac{\partial}{\partial_\mu} \frac{k^\nu k^\alpha k^\beta}{(k^2-\mu^2)^3} \\ &= 24\Big((g^{\mu\nu} g^{\alpha\beta} + g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha})\frac{I_{log}(\mu^2)}{24} - I_{log}^{\mu\nu\alpha\beta}(\mu^2)\Big),\end{aligned} \tag{2.16}$$

and

$$\Gamma_2^{\mu\nu} = \int_k \frac{\partial}{\partial_\mu} \frac{k^\nu}{(k^2-\mu^2)} = 2\Big(\frac{g_{\mu\nu}}{2} I_{quad}(\mu^2) - I_{quad}^{\mu\nu}(\mu^2)\Big). \tag{2.17}$$

Setting the surface terms to zero we have $\Gamma_0^{\mu\nu} = \Gamma_0^{\mu\nu\alpha\beta} = \Gamma_2^{\mu\nu} = 0$, and we have the following relations

$$I^{\mu\nu}_{log}(\mu^2) = \frac{g_{\mu\nu}}{4} I_{log}(\mu^2), \tag{2.18}$$

$$I^{\mu\nu\alpha\beta}_{log}(\mu^2) = (g^{\mu\nu}g^{\alpha\beta} + g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha}) \frac{I_{log}(\mu^2)}{24}, \tag{2.19}$$

and

$$I^{\mu\nu}_{quad}(\mu^2) = \frac{g_{\mu\nu}}{2} I_{quad}(\mu^2). \tag{2.20}$$

As previously stated, the BDI's with Lorenz indices can be expressed as a linear combination of BDI's containing only scalar integrals with the same order of divergence multiplying combinations of metric tensors.

2.4.4 Renormalization scale

In a massless theory, μ represents an infrared regulator that is set to zero in the end of the calculation, therefore we need to introduce a positive arbitrary constant λ, mass independent scale, which will play the role of the renormalization group scale. Renormalization functions can be computed using the following regularisation independent identity

$$\lambda^2 \frac{\partial I_{log}(\lambda^2)}{\partial \lambda^2} = \frac{i}{(4\pi)^2}, \tag{2.21}$$

with the following solution

$$I_{log}(\mu^2) = I_{log}(\lambda^2) + b \ln\left(\frac{\lambda^2}{\mu^2}\right), \tag{2.22}$$

where the constant is $b = \dfrac{i}{(4\pi)^2}$. It's worth noting that a minimal subtraction renormalization scheme emerges naturally from this formalism, in which the infinite divergences that depend only on the internal momentum are subtracted from the theory. This means that the $I_{log}(\lambda^2)$ will be subtracted via renormalization whereas the IR divergent part $\ln(\mu^2)$ will cancel in the final amplitude for infrared safe processes and in the cross section/decay rate otherwise, which will be finite.

2.4.5 UV finite integrals

As in previous work, the result of the finite integrals can be determined by hand but since we have a lot of different forms of integrals, we will use here the *Package-X* of software *Mathematica*, [6]. We will explain here the procedure to the particular case of one-loop integrals. To start, we do the input of the one-loop integral. We define the numerator and the integration variable in terms of Passarino-Veltman functions, [17]. The software will then convert the Passarino-Veltman coefficient functions into analytic expressions. We collect the output and do an expansion in a dimensionless constant μ_0 around zero which allows to isolate all terms that are divergent and finite in the limit, and neglect all the higher order terms. This gives the final answer to the evaluation of the integrals.

2.5 Summarizing the procedure

For the sake of simplicity, once all of the above concepts have been introduced, we put down a set of rules in order to manipulate the amplitudes of the Feynman diagrams in the framework of IReg.

- Use Feynman rules of the theory to compute the amplitudes of the theory which consist of some collection of integrals that can depend on internal or external momenta;
- Introduce a regulator μ in all the denominators; when IR divergences do appear, this parameter will allow us to express all IR divergences in terms of $\ln\mu$. On the other hand, if the amplitude is IR safe, it will assure that no spurious IR divergences arise in intermediate steps;
- Apply an algebraic identity as many times as necessary until the UV divergent integrals in the amplitude are entirely separated from the finite ones;
- Express the UV divergent integrals in terms of tensorial BDI's;
- Set the surface terms to zero, and by doing this, the tensorial BDI's can be expressed as scalar integrals with the same order of divergence multiplied by products of metric tensors containing the Lorentz indices;
- By the end of the calculation introduce a renormalization scale λ and set the regulator to zero $\mu \longrightarrow 0$;
- Calculate the value of UV finite integrals, either by hand or with software. In this work we will use the *Package-X* of the software *Mathematica*.

Chapter 3

Spinor-helicity formalism

In this chapter we start by discussing the motivation to introduce modern ways of computing amplitudes and then provide an introduction to the spinor-helicity formalism. The main purpose is to provide some basic notions for dealing with color and spin quantum numbers in order to make the computation of tree-level diagram amplitudes simpler.

3.1 Motivation for the spinor-helicity formalism

The usual method to compute decay rates or cross sections involves computing the amplitude of a Feynman diagram, square it and then sum over the spins and colors (if present) of the external states. This can easily become complex if many Feynman diagrams are present, [18]. This is because the physical theories must be gauge invariant, therefore transformations of the type

$$A_\mu \longrightarrow U A_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger, \tag{3.1}$$

with A_μ as a gauge field, U the unitary matrix and g the coupling constant, must leave the theory invariant. Then when computing the Feynman rules there will be redundancies and although we end up with a simple and compact expression, the intermediate steps are very extensive. This means that we have many Feynman diagrams that are related by gauge invariance, [18]. This redundancy is also reflected in the Lorenz condition that requires $\varepsilon_{p_i} \cdot p_i = 0$, if we do a gauge transformation in the polarization ε_i, the equation above must still hold and the amplitude must be invariant, therefore the gauge transformation $\varepsilon_i \longrightarrow \varepsilon_i + a p_i$ where a is the gauge parameter, also leads to redundancies. Additionally, in non-abelian theories there are to many terms in the Feynman diagrams. As a result, intermediate expressions may become much more complex than the final result and the process of computation becomes difficult. Therefore, one searches a formalism that reduces the complexity of the calculation, [19].

If we consider only the outgoing particles and consider them on-shell, we reduce considerably the number of degrees of freedom and also eliminate the connection to quantised fields and therefore gauge redundancies, [19]. This motivates the usage of an on-shell formalism.

We will explore here how to organize the spin (and helicity) and color in order to simplify the amplitude into gauge invariant pieces.

3.2 Spinor-helicity formalism

In high-energy collision processes, if we consider ultra-relativistic particles, they will behave as if they were massless for whom it is known that chirality and helicity are the same and we have conservation of helicity through the interaction of the particles,[7], [18]. In this context, the helicity basis becomes useful.

The next step is to determine which kinematic variables may be used to write the scattering amplitudes. The Lorentz group is a Lie group of symmetries of space-time of special relativity and can have a variety of representations. The four-vectors p^μ are represented in the $\left(\frac{1}{2},\frac{1}{2}\right)$ Lorentz space and are the usual choice of kinematic variables to define the amplitudes. But there is a smaller representation of the Lorentz group which is the representation $\left(\frac{1}{2},0\right)\bigoplus\left(0,\frac{1}{2}\right)$. This representation of the Lorentz group is the Dirac representation, [7]. The main idea behind this is using constant spinor that transform under the finite dimensional representations of the Lorentz group, so we define the helicity spinor as real or complex doublets that transform under the $\left(\frac{1}{2},0\right)$ and $\left(0,\frac{1}{2}\right)$ representations of the Lorentz group, [20]. So what we do in spinor-helicity formalism is trading our four vector momentum p_i^μ for a pair of spinors as follows

$$\begin{aligned} u_+(p_i) &= |i^+\rangle = \lambda_i^\alpha, \\ u_-(p_i) &= |i^-\rangle = \lambda_i^{\dot\alpha} \end{aligned} \tag{3.2}$$

For massless vectors, these are two-dimensional Weyl spinor and they obey the Dirac massless equation

$$\not{p}u_\pm(p) = 0, \tag{3.3}$$

where $u_+(p_i) = \frac{(1+\gamma_5)}{2}u(p_i)$ and $u_-(p_i) = \frac{(1-\gamma_5)}{2}u(p_i)$ are respectively a right-handed and left-handed 4-component spinors in the Dirac notation and λ_i^α and $\lambda_i^{\dot\alpha}$ are the respective 2 component versions with $\alpha = 1,2$.

To raise and lower the indices we use anti-symmetric tensors of the type

$$\varepsilon^{\alpha\beta} = -\varepsilon_{\alpha\beta} = \varepsilon^{\dot\alpha\dot\beta} = -\varepsilon_{\dot\alpha\dot\beta} = i\sigma_\mu\sigma^\mu = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.4}$$

We can use the Pauli matrices σ to write the momenta as bispinors. For example, for the positive energy solution we have

$$u_+(p)u_+^\dagger(p) = p^{\alpha\dot{\alpha}} = \sigma_\mu^{\alpha\dot{\alpha}} p^\mu = \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & p^0 + p^3 \end{pmatrix}, \tag{3.5}$$

from where we infer the following relations

$$p^{\alpha\dot{\alpha}} = \sigma_\mu^{\alpha\dot{\alpha}} p^\mu, \tag{3.6}$$

$$p_{\alpha\dot{\alpha}} = \bar{\sigma}^\mu_{\dot{\alpha}\alpha} p^\mu. \tag{3.7}$$

The determinant of the matrix of the bispinor is

$$det(p^{\alpha\dot{\alpha}}) = p_0^2 - p_1^2 - p_2^2 - p_3^2 = p_0^\mu p_{0\mu} = m^2. \tag{3.8}$$

If the particles are massless then $det(p^{\alpha\dot{\alpha}}) = 0$. This means that we can factorize the matrix into vectors, which is precisely what we have obtained before by choosing the Weyl basis. In other notation we have

$$\lambda^\alpha = p\rangle, \qquad \lambda_\alpha = \langle p, \qquad \tilde{\lambda}_{\dot{\alpha}} = p], \qquad \tilde{\lambda}^{\dot{\alpha}} = [p, \tag{3.9}$$

where

$$p^{\alpha\dot{\alpha}} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} = p\rangle[p \tag{3.10}$$

and

$$p_{\alpha\dot{\alpha}} = \tilde{\lambda}_{\dot{\alpha}} \lambda_\alpha = p]\langle p. \tag{3.11}$$

The momentum conservation in terms of the spinors using the ket notation can be written as

$$\sum p_i^\mu = \sum p_i^{\alpha\dot{\alpha}} = \sum_j \lambda_j^\alpha \lambda_j^{\dot{\alpha}} = \sum_j j\rangle[j = 0. \tag{3.12}$$

Using the tensor in equation 3.4, we can define the following objects

$$\bar{u}_-(p_i)u_+(p_j) = \varepsilon^{\alpha\beta}(\lambda_i)_\alpha(\lambda_j)_\beta \equiv \langle ij\rangle \tag{3.13}$$

and

$$\bar{u}_+(p_i)u_-(p_j) = \varepsilon^{\dot{\alpha}\dot{\beta}}(\tilde{\lambda}_i)_{\dot{\alpha}}(\tilde{\lambda}_j)_{\dot{\beta}} \equiv [ij]. \tag{3.14}$$

The relation between these objects and the usual Mandelstam variables which we use to compute amplitudes is

$$\langle ij\rangle[ji] = 2p_ip_j = (p_i + p_j)^2 = s_{ij} \equiv s \tag{3.15}$$

and a similar reasoning for the other channels. These are our invariant quantities in terms of the spinors.

We consider now that we are only working with real momenta. Notice that this equation does not completely define the vectors λ, since in the real Minkowski space they are complex conjugates and can differ by a complex phase, [21]. The complex conjugate of the spinors is $[ij] = \langle ij\rangle^*$ which we notice is just a parity transformation. This happens because for the case of real momenta, the complex conjugation is obtained by transposing the matrix $\not{p}_i$, meaning that we exchange left and right handed spinors, [7]. By looking at equation 3.15 we notice the bispinors can be written as follows

$$\langle ij\rangle = \sqrt{s_{ij}}e^{i\phi} \tag{3.16}$$

and

$$[ij] = \sqrt{s_{ij}}e^{-i\phi}, \tag{3.17}$$

and the only difference between them is just a complex phase.

The spinor products also respect anti-symmetry, [7]

$$\langle ij\rangle = -\langle ji\rangle, \qquad [ij] = -[ji], \qquad \langle ii\rangle = [ii] = 0. \tag{3.18}$$

After having defined a notation for the momenta and their invariant scalar products, we now write the polarization vector for a massless gauge boson

$$[\varepsilon_p^-(r)]^{\alpha\dot{\alpha}} = \sqrt{2}\frac{p\rangle[r}{[pr]}, \tag{3.19}$$

$$[\varepsilon_p^+(r)]^{\alpha\dot{\alpha}} = \sqrt{2}\frac{r\rangle[p}{\langle rp\rangle}. \tag{3.20}$$

where we defined a reference momentum r that needs to be chosen so that it can obey the following properties,[22]

$$\begin{cases} \sum_{i=1}^{n} r_i = 0 \\ r_i \cdot r_j = 0 \\ p_i \cdot r_i = 0, \text{ for each i} \end{cases} \tag{3.21}$$

The choice of these reference momentum r is arbitrary, which reflects the freedom of gauge. This means that the amplitudes should be unchanged when the polarization vector is shifted by an amount proportional to the momentum, [7].

We have now established a new notation for our objects and, as the amplitudes should be functions of invariant quantities, the simplest way to do it is to compute scalar products between momenta and polarizations.

All the above formalism is derived in the basis that helicity is conserved and particles are massless. But it is know that vector particles like photons and gluons do not have a conserved helicity. Despite this, in the tree-level reactions, there is conservation of helicity because all the processes that do not conserve it are zero.

3.3 Color decomposition

Working in QCD we have the presence of the color degrees of freedom in the theory. When computing QCD amplitudes using the Feynman rules, we find the presence of the Lie algebra structure constants f^{abc}. These structure constants are defined by the commutator

$$[T^a, T^b] = i\sqrt{2} f^{abc} T^c, \tag{3.22}$$

which allows us to relate the constant structures with the matrices of the fundamental representation T^a

$$f^{abc} = \frac{i}{\sqrt{2}}(Tr(T^a T^b T^c) - Tr(T^a T^c T^b)) \tag{3.23}$$

We can now use this equation to eliminate the structure constants in the amplitude in favour of these traces of the matrices T^i. Furthermore, products of traces can be simplified using the Fierz identity, [18]

$$\sum_a (T^a)_{i_1}^{j_1} (T^a)_{i_2}^{j_2} = \delta_{i_1}^{j_2} \delta_{i_2}^{j_1} - \frac{1}{N_c} \delta_{i_1}^{j_1} \delta_{i_2}^{j_2} \tag{3.24}$$

This equation means that the generators T^a of the $SU(N_C)$ group form a complete set of traceless hermitian matrices $N_C \times N_C$, [7], [18]. More details and discussion are provided in the appendices.

3.4 The amplitude in the spinor-helicity formalism

After the algebraic manipulation of the colour algebra, we have an amplitude that is a function of the momenta p_i and the polarization of the particles ε_i. In the spinor helicity formalism what we do is a transformation to a space where the amplitudes are a function of spinor $u(p)$ and helicity h_i. It is here that the major simplification is done as we eliminate the redundancies coming from the Lorenz condition and the gauge invariance of the theories.

Consider a process involving the Higgs plus some number n of gluons. We will see that the Feynman rules of the vertices involving three or four gluons translate to the Yang-Mills' multiplied by some constant. Therefore, the form of the amplitudes will be the same as for the pure YM theory. For this case, the color decomposition of the $Higgs-n-gluon$ tree amplitude is

$$M_n^{tree}(p_i, h_i, a_i) = g^{n-2} \sum_{\varepsilon \in S_n/Z_n} \mathrm{Tr}(T_{\varepsilon_1}^{a_{\varepsilon_1}} \ldots T_{\varepsilon_n}^{a_{\varepsilon_n}}) M_n(\varepsilon_1^1 \ldots \varepsilon_n^n). \tag{3.25}$$

M_n^{tree} is the partial amplitude with the kinematic variables. The notation is g as the gauge coupling, $p_i \equiv i$ as the momenta, h_i the helicity, a_i the color and ε the polarization, S_n is the set of all permutations of n objects and Z_n is the subset of cyclic permutations that preserves the trace. By doing the sum of all polarization that belong to S_n/Z_n, what we are doing is sweeping out all cyclically nonequivalent orderings in the trace. This means that the amplitude is color ordered and is very important, since contributions from some diagrams with a particular cyclic ordering of the gluons are taken into account, [7], [18].

3.5 Little group covariance

The little group is the ISO(2) group of Lorentz transformations which leaves the outgoing particle momenta invariant. In terms of bispinors notation we have that the objects $p\rangle[p$ and $p]\langle p$ are invariant.

This can also be interpreted as the overall phases of the spinors not being determined by the Dirac equation, so they can be arbitrarily chosen. Due to this freedom of choice, in terms of bispinors, we have that the above expressions are invariant under the transformations

$$p\rangle \longrightarrow zp\rangle, \quad [p \longrightarrow \frac{1}{z}[p \tag{3.26}$$

The requirement of the little group covariance in a scattering amplitude takes the generic form

$$M(p_1^{h_1}...p_n^{h_n}) \longrightarrow \Pi_{j=1}^{n} z_j^{-2h_j} M(p_1^{h_1}...p_n^{h_n}). \tag{3.27}$$

Notice that the amplitude is covariant and not invariant under the little group. Nevertheless z_j are phases that cancel when we do the modulus squared of the amplitudes, which will remain invariant.

Since momenta and reference momenta are invariant under little group, only the polarizations play a part in determining the amplitude here, since we have

$$\varepsilon_p^- = \sqrt{2}\frac{p\rangle[r}{[pr]} \longrightarrow z^2 \varepsilon_p^-(r) \tag{3.28}$$

and

$$\varepsilon_p^+ = \sqrt{2}\frac{r\rangle[p}{\langle rp\rangle} \longrightarrow z^{-2} \varepsilon_p^+(r). \tag{3.29}$$

This will therefore diminish the range of forms of the amplitudes we can accept for a certain process with certain helicities.

Chapter 4

The effective model for the decay H⟶gg

4.1 Effective Field Theories

An EFT is a quantum theory that has a regularization and renormalization scheme. Its key idea is that the dynamics at low energies given by the scale m does not depend on the dynamics at high energies given by the scale Λ. As a result, we can define a power counting parameter $\delta = \frac{m^2}{\Lambda^2}$ and perform the computations as an expansion to some order n in δ. To better understand the power counting we start by defining the EFT functional integral as

$$F = \int D\phi e^{iS}, \tag{4.1}$$

with the action being

$$S = \int d^d x L(x). \tag{4.2}$$

L is the Lagrangian density with energy dimension d, $[L(x)] = d$ given by the sum of the products of coefficients c_i of dimension d-D with Lorentz invariant operators O_i of dimension D

$$L(x) = \sum_i c_i O_i(x). \tag{4.3}$$

Now following the same logic as for the Lagrangian density, the EFT Lagrangian can be expanded as

$$L_{EFT} = \sum_{D \geqslant 0, i} \frac{c_i(D) O_i(D)}{\Lambda^{D-d}} = \sum_{D \geqslant 0} \frac{L_D}{\Lambda^{D-d}}, \tag{4.4}$$

where D is the dimension of the higher order operator, d is the dimension of the Lagrangian of the theory and c_i are Wilson coefficients that are the coefficients of the expansion and contain all information about short-distance physics above the scale m. The energy scale Λ is the scale at which new physics occurs and guarantees that even if higher order operators are added to the theory, the full Lagrangian remains at dimension d, which means that the $c_i^{'}s$ are dimensionless. For $d = 4$ we have the following

$$L_{EFT} = L_{D\leqslant 4} + \frac{L_5}{\Lambda} + \frac{L_6}{\Lambda^2} + ... \tag{4.5}$$

We notice that one does not stop at $D = d$, but includes operators of arbitrarily higher dimension.

EFT can be of two distinct types: bottom-up and top-down. The first is applied when the underlying theory is known and we write the most general Lagrangian that is consistent with the symmetries of the system. The top-down EFT is applied when the full theory is known and we wish to have a Lagrangian to describe a low energy system; in this case we integrate out the heavy particles.

There are advantages in using EFT, such as the simplification of the computations, as we are only dealing with the relevant interactions to the energy scale that we wish to study. Also, when constructing an EFT, the EFT Lagrangian reproduces the same S-matrix as the original theory and therefore the observables in both theories must coincide.

So as we have seen EFT provide an efficient method to characterize new physics using operators of higher dimension, so we will look for a way to construct a Lagrangian that provides us an accurate form to study the system of this work: the Higgs decay into gluons. This discussion is extensively exposed in [23].

4.2 The effect of heavy quarks in Higgs boson decay

We are now in a position to derive the Lagrangian of our model. In general

$$L = L_{YM} + L_{int} + L_{Higgs} \tag{4.6}$$

where L_{YM} is the Yang-Mills Lagrangian, L_{Higgs} represents the kinetic part of the Higgs Lagrangian and L_{int} describes the effective interaction of the Higgs with the gluons, where the heavier degrees of freedom (quarks) have been integrated out. For our discussion here the relevant parts will be the YM and the effective interaction. We can write the interaction part as an expansion in a power of series as in [24],

$$L_{int} = -H \sum_i c_i O_i. \tag{4.7}$$

Here, H stands for the Higgs field, c_i are the coefficient of the expansion and O_i are local operators. The problem now is to find the local operators with the correct dimensions and their corresponding coefficients. This problem has been studied by *Kazama and Yao*, [25] and the results for this case have been reported in [24]. So we have the contribution of the following operator

$$O_1 = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a \tag{4.8}$$

and adding its dimension with the Higgs field dimension, we have a total operator of dimension 5.

4.3 Higgs decays

The Higgs boson's decay branching ratios are dependent on the energy scale being considered, i.e. the Higgs mass. The Higgs boson can decay into a fermion and an anti-fermion, with the strength of the interaction proportional to the fermion mass. The final decay rate is proportional to the square of the strength of the interaction and grows linearly with the Higgs mass. In this case, the most important final states will be $b\bar{b}$, $\tau\tau$ and $c\bar{c}$, the first being the largest one in the range of mass $80 GeV$ to $200 GeV$ as we can see in figure 4.1. Also, there are decays of the Higgs into bosons W^+W^- and ZZ, which are proportional to the square of the coupling. Loop-induced decays, such as gg, $\gamma\gamma$, and $Z\gamma$, are also present. The $h \longrightarrow \gamma\gamma$ and the $h \longrightarrow Z\gamma$ are dominated by the W boson loop, whereas the $h \longrightarrow gg$ decay is dominated by the top quark loop, with a small contribution from the bottom quark loop, which will be the focus of this work. All this discussion is extensively exposed in [26].

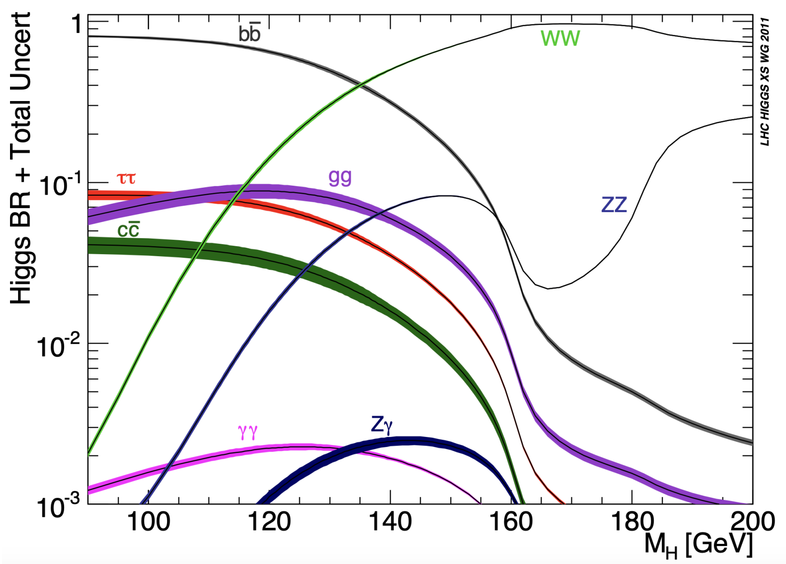


Figure 4.1: Higgs branching ratios and total uncertainty at low mass range. We study the limit where the Higgs mass is between $80 GeV$ and $200 GeV$. SOURCE: [1]

4.4 Effective model to describe the Higgs coupling to gluons

Consider the Higgs decay into two jets of gluons in the mass range $80GeV < m_H < 200GeV$. We have seen that this occurs via a top quark loop. But if we take the top quark mass to be infinite, this limit allows us to integrate out the top quark degree of freedom, replacing the one-loop coupling of the Higgs to the gluons with a top quark loop by an effective local, gauge invariant operator $HG^a_{\mu\nu}G^{a,\mu\nu}$. This will reduce the number of loops by one at each order, [27] and also, by taking an infinite mass for the quarks we may have underestimated the decay rate, but the effective Lagrangian may still give a reliable estimate for these corrections in the range of mass of the Higgs we are considering, [28]. The effective interaction of one Higgs with two, three and four gluons is described by the Lagrangian,

$$L_{eff} = -\frac{1}{4}AHG^a_{\mu\nu}G^{a,\mu\nu} \tag{4.9}$$

that can also be found in [8], [27] and [29].

$G^a_{\mu\nu}$ is the field strength of the $SU(3)$ gluon field given by

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \tag{4.10}$$

and f^{abc} are the anti-symmetric $SU(3)$ structure constants.

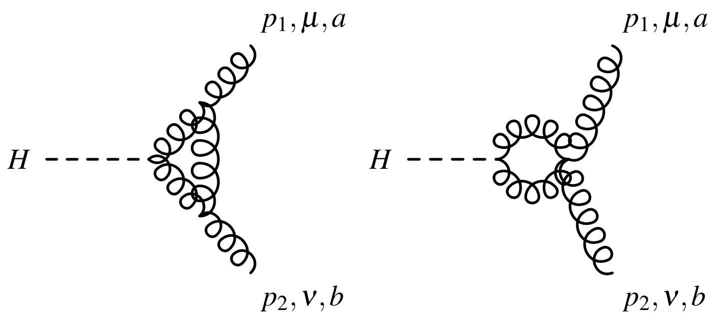
The effective coupling A is given by

$$A = \frac{\alpha_s}{3\pi v}\left(1+\frac{11}{4}\frac{\alpha_s}{\pi}\right), \tag{4.11}$$

where H represents the Higgs boson field and v is the vacuum expectation value, $v^2 = \frac{1}{G_f\sqrt{2}}$. We can relate the effective coupling with the strong coupling constant, $\alpha_s = \frac{g^2}{4\pi}$.

The virtual diagrams that arise from this effective Lagrangian are generated using the package *FeynArts* of the software *Mathematica*, [30]. There are 5 diagrams that contribute to the one-loop order correction, which are represented in figure 4.2.

Virtual diagrams



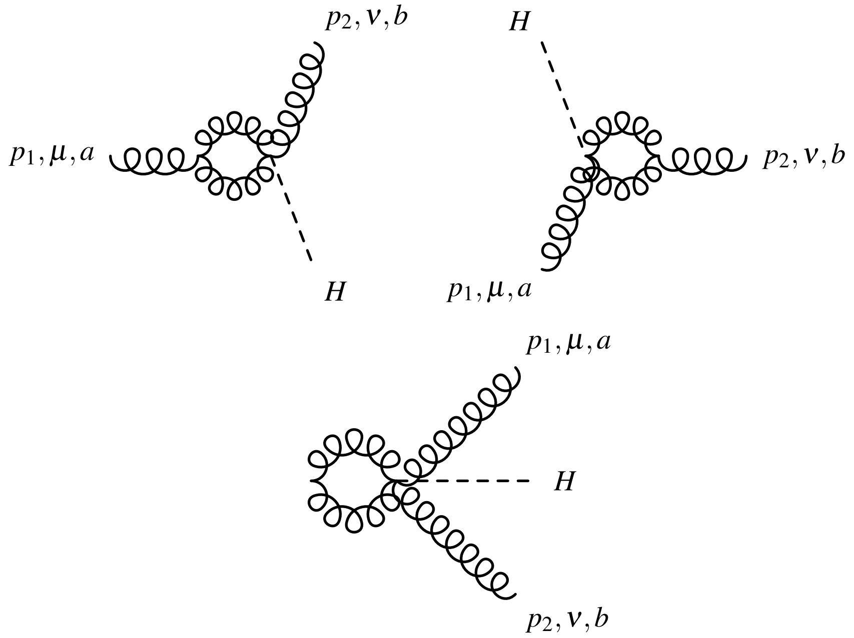


Figure 4.2: Virtual diagrams contribution to the decay rate $H \longrightarrow gg(g)$. From left to right they are respectively V_1,V_2,V_3,V_4,V_5. The dashed line represents the Higgs field, the curly lines represent the gluon field.

Real diagrams

The diagrams that will contribute for the real decay until the α_s order are represented in figure 4.3.

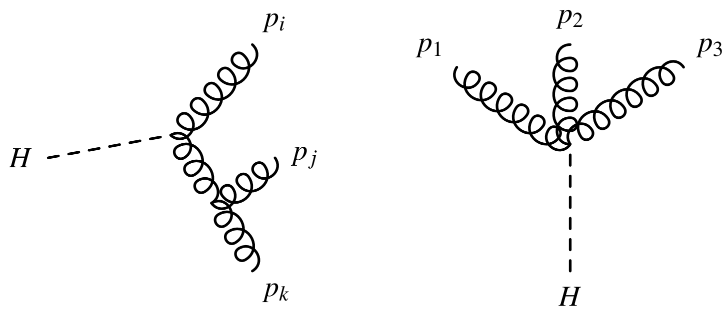


Figure 4.3: Real diagrams contribution to the decay $H \longrightarrow ggg$. From left to right they are respectively R_1 and R_2. The dashed line represents the Higgs field and the curly lines represent the gluon field. The $\{p_i, p_j, p_k\}$ correspond to the three permutations of p_i, p_j and p_k, so R_1 stands for 3 diagrams.

The diagrams V_1 to V_5 contribute to the virtual amplitude and the diagrams R_1 and R_2 (which corresponds to three diagrams) contribute to the real amplitude at the same order of expansion.

4.5 Feynman rules for the effective Lagrangian

By looking at the previous diagrams, notice that the interactions between the Higgs boson and the gluons arise from the effective interaction while the interactions between only gluons arise from the YM pure lagrangian. In order to evaluate all the previous diagrams, we need to compute the Feynman rules for the effective interaction *Hgg* and the pure YM lagrangian that will be the building blocks for the amplitudes. We will compute the Feynman rules for the *2-gluon-Higgs*, *3-gluon-Higgs* and *4-gluon-Higgs* with the effective interaction and the *3-gluon*, *4-gluon* interactions and the ghost vertex with the YM lagrangian. We start by deriving the Feynman rules for the effective interaction. The S-matrix can be written as

$$S = \langle 0|T(e^{-i\int d^4x L_{eff}(x)})|q p_1^{\alpha,a'} p_2^{\beta,b'}\rangle, \tag{4.12}$$

where $|p_1^{\alpha,a'}\rangle, |p_2^{\beta,b'}\rangle$ are the states of the external gluons with color indices a' and b' and $|q\rangle$ represents the state of the Higgs boson. To derive each Feynman rule we extract from the Lagrangian the interaction term that describes the diagram we are interested in and we expand the exponential in the S-matrix until the first order, and perform the contractions between operators and states. Also, for convention all the external momenta are going inwards.

2-gluon-Higgs Feynman rule

We start by computing the *2-gluon-Higgs* Feynman rule represented in figure 4.4.

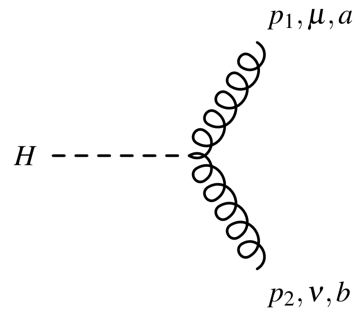


Figure 4.4: *2-gluon-Higgs* Feynman rule.

The part of the lagrangian that gives origin to this rule is

$$\begin{aligned} L_{eff}^{2gH} &= -\frac{1}{4} AH(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}) \\ &= -\frac{1}{2} AH(\partial_\mu A_\nu^a \partial^\mu A^{\nu a} - \partial_\nu A_\mu^a \partial^\mu A^{\nu a}). \end{aligned} \tag{4.13}$$

Expanding the exponential until first order we get

$$\langle 0|T(-i(-\frac{1}{4}AH)2(\partial_\mu A^a_\nu \partial^\mu A^{\nu a} - \partial_\nu A^a_\mu \partial^\mu A^{\nu a}))|e^{-ix(q+p_1+p_2)}\varepsilon^{a\prime}_\alpha(p_1)\varepsilon^{b\prime}_\beta(p_2)\rangle. \tag{4.14}$$

We sum for all the possibilities of the operators on the effective Lagrangian acting on the polarizations $\varepsilon(p_i)$. We get the following

$$= -\frac{1}{4}AH\delta^{aa'}\delta^{ab'}(4g_{\nu\alpha}g_{\nu\beta}(-ip_{1\mu})(-ip_{2\mu}) - 4g_{\mu\alpha}g_{\nu\beta}(-ip_{1\mu})(-ip_{2\nu})). \tag{4.15}$$

Contracting all the indices we get the *2-gluon-Higgs Feynman rule*

$$\delta^4(q+p_1+p_2)iA\delta^{a'b'}H^{\alpha\beta}(p_1,p_2) \tag{4.16}$$

where the tensor is $H^{\alpha,\beta}(p_1,p_2) = -p_1^\beta p_2^\alpha + g^{\alpha\beta}p_1\cdot p_2$.

3-gluon-Higgs Feynman rule

The *3-gluon-Higgs Feynman rule* is represented in figure 4.5.

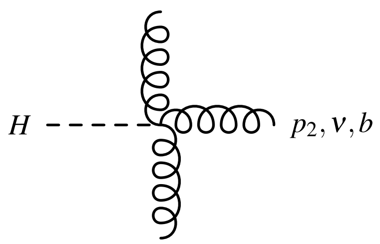


Figure 4.5: *3-gluon-Higgs* Feynman rule.

The S matrix is written as follows

$$\langle 0|T(e^{-i\int d^4xL_{eff}(x)})|qp_1^{\alpha,a'}p_2^{\beta,b'}p_3^{\gamma,c'}\rangle \tag{4.17}$$

up to the normalization with $|p_1^{\alpha,a'}\rangle, |p^{\beta,b'_2}\rangle, |p^{\gamma,c'_3}\rangle$ as the states of the external gluons with color indices a', b' and c' and $|q\rangle$ as the Higgs boson. The part of the effective Lagrangian that gives origin to this rule is

$$L^{3gH}_{eff} = -gAHf^{abc}(\partial_\mu A^a_\nu)A^{\mu,b}A^{\nu,c}. \tag{4.18}$$

Again we expand the exponential until first order and we get

$$\langle 0|T(-i(-gAHf^{abc}(\partial_\mu A^a_\nu)A^{\mu,b}A^{\nu,c})|e^{-ix(q+p_1+p_2+p_3)}\varepsilon^{a\prime}_\alpha(p_1)\varepsilon^{b\prime}_\beta(p_2)\varepsilon^{c\prime}_\gamma(p_3)\rangle. \tag{4.19}$$

In this case we have $3! = 6$ different ways of the operators acting on the polarizations $\varepsilon(p_i)$. By summing for all the possibilities we get

$$\begin{aligned}
&= e^{-i(q+p_1+p_2+p_3)} f^{abc} A \times \\
&\big(\delta^{aa'} g^{\nu\alpha}(-ip_1^{\mu})(\delta^{bb'}\delta^{cc'} g^{\nu\beta} g^{\nu\gamma} + \delta^{bc'}\delta^{cb'} g^{\mu\gamma} g^{\nu\beta}) \\
&+\delta^{ab'} g^{\nu\beta}(-ip_2^{\mu})(\delta^{ba'}\delta^{cc'} g^{\mu\alpha} g^{\nu\gamma} + \delta^{bc'}\delta^{ca'} g^{\mu\gamma} g^{\nu\alpha}) \\
&+\delta^{ac'} g_{\nu\gamma}(-ip_3^{\mu})(\delta^{ba'}\delta^{cb'} g^{\mu\alpha} g^{\nu\beta} + \delta^{bb'}\delta^{ca'} g^{\mu\beta} g^{\nu\alpha})\big)
\end{aligned} \tag{4.20}$$

By contracting all the indices we get the *3-gluon-Higgs* Feynman rule

$$\delta^4(q+p_1+p_2+p_3)(-Agf^{abc}V^{\alpha\beta\gamma}(p_1,p_2,p_3)), \tag{4.21}$$

where the tensor is

$$V^{\alpha\beta\gamma}(p_1,p_2,p_3) = (p_1-p_2)^{\gamma} g^{\alpha\beta} + (p_2-p_3)^{\alpha} g^{\beta\gamma} + (p_3-p_1)^{\beta} g^{\alpha\gamma}. \tag{4.22}$$

4-gluon-Higgs Feynman rule

The *4-gluon-Higgs Feynman rule* is represented in figure 4.6.

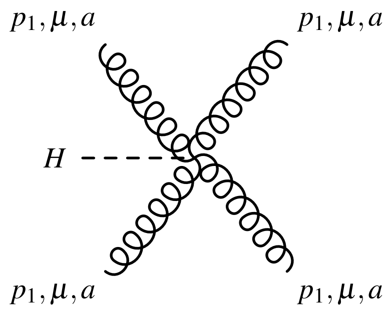


Figure 4.6: *4-gluon-Higgs* Feynman rule.

$$\langle 0|T(e^{-i\int d^4x L_{eff}(x)})|qp^{a'}_{1\mu'}p^{b'}_{2\nu'}p^{c'}_{3\rho'}p^{d'}_{4\tau'}\rangle \tag{4.23}$$

where $|p^{a'}_{1\mu'}\rangle, |p^{b'}_{2\nu'}\rangle, |p^{c'}_{3\rho'}\rangle, |p^{d'}_{4\tau'}\rangle$ are the states of the external gluons with color indices a', b', c', and d' and $|q\rangle$ represents Higgs boson field. The part of the Lagrangian that gives origin to this rule is

$$L^{4gH}_{eff} = -\frac{1}{4} g^2 AH(f^{eab}A^a_{\mu}A^b_{\nu})(f^{ecd}A^c_{\mu}A^d_{\nu}). \tag{4.24}$$

By expanding the exponential in the S-matrix until first order follows

$$
\begin{aligned}
&\langle 0|T(-i\int d^4x(-\frac{1}{4}g^2AH(f^{eab}A^a_\mu A^b_\nu))(f^{ecd}A^c_\mu A^d_\nu) \\
&|e^{-iqx}e^{-ip_1x}e^{-ip_2x}e^{-ip_3x}e^{-ip_4x}\varepsilon^{a'}_{\mu'}(p_1)\varepsilon^{b'}_{\nu'}(p_2)\varepsilon^{c'}_{\rho'}(p_3)\varepsilon^{d'}_{\tau'}(p_4)\rangle.
\end{aligned}
\tag{4.25}
$$

In this case we have $4! = 24$ different ways for the operators to act on the states and we sum for all the possibilities which gives the following

$$
\begin{aligned}
= f^{eab}f^{edc}(&g^{aa'}_{\mu\mu'}g^{bb'}_{\nu\nu'}g^{cc'}_{\mu\rho'}g^{dd'}_{\nu\tau'} + g^{aa'}_{\mu\mu'}g^{bb'}_{\nu\nu'}g^{cd'}_{\mu\tau'}g^{dc'}_{\nu\rho'} + g^{aa'}_{\mu\mu'}g^{bc'}_{\nu\rho'}g^{cb'}_{\mu\nu'}g^{dd'}_{\nu\tau'} \\
&+g^{aa'}_{\mu\mu'}g^{bc'}_{\nu\rho'}g^{cd'}_{\mu\tau'}g^{db'}_{\nu\nu'} + g^{aa'}_{\mu\mu'}g^{bd'}_{\nu\tau'}g^{cb'}_{\mu\nu'}g^{dc'}_{\nu\rho'} + g^{aa'}_{\mu\mu'}g^{bd'}_{\nu\tau'}g^{cc'}_{\mu\rho'}g^{db'}_{\nu\nu'} \\
&+g^{ab'}_{\mu\nu'}g^{ba'}_{\nu\mu'}g^{cc'}_{\mu\rho'}g^{dd'}_{\nu\tau'} + g^{ab'}_{\mu\nu'}g^{ba'}_{\nu\mu'}g^{cd'}_{\mu\tau'}g^{dc'}_{\nu\rho'} + g^{ab'}_{\mu\nu'}g^{bc'}_{\nu\rho'}g^{cd'}_{\mu\tau'}g^{da'}_{\nu\mu'} \\
&+g^{ab'}_{\mu\nu'}g^{bc'}_{\nu\rho'}g^{ca'}_{\mu\mu'}g^{dd'}_{\nu\tau'} + g^{ab'}_{\mu\nu'}g^{bd'}_{\nu\tau'}g^{ca'}_{\mu\mu'}g^{dc'}_{\nu\rho'} + g^{ab'}_{\mu\nu'}g^{cd'}_{\nu\tau'}g^{cc'}_{\mu\rho'}g^{da'}_{\nu\mu'} \\
&+g^{ac'}_{\mu\rho'}g^{ba'}_{\nu\mu'}g^{cb'}_{\mu\nu'}g^{dd'}_{\nu\tau'} + g^{ac'}_{\mu\rho'}g^{ba'}_{\nu\mu'}g^{cd'}_{\mu\tau'}g^{db'}_{\nu\nu'} + g^{ac'}_{\mu\rho'}g^{bb'}_{\nu\nu'}g^{ca'}_{\mu\mu'}g^{dd'}_{\nu\tau'} \\
&+g^{ac'}_{\mu\rho'}g^{bb'}_{\nu\nu'}g^{cd'}_{\mu\tau'}g^{da'}_{\nu\mu'} + g^{ac'}_{\mu\rho'}g^{bd'}_{\nu\tau'}g^{ca'}_{\mu\mu'}g^{db'}_{\nu\nu'} + g^{ac'}_{\mu\rho'}g^{bd'}_{\nu\tau'}g^{cb'}_{\mu\nu'}g^{da'}_{\nu\mu'} \\
&+g^{ad'}_{\mu\tau'}g^{ba'}_{\nu\mu'}g^{cb'}_{\mu\nu'}g^{dc'}_{\nu\rho'} + g^{ad'}_{\mu\tau'}g^{ba'}_{\nu\mu'}g^{cc'}_{\mu\rho'}g^{db'}_{\nu\nu'} + g^{ad'}_{\mu\tau'}g^{bb'}_{\nu\nu'}g^{ca'}_{\mu\mu'}g^{dc'}_{\nu\rho'} \\
&+g^{ad'}_{\mu\tau'}g^{bb'}_{\nu\nu'}g^{cc'}_{\mu\rho'}g^{da'}_{\nu\mu'} + g^{ad'}_{\mu\tau'}g^{bc'}_{\nu\rho'}g^{ca'}_{\mu\mu'}g^{db'}_{\nu\nu'} + g^{ad'}_{\mu\tau'}g^{bc'}_{\nu\rho'}g^{cb'}_{\mu\nu'}g^{da}_{\nu\mu'}).
\end{aligned}
\tag{4.26}
$$

By contracting all the indices we get the 4-gluon-Higgs Feynman rule

$$
\delta^4(q+p_1+p_2+p_3)(-ig^2)X^{a'b'c'd'}_{\mu'\nu'\rho'\tau'},
\tag{4.27}
$$

with the tensor being

$$
\begin{aligned}
X^{a'b'c'd'}_{\mu'\nu'\rho'\tau'} =& f^{a'b'e}f^{c'd'e}(g_{\mu'\rho'}g_{\nu'\tau'} - g_{\mu'\tau'}g_{\nu'\rho'}) \\
&+f^{a'c'e}f^{b'd'e}(g_{\mu'\nu'}g_{\rho'\tau'} - g_{\mu'\tau'}g_{\rho'\nu'}) \\
&+f^{a'd'e}f^{b'c'e}(g_{\mu'\nu'}g_{\tau'\rho'} - g_{\mu'\rho'}g_{\nu'\tau'}).
\end{aligned}
\tag{4.28}
$$

4.6 Yang-Mills Feynman rules

We find the vertices by writing explicitly the nonlinear terms in the YM Lagrangian, which can be found in [31]

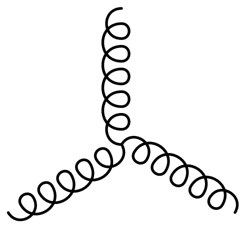
$$
L = L_0 - gf^{abc}(\partial_k A^a_\lambda)A^{kb}A^{\lambda c} - \frac{1}{4}g^2(f^{eab}A^a_k A^b_\lambda)(f^{ecd}A^c_k A^d_\lambda)
\tag{4.29}
$$

where L_0 is the free field Lagrangian and we choose the convention of all momenta going inwards. The procedure to compute the Feynman rules is very similar to the previous one and they read as follows.

3-gluon Feynman rule

The *3-gluon Feynman rule* is represented in figure 4.7

p_3, γ, c



p_1, μ, a p_2, ν, b

Figure 4.7: 3-gluon Feynman rule.

and can be read.

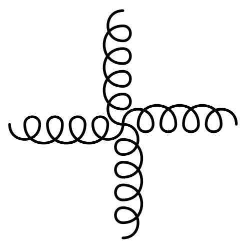
$$V^{3g} = gf^{abc}(g^{\mu\nu}(p_1 - p_2)^\rho + g^{\nu\rho}(p_2 - p_3)^\mu + g^{\rho\mu}(p_3 - p_1)^\nu) \tag{4.30}$$

4-gluon Feynman rule

The *4-gluon Feynman rule* is represented in figure 4.8

p_1, μ, a

p_4, σ, d



p_2, ν, b

p_3, ρ, c

Figure 4.8: 4-gluon Feynman rule.

and reads as follows

$$\begin{aligned} V^{4g} = ig^2(&f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ &f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ &f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})). \end{aligned} \tag{4.31}$$

Feynman rules as the building blocks for the amplitudes

We notice that an advantage of using this effective model is that the structure of the Feynman rules of *3-gluon-Higgs* and *4-gluon-Higgs* is identical to the Feynman rules for *3-gluon* and *4-gluon* derived from pure YM theory.

The Feynman rules derived above from the effective Lagrangian and pure YM Lagrangian are the building blocks for the the Feynman diagrams of the theory. They allow to compute the amplitudes of the virtual and real diagrams exposed above, as we will see in the next chapters.

Chapter 5

Virtual decay rate H⟶gg

In this chapter we compute the one-loop amplitude of each of the virtual diagrams, from V_1 to V_5, using the Feynman rules of the effective lagrangian derived in chapter 4. After this we use the fundamental identity of IReg to completely separate the UV divergent integrals from the UV finite ones. The UV divergent integrals are written as BDI's and the UV finite ones are evaluated using *Package-X* of the software *Mathematica*. We perform renormalization of the effective theory and after that we compute the virtual decay rate of the process $H \longrightarrow gg$.

5.1 Computation of the amplitudes and separation of the UV divergent content

For all the diagrams we choose the external momenta of the two gluons to be p_1 and p_2 and the momentum of the Higgs boson to be q and the internal momentum of the loop to be k. All the external momenta are inwards, therefore we can write the equation of momentum-energy conservation as $p_1 + p_2 + q = 0$ which is valid for any of the diagrams. Because the Feynman rules don't provide us with the symmetry factor of the diagrams, they were extracted from the package *FeynArts* of the software *Mathematica*. In the end of the computation we apply the on-shell conditions and impose $p_1^2 = p_2^2 = 0$. All the diagrams that we analyse are of order one-loop, therefore we use the algebraic identity of IReg given in equation 2.8. As we have two external momenta we have

$$\frac{1}{(k-p_i)^2-\mu^2} = \frac{1}{(k^2-\mu^2)} + \frac{(p_i^2-2p_i\cdot k)}{(k^2-\mu^2)[(k-p_i)^2]-\mu^2} \tag{5.1}$$

where $i = 1,2$, will be used to separate the UV divergent content from the finite in all the amplitudes of the diagrams V_1 to V_5.

We will use the Feynman Gauge and the Feynman rules used are the ones derived in chapter 4. The propagators have the form $\Delta(k)^{ab}_{\mu\nu} = \dfrac{-i\delta^{ab}g^{\mu\nu}}{k^2}$ with *a,b* being color indices and μ, ν being Lorenz indices and k as the momentum in the loop.

Diagram V_1

To obtain to total amplitude of the diagram V_1 we notice that we need two *3-gluon* vertices and the *2-gluon-Higgs* vertex. The propagators are $\Delta(k)^{cc'}_{\rho\rho'}$, $\Delta(k+p_2)^{d''d'''}_{\tau''\tau'''}$ and $\Delta(k-p_1)^{dd'}_{\tau\tau'}$. The symmetry factor is 1, so we have

$$V_1 = V^{3g}_{subV_1} V^{3g}_{subV_1} V^{2g-H}_{subV_1} \Delta(k)^{cc'}_{\rho\rho'} \Delta(k+p_2)^{d''d'''}_{\tau''\tau'''} \Delta(k-p_1)^{dd'}_{\tau\tau'} \tag{5.2}$$

for the *3-gluon* vertices we have respectively

$$V^{3g}_{subV_1} = gf^{cad}(g^{\mu\rho}(-k-p_1)^{\tau} + g^{\mu\tau}(2p_1-k)^{\rho} + g^{\rho\tau}(2k-p_1)^{\mu}), \tag{5.3}$$

$$V^{3g}_{subV_1} = gf^{c'bd''}(g^{\rho'\nu}(k-p_2)^{\tau''} + g^{\nu\tau''}(2p_2+k)^{\rho'} + g^{\tau''\rho''}(-2k-p_2)^{\nu}) \tag{5.4}$$

and for the *2-gluon-Higgs* vertex

$$V^{2g-H}_{subV_1} = iA\delta^{d'd'''}(g^{\tau'\tau'''}(k-p_1)\cdot(k+p_2) - (k-p_1)^{\tau'''}(k+p_2)^{\tau'}). \tag{5.5}$$

Expanding the expression and contracting all indices we obtain

$$\begin{aligned}
V_1 = -Ag^2 C_A \delta^{ab} \int_k \frac{1}{k^2(k-p_1)^2(k+p_2)^2} \\
\Big(& k^4 g^{\mu\nu} + 2(k\cdot p_1)^2 g^{\mu\nu} + 2(k\cdot p_2)^2 g^{\mu\nu} + 4(p_1\cdot p_2)^2 g^{\mu\nu} + 11k^{\mu}k^{\nu}k^2 - 6p_1^{\mu}k^{\nu}k^2 \\
& - k^2 p_2^{\mu} k_{\nu} + k^2 k^{\mu} p_1^{\nu} + k^2 p_1^{\mu} p_1^{\nu} + 9k^2 p_2^{\mu} p_1^{\nu} + 6k^2 k^{\mu} p_2^{\nu} - 3k^2 p_1^{\mu} p_2^{\nu} \\
& + k^2 p_2^{\mu} p_2^{\nu} - 10(k\cdot p_1)k^{\mu}k^{\nu} + (k\cdot p_1)p_1^{\mu}k^{\nu} - 6(k\cdot p_1)p_2^{\mu}k^{\nu} - 2(k\cdot p_1)k^{\mu}p_1^{\nu} \\
& - 6(k\cdot p_1)p_2^{\mu}p_1^{\nu} - 6(k\cdot p_1)k^{\mu}p_2^{\nu} + (k\cdot p_1)p_1^{\mu}p_2^{\nu} - 3(k\cdot p_1)p_2^{\mu}p_2^{\nu} \\
& - 3k^2(k\cdot p_1)g^{\mu\nu} + 10(k\cdot p_2)k^{\mu}k^{\nu} - 6(k\cdot p_2)p_1^{\mu}k^{\nu} - 2(k\cdot p_2)p_2^{\mu}k^{\nu} - 6(k\cdot p_2)k^{\mu}p_1^{\nu} \\
& + 3(k\cdot p_2)p_1^{\mu}p_1^{\nu} + 6(k\cdot p_2)p_2^{\mu}p_1^{\nu} + (k\cdot p_2)k^{\mu}p_2^{\nu} - (k\cdot p_2)p_1^{\mu}p_2^{\nu} + 3k^2(k\cdot p_2)g^{\mu\nu} \\
& + 4p_1^2 k^{\mu}k^{\nu} + 4p_1^2 p_2^{\mu}k^{\nu} + 2p_1^2 k^{\mu}p_2^{\nu} + 2p_1^2 p_2^{\mu}p_2^{\nu} - (p_1\cdot p_2)p_1^{\mu}k^{\nu} \\
& - 4(p_1\cdot p_2)p_2^{\mu}p_1^{\nu} + (p_1\cdot p_2)k^{\mu}p_2^{\nu} - (p_1\cdot p_2)p_1^{\mu}p_2^{\nu} - 9(p_1\cdot p_2)k^2 g^{\mu\nu} + 6(p_1\cdot p_2)(k\cdot p_1)g_{\mu\nu} \\
& - 6(p_1\cdot p_2)(k\cdot p_2)g_{\mu\nu} + 4p_2^2 k^{\mu}k^{\nu} - 2p_2^2 p_1^{\mu}k^{\nu} - 4p_2^2 k^{\mu}p_1^{\nu} + 2p_2^2 p_1^{\mu}p_1^{\nu} \Big).
\end{aligned} \tag{5.6}$$

Equation 5.6 gives us the amplitude before any treatment of divergences. We now apply the identity to separate the divergences. Despite having many terms, the integrals in the previous equation can all be reduced to 7 types that are given explicitly and computed in appendix A. We obtain the following regularised amplitude:

$$
\begin{aligned}
V_1 &= -Ag^2C_A\delta^{ab}\int_k \Bigg(-10(k\cdot(p_1-p_2)k^\mu k^\nu I^1_{V_1}(k,p_1,p_2,\mu) \\
&+ \Big(2g^{\mu\nu}((k\cdot p_1)^2+(k\cdot p_2)^2)+(k\cdot p_1)p_1^\mu k^\nu - 6(k\cdot p_1)(p_2^\mu k^\nu + k^\mu p_2^\nu) - 2(k\cdot p_1)k^\mu p_1^\nu \\
&- 6(k\cdot p_2)(p_1^\mu k^\nu + k^\mu p_1^\nu) - 2(k\cdot p_2)p_2^\mu k^\nu + (k\cdot p_2)k^\mu p_2^\nu\Big) I^2_{V_1}(k,p_1,p_2,\mu) \\
&+ \Big(-6(k\cdot p_1)p_2^\mu p_1^\nu + (k\cdot p_1)p_1^\mu p_2^\nu - 3(k\cdot p_1)p_2^\mu p_2^\nu + 3(k\cdot p_2)p_1^\mu p_1^\nu \\
&+ 6(k\cdot p_2)p_2^\mu p_1^\nu - (k\cdot p_2)p_1^\mu p_2^\nu - (p_1\cdot p_2)p_1^\mu k^\nu + (p_1\cdot p_2)k^\mu p_2^\nu \\
&+ 6(p_1\cdot p_2)(k\cdot(p_1-p_2))g_{\mu\nu}\Big) I^3_{V_1}(k,p_1,p_2,\mu) \\
&+ \Big(4(p_1\cdot p_2)^2 g^{\mu\nu} - 4(p_1\cdot p_2)p_2^\mu p_1^\nu - (p_1\cdot p_2)p_1^\mu p_2^\nu\Big) I^4_{V_1}(k,p_1,p_2,\mu) \\
&+ (11k^\mu k^\nu + k^2 g^{\mu\nu}) I^5_{V_1}(k,p_1,p_2,\mu) \\
&(-6p_1^\mu k^\nu - p_2^\mu k_\nu + k^\mu p_1^\nu + 6k^\mu p_2^\nu - 3(k\cdot(p_1-p_2))g^{\mu\nu} I^6_{V_1}(k,p_1,p_2,\mu) \\
&+ (p_1^\mu p_1^\nu + 9p_2^\mu p_1^\nu - 3p_1^\mu p_2^\nu + p_2^\mu p_2^\nu - 9(p_1\cdot p_2)g_{\mu\nu}) I^7_{V_1}(k,p_1,p_2,\mu) \Bigg),
\end{aligned}
\tag{5.7}
$$

where $I^i_{V_1}$, $(i = 1...7)$ stand for different combinations of denominators appearing in the decomposition of the integral related to the diagram V_1, and given in the appendix A. Similar notations are used below for the other amplitudes. To simplify the expression we have already applied the conditions on-shell, $p_i^2 = 0$, $i = 1,2$.

Diagram V_2

To obtain the total amplitude of the diagram V_2 we notice that the sub-diagrams are the *4-gluon* vertex and the *2-gluon-Higgs* vertex. The propagators are $\Delta(k)^{cc'}_{\rho\rho'}$ and $\Delta(k-p_1-p_2)^{dd'}_{\sigma\sigma'}$. The symmetry factor is $\frac{1}{2}$, so the amplitude is

$$V_2 = \frac{1}{2} V^{gggg}_{sub_{V_2}} V^{2g-H}_{sub_{V_2}} \Delta(k)^{cc'}_{\rho\rho'} \Delta(k-p_1-p_2)^{dd'}_{\sigma\sigma'} \tag{5.8}$$

For the *4-gluon* vertex we have

$$
\begin{aligned}
V^{gggg}_{sub_{V_2}} = -ig^2(&f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\
&f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\
&f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}))
\end{aligned} \tag{5.9}
$$

and for the *2-gluon-Higgs* vertex

$$
V^{2g-H}_{sub_{V_2}} = iA\delta^{c'd'}(g^{\sigma'\rho'}(k-p_1).(k+p_2) - (k-p_1)^{\sigma'}(k+p_2)^{\rho'}). \tag{5.10}
$$

Expanding the expression and contracting the indices we obtain

$$
\begin{aligned}
V_2 =& Ag^2C_A\delta^{ab}\frac{1}{2}\int_k \frac{1}{k^2(k-p_1-p_2)^2} \\
&\Big(4k^2g^{\mu\nu} - 4k\cdot(p_1+p_2)g^{\mu\nu} + 2k^\mu k^\nu - k^\mu(p_1^\nu + p_2^\nu) - k^\nu(p_1^\mu + p_2^\nu)\Big).
\end{aligned} \tag{5.11}
$$

The expression in 5.11 is still unregularized. In general we have to analyse 3 types of integrals to which any of the integrals above can be matched given in appendix A. The regularizes amplitude for V_2 is then given by

$$
\begin{aligned}
V_2 =& Ag^2C_A\delta^{ab}\frac{1}{2}\int_k \Big(4k^2g^{\mu\nu}I^1_{V_2}(k,p_1,p_2,\mu) \\
&-(4k\cdot(p_1+p_2)g^{\mu\nu})I^2_{V_2}(k,p_1,p_2,\mu) + 2k^\mu k^\nu I^3_{V_2}(k,p_1,p_2,\mu) \\
&-(k^\mu(p_1^\nu+p_2^\nu))I^2_{V_2}(k,p_1,p_2,\mu) - (k^\nu(p_1^\mu+p_2^\nu))I^2_{V_2}(k,p_1,p_2,\mu)\Big).
\end{aligned} \tag{5.12}
$$

Diagram V_3

To obtain to total amplitude of the diagrams V_3 we notice that the sub-diagrams are the *3-gluon* vertex and the *3-gluon-Higgs* vertex. The propagators are given by $\Delta(k)^{cc'}_{\rho\rho'}$ and $\Delta(k-p_2)^{dd'}_{\tau\tau'}$. The symmetry factor is $\frac{1}{2}$, so the amplitude of the diagram V_3 is given by

$$
V_3 = \frac{1}{2}V^{3g}_{sub_{V_3}}V^{3g-H}_{sub_{V_3}}\Delta(k-p_2)^{dd'}_{\tau\tau'}\Delta(k)^{cc'}_{\rho\rho'}. \tag{5.13}
$$

We have for the *3-gluon* and the *3-gluon-Higgs* vertices, respectively

$$
V^{3g}_{sub_{V_3}} = gf^{bcd}(g^{\nu\rho}(p_2+k)^\tau + g^{\rho\tau}(-2k+p_2)^\nu + g^{\nu\tau}(k-2p_2)^\rho), \tag{5.14}
$$

$$V^{3g-H}_{sub_{V_3}} = Agf^{d'c'a}(g^{\tau'\rho'}(p_2-2k)^{\mu} + g^{\rho'\mu}(k-p_1)^{\tau'} + g^{\tau'\mu}(p_1+k-p_2)^{\rho'}). \tag{5.15}$$

Expanding the expression and contracting all the indices, we obtain

$$\begin{aligned} V_3 =& \frac{1}{2}Ag^2C_A\delta^{ab}\int_k \frac{1}{k^2(k-p_2)^2}\Big(2k^2g^{\mu\nu} - 2k\cdot p_2 g^{\mu\nu} - 3p_1\cdot p_2 g^{\mu\nu} \\ &+ 2p_2^2 g^{\mu\nu} + 10k^{\mu}k^{\nu} - 5k^{\nu}p_2^{\mu} - 5k^{\mu}p_2^{\nu} + 3p_1^{\nu}p_2^{\mu} + p_2^{\mu}p_2^{\nu}\Big). \end{aligned} \tag{5.16}$$

Here we have to analyse 4 types of integrals analysed in the appendix A. For the diagram V_3 we obtain

$$\begin{aligned} V_3 =& \frac{1}{2}Ag^2C_A\delta^{ab}\int_k \Big(2g^{\mu\nu}I^1_{V_3}(k,p_2,\mu) - 2k\cdot p_2 g^{\mu\nu}I^2_{V_3}(k,p_2,\mu) \\ &- 3p_1\cdot p_2 g^{\mu\nu}I^3_{V_3}(k,p_2,\mu) + 10k^{\mu}k^{\nu}I^7_{V_3}(k,p_2,\mu) \\ &- (5k^{\nu}p_2^{\mu} + 5k^{\mu}p_2^{\nu})I^2_{V_3}(k,p_2,\mu) + (3p_1^{\nu}p_2^{\mu} + p_2^{\mu}p_2^{\nu})I^3_{V_3}(k,p_2,\mu)\Big). \end{aligned} \tag{5.17}$$

Diagram V_4

The diagram V_4 is of the same type as the V_3 diagram, the difference being that the external momentum p_2 is now coming from the effective interaction and p_1 from the pure Yang-Mills interaction. The symmetry factor is also $\frac{1}{2}$ and the propagators the same as in V_3 by making $p_1 \Longleftrightarrow p_2$. For the *3-gluon* vertex we have

$$V^{3g}_{sub_{V_4}} = gf^{c'ad'}(g^{\rho'\mu}(-k-p_1)^{\tau'} + g^{\mu\tau'}(2p_1-k)^{\rho'} + g^{\rho'\tau'}(2k-p_1)^{\mu}) \tag{5.18}$$

and for the *3-gluon-Higgs* vertex

$$V^{3g-H}_{sub_{V_4}} = -Agf^{cdb}(g^{\rho\tau}(2k-p_1)^{\nu} + g^{\tau\nu}(-k+p_1-p_2)^{\rho} + g^{\nu\rho}(p_2-k)^{\tau}). \tag{5.19}$$

Expanding the expression and contracting all the indices, we obtain

$$\begin{aligned} V_4 =& \frac{1}{2}Ag^2C_A\delta^{ab}\int_k \frac{1}{k^2(k-p_1)^2}\Big(2k^2g^{\mu\nu} - 2k\cdot p_1 g^{\mu\nu} - 3p_1\cdot p_2 g^{\mu\nu} \\ &+ 2p_1^2 g^{\mu\nu} + 10k^{\mu}k^{\nu} - 5k^{\nu}p_1^{\mu} - 5k^{\mu}p_1^{\nu} + 3p_1^{\nu}p_2^{\mu} + p_1^{\mu}p_1^{\nu}\Big). \end{aligned} \tag{5.20}$$

We use the integrals in the appendix A to separate the divergences the same way as for V_3 and we obtain the regularized amplitude

$$
\begin{aligned}
V_4 = \frac{1}{2}Ag^2C_A\delta^{ab}\int_k \Big(& 2g^{\mu\nu}I^1_{V_4}(k,p_1,\mu) - 2k\cdot p_1 g^{\mu\nu}I^2_{V_4}(k,p_1,\mu) \\
& - 3p_1\cdot p_2 g^{\mu\nu}I^3_{V_4}(k,p_1,\mu) + 10k^\mu k^\nu I^7_{V_4}(k,p_1,\mu) \\
& - (5k^\nu p_1^\mu + 5k^\mu p_1^\nu)I^2_{V_4}(k,p_1,\mu) + (3p_1^\nu p_2^\mu + p_1^\mu p_1^\nu)I^3_{V_4}(k,p_1,\mu)\Big).
\end{aligned} \tag{5.21}
$$

Diagram V_5

To obtain the total amplitude of the diagram V_5 we notice that the it is a *4-gluon-Higgs* diagram with a loop. The symmetry factor is $\frac{1}{2}$. We obtain

$$
V_5 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{ig\delta^{cd}}{k^2}(-iAg^2)X^{\mu\nu\rho\sigma}_{abcd}, \tag{5.22}
$$

where $\frac{1}{2}$ is the symmetry factor of the diagram. Contracting the indices we obtain

$$
V_5 = -3Ag^2C_A\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\delta^{ab}g^{\mu\nu}. \tag{5.23}
$$

5.2 UV divergent integrals as Basic Divergent Integrals

We use equations 2.10 and 2.11 to write all the UV divergent integrals coming from the amplitudes V_1 to V_5 in terms of BDI's. Setting the surface terms to zero we notice that all the divergences can be written as 2.12 and 2.13.

For the diagram V_1 we get

$$
\begin{aligned}
& - I_{quad}(\mu^2)(\frac{13}{2}g^{\mu\nu}) \\
& - I_{log}(\mu^2)(-\frac{43}{6}p_1\cdot p_2 g^{\mu\nu} + \frac{1}{4}p_1^\mu p_1^\nu - \frac{1}{6}p_1^\mu p_2^\nu + \frac{29}{6}p_1^\nu p_2^\mu + \frac{1}{4}p_2^\mu p_2^\nu).
\end{aligned} \tag{5.24}
$$

For the diagram V_2 we get

$$
\begin{aligned}
& I_{quad}(\mu^2)(\frac{5}{2}g^{\mu\nu}) \\
& + \frac{1}{2}I_{log}(\mu^2)(-\frac{13}{3}p_1\cdot p_2 g^{\mu\nu} - \frac{1}{3}p_1^\mu p_1^\nu - \frac{1}{3}p_1^\mu p_2^\nu - \frac{1}{3}p_1^\nu p_2^\mu - \frac{1}{3}p_2^\mu p_2^\nu).
\end{aligned} \tag{5.25}
$$

For the diagram V_3 we get

$$
\begin{aligned}
& I_{quad}(\mu^2)(\frac{7}{2}g^{\mu\nu}) \\
& + \frac{1}{2}I_{log}(\mu^2)(3p_1^\nu p_2^\mu - 3p_1\cdot p_2 g^{\mu\nu} - \frac{2}{3}p_2^\mu p_2^\nu)).
\end{aligned} \tag{5.26}
$$

For the diagram V_4 the BDI's take the same form as the ones from diagram V_3 and they take the overall form

$$
\begin{aligned}
&I_{quad}(\mu^2)(\frac{7}{2}g^{\mu\nu}) \\
&+\frac{1}{2}I_{log}(\mu^2)(3p_1^\nu p_2^\mu - 3p_1\cdot p_2 g^{\mu\nu} - \frac{2}{3}p_1^\mu p_1^\nu)).
\end{aligned}
\tag{5.27}
$$

For the diagram V_5 the only integral contributing is an UV divergent integral. In terms of BDI's this integral is classified as quadratic

$$\int_k \frac{-3g^{\mu\nu}}{k^2-\mu^2} = -3I_{quad}(\mu^2)g^{\mu\nu}. \tag{5.28}$$

Joining the UV divergent content of all diagrams

After the classification of the BDI's of the diagrams V_1 to V_5 we joined all their contributions. We notice that each diagram individually contributes with logarithmic and/or quadratic divergences. Despite this, the integrals $I_{quad}(\mu^2)$ from V_1 to V_5 cancelled so there was only left $I_{log}(\mu^2)$. This is because in our effective model we only include the coupling of the Higgs with gluons. The Higgs propagator is not included, so there is only going to be renormalization of the gluon field and the effective coupling

$$L_{ren} \propto AHG_{\mu\nu}G^{\mu\nu} = Z_{\alpha_s}Z_A(AHG_{\mu\nu}G^{\mu\nu}), \tag{5.29}$$

where Z_A is the renormalization constant of the gluon and Z_{α_s} is the renormalization constant of the strong coupling constant that allows us to renormalize the effective coupling (see section below). Therefore, all the UV divergence must be absorbed in $Z_{\alpha_s}Z_A$. In this case, there's no terms in the renormalization to absorb quadratic divergences. As a result, the quadratic divergences were expected to cancel between themselves.

Adding all the contributions from V_1 to V_5 we obtain the amplitude V_{div} coming from the UV divergent contribution of the diagrams

$$V_{div} = Ag^2C_A\delta^{ab}I_{log}(\mu^2)(\frac{3}{4}(p_1^\mu p_1^\nu + p_2^\mu p_2^\nu) - 2(p_1^\nu p_2^\mu - p_1\cdot p_2 g^{\mu\nu}). \tag{5.30}$$

As we have defined before $H^{\mu\nu}(p_1,p_2) = p_1\cdot p_2 g^{\mu\nu} - p_1^\nu p_2^\mu$ is the tree level *2-gluon-Higgs*. After multiplying the previous result with the external polarizations ε_1^μ and ε_2^ν, because of the Lorenz condition we have $\varepsilon_1^\mu p_{1\mu} = 0$ and $\varepsilon_2^\nu p_{2\nu} = 0$ and we eliminate the first terms. The UV divergent contribution is given by

$$V_{div} = Ag^2C_A\delta^{ab}2H^{\mu\nu}(p_1,p_2)I_{log}(\mu^2) \tag{5.31}$$

and defining the tree-level amplitude $V_0 = iA\delta^{ab}H^{\mu\nu}$ and using $b = \frac{i}{(4\pi)^2}$ we can write the UV contribution of the amplitudes in terms of the tree-level amplitude in the following way

$$V_{div} = V_0 \alpha_s \frac{1}{b}\frac{1}{4\pi} C_A 2 I_{log}(\mu^2). \tag{5.32}$$

5.3 Renormalization

Our result in equation 5.32 is still UV and IR divergent. Before we proceed to compute the virtual decay rate, we need to perform renormalization. Our Lagrangian written in equation 4.6 is our bare Lagrangian and therefore we need to perform a redefinition of the variables and find the counterterms to cancel the divergences. We redefine the fields A^μ, the coupling constant g and α_s in the following way,

$$A^0_\mu = Z_A A_\mu, \tag{5.33}$$

$$g^0 = Z_g g, \tag{5.34}$$

and

$$\alpha_s^0 = Z_{\alpha_s} \alpha_s, \tag{5.35}$$

where the subscript $_0$ means that we are referring to the bare parameters. After these redefinitions, our renormalized effective Lagrangian is given by

$$(L_{eff})_{ren} = -\frac{1}{4} Z_{\alpha_s} Z_A A H G_{\mu\nu} G^{\mu\nu}. \tag{5.36}$$

Z_A and Z_{α_s} are given by

$$Z_A = 1 + \delta_A \alpha_s \tag{5.37}$$

and

$$Z_{\alpha_s} = 1 + \delta_{\alpha_s} \alpha_s. \tag{5.38}$$

Substituting Z_A and Z_{α_s} in the Lagrangian in 5.36, we notice that the counterterm for the effective vertex is given by

$$V_{count} = \alpha_s(\delta_{\alpha_s} + \delta_A) V_0. \tag{5.39}$$

Using equations 2.4 and 2.5 we have $Z_A = Z_3$ and Z_{α_s} relates with Z_g.

Our counterterm depends of δ_{α_s} rather than δ_g, but we can relate them using $\alpha_s = \frac{g^2}{4\pi}$ and equations 5.34 and 5.35, and we can write

$$g^0 = Z_{\alpha_s} g \tag{5.40}$$

therefore

$$Z_g = \sqrt{Z_{\alpha_s}}. \tag{5.41}$$

Doing a Taylor expansion in Z_{α_s} we can write the relation

$$\delta_{\alpha_s} = 2\delta_g. \tag{5.42}$$

As we are in a Feynman Gauge, $\zeta = 1$, we obtain the counterterm

$$V_{count} = \alpha_s V_0 \frac{1}{b} \frac{1}{(4\pi)} C_A \Big(\frac{5}{3} I_{log}(\mu^2) - \frac{11}{3} I_{log}(\lambda^2)\Big). \tag{5.43}$$

Adding the counterterm to the UV contribution of the Feynman diagrams, we obtain V_{ren}

$$V_{ren} = V_{div} + V_{count} = \alpha_s V_0 \frac{1}{b} \frac{1}{(4\pi)} C_A \Big(\frac{11}{3} I_{log}(\mu^2) - \frac{11}{3} I_{log}(\lambda^2)\Big). \tag{5.44}$$

Using the equation of the renormalization function, 2.22 and making $C_A = 3$, we obtain

$$V_{ren} = V_0 \frac{\alpha_s}{\pi} \frac{11}{4} \ln\Big(\frac{\lambda^2}{\mu^2}\Big), \tag{5.45}$$

rendering an UV finite result.

5.4 UV Finite integrals

The UV finite integrals from V_1 to V_5 were treated using the *Package-X* of the software *Mathematica*, [6]. The diagram V_5 has no finite integrals, therefore will automatically not contribute to the calculus of the decay rate. The integrals in V_3 and V_4 were evaluated to zero after performing the integration and conveniently applying the on-shell conditions. As a result, only the diagrams V_1 and V_2 will contribute with a UV finite contribution to the decay rate of the Higgs boson.

In order to use the *package-X* to evaluate the amplitude, we start by making an input of the integral in its original form with the internal momentum k, the external momenta p_i and the regulator μ. We then integrate using the package and apply the on-shell conditions that require that $p_1^2 = p_2^2 = 0$. These conditions reflect the fact that we are considering massless gluons in the final state of the decay. We define $\mu_0 = \dfrac{\mu^2}{2p_1 \cdot p_2} = \dfrac{\mu^2}{s_{12}}$. After collecting and summing all the integrals we expand in a power series the final result in μ_0 and neglect all the terms above the order zero. Our final result is given by

$$
\begin{aligned}
V_{fin} = V_1 + V_2 &= -Ag^2 C_A \delta^{ab} \frac{i}{(4\pi)^2} \ln(-\mu_0)^2 (-p_1^\nu p_2^\mu + p_1 \cdot p_2 g^{\mu\nu}) \\
&= -Ag^2 C_A \delta^{ab} \frac{i}{(4\pi)^2} \ln(-\mu_0)^2 H^{\mu\nu}(p_1, p_2).
\end{aligned} \tag{5.46}
$$

We notice that again, as for the UV divergent integrals, the final result for the finite integrals is proportional to the tree level $H \longrightarrow gg$. By doing the expansion of the logarithm the following way

$$
\ln(-\mu_0)^2 = \ln(\mu_0)^2 + 2i\pi \ln(\mu_0) - \pi^2, \tag{5.47}
$$

and substituting 5.47 in equation 5.46, and using the tree-level amplitude we can write the contribution from the UV finite integrals as

$$
V_{fin} = -\frac{3}{4}\frac{\alpha_s}{\pi} V_0 (\ln(\mu_0)^2 + 2i\pi \ln(\mu_0) - \pi^2). \tag{5.48}
$$

5.5 Virtual decay rate

The amplitude with the one-loop virtual correction V, is given by the sum of the tree-level amplitude of the process with the renormalized amplitude coming from the UV divergent integrals and the UV finite contributions

$$
V = V_0 + V_{ren} + V_{fin}. \tag{5.49}
$$

By squaring this result up to the order considered we obtain

$$
|V|^2 = V_0^2 + 2\Re e(V_0 (V_{ren} + V_{fin})^*), \tag{5.50}
$$

where $*$ stands for the complex conjugate. The differential decay rate for the *2-gluon* decay can be found in [32] and is generally given by

$$
d\Gamma = |V_0|^2 \frac{S}{2m_H} \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^4 (p_H - p_1 - p_2), \tag{5.51}
$$

where S is a factor that accounts for the presence of identical particles in the final state and has the form $1/j!$ where j is the number of identical particles in the final state. Here $S = \frac{1}{2}$. We are considering the reference frame where the Higgs is at rest. Therefore the only possible outcome for the final momenta of the gluons is that they must be anti-parallel, $\vec{p}_2 = -\vec{p}_1$. This requirement allows us to integrate over all the final momenta to obtain the decay rate without knowing the explicit form of the amplitude. All these computations can be found in [32]. After the integrations and some simplification the final expression is given by

$$\Gamma = \frac{|V|^2}{32\pi m_H}. \tag{5.52}$$

Substituting 5.45 and 5.48 in the square modulus of the virtual amplitude in equation 5.50, we get

$$|V|^2 = |V_0|^2\Big(1+\frac{\alpha_s}{\pi}\Big(\frac{11}{2}\ln\Big(\frac{\lambda^2}{\mu^2}\Big)+\frac{3}{2}(-\ln(\mu_0)^2+\pi^2)\Big) \tag{5.53}$$

and using $\mu_0 = \dfrac{\mu^2}{m_H^2}$, the virtual decay rate is given by

$$\Gamma_v = \Gamma_0\Big(1+\frac{\alpha_s}{\pi}\Big(\frac{11}{2}\ln\Big(\frac{\lambda^2}{\mu^2}\Big)-\frac{3}{2}\Big(\ln^2\Big(\frac{\mu^2}{m_H^2}\Big)-\pi^2\Big)\Big)\Big). \tag{5.54}$$

Here, Γ^0 is the tree-level decay rate of the process, as we will see in section 6.3.1.

Notice that the virtual decay rate with the one-loop correction is written as a correction to the tree-level decay $H \longrightarrow gg$. Also, there are still present IR divergences parameterized by the regulator μ.

Chapter 6

Real decay rate H⟶gg(g)

In this chapter we use the spinor-helicity formalism to compute the real amplitudes of the process H ⟶ gg(g). We also compute the 3-body phase space in terms of dimensionless variables and do the calculation of the real decay rate.

6.1 Real amplitude

The real emission diagrams represent processes that are of the same order in α_s as the one-loop virtual ones, but notice that we have more final particle states than in the virtual contribution, which will make the calculus more complex. The final states momenta will be $p_1^\mu, p_2^\nu, p_3^\tau$ and the external polarizations will be given by $\varepsilon_1^\mu, \varepsilon_2^\nu, \varepsilon_3^\tau$. There are four tree-level diagrams contributing to the decay rate. R_1 contributes with three diagrams because we have permutations $\{p_1, p_2, p_3\}$, $\{p_3, p_1, p_2\}$, $\{p_2, p_3, p_1\}$ which will correspond to s, t, u channels and R_2 contributes with a one vertex of *3-gluon-Higgs*. We start by computing the amplitudes individually using the Feynman rules and then sum them. After this we apply spinor-helicity formalism in order to simplify the squared amplitude.

Amplitude of R_1

As previously stated, R_1 actually corresponds to three diagrams linked by permutations of external momenta and are represented in figure 6.1. As a result, computing one of the channels is sufficient, and the others can be computed directly from the other using only momenta permutations.

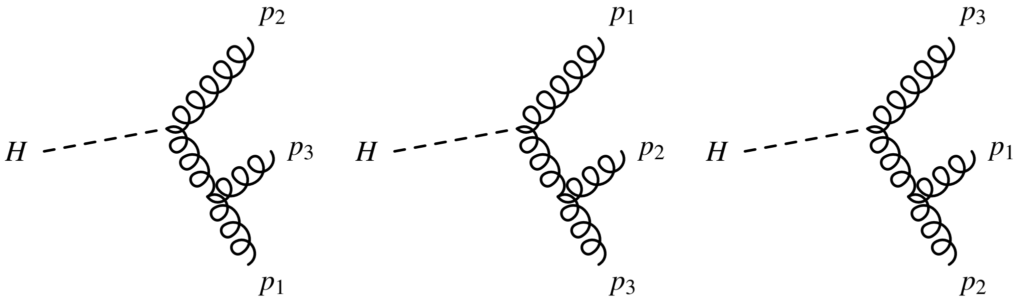


Figure 6.1: Momenta permutations in the real diagram R_1.

The sub-diagrams of the s channel of R_1 are given by the *3-gluon* coming from the pure YM interaction and *2-gluon-Higgs* originated by the effective interaction.

$$\begin{aligned} iM_{R_1} &= gf^{bcd}V^{\nu\tau\delta}(p_2,p_3,-(p_2+p_3))i\frac{g^{\delta\delta'}\delta^{dd'}}{(-(p_2+p_3))^2}iA\delta^{ad'}H^{\delta'\mu}(-(p_2+p_3),p_1)\varepsilon_1^\mu\varepsilon_2^\nu\varepsilon_3^\tau \\ &= -Agf^{bca}\frac{1}{(p_2+p_3)^2}V^{\nu\tau\delta}H^{\delta\mu}\varepsilon_1^\mu\varepsilon_2^\nu\varepsilon_3^\tau \end{aligned} \tag{6.1}$$

where the expanded tensors are given by

$$V^{\nu\tau\delta}(p_2,p_3,-(p_2+p_3)) = (p_2-p_3)^\delta g^{\nu\tau} - p_2^\nu g^{\tau\delta} + p_3^\tau g^{\delta\nu} \tag{6.2}$$

and

$$H^{\delta\mu}((-(p_2+p_3)),p_1) = g^{\mu\delta}p_1\cdot(-(p_2+p_3)) - p_1^\delta(-(p_2+p_3))^\mu. \tag{6.3}$$

Contracting all the indices we get

$$\begin{aligned} iM_{R_1} = \frac{-Agf^{bca}}{(p_2+p_3)^2}\varepsilon_1^\mu\varepsilon_2^\nu\varepsilon_3^\tau\Big(&(p_1\cdot p_2)(g^{\mu\tau}(2p_3+p_2)^\nu - g^{\mu\nu}(2p_2+p_3)^\tau - 2p_3^\mu g^{\nu\tau}) \\ &+(p_1\cdot p_3)(g^{\mu\tau}(2p_3+p_2)^\nu - g^{\mu\nu}(2p_2+p_3)^\tau + 2p_2^\mu g^{\nu\tau}) \\ &-p_1^\tau(p_2+p_3)^\mu(2p_3+p_2)^\nu + p_1^\nu(p_2+p_3)^\mu(2p_2+p_3)^\tau\Big). \end{aligned} \tag{6.4}$$

We can now make use of the Lorenz condition in order to eliminate all the following scalar products

$$\varepsilon^\mu p_\mu = \varepsilon^1 p_1 = \varepsilon^2 p_2 = \varepsilon^3 p_3 = 0, \tag{6.5}$$

and finally we get for the s channel

$$
\begin{aligned}
iM_{R_1} = \frac{-Agf^{bca}}{(p_2+p_3)^2}\varepsilon_1^\mu\varepsilon_2^\nu\varepsilon_3^\tau\Big((p_1\cdot p_2)(2p_3^\nu g^{\mu\tau}-2p_2^\tau g^{\mu\nu}-2p_3^\mu g^{\nu\tau})\\
+(p_1\cdot p_3)(2p_3^\nu g^{\mu\tau}-2p_2^\tau g^{\mu\nu}+2p_2^\mu g^{\nu\tau})\\
-2p_1^\tau(p_2+p_3)^\mu p_3^\nu+2p_1^\nu(p_2+p_3)^\mu p_2^\tau\Big).
\end{aligned}
\tag{6.6}
$$

Notice that we will now purposefully contract the momenta with the polarizations, as it will come in handy when employing the spinor-helicity formalism to obtain

$$
\begin{aligned}
iM_{R_1} =& \frac{-Agf^{bca}}{s_{23}}(s_{12}+s_{13})((\varepsilon_1\cdot\varepsilon_3)(p_3\cdot\varepsilon_2)-(p_2\cdot\varepsilon_3)(\varepsilon_1\cdot\varepsilon_2))\\
&-s_{12}(p_3\cdot\varepsilon_1)(\varepsilon_2\cdot\varepsilon_3)+s_{13}(p_2\cdot\varepsilon_1)(\varepsilon_2\cdot\varepsilon_3)\\
&+2(p_2\cdot\varepsilon_1+p_3\cdot\varepsilon_1)((p_1\cdot\varepsilon_2)(p_2\cdot\varepsilon_3)-(p_1\cdot\varepsilon_3)(p_3\cdot\varepsilon_2)).
\end{aligned}
\tag{6.7}
$$

We now have an amplitude that is determined by scalar products of momenta and polarizations, allowing us to use the spinor-helicity formalism. Firstly we define three auxiliary momenta r_i, $i = 1,2,3$, one for each of the massless gluon. This choice is arbitrary, as choosing a different reference momenta is the equivalent of doing a gauge transformation which leaves the square amplitude invariant. Here, we choose

$$
r(\varepsilon_1) = p_3 \equiv 3, \qquad r(\varepsilon_2) = p_1 \equiv 1, \qquad r(\varepsilon_3) = p_2 \equiv 2. \tag{6.8}
$$

By doing this choice the terms $p_2\cdot\varepsilon_3 = p_1\cdot\varepsilon_2 = p_3\cdot\varepsilon_1 = 0$ are automatically zero, which allows for a great deal of simplification of the amplitude:

$$
\begin{aligned}
iM_{R_1} =& \frac{-Agf^{bca}}{s_{23}}\\
&(s_{12}+s_{13})((\varepsilon_1\cdot\varepsilon_3)(p_3\cdot\varepsilon_2))+s_{13}(p_2\cdot\varepsilon_1)(\varepsilon_2\cdot\varepsilon_3)-2(p_2\cdot\varepsilon_1)(p_1\cdot\varepsilon_3)(p_3\cdot\varepsilon_2)).
\end{aligned}
\tag{6.9}
$$

The t channel is obtained by making $1 \longleftrightarrow 3$, $2 \longleftrightarrow 1$ and $3 \longleftrightarrow 2$ in the s channel and the u channel is obtained by making $1 \longleftrightarrow 2$, $2 \longleftrightarrow 3$ and $3 \longleftrightarrow 1$. We have then for the t and u channels respectively,

$$
\begin{aligned}
iM_{R_1} =& \frac{-Agf^{abc}}{s_{12}}\\
&(s_{13}+s_{23})((\varepsilon_3\cdot\varepsilon_2)(p_2\cdot\varepsilon_1))+s_{23}(p_1\cdot\varepsilon_3)(\varepsilon_1\cdot\varepsilon_2)-2(p_1\cdot\varepsilon_3)(p_3\cdot\varepsilon_2)(p_2\cdot\varepsilon_1)),
\end{aligned}
\tag{6.10}
$$

$$iM_{R_1} = \frac{-Agf^{cab}}{s_{13}} (s_{23}+s_{12})((\varepsilon_1\cdot\varepsilon_2)(p_1\cdot\varepsilon_3)) + s_{12}(p_3\cdot\varepsilon_2)(\varepsilon_3\cdot\varepsilon_1) - 2(p_3\cdot\varepsilon_2)(p_2\cdot\varepsilon_1)(p_1\cdot\varepsilon_3)). \tag{6.11}$$

Amplitude of R_2

The amplitude for R_2 is given by the contraction of the Feynman rule of *3-gluon-Higgs* with the external polarizations,

$$iM_{R_2} = -Agf^{abc}V^{\mu\nu\tau}(p_1,p_2,p_3)\varepsilon_1^{\mu}\varepsilon_2^{\nu}\varepsilon_3^{\tau}, \tag{6.12}$$

the tensor being

$$V^{\mu\nu\tau}(p_1,p_2,p_3) = (p_1-p_2)^{\tau}g^{\mu\nu} + (p_2-p_3)^{\mu}g^{\nu\tau} + (p_3-p_1)^{\nu}g^{\mu\tau}. \tag{6.13}$$

By contracting all the indices we will have an amplitude written in terms of scalar products between momenta and polarizations,

$$iM_{R_2} = -Agf^{abc}\Big((p_1-p_2)\cdot\varepsilon_3(\varepsilon_1\cdot\varepsilon_2) + (p_2-p_3)\cdot\varepsilon_1(\varepsilon_2\cdot\varepsilon_3) + (p_3-p_1)\cdot\varepsilon_2(\varepsilon_1\cdot\varepsilon_3)\Big). \tag{6.14}$$

Applying the spinor-helicity formalism and remembering the choice of reference momenta in equation 6.8 we simplify the expression above to be

$$iM_{R_2} = -Agf^{abc}\Big(p_1\cdot\varepsilon_3(\varepsilon_1\cdot\varepsilon_2) + p_2\cdot\varepsilon_1(\varepsilon_2\cdot\varepsilon_3) + p_3\cdot\varepsilon_2(\varepsilon_1\cdot\varepsilon_3)\Big). \tag{6.15}$$

Notice that we could have made a different choice of reference momenta, but it's crucial that once we made a choice, the same needs to be applied to all diagrams so that we have a consistent result.

Total amplitude M

We will now compute the total amplitude. Summing the s, t and u channels from R_1 with the one vertex diagram from R_2 we have

$$
\begin{aligned}
iM = -Ag\Bigg(& \frac{f^{bca}}{s_{23}}(s_{12}+s_{13})((\varepsilon_1\cdot\varepsilon_3)(p_3\cdot\varepsilon_2))+s_{13}(p_2\cdot\varepsilon_1)(\varepsilon_2\cdot\varepsilon_3)-2(p_2\cdot\varepsilon_1)(p_1\cdot\varepsilon_3)(p_3\cdot\varepsilon_2)) \\
&+\frac{f^{abc}}{s_{12}}(s_{13}+s_{23})((\varepsilon_3\cdot\varepsilon_2)(p_2\cdot\varepsilon_1))+s_{23}(p_1\cdot\varepsilon_3)(\varepsilon_1\cdot\varepsilon_2)-2(p_1\cdot\varepsilon_3)(p_3\cdot\varepsilon_2)(p_2\cdot\varepsilon_1)) \\
&+\frac{f^{cab}}{s_{13}}(s_{23}+s_{12})((\varepsilon_1\cdot\varepsilon_2)(p_1\cdot\varepsilon_3))+s_{12}(p_3\cdot\varepsilon_2)(\varepsilon_3\cdot\varepsilon_1)-2(p_3\cdot\varepsilon_2)(p_2\cdot\varepsilon_1)(p_1\cdot\varepsilon_3)) \\
&+f^{abc}((p_1-p_2)\cdot\varepsilon_3(\varepsilon_1\cdot\varepsilon_2)+(p_2-p_3)\varepsilon_1(\varepsilon_2\cdot\varepsilon_3)+(p_3-p_1)\varepsilon_2(\varepsilon_1\cdot\varepsilon_3))\Bigg).
\end{aligned}
\tag{6.16}
$$

To obtain the square of the amplitude we need to sum to all the possible colors and helicities,

$$
\begin{aligned}
|M|^2 =&|M^{+++}|^2+|M^{+--}|^2+|M^{-+-}|^2+|M^{--+}|^2 \\
&+|M^{---}|^2+|M^{-++}|^2+|M^{+-+}|^2+|M^{++-}|^2.
\end{aligned}
\tag{6.17}
$$

To obtain the square of the amplitude we do for example $|M^{+--}|^2 = (M^{+--})(M^{+--})^*$ where the $(M^{+--})^*$ is obtained from M^{+--} by exchanging $\langle\rangle \longleftrightarrow []$ and adding a minus sign. Following this logic for each helicity we have the following relations

$$
\begin{aligned}
|M^{+++}|^2 &= |M^{---}|^2, \\
|M^{+--}|^2 &= |M^{-++}|^2, \\
|M^{-+-}|^2 &= |M^{+-+}|^2, \\
|M^{--+}|^2 &= |M^{++-}|^2.
\end{aligned}
\tag{6.18}
$$

so we write the unpolarized amplitude as

$$
|\overline{M}|^2 = \frac{1}{4}\sum_{col,polr}|M|^2 = \sum_{col}\frac{2}{4}(|M^{+++}|^2+|M^{+--}|^2+|M^{-+-}|^2+|M^{--+}|^2)
\tag{6.19}
$$

and by permutations of s, t, and u we can obtain $|M^{-+-}|^2$ and $|M^{--+}|^2$ from $|M^{+--}|^2$. So the only amplitudes we need to calculate are M^{+++} and M^{+--} and their squares.

For the helicity $+--$ we use the expressions introduced in the spinor-helicity chapter and write the polarizations as $\varepsilon_1^+ = \sqrt{2}\frac{3\rangle[1}{\langle 31\rangle}$,$\varepsilon_2^- = \sqrt{2}\frac{2\rangle[1}{[21]}$, $\varepsilon_3^- = \sqrt{2}\frac{3\rangle[2}{[32]}$ and the momenta as $p_1 = 1\rangle[1$, $p_2 = 2\rangle[2$, $p_3 = 3\rangle[3$ where for simplicity we used the notation $p_i \equiv i$ and $r_j \equiv j$. We are now in conditions to compute the scalar products between polarizations as

$$\varepsilon_1^+ \cdot \varepsilon_2^- = 0, \qquad \varepsilon_1^+ \cdot \varepsilon_3^- = 0, \qquad \varepsilon_2^- \cdot \varepsilon_3^- = \frac{\langle 23\rangle}{[32]}, \tag{6.20}$$

and the products between momenta and polarizations as

$$p_2 \cdot \varepsilon_1^+ = \frac{1}{\sqrt{2}}\frac{\langle 23\rangle[12]}{\langle 31\rangle}, \quad p_1 \cdot \varepsilon_3^- = \frac{1}{\sqrt{2}}\frac{\langle 13\rangle[21]}{[32]}, \quad p_3 \cdot \varepsilon_2^- = \frac{1}{\sqrt{2}}\frac{\langle 32\rangle[13]}{[21]}. \tag{6.21}$$

The amplitude M^{+--} is obtained by making the previous substitutions of the scalar products in the expression of the total amplitude, after which we obtain

$$\begin{aligned} M^{+--} =& -Agf^{abc}\Big(\frac{\langle 23\rangle}{[32]}\frac{\langle 23\rangle[12]}{\langle 31\rangle}\Big)\Big(\frac{s_{13}}{s_{23}}+\frac{s_{13}}{s_{12}}+\frac{s_{23}}{s_{12}}+1\Big) \\ & -2\Big(\frac{1}{\sqrt{2}}\frac{\langle 23\rangle[12]}{\langle 31\rangle}\Big)\Big(\frac{1}{\sqrt{2}}\frac{\langle 13\rangle[21]}{[32]}\Big)\Big(\frac{1}{\sqrt{2}}\frac{\langle 32\rangle[13]}{[21]}\Big)\Big(\frac{1}{s_{23}}+\frac{1}{s_{12}}+\frac{1}{s_{13}}\Big) \end{aligned} \tag{6.22}$$

and by squaring the amplitude we obtain

$$|M^{+--}|^2 = A^2g^2|f^{abc}|^2\frac{s_{23}^3}{s_{12}s_{13}}. \tag{6.23}$$

By making permutations of the indices we obtain the amplitudes for the helicities $-+-$ and $--+$

$$|M^{-+-}|^2 = A^2g^2|f^{abc}|^2\frac{s_{12}^3}{s_{23}s_{13}}, \tag{6.24}$$

$$|M^{--+}|^2 = A^2g^2|f^{abc}|^2\frac{s_{13}^3}{s_{23}s_{12}}. \tag{6.25}$$

For the helicity $+++$ we have already defined ε_1^+ but we still need to define $\varepsilon_2^+ = \sqrt{2}\frac{1\rangle[2}{\langle 12\rangle}$ and $\varepsilon_3^+ = \sqrt{2}\frac{2\rangle[3}{\langle 23\rangle}$. We can now construct the scalar products between the polarizations as

$$\varepsilon_1^+ \cdot \varepsilon_2^+ = \frac{[12]}{\langle 21\rangle}, \qquad \varepsilon_1^+ \cdot \varepsilon_3^+ = \frac{[13]}{\langle 31\rangle}, \qquad \varepsilon_2^+ \cdot \varepsilon_3^+ = \frac{[23]}{\langle 32\rangle}. \tag{6.26}$$

and the scalar product between polarizations and momenta as

$$p_1 \cdot \varepsilon_3^+ = \frac{1}{\sqrt{2}} \frac{\langle 12 \rangle [31]}{\langle 23 \rangle}, \quad p_2 \cdot \varepsilon_1^+ = \frac{1}{\sqrt{2}} \frac{\langle 23 \rangle [12]}{\langle 31 \rangle}, \quad p_3 \cdot \varepsilon_2^+ = \frac{1}{\sqrt{2}} \frac{\langle 31 \rangle [23]}{\langle 12 \rangle}. \tag{6.27}$$

We do these substitutions in the general equation for the total amplitude and obtain the expression for the amplitude for the $+++$ helicity

$$\begin{aligned} M^{+++} = & -Agf^{abc} \\ & \Big(\frac{[13]}{\langle 31 \rangle} \frac{1}{\sqrt{2}} \frac{\langle 31 \rangle [23]}{\langle 12 \rangle} \big(\frac{s_{12}+s_{13}}{s_{23}} + \frac{s_{12}}{s_{13}} + 1 \big) \\ & + \frac{[23]}{\langle 32 \rangle} \frac{1}{\sqrt{2}} \frac{\langle 23 \rangle [12]}{\langle 31 \rangle} \big(\frac{s_{13}+s_{23}}{s_{12}} + \frac{s_{13}}{s_{23}} + 1 \big) \\ & + \frac{[12]}{\langle 21 \rangle} \frac{1}{\sqrt{2}} \frac{\langle 12 \rangle [31]}{\langle 23 \rangle} \big(\frac{s_{23}+s_{12}}{s_{13}} + \frac{s_{23}}{s_{12}} + 1 \big) \\ & + \frac{1}{\sqrt{2}} \frac{\langle 23 \rangle [12]}{\langle 31 \rangle} \frac{1}{\sqrt{2}} \frac{\langle 12 \rangle [31]}{\langle 23 \rangle} \frac{1}{\sqrt{2}} \frac{\langle 31 \rangle [23]}{\langle 12 \rangle} \big(\frac{1}{s_{12}} + \frac{1}{s_{13}} + \frac{1}{s_{23}} \big) \Big). \end{aligned} \tag{6.28}$$

By squaring the amplitude we obtain

$$\begin{aligned} |M^{+++}|^2 = & A^2 g^2 |f^{abc}|^2 \Big(\frac{s_{12}^3}{s_{23}s_{13}} + \frac{s_{13}^3}{s_{12}s_{23}} + \frac{s_{23}^3}{s_{12}s_{13}} \\ & + 4 \big(\frac{s_{12}^2}{s_{23}} + \frac{s_{12}^2}{s_{13}} + \frac{s_{13}^2}{s_{12}} + \frac{s_{13}^2}{s_{23}} + \frac{s_{23}^2}{s_{12}} + \frac{s_{23}^2}{s_{13}} \big) \\ & + 6 \big(\frac{s_{12}s_{13}}{s_{23}} + \frac{s_{12}s_{23}}{s_{13}} + \frac{s_{13}s_{23}}{s_{12}} \big) \\ & + 12 \big(s_{12} + s_{13} + s_{23} \big) \Big). \end{aligned} \tag{6.29}$$

The conservation of energy is expressed by the condition $m_H^2 = s_{12} + s_{13} + s_{23}$ and we can express the M^{+++} amplitude in terms of the Higgs mass,

$$|M^{+++}|^2 = A^2 g^2 |f^{abc}|^2 \frac{m_H^8}{s_{12}s_{13}s_{23}}. \tag{6.30}$$

We are in conditions to write the full unpolarized amplitude because we already know the amplitude for each possible helicity. The structure constants can be written as $|f^{abc}|^2 = 2C_A^2 C_F$ with $C_F = \frac{N^2-1}{2N}$ and $C_A = N$ with $N = 3$ for the $SU(3)$ group. We also have defined $\alpha_s = \frac{g^2}{4\pi}$. More details are given in the appendices. We get the final result for the dimensionless real unpolarized squared amplitude which is in accordance with [8],

$$
\begin{aligned}
|\overline{M}|^2 &= A^2 192\pi\alpha_s\Big(\frac{s_{12}^3}{s_{23}s_{13}}+\frac{s_{13}^3}{s_{12}s_{23}}+\frac{s_{23}^3}{s_{12}s_{13}}+2\big(\frac{s_{12}^2}{s_{23}}+\frac{s_{12}^2}{s_{13}}+\frac{s_{13}^2}{s_{12}}+\frac{s_{13}^2}{s_{23}}+\frac{s_{23}^2}{s_{12}}+\frac{s_{23}^2}{s_{13}}\big)\\
&+3\big(\frac{s_{12}s_{13}}{s_{23}}+\frac{s_{12}s_{23}}{s_{13}}+\frac{s_{13}s_{23}}{s_{12}}\big)+6(s_{12}+s_{13}+s_{23})\Big)\\
&= A^2 192\pi\alpha_s\frac{1}{s_{12}s_{13}s_{23}}(s_{12}^4+s_{13}^4+s_{23}^4+m_H^8).
\end{aligned}
\tag{6.31}
$$

6.2 Phase Space

After evaluating the unpolarized amplitude, we now need to compute the phase space for the three gluons that result of the decay of the Higgs boson,

$$
\rho = \int \frac{d^3p_1}{(2\pi)^3 2\omega_1}\frac{d^3p_2}{(2\pi)^3 2\omega_2}\frac{d^3p_3}{(2\pi)^3 2\omega_3}(2\pi)^4\delta^4(q-p_1-p_2-p_3) \tag{6.32}
$$

where q denotes the Higgs momenta and $p_i, i=1,2,3$ are the momenta of the gluons. The phase space ρ has dimension of mass square. Choosing the reference frame where the Higgs 3-momentum is zero, $\vec{q}=0$ we get the equation for the conservation of momentum $\vec{p}_1+\vec{p}_2+\vec{p}_3=0$. The gluon energies are defined by

$$
\begin{cases}
\omega_1^2 = \mu^2+|\vec{p}_1|^2 = \mu^2+|\vec{p}_2|^2+|\vec{p}_3|^2+2|\vec{p}_2||\vec{p}_3|\cos\theta_{23} = 2|\vec{p}_2||\vec{p}_3|\cos\theta_{23}\\
\omega_2^2 = \mu^2+|\vec{p}_2|^2 = \mu^2+|\vec{p}_1|^2+|\vec{p}_3|^2+2|\vec{p}_1||\vec{p}_3|\cos\theta_{13} = 2|\vec{p}_1||\vec{p}_3|\cos\theta_{13}\\
\omega_3^2 = \mu^2+|\vec{p}_3|^2 = \mu^2+|\vec{p}_1|^2+|\vec{p}_2|^2+2|\vec{p}_1||\vec{p}_2|\cos\theta_{12} = 2|\vec{p}_1||\vec{p}_2|\cos\theta_{12}
\end{cases}
\tag{6.33}
$$

where μ is the regulator mass and $|\vec{p}_1|^2=|\vec{p}_2|^2=|\vec{p}_3|^2=0$ and θ_{ij} is the angle between the gluons i and j. Integrating ρ over d^3p_3 and changing to spherical coordinates we get

$$
\frac{d^3p_i}{2\omega_i} = \frac{1}{2}\omega_i d\omega_i d\Omega_i, i=1,2 \tag{6.34}
$$

and where the solid angle Ω is given by $d\Omega_i = d\cos\theta_i d\phi_i$ so we get

$$
\begin{aligned}
\rho &= \int \frac{d^3p_1 d^3p_2}{(2\pi)^5(2\omega_1)(2\omega_2)(2\omega_3)}\delta(q_0-\omega_1-\omega_2-\omega_3)\\
&= \int \frac{1}{(2\pi)^5}\frac{1}{2\omega_3}\big(\frac{\omega_1 d\omega_1 d\Omega_1}{2}\big)\big(\frac{\omega_2 d\omega_2 d\Omega_2}{2}\big)\delta(q_0-\omega_1-\omega_2-\omega_3).
\end{aligned}
\tag{6.35}
$$

The integral over one of the variables Ω_i is trivial because we can choose any direction, so integrating we get $\Omega_1 = 4\pi$ and for the other variable we have $\int\Omega_2 =$

2πdcos θ_{12} where θ_{12} represents the angle between the direction of the gluons 1 and 2. Differentiating the gluon energies expression in 6.33 we have $\frac{\omega_3 d\omega_3}{p_1 p_2}$ =dcos θ_{12}. The phase space integral simplifies to be

$$\rho = \int \frac{1}{32\pi^3} d\omega_1 d\omega_2 d\omega_3 \delta(q_0 - \omega_1 - \omega_2 - \omega_3) \tag{6.36}$$

and finally integrating over ω_3 we obtain

$$\rho = \frac{1}{32\pi^3} \int_{w_{1min}}^{w_{1max}} d\omega_1 \int_{w_{2min}}^{w_{2max}} d\omega_2, \tag{6.37}$$

where the conservation of energy is given by $q_0 - \omega_1 - \omega_2 - \omega_3 = 0$.

6.2.1 Dimensionless variables

It is useful to introduce dimensionless variables instead, following [3]. This will provide us with a dimensionless ρ and also allows us to use the same regulator μ as in the virtual decay. We define the new variables χ_i as

$$\chi_i = \frac{(p_i - q)^2}{q^2} - \frac{\mu^2}{q^2} \tag{6.38}$$

with $i = 1, 2, 3$. By expanding the equation we get

$$\chi_i = 1 - \frac{2p_i \cdot q}{q^2} = 1 - \frac{2(p_i^0 q^0 - \vec{p}_i \cdot \vec{q})}{q_0^2 - |\vec{q}|^2}. \tag{6.39}$$

In the referential where the Higgs 3-momentum is zero $\vec{q} = 0$ we have

$$\chi_i = 1 - 2\frac{\omega_i}{q_0}, \tag{6.40}$$

and differentiating both sides of the equation we get

$$d\chi_i = -\frac{2}{q_0} d\omega_i, \tag{6.41}$$

and the phase space integral in terms of dimensionless variables is then

$$\rho = \frac{1}{128\pi^3} q_0^2 \int_{\chi_{1min}}^{\chi_{1max}} \int_{\chi_{2min}}^{\chi_{2max}} d\chi_1 d\chi_2. \tag{6.42}$$

As before we could relate the three gluons energies with the equation of conservation of energy, we can now write down an equation that relates the three new dimensionless variables χ_i. Substituting 6.40 in the equation of conservation of energy we get

$$\chi_1 + \chi_2 + \chi_3 = 1. \tag{6.43}$$

The limits of integration for this case will be for χ_2

$$\chi_{2min} = \frac{1-\chi_1}{2} - \sqrt{\frac{(\chi_1 - 3\mu_0)((1-\chi_1)^2 - 4\mu_0)}{4(\chi_1 + \mu_0)}} \tag{6.44}$$

and

$$\chi_{2max} = \frac{1-\chi_1}{2} + \sqrt{\frac{(\chi_1 - 3\mu_0)((1-\chi_1)^2 - 4\mu_0)}{4(\chi_1 + \mu_0)}}, \tag{6.45}$$

and for χ_1

$$\chi_{1min} = 3\mu_0 \tag{6.46}$$

and

$$\chi_{1max} = 1 - 2\sqrt{\mu_0}. \tag{6.47}$$

6.3 Computation of the real decay rate

We start by computing the tree level decay rate $H \longrightarrow gg$ and then proceed to show that the $H \longrightarrow gg(g)$ real decay can be written in terms of the $H \longrightarrow gg$.

6.3.1 *H⟶gg* real decay rate

The tree-level amplitude is given by

$$M_{Hgg} = iA\delta^{ab}((p_1 \cdot p_2)(\varepsilon_1 \cdot \varepsilon_2) - (p_1 \cdot \varepsilon_2)(p_2 \cdot \varepsilon_1)) \tag{6.48}$$

and making the choice of reference momenta to be

$$r(\varepsilon_1) = p_2, \qquad r(\varepsilon_2) = p_1, \tag{6.49}$$

eliminates the last term in the amplitude. Also, this choice only allows a non-zero $++$ and $--$ amplitudes. Also, as we have stated before, $|M_{Hgg}^{++}| = |M_{Hgg}^{--}|$ so we only compute M_{Hgg}^{++}. We have $\varepsilon_1^+ \cdot \varepsilon_2^+ = \frac{[21]}{\langle 12 \rangle}$ and that $p_1 \cdot p_2 = \frac{s_{12}}{2}$. Substituting these and squaring the amplitude,

$$|M_{Hgg}^{++}|^2 = 4A^2 s_{12}^2 = 4A^2 m_H^4 \tag{6.50}$$

where in the last step we used the conservation of energy-momentum $p_1 + p_2 = p_H$ which means that $s_{12}^2 = m_H^4$. The decay rate for the 2 massless gluons can be found in [32] and we obtain

$$\Gamma_0 = \frac{|M_{Hgg}|^2}{32\pi m_H} = \frac{A^2 m_H^3}{8\pi}. \tag{6.51}$$

6.3.2 *H⟶gg(g)* real decay rate

For the $H \longrightarrow gg(g)$ real decay the decay rate has the form

$$\Gamma_r = \int |\overline{M}|^2 \frac{S}{2m_H} \rho \tag{6.52}$$

and can be found in [32], where for this case $S = \frac{1}{3!}$ because we have 3 identical particles in the final state. We have already computed the unpolarized amplitude given in equation 6.31 and the phase space in terms of dimensionless variables in equation 6.42. We will now perform the integration in equation 6.52 to obtain the real decay rate.

Whereas for the calculus of the virtual amplitude we applied the on-shell conditions and considered $p_1^2 = p_2^2 = 0$, we will here integrate the amplitude in a massive phase space where $p_1^2 = p_2^2 = \mu^2$ for the external particles and internal particles stay massless. This means that where before we had $s_{ij} = 2p_i \cdot p_j$, now we need to make the substitution $s_{ij} = (p_i + p_j)^2$ in the denominators, [8]. We can rewrite this in a more straightforward way to use our new variables using the energy-momentum conservation equation $p_1 + p_2 + p_3 - q = 0$,

$$s_{ij} = (p_i + p_j)^2 = (p_k - q)^2 \tag{6.53}$$

and substituting the old variables with the dimensionless ones and defining $\mu_0 = \frac{\mu^2}{q^2}$ we obtain

$$s_{ij} = q^2(\chi_k + \mu_0). \tag{6.54}$$

We have then the following relations using the equation of conservation of energy-momentum for the dimensionless variables

$$\begin{cases} s_{12} = q^2(\chi_3 + \mu_0) = q^2(1 - \chi_1 - \chi_2 + \mu_0) \\ s_{13} = q^2(\chi_2 + \mu_0) \\ s_{23} = q^2(\chi_1 + \mu_0) \end{cases} \tag{6.55}$$

We substitute the variables in 6.55 in the unpolarized amplitude 6.31 and then do all these substitutions in the decay rate in 6.52. After some manipulation of the mute variables of integrations we get the result

$$\Gamma_r = \int 192\pi\alpha_s A^2 \frac{1}{12m_H} \frac{1}{128\pi^3} q^2 \Big(2 + 3\chi_2 - \frac{4}{\chi_2} + \frac{5\chi_1}{\chi_2} - \frac{\chi_1^2}{\chi_2} + \frac{1}{\chi_1\chi_2}\Big). \tag{6.56}$$

The integrals were evaluated using the following equations which can be found in [8], [3],

$$I(s) = \int d\chi_1 d\chi_2 \frac{1}{(\chi_1 + \mu_0)(\chi_2 + \mu_0)} \tag{6.57}$$

and

$$J_p(s) = \int d\chi_1 d\chi_2 \frac{\chi_1^p}{(\chi_2 + \mu_0)} \tag{6.58}$$

with $p \geqslant 0$. Using our limit of integrations from 6.44 to 6.47 the integrals are evaluated to be

$$I(s) = \frac{\ln^2(\mu_0) - \pi^2}{2} \tag{6.59}$$

and

$$\begin{aligned} J_p(s) &= -\frac{1}{p+1}\ln(\mu_0) + \int_0^1 d\chi_1 \chi_1^p[\ln(\chi_1) + 2\ln(1-\chi_1)] \\ &= -\frac{1}{p+1}\ln(\mu_0) - \frac{1}{p+1}\Big[\frac{1}{p+1} + 2\sum_{n=1}^{p+1}\frac{1}{n}\Big]. \end{aligned} \tag{6.60}$$

We are now in conditions to perform the integration in equation 6.56 using 6.59 and 6.60. The effective coupling A is given in equation 4.11. We have defined $\mu_0 = \frac{\mu^2}{m_H^2}$, so we obtain

$$\Gamma_r = \int 192\pi\alpha_s A^2 \frac{1}{12m_H}\frac{1}{128\pi^3}\frac{3}{2}q^2\Big(\frac{73}{6} + \frac{11}{3}\ln(\frac{\mu^2}{m_H^2}) + \ln^2(\frac{\mu^2}{m_H^2}) - \pi^2\Big). \tag{6.61}$$

In the reference frame where the Higgs is at rest, $\vec{q} = 0$, we have $q^2 = q_0^2 = m_H^2$. Making use of equation 6.51 we have the final real decay rate written in function of tree level decay

$$\Gamma_r = \Gamma_0 \frac{\alpha_s}{\pi}\Big(\frac{73}{4} + \frac{11}{2}\ln\Big(\frac{\mu^2}{m_H^2}\Big) + \frac{3}{2}\Big(\ln^2\Big(\frac{\mu^2}{m_H^2}\Big) - \pi^2\Big)\Big). \tag{6.62}$$

Notice the real decay rate is also IR divergent with the divergences parameterized by the regulator μ.

Chapter 7

Discussion and conclusion

In previous work,[8] and [27], the decay rate using this effective model has been computed and has the form

$$\Gamma(H \longrightarrow gg(g)) = \Gamma_0(H \longrightarrow gg)\Big(1 + \delta\frac{\alpha_s}{\pi}\Big), \tag{7.1}$$

where $\delta = \frac{95}{4} - \frac{7}{6}N_F$, but as we are only considering here interactions between gluons and the Higgs boson we can set $N_F = 0$ and the decay rate is the following

$$\Gamma(H \longrightarrow gg(g)) = \Gamma_0(H \longrightarrow gg)\Big(1 + \frac{95}{4}\frac{\alpha_s}{\pi}\Big). \tag{7.2}$$

By combining our previous results in equations 5.54 and 6.62, we get the final decay rate for the process $H \longrightarrow gg(g)$ modelled by the effective model with the one-loop order correction

$$\Gamma_T(H \longrightarrow gg(g)) = \Gamma_v + \Gamma_r = \Gamma_0(H \longrightarrow gg)\Big(1 + \frac{\alpha_s}{\pi}\Big(\frac{73}{4} + \frac{11}{2}\ln\Big(\frac{\lambda^2}{m_H^2}\Big)\Big)\Big) \tag{7.3}$$

We need now to take into account the correction of $A = \frac{\alpha_s}{3\pi v}(1 + \frac{11}{4}\frac{\alpha_s}{\pi})$. We have that $\Gamma^0 \propto A^2$ and only keeping the terms until order α_s^3 we get the following correction in the decay rate

$$\Gamma_T(H \longrightarrow gg(g)) = \Gamma_0(H \longrightarrow gg)\Big(1 + \frac{\alpha_s}{\pi}\Big(\frac{95}{4} - \frac{11}{2}\ln\Big(\frac{m_H^2}{\lambda^2}\Big)\Big)\Big) \tag{7.4}$$

Notice that the cancellation of all logarithms depending on the regularization parameter μ happens only when combining the virtual and real decay rates. The logarithm that appears on the final result is parameterized only by the renormalization scale λ.

By choosing $\lambda = m_H$, we see that the total decay rate taking into account the one-loop virtual diagrams writes as

$$\Gamma_T(H \longrightarrow gg(g)) = \Gamma_0(H \longrightarrow gg)\left(1+\frac{95}{4}\frac{\alpha_s}{\pi}\right). \tag{7.5}$$

Looking at the previous equation we see that the correction is $\frac{95}{4}\frac{\alpha_s}{\pi}$. In [33] we find that the value $\alpha_s(m_Z) = 0.1175$ with a $\pm 0.9\%$ uncertainty and we can find the contribution of our correction to be an increment of $88,8\%$ in the final result, emphasizing the importance of the one-loop corrections in this decay rate.

Summarizing, we have concluded that the use of IReg in the effective decay $H \longrightarrow gg(g)$ verifies the KLN theorem. We also show that by adopting an effective theory to describe our system, we verify a cancellation of all the quadratic divergences ($I_{quad}(\mu^2)$) when summing the contributions of all the virtual diagrams and the $I_{log}(\mu^2)$ divergences were absorbed in the process of renormalization. Additionally, we verify that the UV behaviour of the amplitudes displayed in terms of BDI's should not be disregarded in IReg, as they are essential in the final decay rate so that all the divergences are cancelled and we get a result independent of the regulator μ. It should be noticed that along this computation we did not make any changes on the Lagrangian, or the amplitudes or in the dimension of the underlying theory. We simply have assumed an implicit regulator in an algebraic identity that allowed us to separate divergences and to make the amplitude have a more convenient form to be treated. We have also showed that spinor-helicity is a good framework to compute the Feynman amplitudes, as the task of computation of the amplitudes became much more easier. We showed that we can apply this formalism in Feynman amplitudes of an effective theory involving gluons and a massive scalar.

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Appendices

Appendix A

Divergence separation in the integrals

Diagram V_1

For the diagram V_1 all the integrals have the structure of one of the following and after applying the separation identity and collecting the terms we obtain

$$
\begin{aligned}
&\int_k \frac{k^\alpha k^\beta k^\sigma p^\delta}{k^2(k-p_1)^2(k+p_2)^2} = \\
&\int_k \frac{k^\alpha k^\beta k^\sigma p^\delta}{(k^2-\mu^2)^3} + \int_k \frac{k^\alpha k^\beta k^\sigma p^\delta(-2k\cdot p_2)}{(k^2-\mu^2)^4} + \int_k \frac{k^\alpha k^\beta k^\sigma p^\delta k^\nu(2k\cdot p_1)}{(k^2-\mu^2)^4} \\
&+ \int_k \frac{k^\alpha k^\beta k^\sigma p^\delta(2k\cdot p_1)(-2k\cdot p_2)}{(k^2-\mu^2)^4((k+p_2)^2-\mu^2)} + \int_k \frac{k^\alpha k^\beta k^\sigma p^\delta(-2k\cdot p_2)^2}{(k^2-\mu^2)^4((k+p_2)^2-\mu^2)} \\
&+ \int_k \frac{k^\alpha k^\beta k^\sigma p^\delta(2k\cdot p_1)^2}{(k^2-\mu^2)^3((k-p_1)^2-\mu^2)((k+p_2)^2-\mu^2)} \\
&= \int_k k^\alpha k^\beta k^\sigma k^\delta I^1_{V_1}(k,p_1,p_2,\mu)
\end{aligned}
\tag{A.1}
$$

The 1º-3º integrals are UV divergent. The 4º-5º integrals are IR divergent as $\frac{k^9}{k^9}$ and the 6º integral is IR finite.

$$
\begin{aligned}
&\int_k \frac{k^\alpha k^\beta p^\sigma p^\delta}{k^2(k-p_1)^2(k+p_2)^2} = \\
&\int_k \frac{k^\alpha k^\beta p^\sigma p^\delta}{(k^2-\mu^2)^3} + \int_k \frac{k^\alpha k^\beta p^\sigma p^\delta(-2k\cdot p_2)}{(k^2-\mu^2)^3((k+p_2)^2-\mu^2)} \\
&+ \int_k \frac{k^\alpha k^\beta p^\sigma p^\delta(2k\cdot p_1)}{(k^2-\mu^2)^2((k-p_1)^2-\mu^2)((k+p_2)^2-\mu^2)} \\
&= \int_k k^\alpha k^\beta p^\sigma p^\delta I^2_{V_1}(k,p_1,p_2,\mu)
\end{aligned}
\tag{A.2}
$$

The 1º integral is UV divergent, the 2º integral is IR divergent as $\frac{k^7}{k^7}$, the 3º integral is IR finite.

$$
\begin{aligned}
&\int_k \frac{k^\alpha p^\beta p^\sigma p^\delta}{k^2(k-p_1)^2(k+p_2)^2} \\
&= k^\alpha p^\beta p^\sigma p^\delta I^3_{V_1}(k,p_1,p_2,\mu)
\end{aligned}
\tag{A.3}
$$

The integral is UV finite and IR finite.

$$
\begin{aligned}
&\int_k \frac{p^\alpha p^\beta p^\sigma p^\delta}{k^2(k-p_1)^2(k+p_2)^2} \\
&= p^\alpha p^\beta p^\sigma p^\delta I^4_{V_1}(k,p_1,p_2,\mu)
\end{aligned}
\tag{A.4}
$$

The integral is UV finite and IR divergent as $\frac{k^4}{k^4}$.

$$
\begin{aligned}
&\int_k \frac{k^\alpha k^\beta k^2}{k^2(k-p_1)^2(k+p_2)^2} = \\
&\int_k \frac{k^\alpha k^\beta}{(k^2-\mu^2)^2} + \int_k \frac{k^\alpha k^\beta(-2k\cdot p_2)}{(k^2-\mu^2)^3} + \int_k \frac{k^\alpha k^\beta(2k\cdot p_1)}{(k^2-\mu^2)^3} \\
&+ \int_k \frac{k^\alpha k^\beta(-2k\cdot p_2)^2}{(k^2-\mu^2)^4} + \int_k \frac{k^\alpha k^\beta(2k\cdot p_1)(-2k\cdot p_2)}{(k^2-\mu^2)^4} + \int_k \frac{k^\alpha k^\beta(2k\cdot p_1)^2}{(k^2-\mu^2)^4} \\
&+ \int_k \frac{k^\alpha k^\beta(-2k\cdot p_2)^3}{(k^2-\mu^2)^4((k+p_2)^2-\mu^2)} + \int_k \frac{k^\alpha k^\beta(2k\cdot p_1)(-2k\cdot p_2)^2}{(k^2-\mu^2)^4((k+p_2)^2-\mu^2)} \\
&+ \int_k \frac{k^\alpha k^\beta(2k\cdot p_1)^2(-2k\cdot p_2)}{(k^2-\mu^2)^4((k+p_2)^2-\mu^2)} + \int_k \frac{k^\alpha k^\beta(2k\cdot p_1)^3}{(k^2-\mu^2)^3((k-p_1)^2-\mu^2)((k+p_2)^2-\mu^2)} \\
&= k^\alpha k^\beta I^5_{V_1}(k,p_1,p_2,\mu)
\end{aligned}
\tag{A.5}
$$

The 1º-6º integral are UV finite. The 7º-9º integrals are IR divergent as $\frac{k^9}{k^9}$ and the 10º integral is IR finite.

$$\begin{aligned}
&\int_k \frac{k^\alpha p^\beta}{(k-p_1)^2(k+p_2)^2} = \\
&\int_k \frac{k^\alpha p^\beta}{(k^2-\mu^2)^2} + \int_k \frac{k^\alpha p^\beta(-2k\cdot p_2)}{(k^2-\mu^2)^3} + \int_k \frac{k^\alpha p^\beta(2k\cdot p_1)}{(k^2-\mu^2)^3} \\
&+ \int_k \frac{k^\alpha p^\beta(2k\cdot p_1)(-2k\cdot p_2)}{(k^2-\mu^2)^3((k+p_2)^2-\mu^2)} + \int_k \frac{k^\alpha p^\beta(-2k\cdot p_2)^2}{(k^2-\mu^2)^3((k+p_2)^2-\mu^2)} \\
&+ \int_k \frac{k^\alpha p^\beta(2k\cdot p_1)^2}{(k^2-\mu^2)^2((k-p_1)^2-\mu^2)((k+p_2)^2-\mu^2)} \\
&= k^\alpha p^\beta I^6_{V_1}(k,p_1,p_2,\mu)
\end{aligned} \tag{A.6}$$

The 1º-3º integrals are UV finite. The 4º-5º integrals are IR divergent as $\frac{k^7}{k^7}$ and the 6º integral is IR finite.

$$\begin{aligned}
&\int_k \frac{p_a^\alpha p_b^\beta}{(k-p_1)^2(k+p_2)^2} = \\
&\int_k \frac{p_a^\alpha p_b^\beta}{(k^2-\mu^2)^2} + \int_k \frac{p_a^\alpha p_b^\beta(-2k\cdot p_2)}{(k^2-\mu^2)^2((k+p_2)^2-\mu^2)} \\
&+ \int_k \frac{p_a^\alpha p_b^\beta(2k\cdot p_1)}{(k^2-\mu^2)((k-p_1)^2-\mu^2)((k+p_2)^2-\mu^2)} \\
&= p^\alpha p^\beta I^7_{V_1}(k,p_1,p_2,\mu)
\end{aligned} \tag{A.7}$$

The 1º integral is UV divergent, the 2º integral is IR divergent as $\frac{k^5}{k^5}$ and the 3º integral is IR finite.

Diagram V_2

$$\begin{aligned}
&\int_k \frac{1}{(k-p_1-p_2)^2} = \\
&\int_k \frac{1}{k^2-\mu^2} + \int_k \frac{2k\cdot(p_1+p_2)}{(k^2-\mu^2)^2} + \int_k \frac{-2p1\cdot p2}{(k^2-\mu^2)^2} + \int_k \frac{(2k\cdot(p_1+p_2))^2}{(k^2-\mu^2)^3} \\
&+ \int_k \frac{2(-2p_1\cdot p_2)(2k\cdot(p_1+p_2))}{(k^2-\mu^2)^2((k-p_1-p_2)^2-\mu^2)} + \int_k \frac{(-2p_1\cdot p_2)^2}{(k^2-\mu^2)^2((k-p_1-p_2)^2-\mu^2)} \\
&+ \int_k \frac{(2k\cdot(p_1+p_2))^3}{(k^2-\mu^2)^3((k-p_1-p_2)^2-\mu^2)} + \int_k \frac{2(-2p_1\cdot p_2)(2k\cdot(p_1+p_2))^2}{(k^2-\mu^2)^3((k-p_1-p_2)^2-\mu^2)} \\
&= I^1_{V_2}(k,p_1,p_2,\mu)
\end{aligned} \tag{A.8}$$

The integral 1º-4º integrals are UV divergent. The 5º integral is IR divergent as $\frac{k^5}{k^5}$; the 6º integral is IR divergent as $\frac{k^4}{k^5}$; the 7º integral is IR divergent as $\frac{k^7}{k^7}$ and the 8º integral is IR divergent as $\frac{k^6}{k^7}$.

$$\begin{aligned}
&\int_k \frac{k^\alpha p^\beta}{k^2(k-p_1-p_2)^2} = \\
&\int_k \frac{k^\alpha p^\beta}{(k^2-\mu^2)^2} + \int_k \frac{k^\alpha p^\beta(2k\cdot(p_1+p_2)}{(k^2-\mu^2)^3} + \int_k \frac{k^\alpha p^\beta(-2p_1\cdot p_2)}{(k^2-\mu^2)^2((k-p_1-p_2)^2-\mu^2)} \\
&+ \int_k \frac{k^\alpha p^\beta(2k\cdot(p_1+p_2))^2}{(k^2-\mu^2)^3((k-p_1-p_2)^2-\mu^2)} + \int_k \frac{k^\alpha p^\beta(2k\cdot(p_1+p_2))(-2p_1\cdot p_2)}{(k^2-\mu^2)^3((k-p_1-p_2)^2-\mu^2)} \\
&= k^\alpha p^\beta I^2_{V_2}(k,p_1,p_2,\mu)
\end{aligned} \tag{A.9}$$

The 1º-2º integrals are UV divergent. The 3º-5º integrals are IR divergent as respectively: $\frac{k^5}{k^5}, \frac{k^7}{k^7}, \frac{k^6}{k^7}$.

$$\begin{aligned}
&\int_k \frac{k^\alpha k^\beta}{k^2(k-p_1-p_2)^2} = \\
&\int_k \frac{k^\alpha k^\beta}{(k^2-\mu^2)^2} + \int_k \frac{k^\alpha k^\beta(2k\cdot(p_1+p_2))}{(k^2-\mu^2)^3} + \int_k \frac{k^\alpha k^\beta(-2p_1\cdot p_2)}{(k^2-\mu^2)^3} \\
&+ \int_k \frac{k^\alpha k^\beta(2k\cdot(p_1+p_2))^2}{(k^2-\mu^2)^4} + \int_k \frac{2k^\alpha k^\beta(2k\cdot(p_1+p_2))(-2p_1\cdot p_2)}{(k^2-\mu^2)^3((k-p_1-p_2)^2-\mu^2)} \\
&+ \int_k \frac{k^\alpha k^\beta(-2p_1\cdot p_2)^2}{(k^2-\mu^2)^3((k-p_1-p_2)^2-\mu^2)} + \int_k \frac{k^\alpha k^\beta(2k\cdot(p_1+p_2))^3}{(k^2-\mu^2)^2((k-p_1-p_2)-\mu^2)} \\
&+ \int_k \frac{k^\alpha k^\beta(2k\cdot(p_1+p_2))^2(-2p_1\cdot p_2)}{(k^2-\mu^2)^4((k-p_1-p_2)^2-\mu^2)} \\
&= k^\alpha k^\beta I^3_{V_2}(k,p_1,p_2,\mu)
\end{aligned} \tag{A.10}$$

Diagrams V_3 and V_4

$$\begin{aligned}
&\int_k \frac{1}{(k-p_2)^2} = \\
&\int_k \frac{1}{k^2-\mu^2} + \int_k \frac{2k\cdot p_2}{(k^2-\mu^2)^2} + \int_k \frac{(2k\cdot p_2)^2}{(k^2-\mu^2)^3} + \int_k \frac{(2k\cdot p_2)^3}{(k^2-\mu^2)^3((k-p_2)^2-\mu^2)} \\
&= I^1_{V_i}(k,p_2,\mu)
\end{aligned} \tag{A.11}$$

The 1º-3º integrals are UV divergent and the 4º integral is IR divergent as $\frac{k^7}{k^7}$.

$$\begin{aligned}
&\int_k \frac{k^\alpha p^\beta}{k^2(k-p_2)^2} = \\
&\int_k \frac{k^\alpha p^\beta}{(k^2-\mu^2)^2} + \int_k \frac{k^\alpha p^\beta(2k\cdot p_2)}{(k^2-\mu^2)^3} + \int_k \frac{k^\alpha p^\beta(2k\cdot p_2)^2}{(k^2-\mu^2)^3((k-p_2)^2-\mu^2)} \\
&= k^\alpha p^\beta I^2_{V_i}(k,p_2,\mu)
\end{aligned} \tag{A.12}$$

The 1º-2º integrals are UV divergent and the 3º integral is IR divergent as $\frac{k^7}{k^7}$.

$$\begin{aligned}
&\int_k \frac{p^\alpha p^\beta}{k^2(k-p_2)^2} = \\
&\int_k \frac{p^\alpha p^\beta}{(k^2-\mu^2)^2} + \int_k \frac{(p^\alpha p^\beta)(2k\cdot p_2)}{(k^2-\mu^2)^2((k-p_2)^2-\mu^2)} \\
&= p^\alpha p^\beta I^3_{V_i}(k,p_2,\mu)
\end{aligned} \tag{A.13}$$

The 1º integral is UV divergent and the 2º integral is IR divergent as $\frac{k^5}{k^5}$.

$$
\begin{aligned}
&\int_k \frac{k^\alpha k^\beta}{k^2(k-p_2)^2} = \\
&\int_k \frac{k^\alpha k^\beta}{(k^2-\mu^2)^2} + \int_k \frac{k^\alpha k^\beta (2k\cdot p_2)}{(k^2-\mu^2)^3} + \int_k \frac{k^\alpha k^\beta (2k\cdot p_2)^2}{(k^2-\mu^2)^4} \\
+&\int_k \frac{k^\alpha k^\beta (2k\cdot p_2)^3}{(k^2-\mu^2)^4((k-p_2)^2-\mu^2)} \\
&= k^\alpha k^\beta I^7_{V_i}(k,p_2,\mu)
\end{aligned}
\tag{A.14}
$$

The 1º-3º integrals are UV divergent and the 4º integral is IR divergent as $\frac{k^9}{k^9}$. As the structure of the diagrams V_3 and V_4 is identical we can have $i = 3,4$.

Appendix B

Color factors

The fundamental representation of a group is the algebra smallest, non-trivial representation. The generators of the fundamental representation of SU(N) is a set of $N \times N$ Hermitian matrices with determinant equal to 1 that can be multiplied. For the SU(3) the generators can be written as

$$T^a = \frac{1}{2}\lambda^a \tag{B.1}$$

where λ^i are the Gell-Mann matrices with a=1,...8. They can be found explicitly computed in [31]. We normalize the generators conveniently in the following way

$$\sum_{cd} f^{acd} f^{bcd} = N\delta^{ab} \tag{B.2}$$

The adjoint representation is defined as the following

$$(T^a_{adj})^{bc} = -if^{abc} \tag{B.3}$$

and is given by 8×8 matrices in the SU(3) representation.