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Eulerian ideals and beyond

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Abstract

The polynomial ring $\mathbb{K}[x_1,\ldots,x_n]$, with $\mathbb{K}$ a field, is an important concept in commutative algebra. Mathematicians have been working with polynomial rings and their ideals since the late XIX century, but commutative algebra itself only came alive, as a field of mathematics, in the XX century. It was in 1921, with the work of Emmy Noether, that many of the current abstract concepts we study in commutative algebra drew the attention of the mathematical community.

Nowadays there is a new area of research that combines commutative algebra and combinatorics through the polynomial ring. In this work we will study some of the theory necessary to comprehend many concepts of this field of mathematics, now called combinatorial commutative algebra. We begin by studying general properties of modules and other related concepts, such as exact sequences and syzygy modules. We explain how to construct a free resolution of a module and enunciate the Hilbert's Syzygy Theorem. Then we move on to the theory of graded modules. We show syzygy modules can be seen as graded submodules, and define graded resolutions. For these we will also give the construction, and then enunciate the graded version of the Syzygy Theorem of Hilbert. We end the chapter of the preliminary theory by defining the Hilbert function, giving examples, and showing it is a function of polynomial type.

Regarding combinatorial commutative algebra, we will present one construction that connects the algebraic tools we mentioned before to the theory of graphs, *the Eulerian ideal of a graph*. We will present the results and proofs of Neves, Vaz Pinto, and Villarreal. We first characterize the generators of the ideal using the Eulerian subgraphs of the graph. We prove that the Hilbert polynomial of the quotient module by the Eulerian ideal is constant, and study the regularity index of this module. Then we present a characterization of this regularity index, for bipartite graphs, using the joins of the graph. After that, we study T-joins and present the connection between join and T-join. These results are then used to explicitly calculate the regularity index for the complete bipartite graphs, and Hamiltonian bipartite graphs. Afterwards, we generalize the construction of the Eulerian ideal for hypergraphs. We focus on k-uniform hypergraphs, and generalize for these the results presented for graphs. In particular, we characterize the regularity index for k-partite k-uniform hypergraphs, and calculate it for the complete k-partite case.

Resumo

O anel de polinómios $\mathbb{K}[x_1,\ldots,x_n]$, com $\mathbb{K}$ um corpo, é um conceito importante na Álgebra Comutativa. Os matemáticos têm trabalhado com anéis de polinómios e os seus ideais desde o final do século XIX, mas a Álgebra Comutativa apenas se concretizou como um ramo da matemática no século XX. Foi em 1921, com o trabalho de Emmy Noether, que muitos dos atuais conceitos abstratos que estudamos em Álgebra Comutativa, ganharam a atenção da comunidade matemática.

Hoje em dia, há uma nova área de investigação que combina a Álgebra Comutativa com a Combinatória, através do anel de polinómios. Neste trabalho, vamos estudar alguma da teoria necessária para compreender alguns conceitos deste ramo da matemática, que tem hoje o nome de Álgebra Comutativa Combinatória. Começamos por estudar propriedades gerais de módulos e de outros conceitos relacionados, como sequências exactas e módulos de sizígias. Explicamos como construir resoluções livres de um módulo e enunciamos o Teorema das Sizígias de Hilbert. Depois passamos para a teoria dos módulos graduados. Mostramos que os módulos de sizígias podem ser vistos como submódulos graduados, e definimos resoluções graduadas. Apresentamos também a sua construção, e de seguida enunciamos a versão graduada do Teorema das Sizígias de Hilbert. Terminamos o capítulo da teoria preliminar definindo a função de Hilbert, dando exemplos, e mostrando que esta é de tipo polinomial.

Relativamente à Álgebra Comutativa Combinatória, vamos apresentar uma construção que liga as ferramentas algébricas mencionadas à teoria dos grafos, *o ideal Euleriano de um grafo.* Vamos apresentar os resultados e as demonstrações de Neves, Vaz Pinto, e Villarreal. Primeiro caracterizamos os geradores do ideal usando os subgrafos Eulerianos do grafo. Mostramos que o polinómio de Hilbert do módulo quociente pelo ideal Euleriano é constante, e estudamos o índice de regularidade deste módulo. Nesse estudo caracterizamos o índice de regularidade para grafos bipartidos, através das junções do grafo. De seguida estudamos T-junções e apresentamos a relação entre junção e T-junção. Estes resultados são depois usados para calcular, de forma explícita, o índice de regularidade para os grafos bipartidos completos, e Hamiltonianos bipartidos. Depois generalizamos a construção do ideal Euleriano para hipergrafos. Focamo-nos em hipergrafos k-uniformes, e generalizamos para estes os resultados apresentados para grafos. Em particular, caracterizamos o índice de regularidade para hipergrafos k-uniformes k-partidos, calculando-o para o caso k-partido completo.

Contents

Chapter 1

Introduction

The ring of polynomials

The concept of *polynomial ring* was developed in the late XIX century by mathematicians working in number theory, algebraic geometry and invariant theory. Many of their studies involved sets of polynomials in several variables that were closed under certain operations, what we now call rings and ideals of polynomials. These concepts gained strength when David Hilbert presented an initial version of what would be called his "Basis Theorem" in his "Über die Theorie der algebraischen Formen" ([10]). This result solved, without the hard computations that were then expected, the most important problem in invariant theory at the time, see chapter 25 in [7].

At this moment in history, the term 'ring' had only been used in the context of rings of algebraic integers, and only by Dedekind and Hilbert ([11]). The definition of ring that we use today is due to Emmy Noether, which she gave in her "Idealtheorie in Ringbereichen" ([14]). Noether was not the first to give an abstract definition of ring, but it was the importance of Noether's paper that made her definition popular. Noether had considered, in this abstract context, rings that only had finitely generated ideals and showed that this was equivalent to the property that every ascending chain of ideals (with respect to inclusion) is eventually stationary, which is called the *ascending chain condition* ([7]). She then showed that in every commutative ring with this property, every ideal is a finite intersection of primary ideals; a result Hilbert, Lasker and Macaulay had previously proved for the polynomial ring over a field with a finite number of variables ([11]). The rings that satisfy the ascending chain condition are now called Noetherian rings in her honor. Noether's abstract notions and results became popular and from them grew the field now known as *commutative algebra*.

We define the polynomial ring as follows. Let $x_1,\dots,x_n$ be variables. If $(\alpha_1,\dots,\alpha_n) \in \mathbb{N}_0^n$, the associated monomial is defined as $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, with 1 also denoting $x_1^0\cdots x_n^0$. Multiplication of monomials is defined by $x_1^{\alpha_1}\cdots x_n^{\alpha_n}\cdot x_1^{\beta_1}\cdots x_n^{\beta_n} = x_1^{\alpha_1+\beta_1}\cdots x_n^{\alpha_n+\beta_n}$. The ring of polynomials with variables $x_1,\dots,x_n$ and coefficients in a field $\mathbb{K}$, denoted by $\mathbb{K}[x_1,\dots,x_n]$, is then the $\mathbb{K}$-vector space with basis the set of monomials and multiplication induced by the multiplication of monomials.

In a trend that began less than 50 years ago with R. Stanley and others ([18]), the polynomial ring $\mathbb{K}[x_1,\dots,x_n]$ is used to transport techniques and concepts from commutative algebra to combinatorics. This new area of interest is called combinatorial commutative algebra and will be one of the focuses of this work. Let us illustrate this idea with three examples.

Stanley–Reisner rings. A finite simplicial complex Δ on the vertex set $V = \{1,\dots,n\}$ is a collection of subsets of V such that Δ contains all singletons $\{i\}$ and every subset of a member of Δ also belongs to Δ. The Stanley-Reisner ring of Δ is the quotient of $\mathbb{K}[x_1,\dots,x_n]$ by the ideal generated by the monomials $x_{i_1}\cdots x_{i_k}$, such that $\{i_1,\dots,i_k\} \notin \Delta$. The great impact of this construction is in a paper of 1975 ([17]), in which Stanley proves an important conjecture for topological simplicial complexes, the Upper Bound Conjecture for Spheres. For more on this topic, see also [18].

Edge ideal and edge subring. Let G be a simple graph (i.e., G is undirected, without loops and multiple edges). Identify the vertex set of G with $\{1,\dots,n\}$. In $\mathbb{K}[x_1,\dots,x_n]$ the ideal generated by x_ix_j, for every edge $\{i,j\}$ of G, is called the *edge ideal of* G. The subring generated by the same monomials is called the *edge subring of* G. More on this ideal and on this subring can be found in [16], or in the chapter 5 of [9].

Binomial edge ideals. With G a simple graph with vertex set $\{1,\dots,n\}$, consider the variables $x_1,\dots,x_n,y_1,\dots,y_n$. The ideal of $\mathbb{K}[x_1,\dots,x_n,y_1,\dots,y_n]$ generated by the binomials $x_iy_j - x_jy_i$, for every edge $\{i,j\}$ of G, is called the *binomial edge ideal of* G. For more on this see the chapter 7 of [9].

Eulerian ideals

In Chapter 3 we will define a new class of ideals of a polynomial ring that one can associate to a simple graph. The construction is closely related to the *toric ideal* of the edge subring of a graph. Consider a polynomial ring $\mathbb{K}[t_{ij} : \{i,j\} \in E_G]$ where E_G is the edge set of G. Mapping each variable t_{ij} to x_ix_j defines a ring homomorphism $\varphi\colon \mathbb{K}[t_{ij} : \{i,j\} \in E_G] \to \mathbb{K}[x_1,\dots,x_n]$. The toric ideal of G is then $P_G = \ker\varphi$ and it follows that the edge subring of G is isomorphic to $\mathbb{K}[t_{ij} : \{i,j\} \in E_G]/P_G$. Also, it was proved by Villarreal, in [19], that P_G has a generating set consisting only of binomials, and that these are related to the even closed walks in G. The *Eulerian ideal of* G is defined as $\varphi^{-1}((x_i^2 - x_j^2 : i,j \in V_G))$, where V_G is the vertex set of G. This ideal contains the toric ideal of G and its generators can be described explicitly. Moreover, there is an algebraic invariant of this ideal, called the regularity index, which bears a close relation to a non-trivial graph invariant.

Structure of the text

In Chapter 2 we lay out some preliminary theory about modules and syzygy modules. We give the construction of free resolutions, and enunciate the Hilbert Syzygy Theorem. Then we present the graded version of this theory, and study the Hilbert function, proving it is of polynomial type.

In chapter 3 we study the Eulerian ideal of a graph, presenting the results of Neves, Vaz Pinto, and Villarreal, from [13]. We show the ideal is homogeneous and binomial, and characterize its generators using the graph. We show that the Hilbert polynomial of the quotient by this ideal is constant, and study the least nonnegative integer for which it is attained by the Hilbert function, the regularity index, characterizing it for bipartite graphs, and calculating it explicitly for some classes of graphs.

In Chapter 4, we present the construction of the Eulerian ideal for hypergraphs, and generalize the results from Chapter 3, about the generators and the regularity index, for k-uniform hypergraphs.

Due to lack of space, we will not present an introduction to the theory of Gröbner bases, nor give the proof of the Cayley-Hamilton theorem. We will assume the reader is familiar with this theory. As a reference for the results we will need, from the theory of Gröbner bases, we will follow [6].

Chapter 2

Graded Modules

This chapter focuses on the theory of free resolutions of graded modules over the polynomial ring. We start with an overview of the related concepts in the general setting of modules over a ring. Throughout, unless otherwise stated, R denotes a commutative ring with identity and $\mathbb{K}$ denotes a field.

2.1 Free Resolutions of Modules

Exact sequences

We will be mostly speaking of R-modules and R-homomorphisms between such modules. So, in order to simplify the notation, every time there is no ambiguity, we will leave out the ring R and write "module" and "homomorphism". A sequence of R-modules and R-homomorphisms is denoted by

$$\cdots \longrightarrow M_{i+1} \xrightarrow{\varphi_{i+1}} M_i \xrightarrow{\varphi_i} M_{i-1} \longrightarrow \cdots .$$

Such a sequence is not assumed to be infinite, it may be bounded on the left, i.e., it may start with a module, it may be bounded on the right or both.

Definition 2.1.1. Given a sequence of R-modules and R-homomorphisms, as above, we say that it is exact at M_i if $\mathrm{Im}(\varphi_{i+1}) = \ker(\varphi_i)$. We say that the entire sequence is exact if it is exact at each M_i for which M_{i-1} and M_{i+1} exist in the sequence.

Remarks 2.1.2. (i) Many properties of homomorphisms can be expressed in terms of exact sequences. For example, a homomorphism $\varphi : M \to N$ is surjective if and only if the sequence $M \xrightarrow{\varphi} N \to 0$ is exact, and it is injective if and only if $0 \to M \xrightarrow{\varphi} N$ is exact. (ii) In general, given $\varphi : M \to N$, the sequence $0 \to \ker(\varphi) \xrightarrow{i} M \xrightarrow{\varphi} N \xrightarrow{\pi} N/\mathrm{Im}(\varphi) \to 0$, where i is the inclusion and π is the canonical surjection, is exact. (iii) If $M \oplus N$ is the direct sum of R-modules M and N, the sequence $0 \to M \xrightarrow{\alpha} M \oplus N \xrightarrow{\beta} N \to 0$, where $\alpha(m) = (m,0)$ for every $m \in M$, and $\beta(m,n) = n$ for every $(m,n) \in M \oplus N$, is exact.

Generators, bases and rank

Definition 2.1.3. Let M be an R-module and $v_1, \ldots, v_m \in M$.

i) $\{v_1, \ldots, v_m\}$ is called a generating set for M if $M = \sum_{i=1}^{m} Rv_i = \{\sum_{i=1}^{m} r_i v_i : r_i \in R\}$.

ii) $\{v_1, \ldots, v_m\}$ is said an R-linearly independent set, if, for every $r_1, \ldots, r_m \in R$,

$$\textstyle\sum_{i=1}^{m} r_i v_i = 0 \implies r_1 = \cdots = r_m = 0.$$

iii) $\{v_1, \ldots, v_m\}$ is called a basis of M if it is an R-linearly independent generating set of M.

iv) M is called a free module if it has a basis.

Remarks 2.1.4. (i) The simplest case of a free R-module is R^m, for some $m > 0$. This module has a basis consisting of the elements: $e_1 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}^T, e_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^T, \ldots, e_m = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T$. We will refer to $\{e_1, e_2, \ldots, e_m\}$ as the *standard basis* of R^m. Note that throughout this work we will regard the elements of R^m as column vectors. (ii) If M is an R-module, choosing a set $\{v_1, \ldots, v_m\}$ of elements of M yields a homomorphism $\varphi \colon R^m \to M$, uniquely defined by $e_i \mapsto v_i$, for every element of the standard basis. Conversely, a homomorphism $\varphi \colon R^m \to M$ yields a set of m elements of M, identified by the images of the elements of the standard basis. (iii) If M is finitely generated, choosing a set of m generators of M is equivalent to choosing φ as above, surjective. If M is free, choosing a basis for M is equivalent to choosing φ an isomorphism.

The notions in Definition 2.1.3 follow closely the case of vector spaces, i.e., modules over a field. However there will be some significant differences in the theory, if not in the results obtained, certainly in the way proofs are carried out. To start with, contrary to the case of vector spaces, not all modules over a ring own a basis, i.e., not all modules are free. The following yields an easy example of this.

Example 2.1.5. Let $R = \mathbb{K}[x_1, \ldots, x_n]$. Then $M \subseteq R$ is a submodule of R if and only if M is an ideal. Assume $M \neq (0)$. Let us show that M is free if and only if M is a principal ideal. It suffices to show that for any generating set of M containing more than one element, we can find nontrivial R-linear combinations of zero. Let f_1, f_2 be two nonzero elements of a generating set of M. Then $f_2 f_1 - f_1 f_2 = 0$ is such an R-linear combination of zero. Thus, if M is free, a basis for M can only have one element. The converse is trivial because R is a domain.

Next we give an example of a nontrivial free submodule of R^3, where R is a polynomial ring, with a basis of cardinality two. The example is meant to illustrate the notions of Definition 2.1.3 and also point out to the notion of syzygy module, which shall be introduced later in the text.

Example 2.1.6. Let $R = \mathbb{K}[x, y]$ and consider the R-homomorphism $\varphi \colon R^3 \to R$ given, for every $u \in R^3$, by $\varphi(u) = \begin{bmatrix} x^2 y & y^3 + x^2 & x^4 \end{bmatrix} u$. Let $M = \ker \varphi$ and let us show that M is a free module with a basis of cardinality 2. As

$$-x^2(x^2 y) + 0(y^3 + x^2) + y(x^4) = -y^2(x^2 y) + x^2(y^3 + x^2) - 1(x^4) = 0,$$

it is clear that $v_1 = \begin{bmatrix} -x^2 & 0 & y \end{bmatrix}^T$ and $v_2 = \begin{bmatrix} -y^2 & x^2 & -1 \end{bmatrix}^T$ belong to M. It is also clear that these elements of M form an R-linearly independent set. Let us show that v_1 and v_2 generate M. Assume that $\begin{bmatrix} f & g & h \end{bmatrix}^T \in M$. Let $a(y), b(y) \in \mathbb{K}[y]$, $c(x) \in \mathbb{K}[x]$ and $g', h' \in \mathbb{K}[x, y]$ be such that

$$g = a(y)x + b(y) + g' x^2 \quad \text{and} \quad h + g' = h' y + c(x).$$

Then $\begin{bmatrix} f & g & h \end{bmatrix}^T - h'v_1 - g'v_2 = \begin{bmatrix} f' & a(y)x+b(y) & c(x) \end{bmatrix}^T$, where $f' = f + h'x^2 + g'y^2$. If we can show that $\begin{bmatrix} f' & a(y)x+b(y) & c(x) \end{bmatrix}^T = 0$ then we will have finished. Since $\begin{bmatrix} f' & a(y)x+b(y) & c(x) \end{bmatrix}^T \in M$ we get:

$$f'x^2y + (a(y) + xb(y))(y^3 + x^2) + c(x)x^4 = 0. \tag{2.1}$$

Setting $y = 0$ in the above and arguing on the degrees of the resulting monomials, we deduce that $c(x) = 0$, $a(y) = ya'(y)$ and $b(y) = yb'(y)$ for suitable $a'(y), b'(y) \in \mathbb{K}[y]$. Then (2.1) becomes:

$$f'x^2y + (ya'(y) + xyb'(y))(y^3 + x^2) = 0. \tag{2.2}$$

Setting $x = 0$ in the above, we get $a'(y) = 0$ and (2.2) becomes:

$$f'x^2y + xyb'(y)(y^3 + x^2) = 0 \implies f'x + b'(y)(y^3 + x^2) = 0. \tag{2.3}$$

Finally, setting $x = 0$ in (2.3) we deduce that $b'(y) = 0$ and then it follows that $f' = 0$. We conclude that $\begin{bmatrix} f' & a(y)x+b(y) & c(x) \end{bmatrix}^T = 0$.

To continue the comparison with the case of vector spaces, let us show that, if M is a free module over R, every basis of M has the same number of elements. By what was said above, this is equivalent to showing that if $R^m \to R^n$ is an isomorphism of R-modules then $m = n$. The proof, as we shall see, does not follow the traditional path of the proof in the case of vector spaces. There, a linearly independent set is shown to extend to a basis, while in the case of modules this is not true. Take, for example, $R = \mathbb{K}[x_1, \ldots, x_n]$ as a module over itself. If $f \in R$ is a nonconstant polynomial, $\{f\}$ is an R-linearly independent set that does not generate R. But as we saw in Example 2.1.5, a basis of R must have cardinality 1, so $\{f\}$ cannot be extended to a basis of R. Instead, the proof that if $R^m \to R^n$ is an isomorphism of R-modules then $m = n$, will involve using the surjectivity and injectivity to show the inequalities $m \geq n$ and $m \leq n$, respectively. To do so we need to recall the Cayley-Hamilton Theorem, which we present next in one of its most general forms, as it can be found in [5].

Proposition 2.1.7 (Cayley-Hamilton Theorem)**.** *Let $I \subseteq R$ be an ideal, and $\varphi\colon M \to M$ an R-homomorphism, with M an R-module generated by n elements. If $\varphi(M) \subseteq IM$, there is a polynomial $f(x) = x^n + r_{n-1}x^{n-1} + \cdots + r_1x + r_0$, with each $r_i \in I^{n-i}$, such that for every $v \in M$,*

$$\varphi(v)^n + r_{n-1}\varphi(v)^{n-1} + \cdots + r_1\varphi(v) + r_0v = 0.$$

The first inequality will follow from the next Proposition, which is a generalization of a known result for vector spaces, that a surjective endomorphism is an isomorphism.

Proposition 2.1.8. *Let M be a finitely generated R-module. An R-homomorphism, $\varphi\colon M \to M$, that is surjective, is an isomorphism.*

Proof. Let $R[x]$ be the polynomial ring in the variable x, with coefficients in the ring R. Consider M as an $R[x]$-module with the multiplication induced by setting the product $x \cdot v = \varphi(v)$. That is, the multiplication is defined by $(\sum_{i=0}^d r_ix^i) \cdot v = \sum_{i=1}^d r_i\varphi^i(v) + r_0v$, for every polynomial $\sum_{i=0}^d r_ix^i$ with coefficients in R, and every $v \in M$, where φ^i is the composition of φ, i times. Also, take the ideal (x) of $R[x]$. Since φ is surjective, for every $v \in M$, there is $u \in M$ such that $v = \varphi(u) = x \cdot u$, which is in

$(x)M$. Therefore $M = (x)M$, and using Proposition 2.1.7 with the identity homomorphism in M, there are $f_1, \dots, f_n \in (x) \subseteq R[x]$ such that, for every $v \in M$, $v + f_{n-1} \cdot v + \cdots + f_1 \cdot v + f_0 \cdot v = 0$. This implies there is a polynomial $g \in R[x]$ such that $v = (gx) \cdot v$, for all $v \in M$. Now, setting $\psi \colon M \to M$ as the R-homomorphism defined by $\psi(u) = g \cdot u$, for all $u \in M$, we obtain that $\mathrm{id}_M(u) = u = g \cdot (x \cdot u) = \psi(\varphi(u))$, for all $u \in M$, so ψ is the inverse of φ and φ is an isomorphism. □

Corollary 2.1.9. *Let $\varphi \colon R^m \to R^n$ be a surjective homomorphism of R-modules. Then $m \geq n$.*

Proof. We argue by contradiction. Suppose that $m < n$, and let $e_1, \dots, e_n$ be the standard basis of R^n. We can define a surjective R-homomorphism, $\psi \colon R^n \to R^m$ by sending each e_i, with $i = 1, \dots, m$, to a different element of the standard basis of R^m, and sending the $e_{m+1}, \dots, e_n$ to zero. Then $\varphi \circ \psi$ is a surjective endomorphism, and by Proposition 2.1.8 it is an isomorphism. However, $(\varphi \circ \psi)(e_n) = 0$, so $\varphi \circ \psi$ is not injective and we have obtained a contradiction. We conclude that $m \geq n$. □

Note that Corollary 2.1.9 is enough to show that if $\varphi \colon R^m \to R^n$ is an isomorphism then $m = n$. From φ being an isomorphism, $\varphi^{-1} \colon R^n \to R^m$ is surjective and also $n \geq m$. However, to continue the analogy between R-modules and vector spaces, we present the next Proposition from which the same conclusion follows.

Proposition 2.1.10. *Let $\varphi \colon R^m \to R^n$ be an injective homomorphism of R-modules. Then $m \leq n$.*

Proof. Arguing by contradiction, assume that $m > n$. Let $\xi \colon R^n \to R^m$ be the injective homomorphism defined by $\begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix}^T \mapsto \begin{bmatrix} f_1 & \cdots & f_n & 0 & \cdots & 0 \end{bmatrix}^T$, for every $\begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix}^T \in R^n$. Now $\psi = \xi \circ \varphi \colon R^m \to R^m$ is an injective R-endomorphism that verifies $\psi(R^m) \subseteq RR^m = R^m$. By Proposition 2.1.7, there exist $r_0, \dots, r_{m-1} \in R$ such that the equation

$$\psi^m(v) + r_{m-1}\psi^{m-1}(v) + \cdots + r_1\psi(v) + r_0 v = 0, \tag{2.4}$$

holds for every $v \in R^m$. Letting $v = e_m$ of the standard basis, (2.4) becomes a system of m equations with the last one being $r_0 = 0$. Now (2.4) becomes $\psi(\psi^{m-1}(v) + r_{m-1}\psi^{m-2}(v) + \cdots + r_1 v) = 0$ for every $v \in R^m$, and since ψ is injective we obtain that $\psi^{m-1}(v) + r_{m-1}\psi^{m-2}(v) + \cdots + r_1 v = 0$. Again choosing $v = e_m$, we can repeat this argument until we obtain that $\psi(v) = 0$, for every $v \in R^m$. This contradicts ψ being injective and therefore we must have that $m \leq n$. □

Corollary 2.1.11. *If $\varphi \colon R^m \to R^n$ is an isomorphism of R-modules then $m = n$, in particular, every basis of a free module has equal cardinality.*

Definition 2.1.12. Let M be a free R-module. Then the *rank of M* is the cardinality of the bases of M.

Remark 2.1.13. As usual, the trivial R-module, $M = \{0\}$, is considered a free module of rank 0 with basis the empty set. For convenience, specially in the constructions below, we will denote M by 0.

Syzygies and free resolutions

Definition 2.1.14. Let M be an R-module. Given $v_1, \dots, v_m$, elements of M, we say $\begin{bmatrix} a_1 & \cdots & a_m \end{bmatrix}^T \in R^m$ is a syzygy of $v_1, \dots, v_m$ if $a_1 v_1 + \cdots + a_m v_m = 0$. The set of all syzygies, which can be seen as a kernel of the homomorphism $\varphi \colon R^m \to M$ defined by $v_1, \dots, v_m$, is a submodule of R^m and is denoted by $\mathrm{Syz}(v_1, \dots, v_m)$.

Example 2.1.15. Let $R = \mathbb{K}[x,y]$, with $\mathbb{K}$ a field. Let $I = (x^2y, y^3+x^2, x^4)$ be an ideal of R, viewed as a submodule of R. Then I is the image of the homomorphism of R-modules, $\varphi\colon R^3 \to R$, of Example 2.1.6. Accordingly, the module $\mathrm{Syz}(x^2y, y^3+x^2, x^4) \subseteq R^3$ is free with basis $v_1 = \begin{bmatrix} -x^2 & 0 & y \end{bmatrix}^T$ and $v_2 = \begin{bmatrix} -y^2 & x^2 & -1 \end{bmatrix}^T$. It is then clear that, in turn, $\mathrm{Syz}(v_1, v_2) \subseteq R^2$ is the zero submodule.

Definition 2.1.16. Let R be a ring, we say that an R-module M is *Noetherian* if every submodule of M is finitely generated. Also, we say that R is Noetherian if it is a Noetherian R-module.

Recall that Noetherian modules can equivalently be defined by the *ascending chain condition*, i.e., a Noetherian module is a module in which every ascending chain of submodules $M_1 \subseteq M_2 \subseteq \cdots$ is eventually stationary. By the Hilbert's basis theorem, the ring of polynomials $\mathbb{K}[x_1, \ldots, x_n]$ with coefficients in a field is Noetherian. For the most of this work , we will focus on finitely generated modules over this ring. A key property will be the fact that the finitely generated free modules over it are also Noetherian, which holds for any Noetherian ring. This is what we will show next in a Proposition based in the Proposition 6.3 of [1].

Proposition 2.1.17. *Let R be a ring, and K, M, and N be R-modules. Given an exact sequence, $0 \longrightarrow K \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$, the module M is Noetherian if and only if N and K are.*

Proof. Let us show that K is Noetherian if M is. We will prove that any ascending chain, $K_1 \subseteq K_2 \subseteq \cdots$, of submodules of K is stationary. Such a chain induces the ascending chain $\alpha(K_1) \subseteq \alpha(K_2) \subseteq \cdots$ of submodules of M. By the assumption on M, there is $k \in \mathbb{N}$ such that $\alpha(K_l) = \alpha(K_k)$ for every $l \geq k$. Since α is injective, for every $l \geq k$, $K_l = \alpha^{-1}(\alpha(K_l)) = \alpha^{-1}(\alpha(K_k)) = K_k$, and we conclude that K is Noetherian. Reasoning alike, it follows from β being surjective that N is Noetherian. Conversely, assume N and K are Noetherian. Let us show that any ascending chain $M_1 \subseteq M_2 \subseteq \cdots$ of submodules of M is stationary. Such a chain induces two ascending chains of submodules, $\alpha^{-1}(M_1) \subseteq \alpha^{-1}(M_2) \subseteq \cdots$, and $\beta(M_1) \subseteq \beta(M_2) \subseteq \cdots$, of K and N respectively. They are stationary and there exists $k \in \mathbb{N}$, such that for every $l \geq k$, $\alpha^{-1}(M_l) = \alpha^{-1}(M_k)$, and $\beta(M_l) = \beta(M_k)$. It now suffices to show that, for $l \geq k$, $M_l \subseteq M_k$. Given v in M_l, $\beta(v)$ is in $\beta(M_l)$ and there is $u \in M_k$ such that $\beta(v) = \beta(u)$. Then $v-u$ is in $\ker(\beta) = \mathrm{Im}(\alpha)$, and so $v-u = \alpha(z)$ for some z in K. Since $v-u$ is in M_l, z is in $\alpha^{-1}(M_l) = \alpha^{-1}(M_k)$, and so $\alpha(z)$ is in M_k. We conclude that $v = u + \alpha(z)$ is in M_k, and so $M_l = M_k$, for every $l \geq k$. Therefore M is Noetherian. $\square$

Corollary 2.1.18. *Let R be a Noetherian ring. Then R^m, for every $m > 1$, is a Noetherian R-module.*

Proof. Consider the exact sequence $0 \to R \xrightarrow{\iota} R^m \xrightarrow{\rho} R^{m-1} \to 0$, where ι is given by $r \mapsto \begin{bmatrix} r & 0 & \ldots & 0 \end{bmatrix}^T$, for every $r \in R$, and ρ is given by $\begin{bmatrix} r_1 & \cdots & r_m \end{bmatrix}^T \mapsto \begin{bmatrix} r_2 & \cdots & r_m \end{bmatrix}^T$, for every $\begin{bmatrix} r_1 & \cdots & r_m \end{bmatrix}^T \in R^m$. Using Proposition 2.1.17, the result now follows by induction on m. $\square$

Corollary 2.1.19. *Let R be a Noetherian ring, and M an R-module. If M is finitely generated, then it is Noetherian.*

Proof. A finite generating set for M, say with m elements, induces a surjective R-homomorphism $\varphi\colon R^m \to M$. Then the sequence $0 \longrightarrow \ker(\varphi) \xrightarrow{i} R^m \xrightarrow{\varphi} M \longrightarrow 0$, with i the inclusion, is exact, and using Proposition 2.1.17 and Corollary 2.1.18, we obtain that M is Noetherian. $\square$

Definition 2.1.20. Assume R is a Noetherian ring and M is a finitely generated R-module. A *presentation for M* is given by a list of generators $v_1,\dots,v_m$ of M and a list of generators, $u_1,\dots,u_n \in R^m$ for $\mathrm{Syz}(v_1,\dots,v_m)$. The *presentation matrix* of a presentation of M is the $m\times n$ matrix (with entries in R) the columns of which are $u_1,\dots,u_n$.

Remarks 2.1.21. (i) By Corollary 2.1.18, any submodule of R^m is finitely generated and, in particular, so is $\mathrm{Syz}(v_1,\dots,v_m)$. (ii) Let M be a finitely generated R-module. Having a presentation for M is equivalent to having an exact sequence

$$R^n \xrightarrow{\psi} R^m \xrightarrow{\varphi} M \to 0. \tag{2.5}$$

Indeed, a list of generators, $v_1,\dots,v_m$, of M yields a surjective homomorphism $\varphi\colon R^m \to M$, and a list of generators of $\ker\varphi = \mathrm{Syz}(v_1,\dots,v_m)$, of size n, yields a homomorphism ψ making (2.5) exact. Conversely, given an exact sequence as (2.5), the images of the standard bases of R^m and R^n by φ and ψ, respectively, give generators $v_1,\dots,v_m$ for M, and generators $u_1,\dots,u_n$ for $\mathrm{Syz}(v_1,\dots,v_m)$. Under this equivalence, the homomorphism ψ is given by multiplication with the presentation matrix. (iii) Note that if A is a presentation matrix of M then, by (2.5), M is isomorphic to R^m/AR^n, where $AR^n = \mathrm{Im}\,\psi$, in particular, any two modules with a common presentation matrix are isomorphic and vice-versa. (iv) If besides the generating sets of M and $\mathrm{Syz}(v_1,\dots,v_m)$, we choose a generating set for $\mathrm{Syz}(u_1,\dots,u_n)$, then we can extend (2.5) to

$$R^s \xrightarrow{\xi} R^n \xrightarrow{\psi} R^m \xrightarrow{\varphi} M \to 0.$$

Continuing in this fashion, we would obtain an infinite exact sequence.

Definition 2.1.22. Let M be an R-module. A free resolution of M is an exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$$

where for all i, F_i is a free R-module R^{m_i}, for some $m_i \geq 0$. If there is an l such that $F_{l+k} = 0$, for all $k \geq 1$, but $F_l \neq 0$, then we say the resolution is finite of length l. In this case we will write the resolution as $0 \to F_l \to F_{l-1} \to \cdots \to F_1 \to F_0 \to M \to 0$.

The next Proposition is the conclusion of the remarks above.

Proposition 2.1.23. *Any finitely generated module over a Noetherian ring has a free resolution.*

Example 2.1.24. Recall the ideal $I = (x^2y, y^3+x^2, x^4)$ of $R = \mathbb{K}[x,y]$ from the Examples 2.1.6 and 2.1.15. It is simple to construct a free resolution of I following the ideas of the last remarks. I is the image of an R-homomorphism $\varphi\colon R^3 \to R$, given by its generators. Also, we showed that there exist $v_1, v_2 \in R^3$ such that $\mathrm{Syz}(x^2y, y^3+x^2, x^4) = Rv_1 + Rv_2$ and $\mathrm{Syz}(v_1,v_2) = 0 \subseteq R^2$. Let ψ be defined by v_1, v_2. We obtain the free resolution $0 \to R^2 \xrightarrow{\psi} R^3 \xrightarrow{\varphi} I \to 0$.

Hilbert's Syzygy Theorem

In the Example 2.1.24, we saw an example of a finite free resolution. This raises the following question: Does a finitely generated module over a Noetherian ring always have a finite free resolution?

The answer is in general negative, in fact there are rings for which only the free modules have finite free resolutions. However, it is important to mention that this will not be the case if the ring is a polynomial ring over a field. This was first shown by Hilbert in [10], and is called the Hilbert's Syzygy Theorem. The following formulation of this result, together with its proof, can be found in [4].

Theorem 2.1.25 (Hilbert's Syzygy Theorem)**.** *Let $R = \mathbb{K}[x_1, \ldots, x_n]$, with $\mathbb{K}$ a field. Every finitely generated R-module has a finite free resolution of length at most n.*

2.2 Graded Modules

Graded rings, modules, and submodules

In a polynomial ring $\mathbb{K}[x_0, x_1, \ldots, x_n]$ over a field $\mathbb{K}$ every polynomial can be written uniquely as a finite sum of homogeneous polynomials. Another way of saying this is that the polynomial ring is a direct sum of the additive subgroups

$$\mathbb{K}[x_0, \ldots, x_n]_k = \{f \in \mathbb{K}[x_0, \ldots, x_n] : \text{every term of } f \text{ has degree } k\} \cup \{0\},$$

for $k \geq 0$. Additionally, when $f \in \mathbb{K}[x_0, \ldots, x_n]_s$ and $g \in \mathbb{K}[x_0, \ldots, x_n]_k$ the product fg belongs to $\mathbb{K}[x_0, \ldots, x_n]_{s+k}$, for any $k, s \geq 0$. These properties define the general notion of a graded ring.

Definition 2.2.1. Let R be a commutative ring with identity. A graded ring structure on R is given by a family $\{R_k : k \geq 0\}$ of subgroups of the additive group of R, such that $R = \bigoplus_{k \geq 0} R_k$ and $R_s R_k \subseteq R_{s+k}$, for every $k, s \geq 0$. If f is in R_k we will say that f is homogeneous of degree k, and write $\deg(f) = k$.

The canonical graded ring structure on $\mathbb{K}[x_0, \ldots, x_n]$ described in the beginning of this section is referred to as the *standard grading* of $\mathbb{K}[x_0, \ldots, x_n]$. We will see in the example below that polynomial rings afford other gradings.

Example 2.2.2. A grading of $\mathbb{K}[x_0, \ldots, x_n]$, potentially different from the standard one, is obtained by setting $\deg(x_i) = d_i$, for some arbitrary integers $d_i \geq 1$. For example, let $R = \mathbb{K}[x_1, x_2, y]$ and set $\deg(x_1) = \deg(x_2) = 1$ and $\deg(y) = 2$. Then, in this grading,

$$R_0 = \mathbb{K}1, \quad R_1 = \mathbb{K}x_1 \oplus \mathbb{K}x_2, \quad R_2 = \mathbb{K}x_1^2 \oplus \mathbb{K}x_1x_2 \oplus \mathbb{K}x_2^2 \oplus \mathbb{K}y, \quad \text{etc.}$$

Polynomial rings endowed with this type of grading are called *weighted polynomial rings*. There are good reasons to study these objects. One being that they appear naturally associated to projective varieties in Algebraic Geometry.

From now on we assume that R is a graded ring.

Definition 2.2.3. A graded module over R is a module M with a family of subgroups $\{M_k : k \in \mathbb{Z}\}$ of the additive group of M, such that $M = \bigoplus_{k \in \mathbb{Z}} M_k$ and $R_s M_k \subseteq M_{s+k}$, for all $s \geq 0$ and $k \in \mathbb{Z}$. The subgroup M_k is called the homogeneous component of degree k of M, and its elements are called the homogeneous elements of degree k of M. If $v \in M$, for every $k \in \mathbb{Z}$, denote by $[v]_k$ the unique element of M_k such that $v = \sum_{k \in \mathbb{Z}} [v]_k$. The element $[v]_k$ is called the homogeneous component of v in degree k.

Remarks 2.2.4. (i) Since R is itself an R-module, extending the standard grading of R by $R_k = \{0\}$, for every $k < 0$, makes R into a graded module over itself. (ii) Note that if M is a graded R-module and $v \in M$ then only finitely many of the homogeneous components of v, $[v]_k$, for $k \in \mathbb{Z}$, are nonzero.

Graded modules appear extensively in the theory we are laying out. The constructions of the induced graded structures on submodules and direct sums also apply, as we define next, thus yielding a wealth of interesting examples.

Definition 2.2.5. (i) If M is a graded R-module and N is a submodule of M then N is said a *graded submodule of* M if $N = \bigoplus_{k\in\mathbb{Z}}(M_k \cap N)$. In this situation we set $N_k = M_k \cap N$, for every $k \in \mathbb{Z}$. (ii) If M and N are graded R-modules, the induced grading on the direct sum $M \oplus N$ is given by $(M \oplus N)_k = M_k \oplus N_k$, for every $k \in \mathbb{Z}$.

Remarks 2.2.6. (i) If R is a polynomial ring with the standard grading, and $I \subseteq R$ is a homogeneous ideal, I is a graded submodule of R. (ii) For every integer $m \geq 1$, R^m is a graded R-module by considering the additive subgroups $(R^m)_k = (R_k)^m$, for every $k \in \mathbb{Z}$. This will be called the *standard grading* on R^m.

Proposition 2.2.7. *Let M be a graded R-module and N a submodule of M. The following are equivalent:*

i) N is a graded submodule of M;

ii) For every $v \in N$, the homogeneous components of v in M are also in N;

iii) N has a generating set consisting of homogeneous elements of M.

Proof. Let us start by showing that (i) implies (ii). Suppose N is a graded submodule of M and take $v \in N$. Since $N = \bigoplus_{k\in\mathbb{Z}}(M_k \cap N)$, for every $k \in \mathbb{Z}$ there exist $v_k \in M_k \cap N$, where at most finitely many are nonzero, such that $v = \sum_{k\in\mathbb{Z}} v_k$. Since $M = \bigoplus_{k\in\mathbb{Z}} M_k$, we deduce that v_k are the homogeneous components of v in M.

To prove that (ii) implies (iii), let $\mathscr{S} \subseteq N$ be a generating set of N, and assume that the homogeneous components in M of elements of N also belong to N. Let $\mathscr{S}'$ be the set of all homogeneous components of elements of $\mathscr{S}$. Then $\mathscr{S}'$ is a generating set for N and consists of homogeneous elements of M.

Finally let us show that (iii) implies (i). Let $\mathscr{S}$ be a generating set of N consisting of homogeneous elements of M. Given $f_1, \ldots, f_r \in R$, and $v_1, \ldots, v_r \in \mathscr{S}$, the R-linear combination $f_1v_1 + \cdots + f_rv_r$ is in N. Then, as $v_1, \ldots, v_r$ are homogeneous elements of M, say of degrees $d_1, \ldots, d_r$, we see that $[f_1v_1 + \cdots + f_rv_r]_k = [f_1]_{k-d_1}v_1 + \cdots + [f_r]_{k-d_r}v_r$ is in $M_k \cap N$. Hence any R-linear combination of elements of $\mathscr{S}$ belongs to $\bigoplus_{k\in\mathbb{Z}}(M_k \cap N)$, which implies that $N = \bigoplus(M_k \cap N)$, in other words, that N is a graded submodule of M. $\square$

Example 2.2.8. Let $M \neq \{0\}$ be a graded R-module and f an element of R. The set $fM = \{fv : v \in M\}$ is a submodule of M. Let us show fM is a graded submodule of M if f is a homogeneous element of R. Assume that f is homogeneous of degree d. For every $v \in M$, we have that $fv = \sum_{k\in\mathbb{Z}} f[v]_k$, with each $f[v]_k$ in M_{d+k}. Now, we see that $\{fv : v \in M_k \text{ and } k \in \mathbb{Z}\}$ is a generating set for fM consisting of homogeneous elements of M. By Proposition 2.2.7 fM is a graded submodule of M.

Remark 2.2.9. Every finitely generated graded R-module, M, has a finite generating set consisting of homogeneous elements. Let us show this. If $\mathscr{S}$ is a finite generating set of M, the set of the homogeneous components of the elements of $\mathscr{S}$ is still finite, and generates M. In particular, if R is a Noetherian ring the syzygy modules are finitely generated and so, if they are graded, they will have a finite generating set consisting of homogeneous elements.

Another well known module that can be endowed with a grading is the quotient module. The next few results will establish its natural grading, but first, let us recall a concept from group theory. The external direct sum $\bigoplus_{k\in\mathbb{Z}} A_k$, of abelian groups A_k, is the abelian group of sequences $(a_k)_{k\in\mathbb{Z}}$, with the component-wise sum, such that $a_k \in A_k$ for all $k \in \mathbb{Z}$, and only finitely many a_k are not the identity element of the respective group.

Proposition 2.2.10. *Let M be an abelian group, and $\{M_k : k \in \mathbb{Z}\}$ a family of subgroups of M such that $M = \bigoplus_{k\in\mathbb{Z}} M_k$. If N is a graded subgroup of M, with $N_k = M_k \cap N$ for all $k \in \mathbb{Z}$, there is a group isomorphism*

$$M/N \cong \bigoplus_{k\in\mathbb{Z}} (M_k/N_k).$$

Proof. Since M is abelian, all its subgroups are normal, and in particular, N_k is a normal subgroup of M_k. This means that the quotient groups M/N and M_k/N_k are well defined. Let $\pi\colon M \to \bigoplus_{k\in\mathbb{Z}}(M_k/N_k)$ be the function given by $\sum_{k\in\mathbb{Z}} m_k \mapsto (m_k + N_k)_{k\in\mathbb{Z}}$, for all $\sum_{k\in\mathbb{Z}} m_k \in M$, and where each m_k is in M_k. In each element $\sum_{k\in\mathbb{Z}} m_k$ of M, only finitely many m_k are not the identity element of M, so $(m_k + N_k)_{k\in\mathbb{Z}}$ is in $\bigoplus_{k\in\mathbb{Z}}(M_k/N_k)$ and π is well defined. Also, π is a surjective group homomorphism with kernel $\bigoplus_{k\in\mathbb{Z}} N_k$, and since $N = \bigoplus_{k\in\mathbb{Z}} N_k$, from the First Isomorphism Theorem the result follows. □

Proposition 2.2.11. *Let M be a graded R-module and N a graded submodule of M. The group $\bigoplus_{k\in\mathbb{Z}}(M_k/N_k)$ can be given the structure of a graded R-module.*

Proof. Let us denote $\bigoplus_{k\in\mathbb{Z}}(M_k/N_k)$ by G, and consider it as an R-module with the multiplication defined, for every $f \in R$ and $(m_k + N_k)_{k\in\mathbb{Z}} \in G$, by

$$f(m_k + N_k)_{k\in\mathbb{Z}} = \left(\left(\sum_{i=0}^{\deg(f)} [f]_i m_{k-i} \right) + N_k \right)_{k\in\mathbb{Z}}.$$

Note that this multiplication is well defined as only finitely many m_k are nonzero. Let us endow G with a grading by setting $G_k = \cdots \times \{0 + N_{k-1}\} \times M_k/N_k \times \{0 + N_{k+1}\} \times \cdots$, for every $k \in \mathbb{Z}$, where 0 is the zero of M. Each G_k is a subgroup of G, and we have that $G = \bigoplus_{k\in\mathbb{Z}} G_k$. Also, by the multiplication follows that $R_i G_k \subseteq G_{i+k}$, for every $i \geq 0$ and $k \in \mathbb{Z}$, so G is now a graded R-module. □

In the next Corollary, $\bigoplus_{k\in\mathbb{Z}}(M_k/N_k)$ is an R-module as in the proof of Proposition 2.2.11.

Corollary 2.2.12. *Let M be a graded R-module and N a graded submodule of M. The graded R-module $\bigoplus_{k\in\mathbb{Z}}(M_k/N_k)$ induces a grading in M/N through group isomorphisms $(M/N)_k \cong M_k/N_k$.*

Proof. Let $\varphi\colon M/N \to \bigoplus_{k\in\mathbb{Z}}(M_k/N_k)$ be the isomorphism given by Proposition 2.2.10. By the First Isomorphism Theorem, φ is given by $m + N \mapsto \pi(m) = ([m]_k + N_k)_{k\in\mathbb{Z}}$, with $\pi\colon M \to \bigoplus_{k\in\mathbb{Z}}(M_k/N_k)$ the homomorphism from Proposition 2.2.10. Consider $G = \bigoplus_{k\in\mathbb{Z}}(M_k/N_k)$ with the graded R-module structure given by Proposition 2.2.11. Let G_k denote the homogeneous component of degree k of G. For every integer k, set $(M/N)_k = \varphi^{-1}(G_k)$, and let us show this defines a grading in M/N. Since φ^{-1} is a group homomorphism, these are subgroups of M/N that decompose it as a direct sum:

$$M/N = \varphi^{-1}(\bigoplus_{k\in\mathbb{Z}}(M_k/N_k)) = \varphi^{-1}(\bigoplus_{k\in\mathbb{Z}} G_k) = \bigoplus_{k\in\mathbb{Z}} \varphi^{-1}(G_k).$$

Now, to show the second condition of a grading, for every $f \in R_s$ and every $m+N \in (M/N)_k$, we have that $fm+N = \varphi^{-1}(\dots,0,fm+N_{k+s},0,\dots) = \varphi^{-1}(f(\dots,0,m+N_k,0,\dots))$. This implies that $f(m+N) = fm+N$ is in $\varphi^{-1}(R_sG_k) \subseteq \varphi^{-1}(G_{s+k})$, and therefore $R_s(M/N)_k \subseteq (M/N)_{s+k}$ for all $s \geq 0$ and $k \in \mathbb{Z}$. This shows that $(M/N)_k = \varphi^{-1}(G_k)$ defines a grading in M/N. To conclude, for every integer k we see from the definition of G_k, in Proposition 2.2.11, that G_k and M_k/N_k are isomorphic groups. This way $(M/N)_k = \varphi^{-1}(G_k) \cong G_k \cong M_k/N_k$. □

Definition 2.2.13. If M is a graded R-module and N is a graded submodule of M, M/N is a graded R-module through the identification $(M/N)_k = M_k/N_k$. We will call this grading the *standard grading of the quotient module*.

Remarks 2.2.14. (i) The group isomorphism $M/N \cong \bigoplus_{k\in\mathbb{Z}}(M_k/N_k)$, considered in Corollary 2.2.12, and its restrictions, $(M/N)_k \cong M_k/N_k$, for every $k \in \mathbb{Z}$, are also R-isomorphisms when $\bigoplus_{k\in\mathbb{Z}}(M_k/N_k)$ has the R-module structure defined in the proof of Proposition 2.2.11, and M/N is the quotient module.
(ii) Let R be a polynomial ring with the standard grading, and $I \subseteq R$ be a homogeneous ideal. The ring R/I, with the quotient module structure, is a graded R-module with the standard grading of the quotient module.

Sygygies of a graded module

Not all submodules of a graded module are graded submodules. For example not all ideals of a polynomial ring are generated by homogeneous polynomials, and therefore, by Proposition 2.2.7, are not graded submodules of the polynomial ring. This situation can occur also with the syzygy module of a set of homogeneous elements of a graded module, as the next example shows. This is of particular relevance to us, since one of the aims of this section is to set the theory of free resolutions of a module over a graded ring within the category of graded modules over the ring.

Example 2.2.15. Consider $R = \mathbb{K}[x,y]$ a polynomial ring in two variables over a field, and the homogeneous ideal $(x^2,y^3,xy^2) \subseteq R$. Let us show that $\mathrm{Syz}(x^2,y^3,xy^2) \subseteq R^3$ is not a graded submodule of the graded module R^3 (endowed with the standard grading). To start, arguing as we did in Example 2.1.6, one can show that $\mathrm{Syz}(x^2,y^3,xy^2)$ is generated by $\{[\,y^2\ 0\ {-x}\,]^T, [\,0\ x\ {-y}\,]^T\}$. By Proposition 2.2.7, to show that $\mathrm{Syz}(x^2,y^3,xy^2)$ is not a graded submodule of R^3 we must show it cannot be generated by a set of homogeneous elements of R^3. Let $v = [\,f_1\ f_2\ f_3\,]^T \in \mathrm{Syz}(x^2,y^3,xy^2)$ be a homogeneous element of R^3. Then the monomials in the polynomials f_1, f_2, f_3 have the same degree. Also, since v is a syzygy of x^2,y^3,xy^2 we have that $f_1x^2 + f_2y^3 + f_3xy^2 = 0$, and arguing on the degree we get that $f_1x^2 = f_2y^3 + f_3xy^2 = 0$. From these equalities we obtain that $f_1 = 0$ and $f_2y + f_3x = 0$, or equivalently, that $f_2y = -f_3x$, and so there exists a homogeneous polynomial g such that $f_2 = gx$, and $-f_3 = gy$. Finally, this yields that $v = [\,f_1\ f_2\ f_3\,]^T = [\,0\ gx\ {-gy}\,]^T$. We conclude that every homogeneous element of $\mathrm{Syz}(x^2, y^3, xy^2)$ belongs to the module generated by $\begin{bmatrix} 0 & x & -y \end{bmatrix}^T$, which is a proper submodule of $\mathrm{Syz}(x^2, y^3, xy^2)$, and hence $\mathrm{Syz}(x^2, y^3, xy^2)$ cannot be generated by homogeneous elements of R^3.

Definition 2.2.16. If M is a graded R-module and d is an integer, we denote by $M(d)$ the module M with a new graded structure defined by $M(d)_k = M_{d+k}$, for every $k \in \mathbb{Z}$. The new graded module is called the graded module obtained from M by shifting by d, or shifted module if there is no ambiguity.

Remarks 2.2.17. (i) Let $d \geq 0$ and consider $R(-d)$. If $k < d$, the degree k homogeneous component of $R(-d)$ is zero, and if $k \geq d$, the degree k homogeneous component of $R(-d)$ is given by the homogeneous elements of R of degree $k - d$. It is as if we look at $R = \sum_{k \in \mathbb{Z}} R_k$ as a string and we shift it to the right d positions:

$$\cdots \oplus 0 \oplus \underset{\underset{0}{\uparrow}}{R_0} \oplus R_1 \oplus R_2 \oplus \cdots \quad \rightsquigarrow \quad \cdots \underset{\underset{0}{\uparrow}}{0} \oplus \cdots \oplus 0 \oplus \underset{\underset{d}{\uparrow}}{R_0} \oplus R_1 \oplus R_2 \oplus \cdots$$

(ii) If $d_1, \ldots, d_m$ are integers, then, likewise, we see that the homogeneous elements of degree k of the graded R-module $R(-d_1) \oplus \cdots \oplus R(-d_m)$ are the vectors $\begin{bmatrix} f_1 & \cdots & f_m \end{bmatrix}^T$, such that $\deg(f_i) = k - d_i$ for all nonzero f_i. (iii) If $M = R(-d_1) \oplus \cdots \oplus R(-d_m)$ and d is an integer, then shifting by d yields the module $M(d) = R(d - d_1) \oplus \cdots \oplus R(d - d_m)$. This is shown by computing the degree k homogeneous components $M(d)_k = M_{k+d} = R_{k+d-d_1} \oplus \cdots \oplus R_{k+d-d_m}$, which are the homogeneous components of degree k in $R(d - d_1) \oplus \cdots \oplus R(d - d_m)$. (iv) Let M be a graded R-module and N a submodule of M. Let us show that N is a graded submodule of M if and only if it is a graded submodule of $M(d)$, for every integer d. If N is a graded submodule of M, by Proposition 2.2.7, it has a homogeneous generating set $\{v_1, \ldots, v_n\}$, where for each i there is an integer k_i such that $v_i \in M_{k_i}$. But $M_{k_i} = M(d)_{k_i - d}$ and so $\{v_1, \ldots, v_n\}$ is a set of homogeneous elements of $M(d)$ that generates N, which by Proposition 2.2.7 means that N is a graded submodule of $M(d)$. The converse is obtained by considering $d = 0$.

In view of Proposition 2.2.7, and the remarks above, the next result has a straightforward proof.

Proposition 2.2.18. *Let $d_1, \ldots, d_m$ be integers. Then $N \subseteq R(-d_1) \oplus \cdots \oplus R(-d_m)$ is a graded submodule if and only if N has a generating set $\{v_1, \ldots, v_r\}$ such that, for every $i = 1, \ldots, r$, the components of $v_i = \begin{bmatrix} f_1^i & \cdots & f_m^i \end{bmatrix}^T$ are homogeneous elements of R and $\deg(f_j^i) + d_j = \deg(f_\ell^i) + d_\ell$, for every j, ℓ with $f_j^i, f_\ell^i \neq 0$.*

Example 2.2.19. Let $R = \mathbb{K}[x, y]$, with $\mathbb{K}$ a field. In the Example 2.2.15 we saw that $\mathrm{Syz}(x^2, y^3, xy^2)$ was not a graded submodule of R^3. However, one could ask if there are integers d_1, d_2, d_3 that make it a graded submodule of $R(-d_1) \oplus R(-d_2) \oplus R(-d_3)$. Attending to Proposition 2.2.18, with the generating set $\{\begin{bmatrix} y^2 & 0 & -x \end{bmatrix}^T, \begin{bmatrix} 0 & x & -y \end{bmatrix}^T\}$, it is necessary and sufficient that the integers satisfy the equations $2 + d_1 = 1 + d_3$ and $1 + d_2 = 1 + d_3$. These simplify to $d_1 = d_3 - 1$ and $d_2 = d_3$, and we obtain that for every integer d, $\mathrm{Syz}(x^2, y^3, xy^2)$ is a graded submodule of $R(1 - d) \oplus R(-d) \oplus R(-d)$. With this grading, for each d, the elements of the chosen generating set are homogeneous, and both have degree $d + 1$.

Remark 2.2.20. As the Example 2.2.19 shows, if we have a generating set for a syzygy module that consists of vectors of homogeneous elements, Proposition 2.2.18 gives a way of trying to make the syzygy module into a graded submodule by solving a linear system with integer solutions. This raises two questions about existence. One for the generating set that must consist only of vectors of homogeneous elements, and other for the solution of the system. Instead of adressing these questions to know if all syzygy modules can be made into graded submodules by choice of a proper grading, the answer will follow from the next definition and results.

Definition 2.2.21. Let M and N be graded R-modules. A homomorphism $\varphi\colon M \to N$ is a *graded homomorphism of degree d* if $\varphi(M_k) \subseteq N_{k+d}$, for all $k \in \mathbb{Z}$.

Remark 2.2.22. (i) An R-homomorphism $\varphi\colon R \to R$ is given by $f \mapsto f\varphi(1)$, so φ is graded if and only if $\varphi(1)$ is a homogeneous element of R. Also, the degree of φ is the degree of $\varphi(1)$. (ii) Assume now that φ has degree d. To consider φ with a different degree we only need to change the grading in the domain or codomain of φ. For example, $\varphi\colon R(-d) \to R$ is graded of degree 0. (iii) For the general case, let M and N be graded R-modules, and $\varphi\colon M \to N$ be a graded R-homomorphism of degree d_1. If d_2 is an integer, changing the grading in the domain of φ, $M(d_2) \to N$, makes it of degree $d_1 + d_2$, while changing the grading in the codomain, $M \to N(d_2)$, makes it of degree $d_1 - d_2$. (iv) Let $d, d_1, \ldots, d_n, c_1, \ldots, c_m$ be integers, and consider an R-homomorphism, $\varphi\colon R(-d_1) \oplus \cdots \oplus R(-d_n) \to R(-c_1) \oplus \cdots \oplus R(-c_m)$, defined by the matrix A. Then φ is graded of degree d if and only if the element ij of A is homogeneous of degree $d + d_j - c_i$ in R. (v) The R-isomorphism $M/N \cong \bigoplus_{k\in\mathbb{Z}}(M_k/N_k)$, considered in Corollary 2.2.12, is by Definition 2.2.13 graded of degree 0.

Proposition 2.2.23. *Let M and N be graded R-modules and $\varphi\colon M \to N$ be a graded homomorphism of degree d. Then $\ker(\varphi)$ and $\operatorname{Im}(\varphi)$ are graded submodules of M and N, respectively.*

Proof. To show that $\ker(\varphi)$ is a graded submodule of M, take any $v \in \ker(\varphi)$. By the grading in M we have that $v = \sum_{k\in\mathbb{Z}}[v]_k$, and applying φ to both sides yields $0 = \varphi(v) = \sum_{k\in\mathbb{Z}}\varphi([v]_k)$. This implies for every $k \in \mathbb{Z}$ that $\varphi([v]_k) = 0$, and so $[v]_k$ is in $\ker(\varphi)$. We conclude $\ker(\varphi)$ contains the homogeneous components of its elements, and by Proposition 2.2.7 it is a graded submodule. Now let us show that $\operatorname{Im}(\varphi)$ is a graded submodule of N. If $\mathscr{S} \subseteq M$ is a generating set of M that consists only of homogeneous elements, then $\varphi(\mathscr{S})$ is a generating set of $\operatorname{Im}(\varphi)$ that consists only of homogeneous elements of N. By Proposition 2.2.7, $\operatorname{Im}(\varphi)$ is a graded submodule of N. □

Proposition 2.2.24. *Let M be a finitely generated graded R-module. Choosing a generating set $\{v_1, \ldots, v_m\}$ of M consisting of homogeneous elements of degrees $d_1, \ldots, d_m$ respectively, is equivalent to choosing a surjective graded R-homomorphism $\varphi\colon R(-d_1) \oplus \cdots \oplus R(-d_m) \to M$ of degree 0.*

Proof. Choosing a generating set of M, $\{v_1, \ldots, v_m\}$, yields a surjective R-homomorphism $R^m \to M$ defined by $e_i \mapsto v_i$, for $i = 1, \ldots, m$. Let us show that if each v_i is homogeneous of degree d_i in M, $\varphi\colon R(-d_1) \oplus \cdots R(-d_m) \to M$, with the same definition, is graded of degree 0. Let $v = [\,f_1 \cdots f_m\,]^T$ be a homogeneous element of degree k in $R(-d_1) \oplus \cdots \oplus R(-d_m)$. Each f_i has degree $k - d_i$ in R, so $\varphi(v) = \sum_{i=1} f_i\varphi(e_i) = \sum_{i=1} f_i v_i$ is an element of M_k. This shows that φ is graded of degree 0. Conversely, assume $\varphi\colon R(-d_1) \oplus \cdots R(-d_m) \to M$ is a surjective graded R-homomorphism of degree 0. Then $\{\varphi(e_1), \ldots, \varphi(e_m)\}$ is a generating set of M, and since each e_i is homogeneous of degree d_i, so is $\varphi(e_i)$, for all $i = 1, \ldots, m$. □

We will now conclude that every syzygy module can be made into a graded submodule.

Corollary 2.2.25. *Let M be a finitely generated graded R-module, and $\{v_1,\dots,v_m\}$ a generating set of M consisting of homogeneous elements of degrees $d_1,\dots,d_m$. Then $\mathrm{Syz}(v_1,\dots,v_m)$ is a graded submodule of $R(-d_1)\oplus\cdots\oplus R(-d_m)$.*

Proof. Let $\varphi\colon R(-d_1)\oplus\cdots\oplus R(-d_m)\to M$ be the surjective graded homomorphism of degree 0 given by Proposition 2.2.24 from choosing $\{v_1,\dots,v_m\}$. Since $\mathrm{Syz}(v_1,\dots,v_m)=\ker(\varphi)$, it follows from Proposition 2.2.23 that $\mathrm{Syz}(v_1,\dots,v_m)$ is a graded submodule of $R(-d_1)\oplus\cdots\oplus R(-d_m)$. □

Example 2.2.26. Let $R=\mathbb{K}[x,y]$, with $\mathbb{K}$ a field. Consider the ideal (x^2,y^3,xy^2) of R and the R-homomorphism $\varphi\colon R^3\to(x^2,y^3,xy^2)$ given by $\begin{bmatrix} f_1 & f_2 & f_3\end{bmatrix}^T\mapsto f_1x^2+f_2y^3+f_3xy^2$. With the standard grading in R, and the submodule grading in the ideal, φ is not a graded homomorphism since $\varphi(\begin{bmatrix} x & x & x\end{bmatrix}^T)$ is not a homogeneous polynomial. However, by changing the grading in the domain, Proposition 2.2.24 says that $\varphi\colon R(-2)\oplus R(-3)\oplus R(-3)\to(x^2,y^3,xy^2)$ is a graded homomorphism of degree 0.

We can now consider free resolutions in the category of graded R-modules with graded R-homomorphisms of degree 0.

Definition 2.2.27. If M is a finitely generated graded R-module, with R a Noetherian ring, a graded resolution of M is a free resolution of the form $\cdots\to F_2\xrightarrow{\varphi_2}F_1\xrightarrow{\varphi_1}F_0\xrightarrow{\varphi_0}M\to 0$ such that each F_l is a shifted free graded R-module $F_l=R(-d_{l1})\oplus\cdots\oplus R(-d_{lp})$ and each R-homomorphism φ_l is a graded R-homomorphism of degree zero.

Remark 2.2.28. The construction of graded resolutions is similar to the one done for free resolutions. Take M a finitely generated graded R-module, with R a Noetherian ring. Choose a finite generating set of M consisting of homogeneous elements, say m, of degrees $d_1,\dots,d_m$. By Proposition 2.2.24, this yields a graded homomorphism of degree zero, $\varphi_0\colon R(-d_1)\oplus\cdots R(-d_m)\to M$. Now, by Corollary 2.2.25, $\ker(\varphi_0)$ is a graded submodule of $R(-d_1)\oplus\cdots R(-d_m)$, and because R is Noetherian, it has a finite generating set consisting of homogeneous elements. Like before, this generating set yields a graded homomorphism of degree 0, and we repeat the process to obtain a graded resolution of M.

Example 2.2.29. Let $R=\mathbb{K}[x,y]$, with $\mathbb{K}$ a field, and consider the ideal $(x^2,y^3,xy^2)\subseteq R$ from the Example 2.2.26. Let $\varphi\colon R(-2)\oplus R(-3)\oplus R(-3)\to(x^2,y^3,xy^2)$ be the graded homomorphism of degree 0 obtained by choosing the generators of the ideal. In Example 2.2.15, we saw that $\ker(\varphi)=\mathrm{Syz}(x^2,y^3,xy^2)$ is generated by $\{\begin{bmatrix} y^2 & 0 & -x\end{bmatrix}^T,\begin{bmatrix} 0 & x & -y\end{bmatrix}^T\}$. Moreover, this set is a basis for $\mathrm{Syz}(x^2,y^3,xy^2)$, and its elements are homogeneous of degree 4 in $R(-2)\oplus R(-3)\oplus R(-3)$. Setting ψ as the graded homomorphism of degree 0 given by the choice of $\{\begin{bmatrix} y^2 & 0 & -x\end{bmatrix}^T,\begin{bmatrix} 0 & x & -y\end{bmatrix}^T\}$, we obtain a graded resolution of (x^2,y^3,xy^2):

$$0\to R(-4)\oplus R(-4)\xrightarrow{\psi}R(-2)\oplus R(-3)\oplus R(-3)\xrightarrow{\varphi}(x^2,y^3,xy^2)\to 0.$$

We now present the version of the Hilbert's Syzygy Theorem for graded modules. The proof follows from the Hilbert's Sygyzy Theorem and can be found in [4].

Theorem 2.2.30 (Graded Hilbert Syzygy Theorem)**.** *Let $R=\mathbb{K}[x_0,x_1,\cdots,x_n]$, then every finitely generated graded R-module has a finite graded resolution of length at most $n+1$.*

2.3 Hilbert Function and Polynomial

Throughout we will use R to denote a polynomial ring $\mathbb{K}[x_0,\dots,x_n]$ over a field $\mathbb{K}$, with the standard grading. If M is a graded R-module, since $R_0 \cong \mathbb{K}$ and $R_0M_k \subseteq M_k$, for every $k \in \mathbb{Z}$, the M_k are also vector spaces over $\mathbb{K}$.

Proposition 2.3.1. *Let M be a finitely generated graded R-module. The $\mathbb{K}$-vector spaces M_k, with $k \in \mathbb{Z}$, have finite dimension.*

Proof. Let $\{v_1,\dots,v_m\}$ be a generating set of M consisting of nonzero homogeneous elements. Assume each v_i has degree $d_i \in \mathbb{Z}$, and let l be the least of these degrees. For every integer k, let v be an element of M_k. Writing v as an R-linear combination of the $v_1,\dots,v_m$ yields $v = \sum_{i=1}^m f_iv_i$, for some $f_1,\dots,f_m \in R$. Since v has degree k, we can say

$$v = \sum_{i=1}^{m}\Big(\sum_{j\in\mathbb{Z}}[f_i]_j\Big)v_i = \sum_{i=1}^{m}[f_i]_{k-d_i}v_i. \tag{2.6}$$

Now, if $k < l$, then $k-d_i < 0$ and $[f_i]_{k-d_i} = 0$ for every i. This means $v = 0$, and so $M_k = \{0\}$ and has finite dimension. Otherwise, if $k \geq l$, each $[f_i]_{k-d_i}$ in (2.6) is zero if $k-d_i < 0$, or is a $\mathbb{K}$-linear combination of monomials of degree $k-d_i$. It follows that v is a $\mathbb{K}$-linear combination of elements uv_i for all i such that $k-d_i \geq 0$, where u is a monomial of degree $k-d_i$. We conclude that M_k is generated by these elements, which are in finite number, so M_k is a finite dimensional vector space. □

Definition 2.3.2. If M is a finitely generated graded R-module, the function $H_M\colon \mathbb{Z}\to\mathbb{Z}$ defined by $H_M(k) = \dim_{\mathbb{K}} M_k$, for all $k\in\mathbb{Z}$, is called the *Hilbert function*. Here $\dim_{\mathbb{K}} M_k$ is the dimension of M_k as a vector space over $\mathbb{K}$.

Remark 2.3.3. If M is a finitely generated graded R-module, by Proposition 2.3.1, the Hilbert function H_M is well defined.

Example 2.3.4. Let us calculate the Hilbert function of R. For every integer $k \geq 0$, the set of monomials of degree k is a basis for R_k, so $\dim_{\mathbb{K}} R_k = \frac{(k+n)!}{k!n!}$. Also $\dim_{\mathbb{K}} R_k = 0$ for every $k < 0$, so using the binomial coefficient we obtain, for every $k\in\mathbb{Z}$,

$$H_R(k) = \binom{k+n}{n}.$$

Example 2.3.5. Let M be a finitely generated R-module and d an integer. For all $k\in\mathbb{Z}$, we have that $H_{M(d)}(k) = \dim_{\mathbb{K}} M(d)_k = \dim_{\mathbb{K}} M_{k+d} = H_M(k+d)$. In particular if $M = R$ we have

$$H_{R(d)}(k) = H_R(k+d) = \binom{k+d+n}{n}.$$

Example 2.3.6. Let M and N be finitely generated graded R-modules. If there is a graded R-homomorphism of degree 0, $\varphi\colon M\to N$, that is also an isomorphism, then $H_M = H_N$. This follows from the fact that, for every $k\in\mathbb{Z}$, the restriction of φ to M_k and N_k, denoted $\varphi_k\colon M_k\to N_k$, is an isomorphism of $\mathbb{K}$-vector spaces. To show this it suffices to show that φ_k is surjective. For every $v \in N_k$, there exists $u \in M$ such that $v = \varphi(u)$. Writing u as a sum of homogeneous components yields that the sum $\sum_{l \in \mathbb{Z}} \varphi([u]_l)$ is in N_k. Since φ has degree 0, for every $l \neq k$ we must have $\varphi([u]_l) = 0$ and so $[u]_l = 0$. We conclude that $u = [u]_k$ belongs in M_k, and since $v = \varphi(u) = \varphi_k(u)$, φ_k is surjective.

Proposition 2.3.7. *Given an exact sequence* $0 \to M \xrightarrow{\alpha} P \xrightarrow{\beta} N \to 0$*, with* M*,* N*,* P *finitely generated graded* R*-modules, and* α, β *graded homomorphisms of degree* 0*, we have* $H_P(k) = H_M(k) + H_N(k)$*, for every* $k \in \mathbb{Z}$*.*

Proof. For any integer k, let $\alpha_k \colon M_k \to P_k$, and $\beta_k \colon P_k \to N_k$ be the restrictions of α, and β, to the homogeneous components of degree k of their respective domains and codomains. Since α, and β, are graded R-homomorphisms of degree 0, these restrictions are well defined $\mathbb{K}$-homomorphisms. These yield, for each k, an exact sequence, $0 \to M_k \xrightarrow{\alpha_k} P_k \xrightarrow{\beta_k} N_k \to 0$, of $\mathbb{K}$-vector spaces. It follows that $\dim_{\mathbb{K}} P_k = \dim_{\mathbb{K}} \ker(\beta_k) + \dim_{\mathbb{K}} \operatorname{Im}(\beta_k) = \dim_{\mathbb{K}} M_k + \dim_{\mathbb{K}} N_k$, and therefore we conclude that $H_P(k) = H_M(k) + H_N(k)$ for every $k \in \mathbb{Z}$. □

Remark 2.3.8. Let M, N and P be finitely generated graded R-modules. If the Hilbert functions of these modules satisfy the conclusion of the previous theorem, we will just write $H_P = H_M + H_N$. Following the usual arithmetic operations, we may also write $H_M = H_P - H_N$, or $H_N = H_P - H_M$.

Example 2.3.9. Let M be a finitely generated graded R-module, and N a graded submodule of M. Consider the R-module M/N with the standard grading. It is finitely generated since M is, and because R is a Noetherian ring, by Corollary 2.1.19, N is also finitely generated. This means we can define the Hilbert functions for the modules N and M/N. From the exact sequence $0 \to N \xrightarrow{i} M \xrightarrow{\pi} M/N \to 0$, with i the inclusion and π the canonical surjection, Proposition 2.3.7 implies $H_{M/N} = H_M - H_N$.

Example 2.3.10. Suppose M, and N, are finitely generated graded R-modules. Then the same holds for the module $M \oplus N$. Let $\alpha \colon M \to M \oplus N$, and $\beta \colon M \oplus N \to N$, be the R-homomorphisms given by $\alpha(m) = (m, 0)$ for every $m \in M$, and $\beta(m, n) = n$ for every $(m, n) \in M \oplus N$. Clearly α, and β, are graded of degree 0, and the sequence $0 \to M \xrightarrow{\alpha} M \oplus N \xrightarrow{\beta} N \to 0$ is exact. By Proposition 2.3.7, we have $H_{M \oplus N} = H_M + H_N$.

Remark 2.3.11. This example easily generalizes. Let $M_1, \ldots, M_m$ be finitely generated graded R-modules, and consider $N = M_1 \oplus \cdots \oplus M_{m-1}$, and $M = M_1 \oplus \cdots \oplus M_m$ with the grading induced by the direct sum. Clearly $H_M = H_{N \oplus M_m} = H_N + H_{M_m}$, and it follows by induction that $H_M = \sum_{i=1}^{m} H_{M_i}$.

The next Proposition tells us how to calculate the Hilbert function from a finite graded resolution.

Proposition 2.3.12. *Let* M *be a finitely generated graded* R*-module. Given a finite graded resolution of* M*,* $0 \longrightarrow F_s \xrightarrow{\varphi_s} F_{s-1} \xrightarrow{\varphi_{s-1}} \cdots \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0$*, we have that* $H_M = \sum_{i=0}^{s} (-1)^i H_{F_i}$*.*

Proof. If $s = 0$, φ_0 is an isomorphism and $H_M = H_{F_0}$. The case $s = 1$ follows from the Proposition 2.3.7 which says that $H_M = H_{F_0} - H_{F_1}$. Suppose now that $s \geq 2$, and take any integer k. Consider, for $i = 0, \ldots, s$, the restriction $\varphi_{ik} \colon (F_i)_k \to (F_{i-1})_k$ of φ_i, where $F_{-1} = M$. These are $\mathbb{K}$-homomorphisms and form, for each k, an exact sequence of $\mathbb{K}$-vector spaces,

$$0 \longrightarrow (F_s)_k \xrightarrow{\varphi_{sk}} (F_{s-1})_k \xrightarrow{\varphi_{(s-1)k}} \cdots \xrightarrow{\varphi_{1k}} (F_0)_k \xrightarrow{\varphi_{0k}} M_k \longrightarrow 0.$$

For $i = 0,\dots,s-1$, the equality $\dim_{\mathbb{K}}(F_i)_k = \dim_{\mathbb{K}}\ker(\varphi_{ik}) + \dim_{\mathbb{K}}\operatorname{Im}(\varphi_{ik})$, together with exactness at $(F_i)_k$, implies that $\dim_{\mathbb{K}}\operatorname{Im}(\varphi_{ik}) = \dim_{\mathbb{K}}(F_i)_k - \dim_{\mathbb{K}}\operatorname{Im}(\varphi_{(i+1)k})$. Substituting this equation with $i = 1$ in the one with $i = 0$ yields that

$$\dim_{\mathbb{K}}\operatorname{Im}(\varphi_{0k}) = \dim_{\mathbb{K}}(F_0)_k - \dim_{\mathbb{K}}\operatorname{Im}(\varphi_{1k}) = \dim_{\mathbb{K}}(F_0)_k - \dim_{\mathbb{K}}(F_1)_k + \dim_{\mathbb{K}}\operatorname{Im}(\varphi_{2k}).$$

Now, use in this equation the same substitution as before but with $i = 2$, and continue applying these substitutions until using the one with $i = s-1$. We will obtain the equation

$$\dim_{\mathbb{K}}\operatorname{Im}(\varphi_{0k}) = \sum_{i=0}^{s-1}(-1)^i\dim_{\mathbb{K}}(F_i)_k + (-1)^s\dim_{\mathbb{K}}\operatorname{Im}(\varphi_{sk}). \tag{2.7}$$

Finally, since $\dim_{\mathbb{K}}\operatorname{Im}(\varphi_{0k}) = \dim_{\mathbb{K}} M_k$ and $\dim_{\mathbb{K}}\operatorname{Im}(\varphi_{sk}) = \dim_{\mathbb{K}}(F_s)_k$, from (2.7) we obtain, for every integer k, that $\dim_{\mathbb{K}} M_k = \sum_{i=0}^{s}(-1)^i\dim_{\mathbb{K}}(F_i)_k$. Therefore $H_M = \sum_{i=0}^{s}(-1)^i H_{F_i}$. □

An important property of the Hilbert function is that it is eventually given by a polynomial.

Definition 2.3.13. We say a function $F : \mathbb{Z} \to \mathbb{Z}$ is *of polynomial type* if there exists a polynomial $P \in \mathbb{Q}[x]$, and an integer r, such that for all $k \geq r$, $F(k) = P(k)$.

Remark 2.3.14. Note that if a function $F : \mathbb{Z} \to \mathbb{Z}$ is of polynomial type, then the polynomial $P \in \mathbb{Q}[x]$ with which it coincides, for all large enough integers, is unique. If there are two such polynomials $P_1, P_2 \in \mathbb{Q}[x]$, then $P_1 - P_2$ is a polynomial with infinitely many roots and therefore must be zero. We conclude that $P_1 = P_2$.

Example 2.3.15. Let d be an integer, and $P = \frac{1}{n!}(x+d+n)(x+d+n-1)\cdots(x+d+1)$ be a polynomial in $\mathbb{Q}[x]$. By the Example 2.3.5, for every integer $k \geq -d$, the Hilbert function $H_{R(d)}$ is given by

$$\frac{(k+d+n)!}{(k+d)!n!} = \frac{(k+d+n)(k+d+n-1)\cdots(k+d+1)}{n!}.$$

Then, for all $k \geq -d$, we have that $H_{R(d)}(k) = P(k)$, and so $H_{R(d)}$ is a function of polynomial type.

Remark 2.3.16. Note that the polynomial with which the Hilbert function of $R(d)$ coincides for all large enough integers has degree equal to the number of variables minus 1.

Example 2.3.17. Let $d_1,\dots,d_m$ be integers, and let M denote the R-module $R(d_1)\oplus\cdots\oplus R(d_m)$. For each $i = 1,\dots,m$, and every $k \geq -d_i$, the Hilbert function $H_{R(d_i)}$ coincides with the polynomial

$$P_i = \frac{1}{n!}(x+d_i+n)(x+d_i+n-1)\cdots(x+d_i+1).$$

Since $H_M = \sum_{i=1}^{m} H_{R(d_i)}$, for $k \geq \max\{-d_1,\dots,-d_m\}$, the Hilbert function H_M coincides with the polynomial $P = \sum_{i=1}^{m} P_i$, and so it is of polynomial type.

Theorem 2.3.18. *The Hilbert function of a finitely generated graded R-module is of polynomial type.*

Proof. Let M be a finitely generated graded R-module. By the Graded Hilbert Syzygy Theorem (2.2.30), M has a finite graded resolution, $0 \to F_s \to \cdots \to F_0 \to M \to 0$. For every $i = 0,\dots,s$, let

$F_i = R(d_{i1}) \oplus \cdots R(d_{im_i})$, for some integers $d_{i_1}, \ldots, d_{im_i}$. By Example 2.3.17, the Hilbert function H_{F_i} coincides with $\sum_{j=1}^{m_i} P_{i_j}$, for integers greater than or equal to $\max\{-d_{i_1}, \ldots, -d_{im_i}\}$, where each $P_{i_j} = \frac{1}{n!}(x + d_{i_j} + n)(x + d_{i_j} + n - 1) \cdots (x + d_{i_j} + 1)$. Now, it follows from Proposition 2.3.12, that $H_M = \sum_{i=0}^{s} (-1)^i H_{F_i}$, and therefore, for $k \geq \max\{-d_{ij} : j = 1, \ldots, m_i, i = 0, \ldots, s\}$ we obtain that $H_M(k) = \sum_{i=0}^{s} (-1)^i \sum_{j=1}^{m_i} P_{i_j}(k)$. This means that H_M is a function of polynomial type. □

Definition 2.3.19. Let M be a finitely generated graded R-module. We define *the Hilbert polynomial of M* as the unique polynomial in $\mathbb{Q}[x]$ that coincides, for all large enough integers, with the Hilbert function of M. It is denoted by HP_M.

Definition 2.3.20. Let M be a finitely generated graded R-module. We define the *regularity index of M*, denoted $\mathrm{ri}(M)$, as the least nonnegative integer for which the Hilbert function of M coincides with the Hilbert polynomial of M.

Example 2.3.21. Given a nonnegative integer d, let us to calculate the regularity index of $R(-d)$. Attending to the Example 2.3.5, we know that for every integer k

$$H_{R(-d)}(k) = \binom{k-d+n}{n}.$$

And by Example 2.3.15, the Hilbert polynomial of $R(-d)$ is $\frac{1}{n!}(x-d+n)(x-d+n-1)\cdots(x-d+1)$, which coincides with $H_{R(-d)}$ for all $k \geq d$. Also, note that $H_{R(-d)}(k) = 0 = HP_{R(-d)}(k)$ for all k such that $d-n \leq k \leq d-1$, and $H_{R(-d)}(d-n-1) \neq 0$. This shows that the Hilbert function coincides with the Hilbert polynomial exactly from $k = d-n$, and so $\mathrm{ri}(R(d)) = \max\{0, d-n\}$.

Chapter 3

Eulerian ideals of graphs

3.1 The ideals

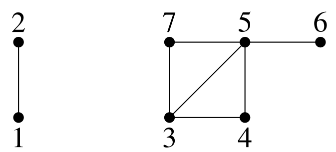
Let $G = (V_G, E_G)$ be a graph. We will assume that V_G, the set of vertices, is equal to $\{1, \ldots, n\}$, for some $n \in \mathbb{N}$, and the set of edges, E_G, is a subset of $\binom{V_G}{2}$, the set of subsets of V_G with cardinality two; in particular we will only consider simple graphs, i.e., graphs without multiple edges or loops and whose edges have no orientation. Below, the notion of *cycle* plays an important role. In this work a cycle in G is any subgraph isomorphic to the graph associated to a regular polygon, in other words, any connected subgraph whose vertices have degree two.

We will work with two polynomial rings over a field $\mathbb{K}$, that one can associate to the vertices and edges of a graph. More precisely, in one polynomial ring the variables are indexed by the vertex set, and in the other they are indexed by the edge set. If $\{i, j\}$ is an edge of G and $i < j$, we will use t_{ij} as shorthand notation for the variable $t_{\{i,j\}}$, indexed by $\{i, j\}$. We denote $\mathbb{K}[V_G] = \mathbb{K}[x_i : i \in V_G]$ and $\mathbb{K}[E_G] = \mathbb{K}[t_{ij} : \{i, j\} \in E_G]$.

Definition 3.1.1. Let G be a graph with $E_G \neq \emptyset$. Consider the ideal $(x_i^2 - x_j^2 : i, j \in V_G)$ of $\mathbb{K}[V_G]$, and the ring homomorphism, $\varphi\colon \mathbb{K}[E_G] \to \mathbb{K}[V_G]$, uniquely given by $t_{ij} \mapsto x_i x_j$, for all $\{i, j\} \in E_G$. We define the *Eulerian ideal of* G as the ideal $\varphi^{-1}(x_i^2 - x_j^2 : i, j \in V_G)$ of $\mathbb{K}[E_G]$, and denote it by $I(G)$.

The interest in these particular ideals comes from a study, done by Rentería, Simis and Villarreal in [15], that the authors then apply to Algebraic Coding Theory.

Example 3.1.2. Consider the graph G presented below:



We have $\mathbb{K}[V_G] = \mathbb{K}[x_1, x_2, x_3, x_4, x_5, x_6, x_7]$ and $\mathbb{K}[E_G] = \mathbb{K}[t_{12}, t_{34}, t_{35}, t_{37}, t_{45}, t_{56}, t_{57}]$. Using the software Macaulay2, [8], we obtained a generating set for $I(G)$ with the following polynomials:

$$t_{37}^2 - t_{35}^2, \quad t_{57}^2 - t_{35}^2, \quad t_{56}^2 - t_{35}^2, \quad t_{45}^2 - t_{35}^2, \quad t_{34}^2 - t_{35}^2, \quad t_{12}^2 - t_{35}^2,$$

$$t_{45}t_{57} - t_{34}t_{37}, \quad t_{34}t_{57} - t_{45}t_{37}, \quad t_{34}t_{45} - t_{57}t_{37}.$$

Remarks 3.1.3. Let G be a graph with $E_G \neq \emptyset$. (i) The ideal $I(G)$ does not contain polynomials with degree 1 terms. To see this suppose $f \in I(G)$ is a polynomial with some degree 1 term. Then $\varphi(f)$ has a term of degree 2 that is not divisible by the square of any variable. However, $\varphi(f)$ being in $(x_i^2 - x_j^2 : i, j \in V_G)$ means that $\varphi(f) = \sum_{i,j=1}^{|V_G|} f_{ij}(x_i^2 - x_j^2)$, for some polynomials $f_{ij} \in \mathbb{K}[V_G]$, and therefore every term of $\varphi(f)$ must be divisible by the square of a variable. This is a contradiction and therefore $I(G)$ does not contain polynomials with terms of degree 1. (ii) In particular $I(G)$ has no degree 1 polynomials. (iii) For every $t_{ij}, t_{kl} \in \mathbb{K}[E_G]$, $t_{ij}^2 - t_{kl}^2$ is in $I(G)$ as

$$\varphi(t_{ij}^2 - t_{kl}^2) = x_i^2 x_j^2 - x_k^2 x_l^2 = x_i^2(x_j^2 - x_k^2) + x_k^2(x_i^2 - x_l^2).$$

Graded ring homomorphisms and binomial ideals

Definition 3.1.4. Consider the ring $\mathbb{K}[x_1, \ldots, x_n]$, we say an ideal I of $\mathbb{K}[x_1, \ldots, x_n]$ is binomial if there is a generating set for I that consists only of binomials.

Recall that a $\mathbb{K}$-algebra homomorphism is a function between $\mathbb{K}$-algebras that is both a $\mathbb{K}$-homomorphism and a ring homomorphism. We will now consider two polynomial rings, $\mathbb{K}[y_1, \ldots, y_s]$ and $\mathbb{K}[x_1, \ldots, x_n]$, and certain $\mathbb{K}$-algebra homomorphisms between them. Also, we will always assume that a $\mathbb{K}$-algebra homomorphism fixes the elements of $\mathbb{K}$.

Definition 3.1.5. A $\mathbb{K}$-algebra homomorphism $\theta\colon \mathbb{K}[y_1, \ldots, y_s] \to \mathbb{K}[x_1, \ldots, x_n]$ is said to be *graded of degree d* if, for every nonnegative integer k, $\theta(\mathbb{K}[y_1, \ldots, y_s]_k) \subseteq \mathbb{K}[x_1, \ldots, x_n]_{dk}$.

Example 3.1.6. Given a graph G, the homomorphism φ used to define the ideal $I(G)$ is a $\mathbb{K}$-algebra homomorphism. Since every monomial of degree k in $\mathbb{K}[E_G]$ is sent by φ to a monomial of degree $2k$, for all nonnegative integer k, $\varphi(\mathbb{K}[E_G]_k) \subseteq \mathbb{K}[V_G]_{2k}$. Therefore φ is graded of degree 2.

From now on, $\theta\colon \mathbb{K}[y_1, \ldots, y_s] \to \mathbb{K}[x_1, \ldots, x_n]$ is a graded $\mathbb{K}$-algebra homomorphism of degree d.

Proposition 3.1.7. *If $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ is a homogeneous ideal, $\theta^{-1}(I)$ is a homogeneous ideal of $\mathbb{K}[y_1, \ldots, y_s]$.*

Proof. It suffices to show that $\theta^{-1}(I)$ contains every homogeneous component of its polynomials. Take $f \in \theta^{-1}(I)$, and let $f = [f]_0 + [f]_1 + \cdots + [f]_l$ be its decomposition in homogeneous polynomials. Since θ is graded of degree d, $\theta(f) = \theta([f]_0) + \cdots + \theta([f]_l)$ is the unique decomposition of $\theta(f)$ in homogeneous polynomials. And because I is a homogeneous ideal, it must contain each $\theta([f]_k)$, with $k = 0, \ldots, l$. We conclude that each $[f]_k$ is in $\theta^{-1}(I)$ and so this ideal is homogeneous. $\square$

It follows from Proposition 3.1.7 that, for every graph G, the ideal $I(G)$ is homogeneous. We now present the proof of Proposition 2.2 of [13], split in the next two results.

Proposition 3.1.8. *Let $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ be a homogeneous ideal. In the ring $\mathbb{K}[x_1, \ldots, x_n, z, y_1, \ldots, y_s]$ consider the ideal $J = (\{y_i - \theta(y_i)z : i = 1, \ldots, s\} \cup I)$. If $I \neq \mathbb{K}[x_1, \ldots, x_n]$, $\theta^{-1}(I) = J \cap \mathbb{K}[y_1, \ldots, y_s]$.*

Proof. Let us show first that $\theta^{-1}(I) \subseteq J \cap \mathbb{K}[y_1,\ldots,y_s]$. By Proposition 3.1.7, $\theta^{-1}(I)$ is homogeneous, so it suffices to prove that the homogeneous polynomials of $\theta^{-1}(I)$ are in J. Let $f \in \theta^{-1}(I)$ be a homogeneous polynomial of degree k, consider it as an element of $\mathbb{K}[x_1,\ldots,x_n,z,y_1,\ldots,y_s]$ and let us show $f \in J$. If f is a constant polynomial, $\theta(f) = f$ is a constant in I, so $f = 0$ and is in J. Otherwise, consider a nonconstant term $cy_1^{\alpha_1}\cdots y_s^{\alpha_s}$ of f, with $c \in \mathbb{K}$ nonzero. Applying the equality $y_i = (y_i - \theta(y_i)z) + \theta(y_i)z$ to each variable y_i of this term we obtain, by the binomial theorem, that

$$cy_1^{\alpha_1}\cdots y_s^{\alpha_s} = c\prod_{i=1}^{s}\left(\sum_{j=0}^{\alpha_i}\binom{\alpha_i}{j}(y_i-\theta(y_i)z)^{\alpha_i-j}(\theta(y_i)z)^j\right). \tag{3.1}$$

The right side of (3.1) can be rewritten as $\sum_{i=1}^{s} h_i(y_i - \theta(y_i)z) + z^k\theta(cy_1^{\alpha_1}\cdots y_s^{\alpha_s})$, for some polynomials $h_1,\ldots,h_s \in \mathbb{K}[x_1,\ldots,x_n,z,y_1,\ldots,y_s]$. Repeating this for every term of f, we see there are polynomials $g_1,\ldots,g_s \in \mathbb{K}[x_1,\ldots,x_n,z,y_1,\ldots,y_s]$ such that $f = \sum_{i=1}^{s} g_i(y_i - \theta(y_i)z) + z^k\theta(f)$. Now $\theta(f) \in I$ implies $f \in J$, which shows that $\theta^{-1}(I) \subseteq J \cap \mathbb{K}[y_1,\ldots,y_s]$. Conversely, let us show that $J \cap \mathbb{K}[y_1,\ldots,y_s] \subseteq \theta^{-1}(I)$. Let $\{g_1,\ldots,g_m\}$ be a generating set for I. Then J is generated by $\{y_i - \theta(y_i)z : i = 1,\ldots,s\} \cup \{g_1,\ldots,g_m\}$, and for every $f \in J \cap \mathbb{K}[y_1,\ldots,y_s]$, there are polynomials $h_1,\ldots,h_s,r_1,\ldots,r_m \in \mathbb{K}[x_1,\ldots,x_n,z,y_1,\ldots,y_s]$ such that $f = \sum_{i=1}^{s} h_i(y_i - \theta(y_i)z) + \sum_{j=1}^{m} r_jg_j$. Making the substitutions $z \mapsto 1$, $y_i \mapsto \theta(y_i)$, for $i = 1,\ldots,s$, yields $f(\theta(y_1),\ldots,\theta(y_s)) = \sum_{j=1}^{m} \hat{r}_jg_j$, for some polynomials $\hat{r}_1,\ldots,\hat{r}_m \in \mathbb{K}[x_1,\ldots,x_n]$. It follows that $\theta(f) = f(\theta(y_1),\ldots,\theta(y_s))$ is in I, and f is in $\theta^{-1}(I)$. This shows that $J \cap \mathbb{K}[y_1,\ldots,y_s] \subseteq \theta^{-1}(I)$, and concludes the proof. □

Corollary 3.1.9. *Let $I \subseteq \mathbb{K}[x_1,\ldots,x_n]$ be a binomial ideal that is also homogeneous. If θ sends all variables to monomials, $\theta^{-1}(I)$ is a binomial ideal of $\mathbb{K}[y_1,\ldots,y_s]$. Also, for every monomial order in $\mathbb{K}[y_1,\ldots,y_s]$, there is a Gröbner basis of $\theta^{-1}(I)$ consisting of homogeneous binomials.*

Proof. Let $\{g_1,\ldots,g_m\}$ be a set of binomials that generates I, and in $\mathbb{K}[x_1,\ldots,x_n,z,y_1,\ldots,y_s]$ consider the ideal $J = (\{y_i - \theta(y_i)z : i = 1,\ldots,s\} \cup I)$. Since $\theta(y_i)$ is a monomial, the generating set of J, $H = \{y_i - \theta(y_i)z : i = 1,\ldots,s\} \cup \{g_1,\ldots,g_m\}$, only has binomials, so J is a binomial ideal. Consider a monomial order in $\mathbb{K}[y_1,\ldots,y_s]$ and another in $\mathbb{K}[x_1,\ldots,x_n,z]$. Let $\geq$ be the product order on $\mathbb{K}[x_1,\ldots,x_n,z,y_1,\ldots,y_s]$, between these two orders, such that $y_1,\ldots,y_s$ are the least variables. Then $\geq$ is an elimination order on $\mathbb{K}[x_1,\ldots,x_n,z,y_1,\ldots,y_s]$ for $x_1,\ldots,x_n,z$. Using Buchberger's algorithm, see Section 2.5 of [6], we obtain a Gröbner basis $\mathscr{G}$ of J, with respect to $\geq$, that contains H. Note that $\mathscr{G}$ only has binomials, because the S-polynomial of two binomials is a binomial, and the remainder of the division algorithm of a binomial, with respect to a set of binomials, is either zero or a binomial. By the elimination theorem, Theorem 3.3 of [6], $\mathscr{G} \cap \mathbb{K}[y_1,\ldots,y_s]$ is a Gröbner basis for $J \cap \mathbb{K}[y_1,\ldots,y_s]$, with the monomial order initially chosen for $\mathbb{K}[y_1,\ldots,y_s]$. Combining this with Proposition 3.1.8, $\mathscr{G} \cap \mathbb{K}[y_1,\ldots,y_s]$ is a Gröbner basis for $\theta^{-1}(I)$ consisting of binomials, so $\theta^{-1}(I)$ is a binomial ideal. All we need to show now is that every binomial in $\mathscr{G} \cap \mathbb{K}[y_1,\ldots,y_s]$ is homogeneous. For each binomial $\mathbf{y}^\alpha - \mathbf{y}^\beta \in \mathscr{G} \cap \mathbb{K}[y_1,\ldots,y_s]$, being in $J \cap \mathbb{K}[y_1,\ldots,y_s]$ means there are polynomials $h_1,\ldots,h_s,r_1,\ldots,r_m \in \mathbb{K}[x_1,\ldots,x_n,z,y_1,\ldots,y_s]$ such that

$$\mathbf{y}^\alpha - \mathbf{y}^\beta = \sum_{i=1}^{s} h_i(y_i - \theta(y_i)z) + \sum_{j_1}^{m} r_jg_j. \tag{3.2}$$

Each g_j being a binomial means that $g_j(1,\dots,1)=0$. Therefore, the substitution $x_1=\cdots=x_n=1$ transforms (3.2) into $\mathbf{y}^\alpha-\mathbf{y}^\beta=\sum_{i=1}^s \hat{h}_i(y_i-z)$, for some $\hat{h}_i\in\mathbb{K}[z,y_1,\dots,y_s]$. Finally, the substitution $y_i=z$, for every $i=1,\dots,s$, gives that $z^{\alpha_1+\cdots\alpha_s}-z^{\beta_1+\cdots\beta_s}=0$ and $\alpha_1+\cdots\alpha_s=\beta_1+\cdots\beta_s$. This shows that $\mathbf{y}^\alpha-\mathbf{y}^\beta$ is homogeneous, and so is every binomial in $\mathscr{G}\cap\mathbb{K}[y_1,\dots,y_s]$. □

Remarks 3.1.10. (i) In the proof of Corollary 3.1.9, for each monomial order $\geq$ in $\mathbb{K}[y_1,\dots,y_s]$, the set of homogeneous binomials obtained, that is a Gröbner basis for $\theta^{-1}(I)$, with respect to $\geq$, has the same binomials independently of the choice of $\mathbb{K}$. (ii) Given a graph G, with $E_G\neq\emptyset$, $I(G)$ is a binomial ideal. Also, for every monomial order in $\mathbb{K}[E_G]$, there is a Gröbner basis of $I(G)$ consisting of homogeneous binomials, and these binomials do not depend on the choice of $\mathbb{K}$.

Generators of $I(G)$

Definition 3.1.11. Let G be a graph. (i) For a vertex i of G, the number of edges of G that contain i is *the degree of i in G*, which we denote by $\deg_G(i)$. (ii) G is called *Eulerian* if all of its vertices have even degree. (iii) We will say a graph H is an *Eulerian subgraph of* G, if it is an Eulerian graph and a subgraph of G.

Remark 3.1.12. Many authors define an Eulerian graph to be a graph with an Eulerian circuit, that is, a circuit that contains all the edges of the graph. This definition coincides for connected graphs with the one we use, as per the Euler's Theorem, a connected graph with nonempty edge set has an Eulerian circuit if and only if every vertex has even degree. For a proof see the Theorem 12 of [2].

The next two results give the proof of Proposition 2.5 from [13].

Proposition 3.1.13. *Take monomials $\mathbf{x}^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ and $\mathbf{x}^\beta=x_1^{\beta_1}\cdots x_n^{\beta_n}$ of $\mathbb{K}[x_1,\dots,x_n]$, with α and β the vectors $\begin{bmatrix}\alpha_1 & \cdots & \alpha_n\end{bmatrix}^T$ and $\begin{bmatrix}\beta_1 & \cdots & \beta_n\end{bmatrix}^T$ in $\mathbb{N}_0^n$. If $\mathbf{x}^\alpha-\mathbf{x}^\beta$ is a homogeneous binomial with degree greater than 1, $\mathbf{x}^\alpha-\mathbf{x}^\beta\in(x_i^2-x_j^2:i,j=1,\dots,n)$ if and only if $\alpha_i+\beta_i$ is even, for every $i=1,\dots,n$.*

Proof. Suppose that $\alpha_i+\beta_i$ is even, for every $i=1,\dots,n$, and let us show that $\mathbf{x}^\alpha-\mathbf{x}^\beta$ is in the ideal $(x_i^2-x_j^2:i,j=1,\dots,n)$. If $\mathbf{x}^\alpha-\mathbf{x}^\beta=0$ we are done, otherwise, $\alpha_k>\beta_k$ for some $k\in\{1,\dots,n\}$, which by hypothesis means that $\alpha_k\geq\beta_k+2$. We will argue by induction on the degree of $\mathbf{x}^\alpha-\mathbf{x}^\beta$. If $\mathbf{x}^\alpha-\mathbf{x}^\beta$ has degree 2, $\mathbf{x}^\alpha=x_k^2$, and by our assumption, $\mathbf{x}^\beta=x_l^2$ for some $l\neq k$, so $\mathbf{x}^\beta-\mathbf{x}^\gamma=x_k^2-x_l^2$ is in the ideal. Assume now that $\mathbf{x}^\alpha-\mathbf{x}^\beta$ has degree $d>2$, and the result holds for homogeneous binomials of degree $d-1$. Take l such that $\beta_l>0$, letting $\mathbf{x}^{\alpha'}$ and $\mathbf{x}^{\beta'}$ be the monomials for which $\mathbf{x}^\alpha=x_k^2\mathbf{x}^{\alpha'}$ and $\mathbf{x}^\beta=x_l\mathbf{x}^{\beta'}$, we have that

$$\mathbf{x}^\alpha-\mathbf{x}^\beta=(x_k^2-x_l^2)\mathbf{x}^{\alpha'}+x_l(x_l\mathbf{x}^{\alpha'}-\mathbf{x}^{\beta'}). \tag{3.3}$$

Letting $\mathbf{x}^\mu=x_l\mathbf{x}^{\alpha'}$ and $\mathbf{x}^\nu=\mathbf{x}^{\beta'}$, the binomial $\mathbf{x}^\mu-\mathbf{x}^\nu$ is homogeneous of degree $d-1$, and $\mu_i+\nu_i$ is even for all $i=1\dots,n$, because $\mu_k+\nu_k=\alpha_k-2+\beta_k$, $\mu_l+\nu_l=\alpha_l+1+\beta_l-1$, and $\mu_i+\nu_i=\alpha_i+\beta_i$, for all $i\neq k,l$. By the induction hypothesis $\mathbf{x}^\mu-\mathbf{x}^\nu\in(x_i^2-x_j^2:i,j=1,\dots,n)$, and from (3.3) follows that $\mathbf{x}^\alpha-\mathbf{x}^\beta\in(x_i^2-x_j^2:i,j=1,\dots,n)$. Conversely, assume $\mathbf{x}^\alpha-\mathbf{x}^\beta$ is in the ideal, and let us show that $\alpha_i+\beta_i$ is even for every $i=1,\dots,n$. We know $\mathbf{x}^\alpha-\mathbf{x}^\beta=\sum_{i,j=1}^n f_{ij}(x_i^2-x_j^2)$, for certain polynomials $f_{ij}\in\mathbb{K}[x_1,\dots,x_n]$. For every i, applying to the equality the substitution $x_j=1$, for every

$j \neq i$, yields $x_i^{\alpha_i} - x_i^{\beta_i} = g_i(x_i^2 - 1)$, for some polynomial $g_i \in \mathbb{K}[x_i]$. If $\alpha_i = \beta_i$, $\alpha_i + \beta_i$ is even and we are done. Otherwise, assume $\alpha_i > \beta_i$, as the other case is analogous. Note that the division algorithm of $x_i^{\alpha_i} - x_i^{\beta_i}$ by $x_i^2 - 1$ has quotient g_i, and remainder zero. Eyeing a contradiction, suppose that $\alpha_i + \beta_i$ is odd. Then $\alpha_i - \beta_i$ is also odd, and there is some $k \in \mathbb{Z}$ for which $\alpha_i = \beta_i + 2k + 1$. Let us apply the division algorithm of $x_i^{\alpha_i} - x_i^{\beta_i} = x_i^{\beta_i + 2k+1} - x_i^{\beta_i}$ by $x_i^2 - 1$. In each division step, the next dividend is the binomial obtained from the previous dividend by subtracting 2 from the exponent of its initial monomial. This way, after k steps, the dividend is $x_i^{\beta_i+1} - x_i^{\beta_i}$. Continuing the algorithm, we will eventually obtain remainder equal to $(-1)^{\beta_i}(x_i - 1)$. This is a contradiction because the remainder of the division algorithm, in one variable, is unique. Therefore $\alpha_i + \beta_i$ is even, for every $i = 1, \ldots, n$. □

Corollary 3.1.14. *Let G be a graph, with $E_G \neq \emptyset$, and $\mathbf{t}^\alpha - \mathbf{t}^\beta \in \mathbb{K}[E_G]$ a nonzero homogeneous binomial with $\gcd(\mathbf{t}^\alpha, \mathbf{t}^\beta) = 1$. Let H be a subgraph of G the edges of which index the odd power variables of $\mathbf{t}^\alpha - \mathbf{t}^\beta$. Then $\mathbf{t}^\alpha - \mathbf{t}^\beta \in I(G)$ if and only if H is an Eulerian subgraph of G.*

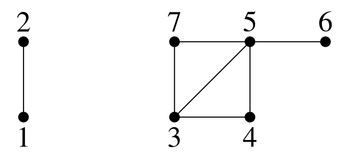
Proof. We begin by writing $\mathbf{x}^\delta - \mathbf{x}^\gamma = \varphi(\mathbf{t}^\alpha - \mathbf{t}^\beta)$. For every $i \in V_H$, let $a_{i1}, \ldots, a_{ir_i}$ and $k_{i1}, \ldots, k_{ip_i}$ be respectively the even and odd exponents of the variables of $\mathbf{t}^\alpha$ that have i in the index, and also, let $b_{i1}, \ldots, b_{is_i}$ and $l_{i1}, \ldots, l_{iq_i}$ be respectively the even and odd exponents of the variables of $\mathbf{t}^\beta$ that have i in the index. Note that $\delta_i = a_{i1} + \cdots + a_{ir_i} + k_{i1} + \cdots + k_{ip_i}$, $\gamma_i = b_{i1} + \cdots + b_{is_i} + l_{i1} + \cdots + l_{iq_i}$ and, because $\gcd(\mathbf{t}^\alpha, \mathbf{t}^\beta) = 1$, $\deg_H(i) = p_i + q_i$. From the first two equalities, δ_i has the same parity as p_i, and γ_i has the same parity as q_i, so $\delta_i + \gamma_i - \deg_H(i) = (\delta_i - p_i) + (\gamma_i - q_i)$ is always an even number. Now, if $\mathbf{t}^\alpha - \mathbf{t}^\beta \in I(G)$, $\mathbf{x}^\delta - \mathbf{x}^\gamma$ is in $(x_i^2 - x_j^2 : i, j \in V_G)$, and by Proposition 3.1.13, $\delta_i + \gamma_i$ is even for every $i \in V_G$. This implies the $\deg_H(i)$ is even, for every $i \in V_H$, and therefore H is an Eulerian subgraph of G. Conversely, if the $\deg_H(i)$ is even, $\delta_i + \gamma_i$ is even for all $i \in V_H$. And for every $i \in V_G \setminus V_H$, appearing in the index of a variable of $\mathbf{t}^\alpha - \mathbf{t}^\beta$, i is only in even exponent variables. This implies, for all $i \in V_G \setminus V_H$, that $\delta_i + \gamma_i$ is even. Therefore $\delta_i + \gamma_i$ is even, for every $i \in V_G$, which by Proposition 3.1.13 means that $\mathbf{x}^\delta - \mathbf{x}^\gamma$ is in $(x_i^2 - x_j^2 : i, j \in V_G)$, and so $\mathbf{t}^\alpha - \mathbf{t}^\beta \in I(G)$. □

Corollary 3.1.14 is the reason why $I(G)$ is called the Eulerian ideal of G.

Remarks 3.1.15. Let G be a graph, with $E_G \neq \emptyset$, and $\mathbf{t}^\alpha - \mathbf{t}^\beta \in I(G)$ a nonzero homogeneous binomial with $\gcd(\mathbf{t}^\alpha, \mathbf{t}^\beta) = 1$. (i) The subgraph H of G, identified by the indeces of the odd power variables of $\mathbf{t}^\alpha - \mathbf{t}^\beta$, must have an even number of edges. To see this, regarding the variables of $\mathbf{t}^\alpha - \mathbf{t}^\beta$, let p be the sum of the odd exponents, and q the sum of the even exponents. Then $2\deg(\mathbf{t}^\alpha - \mathbf{t}^\beta) = p + q$ and so p must be even. As p is the sum of $|E_H|$ odd numbers, $|E_H|$ must be even. (ii) The odd exponent variables of $\mathbf{t}^\alpha - \mathbf{t}^\beta$ identify a unique subgraph of G, up to isolated vertices. (iii) If H is an Eulerian subgraph of G, with $E_H \neq \emptyset$ and $|E_H|$ even, each partition of E_H in two sets of equal cardinality gives a different homogeneous binomial of $I(G)$. If $A, B \subseteq E_H$ form such a partition of E_H, set $\mathbf{t}^\alpha$ as the product of the variables indexed by the edges of A, and $\mathbf{t}^\beta$ as the product of the variables indexed by the edges of B. By Corollary 3.1.14, the homogeneous binomial $\mathbf{t}^\alpha - \mathbf{t}^\beta$ is in $I(G)$.

Example 3.1.16. In Example 3.1.2, with the graph G as below, the generating set obtained for $I(G)$ has the homogeneous binomials,

$$t_{45}t_{57} - t_{34}t_{37}, \quad t_{34}t_{57} - t_{45}t_{37}, \quad t_{34}t_{45} - t_{57}t_{37}.$$



As we can see, the indeces of the variables in these binomials identify Eulerian subgraphs of G, with positive even number of edges. In this case the only one, which is the cycle of length 4.

We end this section presenting Corollary 2.7 of [13], that exhibits the Eulerian ideal for subgraphs.

Proposition 3.1.17. *Let G be a graph and H a subgraph of G, with $E_H \neq \emptyset$. Consider $\mathbb{K}[E_H]$ as a subset of $\mathbb{K}[E_G]$. For the ideal $I(H)$, seen as a subset of $\mathbb{K}[E_G]$, we have that $I(H) = I(G) \cap \mathbb{K}[E_H]$.*

Proof. Let $\varphi_G \colon \mathbb{K}[E_G] \to \mathbb{K}[V_G]$ and $\varphi_H \colon \mathbb{K}[E_H] \to \mathbb{K}[V_H]$ be the ring homomorphisms that define $I(G)$ and $I(H)$, respectively, and note that φ_H is the restriction of φ_G to $\mathbb{K}[E_H]$ and $\mathbb{K}[V_H]$. Given $f \in I(H)$, $\varphi_G(f) = \varphi_H(f)$ is in the ideal $(x_i^2 - x_j^2 : i, j \in V_H)$ of $\mathbb{K}[V_H]$, which is a subset of the ideal $(x_i^2 - x_j^2 : i, j \in V_G)$ of $\mathbb{K}[V_G]$. Therefore $f \in I(G)$, and $I(H) \subseteq I(G) \cap \mathbb{K}[E_H]$. Conversely, if f is a polynomial in $I(G) \cap \mathbb{K}[E_H]$, $\varphi_H(f) = \varphi_G(f)$ is in $(x_i^2 - x_j^2 : i, j \in V_G) \cap \mathbb{K}[V_H] = (x_i^2 - x_j^2 : i, j \in V_H)$. This way $f \in I(H)$, and $I(G) \cap \mathbb{K}[E_H] \subseteq I(H)$. □

3.2 Regularity index of $\mathbb{K}[E_G]/I(G)$

In this section we will begin by defining the notion of regular element of a module over a commutative ring. Applying it to $\mathbb{K}[E_G]/I(G)$ we will show that the Hilbert polynomial of $\mathbb{K}[E_G]/I(G)$ is constant.

Regular elements over a module

In the following definition, let R denote any commutative ring.

Definition 3.2.1. Let M be an R-module and $f \in R$. We say that f is *M-regular* if, for every $m \in M$, $fm = 0 \Rightarrow m = 0$.

Remark 3.2.2. Consider $R = \mathbb{K}[x, y]$ and $M = R/I$, where $I = (xy)$ is the vanishing ideal of the union of two lines in the affice plane $\mathbb{A}^2$, over an infinite field. Then $f = x$ is not M-regular, since $x(y+I) = I$ and $y + I \neq I$, and neither is $f = y$. On the other hand, $f = x + y$ is M-regular. To see this, consider any element $m \in M$. Using the generator of I, we can write $m = g_1(x) + g_2(y) + I$, for some $g_1 \in \mathbb{K}[x]$ and $g_2 \in \mathbb{K}[y]$. Note that

$$fm = 0 \iff (x+y)(g_1(x) + g_2(y)) \in I \iff xg_1(x) + xg_2(0) + yg_1(0) + yg_2(y) \in (xy).$$

Setting $x = 0$ we deduce that $g_2(y) = -g_1(0)$, and hence $g_2(0) = -g_1(0)$. Likewise, setting $y = 0$ we deduce that $g_1(x) = -g_2(0)$, so $g_1(x) + g_2(y) = -g_2(0) - g_1(0) = -g_2(0) + g_2(0) = 0$ and so $m = 0$.

Let us apply the notion of M-regular element to our setting. We start by proving that the variables of $\mathbb{K}[V_G]$ are M-regular, where M is $\mathbb{K}[V_G]/(x_i^2 - x_j^2 : i, j \in V_G)$. At this point, this does not depend on G, and we can state the next proposition for a polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$ and the R-module

$$M = R/(x_i^2 - x_j^2 : i, j = 1, \ldots, n).$$

Proposition 3.2.3. *A variable, x_l, is M-regular.*

Proof. Take f in R. To show that x_l is M-regular, we assume $x_l f \in (x_i^2 - x_j^2 : i, j = 1, \ldots, n)$ and show it implies $f \in (x_i^2 - x_j^2 : i, j = 1, \ldots, n)$. Consider any monomial order such that x_n is the least variable, and let $\mathcal{G} = \{x_1^2 - x_n^2, \ldots, x_{n-1}^2 - x_n^2\}$. Since the initial monomials of any two polynomials of $\mathcal{G}$ are relatively prime, by Proposition 2.15 of [6] and Buchberger's criterion, Theorem 2.14 of [6], $\mathcal{G}$ is a Gröbner basis for $(x_i^2 - x_j^2 : i, j = 1, \ldots, n)$. Applying the division algorithm on $x_l f$, with respect to $\mathcal{G}$, we obtain a standard expression, $x_l f = \sum_{j=1}^{n-1} q_j (x_j^2 - x_n^2)$, with zero remainder and $q_1, \ldots, q_{n-1} \in R$. As $\mathcal{G}$ is a Gröbner basis, the order of its elements, as the divisors in the division algorithm, does not change the remainder. We claim that if the algorithm is applied with the polynomials of $\mathcal{G}$ reordered such that $x_l^2 - x_n^2$ is the last divisor, for every $j \neq l$ the polynomials q_j will be multiples of x_l. Using the claim, there are $\hat{q}_j \in R$ such that $x_l f = \sum_{j=1}^{n-1} q_j (x_j^2 - x_n^2) = q_l (x_l^2 - x_n^2) + x_l \sum_{j \neq l} \hat{q}_j (x_j^2 - x_n^2)$, and we see x_l must also divide q_l, thus implying that $f \in (x_i^2 - x_j^2 : i, j = 1, \ldots, n)$. Therefore, to conclude that x_l is M-regular, it suffices to prove the claim, which we do now. The division algorithm, with respect to $\mathcal{G}$, gives a sequence of polynomials $h_0 = x_l f, h_1, \ldots, h_{k-1}, h_k = 0$ by the inductive rule

$$h_{m+1} = h_m - c_m w_m (x_j^2 - x_n^2), \tag{3.4}$$

where w_m is the unique monomial such that $w_m x_j^2 = \mathrm{in}(h_m)$, for some $j \neq n$, and c_m is the leading coefficient of h_m. If the algorithm is applied with $x_l^2 - x_n^2$ as the last divisor, in each step we only use $x_l^2 - x_n^2$ if no other x_j^2, with $j \neq l, n$, divides $\mathrm{in}(h_m)$. By the algorithm, $q_1, \ldots, q_{n-1}$ are $\mathbb{K}$-linear combinations of the monomials w_m from (3.4). Let us show that, for $j \neq l$, these monomials w_m that form q_j are multiples of x_l, by proving that when $\mathrm{in}(h_m)$ is divisible by x_j^2, with $j \neq l$, $\mathrm{in}(h_m) = w_m x_j^2$ is also divisible by x_l. It suffices to show that, for every $m \in \{0, \ldots, k-1\}$, every term of h_m divisible by some x_j^2, with $j \neq l, n$, is also divisible by x_l. Using (3.4), we argue by induction on m. For $h_0 = x_l f$ this is clear, so assume that, for some $m \in \{0, \ldots, k-2\}$, every term of h_m divisible by some x_j^2, with $j \neq l, n$, is also divisible by x_l, and let us show that the same holds for h_{m+1}. In each division step, (3.4), h_{m+1} is obtained from h_m by substituting $\mathrm{in}(h_m) = w_m x_j^2$ by $w_m x_n^2$, so we only need to consider what happens with this monomial $w_m x_n^2$. If we use $j = l$ in (3.4), $\mathrm{in}(h_m) = w_m x_l^2$ is not divisible by any x_j^2 with $j \neq l$, so neither is $w_m x_n^2$, and we are done. And if we use $j \neq l$ in (3.4), by the induction hypothesis, x_l divides $\mathrm{in}(h_m) = w_m x_j^2$, and so x_l divides $w_m x_n^2$. This shows that, for every $m \in \{0, \ldots, k-1\}$, every term of h_m divisible by some x_j^2, with $j \neq l, n$, is also divisible by x_l. Therefore the monomials w_m that form each q_j, with $j \neq l$, are multiples of x_l, so all monomials of the polynomials q_j, for $j \neq l$, are multiples of x_l, thus proving the claim. □

Regular elements over a graded module

Let R be a polynomial ring over a field and M a graded R-module.

Example 3.2.4. Take an integer d, and a polynomial $f \in R$. Consider the R-homomorphism $\psi\colon M(-d) \to M$ defined by $\psi(m) = fm$, for every $m \in M(-d)$. ψ is injective if and only if $\ker(\psi) = \{0\}$, if and only if, for every $m \in M$, $fm = \psi(m) = 0 \Rightarrow m = 0$. So ψ is injective if and only if f is M-regular. Also, if the polynomial f is homogeneous of degree d, ψ is a graded homomorphism of degree 0.

An element being M-regular is related to the exactness of a certain sequence, as we will now see.

Proposition 3.2.5. *Take $d \in \mathbb{Z}$, and a polynomial $f \in R$. Consider the R-homomorphisms ψ, from the Example 3.2.4, and $\xi : M \to M/fM$ defined by $\xi(m) = m + fM$, for all $m \in M$. Then f is M-regular if and only if the next sequence is exact*

$$0 \to M(-d) \xrightarrow{\psi} M \xrightarrow{\xi} M/fM \to 0.$$

Proof. First let us show that the sequence is exact at M. Take any element of $\mathrm{Im}(\psi)$. It can be written as fm, for some $m \in M(-d)$, and because $\xi(fm) = fm + fM = fM$, it is in $\ker(\xi)$. Therefore $\mathrm{Im}(\psi) \subseteq \ker(\xi)$. Conversely, take $m \in \ker(\xi)$, then $fM = \xi(m) = m + fM$, which implies $m \in fM$, and so there is $v \in M$ such that $m = fv = \psi(v)$. This means that $m \in \mathrm{Im}(\psi)$, so $\mathrm{Im}(\psi) = \ker(\xi)$ and the sequence is exact at M. Now, since ξ is surjective, and by the Example 3.2.4, ψ is injective if and only if f is M-regular, the result follows. $\square$

Remark 3.2.6. If in Proposition 3.2.5, $f \in R$ is homogeneous of degree d, Example 2.2.8 says that fM is a graded submodule of M. Also, by Definition 2.2.13 we can consider M/fM as a graded module through the identification $(M/fM)_k = M_k/(fM)_k = M_k/fM_{k-d}$, for every integer k. This way ξ becomes a graded homomorphism of degree 0.

Let I be a homogeneous ideal of R and $f \in R$ a homogeneous polynomial. Setting M as the R-module R/I yields $fM = (f,I)/I$, where (f,I) is the ideal generated by f and I. By the Third Isomorphism Theorem we have that $M/fM = (R/I)/((f,I)/I) \cong R/(f,I)$, which is how we will apply this theory to the Eulerian ideal of a graph G: by having $R = \mathbb{K}[E_G]$, $I = I(G)$, and f a variable of R. However, there are still some results, mainly about the regularity index of R/I, that are advantageous to present in the more general setting for R and I. These will later be applied to the study of the Eulerian ideal of a graph, and also to its generalization for hypergraphs in Chapter 4.

Lemma 3.2.7. *Let $R = \mathbb{K}[x_1,\ldots,x_n]$ and I be a homogeneous ideal of R. Take a variable x_i, and consider the ideal (I,x_i) as a graded submodule of R, with the standard grading. If (I,x_i) contains every square of a variable of R, for every integer $k \geq n+1$, $(I,x_i)_k = R_k$.*

Proof. Recall that $(I,x_i)_k = (I,x_i) \cap R_k$, for every integer k. Assuming that $k \geq n+1$, we only need to show that $R_k \subseteq (I,x_i)$, for which suffices that the monomials of degree k of R be in (I,x_i). Let $\mathbf{x}^\alpha \in R$ be a monomial of degree $k \geq n+1$. It is divisible by some x_j^2, as at least one variable must appear twice in $\mathbf{x}^\alpha$. By hypothesis $x_j^2 \in (I,x_i)$, therefore so is $\mathbf{x}^\alpha$, and we conclude that $R_k \subseteq (I,x_i)$. $\square$

Theorem 3.2.8. *Let $R = \mathbb{K}[x_1,\ldots,x_n]$ and I be a homogeneous ideal of R. Take a variable x_i and assume that it is R/I-regular. Let $H_{R/I}$ be the Hilbert function of R/I. If (I,x_i) contains every square of a variable of R, for every integer $k \geq n$, $H_{R/I}(k) = H_{R/I}(n)$.*

Proof. Since x_i is R/I-regular, Proposition 3.2.5 says that the following sequence is exact,

$$0 \to (R/I)(-1) \xrightarrow{\psi} R/I \xrightarrow{\xi} R/(I,x_i) \to 0, \tag{3.5}$$

where ψ and ξ are given by $\psi(g+I) = x_i g + I$ and $\xi(g+I) = g + (I,x_i)$, for every $g \in R$. Also, ψ and ξ are graded R-homomorphisms of degree 0. Consider now, for every integer k, the restrictions of ψ and ξ to the homogeneous components of degree k, of their domain and codomain. These restrictions, ψ_k and ξ_k, are $\mathbb{K}$-homomorphisms that form, from (3.5), an exact sequence of $\mathbb{K}$-vector spaces:

$$0 \to (R/I)_{k-1} \xrightarrow{\psi_k} (R/I)_k \xrightarrow{\xi_k} (R/(I,x_i))_k \to 0.$$

Assume now that $k \geq n+1$. By Lemma 3.2.7, we have that $(I,x_i)_k = R_k$, which implies that $(R/(I,x_i))_k = R_k/(I,x_i)_k = (I,x_i)_k/(I,x_i)_k$. This is the zero $\mathbb{K}$-vector space, and so ξ_k is the null $\mathbb{K}$-homomorphism. By exactness, it follows that $\mathrm{Im}(\psi_k) = \ker(\xi_k) = (R/I)_k$, so ψ_k is an isomorphism between $(R/I)_{k-1}$ and $(R/I)_k$. Finally, being isomorphic $\mathbb{K}$-vector spaces, $(R/I)_{k-1}$ and $(R/I)_k$ have the same dimension. This means that $H_{R/I}(k-1) = H_{R/I}(k)$, for every $k \geq n+1$. □

Lemma 3.2.9. *Let $R = \mathbb{K}[x_1,\dots,x_n]$ and I be a homogeneous ideal of R. If some variable $x_i \in R$ is R/I-regular, the Hilbert function of R/I, $H_{R/I}$, is nondecreasing.*

Proof. Consider the R-homomorphism $\psi\colon (R/I)(-1) \to R/I$, defined by $\psi(g+I) = x_i g + I$, for every $g \in R$. By the Example 3.2.4 ψ is injective, and for every $k \in \mathbb{Z}$, the restrictions of ψ, $\psi_k\colon (R/I)_{k-1} \to (R/I)_k$, are injective $\mathbb{K}$-homomorphisms, so $H_{R/I}(k-1) \leq H_{R/I}(k)$. □

The next Proposition will allow us to obtain estimates for the regularity index, see Definition 2.3.20, for when we work with the Eulerian Ideal of a graph. It is a generalization of the Proposition 3.4 from [13], from which we follow the proof.

Proposition 3.2.10. *Let $R = \mathbb{K}[x_1,\dots,x_n]$ and I be a homogeneous ideal of R. Consider a monomial $\mathbf{x}^\delta \in R$ of degree d. If every x_i is R/I-regular, and for some x_i, (I,x_i) contains every square of a variable of R, the homogeneous component $(R/(I,\mathbf{x}^\delta))_k$ is the zero $\mathbb{K}$-vector space if and only if $k \geq \mathrm{ri}(R/I) + d$.*

Proof. First, fix an integer k, and let us see that $(R/(I,\mathbf{x}^\delta))_k$ is the zero $\mathbb{K}$-vector space if and only if, for every $i \geq k$, $(R/(I,\mathbf{x}^\delta))_i$ is also the zero $\mathbb{K}$-vector space. If $(R/(I,\mathbf{x}^\delta))_k$ is the zero $\mathbb{K}$-vector space, $R_k = (I,\mathbf{x}^\delta)_k$, and every monomial in R_k is in $(I,\mathbf{x}^\delta)$. Therefore every monomial of degree $i \geq k$, being divisible by a monomial of degree k, must also be in $(I,\mathbf{x}^\delta)$. Then, for $i \geq k$, $R_i \subseteq (I,\mathbf{x}^\delta)$, and $(R/(I,\mathbf{x}^\delta))_i$ is the zero $\mathbb{K}$-vector space. The converse is clear. Now, consider the sequence

$$0 \to (R/I)(-d) \xrightarrow{\hat{\psi}} R/I \xrightarrow{\hat{\xi}} (R/(I,\mathbf{x}^\delta) \to 0,$$

with $\hat{\psi}$ and $\hat{\xi}$ defined by $\hat{\psi}(g+I) = \mathbf{x}^\delta g + I$ and $\hat{\xi}(g+I) = g + (I,\mathbf{x}^\delta)$, for every $g+I \in R/I$. As every variable is R/I-regular, so is $\mathbf{x}^\delta$. And by Proposition 3.2.5 this sequence is exact, and induces, for every integer i, an exact sequence of $\mathbb{K}$-vector spaces, and $\mathbb{K}$-homomorphisms:

$$0 \to (R/I)_{i-d} \xrightarrow{\hat{\psi}_i} (R/I)_i \xrightarrow{\hat{\xi}_i} (R/(I,\mathbf{x}^\delta))_i \to 0. \tag{3.6}$$

For every integer i, we deduce from (3.6) that $(R/(I,\mathbf{x}^\delta))_i$ is the zero vector space if and only if $H_{R/I}(i) = H_{R/I}(i-d)$. It follows that $(R/(I,\mathbf{x}^\delta))_k$ being the zero vector space is equivalent to $H_{R/I}(i) = H_{R/I}(i-d)$ for every $i \geq k$. Also, if we assume that $H_{R/I}(i) = H_{R/I}(i-d)$ for every $i \geq k$, induction yields that $H_{R/I}(k+jd) = H_{R/I}(k-d)$, for every integer $j \geq -1$. By Lemma 3.2.9, $H_{R/I}$ is a nondecreasing function, so $H(i) = H(k-d)$ for every $i \geq k-d$, and $\mathrm{ri}(R/I) \leq k-d$. This shows that $H_{R/I}(i) = H_{R/I}(i-d)$ for every $i \geq k$ implies that $\mathrm{ri}(R/I) \leq k-d$. The converse also holds because, by Theorem 3.2.8, the Hilbert polynomial of R/I is constant, so we obtain that $H_{R/I}(i) = H_{R/I}(i-d)$ for every $i \geq k$ is equivalent to $\mathrm{ri}(R/(I) \leq k-d$. It follows that $(R/(I,\mathbf{x}^\delta))_k$ is the zero vector space if and only if $\mathrm{ri}(R/I) \leq k-d$. □

We now present another result that we will need to study the regularity index of the Eulerian ideal of a graph, and its generalization in Chapter 4. The result and its proof are based on the Proposition 3.5 of [13], which concern the Eulerian ideal of a graph G, and the ideal $(I(G),\mathbf{t}^\delta)$ with $\mathbf{t}^\delta$ a monomial in $\mathbb{K}[E_G]$. For simplicity, and to extend its applicability to Chapter 4, we present it in a more general context, but for the simpler case where $\mathbf{t}^\delta$ is a variable.

Proposition 3.2.11. *Let $R = \mathbb{K}[x_1,\dots,x_n]$ and $I \subseteq R$ be an ideal. Assume all variables of R are R/I- regular, and that there is a monomial order $\geq$ in R such that I has a Gröbner basis with only homogeneous binomials. With x_i the least variable in $\geq$, a monomial $\mathbf{x}^\alpha \in R$ is in (I,x_i) if and only if there is a monomial $\mathbf{x}^\beta \in R$, such that $\mathbf{x}^\alpha - \mathbf{x}^\beta x_i$ is homogeneous and belongs in I.*

Proof. Let $\mathscr{G}$ be a Gröbner basis of I, with respect to $\geq$, with only homogeneous binomials. Assume every binomial $\mathbf{x}^\alpha - \mathbf{x}^\beta \in \mathscr{G}$ is such that $\mathrm{in}(\mathbf{x}^\alpha - \mathbf{x}^\beta) = \mathbf{x}^\alpha$, where, for every $f \in R$, $\mathrm{in}(f)$ denotes the initial monomial of f with respect to $\geq$. We claim that the binomials $\mathbf{x}^\alpha - \mathbf{x}^\beta \in \mathscr{G}$ can be chosen to satisfy that $\gcd(\mathbf{x}^\alpha,\mathbf{x}^\beta) = 1$, let us show this. For every $\mathbf{x}^\alpha - \mathbf{x}^\beta \in \mathscr{G}$, let $\mathbf{x}^\gamma$ and $\mathbf{x}^\delta$ be the monomials for which $\mathbf{x}^\alpha = \gcd(\mathbf{x}^\alpha,\mathbf{x}^\beta)\mathbf{x}^\gamma$ and $\mathbf{x}^\beta = \gcd(\mathbf{x}^\alpha,\mathbf{x}^\beta)\mathbf{x}^\delta$. Since the variables of R are R/I-regular, so is $\gcd(\mathbf{x}^\alpha,\mathbf{x}^\beta)$, and $\mathbf{x}^\alpha - \mathbf{x}^\beta$ being in I implies $\mathbf{x}^\gamma - \mathbf{x}^\delta$ is in I. Let $\mathscr{G} = \{\mathbf{x}^\alpha - \mathbf{x}^\beta, g_2,\dots,g_m\}$ and $\mathscr{G}' = (\mathscr{G} \setminus \{\mathbf{x}^\alpha - \mathbf{x}^\beta\}) \cup \{\mathbf{x}^\gamma - \mathbf{x}^\delta\}$. By definition of Gröbner basis, $\mathrm{in}(I) = (\mathbf{x}^\alpha, \mathrm{in}(g_2),\dots,\mathrm{in}(g_m))$, where $\mathrm{in}(I)$ denotes the initial ideal of I. Therefore, as $\mathbf{x}^\gamma \in \mathrm{in}(I)$ and $\mathbf{x}^\gamma$ divides $\mathbf{x}^\alpha$, we deduce that $(\mathbf{x}^\gamma, \mathrm{in}(g_2),\dots,\mathrm{in}(g_m)) = (\mathbf{x}^\alpha, \mathrm{in}(g_2),\dots,\mathrm{in}(g_m)) = \mathrm{in}(I)$. This shows that $\mathscr{G}'$ is also a Gröbner basis for I, with respect to $\geq$. Repeating this argument for the other elements of $\mathscr{G}'$, we obtain a Gröbner basis of I with the desired property, thus proving the claim. Assume then that every $\mathbf{x}^\alpha - \mathbf{x}^\beta \in \mathscr{G}$ satisfies that $\gcd(\mathbf{x}^\alpha,\mathbf{x}^\beta) = 1$. Then x_i, being the least variable, must not divide the initial monomial of any binomial of $\mathscr{G}$, which by Proposition 2.15 of [6] and Buchberger's criterion, Theorem 2.14 of [6], implies that $\mathscr{G} \cup \{x_i\}$ is a Gröbner basis for (I,x_i). Now, take a monomial $\mathbf{x}^\alpha \in (I,x_i)$. The remainder of the division algorithm of $\mathbf{x}^\alpha$, with respect to $\mathscr{G} \cup \{x_i\}$, is zero. And in each step of the division algorithm, using a binomial of $\mathscr{G}$ produces a monomial with the same degree as $\mathbf{x}^\alpha$. This way, the algorithm stops when and only when x_i is used for the division step. The algorithm then produces a polynomial $h \in I$ and $\mathbf{x}^\beta \in R$, such that $\mathbf{x}^\alpha = h + \mathbf{x}^\beta x_i$. Finally, $\mathbf{x}^\alpha - \mathbf{x}^\beta x_i$ is in I and is homogeneous. This shows one implication and the other one is trivial, which concludes the proof. □

We will now apply this theory to the Eulerian ideal of a graph G. Consider the quotient rings $\mathbb{K}[V_G]/(x_i^2 - x_j^2 : i,j \in V_G)$ and $\mathbb{K}[E_G]/I(G)$ as graded quotient modules, as in Definition 2.2.13.

Lemma 3.2.12. *Let G be a graph with $E_G \neq \emptyset$. The variables t_{ij} in $\mathbb{K}[E_G]$ are $\mathbb{K}[E_G]/I(G)$-regular.*

Proof. Take a variable $t_{ij} \in \mathbb{K}[E_G]$. For every $f + I(G) \in \mathbb{K}[E_G]/I(G)$ such that $t_{ij}(f + I(G)) = I(G)$, we will show that $f \in I(G)$. Since $t_{ij}f \in I(G)$, $\varphi(t_{ij}f) = x_i x_j \varphi(f)$ is in $(x_i^2 - x_j^2 : i, j \in V_G)$. By Proposition 3.2.3, x_i and x_j are $\mathbb{K}[V_G]/(x_i^2 - x_j^2 : i, j \in V_G)$-regular, from which follows that $\varphi(f)$ is in $(x_i^2 - x_j^2 : i, j \in V_G)$, and so f is in $I(G)$. We conclude that t_{ij} is $\mathbb{K}[E_G]/I(G)$-regular. □

Lemma 3.2.13. *Let G be a graph with $E_G \neq \emptyset$. For each variable $t_{ij} \in \mathbb{K}[E_G]$, the ideal $(I(G), t_{ij})$ contains every square of a variable of $\mathbb{K}[E_G]$.*

Proof. Take any $t_{kl}^2 \in \mathbb{K}[E_G]$. Since $t_{kl}^2 = (t_{kl}^2 - t_{ij}^2) + t_{ij}^2$, and $t_{kl}^2 - t_{ij}^2$ is in $I(G)$, $t_{kl}^2 \in (I(G), t_{ij})$. □

To begin the study of the regularity index $\mathrm{ri}(\mathbb{K}[E_G]/I(G))$, we present some conclusions of the theory above. The next result follows immediately from the previous lemmas, and Theorem 3.2.8.

Theorem 3.2.14. *Let G be a graph such that $s = |E_G| > 0$. Denote $M = \mathbb{K}[E_G]/I(G)$, and let H_M be the Hilbert function of M. Then, for every $k \geq s$, $H_M(k) = H_M(s)$.*

Remarks 3.2.15. Let G be a graph such that $s = |E_G| > 0$. (i) From Theorem 4.2.3 we see that the Hilbert polynomial of $\mathbb{K}[E_G]/I(G)$ is constant, and that $\mathrm{ri}(\mathbb{K}[E_G]/I(G)) \leq s$. (ii) The first two nonzero values of the Hilbert function of $M = \mathbb{K}[E_G]/I(G)$, H_M, are easy to compute. As $I(G)$ does not contain any nonzero polynomials of degree 0, or 1, $H_M(0) = H_{\mathbb{K}[E_G]}(0) - H_{I(G)}(0) = 1 - 0 = 1$, and $H_M(1) = H_{\mathbb{K}[E_G]}(1) - H_{I(G)}(1) = s - 0 = s$. (iii) By Lemma 3.2.9, H_M is a nondecreasing function.

3.3 Joins and regularity

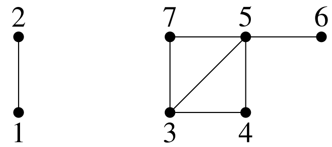
In this section we will present the results from [13] of Neves, Vaz Pinto and Villarreal, about the regularity index $\mathrm{ri}(\mathbb{K}[E_G]/I(G))$, for a graph G. To do so we will first need to introduce the notion of join of G, a subset of E_G that intersects the edge set of each cycle of G in at most half its edges. The proofs of the results of this section will mostly follow the proofs given by the authors in [13]. Later, in Chapter 4, we will generalize the construction of the Eulerian ideal, and the results of this section, for hypergraphs. There, different proofs of these results will be given. The main result of this section is the Theorem 4.5 of [13], that characterizes the regularity index using the joins of G, if G is a bipartite graph. Recall that bipartite graphs are graphs for which V_G is the union of two disjoint sets of vertices, $V_1, V_2 \subseteq V_G$, such that every edge of G contains a vertex of V_1 and a vertex of V_2. In this case we will say that G has *bipartition* (V_1, V_2). For other notions and results on graph theory we refer the reader to [2], however, we will need one particular result, related to the Euler's Theorem, that we present below.

Proposition 3.3.1. *Let G be a graph with $E_G \neq \emptyset$. G is an Eulerian graph if and only if E_G is the union of the edge sets of edge-disjoint cycles of G.*

Proof. See Theorem 1 of [2]. □

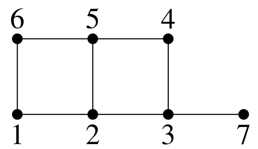
Definition 3.3.2. Let G be a graph. A set $J \subseteq E_G$ is a *join of G* if, for every Eulerian subgraph H of G, $|J \cap E_H| \leq \frac{|E_H|}{2}$.

Example 3.3.3. Consider the graph G of the Example 3.1.2, that is represented below.



The only Eulerian subgraphs of G, with nonempty edge set, and up to isolated vertices, are the three cycles, so a join of G is any set $J \subseteq E_G$, that does not contain more than 1 edge from each of the cycles of length 3, and more than 2 edges from the cycle of length 4. For example, $\{\{1,2\},\{3,4\},\{5,6\},\{5,7\}\}$ is a join of G.

Example 3.3.4. Let G be the graph presented below.



The Eulerian subgraphs of G, with nonempty edge set, and up to isolated vertices, are the two cycles of length 4, and the cycle of length 6, so a join of G is any set $J \subseteq E_G$, that contains no more than 2 edges from each cycle of length 4, and no more than 3 edges from the cycle of length 6. The sets $\{\{1,6\},\{2,5\},\{3,4\},\{3,7\}\}$ and $\{\{1,2\},\{1,6\},\{2,3\},\{3,7\}\}$ are joins of G.

Remarks 3.3.5. (i) Every graph G has a nonempty join, since any subset of E_G, with cardinality 1, is a join of G. (ii) If G is a graph and $J \subseteq E_G$ is a join of G, every subset of J is also a join of G.

We now give Definition 1.2 of [13].

Definition 3.3.6. Let G be a graph. We define *the maximum vertex join number* as the maximum cardinality of a join of G. We will denote this number by $\mu(G)$.

To first relate $\mathrm{ri}(\mathbb{K}[E_G]/I(G))$ with the joins of G, we present Theorem 4.2 of [13], and its proof.

Theorem 3.3.7. *If G is a graph with $E_G \neq \emptyset$,* $\mathrm{ri}(\mathbb{K}[E_G]/I(G)) \geq \mu(G) - 1$.

Proof. Let $J \subseteq E_G$ be a join of G, and let us show that $\mathrm{ri}(\mathbb{K}[E_G]/I(G)) \geq |J|-1$. Fix an edge $e \in J$, by Proposition 3.2.10, it suffices to show that $(\mathbb{K}[E_G]/(I(G),t_e))_{|J|-1}$ is not the zero vector space, which we will prove by showing that there is a monomial of $\mathbb{K}[E_G]_{|J|-1}$ that is not in $(I(G),t_e)$. Consider a join $J' \subseteq J \setminus \{e\}$, and let us show by induction on $|J'|$, that the product of the variables indexed by the edges of J' is not in $(I(G),t_e)$. If $|J'| = 1$, say $J' = \{a\}$, no polynomial in $I(G)$ has a term with degree 1, so t_a is not in $(I(G),t_e)$. Assume now that $|J'| \geq 2$, and that the result holds for subsets of $J \setminus \{e\}$ with cardinality lower than or equal to $|J'|-1$, that is, if a monomial is the product of variables indexed by at most $|J'|-1$ edges of $J \setminus \{e\}$, then it is not in $(I(G),t_e)$. Set $\mathbf{t}^\alpha \in \mathbb{K}[E_G]$ as the product of the variables indexed by the edges of J'. Eyeing a contradiction, suppose that $\mathbf{t}^\alpha \in (I(G),t_e)$. By Proposition 3.2.11 there is a monomial $\mathbf{t}^\beta \in \mathbb{K}[E_G]$ such that $\mathbf{t}^\alpha - \mathbf{t}^\beta t_e \in I(G)$, and is homogeneous. Let $\mathbf{t}^\gamma$ and $\mathbf{t}^\delta$ be the monomials such that $\mathbf{t}^\alpha = \mathbf{t}^\gamma \gcd(\mathbf{t}^\alpha,\mathbf{t}^\beta)$ and $\mathbf{t}^\beta t_e = \mathbf{t}^\delta \gcd(\mathbf{t}^\alpha,\mathbf{t}^\beta)$. By Lemma 3.2.12, $\gcd(\mathbf{t}^\alpha,\mathbf{t}^\beta)$ is

$\mathbb{K}[E_G]/I(G)$-regular, and so $\mathbf{t}^\gamma - \mathbf{t}^\delta \in I(G)$, and is homogeneous. By definition, $\mathbf{t}^\alpha$ is not divisible by t_e, therefore neither is $\gcd(\mathbf{t}^\alpha, \mathbf{t}^\beta t_e)$ and t_e must divide $\mathbf{t}^\delta$ so, by Proposition 3.2.11, $\mathbf{t}^\gamma$ belongs in $(I(G), t_e)$. If $\mathbf{t}^\alpha \neq \mathbf{t}^\gamma$, $\mathbf{t}^\gamma$ is the product of less than $|J'|$ variables indexed by the edges in $J \setminus \{e\}$ so, by the induction hypothesis, we have that $\mathbf{t}^\gamma$ is not in $(I(G), t_e)$. This is a contradiction so we must have that $\mathbf{t}^\alpha = \mathbf{t}^\gamma$ and $\gcd(\mathbf{t}^\alpha, \mathbf{t}^\beta t_e) = 1$. Let H be the subgraph of G the edges of which are indexed by the odd power variables of $\mathbf{t}^\alpha - \mathbf{t}^\beta t_e$. By Corollary 3.1.14, every vertex of H has even degree. Using that J' is a join of G, and is contained in E_H, $\deg(\mathbf{t}^\alpha) = |J'| = |J' \cap E_H| \leq \frac{1}{2}|E_H|$. But as $\mathbf{t}^\alpha - \mathbf{t}^\beta t_e$ is homogeneous, $|E_H| \leq 2\deg(\mathbf{t}^\alpha)$, and therefore $|E_H| = 2\deg(\mathbf{t}^\alpha)$. This implies that every variable in $\mathbf{t}^\beta t_e$ has exponent equal to 1, which by definition of H means that $e \in E_H$. Finally, $J' \cup \{e\}$, being a subset of J, is a join, and yields that $\deg(\mathbf{t}^\alpha) + 1 = |J' \cup \{e\}| = |(J' \cup \{e\}) \cap E_H| \leq \frac{1}{2}|E_H| = \deg(\mathbf{t}^\alpha)$. We have obtained a contradiction, so we must have that $\mathbf{t}^\alpha \notin (I(G), t_e)$. This concludes the induction step, so $(\mathbb{K}[E_G]/(I(G), t_e))_{|J|-1}$ is not the zero vector space and $\mathrm{ri}(\mathbb{K}[E_G]/I(G)) \geq |J| - 1$, for every join J of G. □

In [13], this inequality was shown to become an equality, if G is a bipartite graph. Before showing this result, we will need the following Lemmas.

Lemma 3.3.8. *If C is an even cycle,* $\mathrm{ri}(\mathbb{K}[E_G]/I(G)) = \frac{|E_C|}{2} - 1$.

Proof. Clearly $\mu(C) = \frac{|E_C|}{2}$, so Theorem 3.3.7 says that $\mathrm{ri}(\mathbb{K}[E_C]/I(C) \geq \frac{|E_C|}{2} - 1$, and we just need to show the other inequality. Fix an edge $e \in E_C$, using Proposition 3.2.10, it suffices to show that the homogeneous component of $\mathbb{K}[E_C]/(I(C), t_e)$, of degree $\frac{|E_C|}{2}$, is the zero $\mathbb{K}$-vector space. To do so, take a monomial $\mathbf{t}^\alpha \in \mathbb{K}[E_C]$ of degree $\frac{|E_C|}{2}$, and let us show that $\mathbf{t}^\alpha \in (I(C), t_e)$. If t_e divides $\mathbf{t}^\alpha$, we are done. And if there is an edge $l \in E_C$ such that t_l^2 divides $\mathbf{t}^\alpha$, let $\mathbf{t}^\gamma$ be the monomial such that $\mathbf{t}^\alpha = t_l^2 \mathbf{t}^\gamma$. Since $\mathbf{t}^\alpha = (t_l^2 - t_e^2)\mathbf{t}^\gamma + t_e^2 t^\gamma$ we see that $\mathbf{t}^\alpha \in (I(C), t_e)$. Suppose now that $\mathbf{t}^\alpha$ is neither divisible by t_e nor by the square of a variable of $\mathbb{K}[E_C]$. Then $\mathbf{t}^\alpha$ identifies half the edges of C. Let $\mathbf{t}^\beta$ be the product of the variables, indexed by the edges of C, that do not index any variable of $\mathbf{t}^\alpha$. Then $\mathbf{t}^\alpha - \mathbf{t}^\beta$ is a homogeneous binomial of $\mathbb{K}[E_C]$, with $\gcd(\mathbf{t}^\alpha, \mathbf{t}^\beta) = 1$, that identifies the edges of C, an Eulerian graph. By Corollary 3.1.14, $\mathbf{t}^\alpha - \mathbf{t}^\beta \in I(C)$, and since t_e must divide $\mathbf{t}^\beta$, it follows that $\mathbf{t}^\alpha = (\mathbf{t}^\alpha - \mathbf{t}^\beta) + \mathbf{t}^\beta$ is in $(I(C), t_e)$. Therefore $(\mathbb{K}[E_C]/(I(C), t_e))_{\frac{|E_C|}{2}}$ is the zero vector space and $\mathrm{ri}(\mathbb{K}[E_C]/I(C)) = \frac{|E_C|}{2} - 1$. □

Lemma 3.3.9. *Let C be an even cycle, and C' the graph obtained by adding an edge l to C. Then* $\mathrm{ri}(\mathbb{K}[E_{C'}]/I(C')) \leq \frac{|E_C|}{2}$.

Proof. Fix an edge $e \in E_C$, by Proposition 3.2.10, it suffices to show that the homogeneous component of $\mathbb{K}[E_{C'}]/(I(C'), t_e)$, of degree $\frac{|E_C|}{2} + 1$, is the zero vector space. Take a monomial $\mathbf{t}^\alpha \in \mathbb{K}[E_{C'}]$ of degree $\frac{|E_C|}{2} + 1$, and let us show it is in $(I(C'), t_e)$. If $\mathbf{t}^\alpha$ is divisible by t_e, or by the square of a variable, we are done. Otherwise, at least half the edges of C index the variables of $\mathbf{t}^\alpha$. Consider a monomial $\mathbf{t}^\beta$ and a variable t_k such that $\mathbf{t}^\alpha = \mathbf{t}^\beta t_k$, and the variables of $\mathbf{t}^\beta$ are all indexed by edges of C. Let $\mathbf{t}^\gamma$ be the product of the variables the index of which are the edges of C that do not index the variables of $\mathbf{t}^\beta$. Then $\mathbf{t}^\beta - \mathbf{t}^\gamma$ identifies the edges of C, and by Proposition 4.1.7, $\mathbf{t}^\beta - \mathbf{t}^\gamma \in I(C')$. Since $\mathbf{t}^\beta$ is not divisible by t_e, $\mathbf{t}^\gamma$ must be, so the equality $\mathbf{t}^\alpha = t_k(\mathbf{t}^\beta - \mathbf{t}^\gamma) + t_k \mathbf{t}^\gamma$ implies that $\mathbf{t}^\alpha \in (I(C'), t_e)$. This shows that $\mathrm{ri}(\mathbb{K}[E_{C'}]/I(C')) \leq \frac{|E_C|}{2}$. □

We now present the main result of this section, the Theorem 4.5 of [13], that states that for bipartite graphs, the inequality in Theorem 3.3.7 is an equality. For this purpose, recall that a graph is bipartite if and only if it does not contain an odd cycle, see Theorem 4 of [2]. The proof below follows the one given in [13]. However, we will use both Lemma 3.3.8 and Lemma 3.3.9, while in [13] a stronger result is first shown, the characterization of the regularity index for Hamiltonian bipartite graphs. We will present a proof of this result below in Section 3.4.

Theorem 3.3.10. *If G is a bipartite graph with $E_G \neq \emptyset$,* $\mathrm{ri}(\mathbb{K}[E_G]/I(G)) = \mu(G) - 1$.

Proof. By Theorem 3.3.7, we only need to show that $\mathrm{ri}(\mathbb{K}[E_G]/I(G)) \leq \mu(G) - 1$. Fix an edge $e \in E_G$, using Proposition 3.2.10, it suffices to show that the homogeneous component of $\mathbb{K}[E_G]/(I(G), t_e)$, of degree $\mu(G)$, is the zero vector space. Let $\mathbf{t}^\alpha \in \mathbb{K}[E_G]$ be a monomial of degree $\mu(G)$, and let us show it is in $(I(G), t_e)$. Assume that $\mathbf{t}^\alpha$ is not divisible neither by t_e nor by the square of a variable, as otherwise we are done. Let H be the subgraph of G the edges of which are indexed by the variables of $\mathbf{t}^\alpha$. Then $|E_H \cup \{e\}| = \mu(G) + 1$, so $E_H \cup \{e\}$ is not a join of G, and there is an Eulerian subgraph C of G such that

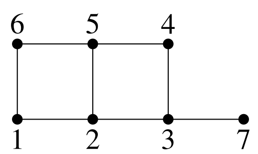
$$|(E_H \cup \{e\}) \cap E_C| > \frac{|E_C|}{2}. \tag{3.7}$$

By Proposition 3.3.1, we may assume that C is a cycle of G, and so C is an even cycle, because G is bipartite. Let us consider two cases. If $e \in E_C$, then $(E_H \cup \{e\}) \cap E_C = (E_H \cap E_C) \cup \{e\}$ and (3.7) implies $|E_H \cap E_C| \geq \frac{|E_C|}{2} = \mathrm{ri}(\mathbb{K}[E_C]/I(C) + 1$, where the last equality follows from Lemma 3.3.8. Now, Proposition 3.2.10 says that any monomial in $\mathbb{K}[E_C]$ of degree $|E_H \cap E_C|$ belongs to $(I(C), t_e)$, and so, the product of the variables indexed by the edges of $E_H \cap E_C$, $\mathbf{t}^\beta$, is in $(I(C), t_e)$. By Proposition 3.1.17, $I(C) \subseteq I(G)$, so $\mathbf{t}^\beta \in (I(G), t_e)$, and since $\mathbf{t}^\beta$ divides $\mathbf{t}^\alpha$, $\mathbf{t}^\alpha$ is in $(I(G), t_e)$. For the second case assume e is not in E_C. Then (3.7) means that

$$|E_H \cap E_C| = |(E_H \cup \{e\}) \cap E_C| \geq \frac{|E_C|}{2} + 1 \geq \mathrm{ri}(\mathbb{K}[E_{C'}]/I(C') + 1, \tag{3.8}$$

where C' is the subgraph of G obtained by adding the edge e to C, and the last inequality comes from Lemma 3.3.9. Combining $|E_H \cap E_C| \geq \mathrm{ri}(\mathbb{K}[E_{C'}]/I(C')) + 1$ with Proposition 3.2.10, we get that any monomial of $\mathbb{K}[E_{C'}]$, of degree $|E_H \cap E_C|$, is in $(I(C'), t_e)$. Again, taking $\mathbf{t}^\beta$ as the product of the variables indexed by the edges of $E_H \cap E_C$, $\mathbf{t}^\beta \in (I(C'), t_e) \subseteq (I(G), t_e)$, and since it divides $\mathbf{t}^\alpha$, $\mathbf{t}^\alpha \in (I(G), t_e)$. This shows that $(\mathbb{K}[E_G]/(I(G), t_e))_{\mu(G)}$ is the zero vector space and therefore $\mathrm{ri}(\mathbb{K}[E_G]/I(G) = \mu(G) - 1$. □

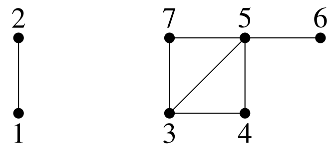
Example 3.3.11. Consider the graph G from the Example 3.3.4.



G is a bipartite graph since it does not contain any odd cycles. Let us use Theorem 3.3.10 to show that $\mathrm{ri}(\mathbb{K}[E_G]/I(G)) = 3$, by proving that $\mu(G) = 4$. Let H be a subgraph of G with edge set

$E_G \setminus \{\{3,7\}\}$, and let J be a join of H. If J does not contain the edge $\{2,5\}$, J is contained in the cycle of length 6, and $|J| \leq 3$. And if J contains the edge $\{2,5\}$, for each of the cycles of length 4, J can only have one other edge, so again we obtain that $|J| \leq 3$. It follows that $\mu(H) = 3$, and since the joins of G are either joins of H, or obtained from joins of H by adding the edge $\{3,7\}$, we deduce that $\mu(G) = 4$. This means that the Hilbert function of $\mathbb{K}[E_G]/I(G)$ is constant for every integer greater than or equal to 3. Using the software Macaulay2, [8], and in order to illustrate this, we obtained for the integers $0,1,\ldots,6$, the values of the Hilbert function of $\mathbb{K}[E_G]/I(G)$, $1,8,23,32,32,32,32$.

Example 3.3.12. The equality of Theorem 3.3.10 may not hold if G is not a bipartite graph. Take the graph G of the Example 3.1.2, that is not bipartite since it contains odd cycles.



Let us calculate $\mu(G)$. Take a join J, of G, that contains the edges $\{1,2\}$ and $\{5,6\}$. If $\{3,5\} \in J$, no other edge of a cycle of G can be in J, and so $|J| = 3$. Otherwise, if $\{3,5\} \notin J$, J may have one edge from each cycle of length 3, so $|J| \leq 4$. It follows that $\mu(G) = 4$. However, by using the software Macaulay2, [8], we obtained that $\mathrm{ri}(\mathbb{K}[E_G]/I(G)) = 4 \neq 3 = \mu(G) - 1$.

3.4 Applications

To conclude the study of the Eulerian ideal of a graph G, and as an application of Theorem 3.3.10, we will calculate the regularity index $\mathrm{ri}(\mathbb{K}[E_G]/I(G))$ for two classes of graphs: Complete bipartite graphs, which are the bipartite graphs, with bipartition (V_1, V_2), for which the edge set contains all possible edges between the vertices of V_1 and the vertices of V_2; and Hamiltonian bipartite graphs, that is, bipartite graphs with a *Hamiltonian cycle*, i. e., a cycle that contains all vertices of G. For this purpose, we will need to introduce the notion of T-join of a graph. This is an important notion in Combinatorial Optimization, and can be found in Chapter 12 of [12]. As we will see below, there is a connection between the T-joins and the joins of a graph.

Joins and T-joins

Definition 3.4.1. Let G be a graph. A set $J \subseteq E_G$ is said to be a T-join if there is a set $T \subseteq V_G$ such that, in the graph (V_G, J), T is the set of odd degree vertices.

Remarks 3.4.2. Let G be a graph. (i) To find a T-join of G, it is enough to choose $J \subseteq E_G$, and set T as the set of odd degree vertices in (V_G, J). (ii) The edge sets of the Eulerian subgraphs of G are $\emptyset$-joins of G. In particular, the $\emptyset$ is an $\emptyset$-join.

Example 3.4.3. Consider the graph G from the Example 3.3.4, which is the leftmost graph below. If $T = V_G \setminus \{3\}$, the set $J = \{\{1,6\},\{2,5\},\{3,4\},\{3,7\}\}$ is a T-join of G, since in the graph (V_G, J), T is the set of vertices with odd degree, or equivalently, 3 is the only vertex with even degree. The set $J' = E_G \setminus \{\{1,6\},\{3,4\}\}$ is also a T-join for the same T.

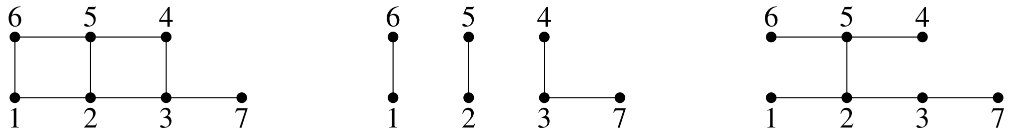


Figure 3.1 The graphs G, (V_G, J), and (V_G, J').

Definition 3.4.4. Given two sets A and B, the *symmetric difference of A and B* is defined as the set $A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cap B^c) \cup (B \cap A^c)$.

Proposition 3.4.5. *Let G be a graph. If $J_1, J_2 \subseteq E_G$ are such that J_1 is a T_1-join and J_2 is a T_2-join, for some $T_1, T_2 \subseteq V_G$, then $J_1 \triangle J_2$ is a $(T_1 \triangle T_2)$-join.*

Proof. We must show that a vertex $i \in V_G$ is in $T_1 \triangle T_2$ if and only if it has odd degree in the graph $(V_G, J_1 \triangle J_2)$. Let $\delta \subseteq E_G$ be the set of edges of G that contain i. We will now use the equality

$$|\delta \cap (J_1 \triangle J_2)| = |\delta \cap J_1| + |\delta \cap J_2| - 2|\delta \cap J_1 \cap J_2|, \tag{3.9}$$

to show that $i \in T_1 \triangle T_2$ if and only if $|\delta \cap (J_1 \triangle J_2)|$ is odd, that is, i has odd degree in $(V_G, J_1 \triangle J_2)$. Supposing $i \in T_1 \triangle T_2$, either i is in $T_1 \setminus T_2$, or i is in $T_2 \setminus T_1$. As the other case is analogous assume $i \in T_1 \setminus T_2$. Since J_1 is a T_1-join and J_2 is a T_2-join, i has odd degree in (V_G, J_1) and even degree in (V_G, J_2), that is, $|\delta \cap J_1|$ is odd and $|\delta \cap J_2|$ is even. Using (3.9) follows that $|\delta \cap (J_1 \triangle J_2)|$ is odd. Conversely, eyeing a contradiction assume that $|\delta \cap (J_1 \triangle J_2)|$ is odd but i is not in $T_1 \triangle T_2$. Then $i \in T_1^c \cap T_2^c$ or $i \in T_1 \cap T_2$, and in either case we see that, in (V_G, J_1) and (V_G, J_2), the degree of i has the same parity. Therefore $|\delta \cap J_1| + |\delta \cap J_2|$ is even, which by (3.9) contradicts $|\delta \cap (J_1 \triangle J_2)|$ being odd, so $i \in T_1 \triangle T_2$. This shows that $J_1 \triangle J_2$ is a $T_1 \triangle T_2$-join. ☐

Proposition 3.4.6 (Guan's Lemma). *Let G be a graph. If $J \subseteq E_G$ is a T-join with the least cardinality among all T-joins of G, it is a join of G. And if $J \subseteq E_G$ is both a join and a T-join of G, it has the least cardinality among all T-joins of G.*

Proof. Let $J \subseteq E_G$ be a T-join with the least cardinality among all T-joins of G. To show that J is a join, let H be an Eulerian subgraph of G. Proposition 3.4.5 says that $J \triangle E_H$ is also a T-join of G, so $|J| \leq |J \triangle E_H| = |J| + |E_H| - 2|J \cap E_H|$. It follows that $|J \cap E_H| \leq \frac{|E_H|}{2}$, and J is a join of G. Assume now that $J \subseteq E_G$ is both a join and a T-join of G. Let J' be a T-join with the least cardinality among all T-joins of G, and let us show that $|J| = |J'|$. By Proposition 3.4.5, $J \triangle J'$ is an $\emptyset$-join. Therefore the graph $(V_G, J \triangle J')$ is Eulerian, and since J is a join of G,

$$|J| - |J \cap J'| = |J \cap (J \triangle J')| \leq \frac{|J \triangle J'|}{2} = \frac{|J|}{2} + \frac{|J'|}{2} - |J \cap J'|,$$

and so $|J| \leq |J'|$. As J' has the least cardinality among T-joins of G, $|J'| = |J|$. This ends the proof. ☐

To prove the main results of this section, we will need to be able to compare the regularity index for a bipartite graph G, and for a subgraph H. By Theorem 3.3.10, we only need to compare $\mu(G)$ and $\mu(H)$, the greatest cardinalities of the joins of G and H. If H has less edges than G, one might expect that $\mu(H) \le \mu(G)$, as it happens if G is a forest. However, if by excluding edges of G, we make H have less cycles than G, we are decreasing the number of inequalities in the definition of join, Definition 3.3.2, and may actually obtain that $\mu(H) \ge \mu(G)$. This inequality is stated by Proposition 3.2 (i), of [13], under certain conditions. A possible difficulty in proving this is that a join of H need not be a join of G. This is where, through Proposition 3.4.6, T-joins come in handy. As we will see, T-joins of H are T-joins of G. Also, to find a join of G with greatest cardinality, we must choose T, among all sets $T \subseteq V_G$, for which there is a T-join, in a way that maximizes the cardinality of the T-joins of least cardinality among all T-joins of G. This motivates the next definition.

Definition 3.4.7. Given a graph G, we define $\mathscr{E}_G$ as the set $\{T \subseteq V_G : \ G \text{ has a } T\text{-join}\}$.

Remarks 3.4.8. Let G be a graph and H a subgraph of G. (i) If $J \subseteq E_H$ is a T-join of H, J is also a T-join of G. To show this we must prove that T is the set of odd degree vertices of the graph (V_G, J). Take any vertex $i \in V_G$, if $i \notin V_H$, it is not in T, and has degree zero in the graph (V_G, J). And if $i \in V_H$, it has odd degree in (V_G, J) if and only if it has odd degree in (V_H, J) if and only if $i \in T$. Therefore J is a T-join of G. (ii) In particular $\mathscr{E}_H \subseteq \mathscr{E}_G$. (iii) The other inclusion need not hold. For example, let $G = (\{1,2\},\{1,2\})$ and $H = (\{1,2\},\emptyset)$. G only has two T-joins, the $\emptyset$ which is an $\emptyset$-join, and $\{\{1,2\}\}$ which is an $\{1,2\}$-join. It follows that $\mathscr{E}_G = \{\emptyset,\{1,2\}\}$, but $\mathscr{E}_H = \{\emptyset\}$.

Proposition 3.4.9. *Let G be a bipartite graph, and H a subgraph of G with $E_H \neq \emptyset$. If $\mathscr{E}_G = \mathscr{E}_H$,* $\mathrm{ri}(\mathbb{K}[E_H]/I(H)) \ge \mathrm{ri}(\mathbb{K}[E_G]/I(G))$.

Proof. Subgraphs of bipartite graphs are still bipartite so, by Theorem 3.3.10, it suffices to show that $\mu(H) \ge \mu(G)$. For each $T \in \mathscr{E}_H$, let J be a T-join of H and J' a T-join of G, such that, J and J' have the least cardinality among the T-joins of H and G, respectively. Since T-joins of H are T-joins of G, we see that $|J| \ge |J'|$, and it follows that

$$\max_{T\in\mathscr{E}_H} |J| \ge \max_{T\in\mathscr{E}_H} |J'| = \max_{T\in\mathscr{E}_G} |J'|.$$

Using Proposition 3.4.6 we obtain that $\mu(H) \ge \mu(G)$ and therefore $\mathrm{ri}(\mathbb{K}[E_H]/I(H)) \ge \mathrm{ri}(\mathbb{K}[E_G]/I(G))$. □

In [13], a more general result was shown, with a different proof. Proposition 3.2 (i), of [13], states that if G is bipartite, or if both G and H are non-bipartite, $V_G = V_H$ and G and H having the same number of connected components implies that $\mathrm{ri}(\mathbb{K}[E_H]/I(H)) \ge \mathrm{ri}(\mathbb{K}[E_G]/I(G))$. Now, Proposition 3.4.9 asks the question: When does the equality $\mathscr{E}_G = \mathscr{E}_H$ hold? To answer this we need the following useful characterization of the sets $T \in \mathscr{E}_G$.

Proposition 3.4.10. *Let G be a graph. For every $T \subseteq V_G$, there is a T-join of G if and only if, for every connected component H of G, $|T \cap V_H|$ is even.*

Proof. See Proposition 12.6 of [12]. □

Proposition 3.4.11. *Let G be a graph and H a subgraph of G. If $V_G = V_H$, and G and H have the same number of connected components, $\mathscr{E}_G = \mathscr{E}_H$. The converse holds if G does not have isolated vertices.*

Proof. Assume that $V_G = V_H$, and G and H have the same number of connected components. It suffices to prove that $\mathscr{E}_G \subseteq \mathscr{E}_H$. Take any $T \in \mathscr{E}_G$, by Proposition 3.4.10, T intersects the vertex set of every connected component of G in an even number of vertices. Each connected component H' of H is a subgraph of some connected component G' of G, let us see that $V_{H'} = V_{G'}$. If a vertex of G' is in a connected component of H other than H', that component is also a subgraph of G'. And because H and G have the same number of connected components, we deduce that some connected component of G does not have any connected component of H as a subgraph, contradicting that $V_G = V_H$. Therefore $V_{H'} = V_{G'}$, and so, each connected component of H must have the same vertex set as some connected component of G. This implies T also intersects the vertex set of every connected component of H in an even number of vertices. By Proposition 3.4.10, $T \in \mathscr{E}_H$, which shows that $\mathscr{E}_G \subseteq \mathscr{E}_H$. Suppose now that G does not have isolated vertices, and $\mathscr{E}_G = \mathscr{E}_H$. Then every vertex $i \in V_G$ is in some edge $\{i, j\} \in E_G$. Since $\{\{i, j\}\}$ is an $\{i, j\}$-join of G, $\{i, j\}$ is in $\mathscr{E}_G = \mathscr{E}_H$, and so $i \in V_H$. This means that $V_G = V_H$. And, as each connected component of H is a subgraph of some connected component of G, $V_G = V_H$ implies that H has at least as much connected components as G. Also, if H has more connected components than G, there is a connected component of G with two vertices i and j, in different connected components of H. By Proposition 3.4.10, $\{i, j\} \in \mathscr{E}_G$ but not in $\mathscr{E}_H$. This contradicts $\mathscr{E}_G = \mathscr{E}_H$, so G and H must have the same number of connected components. □

Remark 3.4.12. Given a graph G and a subgraph H, $V_G = V_H$, and G and H having the same number of connected components, is equivalent to H be obtained from G by excluding no more than one edge from each cycle of G. Therefore, Proposition 3.4.9 and Proposition 3.4.11 say that, as expected, if H is obtained from G in this fashion, $\mu(H) \geq \mu(G)$.

Regularity index for Hamiltonian bipartite graphs

Before calculating $\mathrm{ri}(\mathbb{K}[E_G]/I(G))$, for a Hamiltonian bipartite graph G, we will calculate it for the complete bipartite graphs. For every integers $a, b \geq 1$, there is only a unique complete bipartite graph, up to isomorphism, with bipartition (V_1, V_2) such that $|V_1| = a$ and $|V_2| = b$, we denote it by $K_{a,b}$.

Proposition 3.4.13. *If $G = K_{a,b}$, for some integers $a, b \geq 1$, $\mathrm{ri}(\mathbb{K}[E_G]/I(G)) = \max\{a, b\} - 1$.*

Proof. According to Theorem 3.3.10, it suffices to show that the greatest cardinality of a join of G is $\max\{a, b\}$. We begin by showing that G has a join with this cardinality. Assume $a \geq b$ and choose a vertex $j \in V_2$. The set $J = \{\{i, j\} : i \in V_1\} \subseteq E_G$ has cardinality $|V_1| = a = \max\{a, b\}$, let us show it is a join of G. Let H be an Eulerian subgraph of G. For every edge $\{i, j\} \in J \cap E_H$, there is another edge of H containing i. This implies H has at least twice as much edges as $|J \cap E_H|$, that is, $|J \cap E_H| \leq \frac{|E_H|}{2}$. This shows that J is a join of G with cardinality $\max\{a, b\}$. If $b \geq a$, analogously G has a join with cardinality $\max\{a, b\}$. All that is left to show now is that G does not have joins of cardinality greater than $\max\{a, b\}$. If $a = 1$ or $b = 1$, $|E_G| = \max\{a, b\}$ and we are done. Suppose then that $a, b > 1$, and there is a join of G, J, with $|J| > \max\{a, b\}$. This means there are at least two edges in J with the same vertex $i_1 \in V_1$. Let j_1, j_2 be vertices of V_2 such that $\{i_1, j_1\}$ and $\{i_1, j_2\}$ are in J. If there is a vertex $i_2 \neq i_1$ in V_1 such that $\{i_2, j_k\} \in J$, for some $k \in \{1, 2\}$, because G is complete bipartite, there is an edge $\{i_2, j_l\} \in G$, with $l \in \{1, 2\} \setminus \{k\}$. Then, the edges $\{i_1, j_k\}, \{j_k, i_2\}, \{i_2, j_l\}, \{j_l, i_1\}$ form a cycle C with $|J \cap E_C| = 3 > 2 = \frac{|E_C|}{2}$. This would contradict J being a join, so there cannot exist such a vertex $i_2 \neq i_1$ in V_1, such that $\{i_2, j_k\} \in J$, for some $k \in \{1,2\}$. This means that $\{i_1, j_1\}$ and $\{i_1, j_2\}$ are the only edges in J that contain the vertices j_1 and j_2, and that $b \geq 3$, because $|J| > b$. Now $|J \setminus \{\{i_1, j_1\}, \{i_1, j_2\}\}|$ is greater than $|V_2 \setminus \{j_1, j_2\}| = b-2$. Therefore there are at least two edges in $J \setminus \{\{i_1, j_1\}, \{i_1, j_2\}\}$ with the same vertex $j_3 \in V_2 \setminus \{j_1, j_2\}$. Let $\{i_3, j_3\}$ and $\{i_4, j_3\}$ be such edges, for some vertices $i_3, i_4 \in V_1$, with $i_3 \neq i_4$. There are two cases to consider, if either $i_3 = i_1$ or $i_4 = i_1$, say $i_3 = i_1$, then $\{i_1, j_1\}, \{j_1, i_4\}, \{i_4, j_3\}, \{j_3, i_1\}$ forms a cycle of length 4 with 3 edges of J, contradicting that J is a join. Otherwise, if $i_3 \neq i_1$ and $i_4 \neq i_1$, then $\{i_1, j_1\}, \{j_1, i_3\}, \{i_3, j_3\}, \{j_3, i_4\}, \{i_4, j_2\}, \{j_2, i_1\}$ forms a cycle of length 6 with 4 edges of J. Again, this contradicts that J is a join, so we conclude that there is no join of G with cardinality greater than $\max\{a, b\}$. □

Proposition 3.4.13 was first shown in [13] with a different proof, not relying in the characterization of Theorem 3.3.10 for the regularity index of a bipartite graph, see Proposition 3.2 (iii) of [13]. We will now present the main result of this section, the formula for the regularity index for a Hamiltonian bipartite graph. The proof we present is the one of Proposition 3.2 (iv) of [13].

Corollary 3.4.14. *If G is a Hamiltonian bipartite graph,* $\mathrm{ri}(\mathbb{K}[E_G]/I(X_G)) = \frac{|V_G|}{2} - 1$.

Proof. Let C be a Hamiltonian cycle of G, and let G' denote the complete bipartite graph $K_{a,a}$, with $a = \frac{|V_G|}{2}$. We have that C is a subgraph of G and G is a subgraph of G'. Using Proposition 3.4.9 together with Proposition 3.4.13 and Lemma 3.3.8, we get that

$$\frac{|V_G|}{2} - 1 = \mathrm{ri}(\mathbb{K}[E_C]/I(C)) \geq \mathrm{ri}(\mathbb{K}[E_G]/I(G)) \geq \mathrm{ri}(\mathbb{K}[E_{G'}]/I(G')) = \frac{|V_G|}{2} - 1.$$

Therefore we conclude that $\mathrm{ri}(\mathbb{K}[E_G]/I(G)) = \frac{|V_G|}{2} - 1$. □

Chapter 4

Eulerian ideals of hypergraphs

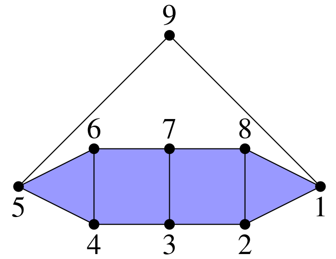
In Chapter 3, we studied the Eulerian ideal of a graph G, and presented the results of [13] about the regularity index $\mathrm{ri}(\mathbb{K}[E_G]/I(G))$. As one may have noticed, in many of the results from Chapter 3, it is not clear if the cardinality of the edges being 2 is important. This raises the question of whether those results could be generalized for graphs with edges of other cardinalities, that is, hypergraphs. A hypergraph H is a pair (V_H, E_H), where $V_H = \{1, \dots, n\}$, for some positive integer n, and E_H is a collection of subsets of V_H. Like for graphs, V_H is the vertex set of H, and E_H is the edge set of H. In this chapter we generalize, for hypergraphs, the construction of the Eulerian ideal. To use the results from Section 3.1 and Section 3.2, from early on we will restrict ourselves to hypergraphs with edges of a fixed cardinality k, that is, k-uniform hypergraphs. After that, we show that the results of [13], about the generators of the ideal, and the regularity index, also generalize for these hypergraphs.

From now on let $\mathbb{K}$ be a field. If H is a hypergraph, we will work with the polynomial rings $\mathbb{K}[V_H] = \mathbb{K}[x_i : i \in V_H]$ and $\mathbb{K}[E_H] = \mathbb{K}[t_A : A \in E_H]$. As we did for graphs, if $\{i_1, \dots, i_l\}$ is an edge of H, we will denote the variable $t_{\{i_1,\dots,i_l\}}$ by $t_{i_1\cdots i_l}$.

4.1 Preliminaries

Definition 4.1.1. Let H be a hypergraph with $E_H \neq \emptyset$, and consider the ring homomorphism $\varphi\colon \mathbb{K}[E_H] \to \mathbb{K}[V_H]$ given by $t_A \mapsto \prod_{i\in A} x_i$. The *Eulerian ideal of* H is the ideal $\varphi^{-1}(x_i^2 - x_j^2 : i, j \in V_H)$, and will be denoted by $I(H)$.

Example 4.1.2. Consider the hypergraph H, with vertex set $V_H = \{1,2,3,4,5,6,7,8,9\}$, and edge set $E_H = \{\{1,9\},\{5,9\},\{1,2,8\},\{4,5,6\},\{2,3,7,8\},\{3,4,6,7\}\}$, presented below.



We have $\mathbb{K}[V_H] = \mathbb{K}[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9]$, and $\mathbb{K}[E_H] = \mathbb{K}[t_{19}, t_{59}, t_{128}, t_{456}, t_{2378}, t_{3467}]$. Using the software Macaulay2, [8], we obtained the reduced Gröbner basis of $I(H)$ with respect to the grevlex monomial order, which consists in the following binomials:

$$t_{2378}^2 - t_{3467}^2, \quad t_{19}^2 - t_{59}^2, \quad t_{128}^2 - t_{456}^2, \quad t_{456}^4 - t_{59}^2 t_{3467}^2,$$

$$t_{59}t_{128}t_{2378} - t_{19}t_{456}t_{3467}, \quad t_{19}t_{456}t_{2378} - t_{59}t_{128}t_{3467},$$

$$t_{59}t_{456}t_{2378} - t_{19}t_{128}t_{3467}, \quad t_{19}t_{128}t_{2378} - t_{59}t_{456}t_{3467}, \quad t_{128}t_{456}^3 - t_{19}t_{59}t_{2378}t_{3467},$$

$$t_{19}t_{59}^3 t_{456} - t_{128}t_{2378}t_{3467}, \quad t_{19}t_{59}^3 t_{128} - t_{456}t_{2378}t_{3467}, \quad t_{59}^4 - t_{3467}^2.$$

Remarks 4.1.3. (i) In Example 4.1.8, not all polynomials of the reduced Gröbner basis of $I(H)$ are homogeneous, e.g., the polynomial $t_{19}t_{59}^3 t_{456} - t_{128}t_{2378}t_{3467}$. If $I(H)$ were homogeneous, using Buchberger's algorithm, the reduced Gröbner basis would be homogeneous, so we conclude that the ideal $I(H)$ is not homogeneous. This contrasts with the Eulerian ideal of a graph, which by Proposition 3.1.7, is always a homogeneous ideal. The difference here is that the ring homomorphism that defines $I(H)$ is not graded, for example it sends the variables t_{19} and t_{2378} to monomials of different degrees. (ii) Another difference between the Eulerian ideal for a graph and for a hypergraph, is that for a hypergraph it need not contain every difference of squares of any two variables. This is also illustrated by the Example 4.1.8, the binomial $t_{59}^2 - t_{456}^2$ is not in $I(H)$ as its initial monomial, t_{59}^2, is not divisible by any initial monomial of the elements of the Gröbner basis obtained in Example 4.1.8. (iii) As for graphs, if H is a hypergraph, no polynomial in the ideal $I(H)$ has a term of degree 1. If some $f \in I(H)$ had a term of degree 1, $\varphi(f)$ would have a term not divisible by the square of any variable x_i. This would contradict $\varphi(f)$ being in $(x_i^2 - x_j^2 : i, j \in V_H)$, so no term of f has degree 1. In particular $I(H)$ does not have degree 1 polynomials.

k-Uniform Hypergraphs

Definition 4.1.4. Let H be a hypergraph. The maximum cardinality of an edge is called the *rank of* H. We say that H is uniform of rank k (or simply k-uniform) if all edges have cardinality k.

Remarks 4.1.5. (i) The cases of k-uniform hypergraphs, with $k = 0, 1$, are of no interest to us. We will always assume $k \geq 2$. (ii) A 2-uniform hypergraph is a graph. (ii) Given a k-uniform hypergraph H, the ring homomorphism φ, that defines the Eulerian ideal, is a graded ring homomorphism of degree k, as every variable of $\mathbb{K}[E_H]$ is transformed by φ into a degree k monomial. (iv) Attending to Proposition 3.1.7, and Corollary 3.1.9, the Eulerian ideal of a k-uniform hypergraph is a homogeneous binomial ideal. Also, for every monomial order in $\mathbb{K}[E_H]$, there is a Gröbner basis of $I(H)$ consisting of homogeneous binomials, and these binomials do not depend on the choice of $\mathbb{K}$. (v) If H is a k-uniform hypergraph, $I(H)$ contains every difference of squares of any two variables. To see this, take edges $A, B \in E_H$, and let us show that $t_A^2 - t_B^2 \in I(H)$. Assume $|V_H| = n$, and fix a monomial order in $\mathbb{K}[V_H]$ such that x_n is the least variable. By Proposition 2.15 of [6] and Buchberger's criterion, Theorem 2.14 of [6], $\mathscr{G} = \{x_1^2 - x_n^2, \ldots, x_{n-1}^2 - x_n^2\}$ is a Gröbner basis for $(x_i^2 - x_j^2 : i, j \in V_H)$. Applying to $\varphi(t_A^2 - t_B^2)$ the division algorithm with respect to $\mathscr{G}$, each step of the algorithm changes some x_i^2, in the initial monomial of the dividend, to x_n^2. It follows that the last step in the algorithm transforms $x_i^2 x_n^{2k-2} - x_n^{2k}$ into zero, and so the remainder is zero. Therefore $\varphi(t_A^2 - t_B^2) \in (\mathscr{G})$ and $t_A^2 - t_B^2 \in I(H)$.

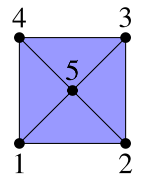
In Proposition 2.5 of [13], Neves, Vaz Pinto, and Villarreal characterized the generators of the Eulerian ideal of a graph G in terms of the Eulerian subgraphs of G, that is, subgraphs in which every vertex has even degree. The proof of this result also works for k-uniform hypergraphs, however, the notion of subgraph does not have a straightforward generalization to hypergraphs. There are several nonequivalent notions of subhypergraph in the literature. For one, an edge of a subhypergraph can be a subset of an edge of the hypergraph and not the full set. If we insist to stay in the class of k-uniform hypergraphs then there is only one reasonable definition of subhypergraph.

Definition 4.1.6. Let H be a k-uniform hypergraph. We say that a k-uniform hypergraph L is a *subhypergraph of* H if $V_L \subseteq V_H$ and $E_L \subseteq E_H$.

Proposition 4.1.7. *Let H be a k-uniform hypergraph with $E_H \neq \emptyset$, and $\mathbf{t}^\alpha - \mathbf{t}^\beta \in \mathbb{K}[E_H]$ a nonzero homogeneous binomial with* $\gcd(\mathbf{t}^\alpha, \mathbf{t}^\beta) = 1$. *Let L be a subhypergraph of H the edges of which index the odd power variables of $\mathbf{t}^\alpha - \mathbf{t}^\beta$. Then $\mathbf{t}^\alpha - \mathbf{t}^\beta \in I(H)$ if and only if every vertex of L has even degree.*

Proof. We begin by writing $\mathbf{x}^\delta - \mathbf{x}^\gamma = \varphi(\mathbf{t}^\alpha - \mathbf{t}^\beta)$. For every $i \in V_L$, let $a_{i1}, \dots, a_{ir_i}$ and $k_{i1}, \dots, k_{ip_i}$ be respectively the even and odd exponents of the variables of $\mathbf{t}^\alpha$ that have i in the index, and also, let $b_{i1}, \dots, b_{is_i}$ and $l_{i1}, \dots, l_{iq_i}$ be respectively the even and odd exponents of the variables of $\mathbf{t}^\beta$ that have i in the index. Note that $\delta_i = a_{i1} + \cdots + a_{ir_i} + k_{i1} + \cdots + k_{ip_i}$, $\gamma_i = b_{i1} + \cdots + b_{is_i} + l_{i1} + \cdots + l_{iq_i}$ and, because $\gcd(\mathbf{t}^\alpha, \mathbf{t}^\beta) = 1$, $\deg_L(i) = p_i + q_i$. From the first two equalities, δ_i has the same parity as p_i, and γ_i has the same parity as q_i, so $\delta_i + \gamma_i - \deg_L(i) = (\delta_i - p_i) + (\gamma_i - q_i)$ is always an even number. Now, if $\mathbf{t}^\alpha - \mathbf{t}^\beta \in I(H)$, $\mathbf{x}^\delta - \mathbf{x}^\gamma$ is in $(x_i^2 - x_j^2 : i, j \in V_H)$, and by Proposition 3.1.13, $\delta_i + \gamma_i$ is even for every $i \in V_H$. This implies the $\deg_L(i)$ is even, for every $i \in V_L$, and therefore every vertex of L has even degree. Conversely, if the $\deg_L(i)$ is even, $\delta_i + \gamma_i$ is even for all $i \in V_L$. And for every $i \in V_H \setminus V_L$, appearing in the index of a variable of $\mathbf{t}^\alpha - \mathbf{t}^\beta$, i is only in even exponent variables. This implies, for all $i \in V_H \setminus V_L$, that $\delta_i + \gamma_i$ is even. Therefore $\delta_i + \gamma_i$ is even, for every $i \in V_H$, which by Proposition 3.1.13 means that $\mathbf{x}^\delta - \mathbf{x}^\gamma$ is in $(x_i^2 - x_j^2 : i, j \in V_H)$, and so $\mathbf{t}^\alpha - \mathbf{t}^\beta \in I(H)$. □

Example 4.1.8. Consider the 3-uniform hypergraph H, with vertex set $V_H = \{1,2,3,4,5\}$ and edge set $E_H = \{\{1,2,5\},\{1,4,5\},\{2,3,5\},\{3,4,5\}\}$.



We have $\mathbb{K}[V_H] = \mathbb{K}[x_1,x_2,x_3,x_4,x_5]$, and $\mathbb{K}[E_H] = \mathbb{K}[t_{125},t_{145},t_{235},t_{345}]$. Using the software Macaulay2, [8], we obtained a generating set for $I(H)$ consisting of the following binomials:

$$t_{235}^2 - t_{345}^2, \quad t_{145}^2 - t_{345}^2, \quad t_{125}^2 - t_{345}^2,$$

$$t_{145}t_{235} - t_{125}t_{345}, \quad t_{125}t_{235} - t_{145}t_{345}, \quad t_{125}t_{145} - t_{235}t_{345}.$$

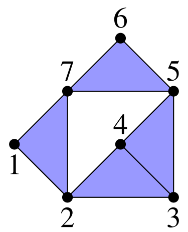
Note that the last three generators are the ones mentioned by Proposition 4.1.7 as they identify H, the only subhypergraph of H, with nonempty edge set, in which every vertex has even degree.

Remark 4.1.9. Let H be a k-uniform hypergraph, with $E_H \neq \emptyset$, and $\mathbf{t}^\alpha - \mathbf{t}^\beta \in I(H)$ a nonzero homogeneous binomial with $\gcd(\mathbf{t}^\alpha, \mathbf{t}^\beta) = 1$. The subhypergraph L of H, identified by the indeces of the odd power variables of $\mathbf{t}^\alpha - \mathbf{t}^\beta$, must have an even number of edges. To see this, regarding the variables of $\mathbf{t}^\alpha - \mathbf{t}^\beta$, let p be the sum of the odd exponents, and q the sum of the even exponents. Then $2\deg(\mathbf{t}^\alpha - \mathbf{t}^\beta) = p + q$, so p must be even. As p is the sum of $|E_H|$ odd numbers, $|E_H|$ must be even.

Definition 4.1.10. Let H be a k-uniform hypergraph and L a k-uniform subhypergraph of H, with $E_L \neq \emptyset$. We say that L *is even* if every vertex of L has even degree.

In Chapter 3 we defined the notion of Eulerian subgraph of a graph as a subgraph in which every vertex has even degree. As mentioned before, if the subgraph is connected, by a celebrated Theorem of Euler, this is the same as saying it admits an Eulerian circuit. When considering hypergraphs, even restricting to k-uniform hypergraphs, a hypergraph with an Eulerian circuit need not have every vertex of even degree, as the next example shows.

Example 4.1.11. Let H be the 3-uniform hypergraph H, with vertex set $V_H = \{1,2,3,4,5,6,7\}$ and edge set $E_H = \{\{1,2,7\},\{2,3,4\},\{3,4,5\},\{5,6,7\}\}$.



The sequence $2,\{2,3,4\},3,\{3,4,5\},5,\{5,6,7\},7,\{1,2,7\},2$ is an Eulerian circuit in the sense that it is an alternating sequence of vertices and edges; beginning and ending in the same vertex; in which every edge contains its two adjacent vertices in the sequence; and such that every edge of H is in the sequence. However, not all vertices of H have even degree, e.g., the vertices 1 and 6. In fact, as we will explain below, the sequence above is often called a cycle in the theory of hypergraphs.

We now generalize Proposition 3.1.17, characterizing the Eulerian ideal for subhypergraphs.

Proposition 4.1.12. *Let H be a k-uniform hypergraph and L a subhypergraph of H, with $E_L \neq \emptyset$. Consider $\mathbb{K}[E_L]$ as a subset of $\mathbb{K}[E_H]$. For the ideal $I(L)$, seen as a subset of $\mathbb{K}[E_H]$, we have that $I(L) = I(H) \cap \mathbb{K}[E_L]$.*

Proof. Let $\varphi_H\colon \mathbb{K}[E_H] \to \mathbb{K}[V_H]$ and $\varphi_L\colon \mathbb{K}[E_L] \to \mathbb{K}[V_L]$ be the ring homomorphisms that define $I(H)$ and $I(L)$, respectively, and note that φ_L is the restriction of φ_H to $\mathbb{K}[E_L]$ and $\mathbb{K}[V_L]$. Given $f \in I(L)$, $\varphi_H(f) = \varphi_L(f)$ is in the ideal $(x_i^2 - x_j^2 : i,j \in V_L)$ of $\mathbb{K}[V_L]$, which is a subset of the ideal $(x_i^2 - x_j^2 : i,j \in V_H)$ of $\mathbb{K}[V_H]$. Therefore $f \in I(H)$, and $I(L) \subseteq I(H) \cap \mathbb{K}[E_L]$. Conversely, if f is a polynomial in $I(H) \cap \mathbb{K}[E_L]$, $\varphi_L(f) = \varphi_H(f)$ is in $(x_i^2 - x_j^2 : i,j \in V_H) \cap \mathbb{K}[V_L] = (x_i^2 - x_j^2 : i,j \in V_L)$. This way $f \in I(L)$, and $I(H) \cap \mathbb{K}[E_L] \subseteq I(L)$. $\square$

4.2 Regularity index of $\mathbb{K}[E_H]/I(H)$

In this section we use the results from Section 3.2, to show that, given a k-uniform hypergraph H, the Hilbert polynomial of $\mathbb{K}[E_H]/I(H)$ is constant.

Lemma 4.2.1. *Let H be a k-uniform hypergraph with $E_H \neq \emptyset$. The variables t_A in $\mathbb{K}[E_H]$ are $\mathbb{K}[E_H]/I(H)$-regular.*

Proof. Take t_A in $\mathbb{K}[E_H]$. For every $f + I(H) \in \mathbb{K}[E_H]/I(H)$ such that $t_A(f + I(H)) = I(H)$, we will show that $f \in I(H)$. Since $t_A f$ is in $I(H)$, $\varphi(t_A f) = (\prod_{i \in A} x_i)\varphi(f)$ is in $(x_i^2 - x_j^2 : i, j \in V_H)$. By Proposition 3.2.3, each x_i is $\mathbb{K}[V_H]/(x_i^2 - x_j^2 : i, j \in V_H)$-regular, so it follows that $\varphi(f)$ is in $(x_i^2 - x_j^2 : i, j \in V_H)$, and therefore f is in $I(H)$. We conclude that t_A is $\mathbb{K}[E_H]/I(H)$-regular. □

Lemma 4.2.2. *Let H be a k-uniform hypergraph with $E_H \neq \emptyset$. For each variable $t_A \in \mathbb{K}[E_H]$, the ideal $(I(H), t_A)$ contains every square of a variable of $\mathbb{K}[E_H]$.*

Proof. Take any $t_B^2 \in \mathbb{K}[E_H]$. Since $t_B^2 = (t_B^2 - t_A^2) + t_A^2$, and $t_B^2 - t_A^2$ is in $I(H)$, $t_B^2 \in (I(H), t_A)$. □

Theorem 4.2.3. *Let H be a k-uniform hypergraph such that $s = |E_H| > 0$. Denote $M = \mathbb{K}[E_H]/I(H)$, and let H_M be the Hilbert function of M. Then, for every $l \geq s$, $H_M(l) = H_M(s)$.*

Proof. By Lemma 4.2.1, the variables of $\mathbb{K}[E_H]$ are M-regular, and by Lemma 4.2.2, given a variable $t_A \in \mathbb{K}[E_H]$, the ideal $(I(H), t_A)$ contains every square of a variable of $\mathbb{K}[E_H]$. The result now follows from Theorem 3.2.8. □

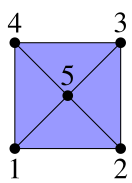
Remark 4.2.4. Let H be a k-uniform hypergraph such that $s = |E_H| > 0$. Just like for graphs, the Hilbert Polynomial of $\mathbb{K}[E_H]/I(H)$ is constant, and $\mathrm{ri}(\mathbb{K}[E_H]/I(H)) \leq s$. Also, with $M = \mathbb{K}[E_H]/I(H)$, we again have that $H_M(0) = 1$, $H_M(1) = s$, and by Lemma 3.2.9, H_M is a nondecreasing function.

4.3 Joins and regularity for k-uniform Hypergraphs

In Section 3.3, we characterized the regularity index, $\mathrm{ri}(\mathbb{K}[E_G]/I(G))$, for a graph G as was first shown in [13], using the notion of join of G. Now we will generalize this result for k-uniform hypergraphs, giving a different proof than the one we gave for graphs. The definition of join for hypergraphs, presented below, is the natural generalization of the one for graphs, see Definition 3.3.2.

Definition 4.3.1. Let H be a k-uniform hypergraph. (i) A set $J \subseteq E_H$ is a join of H if, for every even subhypergraph L of H, $|J \cap E_L| \leq \frac{|E_L|}{2}$. (ii) We set $\mu(H)$ as the greatest cardinality of a join of H.

Example 4.3.2. Consider the 3-uniform hypergraph H, of the Example 4.1.8.



The even subhypergraphs of H are H itself and $(V_H,\emptyset)$. Then $J \subseteq E_H$ is a join of H if and only if $|J \cap E_H| \leq \frac{|E_H|}{2} = 2$, that is, J is any subset of E_H with cardinality $0,1$ or 2. In particular $\mu(H) = 2$.

We will now generalize, for k-uniform hypergraphs, the results from [13] about the regularity index, that we presented in Section 3.3. However, in Section 3.3 we followed the proofs of Neves, Vaz Pinto, and Villarreal, from [13], while now we will use a slightly different approach.

Definition 4.3.3. Let H be a k-uniform hypergraph with $E_H \neq \emptyset$. Given a monomial $\mathbf{t}^\alpha \in \mathbb{K}[E_H]$, we define $\mathfrak{I}(\mathbf{t}^\alpha)$ as the set $\{A \in E_H : t_A \text{ has odd power in } \mathbf{t}^\alpha\}$.

Proposition 4.3.4. *Let H be a k-uniform hypergraph, $J \subseteq E_H$ a join of H, and A an edge in J. If $\mathbf{t}^\alpha$ is the product of the variables the index of which is in $J \setminus \{A\}$, then $\mathbf{t}^\alpha \notin (I(H), t_A)$.*

Proof. Eyeing a contradiction suppose that $\mathbf{t}^\alpha \in (I(H), t_A)$. Then, by Proposition 3.2.11, there is a monomial $\mathbf{t}^\beta \in \mathbb{K}[E_H]$ such that $\mathbf{t}^\alpha - \mathbf{t}^\beta t_A \in I(H)$ and is homogeneous. In particular $\deg(\mathbf{t}^\alpha) \geq 1$. Let $\mathbf{t}^\gamma, \mathbf{t}^\delta \in \mathbb{K}[E_H]$ be the monomials such that $\mathbf{t}^\alpha = \mathbf{t}^\gamma \gcd(\mathbf{t}^\alpha, \mathbf{t}^\beta)$ and $\mathbf{t}^\beta = \mathbf{t}^\delta \gcd(\mathbf{t}^\alpha, \mathbf{t}^\beta)$. By Lemma 4.2.1, it follows that monomials are $\mathbb{K}[E_H]/I(H)$-regular, so $\mathbf{t}^\gamma - \mathbf{t}^\delta t_A \in I(H)$. Also $\mathbf{t}^\gamma - \mathbf{t}^\delta t_A$ is homogeneous and, because t_A does not divide $\mathbf{t}^\alpha$, $\gcd(\mathbf{t}^\gamma, \mathbf{t}^\delta t_A) = 1$. Let L be a subhypergraph of H with edge set $\mathfrak{I}(\mathbf{t}^\gamma) \cup \mathfrak{I}(\mathbf{t}^\delta t_A)$. Since $\deg(\mathbf{t}^\gamma) = |\mathfrak{I}(\mathbf{t}^\gamma)|$ and $\deg(\mathbf{t}^\delta t_A) = |\mathfrak{I}(\mathbf{t}^\delta t_A)| + 2p$, for some nonnegative integer p, we get that

$$|E_L| = |\mathfrak{I}(\mathbf{t}^\gamma) \cup \mathfrak{I}(\mathbf{t}^\delta t_A)| = |\mathfrak{I}(\mathbf{t}^\gamma)| + |\mathfrak{I}(\mathbf{t}^\delta t_A)| = \deg(\mathbf{t}^\gamma) + \deg(\mathbf{t}^\delta t_A) - 2p. \tag{4.1}$$

By Proposition 4.1.7, as $\mathbf{t}^\gamma - \mathbf{t}^\delta t_A \in I(H)$, L is an even subhypergraph of H. Now, if the power of t_A in $\mathbf{t}^\delta t_A$ is even, $p \neq 0$, and by (4.1), $|E_L| < \deg(\mathbf{t}^\gamma) + \deg(\mathbf{t}^\delta t_A) = 2\deg(\mathbf{t}^\gamma) = 2|\mathfrak{I}(\mathbf{t}^\gamma)| = 2|\mathfrak{I}(\mathbf{t}^\gamma) \cap E_L|$. Therefore $\frac{|E_L|}{2} < |\mathfrak{I}(\mathbf{t}^\gamma) \cap E_L| \leq |J \cap E_L|$, contradicting that J is a join of H. Then, the power of t_A in $\mathbf{t}^\delta t_A$ must be odd, and $A \in E_L$. By (4.1), $|E_L| \leq \deg(\mathbf{t}^\gamma) + \deg(t_A \mathbf{t}^\delta) = 2\deg(\mathbf{t}^\gamma) = 2|\mathfrak{I}(\mathbf{t}^\gamma) \cap E_L|$, and since $A \in J \cap E_L$ but not in $\mathfrak{I}(\mathbf{t}^\gamma)$, we obtain that $\frac{|E_L|}{2} \leq |\mathfrak{I}(\mathbf{t}^\gamma) \cap E_L| < |J \cap E_L|$, which again contradicts J being a join of H. With this we conclude that $\mathbf{t}^\alpha \notin (I(H), t_A)$. □

Theorem 4.3.5. *If H is a k-uniform hypergraph with $E_H \neq \emptyset$, $\operatorname{ri}(\mathbb{K}[E_H]/I(H)) \geq \mu(H) - 1$.*

Proof. Let $J \subseteq E_H$ be a join of H and $A \in J$ an edge. We must show that $\operatorname{ri}(\mathbb{K}[E_H]/I(H)) \geq |J| - 1$. By Proposition 3.2.10, it suffices to show that the homogeneous component of $\mathbb{K}[E_H]/(I(H), t_A)$ with degree $|J| - 1$ is not the zero $\mathbb{K}$-vector space. Now, Proposition 4.3.4 says that the product of the variables, indexed by the edges of $J \setminus \{A\}$, is not in $(I(H), t_A)$, so we see that the $\mathbb{K}$-vector space $(\mathbb{K}[E_H]/(I(H), t_A))_{|J|-1}$ is not zero, and so $\operatorname{ri}(\mathbb{K}[E_H]/I(H)) \geq |J| - 1$. □

Our goal now is to prove that the inequality of Theorem 4.3.5 is an equality for certain k-uniform hypergraphs. In Section 3.3, this was shown for bipartite graphs, see Theorem 3.3.10, and the key feature of these graphs is that they do not contain odd cycles, or equivalently, Eulerian subgraphs with odd number of edges. In the theory of hypergraphs there is more than one nonequivalent definition for cycle. Following [3], given a hypergraph H, a cycle is a sequence $i_1, A_1, \ldots, i_l, A_l, i_1$, such that $i_1, \ldots, i_l$ are different vertices of H; $A_1, \ldots, A_l$ are different edges of H such that $i_j, i_{j+1} \in A_j$, for every $j = 1, \ldots, l-1$; and $i_l, i_1 \in A_l$. In Example 4.1.11 we gave an example of a cycle, for the hypergraph H considered, but this cycle is not contained, nor it identifies, any even subhypergraph of H, contrary to what happens for graphs. This way, given a k-uniform hypergraph H, instead of considering cycles in H, we will focus on the property of the even subhypergraphs having edge sets of even cardinality.

Lemma 4.3.6. *Let H be an even k-uniform hypergraph with $E_H \neq \emptyset$. If $|E_H|$ is even, we have that* $\mathrm{ri}(\mathbb{K}[E_H]/I(H)) \leq \frac{|E_H|}{2} - 1$.

Proof. Fix an edge $A \in E_H$, using Proposition 3.2.10, it suffices to show that the homogeneous component of $\mathbb{K}[E_H]/(I(H),t_A)$, with degree $\frac{|E_H|}{2}$, is the zero $\mathbb{K}$-vector space. Take a monomial $\mathbf{t}^\alpha \in \mathbb{K}[E_H]$ of degree $\frac{|E_H|}{2}$, and let us show it is in $(I(H),t_A)$. If $\mathbf{t}^\alpha$ is divisible by t_A, or by the square of a variable of $\mathbb{K}[E_H]$, we are done. Otherwise, the indeces of the variables of $\mathbf{t}^\alpha$ identify half the edges of H. Letting $\mathbf{t}^\beta$ be the product of the variables of $\mathbb{K}[E_H]$ that do not divide $\mathbf{t}^\alpha$, we see that $\mathbf{t}^\alpha - \mathbf{t}^\beta$ is a homogeneous binomial with $\gcd(\mathbf{t}^\alpha,\mathbf{t}^\beta) = 1$, in which the indeces of its variables identify all the edges of H. By Proposition 4.1.7, $\mathbf{t}^\alpha - \mathbf{t}^\beta$ is in $I(H)$, and since t_A divides $\mathbf{t}^\beta$, $\mathbf{t}^\alpha = (\mathbf{t}^\alpha - \mathbf{t}^\beta) + \mathbf{t}^\beta$ is in $(I(H),t_A)$. This shows that $\mathrm{ri}(\mathbb{K}[E_H]/I(H)) \leq \frac{|E_H|}{2} - 1$. □

Remark 4.3.7. Lemma 4.3.6 generalizes the equality in Lemma 3.3.8, that characterizes the regularity index for even cycles in graphs. The inequality missing in Lemma 4.3.6 follows immediately from Theorem 4.3.5, if H does not have have even subhypergraphs other than itself and $(V_H,\emptyset)$, which is the case for even cycles in graphs. More generally, we will show below that the missing inequality holds, if H does not have even subhypergraphs with odd number of edges.

Lemma 4.3.8. *Let H be an even k-uniform hypergraph, with $E_H \neq \emptyset$ and $|E_H|$ even. Given an edge A with k vertices, some of which may be in H, let H' be the k-uniform hypergraph $(V_H \cup A, E_H \cup \{A\})$, obtained by adding A to H. Then* $\mathrm{ri}(\mathbb{K}[E_{H'}]/I(H')) \leq \frac{|E_H|}{2}$.

Proof. Fix an edge $B \in E_H$, by Proposition 3.2.10, it suffices to show that the homogeneous component of $\mathbb{K}[E_{H'}]/(I(H'),t_B)$, of degree $\frac{|E_H|}{2}+1$, is the zero vector space. Take a monomial $\mathbf{t}^\alpha \in \mathbb{K}[E_{H'}]$ of degree $\frac{|E_H|}{2}+1$, and let us show it is in $(I(H'),t_B)$. If $\mathbf{t}^\alpha$ is divisible by t_B, or by the square of a variable, we are done. Otherwise, at least half the edges of H index the variables of $\mathbf{t}^\alpha$. Consider a monomial $\mathbf{t}^\beta$, and a variable t_D, such that $\mathbf{t}^\alpha = \mathbf{t}^\beta t_D$ and the variables of $\mathbf{t}^\beta$ are all indexed by edges of H. Let $\mathbf{t}^\gamma$ be the product of the variables the index of which are the edges of H that do not index the variables of $\mathbf{t}^\beta$. Then $\mathbf{t}^\beta - \mathbf{t}^\gamma$ identifies the edges of H, and by Proposition 4.1.7, $\mathbf{t}^\beta - \mathbf{t}^\gamma \in I(H')$. Since $\mathbf{t}^\beta$ is not divisible by t_B, $\mathbf{t}^\gamma$ must be, so the equality $\mathbf{t}^\alpha = (\mathbf{t}^\beta - \mathbf{t}^\gamma)t_D + \mathbf{t}^\gamma t_D$ implies that $\mathbf{t}^\alpha \in (I(H'),t_B)$. This shows that $\mathrm{ri}(\mathbb{K}[E_{H'}]/I(H')) \leq \frac{|E_H|}{2}$. □

Proposition 4.3.9. *Let H be a k-uniform hypergraph, with $E_H \neq \emptyset$, in which every even subhypergraph of H has an even number of edges. Consider a variable $t_A \in \mathbb{K}[E_H]$ and a monomial $\mathbf{t}^\alpha \in \mathbb{K}[E_H]$. If $\mathbf{t}^\alpha \notin (I(H),t_A)$ then $\mathfrak{J}(\mathbf{t}^\alpha) \cup \{A\}$ is a join of H.*

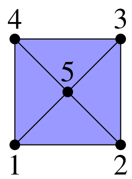
Proof. First note that $\mathbf{t}^\alpha$ is not divisible by t_A, nor by the square of a variable of $\mathbb{K}[E_H]$. Let L be an even subhypergraph of H, and let us show that $|(\mathfrak{J}(\mathbf{t}^\alpha) \cup \{A\}) \cap E_L| \leq \frac{|E_L|}{2}$. Set $\mathbf{t}^\gamma$ as the product of the variables of $\mathbb{K}[E_H]$ the index of which are the edges of $\mathfrak{J}(\mathbf{t}^\alpha) \cap E_L$. There are two cases to consider. If $A \in E_L$, $|(\mathfrak{J}(\mathbf{t}^\alpha) \cup \{A\}) \cap E_L| = |(\mathfrak{J}(\mathbf{t}^\alpha) \cap E_L) \cup \{A\}| = \deg(\mathbf{t}^\gamma)+1$. Using Proposition 4.1.12, $I(L) \subseteq I(H)$, and since $\mathbf{t}^\alpha \notin (I_H,t_A)$, we see that $\mathbf{t}^\gamma \notin (I(L),t_A)$. This in turn gives, by Proposition 3.2.10, that $\deg(\mathbf{t}^\gamma) < \mathrm{ri}(\mathbb{K}[E_L]/I(L))+1$, so we use Lemma 4.3.6 to obtain that

$\deg(\mathbf{t}^{\gamma})+1 \leq \mathrm{ri}(\mathbb{K}[E_L]/I(L))+1 \leq \frac{|E_L|}{2}$. It follows that $|(\mathfrak{J}(\mathbf{t}^{\alpha})\cup\{A\})\cap E_L| \leq \frac{|E_L|}{2}$. For the second case, suppose that $A \notin E_L$, then $|(\mathfrak{J}(\mathbf{t}^{\alpha})\cup\{A\})\cap E_L| = |\mathfrak{J}(\mathbf{t}^{\alpha})\cap E_L| = \deg(\mathbf{t}^{\gamma})$. Let L' be the k-uniform hypergraph $(V_H \cup A, E_H \cup \{A\})$, obtained by adding the edge A to L. Once again, $\mathbf{t}^{\alpha} \notin (I_H, t_A)$ implies that $\mathbf{t}^{\gamma} \notin (I_{L'}, t_A)$, and so, by Proposition 3.2.10, $\deg(\mathbf{t}^{\gamma}) < \mathrm{ri}(\mathbb{K}[E_{L'}]/I(L'))+1$. Now, Lemma 4.3.8 gives that $\deg(\mathbf{t}^{\gamma}) < \mathrm{ri}(\mathbb{K}[E_{L'}]/I(L'))+1 \leq \frac{|E_L|}{2}+1$, so $|(\mathfrak{J}(\mathbf{t}^{\alpha})\cup\{A\})\cap E_L| < \frac{|E_L|}{2}+1$ and therefore $|(\mathfrak{J}(\mathbf{t}^{\alpha})\cup\{A\})\cap E_L| \leq \frac{|E_L|}{2}$. This way we conclude that $\mathfrak{J}(\mathbf{t}^{\alpha})\cup\{A\}$ is a join of H. □

Theorem 4.3.10. *Let H be a k-uniform hypergraph with $E_H \neq \emptyset$. If every even subhypergraph of H has even number of edges,* $\mathrm{ri}(\mathbb{K}[E_H]/I(H)) = \mu(H)-1$.

Proof. According to Theorem 4.3.5, we only need to show that $\mathrm{ri}(\mathbb{K}[E_H]/I(H)) \leq \mu(H)-1$. Fix an edge $A \in E_H$, by Proposition 3.2.10, it suffices to show that the homogeneous component of $\mathbb{K}[E_H]/(I(H),t_A)$, of degree $\mu(H)$, is the zero vector space. Let $\mathbf{t}^{\alpha} \in \mathbb{K}[E_H]$ be a monomial of degree $\mu(H)$, and let us show it is in $(I(H),t_A)$. If $\mathbf{t}^{\alpha}$ is divisible by t_A, or by the square of a variable, we are done, so we assume otherwise. Now, suppose that $\mathbf{t}^{\alpha} \notin (I(H),t_A)$. Then Proposition 4.3.9 says that $\mathfrak{J}(\mathbf{t}^{\alpha})\cup\{A\}$ is a join of H. But since $|\mathfrak{J}(\mathbf{t}^{\alpha})\cup\{A\}| = \deg(\mathbf{t}^{\alpha})+1 = \mu(H)+1$, $\mathfrak{J}(\mathbf{t}^{\alpha})\cup\{A\}$ is a join of H with cardinality greater than $\mu(H)$. By definition of $\mu(H)$, we have obtained a contradiction, so we conclude that $\mathbf{t}^{\alpha} \in (I(H),t_A)$. This shows that $\mathrm{ri}(\mathbb{K}[E_H]/I(H)) \leq \mu(H)-1$. □

Example 4.3.11. In Example 4.3.2, with H the 3-uniform hypergraph below, we saw that $\mu(H)=2$ so, by Theorem 4.3.10, $\mathrm{ri}(\mathbb{K}[E_H]/I(H)) = 1$.



The Example 3.3.12 shows that the equality of Theorem 4.3.10 may not hold without the hypothesis that every even subhypergraph has even number of edges. However, the next result shows that this can only happen when k is even.

Corollary 4.3.12. *Let H be a k-uniform hypergraph with $E_H \neq \emptyset$. If k is an odd number then* $\mathrm{ri}(\mathbb{K}[E_H]/I(H)) = \mu(H)-1$.

Proof. By Theorem 4.3.10, it suffices to show that every even subhypergraph of H has even number of edges. Let L be an even subhypergraph of H, $\mathbf{t}^{\alpha}$ be the product of all variables of $\mathbb{K}[E_L]$, and φ the ring homomorphism that defines $I(L)$. $\varphi(\mathbf{t}^{\alpha})$ has degree $k\deg(\mathbf{t}^{\alpha}) = k|E_L|$, and for each vertex $i \in V_L$, the power of x_i in $\varphi(\mathbf{t}^{\alpha})$ is the number of variables of $\mathbf{t}^{\alpha}$ indexed by edges that contain i, that is, $\deg_L(i)$. It follows that $k|E_L| = \deg(\varphi(\mathbf{t}^{\alpha})) = \sum_{i\in V_L}\deg_L(i)$. As k is odd and every vertex of L has even degree, $|E_L|$ is even. □

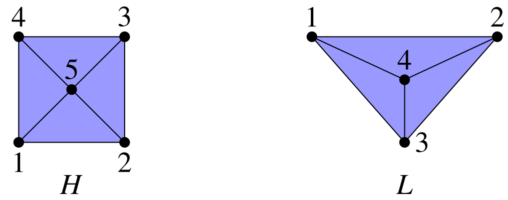
Remark 4.3.13. The equality $k|E_L| = \sum_{i\in V_L}\deg_L(i)$, used in the proof of Corollary 4.3.12, is called the handshaking Lemma in the case of L being a graph, that is, if $k=2$.

k-partite k-uniform hypergraphs

We now consider the more natural generalization of bipartite graphs, k-partite k-uniform hypergraphs.

Definition 4.3.14. Let H be a k-uniform hypergraph with $E_H \neq \emptyset$. We say that H is *k-partite* if there are $V_1, \ldots, V_k \subseteq V_H$ such that $V_H = \bigcup_{i=1}^k V_i$, $V_i \cap V_j = \emptyset$ everytime $i \neq j$, and for every $A \in E_H$, $|A \cap V_i| = 1$ for every $i = 1, \ldots, k$. To $V_1, \ldots, V_k$ we call a *k-partite realization of H*.

Example 4.3.15. Consider the 3-uniform hypergraphs H, from the Example 4.1.8, and L, with vertex set $V_L = \{1,2,3,4\}$ and edge set $E_L = \{\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$, as below.



H is 3-partite by setting $V_1 = \{1,3\}$, $V_2 = \{2,4\}$, and $V_3 = \{5\}$, while L is not 3-partite. Since each two of the vertices $1,2,3,4 \in V_L$ are both in some edge of L, there is no way to distribute the 4 vertices by 3 sets, V_1, V_2, V_3, in a way such that each V_i contains only one of these vertices.

Proposition 4.3.16. *If H is a k-partite k-uniform hypergraph,* $\mathrm{ri}(\mathbb{K}[E_H]/I(H)) = \mu(H) - 1$.

Proof. According to Theorem 4.3.10, and since the subhypergraphs of k-partite k-uniform hypergraphs are still k-partite, we only need to show that every even k-partite k-uniform hypergraph has even number of edges. For this purpose, assume every vertex of H has even degree, and let us show that $|E_H|$ is even. Let $V_1, \ldots, V_k$ be a k-partite realization of H, and take $j \in \{1, \ldots, k\}$. Every edge of H contains one and only one vertex of V_j. Then, counting $|E_H|$ is the same as counting, for every vertex $i \in V_j$, the number of edges of H that contain i, and adding these numbers, that is, $|E_H| = \sum_{i \in V_J} \deg_H(i)$. This shows $|E_H|$ is even and concludes the proof. □

As an application of Proposition 4.3.16, we calculate the regularity index $\mathrm{ri}(\mathbb{K}[E_H]/I(H))$ for the complete k-partite k-uniform hypergraph, which generalizes Proposition 3.4.13.

Definition 4.3.17. Let H be a k-partite k-uniform hypergraph with k-partite realization $V_1, \ldots, V_k$. Given $a_1, \ldots, a_k$ positive integers, H is *complete* if $|V_j| = a_j$ for every $j = 1, \ldots, k$, and for every vertices $i_1, \ldots, i_k$, with each $i_j \in V_j$, $\{i_1, \ldots, i_k\}$ is an edge of H. In this case we denote H by $K^k_{a_1,\ldots,a_k}$.

Remark 4.3.18. Given positive integers $a_1, \ldots, a_k$, there is only one complete k-partite k-uniform hypergraph $K^k_{a_1,\ldots,a_k}$, up to reordering the sets of the k-partite realization, or renaming of the vertices.

Proposition 4.3.19. *If $H = K^k_{a_1,\ldots,a_k}$, for some $k \geq 2$ and $a_1, \ldots, a_k$ positive integers, we have that* $\mathrm{ri}(\mathbb{K}[E_H]/I(H)) = \max\{a_1, \ldots, a_k\} - 1$.

Proof. Attending to Proposition 4.3.16 we just need to show that $\mu(H) = \max\{a_1, \ldots, a_k\}$. Let $V_1, \ldots, V_k$ be the k-partite realization of H. We first show that $\mu(H) \geq \max\{a_1, \ldots, a_k\}$, by proving that H has a join with this cardinality. Assume $a_l = \max\{a_1, \ldots, a_k\}$, and fix vertices $i_j \in V_j$, for each

$j \in \{1,\dots,k\} \setminus \{l\}$. The set $J = \{\{i_1,\dots,i_{l-1},v,i_{l+1},\dots,i_k\} \in E_H : v \in V_l\}$ has cardinality a_l, let us see it is a join of H. If L is an even subhypergraph of H, for every edge $A \in J \cap E_L$, the only vertex of A in V_l has even degree in L, so there must be another edge of L that contains it. Therefore E_L has at least twice as many edges as $J \cap E_L$, that is, $|J \cap E_L| \le \frac{|E_L|}{2}$, and so J is a join of H. Now, to show that $\mu(H) \le \max\{a_1,\dots,a_k\}$, take a join J of H and let us prove that $|J| \le \max\{a_1,\dots,a_k\}$. Let T be the set of odd degree vertices in the hypergraph $L = (V_H, J)$, and let $T_j = T \cap V_j$, for every $j = 1,\dots,k$. For every $j = 1,\dots,k$, $|J| = \sum_{i \in V_j} \deg_L(i)$, from which follows that the number of vertices in V_j with odd degree in L, $|T_j|$, has the same parity as $|J|$. In particular, the numbers $|T_1|,\dots,|T_k|$ have the same parity. We will now construct a certain subhypergraph of H, with edge set containing J, and then use that all $|T_j|$ have the same parity to show that it is even. Set $r_j = |T_j|$ and $r = \max\{r_1,\dots,r_k\}$. For every j such that $r_j > 0$, let $T_j = \{i_{j1},\dots,i_{jr_j}\}$, and for every j such that $r_j = 0$, fix a vertex $i_j \in V_j$. Take the sets $T'_1,\dots,T'_k$, with $T'_j = T_j$, if $r_j > 0$, and $T'_j = \{i_j\}$, if $r_j = 0$, and consider, for $l = 1,\dots,r$, the functions $\varphi_l \colon \{T'_1,\dots,T'_k\} \to V_H$, defined by $T'_j \mapsto i_j$, if $r_j = 0$; $T'_j \mapsto i_{jl}$, if $r_j > 0$ and $l \le r_j$; and $T'_j \mapsto i_{j1}$, if $r_j > 0$ and $l > r_j$. We can view the vertex $\varphi_l(T'_j)$ as the jl entry in the following matrix:

$$\begin{array}{cc}
 & \begin{array}{ccccccccccc} \varphi_1 & \varphi_2 & \cdots & \varphi_{r_1} & \varphi_{r_1+1} & \cdots & \varphi_{r_k} & \varphi_{r_k+1} & \cdots & \varphi_{r-1} & \varphi_r \end{array} \\
\begin{array}{c} T'_1 \\ \vdots \\ T'_{j'} \\ \vdots \\ T'_{j''} \\ \vdots \\ T'_k \end{array} &
\begin{bmatrix}
i_{11} & i_{12} & \cdots & i_{1r_1} & i_{11} & \cdots & i_{11} & i_{11} & \cdots & i_{11} & i_{11} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
i_{j'1} & i_{j'2} & \cdots & i_{j'r_1} & i_{j'r_1+1} & \cdots & i_{j'r_k} & i_{j'r_k+1} & \cdots & i_{j'r-1} & i_{j'r} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
i_{j''} & i_{j''} & \cdots & i_{j''} & i_{j''} & \cdots & i_{j''} & i_{j''} & \cdots & i_{j''} & i_{j''} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
i_{k1} & i_{k2} & \cdots & i_{kr_1} & i_{kr_1+1} & \cdots & i_{kr_k} & i_{k1} & \cdots & i_{k1} & i_{k1}
\end{bmatrix}
\end{array}, \tag{4.2}$$

where j' and j'' are such that $r_{j'} = r$ and $r_{j''} = 0$. Every vertex of T, being some i_{jl}, appears at least once in (4.2), in the entry jl, and the only possible vertices in (4.2) that are not in T are the vertices i_j, for all j such that $r_j = 0$. Also, the vertices in the row j of (4.2) belong to V_j, and since H is complete k-partite, for each $l = 1,\dots,r$, the set $A_l = \{\varphi_l(T'_1),\dots,\varphi_l(T'_k)\}$, containing the vertices in the column l of (4.2), is an edge of H. Consider the subhypergraph of H, $L' = (V_H, J \cup \{A_1,\dots,A_r\})$, and let us show it is even. Take any vertex $i \in V_H$, $\deg_{L'}(i)$ is the sum of $\deg_L(i)$ with the number of times i is an entry of (4.2). If $i \in T$, then $i = i_{jl}$ for some $j = 1,\dots,k$ and some $l = 1,\dots,r_j$. If $l \ne 1$, i_{jl} appears in (4.2) only in the entry jl, so $\deg_{L'}(i) = \deg_L(i) + 1$, which is even. And if $l = 1$, i_{j1} appears in (4.2) in the entry $j1$, and in the last $r - r_j$ entries of the row j, so $\deg_{L'}(i) = \deg_L(i) + 1 + r - r_j$, which is even since r and r_j have the same parity. Otherwise, if $i \notin T$, we must consider two cases. If the numbers $r_1,\dots,r_k$ are odd, each $r_j > 0$, and so (4.2) only has elements of T in its entries, so $\deg_{L'}(i) = \deg_L(i)$ is even. And if the numbers $r_1,\dots,r_k$ are even, either i is not an entry of (4.2) and $\deg_{L'}(i) = \deg_L(i)$, or $i = i_j$, for some j such that $r_j = 0$, and $\deg_{L'}(i) = \deg_L(i) + r$. In either case $\deg_{L'}(i)$ is even, and so L' is an even subhypergraph of H. Finally, we have that $|E_{L'}| = |J| + r - |J \cap \{A_1,\dots,A_r\}|$, and since J is a join of H, $|J| = |J \cap E_{L'}| \le \frac{|E_{L'}|}{2} \le \frac{1}{2}(|J| + r - |J \cap \{A_1,\dots,A_r\}|)$. It follows that $|J| \le r - |J \cap \{A_1,\dots,A_r\}| \le r \le \max\{a_1,\dots,a_k\}$. Therefore $\mu(H) \le \max\{a_1,\dots,a_k\}$.

□

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