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NON-STANDARD RANKS OF MATRICES

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Abstract

In this work we study two matrix rank minimization problems, which lead to two new notions of matrix rank. In the first one, our goal is to minimize the rank of a complex matrix whose absolute values of the entries are given. We call this minimum the *phaseless* rank of the matrix of the entrywise absolute values. In the second, the rank minimization is performed over complex matrices whose entries have prescribed arguments. In this case, the minimum is named as the *phase rank* of the matrix of phases or arguments. Regarding phaseless rank, we extend a classic result of Camion and Hoffman and connect it to the study of amoebas of determinantal varieties and of semidefinite representations of convex sets. As a result, we prove that the set of maximal minors of a matrix of indeterminates forms an amoeba basis for the ideal they define, and we attain a new upper bound on the complex semidefinite extension complexity of polytopes, dependent only on their number of vertices and facets. We also highlight the connections between the notion of phaseless rank and the problem of finding large sets of complex equiangular lines or mutually unbiased bases. The main contributions on phase rank are a new and simpler characterization of the 3×3 case, more specifically that the coamoeba of the 3×3 determinant is completely characterized by the condition of colopsidedness, and a simple upper bound on the phase rank dependent only on the dimensions of the matrix.

Resumo

Nesta tese estudamos dois problemas de minimização de característica matricial, que dão origem eles próprios a dois novos conceitos de característica matricial. No primeiro deles, pretendemos determinar a característica mínima de todas as matrizes complexas cujos valores absolutos das entradas são dados. Este mínimo é chamado característica sem fase da matriz dos valores absolutos. No segundo, a minimização da característica restringe-se às matrizes complexas cujos argumentos estão fixos. Neste caso, o mínimo é designado por característica de fase da matriz dos argumentos. Relativamente à característica sem fase, generalizamos um resultado clássico de Camion e Hoffman que pode ser reinterpretado em termos de amibas de variedades determinantais e ligado às representações semidefinidas de conjuntos convexos. Em particular, provamos que o conjunto dos menores maximais de uma matriz de variáveis constitui uma base da amiba do ideal por eles definido, além de obtermos um novo majorante para a complexidade de extensão complexa semidefinida de polítopos, dependente apenas dos seus números de vértices e facetas. Enfatizamos também as relações entre o conceito de característica sem fase e os problemas das linhas equiangulares complexas e das *mutually unbiased bases*. Quanto à característica de fase, os principais contributos desta tese são uma nova e mais simples caracterização do caso 3×3, nomeadamente que a coamiba do determinante 3×3 é totalmente determinada pela condição de *colopsidedness*, e um majorante simples para a característica de fase que depende apenas das dimensões da matriz.

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Introduction

Minimizing the rank function over a matrix set defines a *rank minimization problem* (RMP). RMPs arise in many research areas, from system control to image reconstruction, and are mostly difficult optimization problems (NP-hard), essentially due to the non-convex nature and discontinuity of the rank function. Even when the feasible set is an affine subspace of matrices, considered to be the simplest case, RMPs remain, in general, highly difficult [63].

In this work, we study two distinct rank minimization problems. In the first one, discussed in Chapter 1, we minimize the rank of a complex matrix whose absolute values of the entries are given. We call this minimum the *phaseless rank* of the matrix of the entrywise absolute values, denoted by $\operatorname{rank}_\theta(A)$. In the second, analyzed in Chapter 4, the rank minimization is performed over complex matrices whose entries share the arguments or phases. In this case, the minimum is named as the *phase rank* of the matrix of phases or arguments, represented by $\operatorname{rank}_{\text{phase}}(A)$.

The study of phaseless rank can be traced back to [15], by Camion and Hoffman, where the problem of characterizing $A \in \mathbb{R}_+^{n\times n}$ for which we have $\operatorname{rank}_\theta(A) = n$ is solved. In that paper, the question is seen as finding a converse for the diagonal dominance, a sufficient condition for nonsingularity of a matrix. This result was further generalized in [48], where a lower bound is derived for $\operatorname{rank}_\theta(A)$ for general A, and some special cases are studied, although the rank itself is never formally introduced. While the result of Camion and Hoffman is well known, there was little, if any, further developments in minimizing the rank of a matrix over an equimodular class. This problem has, however, resurfaced in recent years under different guises in both the theory of semidefinite lifts of polytopes and amoebas of algebraic varieties. In this work we build on the work of these foundational papers, deriving some new results and highlighting the consequences they have in those related areas.

Phase rank, in turn, can be seen as a dual version of phaseless rank. In fact, in the latter, the entrywise absolute value matrix is given and we seek a phase assignment for which the rank is minimum, whereas in the former, the matrix of phases/arguments is specified and we look for positive numbers that, when multiplied by those phases, achieve the minimum rank. Alternatively, phase rank can be introduced as an extension to complex matrices of the notion of sign rank, a well-established topic in matrix theory. Phase rank also provides a natural way of studying an important example of coamoebas of algebraic varities, specifically the coamoebas of determinantal varieties. While the notion of phase rank is new, it is a natural extension of several previous works. Besides the extensive work on sign rank, there are also antecedents to our study of the complex case in what is called the ray nonsingularity problem: when is every matrix with fixed given phases invertible? This question has been exploited and essentially solved for in a series of papers at the turn of the millenium [44, 49, 53] and has seen a few subsequent developments. In this work we will relate this new notion of phase rank with the results in these adjacent topics, and advance towards an understanding of the basic properties of this quantity.

This thesis is organized in the following way. Chapter 1 covers the phaseless rank and contains five sections. In Section 1.1, we introduce formally the notions of phaseless and signless ranks and show some relations between them and other rank notions found in the literature. In Section 1.2, we relate the notion of phaseless rank with questions in amoeba theory and semidefinite representability of sets, providing motivation and intuition to what follows. In Section 1.3, we revisit a result of Camion and Hoffman, reproving it in a language well-suited to our needs, and drawing some simple consequences. Section 1.4 covers our extensions and complements to this classic result. Finally, in Section 1.5, we draw implications from those results to those of the connecting areas. Those include proving that the maximal minors form an amoeba basis for the variety they generate and giving an explicit semialgebraic description for those amoebas, as well as deriving a new upper bound for the complex semidefinite rank of polytopes in terms of their number of facets and vertices. This chapter content is essentially that of [32] (submitted and currently under revision for publication) with only minor changes done to adapt it to the thesis structure.

Chapter 2 uses the case of the 4×4 matrices of phaseless rank at most 2, the first for which the results of Chapter 1 do not give a full characterization, to introduce several tools and techniques to numerically approximate and certify the phaseless rank of matrix. We use tools from optimization to extract certificates of membership or non-membership, and explore how well they work on this case.

In Chapter 3, we discuss some related open questions and report our efforts towards addressing them. We discuss in some detail three of these projects: studying the complexity of phaseless rank computations, in Section 3.1; in Section 3.2, inquiring into when do we have full-dimensionality of the set of $n \times m$ nonnegative matrices with phaseless rank at most k, for given integers n, m and k; and, to close this chapter, in Section 3.3, some phaseless rank variants are proposed and the link between phaseless rank and the geometric problem of finding large sets of equiangular lines is highlighted.

Finally, Chapter 4 is dedicated to the study of phase rank. In Section 4.1, this quantity is motivated and formally defined, and a brief history of the precursors of this definition is presented. Then, in Section 4.2, we introduce the necessary background about both sign rank and coamoebas, two classic objects intrinsically related to phase rank, and explain that relationship. Section 4.3 makes a review of the literature on ray nonsingularity, presenting the relevant results to our work. In section 4.4 we present what is known for the only non-trivial square cases of nonmaximal phase rank, the 3×3 and the 4×4 matrices. In particular we show a new and simpler characterization of the 3×3 case, by proving that the coamoeba of the 3×3 determinant is completely characterized by the condition of colopsidedness. In Section 4.5 we briefly explore when are the phase ranks typical. Lastly, in Section 4.6, we extend two lower bounds from sign rank to phase rank, by slightly adapting its proofs and give a basic upper bound, derived from simple considerations. This chapter will form the basis of a forthcoming paper, still in preparation.

Throughout this work we will use $\mathbb{R}_{+}^{n\times m}$ and $\mathbb{R}_{++}^{n\times m}$ to denote the sets of $n\times m$ real matrices with nonnegative and positive entries, respectively. We will also use $\mathscr{S}^n$, $\mathscr{S}_+^n$, $\mathscr{S}^n(\mathbb{C})$ and $\mathscr{S}_+^n(\mathbb{C})$ to denote, in this order, the sets of $n\times n$ real symmetric matrices, $n\times n$ real positive semidefinite matrices, $n\times n$ complex hermitian matrices and $n\times n$ complex positive semidefinite matrices.

Chapter 1

Phaseless rank

1.1 Notation, definitions and basic properties

Given a matrix in $\mathbb{R}_+^{n\times m}$, we define its phaseless rank as the smallest rank of a complex matrix equimodular with it.

Definition 1.1.1. Given $A \in \mathbb{R}_+^{n\times m}$, the set of matrices equimodular with A is denoted by

$$\Omega(A) = \{B \in \mathbb{C}^{n\times m} : |B| = A \text{ i.e., } |B_{ij}| = A_{ij}, \forall i, j\}$$

and its phaseless rank is defined as

$$\operatorname{rank}_\theta(A) = \min\{\operatorname{rank}(B) : B \in \Omega(A)\}.$$

Equivalently, the phaseless rank of $A \in \mathbb{R}_+^{n\times m}$ can be written as

$$\operatorname{rank}_\theta(A) = \min\{\operatorname{rank}(A \circ B) : B \in \mathbb{C}^{n\times m},\ |B_{ij}| = 1, \forall i, j\},$$

where $\circ$ represents the Hadamard product of matrices. It is obvious that $\operatorname{rank}_\theta(A) \leq \operatorname{rank}(A)$, and it is not hard to see that we can have a strict inequality.

Example 1.1.2. Consider the 4×4 derangement matrix,

$$D_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

We have $\operatorname{rank}(D_4) = 4$ and, for any real θ, the matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & e^{i(\theta+\pi)} & e^{i(\theta+\frac{2\pi}{3})} \\ 1 & e^{i\theta} & 0 & e^{i(\theta+\frac{\pi}{3})} \\ 1 & e^{i(\theta-\frac{\pi}{3})} & e^{i(\theta-\frac{2\pi}{3})} & 0 \end{bmatrix}$$

has rank 2. Since this matrix has as entrywise absolute values the entries of D_4, $\operatorname{rank}_\theta(D_4) \leq 2$, and in fact we have equality. With some extra effort one can show that up to row and column multiplication by complex scalars of absolute value one, and conjugation, this is the only element in the equimodular class of D_4 with rank less or equal than two.

If we restrict ourselves to the real case, we still obtain a sensible definition, and we will denote that quantity by *signless rank*.

Definition 1.1.3. Let $A \in \mathbb{R}_+^{n\times m}$.

$$\operatorname{rank}_\pm(A) = \min\{\operatorname{rank}(B) : B \in \Omega(A) \cap \mathbb{R}^{n\times m}\}.$$

Equivalently, this amounts to minimizing the rank over all possible sign attributions to the entries of A. By construction, it is clear that $\operatorname{rank}_\theta(A) \leq \operatorname{rank}_\pm(A) \leq \operatorname{rank}(A)$ for any nonnegative matrix A and all inequalities can be strict.

Example 1.1.4. Let us revisit Example 1.1.2, and note that the signless rank of D_4 is 4. Indeed, if we expand the determinant of that matrix, we get an odd number of nonzero terms, all 1 or -1, so no possible sign attribution can ever make it sum to zero. Thus, $\operatorname{rank}_\theta(D_4) < \operatorname{rank}_\pm(D_4) = \operatorname{rank}(D_4)$. On the other hand, if we consider matrix

$$B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

it is easy to see that $\operatorname{rank}(B) = 3$ but that flipping the signs of all the 1's to -1's drops the rank to 2, as the matrix rows will then sum to zero, so we have $\operatorname{rank}_\theta(B) = \operatorname{rank}_\pm(B) < \operatorname{rank}(B)$. If we want all inequalities to be strict simultaneously, it is enough to make a new matrix with D_4 and B as its diagonal blocks.

A short remark at the end of [15] points to the fact that the problem seems much harder over the reals, due to the combinatorial nature it assumes in that context. In fact, the signless rank is essentially equivalent to a different quantity, introduced in [36], denoted by the *square root rank* of a nonnegative matrix. In fact, by definition, $\operatorname{rank}_\pm(A) = \operatorname{rank}_{\sqrt{}}(A \circ A)$ or, equivalently, $\operatorname{rank}_{\sqrt{}}(A) = \operatorname{rank}_\pm(\sqrt[\circ]{A})$, where $\circ$ is the Hadamard product and $\sqrt[\circ]{A}$ is the Hadamard square root of A. As such, the complexity results proved in [23] for the square root rank still apply to the signless rank, implying the NP-hardness of the decision problem of checking if an $n \times n$ nonnegative matrix has signless rank equal to n. The proof of that complexity result relies on the combinatorial nature of the signless rank and fails in the more continuous notion of phaseless rank (in fact we will see the analogous result to be false for the phaseless rank), offering some hope that this later quantity will prove to be easier to work with. We will focus most of our attention in this latter notion.

The connection to the square root rank can actually be used to derive some lower bounds for both $\mathrm{rank}_{\pm}$ and rank_{θ}.

Lemma 1.1.5. *Let* $A \in \mathbb{R}_+^{n\times m}$ *and* $r = \mathrm{rank}\,(A \circ A)$. *Then,* $\mathrm{rank}_{\pm}\,(A) \geq \frac{\sqrt{1+8r}-1}{2}$ *and* $\mathrm{rank}_{\theta}\,(A) \geq \sqrt{r}$.

Proof. The basic idea is that if we take a matrix B equimodular with A and a minimal factorization $B = UV^*$, and let u_i and v_j be the i-th and j-th rows of U and V, respectively, we have

$$\langle u_i u_i^*, v_j v_j^* \rangle = |\langle u_i, v_j \rangle|^2 = |b_{ij}|^2 = a_{ij}^2.$$

Now all the $u_i u_i^*$ and $v_j v_j^*$ come from the space of real symmetric matrices of size $\mathrm{rank}_{\pm}\,(A)$, if we are taking real matrices B, and complex hermitian matrices of size $\mathrm{rank}_{\theta}\,(A)$, if we are taking complex matrices B. Since the real dimensions of these spaces are, respectively, $\binom{\mathrm{rank}_{\pm}(A)+1}{2}$ and $\mathrm{rank}_{\theta}\,(A)^2$, and they give real factorizations of $A \circ A$, we get the inequalities

$$\mathrm{rank}\,(A \circ A) \leq \binom{\mathrm{rank}_{\pm}\,(A)+1}{2} \quad \text{and} \quad \mathrm{rank}\,(A \circ A) \leq \mathrm{rank}_{\theta}\,(A)^2,$$

which, when inverted, give us the intended inequalities. □

This result is known in the context of semidefinite rank, and is included here only for the purpose of a unified treatment. An additional very simple property that is worth noting is that a nonnegative matrix has rank one if and only if it has signless rank one, if and only if it has phaseless rank one. This simple fact immediately tells us that the matrices D_4 and B in Example 1.1.4 have phaseless rank 2, since we have proved it is at most 2 and those matrices have rank greater than one.

Besides the problem of computing or bounding the phaseless rank, we will be interested in the geometry of the set of rank constrained matrices. In order to refer to them we will introduce some notation.

Definition 1.1.6. Given positive integers k, n and m we define the following subsets of $\mathbb{R}_+^{n\times m}$:

$$P_k^{n\times m} = \{A \in \mathbb{R}_+^{n\times m} : \mathrm{rank}_{\theta}\,(A) \leq k\},$$

$$S_k^{n\times m} = \{A \in \mathbb{R}_+^{n\times m} : \mathrm{rank}_{\pm}\,(A) \leq k\},$$

and

$$R_k^{n\times m} = \{A \in \mathbb{R}_+^{n\times m} : \mathrm{rank}\,(A) \leq k\}.$$

It is easy to see that these are all semialgebraic sets. Moreover, the set $R_k^{n\times m}$ is well understood, since it is simply the variety of matrices of rank at most k, defined by the $k+1$-minors, intersected with the nonnegative orthant. It is also not too hard to get a grasp on the set $S_k^{n\times m}$, as this is the union of the variety of matrices of rank at most k with all its $2^{n\times m}$ possible reflections attained by flipping the signs of a subset of variables, intersected with the nonnegative orthant. In particular, we have a somewhat simple algebraic description of both these sets, and they have the same dimension, $k(m+n-k)$.

For $P_k^{n\times m}$, all these questions are much more difficult. Clearly we have $R_k^{n\times m} \subseteq S_k^{n\times m} \subseteq P_k^{n\times m}$, which gives us some lower bound on the dimension of the space, but not much else can be immediately derived.

The relations between all these sets are illustrated in Figure 1.1, where we can see a random 2-dimensional slice of the cone of nonnegative 3×3 matrices (in pink) with the corresponding slice of the region of phaseless rank at most 2, highlighted in yellow, while the slices of the algebraic closures of the regions of signless rank at most 2 and usual rank at most 2 are marked in dashed and solid lines, respectively. Note that Figure 1.1 suggests $P_2^{3\times 3}$ is full-dimensional. In fact, $P_k^{n\times n}$ is full-dimensional in $\mathbb{R}_+^{n\times n}$ for any $k \geq \frac{n+1}{2}$. This observation follows from Corollary 1.4.12.

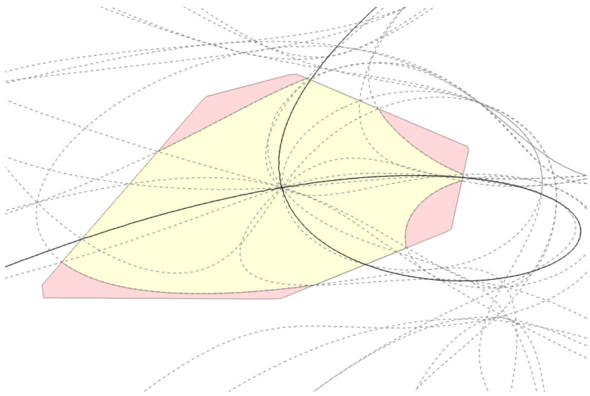

Fig. 1.1 Slice of the cone of nonnegative 3×3 matrices with $P_2^{3\times 3}$, $S_2^{3\times 3}$ and $R_2^{3\times 3}$ highlighted

1.2 Motivation and connections

The concept of phaseless rank is intimately connected to the concept of semidefinite rank of a matrix, used, for instance, to study semidefinite representations of polytopes and amoebas of algebraic varieties. In this section we will briefly introduce each of those areas and establish the connections, as those were the motivating reasons for our study of the subject.

1.2.1 Semidefinite extension complexity of a polytope

The semidefinite rank of a matrix was introduced in [34] to study the semidefinite extension complexity of a polytope. Recall that given a d-polytope P, its *semidefinite extension complexity* its the smallest k for which one can find $A_0, A_1, \dots, A_m \in \mathscr{S}^k$ such that

$$P = \left\{ (x_1, \dots, x_d) \in \mathbb{R}^d : \exists x_{d+1}, \dots, x_m \in \mathbb{R} \text{ s.t. } A_0 + \sum_{i=1}^{m} x_i A_i \succeq 0 \right\}.$$

In other words, it is the smallest k for which one can write P as the projection of a slice of the cone of $k \times k$ real positive semidefinite matrices. In order to study this concept one has to introduce the notion of *slack matrix* of a polytope. If P is a polytope with vertices p_1,..., p_v and facets cut out by the inequalities $\langle a_1, x\rangle \leq b_1$, ..., $\langle a_f, x\rangle \leq b_f$, then we define its slack matrix to be the nonnegative $v \times f$ matrix S_P with entry (i, j) given by $b_j - \langle a_j, p_i\rangle$.

Additionally, the *semidefinite rank* of a nonnegative matrix $A \in \mathbb{R}_+^{n\times m}$, $\operatorname{rank}_{\text{psd}}(A)$, is the smallest k for which one can find $U_1 \dots, U_n, V_1, \dots, V_m \in \mathscr{S}_+^k$ such that $A_{ij} = \langle U_i, V_j\rangle$. By the main result in [34] one can characterize the semidefinite extension complexity of a d-polytope P in terms of the semidefinite rank of its slack matrix.

Proposition 1.2.1. *The semidefinite extension complexity of a polytope P is the same as the semidefinite rank of its slack matrix,* $\operatorname{rank}_{psd}(S_P)$.

For a thorough treatment of the positive semidefinite rank, see [23]. As noted in [33, 46], one can replace real positive semidefinite matrices with complex positive semidefinite matrices and everything still follows through. More precisely, if one defines the *complex semidefinite extension complexity* of P as the smallest k for which one can find $B_0, B_1, \dots, B_m \in \mathscr{S}^k(\mathbb{C})$ such that

$$P = \left\{ (x_1, \dots, x_d) \in \mathbb{R}^d : \exists x_{d+1}, \dots, x_m \in \mathbb{R} \text{ s.t. } B_0 + \sum_{i=1}^{m} x_i B_i \succeq 0 \right\},$$

and the *complex semidefinite rank* of a matrix $A \in \mathbb{R}_+^{n\times m}$, $\operatorname{rank}_{\text{psd}}^{\mathbb{C}}(A)$, as the smallest k for which one can find $U_1 \dots, U_n, V_1, \dots, V_m \in \mathscr{S}_+^k(\mathbb{C})$ such that $A_{ij} = \langle U_i, V_j\rangle$, the analogous of the previous proposition still holds.

Proposition 1.2.2. *The complex semidefinite extension complexity of a polytope P is the same as the complex semidefinite rank of its slack matrix,* $\operatorname{rank}_{psd}^{\mathbb{C}}(S_P)$.

The study of the semidefinite extension complexity of polytopes has seen several important recent breakthroughs, and has brought light to this notion of semidefinite rank. It turns out that the notions of signless and phaseless rank give a natural upper bound for these quantities.

Proposition 1.2.3 ([23, 46])**.** *Given a nonnegative matrix A, we have* $\operatorname{rank}_{psd}^{\mathbb{C}}(A) \leq \operatorname{rank}_{\theta}(\sqrt[\circ]{A})$ *and* $\operatorname{rank}_{psd}(A) \leq \operatorname{rank}_{\pm}(\sqrt[\circ]{A})$.

The proof of this result is essentially the one we used in Lemma 1.1.5, as factorizations of an equimodular matrix with $\sqrt[\circ]{A}$ give rise to semidefinite factorizations to A by taking outer products of the rows of the factors. This bound is particularly important in the study of polytopes, since it fully characterizes polytopes with minimal extension complexity.

Proposition 1.2.4 ([33, 36])**.** *Given a d-polytope P, we have that its complex and real semidefinite extension complexities are at least $d+1$. Moreover, they are $d+1$ if and only if* $\operatorname{rank}_\theta(\sqrt[\circ]{S_P}) = d+1$ *and* $\operatorname{rank}_\pm(\sqrt[\circ]{S_P}) = d+1$*, respectively.*

The characterization of minimal real semidefinite extension complexity in terms of the signless rank of $\sqrt[\circ]{S_P}$ was used to determine which polytopes have minimal real semidefinite extension complexity in $\mathbb{R}^3$ and $\mathbb{R}^4$ [35, 36], while the characterization of the minimal complex semidefinite extension complexity in terms of the phaseless rank of the same matrix yielded an interesting property on the complexity of polygons [33]. One of the main motivations for us to study the phaseless rank comes precisely from this connection.

1.2.2 Amoebas of determinantal varieties

Another way of looking at phaseless rank is through amoeba theory. Amoebas are geometric objects that were introduced by Gelfand, Kapranov and Zelevinsky in [30] to study algebraic varieties. These complex analysis objects have applications in algebraic geometry, both complex and tropical, but are notoriously hard to work with. They are the image of a variety under the entrywise logarithm of the absolute values of the coordinates.

Definition 1.2.5. Given a complex variety $V \subseteq \mathbb{C}^n$, its *amoeba* is defined as

$$\mathscr{A}(V) = \{\operatorname{Log}|z| = (\log|z_1|, \ldots, \log|z_n|) : z \in V \cap (\mathbb{C}^*)^n\}.$$

Deciding if a point is on the amoeba of a given variety, the so called *amoeba membership problem*, is notoriously hard, making even the simple act of drawing an amoeba a definitely nontrivial task. Other questions like computing volumes or even dimensions of amoebas are also hard. A slightly more algebraic version of this object can be defined by simply taking the entrywise absolute values, and omitting the logarithm.

Definition 1.2.6. Given a complex variety $V \subseteq \mathbb{C}^n$, its *algebraic* or *unlog amoeba* is defined as

$$\mathscr{A}_{\text{alg}}(V) = \{|z| = (|z_1|, \ldots, |z_n|) : z \in V\}.$$

Considering this definition, it is clear how it relates to the notion of phaseless rank by way of *determinantal varieties*. These and their corresponding ideals are a central object in both commutative algebra and algebraic geometry, and a great volume of research has been focused on studying them. Given positive integers n, m and k, with $k \leq \min\{n, m\}$, we define the determinantal variety $Y_k^{n,m}$ as the set of all $n\times m$ complex matrices of rank at most k. It is clear that this is simply the variety associated to $I^{n,m}_{k+1}$, the ideal of the $k+1$ minors of an $n\times m$ matrix with distinct variables as entries.

Example 1.2.7. In Figure 1.2 we consider the amoeba of the variety V defined by the following 3×3 determinant:

$$\det\begin{bmatrix} 1 & x & y \\ x & 1 & z \\ y & 0 & 1 \end{bmatrix} = 1 - x^2 + xyz - y^2 = 0.$$

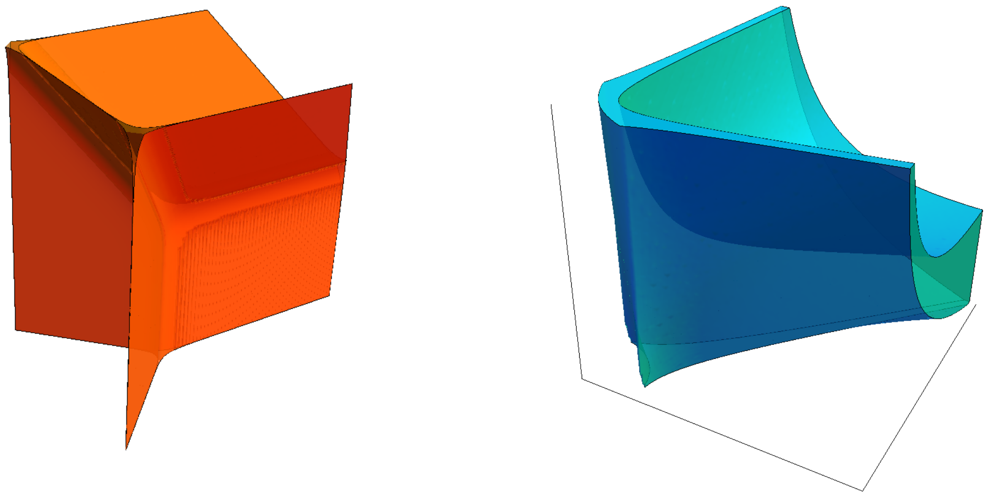

Fig. 1.2 $\mathscr{A}(V)$ and $\mathscr{A}_{\text{alg}}(V)$ of a determinantal variety.

Note that directly from the definition of amoeba, we have that the locus of $n\times m$ matrices of phaseless rank at most k is an algebraic amoeba of a determinantal variety, more precisely,

$$P_k^{n\times m} = \mathscr{A}_{\text{alg}}(Y_k^{n,m}).$$

Example 1.2.8. The blue region in Example 1.2.7 is exactly the region of the values of x, y and z for which

$$\operatorname{rank}_\theta \begin{bmatrix} 1 & x & y \\ x & 1 & z \\ y & 0 & 1 \end{bmatrix} \leq 2.$$

This is not totally immediate, since in the phaseless rank definition we are allowed to freely choose a phase independently to each entry of the matrix, which includes the 1's and also the possibility of different phases for different copies of the same variable, which is not allowed in the amoeba definition. However, since multiplying rows and columns by unitary complex numbers does not change absolute values or rank, we can make any phase attribution into one of the right type, and the regions do coincide.

More generally, computing the phaseless rank of a matrix corresponds essentially to solving the membership problem in the determinantal amoeba, so any result on the phaseless rank can immediately be interpreted as a result about this fundamental object in amoeba theory. Also on the interconnectedness between amoebas and phaseless rank, see Proposition 5.2 from [25], which, in our language, states that the intersection of a fixed number of compactified hyperplane amoebas is empty if and only if the phaseless rank of a specific nonnegative matrix is maximal.

1.3 Camion-Hoffman's Theorem

In this section we set to revisit Camion-Hoffman's Theorem, originally proved in [15]. The main purpose of this section is to set the ideas behind this result in a language and generality that will be convenient for our goals, highlighting the facts that will be most useful, and introducing the necessary notation. For the sake of completeness a proof of the theorem is included. The main idea behind the proof is the simple observation that checking for nonmaximal phaseless rank is simply a linear programming feasibility problem, i.e., checking if a nonnegative matrix has nonmaximal phaseless rank amounts to checking if a specific polytope is nonempty. Here, by nonmaximal phaseless rank we mean that the phaseless rank is less than the minimum of the matrix dimensions.

Inspired by the language of amoeba theory ([61]) we introduce the notion of *lopsidedness*. Simply put, a list of nonnegative numbers is lopsided if one is greater than the sum of all others. It is easy to see geometrically, that a nonlopsided list of numbers can always be realized as the lengths of the sides of a polygon in $\mathbb{R}^2$. Interpreting it in terms of complex numbers we get that a list of nonnegative real numbers $\{a_1,\ldots,a_n\}$ is nonlopsided if and only if there are $\theta_k \in [0,2\pi]$ for which $\sum_{k=1}^{n} a_k e^{\theta_k i} = 0$. This is enough to give us a first characterization of nonmaximal phase rank.

Lemma 1.3.1. *Let $A \in \mathbb{R}_+^{n\times m}$, with $n \leq m$. Then,* $\operatorname{rank}_\theta(A) < n$ *if and only if there is $\lambda \in \mathbb{R}_+^n$ with $\sum_{i=1}^{n} \lambda_i = 1$ such that, for $l = 1,\ldots,m$, $\{A_{1l}\lambda_1,\ldots,A_{nl}\lambda_n\}$ is not lopsided.*

Proof. First note that $\operatorname{rank}_\theta(A) < n$ if and only if there exists a matrix B with $B_{kl} = A_{kl}e^{i\theta_{kl}}$ for all k,l, such that $\operatorname{rank}(B) < n$. This is the same as saying that the rows of B are linearly dependent, and so there exists a nonzero complex vector $z = (z_1,\ldots,z_n)$ such that $\sum|z_j| = 1$ and $\sum_{k=1}^{n} A_{kl}z_k e^{i\theta_{kl}} = 0$, for $l = 1,\ldots,m$. By the observation above, this is equivalent to saying that, for $l = 1,\ldots,m$, $\{A_{1l}|z_1|,\ldots,A_{nl}|z_n|\}$ is not lopsided. $\square$

The previous result tells us essentially that $\operatorname{rank}_\theta(A) < n$ if and only if we can scale rows of A by nonnegative numbers in such a way that the entries on each of the columns verify the generalized triangular inequalities. The conditions for a matrix $A \in \mathbb{R}_+^{n\times m}$, with $n \leq m$, to verify $\operatorname{rank}_\theta(A) < n$

can now be simply stated as checking if there exists $\lambda \in \mathbb{R}^n$ such that

$$\begin{cases} A_{ij}\lambda_i \leq \sum_{k\neq i} A_{kj}\lambda_k,\ j=1,\dots,m,\ i=1,\dots,n \\ \lambda_i \geq 0,\ i=1,\dots,n \\ \sum_{i=1}^n \lambda_i = 1. \end{cases}$$

We have just observed the following result.

Corollary 1.3.2. *Given $A \in \mathbb{R}_+^{n\times m}$, with $n \leq m$, deciding if* $\operatorname{rank}_\theta(A) < n$ *is a linear programming feasibility problem.*

Note that this gives us a polynomial time algorithm (on the encoding length) for checking nonmaximality of the phaseless rank. Equivalently, this gives us a polynomial time algorithm to solve the amoeba membership problem for the determinantal variety of maximal minors.

We are now almost ready to state and prove a version of the result of Camion-Hoffman. We need only to briefly introduce some facts about M-matrices.

Definition 1.3.3. An $n\times n$ real matrix A is an M-matrix if it has nonpositive off-diagonal entries and all its eigenvalues have nonnegative real part.

The class of M-matrices is well studied, and there are numerous equivalent characterizations for them. Of particular interest to us will be the following characterizations.

Proposition 1.3.4. *Let $A \in \mathbb{R}^{n\times n}$ have nonpositive off-diagonal entries. Then the following are equivalent.*

- i *A is a nonsingular M-matrix;*
- ii *There exists $x \geq 0$ such that $Ax > 0$;*
- iii *The diagonal entries of A are positive and there exists a diagonal matrix D such that AD is strictly diagonally dominant;*
- iv *All leading principal minors are positive;*
- v *The diagonal entries of A are positive and all leading principal minors of size at least 3 are positive;*
- vi *Every real eigenvalue of A is positive.*

Remark 1.3.5. Characterizations ii, iii, iv and vi can be found in Theorem 2.3 of [7] and v in Corollary 2.3 of [60].

Finally, recall that given $A \in \mathbb{C}^{n\times n}$, its *comparison matrix*, $\mathscr{M}(A)$, is defined by $\mathscr{M}(A)_{ij} = |A_{ij}|$, if $i = j$, and $\mathscr{M}(A)_{ij} = -|A_{ij}|$, otherwise.

Theorem 1.3.6 (Camion-Hoffman's Theorem)**.** *Given $A \in \mathbb{R}_+^{n\times n}$,* $\operatorname{rank}_\theta(A) = n$ *if and only if there exists a permutation matrix P such that $\mathscr{M}(AP)$ is a nonsingular M-matrix.*

Proof. Let the entries of A be denoted by a_{ij}, $1 \le i, j \le n$. By Corollary 1.3.2, $\operatorname{rank}_\theta(A) = n$, if and only if the linear program

$$M\lambda \le 0, \quad \lambda \ge 0, \quad \sum_{i=1}^{n} \lambda_i = 1$$

is not feasible, where

$$M = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{bmatrix}, \quad \text{with } M_i = \begin{bmatrix} a_{1i} & -a_{2i} & \dots & -a_{ni} \\ -a_{1i} & a_{2i} & \dots & -a_{ni} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1i} & -a_{2i} & \dots & a_{ni} \end{bmatrix} \text{ for } i = 1, \dots, n.$$

By Ville's Theorem, a simple variant of Farkas' Lemma, this is equivalent to the existence of $y \ge 0$ such that $y^T M > 0$. Furthermore, since $y^T M$ is in the convex cone generated by the rows of M, then, by Carathéodory's Theorem, $y^T M$ can be written as a nonnegative combination of n rows of M. Let us call $y'^T M'$ to this representation of $y^T M$, where M' is a submatrix of M containing exactly n rows of M and $y' \ge 0$.

We first observe that each column of M' has exactly one nonnegative entry and all components of y' should be positive. Furthermore, if two rows of M' come from the same M_i, the components of $y'^T M'$ will not be all positive. So, there are $n!$ possibilities for M', given by $M'^T = \mathscr{M}(AP)$, for some permutation matrix P. But then, the existence of $y' \ge 0$ such that $\mathscr{M}(AP)y' > 0$ is equivalent to $\mathscr{M}(AP)$ being a nonsingular M-matrix by Proposition 1.3.4, concluding the proof. □

Note that, while equivalent, this is not the original statement of Camion-Hoffman's result. This precise version can be found, for example, in [13], as a corollary of a stronger result. The way it is originally stated, Camion-Hoffman's Theorem says that, if A is an $n \times n$ matrix with nonnegative entries, every complex matrix in the equimodular class of A, $\Omega(A)$, is nonsingular if and only if there exists a permutation matrix P and a positive diagonal matrix D such that PAD is strictly diagonally dominant. Proposition 1.3.4 immediately gives us the equivalence of both statements. We also highlight Proposition 5.3 from [25], where the authors rediscover Camion-Hoffman's Theorem in an amoeba theory context.

Example 1.3.7. Let us see how Camion-Hoffman's Theorem applies to a 3×3 matrix. Let $X \in \mathbb{R}_+^{3\times 3}$ have entries $[x_{ij}]$. We want to characterize $P_2^{3\times 3}$, that is to say, when is $\operatorname{rank}_\theta(X) \le 2$. By Camion-Hoffman's Theorem, this happens if and only if for every permutation matrix $P \in S_3$, we have that $\mathscr{M}(XP)$ is not a nonsingular M-matrix. By Proposition 1.3.4, checking if $\mathscr{M}(XP)$ is a nonsingular M-matrix amounts to checking if its determinant is positive (since it is a 3×3 matrix).

Hence, $\operatorname{rank}_\theta(X) \leq 2$ if and only if $\det(\mathscr{M}(XP)) \leq 0$ for all $P \in S_3$. There are 6 possible matrices P giving rise to 6 inequalities. For P equal to the identity, for example, we get

$$\det \begin{bmatrix} x_{11} & -x_{12} & -x_{13} \\ -x_{21} & x_{22} & -x_{23} \\ -x_{31} & -x_{32} & x_{33} \end{bmatrix} \leq 0,$$

which means

$$x_{11}x_{22}x_{33} - x_{11}x_{23}x_{32} - x_{12}x_{21}x_{33} - x_{12}x_{23}x_{31} - x_{13}x_{21}x_{32} - x_{13}x_{22}x_{31} \leq 0.$$

It is not hard to check that any other P will result in a similar equality, where one monomial of the terms of the expansion of the determinant of X appears with a positive sign, and all others with a negative sign.

This can be very useful to understand the geometry of the phaseless rank, as seen in a slightly more concrete example.

Example 1.3.8. Building from Example 1.3.7, let us characterize the nonnegative values of x and y for which the circulant matrix

$$\begin{bmatrix} 1 & x & y \\ y & 1 & x \\ x & y & 1 \end{bmatrix}$$

has phaseless rank less than 3. Computing the six polynomials determined in that example, we find that they collapse to just four distinct ones:

$$1 - x^3 - y^3 - 3yx, \quad -1 + x^3 - y^3 - 3yx, \quad -1 - x^3 + y^3 - 3yx, \quad -1 - x^3 - y^3 - yx.$$

For nonnegative x and y, the last one is always negative, so it can be ignored. Furthermore, the other three factor each into a linear term and a nonnegative quadratic term, which can also be ignored, so we are left only with the three linear inequalities

$$1 - x - y \leq 0, \quad -1 + x - y \leq 0, \quad -1 - x + y \leq 0.$$

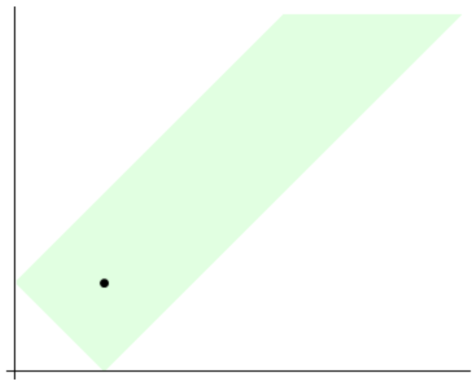

Fig. 1.3 Region where the 3×3 nonnegative circulant matrices have nonmaximal rank_θ

In Figure 1.3 we can observe the region. Note that the only singular matrix in that region is that for which $x = y = 1$, highlighted in the figure, every other one has usual rank equal to three. It is not hard to check that the signless rank drops to two on the boundary of the region.

1.4 Consequences and extensions

In this section, we derive some new results and strengthen some old ones, based on both Camion-Hoffman's result and, more generally, the underlying idea of using linear programming theory to study the phaseless rank.

1.4.1 The rectangular case

While we now have a full characterization for square matrices with nonmaximal phaseless rank, we are interested in extending it to more general settings. In this section we will study the case of rectangular matrices. Note that since transposition preserves the rank, we might restrict ourselves always to the case of $A \in \mathbb{R}^{n \times m}$ with $n \leq m$ for ease of notation. The simplest question one can ask is when does such a matrix have nonmaximal phaseless rank, i.e., when is $\text{rank}_\theta(A) < n$?

Denote by A_I, where I is a set of n distinct numbers between 1 and m, the $n \times n$ submatrix of A of columns indexed by elements of I. It is clear that if A has phaseless rank less than n so does A_I, since the submatrices B_I of a complex matrix B that is equimodular with A and has rank less than n will be, themselves, equimodular to the matrices A_I and have rank less than n. The reciprocal is much less clear, since the existence of singular matrices equimodular with each of the A_I does not seem to imply the existence of a singular matrix globally equimodular with A, since patching together the phases attributions to different submatrices is not trivial. Surprisingly, the result does hold.

Proposition 1.4.1. *Let $A \in \mathbb{R}_+^{n \times m}$, with $n \leq m$. Then,* $\text{rank}_\theta(A) < n$ *if and only if* $\text{rank}_\theta(A_I) < n$ *for all $I \subseteq \{1, \ldots, m\}$ with $|I| = n$.*

Proof. By the above discussion, the only thing that needs proof is the sufficiency of the condition $\mathrm{rank}_\theta(A_I) < n$ for all I, since it is clearly implied by $\mathrm{rank}_\theta(A) < n$. Assume that the condition holds. Then, by Lemma 1.3.1, for each A_I there exists $\lambda^I \in \mathbb{R}^n_+$ with coordinate sum one, such that for each column $l \in I$, $\{A_{1l}\lambda^I_1, \dots, A_{nl}\lambda^I_n\}$ is not lopsided.

Given any $x \in \mathbb{R}^n_+$, denote by $\mathrm{Lop}(x)$ the set of $y \in \mathbb{R}^n_+$ with coordinate sum one such that $\{x_1y_1, \dots, x_ny_n\}$ is not lopsided. This is simply the polyhedral set

$$\mathrm{Lop}(x) = \left\{ y \in \mathbb{R}^n_+, \sum_{i=1}^n y_i = 1 : x_iy_i \le \sum_{k\neq i} x_ky_k,\ i = 1, \dots, n \right\}$$

and, in particular, is convex.

Let a_j denote the jth column of A. The convex sets $\mathrm{Lop}(a_j)$, for $j = 1,\dots,m$, are contained in the hyperplane of coordinate sum one, an $n-1$ dimensional space. Furthermore, by assumption, any n of them intersect, since for any $I = \{i_1, \dots, i_n\}$, we have $\lambda^I \in \bigcap_{j\in I} \mathrm{Lop}(a_j)$. By Helly's Theorem, we must have

$$\bigcap_{j=1}^m \mathrm{Lop}(a_j) \neq \emptyset,$$

which means that we can take λ in the intersection, which will then verify the conditions of Lemma 1.3.1, proving that $\mathrm{rank}_\theta(A) < n$. □

This shows that we can reduce the $n \times m$ case to multiple $n \times n$ cases, so we can still apply Camion-Hoffman's result to study this case.

Example 1.4.2. Consider the family of 3×4 matrices parametrized by

$$\begin{bmatrix} x-y+1 & x-y+1 & x+1 & 1 \\ 1-x & -x+y+1 & 1-y & x+y+1 \\ 1-y & 1-x & 1 & x-y+1 \end{bmatrix}.$$

If we want to study the region where the phaseless rank is two, it is enough to look at the four 3×3 submatrices and use the result of Example 1.3.7 to compute the region for each of them, which are shown in Figure 1.4. The red pentagonal region is the region where the matrix is nonnegative, while the colored region inside is the region of nonmaximal rank for each of the submatrices.

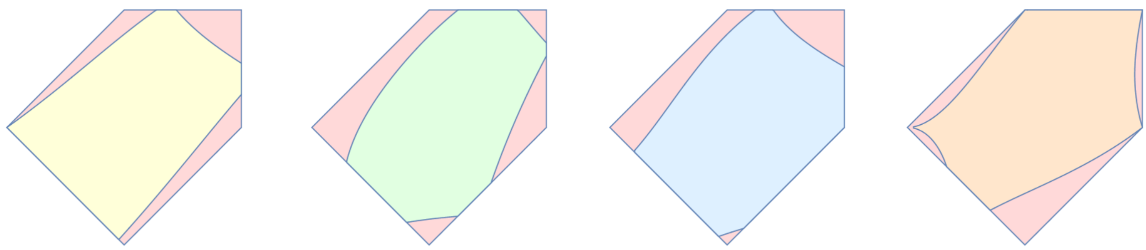

Fig. 1.4 Region of nonmaximal phase rank for each 3×3 submatrix

By Proposition 1.4.1 we then can simply intersect the four regions to observe the region where the phaseless rank of the full matrix is at most 2. The result is shown in Figure 1.5

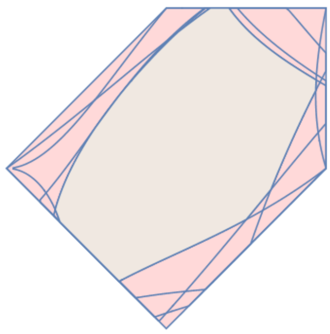

Fig. 1.5 Region of nonmaximal phase rank for the full matrix

1.4.2 Geometric implications

From Camion-Hoffman's Theorem and Proposition 1.4.1 one can also derive results on the geometry of the sets $P_{n-1}^{n\times m}$, of the $n\times m$ matrices of nonmaximal phaseless rank. More precisely, we are interested in the semialgebraic descriptions of such sets, and their boundaries.

Recall that $P_k^{n\times m}$ is always semialgebraic by the Tarski-Seidenberg principle, since it is the projection of a semialgebraic set. However the description can in principle be very complicated. For the square case, Theorem 1.3.6 together with Proposition 1.3.4 give a concrete semialgebraic description of $P_{n-1}^{n\times n}$. Recall that Theorem 1.3.6 states that

$$P_{n-1}^{n\times n} = \bigcap_{P\in S_n} \{A\in\mathbb{R}_+^{n\times n} : \mathscr{M}(AP) \text{ is not a nonsingular } M\text{-matrix}\}.$$

Let $\det_i(X)$ denote the i-th leading principal minor of matrix X. The characterizations of M-matrices given in Proposition 1.3.4 then allow us to write this more concretely as

$$P_{n-1}^{n\times n} = \bigcap_{P\in S_n}\bigcup_{i=3}^{n}\{A\in\mathbb{R}_+^{n\times n} : \det_i(\mathscr{M}(AP))\leq 0\},$$

which is a closed semialgebraic set, but not necessarily basic. For the $n\times m$ case, we just have to intersect the sets corresponding to each of the $n\times n$ submatrices, so we can still write $P_{n-1}^{n\times m}$ explicitly as an intersection of unions of sets described by a single polynomial inequality.

Note that when $n=3$ the unions have a single element, which trivially gives us the following corollary.

Corollary 1.4.3. *The set $P_2^{3\times m}$ is a basic closed semialgebraic set, for $m\geq 3$.*

It is generally not true that we can ignore the size 3 minor when testing a matrix for the property of being a nonsingular M-matrix. However, in our particular application we can get a little more in this direction.

Corollary 1.4.4. *For any $A \in \mathbb{R}_+^{4\times 4}$, we have $\operatorname{rank}_\theta(A) < 4$ if and only if $\det(\mathscr{M}(AP)) \leq 0$ for all permutation matrices $P \in S_4$. In particular, $P_3^{4\times m}$ is a basic closed semialgebraic set for all $m \geq 4$.*

Proof. By Theorem 1.3.6, $\operatorname{rank}_\theta(A) = 4$ if and only if, for some P, $\mathscr{M}(AP)$ is a nonsingular M-matrix, which implies, by Proposition 1.3.4, that all its leading principal minors are positive, including its determinant. This shows that if $\det(\mathscr{M}(AP)) \leq 0$ for all permutation matrices P then $\operatorname{rank}_\theta(A) < 4$.

Suppose now that $\det(\mathscr{M}(AP)) > 0$, for some P. We have to show that that this implies $\operatorname{rank}_\theta(A) = 4$. There exist three different permutation matrices P_1, P_2 and P_3, distinct from P such that

$$\det(\mathscr{M}(AP_1)) = \det(\mathscr{M}(AP_2)) = \det(\mathscr{M}(AP_3)) = \det(\mathscr{M}(AP)) > 0.$$

Namely, P_1, P_2 and P_3 are obtained from P by partitioning its columns in two pairs and transposing the columns in each pair. If we denote the entries of AP by b_{ij}, $i, j \in \{1,2,3,4\}$, we get the four matrices $\mathscr{M}(AP), \mathscr{M}(AP_1), \mathscr{M}(AP_2)$ and $\mathscr{M}(AP_3)$ as presented below in order:

$$\begin{bmatrix} b_{11} & -b_{12} & -b_{13} & -b_{14} \\ -b_{21} & b_{22} & -b_{23} & -b_{24} \\ -b_{31} & -b_{32} & b_{33} & -b_{34} \\ -b_{41} & -b_{42} & -b_{43} & b_{44} \end{bmatrix}, \quad \begin{bmatrix} b_{12} & -b_{11} & -b_{14} & -b_{13} \\ -b_{22} & b_{21} & -b_{24} & -b_{23} \\ -b_{32} & -b_{31} & b_{34} & -b_{33} \\ -b_{42} & -b_{41} & -b_{44} & b_{43} \end{bmatrix},$$

$$\begin{bmatrix} b_{13} & -b_{14} & -b_{11} & -b_{12} \\ -b_{23} & b_{24} & -b_{21} & -b_{22} \\ -b_{33} & -b_{34} & b_{31} & -b_{32} \\ -b_{43} & -b_{44} & -b_{41} & b_{42} \end{bmatrix}, \quad \begin{bmatrix} b_{14} & -b_{13} & -b_{12} & -b_{11} \\ -b_{24} & b_{23} & -b_{22} & -b_{21} \\ -b_{34} & -b_{33} & b_{32} & -b_{31} \\ -b_{44} & -b_{43} & -b_{42} & b_{41} \end{bmatrix}.$$

One can now easily check that $\det(\mathscr{M}(AP))$ can be written as

$$b_{41}\det_3(\mathscr{M}(AP_3)) + b_{42}\det_3(\mathscr{M}(AP_2)) + b_{43}\det_3(\mathscr{M}(AP_1)) + b_{44}\det_3(\mathscr{M}(AP)),$$

which, since all b_{ij} are nonnegative, means that at least one of the size 3 leading principal minors must be positive. By Proposition 1.3.4, the corresponding matrix must be a nonsingular M-matrix, since it has both the 3×3 and the 4×4 leading principal minors positive.

This shows that if $\det(\mathscr{M}(AP)) > 0$ for some permutation matrix, then Camion-Hoffman's Theorem guarantees that $\operatorname{rank}_\theta(A) = 4$, completing the proof. □

Remark 1.4.5. One can extract a little more information from the proof of Corollary 1.4.4. For checking whether a 4×4 nonnegative matrix A has phaseless rank less than four, we just need to check

$\det \mathscr{M}(AP) \leq 0$ for all permutation matrices P. In addition, we also know that each determinant is obtained from four different permutation matrices, leaving only six polynomial inequalities to check.

More concretely, if A has entries a_{ij}, and $\text{perm}(A)$ denotes the permanent of A, we just have to consider the inequalities:

$$2\left(a_{12}a_{23}a_{34}a_{41}+a_{11}a_{24}a_{33}a_{42}+a_{14}a_{21}a_{32}a_{43}+a_{13}a_{22}a_{31}a_{44}\right)-\text{perm}(A)\leq 0,$$

$$2\left(a_{13}a_{22}a_{34}a_{41}+a_{14}a_{21}a_{33}a_{42}+a_{11}a_{24}a_{32}a_{43}+a_{12}a_{23}a_{31}a_{44}\right)-\text{perm}(A)\leq 0,$$

$$2\left(a_{12}a_{24}a_{33}a_{41}+a_{11}a_{23}a_{34}a_{42}+a_{14}a_{22}a_{31}a_{43}+a_{13}a_{21}a_{32}a_{44}\right)-\text{perm}(A)\leq 0,$$

$$2\left(a_{14}a_{22}a_{33}a_{41}+a_{13}a_{21}a_{34}a_{42}+a_{12}a_{24}a_{31}a_{43}+a_{11}a_{23}a_{32}a_{44}\right)-\text{perm}(A)\leq 0,$$

$$2\left(a_{13}a_{24}a_{32}a_{41}+a_{14}a_{23}a_{31}a_{42}+a_{11}a_{22}a_{34}a_{43}+a_{12}a_{21}a_{33}a_{44}\right)-\text{perm}(A)\leq 0,$$

$$2\left(a_{14}a_{23}a_{32}a_{41}+a_{13}a_{24}a_{31}a_{42}+a_{12}a_{21}a_{34}a_{43}+a_{11}a_{22}a_{33}a_{44}\right)-\text{perm}(A)\leq 0.$$

Unfortunately, Corollary 1.4.4 does not extend beyond $n=4$. From $n=5$ onwards, the condition that $\det(\mathscr{M}(AP))\leq 0$ for all permutation matrices is stronger than having phaseless rank less than n, as shown in the next example.

Example 1.4.6. Consider the matrices

$$A=\begin{bmatrix}7&4&9&10&0\\9&2&3&0&3\\3&10&6&4&8\\0&4&1&6&4\\0&3&3&10&2\end{bmatrix}\text{ and }P=\begin{bmatrix}1&0&0&0&0\\0&0&0&1&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&1\end{bmatrix}.$$

We have that $\text{rank}_\theta(A)<5$, by Lemma 1.3.1, since no column is lopsided. However, $\det(\mathscr{M}(AP))=3732>0$, so it does not verify the determinant inequalities for all permutations matrices.

We now turn our attention to the boundary of the set $P_{n-1}^{n\times n}$, which we will denote by $\partial P_{n-1}^{n\times n}$. For $n\leq 4$, the explicit description we got in Corollary 1.4.3 and Corollary 1.4.4 immediately guarantees us that the positive part of the boundary is contained in the set of matrices A such that $\det(\mathscr{M}(AP))=0$ for some permutation matrix P. In particular this tells us that $\partial P_{n-1}^{n\times n}\cap\mathbb{R}_{++}^{n\times n}\subseteq S_{n-1}^{n\times n}$, for $n\leq 4$, the set of signless rank deficient matrices since $\det(\mathscr{M}(AP))=0$ implies $\det(\mathscr{M}(AP)P^{-1})=0$ and $\mathscr{M}(AP)P^{-1}$ is simply A with the signs of some entries switched. What is less clear is that exactly the same is still true for all n.

Proposition 1.4.7. *If $A\in\partial P_{n-1}^{n\times n}\cap\mathbb{R}_{++}^{n\times n}$, then $\det(\mathscr{M}(AP))=0$ for some permutation matrix P.*

Proof. Suppose $A\in\partial P_{n-1}^{n\times n}\cap\mathbb{R}_{++}^{n\times n}$. Since $P_{n-1}^{n\times n}$ is closed, $\text{rank}_\theta(A)<n$ and there must exist a sequence A_k of matrices such that $A_k\to A$ and every A_k is nonnegative and has phaseless rank n.

By Camion-Hoffman's result this implies that for every k we can find a permutation matrix $P_k \in S_n$ such that $\mathscr{M}(A_k P_k)$ is a nonsingular M-matrix or, equivalently, such that all eigenvalues of $\mathscr{M}(A_k P_k)$ have positive real part. Note that since there is a finite number of permutations, there exists a permutation matrix P such that $P_{k_i} = P$ for an infinite subsequence A_{k_i}, and that $\mathscr{M}(A_{k_i} P)$ have all eigenvalues with positive real part.

Since eigenvalues vary continuously, and $\mathscr{M}(A_{k_i} P) \to \mathscr{M}(AP)$, we must have that all eigenvalues of $\mathscr{M}(AP)$ have nonnegative real part, so $\mathscr{M}(AP)$ is an M-matrix. It cannot be a nonsingular M-matrix, as that would imply that $\operatorname{rank}_\theta(A) = n$. Therefore, $\mathscr{M}(AP)$ must be singular, i.e., $\det(\mathscr{M}(AP)) = 0$, as intended. $\square$

So, in spite of needing the smaller leading principal minors to fully describe the region, the boundary of $P_{n-1}^{n\times n}$ will still be contained in the set cut out by the determinants of the comparison matrices of the permutations of the matrices, even for $n > 4$. In the next example we try to illustrate what is happening.

Example 1.4.8. Consider the slice of the nonnegative matrices in $\mathbb{R}_+^{5\times 5}$ that contains the identity, the all-ones matrix and the matrix in Example 1.4.6, all scaled to have row sums 1. By what we saw in Example 1.4.6, we know that in this slice the set of nonnegative matrices, the set of matrices of phaseless rank less than 5 and the set of matrices A verifying $\mathscr{M}(AP) \leq 0$ for all P are all distinct. This can be seen in the first image of Figure 1.6, where we see the sets in light blue, green and yellow, respectively, and the three special matrices mentioned as black dots.

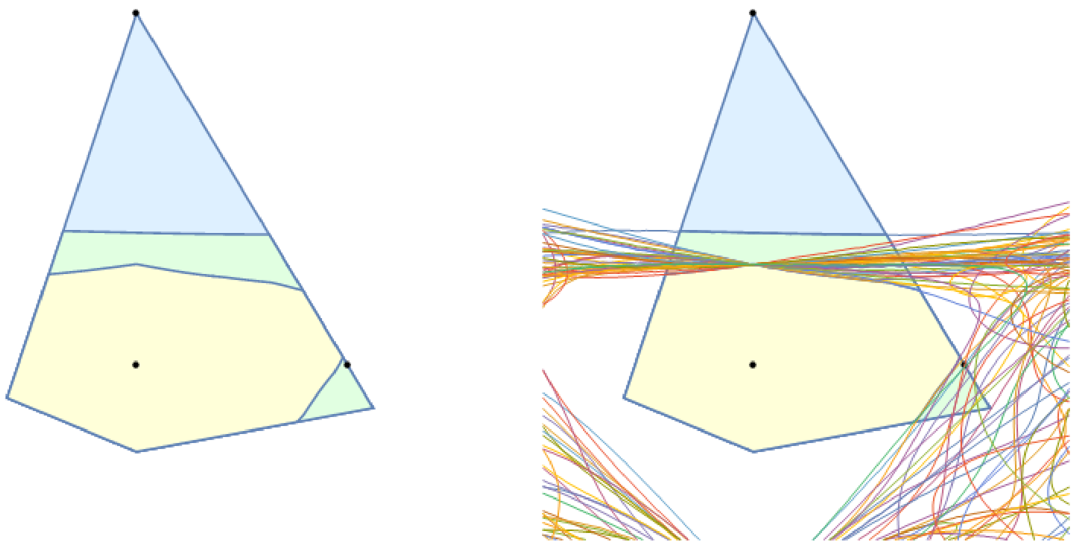

Fig. 1.6 A slice of the cone of 5×5 nonnegative matrices, with the nonmaximal phaseless rank region and its basic closed semialgebraic inner approximation highlighted

In the second image of the same figure we can see the zero sets of the 120 different determinants of the form $\det(\mathscr{M}(AP))$ and check that the extra positive boundary points of $P_4^{5\times 5}$ do indeed come from one of them.

1.4.3 Upper bounds

In Proposition 1.4.1 we have shown that for an $n \times m$ matrix, with $n \leq m$, to have phaseless rank less than n it was enough to check all its $n \times n$ submatrices. A natural question is to ask if a matrix has phaseless rank less than k if and only if the same is true for all its $k \times k$ submatrices, for any positive integer k. This is false, as was shown by Levinger ([48]).

Theorem 1.4.9 ([48]). *Let $A = mI_n + J_n$, where m is an integer with $1 \leq m < n-2$, and I_n and J_n are, respectively, the $n \times n$ identity and all-ones matrices. Then,* $\operatorname{rank}_\theta(A) \geq m+2$.

Note that it is not hard to see that all $(m+2) \times (m+2)$ matrices of the matrix A constructed above have phaseless rank at most $m+1$, so this is indeed a counterexample.

So a perfect generalization of Proposition 1.4.1 is impossible, but we can try to settle for a weaker goal: discovering what having all $k \times k$ submatrices with phaseless rank less than k allows us to conclude about the phaseless rank of the full matrix. This program was carried out in the same paper [48], where the following result was derived.

Proposition 1.4.10 ([48]). *Let $A \in \mathbb{R}_+^{n \times m}$, with $n \leq m$. If all $k \times k$ submatrices of A have nonmaximal phaseless rank, for some $k \leq n$, then*

$$\operatorname{rank}_\theta(A) \leq m - \left\lfloor \frac{m-1}{k-1} \right\rfloor.$$

In this section we use Proposition 1.4.1 to improve on this result. The result we prove is virtually the same, except that we can replace the m in the bound with the smaller n, obtaining a much better bound for rectangular matrices.

Proposition 1.4.11. *Let $A \in \mathbb{R}_+^{n \times m}$, with $n \leq m$. If all $k \times k$ submatrices of A have nonmaximal phaseless rank, for some $k \leq n$, then*

$$\operatorname{rank}_\theta(A) \leq n - \left\lfloor \frac{n-1}{k-1} \right\rfloor.$$

Proof. Let M be an $k \times m$ submatrix of A. By Proposition 1.4.1 the matrix M, has nonmaximal rank. Hence, for every $k \times m$ submatrix M, we can find $B_M \in \Omega(M)$ with rank less than k. Moreover, we are free to pick the first row of B_M to be real, since scaling an entire column of B_M by $e^{\theta i}$ does not change the rank or the equimodular class.

Consider then $k \times m$ submatrices M_i of A, $i = 1, \ldots, \lfloor \frac{n-1}{k-1} \rfloor$ all containing the first row,which we assume nonzero, but otherwise pairwise disjoint. We can then construct a matrix B by piecing together the B_{M_i}'s, since they coincide in the only row they share, and filling out the remaining rows, always less than $k-1$, with the corresponding entries of A.

By construction, in that matrix B we always have in the rows corresponding to B_{M_i} a row different than the first that is a linear combination of the others, and can be erased without dropping the rank of

B. Doing this for all i, we get that the rank of B has at least a deficiency per B_i, so its rank is at most

$$n - \left\lfloor \frac{n-1}{k-1} \right\rfloor,$$

and since B is equimodular with A, $\operatorname{rank}_\theta(A)$ verifies the intended inequality. □

Note that by setting $k = n$ we recover Proposition 1.4.1, so we have a strict extension of that result. Setting $k = 2$, we get that if all 2×2 minors have phaseless rank 1 so does the matrix, which is an obvious consequence of the observation already made in Section 1.1 that $\operatorname{rank}_\theta(A) = 1$ if and only if $\operatorname{rank}(A) = 1$. For every k in-between we get new results, although not necessarily very strong. They are, however, enough to get some further geometric insight. We say that $\operatorname{rank}_\theta(A) = k$ is *typical* in $\mathbb{R}_+^{n\times m}$ if there exists an open set in $\mathbb{R}_+^{n\times m}$ for which all matrices have phaseless rank k.

An interesting question is the study of minimal typical ranks, which in our case corresponds to ask for the minimal k for which $P_k^{n\times m}$ has full dimension. We claim that if k is typical, then we must have $k \geq \left\lceil \frac{n+m-\sqrt{(n-1)^2+(m-1)^2}}{2} \right\rceil$. Take the map which sends each matrix in $(\mathbb{C}^*)^{n\times m}$ to its entrywise absolute value, in $\mathbb{R}_{++}^{n\times m}$. The image under this map of the variety of complex matrices with no zero entries and of rank at most k is $P_k^{n\times m} \cap \mathbb{R}_{++}^{n\times m}$, which is full-dimensional if and only if k is at least the minimal typical phaseless rank. Note that we can assume that every matrix in the domain has real entries in the first row and column, since row and column scaling by complex numbers of absolute value one preserve both the rank and the entrywise absolute value matrix. The real dimension of the variety of complex matrices of rank at most k with real first row and column is $2(n+m-k)k$, twice the number of complex degrees of freedom, minus $m+n-1$, the number of entries forced to be real. This difference should be at least $n \times m$, the dimension of $P_k^{n\times m} \cap \mathbb{R}_{++}^{n\times m}$, since the map is differentiable. Thus, we must have

$$2(n+m-k)k - n - m + 1 \geq nm,$$

which boils down to

$$k \geq \left\lceil \frac{n+m-\sqrt{(n-1)^2+(m-1)^2}}{2} \right\rceil,$$

because k is a positive integer.

Corollary 1.4.12. *For $\mathbb{R}_+^{n\times m}$, with $3 \leq n \leq m$, the minimal typical phaseless rank k must verify*

$$\left\lceil \frac{n+m-\sqrt{(n-1)^2+(m-1)^2}}{2} \right\rceil \leq k \leq \left\lceil \frac{n+1}{2} \right\rceil.$$

Proof. The lower bound comes from the above dimension count. To prove the upper bound, note that the 3×3 all-ones matrix has phaseless rank 1 (less than three), and any small enough entrywise perturbation of it also has phaseless rank less than 3, since it will still have nonlopsided columns. This means that the $n \times m$ all-ones matrix, and any sufficiently small perturbation of it, have all

3×3 submatrices with nonmaximal phaseless rank, which implies, by Proposition 1.4.11, that their phaseless rank is at most $\lceil \frac{n+1}{2} \rceil$. Hence, there exists an open set of $\mathbb{R}_+^{n\times m}$ in which every matrix has phaseless rank less or equal than that number, which implies the smallest typical rank is at most that, giving us the upper bound. $\square$

For m much larger than n the bound is tight, since the lower bound converges to $\lceil \frac{n+1}{2} \rceil$.

1.5 Applications

1.5.1 The amoeba point of view

Many of the results developed in the previous sections have nice interpretations from the viewpoint of amoeba theory. Here, we will introduce some concepts and problems coming from this area of research and show the implications of the work previously developed.

As mentioned before, checking for amoeba membership is a hard problem. Even certifying that a point is not in an amoeba is generally difficult. To that end, several necessary conditions for amoeba membership have been developed. One such condition is the nonlopsidedness criterion. In its most basic form, this gives a necessary condition for a point to be in the amoeba of the principal ideal generated by some polynomial f, $\mathscr{A}(f)$.

Let $f \in \mathbb{C}[z_1,\dots,z_n]$ and $\mathbf{a} \in \mathbb{R}^n$. By writing f as a sum of monomials, $f(\mathbf{z}) = m_1(\mathbf{z}) + \ldots + m_d(\mathbf{z})$, define

$$f\{\mathbf{a}\} := \{|m_1(\mathbf{a})|,\dots,|m_d(\mathbf{a})|\}.$$

It is clear that in order for $\mathbf{a}$ to be the vector of absolute values of some complex root of f, the vector $f\{\mathbf{a}\}$ cannot be lopsided, as it must cancel after the phases are added in. We then define

$$\mathrm{Nlop}(f) = \{\mathbf{a} \in \mathbb{R}^n \ : \ f\{\mathbf{a}\} \text{ is not lopsided}\}.$$

It is clear that $\mathscr{A}(f) \subseteq \mathrm{Log}(\mathrm{Nlop}(f))$, but the inclusion is generally strict. One immediate consequence of Example 1.3.7 is the following.

Proposition 1.5.1. *Let $f = \det(X)$ be the cubic polynomial in variables x_{ij}, $i, j = 1,2,3$. Then*

$$\mathscr{A}(f) = \mathrm{Log}(\mathrm{Nlop}(f)).$$

So, the above proposition gives us an example where nonlopsidedness is a necessary and sufficient condition. There is a general result from amoeba theory that gives sufficiency in some cases: for any polynomial whose support forms the set of vertices of a simplex, it holds that $\mathscr{A}(f) = \mathrm{Log}(\mathrm{Nlop}(f))$. This follows from [25] (see, for instance, Theorem 3.1 of [71] for details). This result is not contained in that family, since the 3×3 determinant is not simple, i.e., its Newton polytope is not a simplex, it is actually the direct sum of two triangles.

Another interesting example that we can extract from our results concerns amoeba bases. Purbhoo shows, in [61], that the amoebas of general ideals can be reduced in a way to the case of principal ideals, since $\mathscr{A}(V(I)) = \bigcap_{f\in I} \mathscr{A}(f)$. The problem is that this is an infinite intersection, which immediately raises the question if a finite intersection may suffice. This suggests the notion of an *amoeba basis*, introduced in [66].

Definition 1.5.2. Given an ideal $I \subseteq \mathbb{C}[z_1,\dots,z_n]$, we call a finite set $B \subset I$ an amoeba basis for I if it generates I and it verifies the property

$$\mathscr{A}(V(I)) = \bigcap_{f\in B} \mathscr{A}(f)$$

while any proper subset of B does not.

Unfortunately, amoeba bases may fail to exist and in fact very few examples of them are known. In [56] it is proved that varieties of a particular kind, those that are *independent complete intersections*, have amoeba bases, and it is conjectured that only union of those can have them (see [56, Conjecture 5.3]). Proposition 1.4.1 gives us a new example of such nice behavior:

Corollary 1.5.3. *Let X be an $n \times m$ matrix of indeterminates. The set of maximal minors of X is an amoeba basis for the determinantal ideal they generate.*

Note that this is just another result in a long line of results about the special properties of the basis of maximal minors of a matrix of indeterminates, notoriously including the fact that they form a universal Groebner basis, as proved in [8]. For $3 \times n$ matrices we actually have that the nonlopsidedness of the generators is enough to guarantee the amoeba membership, an even stronger condition.

All other results automatically translate to amoeba theory, and some have interesting translations. We provide explicit semialgebraic descriptions for the amoeba of maximal minors, adding one example to the short list of amoebas for which such is available, as pointed out in [56, Question 3.7]. Moreover, Proposition 1.4.7 implies that the boundary of the amoeba of the determinant of a square matrix of indeterminates is contained in the image by the entrywise absolute value map of the set of its real zeros, while Corollary 1.4.12 states some conditions for full dimensionality of the amoeba of the variety of bounded rank matrices.

1.5.2 Implications on semidefinite rank

As we saw before, upper bounds on the phaseless rank will immediately give us upper bounds on the complex semidefinite rank. One can use that to improve on some results in the literature, and hopefully to construct examples.

For a simple illustration, recall the following result proved in [46], that gives sufficient conditions for nonmaximality of the complex semidefinite rank of a matrix.

Proposition 1.5.4 ([46]). *Let $A \in \mathbb{R}_+^{n\times m}$. If no column of $\sqrt[\circ]{A}$ has a dominant entry (i.e., if every column of $\sqrt[\circ]{A}$ is not lopsided), then* $\operatorname{rank}_{psd}^{\mathbb{C}}(A) < n$.

We remark that the assumption in the previous result is just a sufficient condition for $\operatorname{rank}_\theta(\sqrt[\circ]{A}) < n$, which implies $\operatorname{rank}_{\mathrm{psd}}^{\mathbb{C}}(A) < n$, by Proposition 1.2.3. This observation easily follows from applying Lemma 1.3.1 to $\sqrt[\circ]{A}$. This means that Proposition 1.5.4 is just a specialization of the following more general statement.

Proposition 1.5.5. *Let $A \in \mathbb{R}_+^{n\times m}$. If* $\operatorname{rank}_\theta(\sqrt[\circ]{A}) < n$, *then* $\operatorname{rank}_{psd}^{\mathbb{C}}(A) < n$.

One can check whether $\operatorname{rank}_\theta(\sqrt[\circ]{A}) < n$ by using both Proposition 1.4.1, if the matrix is not square, and Theorem 1.3.6. More generally, Proposition 1.2.3 dictates that every upper bound for $\operatorname{rank}_\theta(\sqrt[\circ]{A})$ is an upper bound for $\operatorname{rank}_{\mathrm{psd}}^{\mathbb{C}}(A)$. Thus, we have the following corollary of Proposition 1.4.11.

Corollary 1.5.6. *Let $A \in \mathbb{R}_+^{n\times m}$, with $n \leq m$. If all $k\times k$ submatrices of $\sqrt[\circ]{A}$ have nonmaximal phaseless rank,*

$$\operatorname{rank}_{psd}^{\mathbb{C}}(A) \leq n - \left\lfloor \frac{n-1}{k-1} \right\rfloor .$$

One can actually improve on both these results by removing the need to consider the Hadamard square root. To do that, we need an auxiliary lemma, concerning the Hadamard product of matrices:

Lemma 1.5.7. *Let $A \in \mathbb{R}_+^{n\times n}$ and $\alpha \geq 1$. If* $\operatorname{rank}_\theta(A) = n$, *then* $\operatorname{rank}_\theta(A^{\circ\alpha}) = n$, *where $A^{\circ\alpha}$ is obtained from A by taking entrywise powers α.*

Proof. By Theorem 1.3.6, $\operatorname{rank}_\theta(A) = n$ if and only if there exists a permutation matrix P such that $\mathscr{M}(AP)$ is a nonsingular M-matrix, which is equivalent to saying that the minimum real eigenvalue of $\mathscr{M}(AP)$ is positive, according to Proposition 1.3.4, i.e., $\sigma(AP) > 0$.

But then, Theorem 4 from [22] guarantees precisely that we must have

$$\sigma(A^{\circ\alpha}P) = \sigma((AP)^{\circ\alpha}) \geq \sigma(AP)^{\alpha} > 0,$$

proving that $\operatorname{rank}_\theta(A^{\circ\alpha}) = n$. ☐

By specializing $\alpha = 2$ and applying the previous lemma to the Hadamard square root of A we get the following immediate corollary.

Corollary 1.5.8. *Let $A \in \mathbb{R}_+^{n\times n}$. If* $\operatorname{rank}_\theta(A) < n$, $\operatorname{rank}_\theta(\sqrt[\circ]{A}) < n$.

This can be used to get a simpler upper bound on the complex semidefinite rank, testing submatrices of A instead of its square root.

Corollary 1.5.9. *Let $A \in \mathbb{R}_+^{n\times m}$, with $n \leq m$. If all $k\times k$ submatrices of A have nonmaximal phaseless rank,*

$$\operatorname{rank}_{psd}^{\mathbb{C}}(A) \leq n - \left\lfloor \frac{n-1}{k-1} \right\rfloor .$$

This can be used to derive simple upper bounds on the extension complexity of polytopes. Recall that for a d-dimensional polytope, P, its slack matrix, S_P, has rank $d+1$ and its complex semidefinite rank is the complex semidefinite extension complexity of P. Since every $(d+2)\times(d+2)$ submatrix of S_P has rank $d+1$, it also has phaseless rank at most $d+1$. Thus, by applying the previous corollary we obtain the following result.

Corollary 1.5.10. *Let P be a d-dimensional polytope with v vertices and f facets, and $m=\min\{v,f\}$ then*

$$\operatorname{rank}_{psd}^{\mathbb{C}}(S_P) \le m - \left\lfloor \frac{m-1}{d+1} \right\rfloor.$$

For $d=2$, for example, this gives us an upper bound of $\left\lceil \frac{2n+1}{3} \right\rceil$ for the complex extension complexity of an n-gon, which is similar asymptotically to the $4\left\lceil \frac{n}{6} \right\rceil$ bound derived in [37] and slightly better for small n (note that that bound is valid for the real semidefinite extension complexity, and so automatically for the complex case too). Of course it is just linear, so it does not reach the sublinear complexity proved by Shitov in [68] even for the linear extension complexity, but it is applicable in general and can be useful for small polytopes in small dimensions. Moreover, it is, as far as we know, the only non-trivial bound that works for polytopes of arbitrary dimension. As a last remark, we note that such lift can explicitly can be constructed. This can easily be done from an actual rank $m-\left\lfloor \frac{m-1}{d+1} \right\rfloor$ matrix that is equimodular to the Hadamard square root of the slack matrix, and such matrix can, with a small amount of work, be explicitly constructed from our results.

Chapter 2

The set of 4×4 matrices of phaseless rank at most 2

In Chapter 1 we gave a complete characterization of $P_{n-1}^{n\times m}$, for $m \geq n-1$. We also know that $P_1^{n\times m}$ is always trivial. This means that the simplest case that we are yet to cover is $P_2^{4\times 4}$. This is a full dimensional set, i.e., it has dimension 16 (see Table 3.1) and it makes sense to try to characterize membership in it and its borders. In this section we present some of the efforts we carried out towards that goal. This is meant as a case study, so that we can explore general techniques that might be used to tackle any of the outstanding cases, or at least derive some numerical intuition on them.

Given a 4×4 nonnegative matrix, even if all its 3×3 submatrices have nonmaximal phaseless rank, this does not guarantee the full matrix has phaseless rank less than 3. An example can be obtained directly from Theorem 1.4.9, by setting $n = 4$ and $m = 1$:

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

This means that checking membership in $P_2^{4\times 4}$ is likely more delicate than the nonmaximal case. Below we study a simple family of matrices to better illustrate the difficulties involved.

Example 2.0.1. Let

$$A(x) = \begin{bmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{bmatrix}$$

and $S = \{x \geq 0 : A(x) \in P_2^{4\times 4}\}$. On the one hand, S is nonempty and contains $\{0, 1\}$, because $\operatorname{rank}_\theta(A(0)) = 2$ (see Example 1.1.2) and $\operatorname{rank}_\theta(A(1)) = 1$. On the other hand, $2 \notin S$, as we just saw. In fact, $S \subseteq [0, 2[$, as explained below.

If $\operatorname{rank}_\theta(A(x)) \leq 2$, the same holds for all its 3×3 submatrices. A characterization for $P_2^{3\times 3}$ is derived in Example 1.3.7. It follows that all 3×3 submatrices of $A(x)$ have phaseless rank at most 2 if and only if

$$x^3 \leq 2+3x,\ 0 \leq 3x+x^3,\ 0 \leq 2+x+x^3$$
$$x^2 \leq 3+2x,\ 0 \leq 1+2x+x^2,\ 0 \leq 3+x^2.$$

This set of inequalities can be replaced with a single one, namely $0 \leq x \leq 2$. Thus, if x lies outside of this interval, we immediately deduce $\operatorname{rank}_\theta(A(x)) > 2$, i.e., $A(x)$ is not in $P_2^{4\times 4}$.

Now consider

$$B(x) = \begin{bmatrix} x & 1 & 1 & 1 \\ 1 & x & \frac{1}{2}\sqrt{3+2x^2-x^4}+\frac{1}{2}(1-x^2)i & \frac{1}{2}\sqrt{3+2x^2-x^4}-\frac{1}{2}(1-x^2)i \\ 1 & -\frac{1}{2}\sqrt{3+2x^2-x^4}-\frac{1}{2}(1-x^2)i & -\frac{1}{2}x(1-x^2)+\frac{1}{2}x\sqrt{3+2x^2-x^4}i & -i \\ 1 & -\frac{1}{2}\sqrt{3+2x^2-x^4}+\frac{1}{2}(1-x^2)i & i & -\frac{1}{2}x(1-x^2)-\frac{1}{2}x\sqrt{3+2x^2-x^4}i \end{bmatrix},$$

with $0 \leq x \leq \sqrt{3}$ so that the argument in the square root is nonnegative. It is not hard to see that $B(x)$ has rank 2 and $|B(x)| = A(x)$. Thus, $[0,\sqrt{3}] \subseteq S$.

We have shown $[0,\sqrt{3}] \subseteq S \subseteq [0,2[$. Regarding the interval $]\sqrt{3},2[$, it is unclear if it intersects S.

As we have seen above, even a simple family of matrices presents some difficulties. Characterizing $P_2^{4\times 4}$ in terms of polynomial equalities and inequalities seems to be challenging. We can try to focus instead on certifying specific matrices, that is, given $A \in \mathbb{R}_+^{4\times 4}$, how can one check if it is in $P_2^{4\times 4}$?

For convenience and due to repeated use, we state the following remark.

Remark 2.0.2. Any complex matrix $M \in \mathbb{C}^{n\times m}$ such that $\operatorname{rank}(M) \leq k$ can be factorized as M_1M_2, where $M_1 \in \mathbb{C}^{n\times k}$ and $M_2 \in \mathbb{C}^{k\times m}$, or as $(X_1+iY_1)(X_2+iY_2)$, with $X_1,Y_1 \in \mathbb{R}^{n\times k}$ and $X_2,Y_2 \in \mathbb{R}^{k\times m}$. Thus, any $A \in P_k^{n\times m}$ can be written as $|(X_1+iY_1)(X_2+iY_2))|$, with $X_1,Y_1 \in \mathbb{R}^{n\times k}$ and $X_2,Y_2 \in \mathbb{R}^{k\times m}$.

According to it, $A \in P_2^{4\times 4}$ if and only if the system $A = |(X_1+iY_1)(X_2+iY_2))|$ is solvable for some $X_1,Y_1 \in \mathbb{R}^{4\times 2}$ and $X_2,Y_2 \in \mathbb{R}^{2\times 4}$. In alternative, $A \in P_2^{4\times 4}$ if and only if there is a complex matrix $M \in \mathbb{C}^{4\times 4}$ whose 3×3 minors vanish and $A = |M|$.

Equivalently, we can try to solve instead the equations

$$\sqrt{\sum_{1\leq i,j\leq 4} (A_{ij} - |(X_1+iY_1)(X_2+iY_2)|_{ij})^2} = 0,$$

for $X_1,Y_1 \in \mathbb{R}^{4\times 2}$ and $X_2,Y_2 \in \mathbb{R}^{2\times 4}$, in the former case, and, in the latter one,

$$\sqrt{\sum_{1\leq i,j\leq 4} |m(M)_{ij}|^2} = 0,$$

with respect to $M \in \mathbb{C}^{4\times 4}$ such that $|M| = A$, and where $m(M)_{ij}$ denotes the 3×3 minor associated to the submatrix of M obtained by deleting its row i and column j.

2.1 Numerical membership testing

We explore these two approaches numerically. This can be done by minimizing numerically, for each $A \in \mathbb{R}_+^{n\times m}$, the functions

$$\sqrt{\sum_{1\le i,j\le 4} (A_{ij} - |(X_1+iY_1)(X_2+iY_2)|_{ij})^2},$$

over $X_1, Y_1 \in \mathbb{R}^{4\times 2}$, $X_2, Y_2 \in \mathbb{R}^{2\times 4}$, or

$$\sqrt{\sum_{1\le i,j\le 4} |m(M)_{ij}|^2}$$

over $M \in \mathbb{C}^{4\times 4}$ such that $|M| = A$.

Observe that zero is a global minimum for both optimization problems if and only if A is in $P_2^{4\times 4}$. Notice, however, the difference between the two of them: in the former, the goal is to find a matrix in $P_2^{4\times 4}$ which is as close to A as possible, in terms of the absolute values of its entries, while in the latter we search for a complex matrix equimodular with A which is as close to have rank at most 2 as possible.

We illustrate both approaches with concrete examples.

Example 2.1.1. Let us get back to the matrix from Example 2.0.1,

$$A(x) = \begin{bmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{bmatrix},$$

and let $S = \{x \ge 0 : A(x) \in P_2^{4\times 4}\}$, as before. Having in mind what was said in Example 2.0.1, the numerical membership testing can be restricted to $x \in [0,2]$. The two optimization problems to solve are, for each $x \in [0,2]$,

$$\min_{X_1,Y_1,X_2,Y_2} \sqrt{\sum_{1\le i,j\le 4} (A(x)_{ij} - |(X_1+iY_1)(X_2+iY_2)|_{ij})^2}$$

and

$$\min_{B:|B|=A(x)} \sqrt{\sum_{1\le i,j\le 4} |m(B)_{ij}|^2} = \min_{B:|B_{ij}|=1,\forall i,j} \sqrt{\sum_{1\le i,j\le 4} |m(A(x)\circ B)_{ij}|^2}.$$

Note that these are highly nonconvex and thus very hard optimization problems. We numerically minimized both of them for several values of x using the *Mathematica* command `NMinimize` and

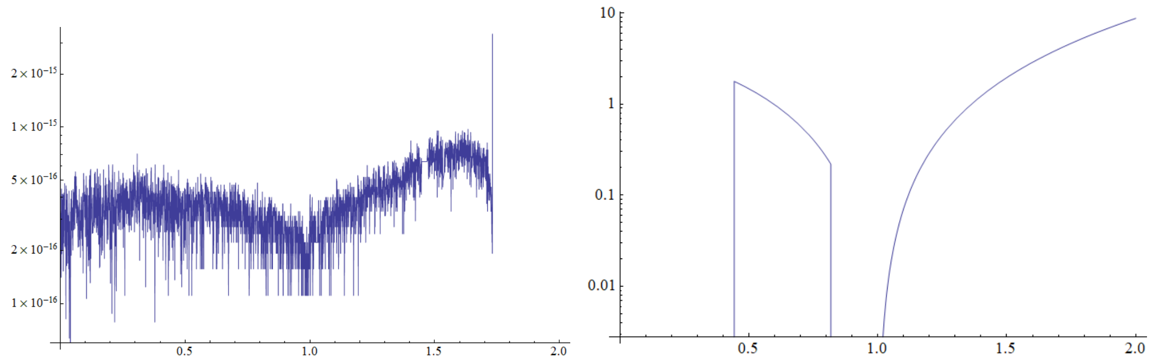

Fig. 2.1 Numerical solution to the first optimization problem, on the left, and to the second one, on the right, on a logarithmic scale.

plotted the attained results, shown in Figure 2.1. Note that due to the nature of the methods there is no guarantee that the global optimum has been achieved, but these are rigorous upper bounds on the true minimum. The results are highly affected by the algorithm picked for performing the numerical minimization. Among the numerical algorithms provided by *Mathematica*, namely Nelder-Mead, Differential Evolution, Simulated Annealing and Random Search, the last one appears to be one which yields the best results. Hence, this was the method chosen for generating all the figures presented in this section.

The first plot suggests clearly that $S = [0, \sqrt{3}]$ (see Example 2.0.1), while the second one shows that at least in this case the second approach is not reliable (as $[0, \sqrt{3}] \subseteq S$). From this moment forward we will therefore focus on the first of the approaches.

Example 2.1.2. Let us consider a slice of $P_2^{4\times 4}$ with a two-dimensional affine space,

$$M(x,y) = A_0 + A_1 x + A_2 y,$$

where A_0, A_1 and A_2 are 4×4 real matrices and x and y are real variables. We want to study the set $T = \{(x,y) : M(x,y) \in P_2^{4\times 4}\}$.

For generating random slices we chose the defining matrices A_0, A_1 and A_2 in the following way. When choosing A_0, we generated random real numbers for the entries of X_1, Y_1, X_2 and Y_2 and considered $A_0 = |(X_1 + iY_1)(X_2 + iY_2)|$, guaranteeing the choice of an interior point of $P_2^{4\times 4}$. Furthermore, we picked

$$A_1 = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} - A_0,$$

and picked the entries of A_2 independently random. This set-up implies, for instance, that T is nonempty, because it contains both the point $(0,0)$ and a neighborhood of it. Moreover, the point

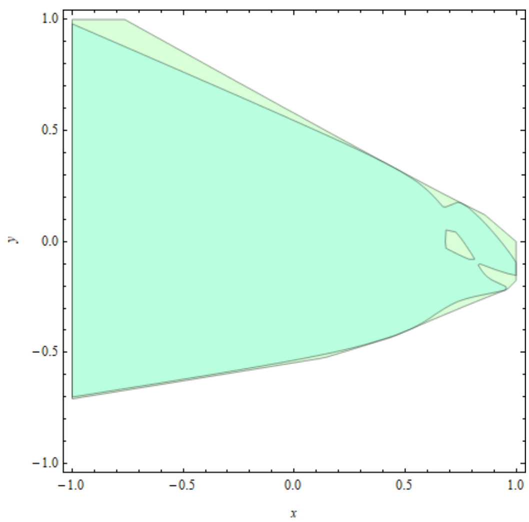

Fig. 2.2 Points (x,y) for which $M(x,y)$ is numerically in $P_2^{4\times 4}$, in darker green, and for which all 3×3 submatrices of $M(x,y)$ have nonmaximal phaseless rank, in lighter green.

$(1,0)$ is not in T, as the associated matrix,

$$M(1,0) = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix},$$

is not in $P_2^{4\times 4}$. We then plotted T using the same method as in the previous example: for each (x,y) we minimized the square root of the sum of the squared entries of $M(x,y) - |(X_1+iY_1)(X_2+iY_2)|$, where the minimization variables are the entries of X_1, Y_1, X_2 and Y_2. Finally, the point (x,y) will appear on the plot if the numerical optimum value of

$$\min_{X_1,Y_1,X_2,Y_2} \sqrt{\sum_{1\le i,j\le 4} (M(x,y)_{ij} - |(X_1+iY_1)(X_2+iY_2)|_{ij})^2}$$

is less than a predefined threshold, which in this case we took as 10^{-8}. A sample slice of $P_2^{4\times 4}$ is illustrated in Figure 2.2.

It is safe to say that the darker region represents matrices that are either in $P_2^{4\times 4}$ or at least very close to it. However, in non-convex functions (like the one we are trying to minimize), we might obtain a local minimum and not necessarily the global one. As a result, the lighter region might contain points whose matrices belong to $P_2^{4\times 4}$.

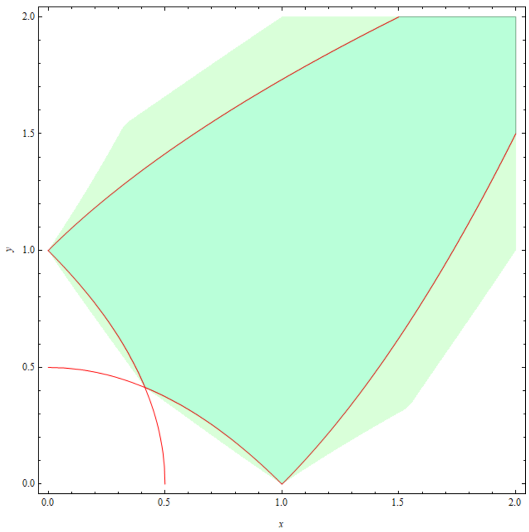

Fig. 2.3 Points (x,y) for which $M(x,y)$ is numerically in $P_2^{4\times 4}$, in darker green, and for which all 3×3 submatrices of $M(x,y)$ have nonmaximal phaseless rank, in lighter green.

Example 2.1.3. As a last example, we will apply the same method as in the previous example to a more structured slice of $P_2^{4\times 4}$. Let

$$M(x,y) = \begin{bmatrix} x & y & 1 & 1 \\ 1 & x & y & 1 \\ 1 & 1 & x & y \\ y & 1 & 1 & x \end{bmatrix},$$

a circulant matrix in two variables. In this case, it is not hard to guess what presumably is the boundary of the set $T = \{(x,y) : M(x,y) \in P_2^{4\times 4}\}$, highlighted in red in Figure 2.3. It is defined by branches of the curves $y^2 - 2x - 1 = 0$, $x^2 - 2y - 1 = 0$, $y^2 + 2x - 1 = 0$ and $x^2 + 2y - 1 = 0$.

2.2 Numerical certificates of non-membership

In the previous section we explored how to create numerical certificates that give evidence of membership in a region of bounded phaseless rank. We noted that not finding a certificate is, however, not very strong evidence for non-membership, as it may be a result from the local nature of the optimization methods employed. To remediate this, we may independently try to certificate non-membership.

There exist some algorithms for outer approximating amoebas in the literature that one could make use of. Most of the them are, however, devised for amoebas of hypersurfaces, i.e, amoebas of varieties whose associated ideals are principal, of the form $\mathscr{A}(f) := \mathscr{A}(V(f))$. Following the notation in Chapter 1,

$$P_2^{4\times 4} = \mathscr{A}_{\text{alg}}(Y_2^{4,4}),$$

where $Y_2^{4,4}$ is the set of all 4×4 complex matrices of rank at most 2, which has codimension 4. Regarding the algorithms for approximation of amoebas of hypersurfaces, we emphasize Forsgård et al. [28] approach (Theorem 6.1), which is mostly based on Purboo's result of the lopsided approximation for $\mathscr{A}(f)$ (Theorem 1 from [61]). A different approach is taken in [72], and allows tackling general amoebas. The approach proposed in this section is similar to the one they propose, but is derived from first principles and tailored to the specific case we are interested in.

Let us write $P_2^{4\times 4}$ membership as the feasibility of a real polynomial system. Take $A = \left[a_{ij}\right]_{1\leq i,j\leq 4} \in \mathbb{R}_+^{4\times 4}$. A is in $P_2^{4\times 4}$ if and only if there exists a complex matrix $Z = \left[z_{ij}\right]_{1\leq i,j\leq 4}$ of rank at most 2 and $|z_{ij}| = a_{ij}$, for $1 \leq i \leq 4, 1 \leq j \leq 4$. So we need to search for a matrix $Z \in \mathbb{C}^{4\times 4}$ such that all its 3×3 minors vanish and

$$|z_{ij}|^2 = a_{ij}^2, \text{ for } 1 \leq i \leq 4, 1 \leq j \leq 4.$$

By writing each $z_{ij} = x_{ij} + iy_{ij}$, where x_{ij} and y_{ij} are real variables, and expanding each minor into its algebraic form (as a complex number), one can rewrite the problem of membership of A in $P_2^{4\times 4}$ as a feasibility problem for a specific polynomial system:

Is there $X = \left[x_{ij}\right]_{1\leq i\leq 4, 1\leq j\leq 4}$ and $Y = \left[y_{ij}\right]_{1\leq i\leq 4, 1\leq j\leq 4}$ in $\mathbb{R}^{4\times 4}$ such that the real and imaginary parts of each 3×3 minor of $X + iY$ vanish and

$$x_{ij}^2 + y_{ij}^2 - a_{ij}^2 = 0 \text{ for } i, j = 1, \ldots, 4?$$

Let $I_A \subset \mathbb{R}[x_{ij}, y_{ij} : 1 \leq i, j \leq 4]$ be ideal generated by the polynomials in this system. The Real Nullstellensatz (see Theorem A.1.11) ensures that either A is in $P_2^{4\times 4}$ or there are polynomials $G \in I_A$ and a sum of squares $H \in \mathbb{R}[x_{ij}, y_{ij} : 1 \leq i, j \leq 4]$ such that

$$G + H + 1 = 0.$$

The Real Nullstellensatz provides a certificate for proving membership in the complement of $P_2^{4\times 4}$, which is not necessarily easy to obtain. In practice, we must search for such a certificate only for some bounded (low) degrees of H, which means we can only derive sufficient conditions for non-membership. Moreover, the numerical nature of the methods we will employ does not immediately provide exact certificates but only offer evidence for their existence, with extra work being needed to turn them into rigorous proofs.

Example 2.2.1. Let $A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$, which we know is not in $P_2^{4\times 4}$. We wish to prove there is no complex matrix Z of rank at most 2 such that $|Z| = A$, entrywise speaking, by exhibiting a Real Nullenstellensatz certificate. We start by simplifying the real polynomial system. Because row and colum scaling do not change the rank of a complex matrix, we assume every entry in the last row and column of Z to be real and positive. This allows us to reduce the number of real variables from 32 to 18. The simplified polynomial system is shown below.

The real and imaginary parts of each 3×3 minor of

$$Z=\begin{bmatrix} x_{11}+iy_{11} & x_{12}+iy_{12} & x_{13}+iy_{13} & 1\\ x_{21}+iy_{21} & x_{22}+iy_{22} & x_{23}+iy_{23} & 1\\ x_{31}+iy_{31} & x_{32}+iy_{32} & x_{33}+iy_{33} & 1\\ 1 & 1 & 1 & 2 \end{bmatrix}$$

must vanish and

$$x_{ii}^2+y_{ii}^2-4=0 \text{ for } i=1,\dots,3 \text{ and } x_{ij}^2+y_{ij}^2-1=0 \text{ for } i,j=1,\dots,3, \text{ and } i\neq j.$$

Let $g_{ij}=x_{ij}^2+y_{ij}^2-A_{ij}^2$, $i,j=1,\dots,3$, and ${f_{ij}}^{\mathrm{re}},{f_{ij}}^{\mathrm{im}}$ denote, in this order, the real and imaginary parts of the minor associated to the submatrix obtained by deleting row i and column j of Z, for $i,j=1,\dots,4$.

For instance,

$$\begin{aligned} g_{11}&=x_{11}^2+y_{11}^2-4,\\ {f_{11}}^{\mathrm{re}}&=-x_{22}+x_{23}+x_{32}-2x_{23}x_{32}-x_{33}+2x_{22}x_{33}+2y_{23}y_{32}-2y_{22}y_{33},\\ {f_{11}}^{\mathrm{im}}&=-y_{22}+2x_{33}y_{22}+y_{23}-2x_{32}y_{23}+y_{32}-2x_{23}y_{32}-y_{33}+2x_{22}y_{33}. \end{aligned}$$

In practice, a Real Nullenstellensatz certificate for the current example means finding polynomials $h_{ij}, i,j=1,\dots,3$, $h_{ij}^{\mathrm{re}},h_{ij}^{\mathrm{im}}, i,j=1,\dots,4$, and $s_i, i=1,\dots,k, \in\mathbb{R}[x_{ij},y_{ij}:1\le i,j\le 3]$ such that

$$\sum_{1\le i,j\le 3} h_{ij}g_{ij}+\sum_{1\le i,j\le 4} h_{ij}^{\mathrm{re}}{f_{ij}}^{\mathrm{re}}+\sum_{1\le i,j\le 4} h_{ij}^{\mathrm{im}}{f_{ij}}^{\mathrm{im}}+\sum_{i=1}^{k} s_i^2+1=0,$$

where k is a nonnegative integer.

Example 2.2.2. Let

$$M(x,y)=\begin{bmatrix} x & y & 1 & 1\\ 1 & x & y & 1\\ 1 & 1 & x & y\\ y & 1 & 1 & x \end{bmatrix},$$

the matrix from Example 2.1.3. In Figure 2.4, as before, the darker region represents points (x,y) for which $M(x,y)$ is numerically in $P_2^{4\times 4}$, and the lighter one the points and for which all 3×3 submatrices of $M(x,y)$ have nonmaximal phaseless rank. Regarding the dots, the gray ones show points for which at least one of the 3×3 submatrices of $M(x,y)$ has maximal phaseless rank (and thus not in $P_2^{4\times 4}$).

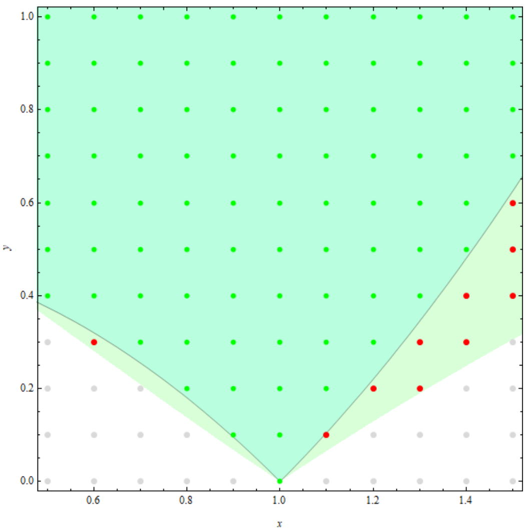

Fig. 2.4 Numerical Real Nullstellensatz certificates for $M(x,y)$.

The red ones represent points for which we were able to find a numerical certificate of the form

$$\sum_{1\le i,j\le 3} h_{ij}g_{ij} + \sum_{1\le i,j\le 4} h_{ij}^{\mathrm{re}}f_{ij}{}^{\mathrm{re}} + \sum_{1\le i,j\le 4} h_{ij}^{\mathrm{im}}f_{ij}{}^{\mathrm{im}} + \sum_{i=1}^{k} s_i^2 + 1 = 0,$$

where $i,j = 1,\ldots,3$ and each term of the sum has degree 4 (and thus not in $P_2^{4\times 4}$, in a numerical sense, according to the Real Nullstellensatz). Here, $g_{ij}, f_{ij}{}^{\mathrm{re}}$ and $f_{ij}{}^{\mathrm{im}}$ are constructed as in the previous example with $A_{ij} = M(x,y)_{ij}$. For the green dots we were not able to find such a certificate. In this case, no conclusion can be made, as we are imposing that each term has degree 4, which does not exclude the existence of certificates of higher degree. For the considered point grid, as hoped, the numerical certificate testing agrees with the numerical membership testing: points for which no certificate was found (green ones) lie inside the numerical approximation for $\{(x,y) : M(x,y) \in P_2^{4\times 4}\}$, whereas the certificated points (red ones) belong to the lighter region.

This fact was observed in all our numerical tests, hence we conjecture that the approximation might actually be tight, i.e., that the answer to the following question is affirmative.

Question 2.2.3. *Let X and Y be 4×4 matrices of indeterminates and $A \in \mathbb{R}_+^{4\times 4}$. Furthermore, let I be the ideal generated by the polynomials $X_{ij}^2 + Y_{ij}^2 - A_{ij}^2$ for all $1 \le i,j \le 4$ and the real and imaginary parts of all 3×3 minors of the matrix $X + iY$. Is it true that a matrix A is not in $P_2^{4\times 4}$ if and only if we can write -1 as a sum of squares of polynomials of degree at most 2 modulo the ideal I?*

More generally, for certifying non-membership in any algebraic amoeba $\mathscr{A}_{\mathrm{alg}}(V)$, one can follow the approach proposed in [72], by Theobald and de Wolff. Let $I := < f_1,\ldots,f_r >$ such that f_i is of the form $\sum_{j=1}^{d_i} b_{ij}\mathbf{z}^{\alpha(i,j)}$, with $\alpha(i,j) \in \mathbb{N}_0^n$, and m_{ij} be the monomial $m_{ij} = \mathbf{z}^{\alpha(i,j)} = z_1^{\alpha(i,j)_1}\cdots z_n^{\alpha(i,j)_n}$. Define the ideal $I^* \subset \mathbb{R}[X,Y]$ generated by the polynomials

$$\{f_i^{\mathrm{re}}, f_i^{\mathrm{im}} : 1 \le i \le r\} \cup \{(m_{ij}^{\mathrm{re}})^2 + (m_{ij}^{\mathrm{im}})^2 - \lambda^{2\alpha(i,j)} : 1 \le i \le r, 1 \le j \le d_i\},$$

where f_i^{re}, f_i^{im}, m_{ij}^{re} and m_{ij}^{im} denote the real and imaginary parts of f_i and m_{ij}, respectively. Those can be obtained by writing $z_j = x_j + iy_j$, where x_j and y_j are real variables, $j = 1,\dots,n$, and then expanding each f_i and m_{ij}. Below, X and Y stand for the variables x_j and y_j, $j = 1,\dots,n$.

Theorem 2.2.4 (de Wolff, Theobald, [72]). *Let $I := < f_1,\dots,f_r >$ and assume that the set $\bigcup_{i=1}^{r}\bigcup_{j=1}^{d_i}\{\alpha(i,j)\}$ spans $\mathbb{R}^n$. Either a point $\lambda \in (0,\infty)^n$ is contained in $\mathscr{A}_{\text{alg}}(V(I))$, or there exist polynomials $G \in I^* \subset \mathbb{R}[X,Y]$ and a sum of squares polynomial $H \in \mathbb{R}[X,Y]$ such that*

$$G + H + 1 = 0.$$

Observe that the authors consider the equations

$$(m_{ij}^{\text{re}})^2 + (m_{ij}^{\text{im}})^2 - \lambda^{2\alpha(i,j)} = 0, 1 \le i \le r, 1 \le j \le d_i$$

instead of

$$x_j^2 + y_j^2 - \lambda_j^2 = 0, j = 1,\dots,n.$$

The assumption that $\bigcup_{i=1}^{r}\bigcup_{j=1}^{d_i}\{\alpha(i,j)\}$ spans $\mathbb{R}^n$ guarantees that the solution set to each is the same.

For $I = Y_2^{4,4}$, the number of generators is 16 and I^* is generated by $2\times 16 + 16\times 6$ polynomials, as each 3×3 minor of a 4×4 matrix with complex entries has 6 monomials. In addition, the number of variables amounts to 2×16. Thus, it becomes extremely expensive (computationally) to approximate $P_2^{4\times 4}$ via Theorem 2.2.4.

2.3 Boundary of $P_2^{4\times 4}$

Since the set $P_2^{4\times 4}$ is semialgebraic, its boundary will be cut minimally by some polynomial equation $R(x_1,\dots,x_n) = 0$. A possible approach to understand the geometry of $P_2^{4\times 4}$ would therefore be to try to obtain such a polynomial.

Take the map in the beginning of Section 3.2 with $n = m = 4$ and $k = 2$:

$$\begin{aligned} F : \mathbb{R}^{4\times 2}\times\mathbb{R}^{4\times 2}\times\mathbb{R}^{2\times 4}\times\mathbb{R}^{2\times 4} &\longrightarrow \mathbb{R}_{++}^{4\times 4} \\ P = (X,Y,A,B) &\longrightarrow F(P) = |(X+iY)(A+iB)| \end{aligned}$$

The first observation is that, since $P_2^{4\times 4}$ is closed, it contains all its boundary points. Hence, any point in $\partial P_2^{4\times 4}$ is of the form $F(P)$, for some P. Furthermore, a necessary condition for $F(P)$ to be a boundary point is that Jacobian of the map at P is rank-deficient (otherwise, $F(P)$ would be an interior point of $P_2^{4\times 4}$). In the present case, the Jacobian is a 16×32 matrix and it will be rank-deficient if all its 16×16 minors vanish.

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 2A_{22} & 0 & 0 & 2B_{22} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2A_{23} & 0 & 0 & 2B_{23} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2A_{24} & 0 & 0 & 2B_{24} \\
\begin{array}{c} 2(1-A_{22})(-X_{31}A_{22} \\ +A_{22}+X_{31}+B_{22}Y_{31}) \\ -2B_{22}(-X_{31}B_{22} \\ +B_{22}-A_{22}Y_{31}+Y_{31}) \end{array} & 0 & 2\begin{array}{c} (1-A_{22})(-X_{31}B_{22} \\ +B_{22}-A_{22}Y_{31}+Y_{31}) \\ +2B_{22}(-X_{31}A_{22} \\ +A_{22}+X_{31}+B_{22}Y_{31}) \end{array} & 0 & \begin{array}{c} 2(1-X_{31})(-X_{31}A_{22} \\ +A_{22}+X_{31}+B_{22}Y_{31}) \\ -2Y_{31}(-X_{31}B_{22} \\ +B_{22}-A_{22}Y_{31}+Y_{31}) \end{array} & 0 & 0 & \begin{array}{c} 2(1-X_{31})(-X_{31}B_{22} \\ +B_{22}-A_{22}Y_{31}+Y_{31}) \\ +2Y_{31}(-X_{31}A_{22} \\ +A_{22}+X_{31}+B_{22}Y_{31}) \end{array} & 0 & 0 \\
\begin{array}{c} 2(1-A_{23})(-X_{31}A_{23} \\ +A_{23}+X_{31}+B_{23}Y_{31}) \\ -2B_{23}(-X_{31}B_{23} \\ +B_{23}-A_{23}Y_{31}+Y_{31}) \end{array} & 0 & 2\begin{array}{c} (1-A_{23})(-X_{31}B_{23} \\ +B_{23}-A_{23}Y_{31}+Y_{31}) \\ +2B_{23}(-X_{31}A_{23} \\ +A_{23}+X_{31}+B_{23}Y_{31}) \end{array} & 0 & 0 & \begin{array}{c} 2(1-X_{31})(-X_{31}A_{23} \\ +A_{23}+X_{31}+B_{23}Y_{31}) \\ -2Y_{31}(-X_{31}B_{23} \\ +B_{23}-A_{23}Y_{31}+Y_{31}) \end{array} & 0 & 0 & \begin{array}{c} 2(1-X_{31})(-X_{31}B_{23} \\ +B_{23}-A_{23}Y_{31}+Y_{31}) \\ +2Y_{31}(-X_{31}A_{23} \\ +A_{23}+X_{31}+B_{23}Y_{31}) \end{array} & 0 \\
\begin{array}{c} 2(1-A_{24})(-X_{31}A_{24} \\ +A_{24}+X_{31}+B_{24}Y_{31}) \\ -2B_{24}(-X_{31}B_{24} \\ +B_{24}-A_{24}Y_{31}+Y_{31}) \end{array} & 0 & \begin{array}{c} 2(1-A_{24})(-X_{31}B_{24} \\ +B_{24}-A_{24}Y_{31}+Y_{31}) \\ +2B_{24}(-X_{31}A_{24} \\ +A_{24}+X_{31}+B_{24}Y_{31}) \end{array} & 0 & 0 & 0 & \begin{array}{c} 2(1-X_{31})(-X_{31}A_{24} \\ +A_{24}+X_{31}+B_{24}Y_{31}) \\ -2Y_{31}(-X_{31}B_{24} \\ +B_{24}-A_{24}Y_{31}+Y_{31}) \end{array} & 0 & 0 & \begin{array}{c} 2(1-X_{31})(-X_{31}B_{24} \\ +B_{24}-A_{24}Y_{31}+Y_{31}) \\ +2Y_{31}(-X_{31}A_{24} \\ +A_{24}+X_{31}+B_{24}Y_{31}) \end{array} \\
0 & \begin{array}{c} 2(1-A_{22})(-X_{41}A_{22} \\ +A_{22}+X_{41}+B_{22}Y_{41}) \\ -2B_{22}(-X_{41}B_{22} \\ +B_{22}-A_{22}Y_{41}+Y_{41}) \end{array} & 0 & \begin{array}{c} 2(1-A_{22})(-X_{41}B_{22} \\ +B_{22}-A_{22}Y_{41}+Y_{41}) \\ +2B_{22}(-X_{41}A_{22} \\ +A_{22}+X_{41}+B_{22}Y_{41}) \end{array} & \begin{array}{c} 2(1-X_{41})(-X_{41}A_{22} \\ +A_{22}+X_{41}+B_{22}Y_{41}) \\ -2Y_{41}(-X_{41}B_{22} \\ +B_{22}-A_{22}Y_{41}+Y_{41}) \end{array} & 0 & 0 & \begin{array}{c} 2(1-X_{41})(-X_{41}B_{22} \\ +B_{22}-A_{22}Y_{41}+Y_{41}) \\ +2Y_{41}(-X_{41}A_{22} \\ +A_{22}+X_{41}+B_{22}Y_{41}) \end{array} & 0 & 0 \\
0 & \begin{array}{c} 2(1-A_{23})(-X_{41}A_{23} \\ +A_{23}+X_{41}+B_{23}Y_{41}) \\ -2B_{23}(-X_{41}B_{23} \\ +B_{23}-A_{23}Y_{41}+Y_{41}) \end{array} & 0 & \begin{array}{c} 2(1-A_{23})(-X_{41}B_{23} \\ +B_{23}-A_{23}Y_{41}+Y_{41}) \\ +2B_{23}(-X_{41}A_{23} \\ +A_{23}+X_{41}+B_{23}Y_{41}) \end{array} & 0 & \begin{array}{c} 2(1-X_{41})(-X_{41}A_{23} \\ +A_{23}+X_{41}+B_{23}Y_{41}) \\ -2Y_{41}(-X_{41}B_{23} \\ +B_{23}-A_{23}Y_{41}+Y_{41}) \end{array} & 0 & 0 & \begin{array}{c} 2(1-X_{41})(-X_{41}B_{23} \\ +B_{23}-A_{23}Y_{41}+Y_{41}) \\ +2Y_{41}(-X_{41}A_{23} \\ +A_{23}+X_{41}+B_{23}Y_{41}) \end{array} & 0 \\
0 & \begin{array}{c} 2(1-A_{24})(-X_{41}A_{24} \\ +A_{24}+X_{41}+B_{24}Y_{41}) \\ -2B_{24}(-X_{41}B_{24} \\ +B_{24}-A_{24}Y_{41}+Y_{41}) \end{array} & 0 & \begin{array}{c} 2(1-A_{24})(-X_{41}B_{24} \\ +B_{24}-A_{24}Y_{41}+Y_{41}) \\ +2B_{24}(-X_{41}A_{24} \\ +A_{24}+X_{41}+B_{24}Y_{41}) \end{array} & 0 & 0 & \begin{array}{c} 2(1-X_{41})(-X_{41}A_{24} \\ +A_{24}+X_{41}+B_{24}Y_{41}) \\ -2Y_{41}(-X_{41}B_{24} \\ +B_{24}-A_{24}Y_{41}+Y_{41}) \end{array} & 0 & 0 & \begin{array}{c} 2(1-X_{41})(-X_{41}B_{24} \\ +B_{24}-A_{24}Y_{41}+Y_{41}) \\ +2Y_{41}(-X_{41}A_{24} \\ +A_{24}+X_{41}+B_{24}Y_{41}) \end{array}
\end{bmatrix}
$$

Fig. 2.5 Jacobian of F after variable reduction.

Each of the entries of $F(P)$ will depend on exactly eight variables. For instance, the expression of $F(P)_{11}$ is

$$\sqrt{(A_{11}Y_{11} + A_{21}Y_{12} + B_{11}X_{11} + B_{21}X_{12})^2 + (A_{11}X_{11} + A_{21}X_{12} - B_{11}Y_{11} - B_{21}Y_{12})^2}.$$

As a result, the Jacobian will exhibit some sparsity, with 24 zeros in each row. Nevertheless, some of its minors possess complicated expressions, making this approach completely ineffective.

Since the Jacobian is a very large matrix, we perform a reduction of the number of variables so that we can show it. Assume every entry in $F(P)$ is nonzero. In fact, if any entry of $F(P)$ is zero, it will be in the boundary of the set of nonnegative matrices, which contains $P_2^{4\times 4}$. Thus, $F(P)$ will be a boundary point of $P_2^{4\times 4}$ if and only if it is in $P_2^{4\times 4}$. We can always scale rows and columns of $F(P)$ in such a way that its first row and column have ones everywhere, an operation which preserves phaseless rank. Now we consider a similar scaling for $(X+iY)(A+iB)$, i.e, with ones in its first row and column. For simplicity, we assume the 2×2 minor associated to the first two rows of $X+iY$ is nonzero. The associated submatrix can then be replaced with an identity matrix if we consider $(X+iY)LL^{-1}(A+iB)$, where L is a suitable invertible matrix. Finally, for the first row and column of $(X+iY)(A+iB)$ to have ones everywhere their entries must obey some conditions. The simplified system is shown below.

$$|(X+iY)(A+iB)| = \left| \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ X_{31}+iY_{31} & 1-X_{31}-iY_{31} \\ X_{41}+iY_{41} & 1-X_{41}-iY_{41} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & A_{22}+iB_{22} & A_{23}+iB_{23} & A_{24}+iB_{24} \end{bmatrix} \right|.$$

We have greatly reduced the number of variables from 32 to 10. When computing the Jacobian of this new map, seven of its rows will be composed of zeros. These rows correspond to the first row and column of $|(X+iY)(A+iB)|$, which are all ones, and can be removed as they are superfluous. The reduced Jacobian is presented in Figure 2.5, a 9×10 matrix. This matrix will be rank-deficient if all its 9×9 minors vanish. This might be useful to derive proofs of membership in the boundary, but direct computations of the boundary equation seem to be computationally out of reach.

Chapter 3

Further work on phaseless rank

In Chapter 1, we established the connection between the classical results of Camion and Hoffman on equimodular classes of matrices with the modern developments in the theories of amoebas and semidefinite extension complexity. This provided a rich field of motivation and applications, and allowed for interesting and new developments. In Chapter 2, we explored the simplest case for which we do not have a full geometric characterization, and used it to illustrate a series of tools that one might use to further expand our characterizations.

However, many other questions remain completely open and are ripe for further explorations. In the subsequent sections, we describe our endeavours - none fully successful or finished - on answering each of the four questions below.

1. What can we say about the complexity of computing phaseless rank?
2. While some work was already carried out here on the dimension of these semialgebraic sets, it should be possible to state more precise results on which values of the phaseless rank are typical.
3. Can we explore the connection between phaseless rank and the geometric problem of equiangular lines?

Note that these are not the only remaining questions from this work. Some other questions were considered but without any work developed that would be sufficiently worthy to include besides what was already discussed before. One that we have not really explored in length, but that we mention here for the interest of cataloguing possible future research directions is described below.

We saw that computing phaseless rank can be seen as a rank minimization problem (RMP), since we want to minimize the rank over a certain set of matrices. Another class of hard optimization problems is that of phase retrieval problems (PRP), which consist in recovering a vector of unknowns from phaseless information, namely, the absolute value of some linear measurements. One can think of phaseless rank as a mix of these two very important classes. In each of these classes, very effective convex relaxations have been developed. One example of such relaxations for the standard RMP consists in replacing the rank objective function with the sum of the singular values of the matrix, the so-called nuclear norm [24, 63]. For the standard PRP there is a similar semidefinite relaxation, known as the PhaseLift [16]. An important avenue of research would be to try to merge some of the approaches for these two classes of optimization problems in order to derive an effective approximation algorithm for phaseless rank.

3.1 Phaseless rank complexity

Here, we address the problem of how hard it is to decide if $A \in \mathbb{R}_+^{n\times m}$ is in $P_k^{n\times m}$ algorithmically, in a deterministic sense and not by means of approximations. Note that such an algorithm enables one to check if $\operatorname{rank}_\theta(A) = k$, by deciding if A is in $P_k^{n\times m}$ but not in $P_{k-1}^{n\times m}$. Recall that, according to the Tarski-Seidenberg theorem [10], $P_k^{n\times m}$ is semialgebraic, which means it is definable in terms of polynomial equalities and inequalities.

More specifically,

$$\begin{aligned} P_k^{n\times m} \text{ is set of all } a_{ij} \in \mathbb{R}, 1 \le i \le n, 1 \le j \le m, \text{ such that}\\ a_{ij} \ge 0 \text{ and } \exists x_{ij}, y_{ij} \in \mathbb{R} \text{ s.t. } x_{ij}^2 + y_{ij}^2 = a_{ij}^2 \text{ and}\\ \operatorname{rank}\left[x_{ij} + \mathrm{i}y_{ij}\right]_{1\le i\le n, 1\le j\le m} \le k \text{ i.e.,} \end{aligned}$$

$$\begin{aligned} P_k^{n\times m} \text{ is a projection of the semialgebraic set of all } a_{ij}, x_{ij}, y_{ij} \in \mathbb{R}, 1 \le i \le n, 1 \le j \le m, \text{ such that}\\ a_{ij} \ge 0\,, x_{ij}^2 + y_{ij}^2 = a_{ij}^2 \text{ and } \operatorname{rank}\left[x_{ij} + \mathrm{i}y_{ij}\right]_{1\le i\le n, 1\le j\le m} \le k, \text{i.e, the real and imaginary parts of all}\\ (k+1)\times(k+1) \text{ minors of } \left[x_{ij} + \mathrm{i}y_{ij}\right]_{1\le i\le n, 1\le j\le m} \text{ vanish.} \end{aligned}$$

While finding a polynomial characterization for $P_k^{n\times m}$ is likely hard, it is always (theoretically) possible. This can be done with any quantifier elimination method, such as the Cylindrical Algebraic Decomposition (CAD) algorithm. According to Theorem 4.1 from [34], originally published in [64], there exists a quantifier elimination method that produces a semialgebraic description for the projection of any semialgebraic set.

As an example, this semialgebraic description for $P_2^{4\times 4}$ is of the form

$$\bigcup_{i=1}^{I} \bigcap_{j=1}^{J_i} \{h_{ij}(x) \triangle_{ij} 0\},$$

where $\triangle_{ij} \in \{>, \ge, =, \ne, \le, <\}$, $h_{ij}(x)$ are real polynomials of degree at most 288^{32K} and $I \le 288^{512K}$, $J_i \le 288^{32K}$, with K a constant.

Alternatively, for deciding whether $A \in \mathbb{R}_+^{n\times m}$ is in $P_k^{n\times m}$, one could resort to the certificate derived from Real Nullstellensatz, discussed in Section 2.2. There, we analyze the particular case of $P_2^{4\times 4}$, but the reasoning can be generalized for $P_k^{n\times m}$ without great difficulty. The corresponding real polynomial system is the following:

$A \in \mathbb{R}_+^{n\times m}$ is in $P_k^{n\times m}$ if and only if there exist $X = \left[x_{ij}\right]_{1\le i\le n, 1\le j\le m}$ and $Y = \left[y_{ij}\right]_{1\le i\le n, 1\le j\le m}$ in $\mathbb{R}^{n\times m}$ such that the real and imaginary parts of each $(k+1)\times(k+1)$ minor of $X+iY$ vanish and

$$x_{ij}^2 + y_{ij}^2 - a_{ij}^2 = 0 \text{ for } i = 1,\ldots,n \text{ and } j = 1,\ldots,m.$$

Let $I_A \subset \mathbb{R}[x_{ij}, y_{ij} : 1 \le i \le n, 1 \le j \le m]$ be, again, the ideal generated by the above polynomials. The Real Nullstellensatz states that either A is in $P_k^{n\times m}$ or there are polynomials $G \in I_A$ and a sum of squares $H \in \mathbb{R}[x_{ij}, y_{ij} : 1 \le i \le n, 1 \le j \le m]$ such that

$$G + H + 1 = 0.$$

Since the certificate includes finding a sum of squares polynomial, whose degree is not known beforehand, there is the need to introduce a maximum value for its degree. We then search for a sum of squares decomposition whose degree is at most the fixed bound. If no sums of squares decomposition is found under these conditions, one accepts that $A \in \mathbb{R}_+^{n\times m}$ is in $P_k^{n\times m}$. Real Nullstellensatz certificates can be obtained via Semidefinite Programming if we fix the degree a priori [57] (see A.1). Typically, the degree bounds that guarantee the existence of a certificate, if there is any, are very large. For instance, Roy and Lombardi [51] propose a bound which is a tower of five exponentials, dependent on, among other quantities, the number of polynomial generators and the number of variables. For the particular case of $P_2^{4\times 4}$ the referred bound is

$$2^{2^{\left(2^{3^{4^{32}}} + (48+1)^{2^{32}} \times 3^{2\times 16^{32}}\right)}}.$$

Due to the huge size of the bounds involved, both methods presented before are impracticable. Still, the advantage of Real Nullstellensatz over any quantifier elimination method is that, if one is lucky enough, one might be able to obtain a certificate in which the sums of squares has a degree much lower than the degree bound.

Finally, in terms of computational complexity, can we claim that deciding if $\text{rank}_\theta(A) < k$ is NP-hard? For instance, for $k = n$ the answer is negative: checking whether $\text{rank}_\theta(A) < n$, for any $A \in \mathbb{R}_+^{n\times m}$, $n \le m$, can be verified in polynomial time (Corollary 1.3.2). Conversely, in [36], the authors prove the NP-hardness of checking if the signless rank of any matrix of the form

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & a_1 \\ 0 & 1 & 0 & \dots & 0 & a_2 \\ 0 & 0 & 1 & \dots & 0 & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & a_{n-1} \\ 1 & 1 & 1 & \dots & 1 & 0 \end{bmatrix},$$

where $a_1,\dots,a_n$ are positive integers, is less than n. This is somehow surprising: a priori, one could expect that, for fixed $A \in \mathbb{R}_+^{n\times m}$, deciding if $\operatorname{rank}_\theta(A) < k$ would be harder than deciding if $\operatorname{rank}_\pm(A) < k$, since we are enlarging the feasible set of the optimization problem, but it turns out not to be the case for $k = n$.

3.2 Full-dimensionality of $P_k^{n\times m}$

In this section we revisit the minimal typical phaseless rank problem. For a fixed matrix dimension $n \times m$, the minimal typical phaseless rank equals the minimum k for which $P_k^{n\times m}$ is full dimensional. We will try to find this number computationally. For that purpose, we consider the map

$$\begin{aligned} F : \mathbb{R}^{n\times k} \times \mathbb{R}^{n\times k} \times \mathbb{R}^{k\times m} \times \mathbb{R}^{k\times m} &\longrightarrow \mathbb{R}_{++}^{n\times m} \\ P = (X,Y,A,B) &\longrightarrow F(P) = |(X+iY)(A+iB)| \end{aligned}$$

It is clear that the image of F is $P_k^{n\times m}$, so F is a parametrization of $P_k^{n\times m}$. Now take any P. If the Jacobian of the map at P is full rank, the inverse function theorem guarantees there are neighborhoods around P and $F(P)$ for which F is surjective. This proves $F(P)$ is an interior point of $P_k^{n\times m}$ and, thus, $P_k^{n\times m}$ is full dimensional. Our goal is, then, to look for points at which the Jacobian is full-rank, if they exist.

The above procedure was converted into a *Mathematica* code. F is a vectorial function with $n\times m$ coordinates and whose variables are the entries of X,Y,A and B. Since the number of such variables is $2\times k\times(n+m)$, the Jacobian of F is an $(n\times m)\times(2\times k\times(n+m))$ matrix. Hence, it will be full rank if its rank equals $\min(n\times m, 2\times k\times(n+m))$. For fixed n,m and k, the code generates random real values for the variables of F and it computes the rank of the Jacobian evaluated at the generated numbers. If the rank is maximal, we know $P_k^{n\times m}$ is full dimensional. A few iterations of the code might be required, as we cannot make sure the rank will be maximal on the first attempt.

A straightforward observation is that if $P_k^{n\times m}$ is full dimensional so is $P_k^{n'\times m'}$, for all $k \le n' \le n$ and $k \le m' \le m$. Using the aforementioned fact, the code can be improved in the following way. First, for any k, we find the largest n, n_k, for which $P_k^{n\times n}$ is full dimensional. For $n_{k-1} < n \le n_k$, k is the minimum l for which $P_l^{n\times n}$ is full dimensional. Now, for fixed k and n, we search for the largest

m, $m_{n,k}$, for which $P_k^{n\times m}$ is full dimensional. For $m_{n,k-1} < m \leq m_{n,k}$, k is the minimum l for which $P_l^{n\times m}$ is full dimensional. The computational results can be seen in Table 3.1 and match the lower bound from Corollary 1.4.12 for the minimum typical phaseless rank. Hence, based on the numerical evidence available, we propose the following conjecture:

Conjecture 3.2.1. *For $\mathbb{R}_+^{n\times m}$, with $3 \leq n \leq m$, the minimal typical phaseless rank equals*

$$\left\lceil \frac{n+m-\sqrt{(n-1)^2+(m-1)^2}}{2} \right\rceil .$$

n \ m	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
4	2	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
5		3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
6			3	3	3	3	3	3	3	3	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
7				3	3	3	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
8					4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	5	5	5	5	5	5
9						4	4	4	4	4	4	4	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
10							4	4	4	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
11								4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	6	6	6	6	6	6
12									5	5	5	5	5	5	5	5	6	6	6	6	6	6	6	6	6	6	6	6
13										5	5	5	5	5	6	6	6	6	6	6	6	6	6	6	6	6	6	6
14											5	5	6	6	6	6	6	6	6	6	6	6	6	6	7	7	7	7
15												6	6	6	6	6	6	6	6	6	7	7	7	7	7	7	7	7
16													6	6	6	6	6	6	7	7	7	7	7	7	7	7	7	7
17														6	6	6	7	7	7	7	7	7	7	7	7	7	7	7
18															6	7	7	7	7	7	7	7	7	7	8	8	8	8
19																7	7	7	7	7	7	7	8	8	8	8	8	8
20																	7	7	7	7	8	8	8	8	8	8	8	8
21																		7	7	8	8	8	8	8	8	8	8	8
22																			8	8	8	8	8	8	8	8	9	9
23																				8	8	8	8	8	9	9	9	9
24																					8	8	9	9	9	9	9	9
25																						9	9	9	9	9	9	9
26																							9	9	9	9	9	9
27																								9	9	9	10	10
28																									9	10	10	10
29																										10	10	10
30																											10	10
31																												10

Table 3.1 For each (n,m), minimum k for which $P_k^{n\times m}$ is full dimensional. Only the upper triangular part is shown, as the table is symmetric, due to the fact that $\operatorname{rank}_\theta(A) = \operatorname{rank}_\theta(A^T)$.

3.3 Variants of phaseless rank

Recall that, given $A \in \mathbb{R}_+^{n\times m}$, its phaseless rank is defined as

$$\operatorname{rank}_\theta(A) = \min\{\operatorname{rank}(B) : B \in \Omega(A)\}.$$

Instead of taking the whole equimodular class as the feasible set of this optimization problem, one can opt for minimizing the rank over some of its subsets, which gives rise to phaseless rank variants. In this sense, the signless rank is itself a variant of phaseless rank, since the feasible set consists only of reals matrices in the equimodular class. Other natural variants arise if one shortens the original feasible set to those of hermitian or positive semidefinite matrices in the equimodular class. These last two quantities will be of particular interest to us and we will call them, respectively, hermitian and psd-phaseless rank:

Definition 3.3.1. Let $A \in \mathbb{R}_+^{n\times n}$ be a symmetric matrix. Its hermitian-phaseless rank and psd-phaseless rank are defined as

$$\operatorname{rank}_\theta{}^{\text{herm}}(A) = \min\{\operatorname{rank}(B) : B \in \Omega(A) \text{ and } B \text{ is hermitian}\}$$

$$\operatorname{rank}_\theta{}^{\text{psd}}(A) = \min\{\operatorname{rank}(B) : B \in \Omega(A) \text{ and } B \succeq 0\}.$$

Their real versions we call, in this order, hermitian and psd-signless rank.

Definition 3.3.2. Let $A \in \mathbb{R}_+^{n\times n}$ be a symmetric matrix. Its hermitian-signless rank and psd-signless rank are defined as

$$\operatorname{rank}_\pm{}^{\text{herm}}(A) = \min\{\operatorname{rank}(B) : B \in \Omega(A) \cap \mathbb{R}^{n\times n} \text{ and } B \text{ is symmetric}\}$$

$$\operatorname{rank}_\pm{}^{\text{psd}}(A) = \min\{\operatorname{rank}(B) : B \in \Omega(A) \cap \mathbb{R}^{n\times n} \text{ and } B \succeq 0\}.$$

By comparing the corresponding feasible sets, we infer the following list of inequalities for these quantities.

$$\operatorname{rank}_\theta(A) \le \operatorname{rank}_\theta{}^{\text{herm}}(A) \le \operatorname{rank}_\theta{}^{\text{psd}}(A), \quad \operatorname{rank}_\pm(A) \le \operatorname{rank}_\pm{}^{\text{herm}}(A) \le \operatorname{rank}_\pm{}^{\text{psd}}(A)$$
$$\operatorname{rank}_\theta{}^{\text{herm}}(A) \le \operatorname{rank}_\pm{}^{\text{herm}}(A), \quad \operatorname{rank}_\theta{}^{\text{psd}}(A) \le \operatorname{rank}_\pm{}^{\text{psd}}(A).$$

Note that these ranks can all be different.

Example 3.3.3. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & \frac{1}{2} \\ 1 & \frac{1}{2} & 1 \end{bmatrix}.$$

Since no column of A is lopsided, $\operatorname{rank}_\theta(A) = 2$.

Moreover,

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & \frac{1}{2}e^{it_1} \\ 1 & \frac{1}{2}e^{-it_1} & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & \frac{1}{2}e^{it_1} \\ 1 & \frac{1}{2}e^{-it_1} & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & \frac{1}{2}e^{it_1} \\ 1 & \frac{1}{2}e^{-it_1} & -1 \end{vmatrix} = -\frac{5}{4} + \operatorname{Cos}[t_1] \le -\frac{1}{4} < 0,$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & \frac{1}{2}e^{it_1} \\ 1 & \frac{1}{2}e^{-it_1} & -1 \end{vmatrix} = \frac{11}{4} + \operatorname{Cos}[t_1] \ge \frac{7}{4} > 0,$$

$$\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & \frac{1}{2}e^{it_1} \\ 1 & \frac{1}{2}e^{-it_1} & 1 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & \frac{1}{2}e^{it_1} \\ 1 & \frac{1}{2}e^{-it_1} & -1 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & \frac{1}{2}e^{it_1} \\ 1 & \frac{1}{2}e^{-it_1} & -1 \end{vmatrix} = \frac{5}{4} + \mathrm{Cos}[t_1] \geq \frac{1}{4} > 0, \text{ and}$$

$$\begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & \frac{1}{2}e^{it_1} \\ 1 & \frac{1}{2}e^{-it_1} & 1 \end{vmatrix} = -\frac{11}{4} + \mathrm{Cos}[t_1] \leq -\frac{7}{4} < 0.$$

This shows that $\mathrm{rank}_\theta{}^{\mathrm{herm}}(A) = 3$. Finally, because $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & \frac{1}{2}e^{it_1} \\ 1 & \frac{1}{2}e^{-it_1} & 1 \end{vmatrix}$ is always negative, there is no $B \succeq 0$ such that $|B| = A$, i.e., $\mathrm{rank}_\theta{}^{\mathrm{psd}}(A) = \infty$.

An interesting problem that arises is to characterize, for each $n \geq 2$, the set of symmetric matrices $A \in \mathbb{R}_+^{n\times n}$ that have finite psd-phaseless rank. In other words, to characterize the set

$$\{|B| : B \text{ is an } n \times n \text{ positive semidefinite matrix}\}.$$

Proposition 3.3.4. *For $n = 2, 3$, $\mathrm{rank}_\theta{}^{psd}(A) < \infty$ is equivalent to saying that A is positive semidefinite.*

Proof. Let Z and W be complex positive semidefinite matrices, hence having real nonnegative diagonal elements. We can always multiply any of their columns by a complex number of absolute value one and the respective row by its conjugate, operation which will preserve both positive semidefiniteness and the entrywise absolute value matrix. Hence, we can assume that both the first columns and rows of Z and W have real nonnegative entries, i.e., we have

$$Z = \begin{bmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{bmatrix} \text{ and } W = \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{12} & w_{22} & w_{23} \\ w_{13} & \overline{w_{23}} & w_{33} \end{bmatrix},$$

where only w_{23} is potentially a non-real number, every other number is a nonnegative real. Since $|Z| = Z$, we conclude that the entrywise absolute value matrix is positive semidefinite. All 2×2 submatrices of $|W|$ are thus also positive semidefinite, and it is enough to show that $\det(|W|) \geq 0$ to conclude that $|W|$ is positive semidefinite, according to Sylvester's criterion. But

$$\det(|W|) = w_{11}w_{22}w_{33} + 2w_{12}w_{13}|w_{23}| - w_{22}w_{13}^2 - w_{11}|w_{23}|^2 - w_{33}w_{12}^2$$

is clearly greater or equal than

$$\det(W) = w_{11}w_{22}w_{33} + 2w_{12}w_{13}\mathrm{Re}(w_{23}) - w_{22}w_{13}^2 - w_{11}|w_{23}|^2 - w_{33}w_{12}^2$$

because $\mathrm{Re}(w_{23}) \leq |w_{23}|$. Therefore, $\det(|W|) \geq \det(W) \geq 0$ and $|W| \succeq 0$, as intended. $\square$

It turns out that for $n = 4$ one can find examples of positive symmetric matrices that have finite psd-phaseless rank but are not themselves positive semidefinite.

Example 3.3.5. Consider the 4×4 positive real symmetric matrix

$$A = \begin{bmatrix} 1 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ \sqrt{\frac{2}{3}} & 1 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 1 & \sqrt{\frac{2}{3}} \\ \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 1 \end{bmatrix}.$$

This matrix is not positive semidefinite. To check that just note that its determinant is negative. However, the matrix

$$\tilde{A} = \begin{bmatrix} 1 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ \sqrt{\frac{2}{3}} & 1 & \sqrt{\frac{2}{3}}e^{-i\frac{\pi}{6}} & \frac{1}{\sqrt{3}}e^{-i\frac{\pi}{6}} \\ \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}}e^{i\frac{\pi}{6}} & 1 & \sqrt{\frac{2}{3}}e^{-i\frac{\pi}{6}} \\ \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}}e^{i\frac{\pi}{6}} & \sqrt{\frac{2}{3}}e^{i\frac{\pi}{6}} & 1 \end{bmatrix}$$

is positive semidefinite and satisfies $|\tilde{A}| = A$. In fact, $\tilde{A}$ has rank 2, so $\operatorname{rank}_\theta{}^{\mathrm{psd}}(A) = 2$.

3.3.1 Equiangular lines

A set of n lines in the vector space $\mathbb{R}^d$ or $\mathbb{C}^d$ is called equiangular if all the lines intersect at a single point and every pair of lines makes the same angle. Without loss of generality, we admit that each line passes through the origin of the vector space. Since each line is a one-dimensional subspace, it can be represented by a unit vector, v_i, $i = 1, \ldots, n$, with $||v_i|| = 1$. Moreover, the condition that every pair of lines makes the same angle is equivalent to imposing the constraint $|\langle v_i, v_j \rangle| = \alpha$, for all $1 \leq i, j \leq n$ s.t. $i \neq j$, where α, a constant between zero and one, is the cosine of the common angle.

Let V denote the matrix whose column i is $v_i, i = 1, \ldots, n$, and A_n^α denote the $n \times n$ matrix with ones on the diagonal and α's everywhere else. We have $|V^*V| = A_n^\alpha$, in the complex case, and $|V^TV| = A_n^\alpha$, in the real case. It is not hard to see that an equiangular set of n lines with common angle $\arccos \alpha$ exists in $\mathbb{R}^d$ (resp. $\mathbb{C}^d$) if and only if there is a real (resp. complex) positive semidefinite matrix B of rank at most d such that $|B| = A_n^\alpha$. This immediately relates the existence of equiangular lines to psd-phaseless rank and psd-signless rank.

Proposition 3.3.6. *For $0 \leq \alpha \leq 1$, $\operatorname{rank}_\theta{}^{psd}(A_n^\alpha)$ (resp. $\operatorname{rank}_\pm{}^{psd}(A_n^\alpha)$) is the smallest dimension d for which there exists an equiangular set of n lines in $\mathbb{C}^d$ (resp. $\mathbb{R}^d$) with common angle $\arccos \alpha$.*

This proposition easily follows from the equalities below.

$$\operatorname{rank}_\theta{}^{\mathrm{psd}}(A) = \min\{\operatorname{rank}(B) : B \in \Omega(A), B \succeq 0\} = \min\{k : X^*X \in \Omega(A), X \in \mathbb{C}^{k\times n}\},$$
$$\operatorname{rank}_\pm{}^{\mathrm{psd}}(A) = \min\{\operatorname{rank}(B) : B \in \Omega(A) \cap \mathbb{R}^{n\times n}, B \succeq 0\} = \min\{k : X^TX \in \Omega(A), X \in \mathbb{R}^{k\times n}\}.$$

Bounding the maximum number of real equiangular lines for a given dimension has long been a popular research problem. Classically, we want bounds on the absolute maximum number of such lines (denoted by $N(d)$) or on the maximum number for a given common angle $\arccos(\alpha)$ (denoted by $N_\alpha(d)$). A somewhat thorough survey on this type of results can be found in [19], while further reading on the real case can be seen in [39], [42], and [47]. The most general bound for $N(d)$, known as the absolute bound, was proved by Gerzon:

Theorem 3.3.7 (Gerzon [47])**.** *For every $d \geq 2$,*

$$N(d) \leq \frac{d(d+1)}{2},$$

and if equality holds then $d = 2, 3$, or $d+2$ is the square of an odd integer.

This bound can be rewritten in terms of the psd-signless rank. In fact, since $\operatorname{rank}_\pm{}^{\text{psd}}(A_n^\alpha)$ is the smallest dimension d for which there exists an equiangular set n lines in $\mathbb{R}^d$ with common angle $\arccos\alpha$, it follows from Theorem 3.3.7 that

$$n \leq N(\operatorname{rank}_\pm{}^{\text{psd}}(A_n^\alpha)) \leq \frac{\operatorname{rank}_\pm{}^{\text{psd}}(A_n^\alpha)(\operatorname{rank}_\pm{}^{\text{psd}}(A_n^\alpha)+1)}{2},$$

which implies, for any positive integer n and $0 \leq \alpha < 1$,

$$\operatorname{rank}_\pm{}^{\text{psd}}(A_n^\alpha) \geq \frac{\sqrt{1+8n}-1}{2}.$$

Conversely, Gerzon's bound can be recovered using the above inequality. For a fixed d, and because $N(d)$ is attained for some α, say α_d, we have

$$\operatorname{rank}_\pm{}^{\text{psd}}\left(A_{N(d)}^{\alpha_d}\right) \geq \frac{\sqrt{1+8N(d)}-1}{2}.$$

By rearranging the inequality in terms of $N(d)$ and using the fact that $\operatorname{rank}_\pm{}^{\text{psd}}\left(A_{N(d)}^{\alpha_d}\right) \leq d$, we obtain

$$N(d) \leq \frac{\operatorname{rank}_\pm{}^{\text{psd}}\left(A_{N(d)}^{\alpha_d}\right)\left(\operatorname{rank}_\pm{}^{\text{psd}}\left(A_{N(d)}^{\alpha_d}\right)+1\right)}{2} \leq \frac{d(d+1)}{2}.$$

Theorem 3.3.8 (Gerzon - rank reformulation)**.** *For any positive integer n and $0 \leq \alpha < 1$,*

$$\operatorname{rank}_\pm{}^{psd}(A_n^\alpha) \geq \frac{\sqrt{1+8n}-1}{2}.$$

Note that, for $0 \leq \alpha < 1$, Gerzon's bound is a particular case of the basic dimension bound for the signless rank presented in Lemma 1.1.5:

$$\operatorname{rank}_\pm{}^{\text{psd}}(A_n^\alpha) \geq \operatorname{rank}_\pm(A_n^\alpha) \geq \frac{\sqrt{1+8r}-1}{2}, \text{ where } r = \operatorname{rank}(A_n^\alpha \circ A_n^\alpha)$$

Because $\det(A_n^\alpha \circ A_n^\alpha) = 0$ implies $\alpha = \pm 1$, then, for $0 \leq \alpha < 1$, $\operatorname{rank}(A_n^\alpha \circ A_n^\alpha) = n$. Note that the rank reformulation lower bound uses only the signless rank, and makes no use of the positive semidefinite requirement.

In [19] one can find a brief overview of other existing bounds for $N_\alpha(d)$ for specific values of α. The same ideas as above allow a translation from these bounds into bounds of the psd-signless rank of certain matrices A_n^α. More interestingly, bounds on the psd-signless rank immediately translate into bounds for the problem of equiangular lines.

Regarding the complex equiangular lines case, it has seen a flurry of recent developments due to its connection to quantum physics (see for instance [4],[38],[65],[67]). The maximum number of complex equiangular lines in $\mathbb{C}^d$, denoted by $N^{\mathbb{C}}(d)$, is bounded from above by d^2:

Proposition 3.3.9 ([31])**.** $N^{\mathbb{C}}(d) \leq d^2$.

Additionally, it is conjectured that $N^{\mathbb{C}}(d) = d^2$ for all $d \geq 2$ [74]. When such a maximum set of d^2 lines exists, one can construct a symmetric, informationally complete, positive operator-valued measure (SIC-POVM), an object that plays an important role in quantum information theory. Recent developments in the construction of large sets of complex equiangular lines can be found in [40] and [41]. Again, the upper bound for $N^{\mathbb{C}}(d)$ can be restated in terms of the psd-phaseless rank:

Proposition 3.3.10 (Proposition 3.3.9 - rank reformulation)**.** *For any positive integer n and* $0 \leq \alpha < 1$*,*

$$\operatorname{rank}_\theta{}^{psd}(A_n^\alpha) \geq \sqrt{n}.$$

This proposition also follows from Lemma 1.1.5. To illustrate this strategy of turning lower bounds on phaseless rank into upper bounds on the number of equiangular lines, we present a simple result derived from our basic bounds on phaseless rank.

Proposition 3.3.11. *For* $\alpha < \frac{1}{d}$*,* $N_\alpha^{\mathbb{C}}(d) = d$*.*

Proof. Fix d and let $\alpha < \frac{1}{d}$. Observe that one can write $N_\alpha^{\mathbb{C}}(d)$ as

$$\max\{n : \operatorname{rank}_\theta{}^{\mathrm{psd}}(A_n^\alpha) \leq d\}.$$

Because A_{d+1}^α is strictly diagonally dominant and is a submatrix of any A_n^α for $n > d$, we have

$$\operatorname{rank}_\theta{}^{\mathrm{psd}}(A_n^\alpha) \geq \operatorname{rank}_\theta(A_n^\alpha) \geq d+1$$

for any $n > d$. Since A_d^α is positive semidefinite and has rank d, the result follows. ☐

While fairly simple, this result highlights the usefulness of deriving effective lower bounds to the phaseless rank, as a means to obtain upper bounds to $N_\alpha^{\mathbb{C}}(d)$.

A related classical concept that can be studied in terms of psd-phaseless rank is that of mutually unbiased bases in $\mathbb{C}^d$ (MUB's). Two orthonormal bases $\{u_1,\ldots,u_d\}$ and $\{v_1,\ldots,v_d\}$ of $\mathbb{C}^d$ are said to be unbiased if $|u_i^* v_j| = \frac{1}{\sqrt{d}}$ for all i and j. A set of orthonormal bases is a set of mutually unbiased bases if all pairs of distinct bases are unbiased. It is known that there cannot exist sets of more than $d+1$ MUB's in $\mathbb{C}^d$, and such sets exist for d a prime power, but the precise maximum number is unknown even for $d=6$, where it is believed to be three (see [21], [6] and [14] for more information and a survey into this rich research area). To translate this in terms of phaseless rank, consider the matrix B_d^k defined as the matrix of $k \times k$ blocks where the blocks in the diagonal are $d \times d$ identities and the off-diagonal ones are constantly equal to $\frac{1}{\sqrt{d}}$. The following simple fact is then clear.

Proposition 3.3.12. *There exists a set of k mutually unbiased bases in* $\mathbb{C}^d$ *if and only if* $\operatorname{rank}_\theta{}^{psd}(B_d^k) = d$.

As in equiangular lines, lower bounds on the phaseless rank have the potential to give upper bounds on the maximum number of MUB's.

Chapter 4

Phase rank

4.1 Notation, definitions, and basic properties

In the previous chapters we studied the problem of minimizing the rank of a complex matrix when we are given the modulus of the entries but not their phases. This immediately suggests a complementary problem, that of minimizing the rank when given only the phases of the entries, but not their modulus. In this chapter we will explore this problem and the notion of phase rank that is associated to it.

We will be interested in *phase matrices*, that is to say, matrices whose entries are complex numbers of modulus one. We now want to minimize the rank among all complex matrices who have this entrywise phases. More precisely, we are interested in the following quantity.

Definition 4.1.1 (Phase rank). Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and Θ be a phase matrix, i.e., with entries in S^1. We define its phase rank as

$$\operatorname{rank}_{\text{phase}}(\Theta) = \min\{\operatorname{rank}(M) : M \in (\mathbb{C}^*)^{n\times m}, \frac{M_{ij}}{|M_{ij}|} = \Theta_{ij}, \forall i, j\}.$$

An alternative way of defining this quantity is

$$\operatorname{rank}_{\text{phase}}(\Theta) = \min\{\operatorname{rank}(M \circ \Theta) : M \text{ is a matrix with positive entries }\},$$

where $\circ$ denotes the Hadamard product of matrices.

Example 4.1.2. Let

$$\Theta = \begin{bmatrix} 1 & 1 & 1 \\ 1 & i & -i \\ 1 & 1 & i \end{bmatrix} \text{ and } M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \sqrt{3}i & -\sqrt{3}i \\ 1 & 3 & \sqrt{3}i \end{bmatrix}.$$

Since $\operatorname{rank}(M) = 2, \operatorname{rank}_{\text{phase}}(\Theta) \leq 2$.

By definition, the matrices that we are considering in our minimization problem have nonzero entries, since zero does not have a well-defined phase. On could consider *generalized phase matrices*, whose entries are allowed to be either complex numbers of modulus one or zero. We can generalize the notion of phase rank to this larger class by saying that for a generalized phase matrix $\bar{\Theta}$ its phase rank is the smallest rank of a matrix with the same support as $\bar{\Theta}$ and whose nonzero entries have the same phase as the corresponding entries of $\bar{\Theta}$. In this chapter, however, we will focus on the phase ranks of proper phase matrices, and not consider the generalized case.

It is clear by construction that $\operatorname{rank}_{\text{phase}}(\Theta) \leq \operatorname{rank}(\Theta)$ is a trivial upper bound on the phase rank. However, unlike what we were able to do in the phaseless rank case, it is much harder to find even a trivial lower bound. We will derive lower bounds in Section 4.6, but they result from a much more involved reasoning than the simple dimensional arguments used in the phaseless rank. This immediately suggests that the phase rank might be even harder to compute than the phaseless rank. The rank one case remains trivial, though.

Proposition 4.1.3. *For any phase matrix* Θ*,* $\operatorname{rank}_{phase}(\Theta) = 1$ *if and only if* $\operatorname{rank}(\Theta) = 1$.

Proof. Since $\operatorname{rank}_{\text{phase}}(\Theta) \leq \operatorname{rank}(\Theta)$, the result follows immediately if one can show that $\operatorname{rank}_{\text{phase}}(\Theta) = 1$ implies that $\operatorname{rank}(\Theta) = 1$. But if M is the rank one matrix with the same phases as Θ, we have $M = vw^t$ for some complex vectors v and w. Let $\bar{v}$ and $\bar{w}$ be the vectors whose entries are the phases of v and w, respectively. It is trivial to see that $\Theta = \bar{v}\bar{w}^t$, which completes the proof. $\square$

This means that the simplest nontrivial case is that of determining if a 3×3 matrix has phase rank two, a case that we will completely characterize in Section 4.3.

A last remark is that we can restrict our study of phase rank to phase matrices whose entries in the first row and column are all ones. This can be done since multiplying a row or a column of a matrix by $e^{i\theta}$, for some θ, preserves the rank. This will be done throughout the chapter, in order to simplify the exposition.

Like in the case of phaseless rank, there is a natural version of phase rank on the reals, the sign rank. The history on that topic goes back to the work in the 1980s in communication complexity [1, 58], but has antecedents in older pioneer work on the problem of *signsolvability*: when is every matrix with a given generalized sign pattern guaranteed to be invertible, a problem that goes back to the 1960s (see for instance [43, 52]). This whole area of research has seen a renewed interest in the last few years in both the mathematical and the computer science communities and several important developments have appeared [2, 5, 9, 62]. The basic facts of this area will be introduced later in this chapter.

There are also antecedents to our study of the complex case. The signsolvability problem has been extended to the complex case to what is called the ray nonsingularity problem: when is every matrix with the same phases of a given square generalized phase matrix (in that literature named a *ray pattern*) invertible? This question has been exploited and essentially solved for (nongeneralized) phase matrices in a series of papers at the turn of the millenium [44, 49, 53] and has seen a few subsequent developments. It seems, however, that the step from nonsingularity to a proper notion of rank has not ever been taken in the literature. In this chapter we will establish the basics of this notion, connect it to the existent literature in sign rank and ray nonsingularity and also to the very relevant topic of algebraic coamoebas. We will also present some basic bounds on phase rank, following the existing work in these topics. To carry this out, we will start by introducing the relevant notions of the theories of sign rank and coamoebas.

4.2 Associated concepts

4.2.1 Sign rank

Sign matrices, with ± 1 entries, arise naturally in many areas of research. For instance, they are used to represent set systems and graphs in combinatorics, hypothesis classes in learning theory, and boolean functions in communication complexity. The minimum rank of a matrix with a given sign pattern has several important interpretations in these fields and has attracted a high level of interest in the last decade.

Definition 4.2.1. For a real matrix M with no zero entries, let $\mathrm{sign}(M)$ denote the sign matrix such that $(\mathrm{sign}(M))_{ij} = \mathrm{sign}(M_{ij})$, for all i, j. The sign rank of a sign matrix S is defined as

$$\text{sign-rank}(S) = \min\{\mathrm{rank}\,(M) : \mathrm{sign}(M) = S\}.$$

Sign matrices are instances of phase matrices and it is straightforward to see that sign rank is the restriction of phase rank to sign matrices:

$$\mathrm{rank}_{\,\mathrm{phase}}(S) = \text{sign-rank}(S), \text{ for any sign matrix } S.$$

Sign rank computation is typically extended to matrices with entries in $\{-1, 0, 1\}$, which we call generalized sign matrices, where the zeros in the sign pattern force the corresponding entries to be zero.

Computing sign rank is hard. More precisely, in [9] it is shown that proving that the sign rank of an $n \times m$ sign matrix is at most k is NP-hard for $k \geq 3$. In particular, computing phase rank is also NP-hard, as the sign rank is the restriction of phase rank to a specific set of matrices. For $k = 2$ they provide a polynomial time algorithm, which is not obviously generalizable to the complex case, suggesting a first question.

Question 4.2.2. *Is there a polynomial time algorithm to decide if an $n \times m$ phase matrix has phase rank at most 2?*

Checking if a generalized sign matrix has maximal sign rank is an interesting problem on its own, known also as the signsolvability problem. It is NP-hard [43] and has a very simple characterization, very similar in spirit to our Lemma 1.3.1.

Lemma 4.2.3 (Remark 1.1. [43])**.** *For an $n \times m$ generalized sign matrix S, with $m \geq n$, sign-rank$(S) = n$ if and only if every scaling of its rows by scalars in $\{-1,0,1\}$ has a unisigned nonzero column, i.e., a column that has only zeros and 1's or -1's but not both.*

In particular, a nongeneralized $n \times m$ sign matrix has maximal sign rank if and only if for any vector $x \in \{-1,1\}^n$ either x or $-x$ appears in its columns, which immediately implies that $n \times m$ sign matrices have nonmaximal sign rank for $m < 2^{n-1}$. Therefore, square $n \times n$ nongeneralized sign matrices cannot be sign-nonsingular for $n \geq 3$. In fact, Alon et al [1] show that the maximum sign rank of an $n \times n$ sign matrix is lower-bounded by $\frac{n}{32}$ and upper-bounded by $\frac{n}{2}(1+o(1))$.

Since computing sign rank is generally hard, a lot of effort was put into devising effective bounds for this quantity. For example, a well-known lower bound on the sign rank of an $m \times n$ sign matrix is due to Forster [29]:

Theorem 4.2.4. *If S is an $m \times n$ sign matrix, then*

$$\textit{sign-rank}(S) \geq \frac{\sqrt{mn}}{||S||},$$

where $||S||$ is the spectral norm of S.

Alon et al. [2] observed that Forster's proof argument works as long as the entries of the matrix are not too close to zero, so we can replace $||S||$ with $||S||^* = \min\{||M|| : M_{ij}S_{ij} \geq 1 \text{ for all } i,j\} = \min\{||M|| : \text{sign}(M) = S \text{ and } |M_{ij}| \geq 1\, \forall i,j\}$ in the above bound, which constitutes an improvement, since $||S||^* \leq ||S||$. Another improvement to Forster's bound, through a different approach, can be found in [50] and it is based on the factorization of linear operators (see A.2 for the meaning of $\gamma_2^*(S)$ and for details on how to compute this quantity).

Theorem 4.2.5 ([50])**.** *For every $m \times n$ sign matrix S,*

$$\textit{sign-rank}(S) \geq \frac{mn}{\gamma_2^*(S)}.$$

Example 4.2.6. Let

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 1 \end{bmatrix}.$$

For this matrix we have $\frac{\sqrt{mn}}{||S||} = 1.7990$ and $\frac{mn}{\gamma_2^*(S)} = 2.0261$. Thus,

$$\text{sign-rank}(S) \geq 3.$$

Given a particular sign matrix S, there are not that many tools to upper bound its sign-rank, besides the ones that are given by any explicit matrix with the given sign pattern. A general bound, based only on the dimension of the matrix, can be obtained using the probabilistic method.

Theorem 4.2.7 (Theorem 13.3.1 [3])**.** *For any $m \times n$ sign matrix S,*

$$\textit{sign-rank}(S) \leq \frac{\min\{m,n\}+1}{2} + \sqrt{\frac{\min\{m,n\}-1}{2}\log(\max\{m,n\})}.$$

One can see that the above inequality is not tight. For $n = m = 3$, for instance, the sign rank of any sign matrix is at most 2, but the bound only guarantees that it is at most 3. It is believed, however, that asymptotically this bound is near optimal.

4.2.2 Coamoebas of determinantal varieties

In Section 1.2.2, we talked about the concept of amoeba of an algebraic variety and its connection to the notion of phaseless rank. Here, we introduce the notion of coamoeba of an algebraic variety, that will play a similar role in the study of phase rank. Just as what we did to move from phaseless rank to phase rank, coamoebas are the notion we get when switching the roles of absolute values and phases in the notion of amoebas.

More precisely, coamoebas are the image of varieties under taking the entrywise arguments, that is, under the map

$$\text{Arg} : (\mathbb{C}^*)^n \longrightarrow (S^1)^n, z = (z_1, \ldots, z_n) \to \left(\frac{z_1}{|z_1|}, \ldots, \frac{z_n}{|z_n|}\right).$$

We will identify S^1 with the interval $[0, 2\pi)$ in the usual way for purposes of plotting.

Definition 4.2.8. Given a complex variety $V \subseteq \mathbb{C}^n$, its *coamoeba* is the set

$$\text{co}\mathscr{A}(V) = \{\text{Arg}(z) = \left(\frac{z_1}{|z_1|}, \ldots, \frac{z_n}{|z_n|}\right) : z \in V \cap (\mathbb{C}^*)^n\}.$$

Coamoebas were introduced by Mikael Passare in talks in the early 2000s as a dual notion to that of amoebas. There has been a growing body of research on these objects, a small selection of influential works are [27, 54, 55]. These tend to be harder objects to grasp than amoebas, but many versions of the results for amoebas hold true for coamoebas.

Example 4.2.9. The simplest example of a coamoeba would be the coamoeba of a line. Consider the variety in $\mathbb{C}^2$ given by $y = 1 + x$. Its points have the form $(z, 1 + z)$, for all $z \in \mathbb{C}$, so its coamoeba is the set

$$\mathrm{co}\mathscr{A}(V) = \{(\mathrm{Arg}(z), \mathrm{Arg}(1+z)), z \in \mathbb{C}\}$$

which is the same as the set

$$\{(\theta, \mathrm{Arg}(1+\rho e^{i\theta})), \rho \in \mathbb{R}_+, \theta \in [0, 2\pi)\}.$$

It is easy to see that if θ is constant and in the interval $(0,\pi)$, then, as ρ varies, the second entry will vary in $(0,\theta)$. If θ is in the interval $(\pi, 2\pi)$, then the second entry will vary in $(\theta, 2\pi)$. Furthermore, if θ is zero, the second entry is zero, while if θ is π, it can be either zero or π, depending on ρ. Putting everything together we obtain the region depicted in Figure 4.1.

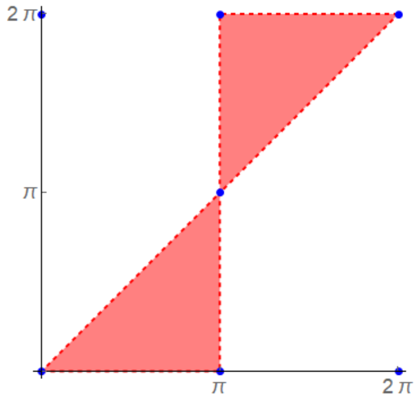

Fig. 4.1 Coamoeba of the line $y = 1 + x$.

When interpreting the figure, one can think of the plotted square as a torus: the left side is identified with the right side and the bottom with the top.

From the above example one can immediately see that, contrary to amoebas, coamoebas are not necessarily closed. However, coamoebas share with amoebas the fact that they can be expressed as the intersection of the coamoebas of all polynomials in the ideal, i.e.,

$$\mathrm{co}\mathscr{A}(V(I)) = \bigcap_{f \in I} \mathrm{co}\mathscr{A}(V(f)).$$

The connection between phase rank and coamoebas is clear and totally analogous to that developed in Section 1.2.2 for phaseless rank. Recall that given positive integers n, m and k, with $k \leq \min\{n, m\}$, we defined the determinantal variety $Y_k^{n,m}$ as the set of all $n \times m$ complex matrices of rank at most k, cut out by the $k+1$ minors of an $n \times m$ matrix of variables. Directly from the definition of coamoeba, we have that the locus of $n \times m$ matrices of phase rank at most k is the coamoeba of the determinantal variety $Y_k^{n,m}$. Computing phase rank is therefore a special case of the problem of checking coamoeba membership.

Both amoeba and coamoeba membership are hard to check. Similarly to lopsidedness condition for amoeba membership, there is a necessary condition for coamoeba membership, called colopsidedness condition ([27],[20],[26]).

Definition 4.2.10. A sequence $z_1, z_2, \ldots, z_k \in \mathbb{C}$ is colopsided if, when considered as points in $\mathbb{R}^2$, 0 is not in the relative interior of their convex hull.

Given a finite multivariate complex polynomial $f(z) = \sum_\alpha b_\alpha z^\alpha \in \mathbb{C}[z_1, \ldots, z_n]$, $b_\alpha \neq 0$, and $\phi \in (S^1)^n$, we say that f is colopsided at ϕ if the sequence

$$\left(\frac{b_\alpha}{|b_\alpha|}\phi^\alpha\right)_\alpha$$

is colopsided.

For $S \subseteq \mathbb{R}^2$, we will denote its convex hull by $\mathrm{Conv}(S)$. In practice, 0 is in the relative interior of $\mathrm{Conv}(S)$ if it can be written as a convex combination of the points in S with all the coefficients strictly positive.

It is easy to see the following relation between colopsidedness and coamoebas.

Lemma 4.2.11. *If $\phi \in co\mathscr{A}(V(I))$, then, for any $f \in I$, f is not colopsided at ϕ.*

Proof. Take $w \in V(I) \cap (\mathbb{C}^*)^n$ such that $\mathrm{Arg}(w) = \phi$ and $f \in I$. Since $f(w) = 0$,

$$\sum_\alpha b_\alpha w^\alpha = \sum_\alpha |w|^\alpha |b_\alpha| \frac{b_\alpha}{|b_\alpha|}\phi^\alpha = 0.$$

By dividing both sides by $\sum_\alpha |w|^\alpha |b_\alpha|$, one concludes that 0 is in the relative interior of the convex hull of $\left(\frac{b_\alpha}{|b_\alpha|}\phi^\alpha\right)_\alpha$. ☐

Note that the converse statement is, in general, not true. Not being colopsided for any $f \in I$ does not necessarily imply membership in the coamoeba $\mathrm{co}\mathscr{A}(V(I))$.

4.3 Nonmaximal phase rank

As mentioned earlier in this chapter, the problem of determining under which conditions is the phase rank of a generalized phase matrix nonmaximal has been studied previously in the literature, under the name of ray nonsingularity [53], with a focus mostly on the square case. In this section we will mainly present a survey of the implications of that work in the study of phase rank, with a few additional insights.

The first important observation is that we still have a version of Lemmas 1.3.1 and 4.2.3.

Lemma 4.3.1 ([53])**.** *An $n \times m$ phase matrix Θ, with $m \geq n$, has $\mathrm{rank}_{phase}(\Theta) < n$ if and only if there is a scaling of its rows by scalars in $S^1 \cup \{0\}$, not all zero, such that no column is colopsided.*

Proof. Suppose $\operatorname{rank}_{\text{phase}}(\Theta) < n$. This means there exists a matrix M with $M_{ij} = |M_{ij}|\Theta_{ij}$ for all i, j and $\operatorname{rank}(M) < n$. So the rows of M are linearly dependent, which implies that there are $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, not all zero, such that $\sum_{i=1}^n \lambda_i M_{ij} = 0$ for all j. Let $\tilde{\Theta}$ be the matrix whose (i, j)th entry is $\frac{\lambda_i}{|\lambda_i|}\Theta_{ij}$ if $\lambda_i \neq 0$ and zero otherwise. Then, for all j,

$$\sum_{i=1}^{n} |M_{ij}||\lambda_i|\tilde{\Theta}_{ij} = \sum_{i=1}^{n} \lambda_i M_{ij} = 0.$$

Therefore, if for each j we divide both sides by $\sum_{i=1}^n |M_{ij}||\lambda_i|$, we conclude that zero is in the relative interior of the convex hull of the entries in column j of $\tilde{\Theta}$, i.e., column j of $\tilde{\Theta}$ is not colopsided.

Suppose now that there is a row scaling of Θ by scalars $\lambda_1, \ldots, \lambda_n$ in $S^1 \cup \{0\}$, not all zero, such that no column is colopsided. This means that 0 is in the relative interior of the scaled columns of Θ, which implies that for every $j = 1, \ldots, m$ there are positive numbers m_{ij} such that

$$\sum_{i=1}^{n} m_{ij}\lambda_i\Theta_{ij} = 0.$$

Hence, the matrix with entries $M_{ij} = m_{ij}\Theta_{ij}$ has linearly dependent rows (and thus rank less than n), which means $\operatorname{rank}_{\text{phase}}(\Theta) < n$. $\square$

Example 4.3.2. Consider the matrix

$$\Theta = \begin{bmatrix} 1 & 1 & 1 \\ i & e^{i\frac{\pi}{4}} & e^{i2\frac{\pi}{3}} \\ -i & e^{i\frac{7\pi}{6}} & e^{i4\frac{\pi}{3}} \end{bmatrix}.$$

The convex hulls of the entries in each column are shown in Figure 4.2. Note that 0 is in the convex hull of the entries in the first column but not in its (relative) interior, so this column is colopsided.

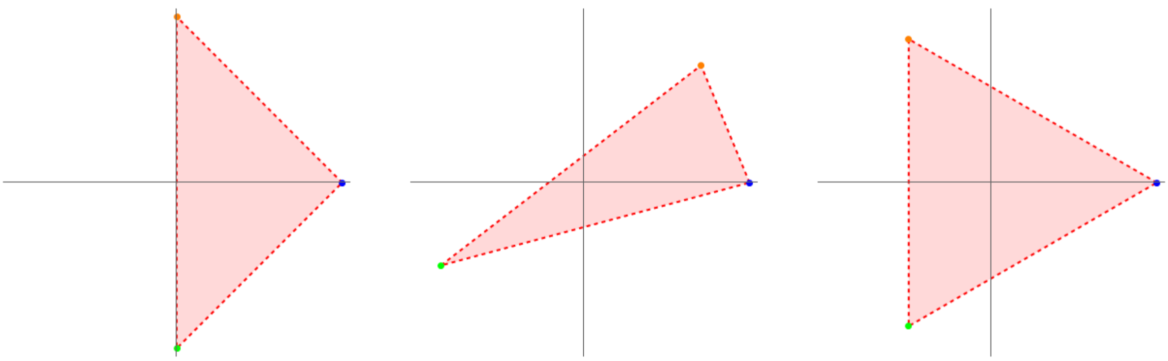

Fig. 4.2 Convex hulls of the entries of the columns of Θ

Scaling a row of Θ by $e^{i\theta}$ ($\theta > 0$) corresponds to rotating counterclockwise the points in the diagrams associated to that row through an angle of θ. If we multiply the second row of Θ by $e^{i\frac{\pi}{4}}$, that is, if we rotate counterclockwise through an angle of $\frac{\pi}{4}$ the points in orange, we see that 0 will be in the interior of all convex hulls (see Figure 4.3).

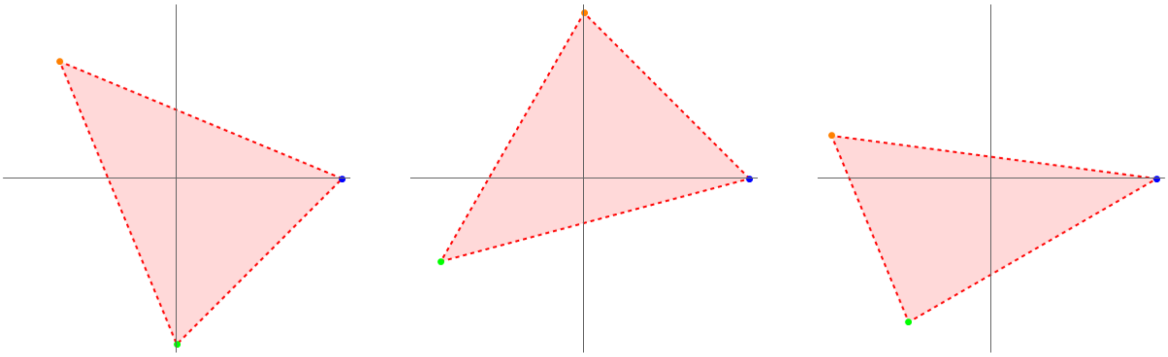

Fig. 4.3 Convex hulls of the entries of the columns of Θ after scaling the second row by $e^{i\frac{\pi}{4}}$.

The criterion in Lemma 4.3.1 is not necessarily easy to use, as searching for such a noncolopsided scaling is a nonconvex problem. There is another geometric way of thinking on this criterion. Consider an $n \times m$ phase matrix Θ and suppose that any submatrix obtained by erasing one of its rows has maximal phase rank (otherwise Θ is phase rank deficient). This is equivalent to asserting that in the row scalings of Lemma 4.3.1 we only need to use scalars in S^1, as we will not zero out any row. Define

$$\mathrm{Colop}(n) = \{y \in (S^1)^n \ : \ (y_1, \ldots, y_n) \text{ is colopsided}\}.$$

The condition that all row scalings have a colopsided column is then easy to characterize geometrically using this set.

Proposition 4.3.3. *Let Θ be an $n \times m$ phase matrix, with $m \geq n$, and assume that any submatrix obtained from Θ by erasing one of its rows has phase rank $n-1$. Then,* $\mathrm{rank}_{phase}(\Theta) = n$ *if and only if*

$$\cup_{j=1}^{m} (Colop(n)/\Theta^j) = (S^1)^n$$

where Θ^j is the j-th column of Θ and the division operation considered is the entrywise division.

Proof. Notice that

$$\mathrm{Colop}(n)/\Theta^j = \{y \in (S^1)^n : (y_1\Theta_{1j}, y_2\Theta_{2j}, \ldots, y_n\Theta_{nj}) \text{ is colopsided}\}$$

is the set of all nonzero scalings that will make column j colopsided. The union of all such sets is the collection of all scalings that will make some column colopsided, so if it does not cover the whole set of possible scalings, there will be a scaling that satisfies the conditions of Lemma 4.3.1. $\square$

Example 4.3.4. When computing the phase rank of a phase matrix one may always assume the matrix first row to be all 1's. Moreover, since multiplying all scalars by a common scalar in S^1 does not change the colopsidedness of columns, we may also assume that the scalars vector in Proposition

4.3.3 has the first entry equal to one. For $n = 3$, we can then think of the elements of Colop(3) as pairs of angles (see Figure 4.4). Although we usually identify S^1 with $[0, 2\pi)$, here we plotted several fundamental domains to better visualize the pattern and the toric nature of this object. Note further that the vertices of the hexagons are actually not in Colop(3).

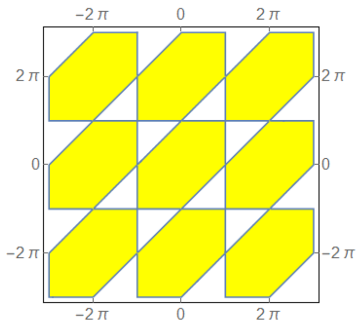

Fig. 4.4 Representation of Colop(3).

Given a $3 \times m$ matrix Θ, we can easily check if there is any $2 \times m$ submatrix of nonmaximal phase rank, since that is equivalent to having usual rank 1, by Proposition 4.1.3. So we can see if we are in the conditions of Proposition 4.3.3, and checking if $\text{rank}_{\text{phase}}(\Theta) = n$ is the same as checking if the geometric translations of the region in Figure 4.4 by the symmetric of the angles of the columns of Θ cover the entire space. Let Θ be the matrix from Example 4.3.2. We have to consider the translations $\text{Colop}(3) - (\pi/2, -\pi/2)$, $\text{Colop}(3) - (\pi/4, 7\pi/6)$ and $\text{Colop}(3) - (2\pi/3, 4\pi/3)$. We get the regions shown in the left of Figure 4.5, which do not cover the whole space, so $\text{rank}_{\text{phase}}(\Theta) < 3$.

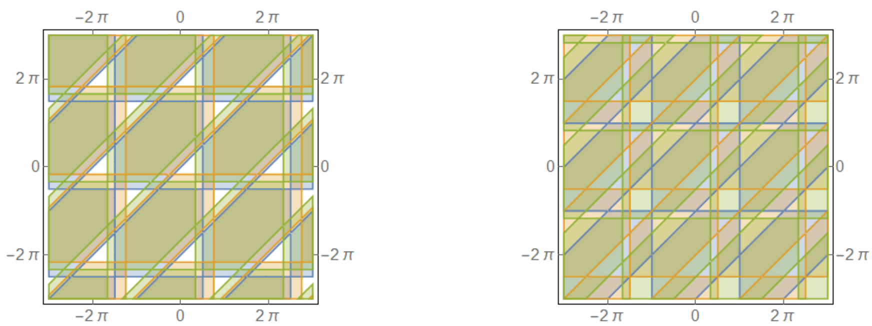

Fig. 4.5 Representation of the translations of Colop(3) for two different matrices.

It is not hard to find examples where the translations do cover the entire space. For instance, for

$$\Theta = \begin{bmatrix} 1 & 1 & 1 \\ 1 & i & e^{i2\frac{\pi}{3}} \\ 1 & -i & e^{i\frac{\pi}{6}} \end{bmatrix},$$

we obtain the region shown in the right of Figure 4.5, that indeed covers the whole space, so this matrix has phase rank 3.

Note that, although in the previous example we saw that Colop(3) occupies a large portion of $(S^1)^3$, this ceases to be true as n increases. In fact, if we think of Colop(n) as a subset of $[0,2\pi)^n$, it is not hard to compute its volume.

Lemma 4.3.5. *For $n \geq 2$, the volume of Colop(n) seen as a subset of $[0,2\pi)^n$ is $2n\pi^n$.*

Proof. A set Γ of n angles (defined up to additions of multiples of 2π) is colopsided (seen as a subset of $(S^1)^n$) if they are all in the same halfspace, i.e., if there is some $\alpha \in \Gamma$ such that every other angle in Γ is in $[\alpha, \alpha+\pi]$. There are n choices for α so we get the volume

$$n \int_0^{2\pi} \int_{[\alpha,\alpha+\pi]^{n-1}} dV d\alpha = 2n\pi^n.$$

□

This immediately leads to a sufficient condition for nonsingularity, based on this lemma and Lemma 4.3.1.

Proposition 4.3.6 ([44])**.** *If $n \leq m < \frac{2^{n-1}}{n}$, then $\text{rank}_{phase}(\Theta) < n$ for every $n \times m$ phase matrix Θ.*

Proof. Let Θ be an $n \times m$ phase matrix. Just note that the volume of $\cup_{j=1}^m(\text{Colop}(n)/\Theta^j)$ is at most m times the volume of Colop(n). If that is not enough to cover $(S^1)^n$, which has volume $(2\pi)^n$, that means there is a set of scalars that do not make any column of Θ colopsided. Therefore, by Lemma 4.3.1, the matrix is phase rank deficient. This happens if

$$m2n\pi^n < (2\pi)^n,$$

which gives us the intended result. □

This result was proven in [44] using a similar but distinct argument that used a combinatorial gadget instead of this one based on volumes. A nice thing about the argument presented here is that it makes clear that the bound is somewhat conservative: in order to have a $n \times \frac{2^{n-1}}{n}$ phase matrix with full rank, we would need $\frac{2^{n-1}}{n}$ translated copies of Colop(n) that covered the entire space without overlapping in any positive volume set. This is a very restrictive condition. In general, one would expect that many more copies than this will be needed to actually cover the space. This is, however, enough to take care of most square cases.

Corollary 4.3.7. *For $n \geq 7$, no $n \times n$ phase matrix has phase rank n.*

Proof. Just note that for $n \geq 7$, $\frac{2^{n-1}}{n} > n$. □

In [44] the authors were able to slightly modify this argument to prove that there are also no 6×6 phase matrices with phase rank 6, while in [49] a very long and technical proof was provided to show that there are also no 5×5 phase rank nonsingular matrices. For $n \leq 4$ such matrices exist and we will come back to these in the next section.

The rectangular case is still not fully characterized. We know from Lemma 4.2.3 on sign rank that the $n \times 2^{n-1}$ matrix whose columns are all the distinct ± 1 vectors starting with a one has sign rank (and thus phase rank) n, and we just saw that for $m < \frac{2^{n-1}}{n}$ these do not exist.

Question 4.3.8. *What is the smallest m for which there is a $n \times m$ phase matrix with phase rank n? Can one at least find better bounds than $\frac{2^{n-1}}{n} \leq m \leq 2^{n-1}$?*

As explained before, this can be seen as the problem of the minimal number of geometric translations of a certain set that are needed to cover $(S^1)^n$, a type of geometrical problem that tends to be hard.

4.4 Small square matrices of maximal phase rank

We have seen that for $n \geq 5$ every $n \times n$ phase matrix has phase rank less than n. So the question of determining if a square matrix is phase nonsingular is only potentially interesting for $n = 2, 3, 4$.

For $n = 2$, due to Proposition 4.1.3, the phase rank coincides with the usual rank, so the question is trivial. For $n = 3$ and $n = 4$, Lemma 4.3.1 gives a potential way to find if the rank is maximal, but it involves either finding a scaling with the required properties or proving that one does not exist, and there is no direct way of doing that. So one would like a simple, preferably semialgebraic, way of describing the sets of $n \times n$ phase matrices of phase rank less than n, for $n = 3$ and 4.

A sufficient certificate for maximal phase rank is given by the colopsided criterion in coamoeba theory, as stated in Lemma 4.2.11. This method is already proposed in [53].

Definition 4.4.1. We say that an $n \times n$ phase matrix Θ is colopsided if the $n \times n$ determinant polynomial is colopsided at Θ.

By Lemma 4.2.11, if Θ is colopsided, Θ does not belong to the coamoeba of the determinant, i.e., it has phase rank n. Let us denote by $\overrightarrow{\det(\Theta)}$ the vector of the monomials of the $n \times n$ determinant evaluated at the phase matrix Θ. We are saying that if 0 is not in the relative interior of the convex hull of $\overrightarrow{\det(\Theta)}$ then Θ has phase rank n.

Example 4.4.2. Let

$$\Theta_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i\frac{3\pi}{4}} & e^{-i\frac{\pi}{2}} \\ 1 & e^{-i\frac{\pi}{2}} & e^{i\frac{3\pi}{4}} \end{bmatrix}, \Theta_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i\frac{\pi}{2}} & e^{i\frac{\pi}{2}} \\ 1 & e^{i\frac{\pi}{2}} & -1 \end{bmatrix} \text{ and } \Theta_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i\frac{\pi}{2}} & e^{i\frac{\pi}{3}} \\ 1 & e^{i\frac{\pi}{3}} & e^{i\frac{\pi}{2}} \end{bmatrix}.$$

We have $\overrightarrow{\det(\Theta_1)} = (e^{i\frac{3\pi}{2}}, e^{-i\frac{\pi}{2}}, e^{-i\frac{\pi}{2}}, -e^{i\frac{3\pi}{4}}, -e^{i\frac{3\pi}{4}}, 1)$, $\overrightarrow{\det(\Theta_2)} = (-e^{i\frac{\pi}{2}}, e^{i\frac{\pi}{2}}, e^{i\frac{\pi}{2}}, -e^{i\frac{\pi}{2}}, 1, 1)$ and $\overrightarrow{\det(\Theta_3)} = (-1, e^{i\frac{\pi}{3}}, e^{i\frac{\pi}{3}}, -e^{i\frac{\pi}{2}}, -e^{i\frac{\pi}{2}}, -e^{i\frac{2\pi}{3}}, -e^{i\frac{\pi}{2}})$. Both Θ_1 and Θ_2 are colopsided, while Θ_3 is not (see Figure

4.6). $\mathrm{Conv}(\overrightarrow{\det(\Theta_1)})$ does not contain 0. $\mathrm{Conv}(\overrightarrow{\det(\Theta_2)})$ does contain it, but its relative interior does not. We can immediately conclude that both Θ_1 and Θ_2 have phase rank 3.

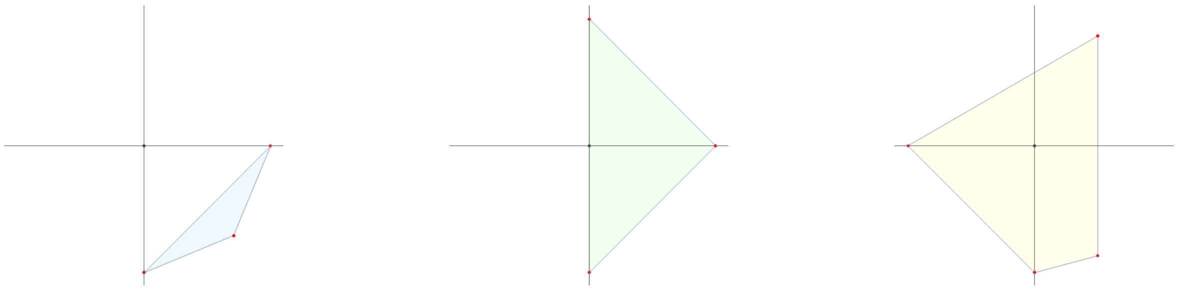

Fig. 4.6 Convex hulls of $\overrightarrow{\det(\Theta_1)}, \overrightarrow{\det(\Theta_2)}, \overrightarrow{\det(\Theta_3)}$, in this order, where Θ_1, Θ_2 and Θ_3 are the matrices from Example 4.4.2.

The colopsidedness criterion was proposed in the ray nonsingularity literature, and was noted not to be necessary in the 4×4 case. In fact, for

$$\Theta = \begin{bmatrix} 1 & 1 & 1 & i \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix},$$

$\overrightarrow{\det(\Theta)}$ contains $1, -1, i$ and $-i$, so 0 is in the interior of its convex hull. Moreover, in [44] it is shown that this matrix has phase rank 4.

The question of necessity of the colopsided criterion for phase nonsingularity is not addressed in the ray nonsingularity literature. In [49], for example, in order to study the 5×5 case, the authors derive an extensive and complicated description for the 3×3 case, without any reference to colopsidedness. It turns out that, as what happened with lopsidedness in the case of the phaseless rank, colopsidedness is a necessary and sufficient condition for phase nonsingularity of 3×3 matrices.

Theorem 4.4.3. *Given a 3×3 phase matrix Θ, $\mathrm{rank}_{phase}\Theta < 3$ if and only if the origin is in the relative interior of the convex hull of* $\overrightarrow{\det(\Theta)}$.

In terms of coamoeba theory, we are saying that the coamoeba of the variety of singular 3×3 matrices is characterized by the noncolopsidedness of the determinant. Coamoebas of simple polynomials are completely characterized by the colopsidedness criterion (see [26]). This is a new nontrivial example of another hypersurface with the same property, since the 3×3 determinant is not simple.

In what follows we will present a proof of Theorem 4.4.3, based only on simple results from linear algebra and convex geometry. We will consider throughout a general 3×3 phase matrix of the following form:

$$\Theta = \begin{bmatrix} e^{i\phi_1} & e^{i\phi_2} & e^{i\phi_3} \\ e^{i\phi_4} & e^{i\phi_5} & e^{i\phi_6} \\ e^{i\phi_7} & e^{i\phi_8} & e^{i\phi_9} \end{bmatrix}.$$

For this matrix we have

$$\overrightarrow{\det(\Theta)} = \left(e^{i(\phi_1+\phi_5+\phi_9)}, e^{i(\phi_2+\phi_6+\phi_7)}, e^{i(\phi_3+\phi_4+\phi_8)}, -e^{i(\phi_1+\phi_6+\phi_8)}, -e^{i(\phi_2+\phi_4+\phi_9)}, -e^{i(\phi_3+\phi_5+\phi_7)}\right).$$

We will prove the equivalence between noncolopsidedness and nonmaximal phase rank for the 3×3 case in two steps. First, we show that nonmaximal phase rank is the same as noncolopsidedness with one additional condition and, then, that this extra restriction can be removed.

Lemma 4.4.4. *Let Θ be a 3×3 phase matrix. Then,* $\operatorname{rank}_{phase}(\Theta) < 3$ *if and only if there exists a coefficient vector $c \in \mathbb{R}^6_{++}$ such that $\overrightarrow{\det(\Theta)} \cdot c = 0$ and $c_1c_2c_3 = c_4c_5c_6$.*

Proof. Let Θ be as above and suppose $\operatorname{rank}_{\text{phase}}(\Theta) < 3$. One can find a real positive matrix M such that $\det(M \circ \Theta) = 0$, where $\circ$ represents the Hadamard product. But $\det(M \circ \Theta)$ can be written as the dot product

$$\overrightarrow{\det(\Theta)} \cdot (M_{11}M_{22}M_{33}, M_{12}M_{23}M_{31}, M_{13}M_{21}M_{32}, M_{11}M_{23}M_{32}, M_{12}M_{21}M_{33}, M_{13}M_{22}M_{31})$$

and the entries of the right-hand side vector do satisfy the intended relations.

Conversely, suppose there exist positive coefficients $c_i, i = 1, \ldots, 6$, satisfying $\overrightarrow{\det(\Theta)} \cdot c = 0$ and $c_1c_2c_3 = c_4c_5c_6$. According to the reasoning above, if we can find a positive matrix M for which $\det(M \circ \Theta) = \overrightarrow{\det(\Theta)} \cdot c = 0$, with $c = (M_{11}M_{22}M_{33}, M_{12}M_{23}M_{31}, M_{13}M_{21}M_{32}, M_{11}M_{23}M_{32}, M_{12}M_{21}M_{33}, M_{13}M_{22}M_{31})$, Θ will have nonmaximal phase rank.

Finding such M_{ij}'s is equivalent to solving the linear system

$$\begin{gathered} M'_{11} + M'_{22} + M'_{23} = c'_1,\, M'_{12} + M'_{23} + M'_{31} = c'_2,\, M'_{13} + M'_{21} + M'_{32} = c'_3 \\ M'_{11} + M'_{23} + M'_{32} = c'_4,\, M'_{12} + M'_{21} + M'_{33} = c'_5,\, M'_{13} + M'_{22} + M'_{31} = c'_6, \end{gathered}$$

where $c'_i = \log c_i$ and $M'_{ij} = \log M_{ij}$. This is solvable for M'_{ij} if and only if c' is in $\mathscr{C}(A)$, the column space of

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix},$$

or, equivalently, c' is orthogonal to Null(A^T), the null space of the transpose of A. Since Null(A^T) is spanned by $(111-1-1-1)^\intercal$, c' is in $\mathscr{C}(A)$ if and only if $c'_1+c'_2+c'_3=c'_4+c'_5+c'_6$, i.e., $c_1c_2c_3=c_4c_5c_6$, which is guaranteed by our hypothesis. □

To get rid of this extra condition on the coefficients we will have to do some extra work. The following lemma is a simple fact in discrete convex geometry and it will be useful later.

Lemma 4.4.5. *Given* 3 *blue points and* 3 *red points in* $\mathbb{R}^2\setminus\{(0,0)\}$ *such that the origin is in the relative interior of their convex hull, there is a proper subset of them containing all the blue points that still has the origin in the relative interior of its convex hull (and similarly for the red points).*

Proof. It is sufficient to prove that we can always erase some red points while keeping the origin in the relative interior, as the statement is clearly symmetric with respect to the colors.

Suppose that there is no triangle with vertices on the six given points containing the origin in its interior. Then there are two possibilities: either all the points are in a line through the origin, or all the points are in exactly two lines through the origin. This is a consequence of a theorem of Steinitz [70] (see Result B in [12]).

If all are in one line, then for each color there must be two red points in the same side of the origin, so we can drop one without losing the origin in the relative interior of the convex hull. If all are in exactly two lines then we must have points in each side of the origin on each line, and if we have more than one point on a side then we can remove one of them. If no red point is redundant, that means that all the blue points are in the same side of the origin in one of the lines, which means we can drop the line with only red points and we will still have the origin on the relative interior.

So we can assume that there is a triangle containing the origin in its interior. If there is such a triangle using a blue point, then we are done, as the set attained by adding any missing blue point to the vertices of the triangle would have the required properties.

The only remaining case would be if there is a triangle that uses only red points. In that case, if we take any of the blue points and consider the three triangles it can define with the red points, the only way that none of them contains the origin in its interior is if the origin is in the interior of the segment between the blue and a red point. If all the blue points are opposite to different red points, then the triangle of blue points would contain the origin, if not, then the set of blue points together with their opposite red points satisfies the intended property. □

We are now ready to prove Theorem 4.4.3 by using this convex geometry fact to remove the extraneous condition on Lemma 4.4.4.

Proof of Theorem 4.4.3. If the origin is not in the relative interior of the convex hull of $\overrightarrow{\det(\Theta)}$, this means the determinant is colopsided at Θ, which implies, by Lemma 4.2.11, that $\text{rank}_{\text{phase}}(\Theta)=3$. So one of the implications is easy.

We will show that if Θ is not colopsided then there exists a coefficient vector $c \in \mathbb{R}^6_{++}$ such that $\overrightarrow{\det(\Theta)} \cdot c = 0$ and $c_1c_2c_3 = c_4c_5c_6$. By Lemma 4.4.4, this will imply that Θ has nonmaximal phase rank, giving us the remaining implication.

Suppose Θ is not colopsided. Then, there exists $a \in \mathbb{R}^6_{++}$ such that $\overrightarrow{\det(\Theta)} \cdot a = 0$. If $a_1a_2a_3 = a_4a_5a_6$, we are done. Thus, we either have $a_1a_2a_3 > a_4a_5a_6$ or $a_1a_2a_3 < a_4a_5a_6$. Without loss of generality we may assume the first, since switching two rows in the matrix will switch the sets $\{a_1, a_2, a_3\}$ and $\{a_4, a_5, a_6\}$.

Note that the entries of $\overrightarrow{\det(\Theta)}$ can be thought of as points in $\mathbb{R}^2$ where the first three are red and the last three are blue, and their convex hull contains the origin in its relative interior. So, by Lemma 4.4.5, the origin is still in the relative interior of the convex hull if we drop some of the a_i's, $i = 1,2,3$. This means that there exists $b \in \mathbb{R}^6_{+}$ for which some of the first three coordinates are zero but none of the last three coordinates is, and such that $\overrightarrow{\det(\Theta)} \cdot b = 0$.

Now note that every 6-uple of the form $c^\lambda = a + \lambda b$, $\lambda \geq 0$, will satisfy $c^\lambda \in \mathbb{R}^6_{++}$ and $\overrightarrow{\det(\Theta)} \cdot c^\lambda = 0$. Furthermore,

$$c_1^\lambda c_2^\lambda c_3^\lambda - c_4^\lambda c_5^\lambda c_6^\lambda = (a_1 + b_1\lambda)(a_2 + b_2\lambda)(a_3 + b_3\lambda) - (a_4 + b_4\lambda)(a_5 + b_5\lambda)(a_6 + b_6\lambda)$$

equals $a_1a_2a_3 - a_4a_5a_6 > 0$ for $\lambda = 0$ but goes to $-\infty$ when λ grows to $+\infty$, since it is a cubic polynomial on λ with the coefficient of λ^3 being $-b_4b_5b_6$. Hence, for some λ we have that $c_1^\lambda c_2^\lambda c_3^\lambda - c_4^\lambda c_5^\lambda c_6^\lambda = 0$ and there is a vector c with the desired properties. $\square$

Example 4.4.6. We saw in Example 4.4.2 that the matrix

$$\Theta_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{i\frac{\pi}{2}} & e^{i\frac{\pi}{3}} \\ 1 & e^{i\frac{\pi}{3}} & e^{i\frac{\pi}{2}} \end{bmatrix}$$

has a noncolopsided determinant. Therefore, $\operatorname{rank}_{\text{phase}}(\Theta_3) < 3$.

Example 4.4.7. The characterization of nonsingular 3×3 phase matrices given by Theorem 4.4.3 is very easy to check. In particular, it can be used to visualize slices of the coamoeba associated to the 3×3 determinant, i.e., of the set of phase singular 3×3 phase matrices. In Figure 4.7 are shown, on the left, the set of triples (t_1, t_2, t_3) for which

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{it_1} & e^{it_2} \\ 1 & e^{it_3} & e^{i\frac{\pi}{3}} \end{bmatrix}$$

has nonmaximal phase rank, and, on the right, its complement.

Since checking if the phase rank of a 3×3 matrix is less or equal than two is so easy, it could be tempting to think that we might be able to leverage the result into some answer to Question 4.2.2,

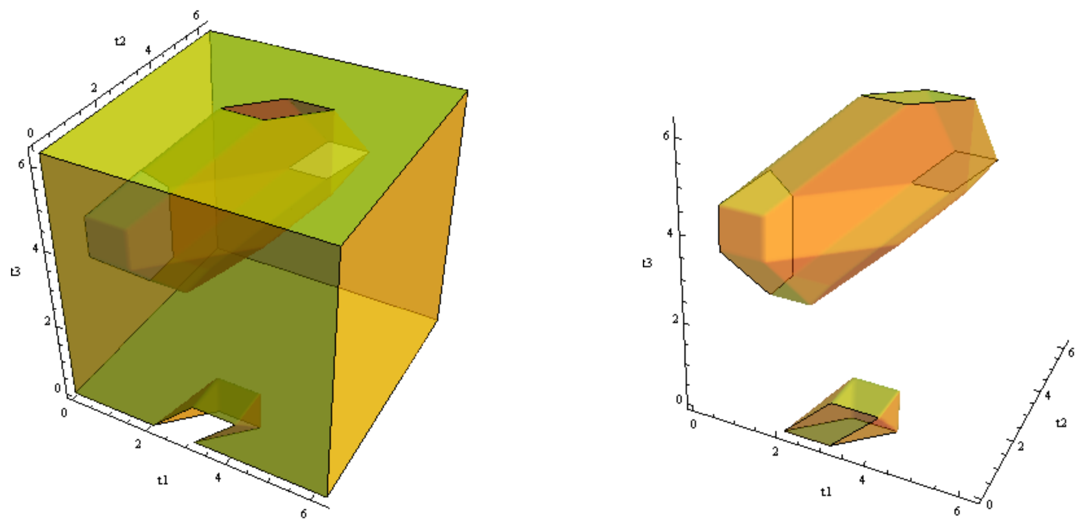

Fig. 4.7 Slice of the 3×3 determinant coamoeba and its complement.

that asks for an algorithm to check if an $n\times m$ matrix has phase rank at most 2. The problem is that, contrary to what happened in the phaseless rank case, not even the $3\times m$ case can be easily derived from the rank of its 3×3 submatrices. In fact, the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

has phase rank 3, although all its 3×3 submatrices have sign rank (and thus phase rank) at most 2.

As mentioned before, for 4×4 matrices we have seen that colopsidedness is not necessary for having phase rank 4. In fact, even generating colopsided examples tends to be hard, but there are some systematic ways of doing it.

Example 4.4.8. The phase matrix

$$\Theta = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & e^{i\frac{\pi}{4}} & e^{i\frac{\pi}{4}} \\ 1 & e^{i\frac{\pi}{4}} & -1 & e^{i\frac{\pi}{4}} \\ 1 & e^{i\frac{\pi}{4}} & e^{i\frac{\pi}{4}} & -1 \end{bmatrix}$$

is colopsided.

$$\mathrm{Conv}(\overrightarrow{\det(\Theta)}) = \mathrm{Conv}(\{e^{i\frac{\pi}{2}}, e^{i\frac{3\pi}{4}}, -1, e^{i\frac{5\pi}{4}}, e^{i\frac{3\pi}{2}}\})$$

contains 0, but its (relative) interior does not (see Figure 4.8).

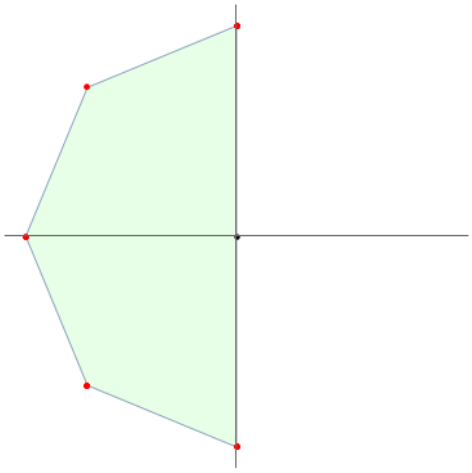

Fig. 4.8 Convex hull of $\overrightarrow{\det(\Theta)}$, where Θ is the matrix from Example 4.4.8.

The underlying idea behind the generation of Example 4.4.8 was the following: we searched for a 3×4 phase matrix for which all 3×3 submatrices are colopsided, namely

$$\Theta = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & e^{i\frac{\pi}{4}} & e^{i\frac{\pi}{4}} \\ 1 & e^{i\frac{\pi}{4}} & -1 & e^{i\frac{\pi}{4}} \end{bmatrix}.$$

This means that for every 3×3 submatrix Θ_i, $\overrightarrow{\det(\Theta_i)}$ is colopsided, i.e., every point is contained in some common half-space H_i. Then, having in mind the Laplace expansion formula for 4×4 determinant along its last row, we can see that $\overrightarrow{\det(\Theta)}$ is just the collection of the sets $\overrightarrow{\det(\Theta_i)}$ each rotated by its complementary entry in the 4×4 matrix. By picking suitable entries we can rotate them in such a way that all the H_i will coincide and ensure the whole matrix is colopsided. This gives us a tool to construct colopsided 4×4 matrices from 3×4 matrices where all 3×3 submatrices have phase rank 3. The problem is that, on the one hand, colopsidedness is not enough to fully characterize 4×4 phase nonsingularity, and, on the other hand, finding 3×4 matrices with this property is also hard. Thus, the problem of finding such a characterization remains completely open.

Question 4.4.9. *Find an* effective *description of the set of* 4×4 *phase matrices with phase rank less than* 4.

4.5 Typical ranks

Let us equip S^1 with the relative topology (i.e, open sets of $\mathbb{C}$ intersected with S^1) and $(S^1)^{n\times m}$ with the product topology (the cartesian product of the relative topologies on S^1).

Definition 4.5.1. We say that $\operatorname{rank}_{\text{phase}}(\Theta) = k$ is *typical* in $(S^1)^{n\times m}$ if there exists an nonempty open set in $(S^1)^{n\times m}$ for which all matrices have phase rank k. To the minimum such k we call *minimal typical phase rank*.

For instance, in $(S^1)^{3\times 3}$ the minimal typical phase rank is 2. It cannot be 1, as a phase matrix has phase rank 1 if and only if it has rank 1 and the set of such matrices is a zero-measure set. Take, for instance,

$$\Theta = \begin{bmatrix} 1 & 1 & 1 \\ 1 & i & -i \\ 1 & 1 & i \end{bmatrix},$$

from Example 4.1.2. This matrix has rank 3 and is not colopsided and $\mathrm{Conv}(\overrightarrow{\det(\Theta)})$ is a rhombus with vertices 1, i, -1 and $-i$ (see Figure 4.9). Since perturbing the entries of Θ will continuously perturb the vertices of $\mathrm{Conv}(\overrightarrow{\det(\Theta)})$, there is an open set containing Θ for which $\overrightarrow{\det(\Theta)}$ will remain not colopsided and the rank of Θ will remain 3. By Theorem 4.4.3, every phase matrix in this open set has phase rank 2.

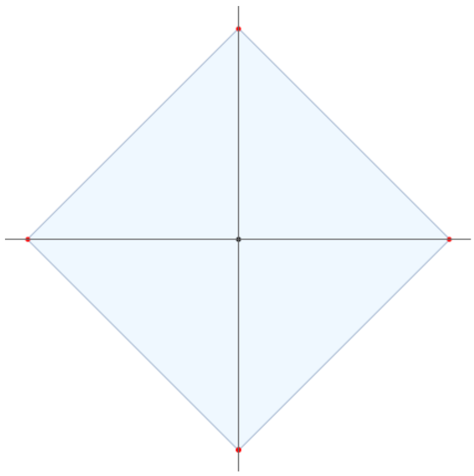

Fig. 4.9 Convex hull of $\overrightarrow{\det(\Theta)}$, where Θ is the matrix from Example 4.1.2.

The minimal typical phase rank in $(S^1)^{n\times m}$, like the minimal typical phaseless rank in $\mathbb{R}_+^{n\times m}$, is at least

$$\left\lceil \frac{n+m-\sqrt{(n-1)^2+(m-1)^2}}{2} \right\rceil,$$

because we can adapt the proof of Corollary 1.4.12 for the lower bound by replacing the entrywise absolute value map with the entrywise argument map. Similar computations to those undertaken in Section 3.2 suggest that the lower bound is tight.

Question 4.5.2. *Is it true that the minimal typical phase rank in $(S^1)^{n\times m}$ is given by*

$$\left\lceil \frac{n+m-\sqrt{(n-1)^2+(m-1)^2}}{2} \right\rceil?$$

4.6 Bounds for phase rank

4.6.1 Lower bounds

In this section we generalize Theorem 4.2.4 and Theorem 4.2.5, by proving they can be extended to any phase matrix. The proofs of the generalizations follow the ones of those results, with minor modifications to adapt them to the complex case.

First, note that if $\operatorname{rank}_{\text{phase}}(\Theta) = k$, with Θ an $n \times m$ phase matrix, there exist $U \in \mathbb{C}^{k\times n}$ and $V \in \mathbb{C}^{k\times m}$ such that $\Theta_{ij} = \frac{\langle u_i, v_j\rangle}{|\langle u_i, v_j\rangle|} = \frac{u_i^* v_j}{|u_i^* v_j|}$, where u_i and v_j denote, respectively, column i of U and column j of V, for $i = 1,\ldots,n,\ j = 1,\ldots,m$. This follows from phase rank definition and the usual matrix rank definition as the minimal size of a two-sided matrix factorization. We will start with an auxiliary lemma.

Lemma 4.6.1. *Let Θ be an $n\times m$ phase matrix. If there exist $u_i,\ v_j \in \mathbb{C}^k$, $i = 1,\ldots,n,\ j = 1,\ldots,m$, such that $\Theta_{ij} = \frac{\langle u_i, v_j\rangle}{|\langle u_i, v_j\rangle|}$ and $||u_i|| = ||v_j|| = 1$, then,*

$$\sum_{j=1}^{m}\left(\sum_{i=1}^{n}|\langle u_i, v_j\rangle|\right)^2 \le n||\Theta||^2,$$

where $||\Theta||$ denotes the spectral norm of Θ.

Proof. For every $1 \le j \le m$, we have

$$\sum_{i=1}^{n}|\langle u_i, v_j\rangle| = \sum_{i=1}^{n}\overline{\Theta}_{ij}\langle u_i, v_j\rangle = \left\langle \sum_{i=1}^{n}\Theta_{ij}u_i, v_j\right\rangle.$$

By Cauchy-Schwarz inequality, as $||v_j|| = 1$, the above quantity is upper bounded by $\left|\left|\sum_{i=1}^{n}\Theta_{ij}u_i\right|\right|$. Therefore,

$$\sum_{j=1}^{m}\left(\sum_{i=1}^{n}|\langle u_i, v_j\rangle|\right)^2 \le \sum_{j=1}^{m}\left\langle \sum_{i=1}^{n}\Theta_{ij}u_i, \sum_{k=1}^{n}\Theta_{kj}u_k\right\rangle = \operatorname{Tr}((U\Theta)^*U\Theta) = \operatorname{Tr}(U^*U\Theta\Theta^*).$$

Since $||\Theta||^2$ is the greatest eigenvalue of $\Theta\Theta^*$, $||\Theta||^2 I_{n\times n} - \Theta\Theta^*$ is positive semidefinite and

$$\operatorname{Tr}(U^*U\Theta\Theta^*) \le \operatorname{Tr}(U^*U||\Theta||^2 I_{n\times n}) = ||\Theta||^2\sum_{i=1}^{n}||u_i||^2 = n||\Theta||^2,$$

completing the proof. □

We need another result in order to prove the first bound. It is a complex version of Theorem 4.1 from [29]. Its proof, which we omit here, is quite long and exactly the same as in the real case.

Theorem 4.6.2 ([29]). *Let $X \subseteq \mathbb{C}^k$, $|X| \geq k$, be a finite set such that all subsets of X with k elements are linearly independent. Then, there is a nonsingular linear transformation $A \in \mathbb{C}^{k\times k}$ such that*

$$\sum_{x\in X} \frac{1}{||Ax||^2} Ax(Ax)^* = \frac{|X|}{k} I_{k\times k}.$$

We are now ready to generalize Theorem 4.2.4 to the complex case.

Theorem 4.6.3. *If Θ is an $n\times m$ phase matrix, then* $\operatorname{rank}_{phase}(\Theta) \geq \frac{\sqrt{nm}}{||\Theta||}$.

Proof. Let $k = \operatorname{rank}_{\text{phase}}(\Theta) \leq \min\{n,m\}$. There exist $u_i,\ v_j \in \mathbb{C}^k$, $i = 1,\dots,n,\ j = 1,\dots,m$, such that $\Theta_{ij} = \frac{\langle u_i, v_j\rangle}{|\langle u_i, v_j\rangle|}$. By perturbing the entries of the $u_i's$, we can assume that none of them lies in any linear span of $k-1$ of the other. This will yield small variations on the entries of Θ. Call this new matrix $\tilde{\Theta}$. $\tilde{\Theta}$ can be chosen arbitrarily close to Θ, which means that if the inequality holds for $\tilde{\Theta}$, it holds for Θ.

Now we apply Theorem 4.6.2: since $n \geq k$ and any k of the $u_i's$ are linearly independent, there is a nonsingular linear transformation $A \in \mathbb{C}^{k\times k}$ such that

$$\sum_{i=1}^{n} \tilde{u}_i\tilde{u}_i^* = \frac{n}{k} I_{k\times k},$$

with $\tilde{u}_i = \frac{Au_i}{||Au_i||}$. Consider now $\tilde{v}_j = \frac{(A^*)^{-1}v_j}{||(A^*)^{-1}v_j||}$, so that

$$\frac{\langle \tilde{u}_i, \tilde{v}_j\rangle}{|\langle \tilde{u}_i, \tilde{v}_j\rangle|} = \frac{\langle u_i, v_j\rangle}{|\langle u_i, v_j\rangle|} = \tilde{\Theta}_{ij}.$$

For every $1 \leq j \leq m$, since $|\langle \tilde{u}_i, \tilde{v}_j\rangle| \leq 1$, we have

$$\sum_{i=1}^{n} |\langle \tilde{u}_i, \tilde{v}_j\rangle| \geq \sum_{i=1}^{n} |\langle \tilde{u}_i, \tilde{v}_j\rangle|^2 = \tilde{v}_j^* \left(\sum_{i=1}^{n} \tilde{u}_i\tilde{u}_i^*\right)\tilde{v}_j = \frac{n}{k}. \tag{4.1}$$

Summing the squares of all these inequalities and using Lemma 4.6.1, we get $m\left(\frac{n}{k}\right)^2 \leq n||\tilde{\Theta}||^2$, and the result follows. □

Analogously to Theorem 4.2.4, we can replace $||\Theta||$ with $||\Theta||^* = \min\{||M|| : \overline{M_{ij}}\Theta_{ij} \geq 1\,\forall i,j\} = \min\{||M|| : \frac{M_{ij}}{|M_{ij}|} = \Theta_{ij} \text{ and } |M_{ij}| \geq 1\,\forall i,j\}$ in the above bound.

Theorem 4.2.5 can also be generalized to the complex case. This bound involves the dual norm of the γ_2 norm, also known as the max-norm. A short introduction to these quantities and how to compute them can be found in the Appendix A.2. For now, recall that for a matrix $A \in \mathbb{C}^{n\times m}$ we have

$$\gamma_2(A) = \min\left\{\max_{i,j} ||x_i||_{l_2}||y_j||_{l_2} : XY^T = A\right\},$$

where $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^m$ are the rows of X and Y, respectively. Therefore, we have

$$\gamma_2^*(A) = \max_{B:\gamma_2(B)\leq 1} |\langle A,B\rangle| = \max_{B:\gamma_2(B)\leq 1} |\operatorname{Tr}(A^*B)|.$$

Theorem 4.6.4. *For any $n\times m$ phase matrix Θ,*

$$\operatorname{rank}_{phase}(\Theta) \geq \frac{nm}{\gamma_2^*(\Theta)}.$$

Proof. Let $\operatorname{rank}_{\text{phase}}(\Theta) = k$ and choose $\tilde{u}^i$ and $\tilde{v}^j$, $i = 1,\dots,n,\ j = 1,\dots,m$, as in the proof of the previous theorem. Summing the inequalities (4.1) for all j we get

$$\sum_{i=1}^n \sum_{j=1}^m |\langle \tilde{u}^i, \tilde{v}^j\rangle| \geq \frac{nm}{k}.$$

Matrix B, defined by $B_{ij} = \langle \tilde{u}^i, \tilde{v}^j\rangle$, has phase pattern equal to $\tilde{\Theta}$, and satisfies $\sum_{i,j} |B_{ij}| \geq \frac{nm}{k}$. Furthermore, $\gamma_2(B) \leq 1$, because $||\tilde{u}^i|| = ||\tilde{v}^j|| = 1$ for any i, j, and

$$\gamma_2^*(\tilde{\Theta}) = \max_{A:\gamma_2(A)\leq 1} |\langle \tilde{\Theta},A\rangle| \geq |\langle \tilde{\Theta},B\rangle| = \sum_{i,j} |B_{ij}| \geq \frac{nm}{k}.$$

Using the fact that $\gamma_2^*(.)$ is continuous and by making $\tilde{\Theta} \to \Theta$,

$$\gamma_2^*(\Theta) \geq \frac{nm}{k}.$$

□

Note that these bounds, being continuous, can only be very rough approximations to the discontinuous notion of phase rank, but can in some cases provide meaningful results and have been used in the real case to prove several results in communication complexity.

Example 4.6.5. Searching randomly for small examples one immediately observes the limitations of these bounds, as it is very hard to obtain effective bounds.

For instance, the bounds from Theorems 4.6.3 and 4.6.4 for the matrix

$$\begin{bmatrix} 1 & 1 & 1 & i \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix},$$

which has phase rank 4, are 1.5751 and 1.6996, respectively. The smallest example that we could find for which one of the bounds is nontrivial has size 8×8 and is shown below. For

$$\begin{bmatrix} -1 & i & -1 & i & i & -1 & 1 & 1 \\ -1 & i & i & i & -i & -i & -i & i \\ 1 & i & 1 & -1 & i & i & -i & -1 \\ 1 & i & 1 & 1 & i & 1 & 1 & i \\ 1 & i & -1 & i & i & -i & -1 & 1 \\ -i & -i & 1 & -i & i & -1 & -i & -1 \\ -1 & i & -1 & -i & i & i & -1 & -i \\ i & -1 & i & -i & 1 & -1 & -i & -1 \end{bmatrix}$$

the bounds values are 1.8383 and 2.0335, so Theorem 4.6.4 guarantees that it has phase rank at least 3. Note that one could maybe conclude the same from checking that some of the 3×3 submatrices are colopsided, but to attain lower bounds larger than 3 these inequalities are our only systematic tool.

4.6.2 Upper bounds

In terms of upper bounds for phase rank, one could aim for a generalization of Theorem 4.2.7. The issue lies in the fact that the proof uses the combinatorial nature of a sign matrix, such as the number of sign changes on each row of the matrix, and it is not obvious how this concept applies to a phase matrix. We can however use Proposition 4.3.6 to create a very simple upper bound in the style of Proposition 1.4.11, that we proved for phaseless rank.

Proposition 4.6.6. *Let Θ be an $n \times m$ phase matrix, with $n \leq m$, and k the smallest positive integer such that $m < \frac{2^{k-1}}{k}$. Then* $\operatorname{rank}_{phase}(\Theta) \leq n - \lfloor \frac{n-1}{k-1} \rfloor$.

Proof. Let k satisfy $m < \frac{2^{k-1}}{k}$. Then, by Proposition 4.3.6, every $k \times m$ submatrix of Θ has phase rank at most $k-1$. Hence, for every $k \times m$ submatrix Γ, we can find B_Γ with the same phases of Γ and rank less than k. Moreover, we are free to pick the first row of B_Γ to have modulus one, since scaling an entire column of B_Γ by a positive integer does not change the rank or the phases.

Consider then $k \times m$ submatrices Γ_i of Θ, $i = 1, \ldots, \lfloor \frac{n-1}{k-1} \rfloor$, all containing the first row but otherwise pairwise disjoint. We can then construct a matrix B by piecing together the B_{Γ_i}'s, since they coincide in the only row they share, and filling out the remaining rows, always less than $k-1$, with the corresponding entries of Θ.

By construction, in that matrix B we always have in the rows corresponding to B_{Γ_i} a row different than the first that is a linear combination of the others, and can be erased without dropping the rank of B. Doing this for all i, we get that the rank of B has at least a deficiency per B_{Γ_i}, so its rank is at most

$$n - \left\lfloor \frac{n-1}{k-1} \right\rfloor,$$

and since B has the same phases as Θ, $\mathrm{rank}_{\mathrm{phase}}(\Theta)$ satisfies the intended inequality. $\square$

It seems that the bound derived above is not very strong. Indeed, for a 100×100 matrix we can only guarantee that the phase rank is at most 91, and for a 1000×1000 matrix that it is at most 929. We can quantify a little how good this bound is. By inverting the function $\frac{2^{x-1}}{x}$ we get that k^*, the smallest k that satisfies the inequality on Proposition 4.6.6 is given by

$$\left\lceil -\frac{W_{-1}\left(-\frac{\ln 2}{2m}\right)}{\ln 2} \right\rceil,$$

where W_{-1} is one of the branches of the Lambert W function, the inverse of $f(x) = xe^x$. Using the bounds for W_{-1} derived in [17], one can conclude that k^* satisfies the inequalities

$$\left\lceil \frac{1+\sqrt{2u}+u}{\ln 2} \right\rceil \geq k^* \geq \left\lceil \frac{1+\sqrt{2u}+\frac{2}{3}u}{\ln 2} \right\rceil$$

where $u = \ln(m) + \ln(2) - \ln(\ln(2)) - 1 \approx \ln(m) - 0.05966$. This means that the best bound we can expect from Proposition 4.6.6 for an $n \times n$ matrix grows as $n\left(1 - \frac{c}{\log_2(n)}\right)$. Recall that for sign rank we have that the rank of an $n \times n$ matrix is at most $\frac{n}{2}(1+o(1))$, so our bound is very far from a result of that type, that one would expect to be true. It is, however, a first step towards a more meaningful bound.

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Appendix A

Appendix

A.1 Sums of squares and Semidefinite Programming

In the first part of this appendix, we briefly explain the connection between sums of squares and semidefinite programmming. We closely follow Chapter 3 from [11] and parts of [18]. Let $\mathbb{R}[x]_{n,d} := \mathbb{R}[x_1,\ldots,x_n]_d$ denote the set of polynomials of reals coefficients of degree d in the n variables $x_1,\ldots,x_n$

Definition A.1.1. We say a polynomial $p(x) := p(x_1,\ldots,x_n) \in \mathbb{R}[x]_{n,2d}$ is a sum of squares (sos) if it can be written as the sum of squares of other polynomials, i.e, if there exist $q_1,\ldots,q_m \in \mathbb{R}[x]_{n,d}$ such that

$$p(x) = \sum_{k=1}^{m} q_k(x)^2.$$

Example A.1.2. Let $q(x_1,x_2) = 4{x_1}^4 + 4{x_1}^3x_2 - 2{x_1}^2{x_2}^2 + 10{x_2}^4$. Since

$$4{x_1}^4 + 4{x_1}^3x_2 - 2{x_1}^2{x_2}^2 + 10{x_2}^4 = (2{x_1}^2 - 3{x_2}^2 + x_1x_2)^2 + ({x_2}^2 + 3x_1x_2)^2,$$

$q(x_1,x_2)$ is a sum of squares.

A combinatorial exercise tells us that, for any polynomial of degree $2d$ in n variables, its number of coefficients is $\binom{n+2d}{2d}$, one for each different monomial. Let $[x]_d := [1,x_1,\ldots,x_n,{x_1}^2,x_1x_2,\ldots,{x_n}^d]^T$ be the vector all $\binom{n+d}{d}$ monomials in $x_1,\ldots,x_n$ of degree at most d. If $p(x) \in \mathbb{R}[x]_{n,2d}$ is an sos, take

$$\begin{bmatrix} q_1(x) \\ q_2(x) \\ \vdots \\ q_m(x) \end{bmatrix} = V[x]_d,$$

where $V \in \mathbb{R}^{m\times\binom{n+d}{d}}$ and whose k-th row contains the coefficients of $q_k(x)$. It follows that

$$p(x) = \sum_{k=1}^{m} q_k(x)^2 = (V[x]_d)^T V[x]_d = [x]_d{}^T V^T V[x]_d = [x]_d{}^T Q[x]_d.$$

We can now state the following theorem:

Theorem A.1.3 ([11])**.** *A polynomial $p(x) \in \mathbb{R}[x]_{n,2d}$ is an sos if and only if there exists a positive semidefinite matrix $Q \in \mathbb{R}^{\binom{n+d}{d}\times\binom{n+d}{d}}$ such that*

$$p(x) = [x]_d{}^T Q[x]_d.$$

Definition A.1.4. A semidefinite program in standard primal form is written as

$$\begin{aligned} &\text{minimize } \langle C, X\rangle \\ &\text{subject to } \langle A_i, X\rangle = b_i, \quad i = 1,\dots,m, \\ &\qquad\qquad X \succeq 0, \end{aligned}$$

where C, A_i are $n\times n$ real symmetric matrices and $\langle X,Y\rangle := \mathrm{Tr}(X^TY)$. The positive semidefinite matrix X is the variable over which the minimization is performed.

YALMIP, SDPT3 and SeDuMi are some examples of software used for solving semidefinite programs. Deciding if a polyomial is a sum of squares is a semidefinite program. Indeed, if $p(x) = \sum_\alpha p_\alpha x^\alpha := \sum_\alpha p_\alpha x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ is a polynomial of degree $2d$ in n variables, where the α's are the exponent vectors, belonging to the set $A = \{(\alpha_1,\dots,\alpha_n) : \alpha_1 + \cdots \alpha_n \le 2d,\, \alpha_i \ge 0, \forall i = 1,\dots,n\}$, the equation

$$p(x) = [x]_d{}^T Q[x]_d$$

is equivalent to the conditions

$$p_\alpha = \sum_{\beta+\gamma=\alpha} Q_{\beta\gamma}, \quad Q \succeq 0.$$

Here, $Q_{\beta\gamma}$ denotes the entry of Q in the row associated to the monomial x^β and in the column associated to the monomial x^γ. Thus, $p(x) = \sum_\alpha p_\alpha x^\alpha$ is an sos if and only if the semidefinite program

$$\begin{aligned} &\text{minimize } \langle \mathbb{O}, Q\rangle \\ &\text{subject to } \langle A^\alpha, Q\rangle = p_\alpha,\ \alpha \in A, \\ &\qquad\qquad Q \succeq 0 \end{aligned}$$

is feasible, where $\mathbb{O}$ is the $\binom{n+d}{d}\times\binom{n+d}{d}$ all-zero matrix and the A^α's are defined as

$$A^\alpha_{\beta\gamma} = \begin{cases} 1, & \beta+\gamma=\alpha \\ 0, & \text{otherwise.} \end{cases}$$

Note that the A^{α}'s are symmetric, because $\beta+\gamma=\gamma+\beta$, and $\langle A^{\alpha},Q\rangle=\sum_{\beta+\gamma=\alpha}Q_{\beta\gamma}$.

Example A.1.5. Consider the polynomial $p(x)=x^4+4x^3+6x^2+4x+5$, for which we want to find an sos decomposition. According to Theorem A.1.3, we wish to solve

$$p(x)=\begin{bmatrix}1\\x\\x^2\end{bmatrix}^T\begin{bmatrix}Q_{11}&Q_{12}&Q_{13}\\Q_{12}&Q_{22}&Q_{23}\\Q_{13}&Q_{23}&Q_{33}\end{bmatrix}\begin{bmatrix}1\\x\\x^2\end{bmatrix}=Q_{33}x^4+2Q_{23}x^3+(Q_{22}+2Q_{13})x^2+2Q_{12}x+Q_{11}$$

where $Q\succeq 0$. Because the coefficients of both polynomials must match, the linear constraints to consider are:

$$Q_{33}=1,\quad 2Q_{23}=4,\quad Q_{22}+2Q_{13}=6,\quad 2Q_{12}=4,\quad Q_{11}=5.$$

Choosing $Q_{13}=0$, we obtain

$$Q=\begin{bmatrix}5&2&0\\2&6&2\\0&2&1\end{bmatrix}=V^TV,\quad V=\begin{bmatrix}0&2&1\\\sqrt{2}&\sqrt{2}&0\\\sqrt{3}&0&0\end{bmatrix},$$

which results in the following sos decomposition

$$p(x)=(2x+x^2)^2+(\sqrt{2}+\sqrt{2}x)^2+\sqrt{3}^2=(2x+x^2)^2+2(1+x)^2+3.$$

It is straightforward to see that any sum of squares in n variables is necessarily nonnegative on $\mathbb{R}^n$. Moreover, sums of squares decompositions can be used for certifying nonnegativity on particular subsets of $\mathbb{R}^n$, such as real varieties.

Definition A.1.6. A subset $I\subseteq\mathbb{R}[x_1,\dots,x_n]$ is an ideal if it satisfies:

$$\begin{aligned}&i)0\in I.\\&ii)\text{If } f,g\in I, f+g\in I.\\&iii)\text{If } f\in I \text{ and } h\in\mathbb{R}[x_1,\dots,x_n],\ \text{then } hf\in I.\end{aligned}$$

Definition A.1.7. Let $f_1,\dots,f_s$ be polynomials in $\mathbb{R}[x_1,\dots,x_n]$. Then,

$$\langle f_1,\dots,f_s\rangle:=\Big\{\sum_{i=1}^{s}h_if_i : h_1,\dots,h_s\in\mathbb{R}[x_1,\dots,x_n]\Big\}.$$

Note that $\langle f_1,\dots,f_s\rangle$ is an ideal. It is called the ideal generated by $f_1,\dots,f_s$.

Theorem A.1.8 (Hilbert Basis Theorem)**.** *Every ideal $I\subseteq\mathbb{R}[x_1,\dots,x_n]$ has a finite generating set. In other words, $I=\langle f_1,\dots,f_s\rangle$, for some $f_1,\dots,f_s$.*

Definition A.1.9. Let $I = \langle f_1, \dots, f_s \rangle \subseteq \mathbb{R}[x_1, \dots, x_n]$ be an ideal. The set

$$V_{\mathbb{R}}(I) = \{p \in \mathbb{R}^n : g(p) = 0, \forall g \in I\} = \{p \in \mathbb{R}^n : f_i(p) = 0, i = 1, \dots, s\}$$

is called the real variety associated to I.

Suppose $I \subseteq \mathbb{R}[x_1, \dots, x_n]$ is an ideal and let $V_{\mathbb{R}}(I)$ be the corresponding real variety. If a polynomial $f(x)$ can be decomposed as $h(x) + g(x)$, where $h(x)$ is a sum of squares and $g(x) \in I$, then, for any $p \in V_{\mathbb{R}}(I)$,

$$f(p) = h(p) + g(p) = h(p) \geq 0,$$

i.e., $f(x)$ is nonnegative on $V_{\mathbb{R}}(I)$.

Definition A.1.10. Let $I \subseteq \mathbb{R}[x_1, \dots, x_n]$ be an ideal. A polynomial $f(x)$ is said to be a sum of squares modulo I (sos mod I) if there exists a sum of squares $h(x)$ such that $f(x) - h(x) \in I$.

Real Nullstellensatz, a well-known characterization for empty real varieties, can be restated in terms of sums squares modulo an ideal.

Theorem A.1.11 (Real Nullstellensatz)**.** *Let $I \subseteq \mathbb{R}[x_1, \dots, x_n]$ be an ideal. Then,*

$$V_{\mathbb{R}}(I) = \emptyset \textit{ if and only if } -1 \textit{ is sos mod } I.$$

Example A.1.12. Let $I = \langle 1 - x^2 - y^2, x - 2 \rangle$. We have $V_{\mathbb{R}}(I) = \{(x, y) \in \mathbb{R}^2 : 1 - x^2 - y^2 = 0, x - 2 = 0\} = \emptyset$ and

$$-1 - ((x-1)^2 + y^2 + 1^2) = (1 - x^2 - y^2) + 2(x - 2) \in I,$$

i.e., -1 is sos mod I.

The issue with finding sums of squares decompositions modulo an ideal for a given polynomial is that the sos decomposition degree is not known in advance. Typically, the procedure is to fix that degree and then solve a semidefinite program. For instance, suppose we want to write a given polynomial $f(x)$ as $h(x) + g(x)$, where $h(x)$ is sos and $g(x) \in I$, with $I = \langle g_1 \dots, g_n \rangle$. If we want $h(x)$ to have degree at most $2d$, we try to find an sos decomposition, through semidefinite programming, for $f(x) - \sum_{i=1}^n s_i(x) g_i(x)$, where each $s_i(x) g_i(x)$ has degree at most $2d$. Since any ideal has more than one generating set, this procedure depends on the generating set chosen, which is fixed beforehand.

Definition A.1.13. Let $I \subseteq \mathbb{R}[x_1, \dots, x_n]$ be an ideal. A polynomial $f(x)$ is said to be a sum of squares modulo I of degree at most $2d$ if there exists a sum of squares $h(x)$ of degree at most $2d$ such that $f(x) - h(x) \in I$.

Note that being an sos modulo an ideal of degree is only a sufficient condition for being an sos modulo an ideal. However, for $2d$ large enough, if f is an sos modulo an ideal is also an sos modulo an ideal of degree at most $2d$.

As a practical example, suppose we want to prove that a list of polynomial equations $g_i(x) = 0, i = 1,\ldots,n$ has no real solutions, i.e, $V_{\mathbb{R}}(I) = \emptyset$, with $I = \langle g_1,\ldots,g_n\rangle$. A possible of way doing it is to keep increasing d until we can find an sos decomposition modulo I of degree at most $2d$ for -1, which guarantees the real variety is empty, as stated in Theorem A.1.11.

A.2 The γ_2 norm

Here, we provide some background, mostly taken from [50], for understanding the bound involved in Theorem 4.2.5.

Recall that any linear operator between vector spaces can be represented by a matrix. Given two norms E_1 and E_2 on $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, the operator norm $||.||_{E_1\to E_2}$ is defined as

$$||A||_{E_1\to E_2} = \sup_{||x||_{E_1}=1} ||Ax||_{E_2},$$

for each $A \in \mathbb{C}^{m\times n}$.

For three normed vector spaces W_1, W_2 and Z and an operator $T : W_1 \to W_2$, the factorization problem deals with representations of the operator T as $T = uv$, where $v : W_1 \to Z$ and $u : Z \to W_2$, such that v and u have small norms. For fixed spaces W_1 and W_2 and $T : W_1 \to W_2$, the factorization constant is defined as

$$\gamma_Z(T) = \inf\{\|v\|_{W_1\to Z}\|u\|_{Z\to W_2} : uv = T\}.$$

It can be shown that if Z is an Hilbert space, then for any W_1 and W_2 the factorization constant is a norm on the space of operators between W_1 and W_2 [73]. In the case which $W_1 = l_1^n$, $W_2 = l_\infty^m$ and $Z = l_2$, the factorization constant is denoted by

$$\gamma_2(A) = \inf\{||X||_{l_2\to l_\infty^m}||Y||_{l_1^n\to l_2} : XY = A\}.$$

If A is an $m\times n$ matrix, $||A||_{l_2\to l_\infty^m}$ and $||A||_{l_1^n\to l_2}$ turn out to be equal to the largest l_2-norm of a row in A and the largest l_2-norm of a column in A, respectively. Thus,

$$\gamma_2(A) = \min\{\max_{i,j} ||x_i||_{l_2}||y_j||_{l_2} : XY^T = A\},$$

where $\{x_i\}_{i=1}^m$ and $\{y_j\}_{j=1}^n$ are the rows of X and Y respectively. The dual norm of $\gamma_2(A)$, $\gamma_2^*(A)$, which shows up in the bound expression of Theorem 4.2.5, is

$$\gamma_2^*(A) = \max_{B:\gamma_2(B)\le 1} |\langle A,B\rangle| = \max_{B:\gamma_2(B)\le 1} |\operatorname{Tr}(A^*B)|.$$

The γ_2 norm can be written as the solution to a semidefinite program [45]:

$$\gamma_2(A) = \min R \quad s.t.$$
$$W = \begin{bmatrix} W_1 & A \\ A^* & W_2 \end{bmatrix} \succeq 0, \quad \operatorname{diag}(W) \leq R.$$

Hence,

$$\gamma_2^*(A) = \max_{B:\gamma_2(B)\leq 1} |\operatorname{Tr}(A^*B)| =$$

$$= \max |\operatorname{Tr}(A^*B)| \quad s.t.$$
$$W = \begin{bmatrix} W_1 & B \\ B^* & W_2 \end{bmatrix} \succeq 0, \quad \operatorname{diag}(W) \leq 1.$$

We can have a better idea of how big is $\gamma_2^*(A)$ by using the famous Grothendieck's Inequality. See for instance [69] for a discussion that gives us the following well-known norm equivalence.

Proposition A.2.1. *For any complex matrix $A \in \mathbb{C}^{n\times m}$ we have*

$$\|A\|_{l_\infty \to l_1} \leq \gamma_2^*(A) \leq K_G^{\mathbb{C}} \|A\|_{l_\infty \to l_1}.$$

Here, the norm $\|A\|_{l_\infty \to l_1}$ is given by

$$\|A\|_{l_\infty \to l_1} = \max_{|s_i|,|t_j|\leq 1} \left| \sum_{i,j} A_{ij} s_i t_j \right|,$$

and $K_G^{\mathbb{C}}$ is the complex Grothendieck's constant, known to be between 1.338 and 1.4049 (see [59], section 4).