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SMOOTHING AND INTERPOLATION ON THE ESSENTIAL MANIFOLD

Tese no âmbito do Programa Interuniversitário de Doutoramento em Matemática, orientada pela Professora Doutora Maria de Fátima da Silva Leite e apresentada ao Departamento de Matemática da Faculdade de Ciências e Tecnologia da Universidade de Coimbra.

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Abstract

Interpolating data in non-Euclidean spaces plays an important role in different areas of knowledge. The main goal of this thesis is to present, in detail, two different approaches for solving interpolation problems on the Generalized Essential manifold $G_{k,n} \times \mathrm{SO}(n)$, consisting of the product of the Grassmann manifold of all k-dimensional subspaces of $\mathbb{R}^n$ and the Lie group of rotations in $\mathbb{R}^n$. The first approach to be considered is a generalization to manifolds of the De Casteljau algorithm and the second is based on rolling motions.

In order to achieve our objective, we first gather information of all the essential topics of Riemannian geometry and Lie theory necessary for a complete understanding of the geometry of the fundamental manifolds involved in this work, with particular emphasis on the Grassmann manifold and on the Normalized Essential manifold.

To perform the De Casteljau algorithm in the manifold $G_{k,n} \times \mathrm{SO}(n)$ we adapt a procedure already developed for connected and compact Lie groups and for spheres, and accomplish the implementation of that algorithm, first for the generation of geometric cubic polynomials in the Grassmann manifold $G_{k,n}$, and then extending it to generate cubic splines in the same manifold. New expressions for the velocity vector field along geometric cubic polynomials and for its covariant derivative are derived in order to obtain admissible curves that also fulfil appropriate boundary conditions.

To solve the interpolation problem using the second approach, we propose an algorithm inspired in techniques that combine rolling/unrolling with unwrapping/wrapping, but accomplishing the objective using rolling motions only. Interpolating curves given in explicit form are obtained for the manifold $G_{k,n} \times \mathrm{SO}(n)$, which also prepares the ground for applications using the Normalized Essential manifold. The definition of rolling map is a crucial tool in this approach. We present a geometric interpretation of all the conditions present in that definition, including a refinement of the non-twist conditions which allows to prove interesting properties of rolling and, consequently, simplifies the study of rolling motions. In particular, the non-twist conditions are rewritten in terms of parallel vector fields, allowing for a clear connection between rolling and parallel transport. When specializing to the rolling manifold $G_{k,n} \times \mathrm{SO}(n)$ the definition of rolling map is adjusted in order to avoid destroying the matrix structure of that manifold. We also address controllability issues for the rolling motion of the Grassmann manifold $G_{k,n}$. In parallel with a theoretical proof, we present a constructive proof of the controllability of the kinematic equations that describe the pure rolling motions of the Grassmann manifold $G_{k,n}$ over the affine tangent space at a point.

We make connections with other known approaches to generate interpolating curves in manifolds and point out some directions for future work.

Resumo

A interpolação de dados em espaços não Euclidianos desempenha um papel importante em diferentes áreas do conhecimento. O objetivo principal desta tese é apresentar, em detalhe, duas abordagens diferentes para resolver problemas de interpolação na variedade Essencial Generalizada $G_{k,n} \times \mathrm{SO}(n)$, que consiste no produto cartesiano da variedade de Grassmann formada por todos os subespaços k-dimensionais de $\mathbb{R}^n$ e o grupo de Lie das rotações em $\mathbb{R}^n$. A primeira abordagem a ser considerada é uma generalização para variedades do algoritmo de De Casteljau e a segunda é baseada em certos movimentos de rolamento.

A fim de alcançar o nosso objetivo, primeiro reunimos informações de todos os tópicos essenciais de geometria Riemanniana e de teoria de Lie necessários para uma completa compreensão da geometria das variedades fundamentais envolvidas neste trabalho, com particular ênfase na variedade de Grassmann e na variedade Essencial Normalizada.

Para implementar o algoritmo de De Casteljau na variedade $G_{k,n} \times \mathrm{SO}(n)$, adaptamos um procedimento já conhecido para grupos de Lie conexos e compactos e para esferas, e realizamos a implementação desse algoritmo, primeiro para a geração de polinómios geométricos cúbicos na variedade de Grassmann $G_{k,n}$, e depois estendemo-lo para gerar *splines* cúbicos na mesma variedade. São deduzidas novas expressões para o campo de vetores velocidade ao longo dessas curvas e para a sua derivada covariante, a fim de obter curvas admissíveis que também satisfaçam condições de fronteiras apropriadas.

Para resolver o problema de interpolação utilizando a segunda abordagem, propomos um algoritmo inspirado em técnicas que combinam *rolling/unrolling* com *unwrapping/wrapping*, mas cumprindo o objetivo utilizando apenas movimentos de rolamento. As curvas de interpolação para a variedade $G_{k,n} \times \mathrm{SO}(n)$ são obtidas de forma explícita, o que também prepara o terreno para aplicações utilizando a variedade Essencial Normalizada. A definição de aplicação rolamento é uma ferramenta crucial nesta abordagem. Apresentamos uma interpretação geométrica de todas as condições presentes nessa definição, incluindo um refinamento das condições de *non-twist* o que permite provar propriedades interessantes de rolamento e, consequentemente, simplifica o estudo dos movimentos de rolamento. Em particular, as condições de *non-twist* são reescritas em termos de campos vectoriais paralelos, permitindo uma ligação clara entre o rolamento e o transporte paralelo. Quando é especificada para a variedade de rolamento $G_{k,n} \times \mathrm{SO}(n)$, a definição de aplicação rolamento é ajustada de forma a evitar destruir a estrutura matricial dessa variedade. Também abordamos questões de controlabilidade para o movimento de rolamento da variedade de Grassmann $G_{k,n}$. Em paralelo com uma prova teórica, apresentamos uma prova construtiva da controlabilidade das equações da cinemática que descrevem os movimentos de rolamento puro da variedade de Grassmann $G_{k,n}$ sobre o espaço afim associado ao espaço tangente num ponto.

Estabelecemos algumas relações com outras abordagens conhecidas para gerar curvas interpoladoras em variedades e apresentamos algumas direções para o trabalho futuro.

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Chapter 1

Introduction

Solving interpolation problems involving data on manifolds is a hot topic in a growing number of applications in different areas of our society, ranging from computer vision and robotics, to industrial and medical applications (see, for instance, Bressan [8]). Research in these areas has been booming over the past few years and is continuously getting new insights from the rigour of mathematics, in particular from methods and technics of Differential Geometry.

Computer vision, for instance, is a challenging topic which is being used in a wide variety of real world applications, such as earth observation, optical character recognition, 3D model building, medical imaging, machine inspection, automotive safety, match move, motion capture, surveillance, fingerprint recognition and biometrics. We refer, for instance, Szeliski [71] and references therein for details concerning multiple applications in this area. The problem of recovering structure and motion from a sequence of images, also known as stereo matching, is also a crucial problem in computer vision and continues to be one of the most active research areas with remarkable progress in imaging and computing hardware (see also Ma et al. [50]).

The real applications previously mentioned served as a motivation for our study about interpolation problems in the Generalized Essential manifold $G_{k,n} \times \mathrm{SO}(n)$, which is the cartesian product of the real Grassmann manifold $G_{k,n}$, consisting of all k-dimensional (linear) subspaces of $\mathbb{R}^n$, and $\mathrm{SO}(n)$, the group of rotations in $\mathbb{R}^n$. One particular case of this is the Normalized Essential manifold, corresponding to $k = 2$ and $n = 3$, which plays an important role in image processing. The classical problem of reconstructing a scene, or a video, from several images of the scene can be formulated as an interpolation problem on that manifold, since it encodes the epipolar constraint. Typically, it is given an ordered set of time-labeled essential matrices, $E_1, \dots, E_n$ relating n different consecutive camera views (*snapshots*), and the objective is to calculate a continuum of additional virtual views by computing a smooth interpolating curve through the E_i's, $i = 1, \dots, n$.

Although the existence in the literature of different methods for solving interpolation problems in manifolds, in this thesis we explore two distinct approaches for the Generalized Essential manifold. The first one, in Chapter 4, is a generalization of the classical De Casteljau algorithm for Riemannian manifolds, which is a powerful tool widely used to, recursively, generate interpolating polynomial curves on manifolds. The second one, in Chapter 6, is based on rolling motions of one manifold over another one which is static, and it is inspired by the work of Hüper and Silva Leite [31]. This approach allows to solve, efficiently, interpolation problems on manifolds using rolling techniques. By choosing the static manifold to behave like an Euclidean space, it is possible to transform a difficult problem on a curved space into an easy problem on a flat space.

For both approaches presented here, the knowledge of explicit formulas for the geodesic that joins two points on the manifolds under study was crucial to solve explicitly the interpolation data problems.

There is a considerable number of papers showing applications of cubic splines in the aerospace and robotics industries, related to air traffic control and path planning. In this context, the manifold $\mathrm{SO}(3)$ plays a crucial role since the configuration space of most mechanical systems, such as airplanes, underwater vehicles and robots, has components represented by rotations in $\mathbb{R}^3$. Most algorithms to generate smooth paths on that manifold are based on the classical De Casteljau approach in $\mathbb{R}^4$, since 3-dimensional rotations can be represented by quaternions. Examples of that are the pioneer work Shoemake [70] and, for instance, Kim et al. [40], Kang and Park [39], Popiel and Noakes [66], just to name a few.

There are other engineering problems where rolling methods have been used successfully. We mention, for instance, Caseiro et al. [12] for an application of rolling to solve multi-class classification problems in computer vision, Batista et al. [4] to solve object recognition problems across dynamically evolving datasets, and Vemulapalli and Chellapa [72] for an application of rolling rotations to recognizing human actions from 3D skeletal data.

This thesis is organized in the following way. After this introductory chapter that started with a motivation based on potential areas of real applications, we proceed with two preliminary chapters containing the necessary background to help fully understanding the main results in subsequent chapters and to make this work more self-contained. These are followed by three chapters where the principal original results are presented, and a final shorter chapter with some remarks to future work and related problems. The three main chapters, always start with a short introduction, followed by some literature review, and our contributions to the problems under study.

Next, we give a more detailed outline about the contents of each chapter after the introduction.

In Chapter 2 we recall some essential topics of Riemannian geometry and Lie theory that will be used throughout the thesis. We start with the notions of Riemannian metric, Riemannian manifold, Riemannian connection, covariant derivative, parallel and normal transport of vector fields. After that we review the concepts of geodesics, distances, geodesic completeness, Lie algebras, Lie groups, group actions, and give particular attention to the description of some properties of matrix Lie groups.

The Chapter 3 is dedicated to study the geometry of the Riemannian manifolds that will play the main role in this thesis. Particular emphasis on the Grassmann manifold and on the Normalized Essential manifold is considered. Although regarded as a preliminary chapter, we also present here some results that, to the best of our knowledge, are new. The importance of the Grassmann manifold in certain applications dealing with nonlinear data based on images is also highlight.

In Chapter 4, using the classical De Casteljau algorithm and its generalization to geodesically complete Riemannian manifolds, we present solutions of an interpolation problem that generalize cubic splines in Euclidean spaces, and pay particular attention to problems with two different types of boundary conditions, the well-known Hermite boundary conditions, and the set of boundary conditions consisting of initial and final points, initial velocity and initial covariant acceleration.

Further, inspired by Crouch et al. [13], a review of the algorithm to generate cubic polynomials in the Euclidean space $\mathbb{R}^m$ is presented, followed by a detailed description and implementation of that algorithm for the Grassmann manifold $G_{k,n}$, using recursive geodesic interpolation. Geometric cubic splines are generated from cubic polynomials for the later manifold. This generation is immediate for the second type of boundary conditions but not for those of Hermite type, which are more natural from the point of view of applications. To overcome this difficulty, we show how each type of boundary conditions are related to each other for the interpolation problem in $G_{k,n}$, and briefly review the interpolation problem in $\mathrm{SO}(n)$, which has been studied in Crouch et al. [13], in order to extend the results to $G_{k,n} \times \mathrm{SO}(n)$ and, in particular, to the Normalized Essential manifold. The results of this chapter have been published recently in Pina and Silva Leite [65] and a shorter version, Pina and Silva Leite [64], has just been submitted.

Chapter 5 looks into the problem of rolling, without slipping and without twisting, two connected and oriented manifolds of the same dimension, both isometrically embedded in the same Riemannian complete manifold. One of the embedded manifolds is the rolling manifold and the other is the static manifold. These rolling motions can be considered as generalizations of rigid body motions, subject to additional holonomic and nonholonomic constraints. The chapter starts with the definition of rolling map for manifolds embedded in a general Riemannian manifold, and with a geometric interpretation of the rolling conditions (holonomic constraints) and of the non-slip and non-twist conditions (nonholonomic constraints). Using the geometric interpretation of the non-twist conditions, a clear connection between rolling and parallel transport is fulfilled. Some interesting properties that allow to reduce the study of any rolling motion to the case where the static manifold is the affine tangent space at a point of the rolling manifold are also presented. Motivated by the particular situation when the embedding manifold is an Euclidean space, the general structure for the kinematic equations of rolling $G_{k,n} \times \mathrm{SO}(n)$ is given. The particular case when $k = 2$ and $n = 3$, i.e., the rolling motion of the Normalized Essential manifold that plays an important role in the area of Vision follows from the previous. Results for the later case have been published in Machado et al. [51]. This chapter ends with the study of the controllability of the kinematic equations describing the rolling motion of Grassmann Manifolds over the affine tangent space at a point. These kinematic equations, which can be seen as a nonlinear nonholonomic control system evolving on a certain Lie group, are proved to be controllable in some subgroup of the group of isometries of the embedding space of the two manifolds. Moreover, a constructive proof of controllability of the rolling motion by showing that all admissible configurations can be recovered by motions that do not violate the non-slip and non-twist constraints is presented. These results have also been published in Pina and Silva Leite [63].

In Chapter 6, taking advantage of the developments and results in the previous chapter, we present a simpler approach to solve interpolation problems based only on rolling motions. The basic idea of this method is projecting the data from the original non-Euclidean manifold to a simpler manifold, where the problem can be easily solved, and then reversing the process by projecting the resulting interpolation curve to the original manifold. This procedure, that combines techniques of unwrapping/wrapping with rolling/unrolling, can be done successfully using rolling motions of the given manifold over its affine tangent space at a point, and it produces an interpolating curve that is given in closed form. For the sake of simplicity, in this chapter, we only give details for a $\mathscr{C}^2$-smooth interpolating curve that solves a two-boundary value problem of Hermite type, but more general problems can be solved in a similar way, although computationally more expensive. Further, we also confirm that the proposed algorithm works for the manifold $G_{k,n} \times \mathrm{SO}(n)$, preparing the ground for applications in the Normalized Essential manifold, and the unwrapping/wrapping techniques of the presented algorithm are performed with rolling maps. So, contrary to previous works, the main contribution in this chapter is that we present an implementation of the interpolation algorithm which is entirely based on rolling motions.

Chapter 7 briefly addresses related work and also points out some open problems and directions for future research.

Chapter 2

Preliminary Concepts

In this chapter we recall some important preliminary concepts, including essential topics of Riemannian geometry and Lie theory that will be used throughout the thesis.

The main references used to construct this chapter were the following: Bishop [7], do Carmo [18], Duistermaat and Kolk [19], Lee [47] and O'Neill [61], for details concerning Riemannian manifolds, Lie groups and Lie algebras; Absil et al. [1] and Horn and Johnson [27], for topics related with matrix analysis.

2.1 Riemannian Metric

Let M be a smooth manifold of finite dimension. If $p \in M$, we denote by T_pM the tangent space of M at p and by TM the tangent bundle of M, which is the disjoint union of all tangent spaces, i.e., $TM = \bigcup_{p\in M} T_pM$, while $\mathfrak{X}(M)$ denotes the set of all smooth vector fields on M.

The tangent space T_pM of M at p is defined as the set of all tangent vectors at the point p, and for different, but equivalent, definitions of the concept of a tangent space we mention, for instance, Jänich [33]. We emphasize that, in this work, we will consider two possible approaches of this concept. An algebraic definition, where briefly, tangent vectors are regarded as derivations acting on real-valued functions on M, and a geometric definition, where the tangent space T_pM of M at p is naturally identified with the vector space of velocity vectors of differentiable curves in M passing through p at $t = 0$, i.e.,

$$T_pM = \{\dot{\gamma}(0) : \ \gamma : J \to M \text{ smooth curve in } M \ \text{ and } \ \gamma(0) = p\},$$

where, $J \subset \mathbb{R}$ is an arbitrary small open interval with $0 \in J$. Whenever convenient, we consider that γ is defined on a closed interval $J = [a,b]$, which means that $\gamma = \widetilde{\gamma}|_{[a,b]}$, where $\widetilde{\gamma}$ is a smooth curve in M defined on an open interval containing $J = [a,b]$.

Let $f : M \to N$ be a smooth map between two smooth manifolds M and N of finite dimension. The *pushforward or differential of f at a point $p \in M$*, will be denoted by d_pf and is the linear map defined as follows

$$\begin{array}{rccl} d_pf: & T_pM & \longrightarrow & T_{f(p)}N \\ & X_p & \longmapsto & d_pf(X_p) = \left.\dfrac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\alpha(t)), \end{array} \tag{2.1}$$

where $t \mapsto \alpha(t)$ is a smooth curve in M, satisfying $\alpha(0) = p$ and $\dot{\alpha}(0) = X_p$.

In the course of this work, unless otherwise stated, the manifolds will always be assumed to be of finite dimension.

Definition 2.1.1 *A Riemannian metric on a smooth manifold M is a covariant 2-tensor field g on M that is symmetric (i.e., $g(X,Y) = g(Y,X)$, for all $X,Y \in \mathfrak{X}(M)$), and positive definite (i.e., $g(X,X) > 0$, if $X \neq 0$).*
A Riemannian manifold is a pair (M,g), where M is a smooth manifold and g is a Riemannian metric on M.

Whenever there is no ambiguity about the Riemannian metric, the Riemannian manifold (M,g) will be simply denoted by M.

A Riemannian metric g on M determines an inner product on each tangent space T_pM, given by the nondegenerate bilinear form

$$\begin{aligned} g_p : T_pM \times T_pM &\longrightarrow \mathbb{R} \\ (X_p, Y_p) &\longmapsto g_p(X_p, Y_p), \end{aligned} \tag{2.2}$$

satisfying

- $g_p(X_p,Y_p) = g_p(Y_p,X_p)$; (Symmetry)
- $g_p(X_p,X_p) \geq 0$, and $g_p(X_p,X_p) = 0$, if and only if, $X_p = 0$. (Positive definiteness)

Therefore, we may use the notation $\langle .,. \rangle_p$ for the inner product on each tangent space T_pM associated with the metric g, that is, $\langle X_p, Y_p \rangle_p := g_p(X_p,Y_p)$, and when it is clear from the context, for simplicity of notation, we will drop the index p.

Similarly to the Euclidean geometry, if p is a point in a Riemannian manifold (M,g), the length or norm of any tangent vector $X_p \in T_pM$ is defined by $\|X_p\| := \langle X_p, X_p \rangle^{\frac{1}{2}}$. Moreover, unless otherwise specified, the angle between two non zero vectors $X_p, Y_p \in T_pM$ is defined as the unique angle $\theta \in [0,\pi]$ satisfying $\cos\theta = \langle X_p,Y_p\rangle / (\|X_p\|\|Y_p\|)$ and the vectors X_p and Y_p are said orthogonal if their angle is $\pi/2$, or equivalently, if $\langle X_p, Y_p \rangle = 0$. The vectors $X_1,\ldots,X_k \in T_pM$ are called orthonormal if they have length one and are pairwise orthogonal, or equivalently, if $\langle X_i, X_j \rangle = \delta_{ij}$, where $\delta_{ij},\ i,j = 1,\ldots,k$, denotes the Kronecker delta function.

Example 2.1.1 *One obvious example of a Riemannian manifold is $\mathbb{R}^m$ with its Euclidean metric, which is just the usual inner product on each tangent space, under the natural identification $T_p\mathbb{R}^m \cong \mathbb{R}^m$.*

Example 2.1.2 *Let $\mathfrak{gl}(n)$ be the set of all $n \times n$ matrices with real entries. This set can be easily identified with $\mathbb{R}^{n^2}$ as follows:*

$$A = \left[\begin{array}{c|c|c|c} a_1 & a_2 & \ldots & a_n \end{array}\right] \in \mathfrak{gl}(n) \longleftrightarrow vec(a) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^{n^2}$$

where $a_1, a_2, \dots, a_n$ denote the columns of A. With this identification, the Euclidean metric $\langle .,. \rangle$ of $\mathbb{R}^{n^2}$ can be rewritten, in matrix notation, by:

$$\langle vec(a), vec(b) \rangle = \mathrm{tr}(A^\top B), \tag{2.3}$$

with $A,\ B \in \mathfrak{gl}(n)$ and where $\mathrm{tr}(.)$ denotes the trace of the matrix within the parenthesis.

According with (2.3), the Euclidean metric on $\mathfrak{gl}(n)$ defined by

$$\langle A, B \rangle = \mathrm{tr}(A^\top B), \tag{2.4}$$

is, usually, called Frobenius metric, and $\mathfrak{gl}(n)$ with this metric is a Riemannian manifold.

Remark 2.1.1 *If (M_1, g_1) and (M_2, g_2) are Riemannian manifolds, the product manifold $M_1 \times M_2$ has a natural Riemannian metric $g = g_1 \oplus g_2$, called the product metric, defined by*

$$g_{(p_1,p_2)}(X_1 + X_2, Y_1 + Y_2) = {g_1}_{p_1}(X_1, Y_1) + {g_2}_{p_2}(X_2, Y_2), \tag{2.5}$$

where $X_i, Y_i \in T_{p_i}M_i$, $i = 1, 2$, under the natural identification $T_{(p_1,p_2)}(M_1 \times M_2) = T_{p_1}M_1 \oplus T_{p_2}M_2$.

If (M_1, g_1) and (M_2, g_2) are two Riemannian manifolds with the same dimension, a diffeomorphism ϕ from M_1 to M_2 is called an isometry if $\phi^* g_2 = g_1$, where ϕ^* denotes the pullback of ϕ. That is, if for all points $p \in M_1$ and any tangent vectors $X_1, Y_1 \in T_p M_1$, one has $g_1(X_1, Y_1) = g_2\left(d_p\phi(X_1), d_p\phi(Y_1)\right)$, where $d_p\phi$ denotes the pushforward or differential of ϕ at p. Also, (M_1, g_1) and (M_2, g_2) are isometric if there exists an isometry between them. It is easy to verify that being isometric is an equivalence relation on the class of Riemannian manifolds. The Riemannian geometry is concerned primarily with properties that are preserved by isometries. An isometric embedding of (M, g) in a manifold $(\overline{M}, \overline{g})$ is a smooth embedding $\phi : M \to \overline{M}$ that preserves the Riemannian metric, that is $\phi^* \overline{g} = g$.

An isometry $\phi : M \to M$ is called an isometry of M. Furthermore, since a composition of isometries and the inverse of an isometry are again isometries, the set of isometries of M forms a group, called the isometry group of M, usually denoted by $\mathsf{Isom(M)}$.

Example 2.1.3 *The isometry group of the Euclidean space $\mathbb{R}^m$ is the special Euclidean group $\mathrm{SE}(m)$. This is the group of rigid motions in $\mathbb{R}^m$ that preserve orientation, and it can be described by rotations and translations, since $\mathrm{SE}(m) = \mathrm{SO}(m) \ltimes \mathbb{R}^m$, where $\ltimes$ denotes the semi-direct product and $\mathrm{SO}(m)$ the group of special orthogonal matrices. We will represent elements of $\mathrm{SE}(m)$ as pairs (R, s), where $R \in \mathrm{SO}(m)$, $s \in \mathbb{R}^m$.*

Remark 2.1.2 *Let M be an (immersed) submanifold of a Riemannian manifold $(\overline{M}, \overline{g})$ and $\iota : M \hookrightarrow \overline{M}$ the inclusion map. The induced metric on M is the 2-tensor $g = \iota^*(\overline{g})$, which is the restriction of $\overline{g}$ to vectors tangent to M. Since the restriction of an inner product is itself an inner product, g defines a Riemannian metric on M. Then (M, g) is a Riemannian submanifold of $(\overline{M}, \overline{g})$ and ι is an isometric embedding.*

2.2 Riemannian Connection and Covariant Derivative

We start this section by introducing some notations that will be necessary for the right understanding of some further concepts. As before, M is a smooth manifold and $\mathfrak{X}(M)$ the set of all smooth vector fields on M. Let $\mathscr{C}^\infty(M)$ denote the algebra of all $\mathscr{C}^\infty$ real-valued functions on M and, furthermore, if $X,Y \in \mathfrak{X}(M)$ and $f \in \mathscr{C}^\infty(M)$, then

(a) Xf denotes the smooth real function on M defined by $(Xf)(p) := X_p(f)$;

(b) fX denotes the smooth vector field on M defined by $(fX)_p := f(p)X_p$;

(c) $[X,Y]$ denotes the smooth vector field on M defined by

$$[X,Y]_p(f) := X_p(Yf) - Y_p(Xf). \tag{2.6}$$

It is known that the set of all smooth vector fields on M, equipped with the Lie bracket $[.,.]$ defined in (2.6) forms a Lie algebra.

Definition 2.2.1 *A linear connection ∇ on a smooth manifold M is a map*

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(M) \\ (X,Y) &\longmapsto \nabla(X,Y) = \nabla_X Y \end{aligned} \tag{2.7}$$

that satisfies the following three properties:

1. *$\nabla_X Y$ is linear over $\mathscr{C}^\infty(M)$ in X, i.e.,*

$$\nabla_{f_1X_1+f_2X_2}Y = f_1\nabla_{X_1}Y + f_2\nabla_{X_2}Y, \qquad \text{for } f_1, f_2 \in \mathscr{C}^\infty(M);$$

2. *$\nabla_X Y$ is $\mathbb{R}$-linear in Y, i.e.,*

$$\nabla_X(a_1Y_1 + a_2Y_2) = a_1\nabla_X Y_1 + a_2\nabla_X Y_2, \qquad \text{for } a_1, a_2 \in \mathbb{R};$$

3. *∇ satisfies the following product rule:*

$$\nabla_X(fY) = f\nabla_X Y + (Xf)Y, \qquad \text{for } f \in \mathscr{C}^\infty(M).$$

The vector field $\nabla_X Y$ is called the covariant derivative of Y in the direction of X.

A linear connection on M is also frequently called *affine connection on M*, or simply *connection on M*. Notice that, although a linear connection on M satisfies the product rule mentioned above, since it is not linear over $\mathscr{C}^\infty(M)$ in Y, it is not a tensor field. By definition, a linear connection on M is a way to compute covariant derivatives of vectors fields on M and, it is possible to show that, every smooth manifold admits a linear connection (see, for instance, Lee [47]).

During this work, given $X \in \mathfrak{X}(M)$ and $f \in \mathscr{C}^\infty(M)$, the notation $\nabla_X f$ will also be used, whenever convenient, for the directional derivative of f in the direction of X, usually denoted by Xf.

Definition 2.2.2 *Let (M,g) be a Riemannian manifold. A linear connection on M is said to be:*

1. *Compatible with the metric g if it satisfies, for all vector fields $X, Y, Z \in \mathfrak{X}(M)$, the product rule:*

$$\nabla_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle. \tag{2.8}$$

2. *Symmetric if, for all vector fields $X, Y \in \mathfrak{X}(M)$,*

$$[X, Y] = \nabla_X Y - \nabla_Y X. \tag{2.9}$$

The next result is a well-known outcome of Riemannian Geometry and its proof can be found, for instance, in do Carmo [18].

Theorem 2.2.1 (Fundamental Theorem of Riemannian Geometry) *Let (M, g) be a Riemannian manifold. There exists a unique linear connection ∇ on M, that is compatible with the metric g and symmetric.*

The connection mentioned in the Theorem 2.2.1 is called the *Riemannian connection* or the *Levi-Civita connection* of g (or, alternatively, on M (with respect to the metric g)).

Let us now consider (U, x) a local coordinate chart of M in $p \in M$, with local coordinates $x = (x_1, \ldots, x_m)$. For the local coordinate vector fields $\dfrac{\partial}{\partial x_i}$, $i = 1, \ldots, m$, it holds that,

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{m} \Gamma_{ij}^{k} \frac{\partial}{\partial x_k}, \quad i, j = 1, \ldots, m, \tag{2.10}$$

where the m^3 smooth functions Γ_{ij}^k defined on U are called the *Christoffel symbols* of the connection in these coordinates. Furthermore,

$$\Gamma_{ij}^{k} = \frac{1}{2} \sum_{l=1}^{m} g^{kl} \left(\frac{\partial}{\partial x_i} g_{jl} + \frac{\partial}{\partial x_j} g_{li} - \frac{\partial}{\partial x_l} g_{ij} \right), \tag{2.11}$$

where, for $i, j = 1, \ldots, m$, $g_{ij} := \langle \dfrac{\partial}{\partial x_i}, \dfrac{\partial}{\partial x_j} \rangle$ and g^{ij} denote the entries of the inverse matrix of the matrix $[g_{ij}]_{1 \leq i,j \leq m}$. Notice that, $g_{ij} g^{jk} = \delta_{ik}$, where δ_{ik}, $i, k = 1, \ldots, m$, denotes the Kronecker delta function.

Therefore, taking into account the properties present in the Definition 2.2.1, if X, $Y \in \mathfrak{X}(U)$ are given by $X = \sum_{i=1}^{m} X^i \dfrac{\partial}{\partial x_i}$ and $Y = \sum_{j=1}^{m} Y^j \dfrac{\partial}{\partial x_j}$, then in local coordinates

$$\nabla_X Y = \sum_{k=1}^{m} \left(X Y^k + \sum_{i,j=1}^{m} X^i Y^j \Gamma_{ij}^{k} \right) \frac{\partial}{\partial x_k}. \tag{2.12}$$

Example 2.2.1 *Let $x_1, \ldots, x_m$ be the standard coordinates of $\mathbb{R}^m$, $X, Y \in \mathfrak{X}(\mathbb{R}^m)$, with $Y = \sum_{i=1}^{m} Y^i \dfrac{\partial}{\partial x_i}$, and $\overline{\nabla}_X Y \in \mathfrak{X}(\mathbb{R}^m)$ the vector field on $\mathbb{R}^m$ defined by*

$$(\overline{\nabla}_X Y)_p = \sum_{i=1}^{m} X_p(Y^i) \frac{\partial}{\partial x_i}\Big|_p,$$

where $X_p(Y^i) = (XY^i)(p)$ is the directional derivative of Y^i in the direction of X at the point p.

The map

$$\begin{aligned}\overline{\nabla} : \mathfrak{X}(\mathbb{R}^m) \times \mathfrak{X}(\mathbb{R}^m) &\longrightarrow \mathfrak{X}(\mathbb{R}^m)\\ (X,Y) &\longmapsto \overline{\nabla}(X,Y) = \overline{\nabla}_X Y = \sum_{i=1}^{m} (XY^i)\frac{\partial}{\partial x_i}\end{aligned}$$

is a linear connection on $\mathbb{R}^m$, called the Euclidean connection. Note that the Christoffel symbols of this connection in standard coordinates are all zero.

Let $\gamma : J \subset \mathbb{R} \longrightarrow M$, be a smooth curve on M. A *vector field along the curve* γ is a smooth map $V : J \longrightarrow TM$ such that $V(t) \in T_{\gamma(t)}M$ for all $t \in J$. We will denote the space of smooth vector fields along γ by $\mathfrak{X}(\gamma)$ and, for $V \in \mathfrak{X}(\gamma)$ and $f \in \mathscr{C}^\infty(J)$, the vector field along γ, defined by $(fV)_t = f(t)V(t)$ will be represented by fV. An obvious example of a vector field along a smooth curve γ is its velocity vector. We are now able to state the following result, whose proof can be found, for instance, in Lee [47].

Proposition 2.2.1 *Let M be a smooth manifold with linear connection ∇. For each smooth curve $\gamma : J \subset \mathbb{R} \longrightarrow M$, the linear connection ∇ determines a unique operator*

$$\begin{aligned}\frac{D}{dt} : \mathfrak{X}(\gamma) &\longrightarrow \mathfrak{X}(\gamma)\\ V &\longmapsto \frac{DV}{dt}\end{aligned}$$

satisfying the following conditions:

1. *Linearity over $\mathbb{R}$:*
$$\frac{D(a_1V_1 + a_2V_2)}{dt} = a_1\frac{DV_1}{dt} + a_2\frac{DV_2}{dt} \qquad (a_1, a_2 \in \mathbb{R});$$
2. *Product Rule:*
$$\frac{D(fV)}{dt} = \frac{df}{dt}V + f\frac{DV}{dt} \qquad (f \in \mathscr{C}^\infty(J));$$
3. *if V is the restriction to $\gamma(t)$ of some vector field $Y \in \mathfrak{X}(M)$, that is, if for all $t \in J$, $V(t) = Y_{\gamma(t)}$, then*
$$\frac{DV}{dt} = \nabla_{\dot\gamma(t)}Y \qquad (t \in J).$$

For any $V \in \mathfrak{X}(\gamma)$, the vector field $\dfrac{DV}{dt}$, defined in Proposition 2.2.1, is called the *covariant derivative of V along γ.*

Example 2.2.2

1. *For the special case when M is the Euclidean space $\mathbb{R}^m$ and ∇ is the Euclidean connection, the covariant derivative coincides with the usual derivative.*

2. *When M is a Riemannian manifold embedded in some Euclidean space $\mathbb{R}^m$, the covariant derivative of V at t, is the projection of the usual derivative of V at t in $\mathbb{R}^m$, $\frac{dV}{dt}(t)$, into the tangent space of M at $\gamma(t)$. For instance, in the particular situation when M is the unit $(m-1)$-sphere S^{m-1}, we have that*

$$\frac{DV}{dt}(t) = \left(I - \gamma(t)\gamma(t)^\top\right)\frac{dV}{dt}(t), \tag{2.13}$$

where I denotes the identity matrix of order m.

During this text we may use the notation $\dot{\gamma}$ to denote the velocity vector field $\frac{d\gamma}{dt}$ along γ and, when convenient, we may also write $\frac{D\gamma}{dt}$ for the same velocity vector, so that the following definition makes sense:

$$\frac{D^k\gamma}{dt^k} := \frac{D}{dt}\left(\frac{D^{k-1}\gamma}{dt^{k-1}}\right), \quad \text{for } k \geq 2. \tag{2.14}$$

Remark 2.2.1 *The Euclidean connection $\overline{\nabla}$ introduced on the Example 2.2.1 is the Levi-Civita connection of the Euclidean space $\mathbb{R}^m$. Hereafter, for each Riemannian manifold, we will use the respective Levi-Civita connection on M and the corresponding covariant derivative associated with this connection.*

2.3 Parallel and Normal Transport

Let (M,g) be a Riemannian submanifold of $(\overline{M},\overline{g})$ (hereafter, written as $M \subset \overline{M}$), and $p \in M$. Each tangent space T_pM is, by definition, a nondegenerate subspace of $T_p\overline{M}$. Consequently, $T_p\overline{M}$ can be decomposed as a direct sum

$$T_p\overline{M} = T_pM \oplus (T_pM)^\perp, \tag{2.15}$$

where $(T_pM)^\perp$ is also nondegenerate and denotes the *orthogonal complement* of T_pM with respect to the metric $\overline{g}$ in $\overline{M}$. Vectors in $(T_pM)^\perp$ are said to be *normal* to M, while those in T_pM are said to be *tangent* to M. Similarly, a vector field Z on $\overline{M}$ is *normal* (respectively, *tangent*) to M provided that each value Z_p, for $p \in M$ belongs to $(T_pM)^\perp$ (respectively, T_pM). Projecting orthogonally, at each $p \in M$, $T_p\overline{M}$ onto the tangent and normal subspaces we get two maps, $\pi^\top$ and $\pi^\perp$, called the *tangential* and the *normal projections*, respectively, defined by

$$\begin{array}{rccc} \pi^\top: & T\overline{M}_{|M} & \to & TM \\ & X & \mapsto & X^\top \end{array} \quad \text{and} \quad \begin{array}{rccc} \pi^\perp: & T\overline{M}_{|M} & \to & (TM)^\perp \\ & X & \mapsto & X^\perp \end{array},$$

where $T\overline{M}_{|M} := \cup_{p\in M} T_p\overline{M}$.

If X,Y are two vector fields on M, we can extend them to $\overline{M}$, apply the ambient connection $\overline{\nabla}$ (the Levi-Civita connection with respect to $\overline{g}$) and then decompose at points of M to get

$$\overline{\nabla}_X Y = (\overline{\nabla}_X Y)^\top + (\overline{\nabla}_X Y)^\perp. \tag{2.16}$$

It turns out that the following formula, known as *Gauss Formula*, holds for vector fields X,Y, tangent to M:

$$\overline{\nabla}_X Y = \underbrace{\nabla_X Y}_{\text{tangent to } M} + \underbrace{(\overline{\nabla}_X Y)^{\perp}}_{\text{normal to } M}, \tag{2.17}$$

where ∇ is the connection with respect to g.

Considering $\gamma : J \subset \mathbb{R} \longrightarrow M$, a smooth curve in $M \subset \overline{M}$ and V a smooth vector field tangent to M along γ, the Gauss formula along the curve γ reduces to

$$\frac{\overline{D}V}{dt} = \underbrace{\frac{DV}{dt}}_{\text{tangent to } M} + \underbrace{\Pi(\dot{\gamma}, V)}_{\text{normal to } M}, \tag{2.18}$$

where $\frac{\overline{D}}{dt}$ (respectively, $\frac{D}{dt}$) denote the *extrinsic* (respectively, *intrinsic*) *covariant derivative along* γ. $\frac{D}{dt}$ is also called the *tangent covariant derivative along* γ, to distinguish from the *normal covariant derivative*, hereafter denoted by $\frac{D^{\perp}}{dt}$.

Definition 2.3.1 *A tangent vector field V along γ is said to be a* tangent parallel vector field along γ *if* $\frac{DV}{dt} \equiv 0$.

Usually, if $M \subset \overline{M}$, the geometry of M is considered that of vectors tangent to M. There is, however, an analogous geometry of vectors normal to M. If X is a tangent vector field to M and Z is a normal vector field to M, as in (2.16), we have

$$\overline{\nabla}_X Z = (\overline{\nabla}_X Z)^{\top} + (\overline{\nabla}_X Z)^{\perp}. \tag{2.19}$$

The *normal connection* of $M \subset \overline{M}$ is the function $\nabla^{\perp}$ that, to each pair (X,Z) of smooth vector fields, X tangent to M and Z normal to M, assigns a vector field $\nabla^{\perp}_X Z$ normal to M, defined by $\nabla^{\perp}_X Z = (\overline{\nabla}_X Z)^{\perp}$.

An analogous to the Gauss formula above, holds for vector fields, X tangent to M and Z normal to M,

$$\overline{\nabla}_X Z = \underbrace{(\overline{\nabla}_X Z)^{\top}}_{\text{tangent to } M} + \underbrace{\nabla^{\perp}_X Z}_{\text{normal to } M}. \tag{2.20}$$

The (tangent) connection ∇ was adapted to tangent vector fields along curves in $M \subset \overline{M}$, to produce the identity (2.18). Similarly, the normal connection $\nabla^{\perp}$ can also be adapted as follows to normal vector fields along curves on M. If Z is a normal vector field along a curve γ on M, then its normal covariant derivative $\frac{D^{\perp}}{dt}Z$ is defined to be the normal component of its $\overline{M}$ covariant derivative $\overline{\nabla}_{\dot{\gamma}}Z$, and the following holds

$$\overline{\nabla}_{\dot{\gamma}} Z = \underbrace{(\overline{\nabla}_{\dot{\gamma}} Z)^{\top}}_{\text{tangent to } M} + \underbrace{\frac{D^{\perp}Z}{dt}}_{\text{normal to } M}. \tag{2.21}$$

Definition 2.3.2 *A normal vector field Z along γ is said to be a* normal parallel vector field along γ *if* $\frac{D^{\perp}Z}{dt} \equiv 0$.

The following result holds, both for tangent and for normal parallel vector fields along curves in M. For more details see, for instance, Lee [47].

Lemma 2.3.1 *Let $\gamma : [t_0, t_1] \to M$ be a smooth curve on $M \subset \overline{M}$.*

1. *If Y_0 is a vector tangent to M at $\gamma(t_0)$, there is a unique tangent parallel vector field Y along γ such that $Y(t_0) := Y(\gamma(t_0)) = Y_0$.*
2. *If Z_0 is a vector normal to M at $\gamma(t_0)$, there is a unique normal parallel vector field Z along γ such that $Z(t_0) = Z_0$.*

Definition 2.3.3 *Let $\gamma : [t_0, t_1] \to M$ be a smooth curve satisfying $\gamma(t_0) = p$ and $\gamma(t_1) = q$. If Y is the tangent parallel vector field along γ that satisfies $Y(t_0) = Y_0$, then*

$$\begin{array}{rccc} P^{\top}_{t_0 t_1}: & T_pM & \longrightarrow & T_qM \\ & Y_0 & \longmapsto & Y(t_1) \end{array} \tag{2.22}$$

defines a linear isomorphism between T_pM and T_qM, which is called the tangent parallel translation of Y_0 along γ, *from the point p to the point q. Similarly, if Z is the normal parallel vector field along γ that satisfies $Z(t_0) = Z_0$, then*

$$\begin{array}{rccc} P^{\perp}_{t_0 t_1}: & (T_pM)^{\perp} & \longrightarrow & (T_qM)^{\perp} \\ & Z_0 & \longmapsto & Z(t_1) \end{array} \tag{2.23}$$

defines a linear isomorphism between $(T_pM)^{\perp}$ and $(T_qM)^{\perp}$, which is called the normal parallel translation of Z_0 along γ, *from the point p to the point q.*

Remark 2.3.1 *Both, tangent and the normal parallel translations are linear isometries. Consequently, tangent (respectively, normal) parallel translation of a tangent (respectively, normal) frame gives a tangent (respectively, normal) parallel frame field along γ.*

Before finishing this section, we introduce the following proposition, whose proof can be found, e.g., in Hüper et al. [29] and that will be useful during this work, namely in Chapter 5.

Proposition 2.3.1 *Let $f : \overline{M} \to \overline{M}$ be an isometry and N be an isometrically embedded submanifold of $\overline{M}$. Then, for each $p \in N$, the following identities hold:*

$$d_pf(T_pN) = T_{f(p)}f(N), \tag{2.24}$$

and

$$d_pf\left((T_pN)^{\perp}\right) = (T_{f(p)}f(N))^{\perp}. \tag{2.25}$$

2.4 Geodesics, Distances and Geodesic Completeness

In this section we present in detail the concepts of geodesics, lengths of curves and distances between points. We also study the property of geodesic completeness of a Riemannian manifold which, briefly, means that all maximal geodesics are defined for all time.

Let (M,g) be a Riemannian manifold and consider $\gamma : J \subset \mathbb{R} \longrightarrow M$ a curve on M, at least $\mathscr{C}^2$-smooth, and $\dot{\gamma}$ its velocity vector field. Then, we can introduce the following notion.

Definition 2.4.1 *A curve γ satisfying the previous conditions is called a* geodesic *in M if its velocity vector field $\dot{\gamma}$ is a tangent parallel vector field along γ, that is, if*

$$\frac{D\dot{\gamma}}{dt} \equiv 0, \quad \text{for all } t \in J. \tag{2.26}$$

The equation (2.26) is called geodesic equation *and the vector field $\dfrac{D\dot{\gamma}}{dt}$ along γ is called the covariant acceleration along γ.*

It is well-known that the geodesics in the Euclidean space $\mathbb{R}^m$ are straight lines defined by $\gamma(t) = c_0 + c_1 t$, with c_0, $c_1 \in \mathbb{R}^m$. Also, for the unit sphere $S^{m-1} \subset \mathbb{R}^m$, with the Riemannian metric induced by the Euclidean metric of $\mathbb{R}^m$, the geodesics curves are great circles. More examples of geodesics for other particular manifolds will be, carefully, presented later in this work.

Considering a coordinate system $x_1, x_2, \ldots, x_m$ on an open set U of M, a curve $\gamma : J \subset \mathbb{R} \longrightarrow U$ is a geodesic in M, if and only if, its component functions $\gamma^i = x_i \circ \gamma$, $i = 1, \ldots, m$, satisfy the second-order differential equation:

$$\ddot{\gamma}^k(t) + \sum_{i,j=1}^{m} \dot{\gamma}^i(t)\dot{\gamma}^j(t)\Gamma_{ij}^k(\gamma(t)) = 0, \tag{2.27}$$

for $k = 1, \ldots, m$, where m denotes the dimension of the manifold M and Γ_{ij}^k the Christoffel symbols corresponding to the local coordinate system.

Taking into account the standard existence and uniqueness results for ordinary differential equations, it is possible to prove the following (see, for instance, O'Neill [61]).

Proposition 2.4.1 *Let (M,g) be a Riemannian manifold. For every point, $p \in M$, and every tangent vector, $v \in T_pM$, there exists an open interval $]-\varepsilon, \varepsilon[$ $(\varepsilon > 0)$ and a geodesic $\gamma :]-\varepsilon, \varepsilon[\subset \mathbb{R} \longrightarrow M$, such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Any two such geodesics coincide on their common domain.*

As a consequence of the uniqueness of the last result, for every point $p \in M$ and every vector $v \in T_pM$, there exists a unique *maximal geodesic*, denoted by γ, such that $\gamma(0) = p$, $\dot{\gamma}(0) = v$, and the domain of γ is the largest possible, that is, cannot be extended. This maximal geodesic is usually called *the geodesic with initial point p and initial velocity v*.

The property of homogeneity for geodesics ensures that if $t \longmapsto \gamma(t)$ is a geodesic in M, then for every constant λ, the curve $t \longmapsto \gamma(\lambda t)$ is also a geodesic in M. Therefore, if $\gamma :]-\varepsilon, \varepsilon[\longrightarrow M$ is a geodesic with initial conditions $\gamma(0) = p$ and $\dot{\gamma}(0) = v \neq 0$, using the chain rule, for any constant $\lambda \neq 0$, the curve $t \longmapsto \gamma(\lambda t)$, is a geodesic in M such that $\dot{\gamma}(0) = \lambda v$.

The notion of distance between two points of a Riemannian manifold is not an immediate concept. In order to introduce this concept, we start to recall the definitions of a piecewise smooth curve and its respective length.

Definition 2.4.2 *A mapping $\gamma : [a,b] \longrightarrow M$ is a* piecewise smooth curve *if γ is continuous and if there is a partition $a = t_0 < t_1 < \cdots < t_{\ell-1} < t_\ell = b$ of the interval $[a,b]$ such that $\gamma_i = \gamma|_{[t_i,t_{i+1}]}$ is smooth for $i = 0, 1, \ldots, \ell - 1$. The map γ_i is usually called* curve segment.

Notice that at any connection point, $\gamma_{i-1}(t_i) = \gamma_i(t_i)$, $i = 1, \ldots, \ell-1$, but there may be a jump in the velocity vector of γ. We define $\dot{\gamma}_{i-1}(t_i) := \dot{\gamma}_{i-1}(t_i^-)$ and $\dot{\gamma}_i(t_i) := \dot{\gamma}_i(t_i^+)$, $i = 1, \ldots, \ell-1$. Also, $\dot{\gamma}_0(t_0) := \dot{\gamma}_0(t_0^+)$ and $\dot{\gamma}_{\ell-1}(t_\ell) := \dot{\gamma}_{\ell-1}(t_\ell^-)$.

Definition 2.4.3 *The length of a piecewise smooth curve $\gamma\colon [a,b] \longrightarrow M$ is denoted by $L(\gamma)$, and is defined as follows*

$$L(\gamma) = \int_a^b \|\dot{\gamma}(t)\| dt = \int_a^b \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))} dt. \tag{2.28}$$

It is well-know that $L(\gamma)$ is invariant under reparametrization, and that γ is said to be parameterized by arc length if $\|\dot{\gamma}(t)\| = 1$. Also, a smooth curve $\gamma\colon J \subset \mathbb{R} \longrightarrow M$ is *regular* if $\dot{\gamma}(t) \neq 0$, for all $t \in J$, and a piecewise smooth curve is *regular* if γ_i is regular for $i = 0, 1, \ldots, \ell-1$.

In what follows, assume that the Riemannian manifold M is connected. Introducing, for each pair of points $p, q \in M$, the path space of M (from p to q),

$$\Omega(p,q) = \{\gamma\colon [0,1] \longrightarrow M : \ \gamma \text{ is piecewise smooth, } \ \gamma(0) = p \ \text{ and } \ \gamma(1) = q\}, \tag{2.29}$$

which is an infinite-dimensional manifold, we can define the *distance between two points p, q in M* by

$$d(p,q) := \inf\{L(\gamma) : \gamma \in \Omega(p,q)\}, \tag{2.30}$$

usually, denominated *Riemannian distance* in M.

A curve γ in M joining two points is said to be *minimizing* if $L(\gamma) \leq L(\widetilde{\gamma})$, for any other curve $\widetilde{\gamma}$ in M, with the same endpoints. Consequently, from (2.30), γ in M is minimizing, if and only if, $L(\gamma)$ is equal to the distance between its endpoints.

Remark 2.4.1 *A minimizing curve is a geodesic when it is given a unit speed parametrization. Also, every geodesic in a Riemannian manifold is locally minimizing (see, for instance, Lee [47]).*

During this work, a geodesic arc γ will be called *minimal* if its length is less than or equal to the length of any other piecewise smooth path joining its endpoints.

Definition 2.4.4 *Let $p, q \in M$ and $\gamma\colon [0,1] \longrightarrow M$ be the minimal geodesic in M such that $\gamma(0) = p$ and $\gamma(1) = q$. The* geodesic distance between the points *p* and *q is denoted by $d(p,q)$ and is equal to $L(\gamma)$.*

We are now in conditions to introduce the following.

Definition 2.4.5 *Let (M, g) be a Riemannian manifold. For every $p \in M$, let $\mathfrak{D}_p$ be the open subset of T_pM given by*

$$\mathfrak{D}_p = \{v \in T_pM : \ \gamma(1) \ \textit{is defined}\,\}, \tag{2.31}$$

where γ is the unique maximal geodesic with initial conditions $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. The map $\exp_p : \mathfrak{D}_p \longrightarrow M$*, defined by* $\exp_p(v) = \gamma(1)$ *is called the exponential map.*

Definition 2.4.6 *A connected Riemannian manifold M is said to be* geodesically complete *if, for all $p \in M$,* $\exp_p$ *is defined for every $v \in T_pM$, i.e., if all geodesics $t \mapsto \gamma(t)$ in M starting at p are defined for all $t \in \mathbb{R}$.*

Remark 2.4.2

1. *The definition of geodesically complete is equivalent to have that* $\mathfrak{D}_p = T_pM$*, for all* $p \in M$.

2. *Considering* $p,q \in M$ *contained in a geodesic ball around the point* p *in* M*, up to reparametrization, the radial geodesic from* p *to* q *is the unique minimizing curve from* p *to* q *in* M*. For more details see, for instance, Lee [47].*

Next we state the main theorem, and its corollaries, of this section related with the notion of completeness, whose proofs can be found, for instance, in do Carmo [18].

Theorem 2.4.1 (Hopf-Rinow) *If* M *is a* m*-dimensional connected Riemannian manifold and* $p \in M$*, then the following statements are equivalent:*

(i) M *is geodesically complete.*

(ii) *The closed and bounded sets of* M *are compact.*

(iii) M *is complete as a metric space.*

Furthermore, any of the above items implies that:

(iv) *for any* $q \in M$ *there exists a geodesic* γ *joining* p *and* q *with* $L(\gamma) = d(p,q)$*, that is any two points of* M *can be joined by a minimal geodesic.*

The following two corollaries are immediate consequences of the above theorem.

Corollary 2.4.1 *If a Riemannian manifold* (M,g) *is complete and connected, then for any* $p \in M$*,* $\exp_p : T_pM \to M$ *is surjective.*

Corollary 2.4.2 *Any connected and compact Riemannian manifold is complete.*

2.5 Lie Algebras, Lie Groups and Group Actions

In this section we present a brief overview about Lie algebras and Lie groups, as well as actions of a Lie group on a manifold. For a recent detailed treatment see, for instance, Duistermaat and Kolk [19].

Definition 2.5.1 *A Lie algebra* $\mathfrak{L}$ *(over a field* $\mathbb{K}$*) is a vector space (over* $\mathbb{K}$*) with a bilinear product* $[.,.] : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$*, called Lie bracket, satisfying:*

(i) $[A,B] = -[B,A]$ *(Antisymmetry),*

(ii) $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$ *(Jacobi Identity),*

for all $A,B,C \in \mathfrak{L}$.

Remark 2.5.1 *Considering* $(\mathfrak{L}_1, [.,.]_{\mathfrak{L}_1})$ *and* $(\mathfrak{L}_2, [.,.]_{\mathfrak{L}_2})$ *two Lie algebras over a field* $\mathbb{K}$*, the direct sum* $\mathfrak{L}_1 \oplus \mathfrak{L}_2$ *of this two Lie algebras is itself a Lie algebra, with Lie bracket defined by*

$$[(X_1,X_2),(Y_1,Y_2)]_{\mathfrak{L}_1\oplus\mathfrak{L}_2} := ([X_1,Y_1]_{\mathfrak{L}_1},[X_2,Y_2]_{\mathfrak{L}_2}) \tag{2.32}$$

where $(X_1,X_2),\ (Y_1,Y_2) \in \mathfrak{L}_1 \oplus \mathfrak{L}_2$.

During this work we will consider $\mathbb{R}$ as the field $\mathbb{K}$, and a clear example of a real Lie algebra is the set of all smooth vector fields on a smooth manifold M, $\mathfrak{X}(M)$, with the Lie bracket defined in (2.6).

Definition 2.5.2 *A Lie group* G *is an algebraic group that is also a differentiable manifold, with the property that the two group operations*

$$\begin{aligned} \mu : G &\to G \\ (g,h) &\mapsto gh \end{aligned} \quad \textit{(Product)} \qquad \textit{and} \qquad \begin{aligned} \nu : G &\to G \\ g &\mapsto g^{-1} \end{aligned} \quad \textit{(Inverse)}$$

are differentiable mappings.

For each $g \in G$, we define the *left translations* on G and *right translations* on G, respectively, by

$$\begin{aligned} L_g : G &\to G \\ h &\mapsto gh \end{aligned} \qquad \text{and} \qquad \begin{aligned} R_g : G &\to G \\ h &\mapsto hg \end{aligned},$$

which, taking into account Definition 2.5.2, are diffeomorphisms of G. Another important diffeomorphism, called *inner automorphism*, is defined for each $g \in G$:

$$\begin{aligned} I_g : G &\to G \\ h &\mapsto ghg^{-1} \end{aligned}.$$

Its differential at the identity e_G is denoted by Ad_g, i.e.,

$$\begin{aligned} Ad_g : T_{e_G}G &\to T_{e_G}G \\ Y &\mapsto d_{e_G}I_g(Y) \end{aligned}.$$

On a Lie group G there are some particular vector fields that play a special role. They are the left-invariant vector fields and the right-invariant vector fields. A vector field X on G is left-invariant if $\forall g,h \in G$, $d_hL_g(X(h)) = X(L_gh)$, and similarly, it is right-invariant if $\forall g,h \in G$, $d_hR_g(X(h)) = X(R_gh)$. It is well-known that both sets, denoted by $\mathfrak{X}_L(G)$ and $\mathfrak{X}_R(G)$, respectively, are closed for the Lie bracket of vector fields on G, so they form Lie algebras. It turns out that both are isomorphic (as vector spaces) to the tangent space to G at the identity e_G. One can define a Lie bracket of two vectors in $T_{e_G}G$ as being the restriction to e_G of the Lie bracket of the corresponding left-invariant vector fields (or, similarly, of the corresponding right-invariant vector fields). For that reason, we take $T_{e_G}G$ as the Lie algebra of G and denote it by $\mathfrak{g}$.

So, Ad_g defines a Lie algebra automorphism (invertible linear transformation from $\mathfrak{g}$ to itself that preserves the Lie bracket). Moreover, the adjoint map

$$\begin{aligned} Ad : G &\to Aut(\mathfrak{g}) \\ g &\mapsto Ad_g \end{aligned}$$

is a group homomorphism, called the adjoint representation of G. By differentiating Ad at the identity e_G, one obtains the adjoint representation of $\mathfrak{g}$

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\to End(\mathfrak{g}) \\ Y &\mapsto \mathrm{ad}_Y \end{aligned},$$

where, for each $Y \in \mathfrak{g}$, ad_Y is the *adjoint operator in* $\mathfrak{g}$, defined by

$$\begin{aligned} \mathrm{ad}_Y : \mathfrak{g} &\to \mathfrak{g} \\ X &\mapsto \mathrm{ad}_Y(X) = [Y,X] \end{aligned}. \tag{2.33}$$

A Riemannian metric $\langle .,. \rangle$ on G is said to be *left-invariant* if it is invariant under all left translations, i.e., if for all v, $w \in T_hG$,

$$\langle d_hL_g(v), d_hL_g(w) \rangle_{T_{gh}G} = \langle v,w \rangle_{T_hG}. \tag{2.34}$$

Similarly, it is said to be *right-invariant* if it is invariant under all right translations, i.e., if for all v, $w \in T_hG$,

$$\langle d_hR_g(v), d_hR_g(w) \rangle_{T_{hg}G} = \langle v,w \rangle_{T_hG}. \tag{2.35}$$

Moreover, a Riemannian metric that is both left and right-invariant is called bi-invariant. It can be proved that a connected and compact Lie group G can be endowed with a bi-invariant Riemannian metric $\langle .,. \rangle$ (see, for instance, Lee [47]).

A Lie group G is said to act on a manifold M through the left action $\phi : G \times M \to M$ if ϕ satisfies

$$\phi(g, \phi(h,p)) = \phi(L_g(h), p) = \phi(gh, p) \quad \text{and} \quad \phi(e_G, p) = p, \tag{2.36}$$

for all g,h in G, and all p in M. We use ϕ_g to denote the diffeomorphism $p \mapsto \phi_g(p)$ on M, and if there is no possibility of confusion we will simply write $g(p)$ for the image of p under ϕ_g. A right action is defined similarly, simply replacing left translations by right translations.

An action of G on M is said to be transitive if for any two points $p_1, p_2 \in M$ there exists an element $g \in G$ such that $g(p_1) = p_2$.

2.6 Matrix Lie Groups

The most common examples of Lie groups, and those which have the greatest application in computer vision problems, robotics, engineering and control theory problems, are the matrix groups. These groups are all subgroups of the General Linear Group $\mathrm{GL}_n(\mathbb{R})$, the group of all real $n \times n$ invertible matrices, whose Lie algebra is $\mathfrak{gl}(n)$, the set of all $n \times n$ matrices with real entries, equipped with the commutator of matrices $[A,B] = AB - BA$ as the Lie bracket.

We denote elements in a matrix Lie group G by capital letters, as well as for elements in the corresponding Lie algebra $\mathfrak{g}$. The distinction of elements in G from those in $\mathfrak{g}$ should be clear from the context. The left translations (similarly, right translations) are expressed simply by matrix multiplication on the left (similarly, on the right), so that for $A \in G$ and $X \in \mathfrak{g}$, the adjoint map simplifies to $Ad_A(X) = AXA^{-1}$.

Let $\mathfrak{gl}(n)$ be equipped with the Euclidean inner product

$$\langle X, Y \rangle = \operatorname{tr}\left(X^\top Y\right), \; X, Y \in \mathfrak{gl}(n). \tag{2.37}$$

Given $X \in \mathfrak{gl}(n)$, the *matrix exponential of* X, denoted by e^X or $\exp(X)$, is the $n \times n$ real matrix given by the sum of the following convergent power series

$$\mathrm{e}^X = \sum_{k=0}^{+\infty} \frac{X^k}{k!},$$

where X^0 is defined to be the identity matrix I_n.

The vector space of $\mathfrak{gl}(n)$ consisting of all symmetric matrices is denoted by $\mathfrak{s}(n)$, while $\mathfrak{so}(n)$ denotes the Lie subalgebra of $\mathfrak{gl}(n)$ consisting of all skew-symmetric matrices. It is well-known that

$$\mathfrak{gl}(n) = \mathfrak{s}(n) \oplus \mathfrak{so}(n), \tag{2.38}$$

is a decomposition of the Lie algebra $\mathfrak{gl}(n)$ and, consequently, any matrix $A \in \mathfrak{gl}(n)$ can be uniquely decomposed as $A = \frac{A+A^\top}{2} + \frac{A-A^\top}{2}$, where $A + A^\top \in \mathfrak{s}(n)$ and $A - A^\top \in \mathfrak{so}(n)$. In particular, one has

$$[\mathfrak{so}(n), \mathfrak{so}(n)] \subset \mathfrak{so}(n), \quad [\mathfrak{so}(n), \mathfrak{s}(n)] \subset \mathfrak{s}(n), \quad [\mathfrak{s}(n), \mathfrak{s}(n)] \subset \mathfrak{so}(n). \tag{2.39}$$

Also, $\mathfrak{so}(n)$ and $\mathfrak{s}(n)$ are orthogonal with respect to the inner product (2.37).

Furthermore, (see, for instance, Higham [26] and Horn and Johnson [27]) the logarithms of an invertible matrix B are the solutions of the matrix equation $\mathrm{e}^X = B$, and when B is real and doesn't have eigenvalues in the closed negative real line, i.e., when $\sigma(B) \cap \mathbb{R}_0^- = \varnothing$, where $\sigma(B)$ denotes the spectrum of B, there exists a unique real logarithm of B whose spectrum lies in the infinite horizontal strip $\{z \in \mathbb{C} : -\pi < Im(z) < \pi\}$ of the complex plane. In this work we will only consider this logarithm, usually called the *principal logarithm* of B and hereafter denoted by $\log B$. When B belongs to the rotation group $\mathrm{SO}(n)$, then $\log B$ belongs to its Lie algebra $\mathfrak{so}(n)$. Further, when $\|B - I\| < 1$, $\log B$ is uniquely defined by the following convergent power series:

$$\log B = \sum_{k=1}^{+\infty} (-1)^{k+1} \frac{(B-I)^k}{k}.$$

This power series defines the principal logarithm for matrices which are close to the identity matrix. However, for $\alpha \in [-1, 1]$, $\log(B^\alpha) = \alpha \log B$, so that, making $\alpha = 1/2^k$, with $k \in \mathbb{Z}$, one has

$$\log\left(B^{\frac{1}{2^k}}\right) = \frac{1}{2^k} \log B.$$

Since $\lim_{k\to+\infty}\left(B^{\frac{1}{2^k}}\right)=I$, the previous expression allows to compute $\log B$ even for matrices B which are not close to the identity. This procedure, that can be found for instance in Higham [26], is called *inverse scaling and squaring method*.

Before proceeding, we recall some significant properties of the matrix exponential and matrix logarithm that will play an important role in the derivation of some results that appear throughout this thesis.

Lemma 2.6.1 *Let A, B, C and D be real square matrices and assume that C is invertible and* $\sigma(B)\cap\mathbb{R}_0^-=\varnothing$. *Then, the following identities hold.*

1. $C^{-1}\mathrm{e}^{A}C=\mathrm{e}^{C^{-1}AC}$;
2. $\mathrm{e}^{A}D\mathrm{e}^{-A}=\mathrm{e}^{\mathrm{ad}_A}(D)=D+[A,D]+\frac{1}{2!}[A,[A,D]]+\cdots$ *(Campbell-Hausdorff Formula)*;
3. $C^{-1}(\log B)C=\log\left(C^{-1}BC\right)$;
4. $\log(B^{-1})=-\log B$ *and* $\log(B^{\top})=(\log B)^{\top}$;
5. $B^{\alpha}=\mathrm{e}^{\alpha\log B}$, *for* $\alpha\in\mathbb{R}$;
6. $\log(e^{A})=A$, *whenever* $\log(e^{A})$ *is defined.*

Geometrically, the invertible matrix B^{α} present in the identity 5. represents the point, corresponding to $t=\alpha$, on the geodesic that passes through the identity (at $t=0$) with initial velocity $\log B$.

In the sequel, we also assume the following notations:

$$f(z)=\frac{\mathrm{e}^{z}-1}{z}\text{ stands for the sum of the series }\sum_{k=0}^{+\infty}\frac{z^k}{(k+1)!},\tag{2.40}$$

and when $|z-1|<1$,

$$g(z)=\frac{\log z}{z-1}\text{ stands for the sum of the series }\sum_{k=0}^{+\infty}(-1)^k\frac{(z-1)^k}{k+1}.\tag{2.41}$$

Note that $f(z)g(e^{z})=1$. Below, we state a few results concerning the derivatives of some particular functions that will be very useful in the development of this work.

Lemma 2.6.2 (Moakher [54]) *Let* $t\longmapsto B(t)$ *be a differentiable matrix valued function and assume that, for each t in the domain,* $B(t)$ *is a non-singular matrix not having eigenvalues in the closed negative real line. Then,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}\left(\log^2 B(t)\right)=2\mathrm{tr}\left(\log(B(t))B^{-1}(t)\dot{B}(t)\right).\tag{2.42}$$

In what follows, we present two equivalent ways of express the derivative of the matrix exponential of a differentiable matrix valued function. Although equivalent, we decided to introduce both alternative methods, because in future chapters we use the one that reveals to be more appropriate to simplify notations and calculations.

Lemma 2.6.3 (Sattinger and Weaver [67]) *Let $t \longmapsto X(t)$ be a differentiable matrix valued function. Then,*

1.
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{X(t)} = \Delta^L_{X(t)}(t)\mathrm{e}^{X(t)}, \tag{2.43}$$

where
$$\Delta^L_{X(t)}(t) = \int_0^1 \mathrm{e}^{u\,\mathrm{ad}_{X(t)}}(\dot{X}(t))du. \tag{2.44}$$

2.
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{X(t)} = f\left(\mathrm{ad}_{X(t)}\right)\left(\dot{X}(t)\right)\mathrm{e}^{X(t)}, \tag{2.45}$$

where f is defined as in (2.40).

In order to confirm that the relations (2.43) and (2.45) of Lemma 2.6.3 are effectively the same, notice that, since
$$\mathrm{e}^{u\,\mathrm{ad}_{X(t)}}(\dot{X}(t)) = \sum_{k=0}^{+\infty}\frac{u^k}{k!}\mathrm{ad}^k_{X(t)}(\dot{X}(t)),$$
we have
$$\begin{aligned}
\Delta^L_{X(t)}(t) &= \int_0^1 \mathrm{e}^{u\,\mathrm{ad}_{X(t)}}(\dot{X}(t))du \\
&= \left[\sum_{k=0}^{+\infty}\frac{u^{k+1}}{(k+1)!}\mathrm{ad}^k_{X(t)}(\dot{X}(t))\right]_{u=0}^{u=1} \\
&= \sum_{k=0}^{+\infty}\frac{\mathrm{ad}^k_{X(t)}}{(k+1)!}(\dot{X}(t)) \\
&= \left.\frac{\mathrm{e}^u - 1}{u}\right|_{u=\mathrm{ad}_{X(t)}}(\dot{X}(t)) = f\left(\mathrm{ad}_{X(t)}\right)\left(\dot{X}(t)\right).
\end{aligned}$$

Remark 2.6.1 *When $X(t) = f(t)A$, with f a real scalar function and A a constant $n\times n$ real matrix, then*
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{f(t)A} = \dot{f}(t)A\mathrm{e}^{f(t)A} = \dot{f}(t)\mathrm{e}^{f(t)A}A.$$
In particular, this holds for $f(t) = t$, giving a well-known formula.

The next three propositions will play an important role in future chapters, namely in Chapter 4.

Proposition 2.6.1 *Let $t \longmapsto A(t)$ be a differentiable matrix valued function. Then*

$$\mathrm{e}^{A(t)}\Delta^L_{-A(t)}(t)\mathrm{e}^{-A(t)} = -\Delta^L_{A(t)}(t), \tag{2.46}$$

where $\Delta^L_{A(t)}(t)$ denotes the operator defined on (2.44).

Proof. It holds that,

$$\begin{aligned}
\mathrm{e}^{A(t)}\Delta^L_{-A(t)}(t)\mathrm{e}^{-A(t)} &= \mathrm{e}^{\mathrm{ad}_{A(t)}}(\Delta^L_{-A(t)}(t)) \\
&= \mathrm{e}^{\mathrm{ad}_{A(t)}}\int_0^1 \mathrm{e}^{u\,\mathrm{ad}_{-A(t)}}(-\dot{A}(t))du \\
&= \mathrm{e}^{\mathrm{ad}_{A(t)}}\int_0^1 \mathrm{e}^{-u\,\mathrm{ad}_{A(t)}}(-\dot{A}(t))du \\
&= -\int_0^1 \mathrm{e}^{\mathrm{ad}_{A(t)}}\mathrm{e}^{-u\,\mathrm{ad}_{A(t)}}(\dot{A}(t))du \\
&= -\int_0^1 \mathrm{e}^{(1-u)\,\mathrm{ad}_{A(t)}}(\dot{A}(t))du.
\end{aligned}$$

Making a change of variable, considering $1-u=z$, we have that $du=-dz$, $u=0$ implies $z=1$ and $u=1$ implies $z=0$. Then,

$$\begin{aligned}
-\int_0^1 \mathrm{e}^{(1-u)\,\mathrm{ad}_{A(t)}}(\dot{A}(t))du &= \int_1^0 \mathrm{e}^{z\,\mathrm{ad}_{A(t)}}(\dot{A}(t))dz \\
&= -\int_0^1 \mathrm{e}^{z\,\mathrm{ad}_{A(t)}}(\dot{A}(t))dz \\
&= -\Delta^L_{A(t)}(t).
\end{aligned}$$

□

Proposition 2.6.2 *Let $t \longmapsto A(t)$ be a differentiable matrix valued function. Then, for $k=0,1$,*

$$\left(\Delta^L_{(t-k)A(t)}(t)\right)\Big|_{t=k} = A(k), \quad \textit{and consequently,} \quad \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=k}\left(\mathrm{e}^{(t-k)A(t)}\right) = A(k). \tag{2.47}$$

Proof. We present here the proof of the statement for $k=0$, since for $k=1$ the proof is similar. Therefore, for $k=0$, we have that

$$\begin{aligned}
\left(\Delta^L_{tA(t)}(t)\right)\Big|_{t=0} &= \left(\int_0^1 \mathrm{e}^{ut\,\mathrm{ad}_{A(t)}}(A(t)+t\dot{A}(t))du\right)\Big|_{t=0} \\
&= \left(\int_0^1 \mathrm{e}^{ut\,\mathrm{ad}_{A(t)}}(A(t))du\right)\Big|_{t=0} + \left(\int_0^1 \mathrm{e}^{ut\,\mathrm{ad}_{A(t)}}(t\dot{A}(t))du\right)\Big|_{t=0} \\
&= \left(\int_0^1 A(t)du\right)\Big|_{t=0} \\
&= A(0).
\end{aligned}$$

Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\left(\mathrm{e}^{tA(t)}\right) = \left(\Delta^L_{tA(t)}(t)\mathrm{e}^{tA(t)}\right)\Big|_{t=0} = A(0).$$

□

Proposition 2.6.3 *Let $t \longmapsto A(t)$ be a differentiable matrix valued function. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\left(\Delta^L_{tA(t)}(t)\right) = 2\dot{A}(0) \quad \textit{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1}\left(\Delta^L_{(t-1)A(t)}(t)\right) = 2\dot{A}(1). \tag{2.48}$$

Proof. To prove the first identity, we have that

$$
\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\Delta^L_{tA(t)}(t)\right) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\int_0^1 \mathrm{e}^{ut\,\mathrm{ad}_{A(t)}}(A(t)+t\dot{A}(t))du\right)\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\int_0^1 \mathrm{e}^{ut\,\mathrm{ad}_{A(t)}}(A(t))du+\int_0^1 \mathrm{e}^{ut\,\mathrm{ad}_{A(t)}}(t\dot{A}(t))du\right)\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\int_0^1 A(t)du\right)+\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\int_0^1 \mathrm{e}^{ut\,\mathrm{ad}_{A(t)}}(t\dot{A}(t))du\right)\\
&= \int_0^1 \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(A(t))\,du+\int_0^1 \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\mathrm{e}^{ut\,\mathrm{ad}_{A(t)}}(t\dot{A}(t))\right)du\\
&= \dot{A}(0)+\int_0^1 \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(\mathrm{e}^{utA(t)}(t\dot{A}(t))\mathrm{e}^{-utA(t)}\right)du\\
&= \dot{A}(0)+\int_0^1 \Big(\Delta^L_{utA(t)}(t)\mathrm{e}^{utA(t)}(t\dot{A}(t))\mathrm{e}^{-utA(t)}\\
&\quad +\mathrm{e}^{utA(t)}(\dot{A}(t)+t\ddot{A}(t))\mathrm{e}^{-utA(t)}+\mathrm{e}^{utA(t)}(t\dot{A}(t))\Delta^L_{-utA(t)}(t)\mathrm{e}^{-utA(t)}\Big)\Big|_{t=0}du\\
&= \dot{A}(0)+\int_0^1 \dot{A}(0)du = 2\dot{A}(0).
\end{aligned}
$$

The proof of the second identity is now immediate, since it is done with similar computations.

□

Notice that, if we differentiate with respect to t the identity $\mathrm{e}^{\log Y(t)} = Y(t)$, using Lemma 2.6.3, we can obtain the corresponding expression for the derivative of the matrix logarithm valued function, as set out in the next result.

Lemma 2.6.4 (Batzies et al. [5]) *Let $t \longmapsto Y(t)$ be a differentiable matrix valued function such that $\log Y(t)$ is defined for all real variable t. Then,*

$$
\frac{\mathrm{d}}{\mathrm{d}t}(\log Y(t)) = \left.\frac{u}{\mathrm{e}^u-1}\right|_{u=\mathrm{ad}_{\log Y(t)}}\left(\dot{Y}(t)Y^{-1}(t)\right), \tag{2.49}
$$

where $\dfrac{u}{\mathrm{e}^u-1} = g(\mathrm{e}^u)$, with g defined as in (2.41).

Chapter 3

The Geometry of our Fundamental Manifolds

In this chapter we review the Riemannian manifolds that will play the main role in the rest of the thesis, with particular emphasis on the Grassmann manifold and on the Normalized Essential manifold. For the first one, we also present some results that to the best of our knowledge are new, namely, Proposition 3.2.1, some identities present in Lemma 3.2.1, Proposition 3.2.2, Proposition 3.2.4, Proposition 3.2.5 and Proposition 3.2.8. Furthermore, we emphasize the importance of the Grassmann manifold in certain applications dealing with nonlinear data based on images and explain how it is possible to represent a set of images by a point in the Grassmann manifold.

The second manifold mentioned, that is the Normalized Essential manifold, turns up as a particular case of the Generalized Essential manifold, the cartesian product of the real Grassmann manifold and the rotation group. For this manifold a geometric formulation is presented, as well as, a detailed description of its Riemannian structure.

The main references used in this chapter were the following: Absil et al. [1] and Batzies et al. [5], for the geometry of the Grassmann manifold; Hartley and Zisserman [23], Helmke et al. [24], Helmke and Moore [25] and Longuet-Higgins [48], for details concerning the Normalized Essential Manifold.

3.1 The Geometry of the Rotation Group $\mathrm{SO}(n)$

Let $\mathrm{GL}_n(\mathbb{R})$ be the set of all real $n \times n$ invertible matrices. Then,

$$\mathrm{SO}(n) = \{\Theta \in \mathrm{GL}_n(\mathbb{R}) : \Theta^\top \Theta = I \text{ and } \det \Theta = 1\},$$

is a smooth compact and connected manifold of dimension $n(n-1)/2$ (see, for instance, Lee [47] and Helmke and Moore [25]). It is well-known that, the tangent space of $\mathrm{SO}(n)$ at a point $R \in \mathrm{SO}(n)$, can be characterized by

$$T_R \mathrm{SO}(n) = \{RY : Y \in \mathfrak{so}(n)\}. \tag{3.1}$$

Furthermore, since $\mathrm{SO}(n)$ is a compact and connected Lie group, the Riemannian metric in $\mathrm{SO}(n)$ induced by the Frobenius inner product (2.37) is bi-invariant, and it is defined by considering, in each tangent space $T_R\,\mathrm{SO}(n)$, the inner product

$$\langle RX, RY\rangle = \langle X, Y\rangle = -\mathrm{tr}\big(XY\big), \qquad X, Y \in \mathfrak{so}(n). \tag{3.2}$$

Therefore, taking into account (3.2), the orthogonal complement of the tangent space to $\mathrm{SO}(n)$ at the point R, can be characterized by

$$(T_R\,\mathrm{SO}(n))^{\perp} = \{RS : S \in \mathfrak{s}(n)\}. \tag{3.3}$$

The geodesics on $\mathrm{SO}(n)$ are translations of one parameter subgroups of $\mathrm{SO}(n)$, i.e., given $\Omega \in \mathfrak{so}(n)$, the Lie algebra of $\mathrm{SO}(n)$, and $t \in \mathbb{R}$, then $\mathrm{e}^{t\Omega}$ is a geodesic of $\mathrm{SO}(n)$, passing through I at $t = 0$, while both $R\mathrm{e}^{t\Omega}$ and $\mathrm{e}^{t\Omega}R$, are geodesics going through $R \in \mathrm{SO}(n)$ at $t = 0$. Therefore, geodesics on $\mathrm{SO}(n)$ are completely characterized by a point in $\mathrm{SO}(n)$ and a vector tangent to $\mathrm{SO}(n)$ at the identity I, that is, by a rotation matrix and a matrix belonging to $\mathfrak{so}(n)$.

In fact, we have the following two results, whose proofs are immediate.

Proposition 3.1.1 *The unique geodesic $t \mapsto \gamma(t)$ in $\mathrm{SO}(n)$, satisfying the initial conditions $\gamma(0) = R$ and $\dot{\gamma}(0) = RY$, where $Y \in \mathfrak{so}(n)$, is given by*

$$\gamma(t) = R\mathrm{e}^{tY}. \tag{3.4}$$

Remark 3.1.1 *We note that any two points in $\mathrm{SO}(n)$ can be joined by a geodesic. This follows from the fact that $\mathrm{SO}(n)$ is connected and compact, so by Corollary 2.4.2 it is complete, and by Theorem 2.4.1 it is geodesically complete. In spite of that, an explicit formula for the geodesic that joins two points requires some restrictions expressed in the following result.*

Proposition 3.1.2 *Let $R_1, R_2 \in \mathrm{SO}(n)$ be such that $R_1^{-1}R_2$ has no negative real eigenvalues, i.e., $\sigma(R_1^{-1}R_2) \cap \mathbb{R}^- = \varnothing$. Then, the minimizing geodesic arc with respect to the Riemannian metric (2.37) that joins R_1, at $t = 0$, to R_2, at $t = 1$, is parameterized explicitly by*

$$\gamma(t) = R_1\mathrm{e}^{tY}, \qquad \textit{where} \qquad Y = \log(R_1^{-1}R_2) \in \mathfrak{so}(n),$$

or, equivalently, (3.5)

$$\gamma(t) = \mathrm{e}^{t\overline{Y}}R_1, \qquad \textit{where} \qquad \overline{Y} = \log(R_2R_1^{-1}) \in \mathfrak{so}(n).$$

Remark 3.1.2 *Notice that, since $\sigma(R_2R_1^{-1}) = \sigma(R_1^{-1}R_2)$, the condition $\sigma(R_1^{-1}R_2) \cap \mathbb{R}^- = \varnothing$ implies that $\sigma(R_2R_1^{-1}) \cap \mathbb{R}^- = \varnothing$, and thus $\overline{Y}$ is well defined.*

The geodesic distance between two points R_1 and R_2 in $\mathrm{SO}(n)$ is the length of the minimal geodesic curve connecting them. Therefore, considering (3.5), it is possible to obtain the next result, whose proof is also immediate.

Proposition 3.1.3 *Let $R_1, R_2 \in \mathrm{SO}(n)$ be such that $\sigma(R_1^{-1}R_2) \cap \mathbb{R}^- = \varnothing$. Then, the geodesic distance between R_1 and R_2 is given explicitly by*

$$d^2(R_1,R_2) = -\mathrm{tr}\big(\log^2(R_1^{-1}R_2)\big),$$

or, equivalently, (3.6)

$$d^2(R_1,R_2) = -\mathrm{tr}\big(\log^2(R_2R_1^{-1})\big).$$

3.2 The Geometry of the Grassmann Manifold $G_{k,n}$

The contents of this section were inspired by the work of Batzies et al. [5]. The real Grassmann manifold $G_{k,n}$, $(0 < k < n)$, consists of all k-dimensional real subspaces of the Euclidean space $\mathbb{R}^n$. Each such subspace can be associated to a unique operator of orthogonal projections onto itself, with respect to the Euclidean metric. It is well-known that these operators (or, equivalently, its matrices, called projection matrices) are symmetric, idempotent, and have rank k. Therefore, $G_{k,n}$ can be defined, alternatively, as:

$$G_{k,n} := \big\{P \in \mathfrak{s}(n) \,:\, P^2 = P \text{ and } \mathrm{rank}(P) = k\big\}. \tag{3.7}$$

It is also known that the real Grassmann manifold $G_{k,n}$ is a smooth compact connected manifold of real dimension $k(n-k)$, and moreover it is an isospectral manifold, where each element has the eigenvalues 1 and 0, with multiplicity k and $n-k$, respectively.

Taking into account this isospectral feature of $G_{k,n}$, another characterization of the real Grassmann manifold $G_{k,n}$ is given by:

$$G_{k,n} = \Big\{P = \Theta E_0 \Theta^\top \in \mathfrak{s}(n) :\ \Theta \in \mathrm{SO}(n)\Big\}, \ \text{ where } \ E_0 = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \tag{3.8}$$

and where I_k denotes the identity matrix of order k. Also, all matrices of $G_{k,n}$ are orthogonally similar, since for each pair of matrices in $G_{k,n}$ it is possible to find an orthogonal matrix that relates them. Indeed, if $P_1 = \Theta_1 E_0 \Theta_1^\top \in G_{k,n}$ and $P_2 = \Theta_2 E_0 \Theta_2^\top \in G_{k,n}$, with $\Theta_1, \Theta_2 \in \mathrm{SO}(n)$, then there exists $\Theta = \Theta_1\Theta_2^\top \in \mathrm{SO}(n)$ such that $P_1 = \Theta P_2 \Theta^\top$.

For P an arbitrary point in the Grassmannian $G_{k,n}$, let us define the following sets of matrices

$$\begin{aligned} \mathfrak{gl}_P(n) &:= \{A \in \mathfrak{gl}(n) \,:\, A = PA + AP\}; \\ \mathfrak{s}_P(n) &:= \mathfrak{s}(n) \cap \mathfrak{gl}_P(n); \\ \mathfrak{so}_P(n) &:= \mathfrak{so}(n) \cap \mathfrak{gl}_P(n). \end{aligned} \tag{3.9}$$

We can now state the following result.

Proposition 3.2.1 *Let $P_1, P_2 \in G_{k,n}$ be such that $P_1 = \Theta P_2 \Theta^\top$ for some $\Theta \in \mathrm{SO}(n)$. Then,*

$$\mathfrak{gl}_{P_1}(n) = \Theta \mathfrak{gl}_{P_2}(n) \Theta^\top. \tag{3.10}$$

Proof. Let $A \in \mathfrak{gl}_{P_1}(n)$. In order to prove that $A \in \Theta\mathfrak{gl}_{P_2}(n)\Theta^\top$ we need to find $B \in \mathfrak{gl}_{P_2}(n)$ such that $A = \Theta B\Theta^\top$. It is enough to consider $B = \Theta^\top A\Theta$. The equality $A = \Theta B\Theta^\top$ is immediately satisfied. To show that $B = \Theta^\top A\Theta \in \mathfrak{gl}_{P_2}(n)$, notice that since $P_2 = \Theta^\top P_1\Theta$, $B = \Theta^\top A\Theta$ and $A \in \mathfrak{gl}_{P_1}(n)$, it holds that

$$\begin{aligned} BP_2 + P_2B &= \Theta^\top A\Theta\Theta^\top P_1\Theta + \Theta^\top P_1\Theta\Theta^\top A\Theta \\ &= \Theta^\top AP_1\Theta + \Theta^\top P_1A\Theta \\ &= \Theta^\top (AP_1 + P_1A)\Theta \\ &= \Theta^\top A\Theta \\ &= B. \end{aligned}$$

To show the other inclusion let $A \in \Theta\mathfrak{gl}_{P_2}(n)\Theta^\top$. We have that $A = \Theta B\Theta^\top$, with $B \in \mathfrak{gl}_{P_2}(n)$. Since $P_1 = \Theta P_2\Theta^\top$, with a few calculations we get that

$$\begin{aligned} AP_1 + P_1A &= \Theta B\Theta^\top\Theta P_2\Theta^\top + \Theta P_2\Theta^\top\Theta B\Theta^\top \\ &= \Theta BAP_2\Theta^\top + \Theta P_2B\Theta^\top \\ &= \Theta (BP_2 + P_2B)\Theta^\top \\ &= \Theta B\Theta^\top \\ &= A, \end{aligned}$$

which proves the result.

□

Moreover, the sets defined in (3.9) will play an important role in this work, particularly, due to their interesting properties listed below.

Lemma 3.2.1 *Let $P \in G_{k,n}$, $A, B, C \in \mathfrak{gl}_P(n)$ and $j \in \mathbb{N}$. Then, the following holds.*

1. $PA^{2j-1}P = 0$;
2. $A^{2j-1} = PA^{2j-1} + A^{2j-1}P$;
3. $PA^{2j} = PA^{2j}P = A^{2j}P$;
4. $[P,[P,A^{2j-1}]] = A^{2j-1}$;
5. $APA^{2j-1} = A^{2j-1}PA$ *and* $A[A^{2j-1},P] = -[A^{2j-1},P]A$;
6. $(I-2P)A^{2j-1} = -A^{2j-1}(I-2P) = [A^{2j-1},P]$;
7. $(I-2P)A^{2j} = A^{2j}(I-2P) = -A[A^{2j-1},P]$;
8. $\operatorname{tr}([A,P]BC) = 0$ *and, consequently,* $\operatorname{tr}([A,P][B,C]) = 0$.

Proof. Let $P \in G_{k,n}$, $A, B, C \in \mathfrak{gl}_P(n)$ and $j \in \mathbb{N}$. The first three properties can be proved by induction on j and the proof of them can be found in Batzies et al. [5]. The proof of the other properties doesn't need to be done by induction on j, since they can be, easily, derived from the three previous ones. We prove these properties in detail bellow.

4. We have

$$\begin{aligned}
[P,[P,A^{2j-1}]] &= P[P,A^{2j-1}]-[P,A^{2j-1}]P \\
&= P\left(PA^{2j-1}-A^{2j-1}P\right)-\left(PA^{2j-1}-A^{2j-1}P\right)P \\
&= PA^{2j-1}-PA^{2j-1}P-PA^{2j-1}P+A^{2j-1}P \\
&\overset{2.}{=} A^{2j-1}.
\end{aligned}$$

5. We start to show the first identity, since it will be used to prove the second one. We have that,

$$\begin{aligned}
APA^{2j-1} &\overset{2.}{=} A\left(A^{2j-1}-A^{2j-1}P\right) \\
&= A^{2j}-A^{2j}P \\
&\overset{3.}{=} A^{2j}-PA^{2j} \\
&= \left(A^{2j-1}-PA^{2j-1}\right)A \\
&\overset{2.}{=} A^{2j-1}PA.
\end{aligned}$$

Then,

$$\begin{aligned}
A[A^{2j-1},P] &= A\left(A^{2j-1}P-PA^{2j-1}\right) \\
&= A^{2j}P-APA^{2j-1} \\
&\overset{3.}{=} PA^{2j}-A^{2j-1}PA \\
&= \left(PA^{2j-1}-A^{2j-1}P\right)A \\
&= -[A^{2j-1},P]A.
\end{aligned}$$

6. Proof of the first identity:

$$\begin{aligned}
(I-2P)A^{2j-1} &= A^{2j-1}-2PA^{2j-1} \\
&\overset{2.}{=} A^{2j-1}-2\left(A^{2j-1}-A^{2j-1}P\right) \\
&= -A^{2j-1}+2A^{2j-1}P \\
&= -A^{2j-1}\left(I-2P\right).
\end{aligned}$$

Proof of the second identity:

$$\begin{aligned}
-A^{2j-1}\left(I-2P\right) &= -A^{2j-1}+2A^{2j-1}P \\
&\overset{2.}{=} -PA^{2j-1}-A^{2j-1}P+2A^{2j-1}P \\
&= -PA^{2j-1}+A^{2j-1}P \\
&= [A^{2j-1},P].
\end{aligned}$$

7. Proof of the first identity:

$$\begin{aligned}(I-2P)A^{2j} &= A^{2j}-2PA^{2j}\\ &\overset{3.}{=} A^{2j}-2A^{2j}P\\ &= A^{2j}\,(I-2P)\,.\end{aligned}$$

Proof of the second identity:

$$\begin{aligned}A^{2j}\,(I-2P) &= AA^{2j-1}\,(I-2P)\\ &\overset{6.}{=} -A[A^{2j-1},P].\end{aligned}$$

8. In order to prove the first identity it is enough to consider the two identities present in the property 6. of this Lemma 3.2.1 and some properties of the trace of a matrix. Indeed, we have that

$$\begin{aligned}\operatorname{tr}([A,P]BC) &= \operatorname{tr}((I-2P)ABC)\\ &= -\operatorname{tr}(A(I-2P)BC)\\ &= \operatorname{tr}(AB(I-2P)C)\\ &= -\operatorname{tr}(ABC(I-2P))\\ &= -\operatorname{tr}((I-2P)ABC)\\ &= -\operatorname{tr}([A,P]BC)\,,\end{aligned}$$

and, therefore, $\operatorname{tr}([A,P]BC)=0$. The second identity is an immediate consequence of the first one.

□

Remark 3.2.1

1. *An immediate consequence of statement* 3. *of the Lemma 3.2.1 is that*

$$\left[P,A^{2j}\right]=0,\quad \textit{for all } P\in G_{k,n},\ A\in\mathfrak{gl}_P(n) \textit{ and } j\in\mathbb{N}. \tag{3.11}$$

2. *The identities present on statements* 4., 5. *and* 6. *of the Lemma 3.2.1, for the particular situation when* $j=1$, *have already appeared in Batzies et al. [5].*

3. *Given* $P\in G_{k,n}$, *from the first identity of the property* 6. *of the Lemma 3.2.1, it is possible to conclude that:*

   (a) $(I-2P)$ *commute with any product of an even number of matrices in* $\mathfrak{gl}_P(n)$, *i.e.,*

$$(I-2P)A_1A_2\ldots A_{2j}=A_1A_2\ldots A_{2j}(I-2P), \tag{3.12}$$

   where $A_1,A_2,\ldots,A_{2j}\in\mathfrak{gl}_P(n)$ *and* $j\in\mathbb{N}$, *or equivalently, that*

$$[P,A_1A_2\ldots A_{2j}]=0,\quad \textit{for all } A_1,A_2,\ldots,A_{2j}\in\mathfrak{gl}_P(n),\ j\in\mathbb{N}. \tag{3.13}$$

(b) $(I-2P)$ *anticommute with any product of an odd number of matrices in* $\mathfrak{gl}_P(n)$*, i.e.,*

$$(I-2P)A_1A_2\dots A_{2j-1} = -A_1A_2\dots A_{2j-1}(I-2P), \tag{3.14}$$

where $A_1, A_2, \dots, A_{2j-1} \in \mathfrak{gl}_P(n)$ *and* $j \in \mathbb{N}$*, or equivalently, that if* $A_1, A_2, \dots, A_{2j-1} \in \mathfrak{gl}_P(n)$*, then also their product* $A_1A_2\dots A_{2j-1} \in \mathfrak{gl}_P(n)$*.*

4. *It follows from the proof of the property* 8. *of the Lemma 3.2.1 that the first equality of this statement still holds if the product BC is replaced by any product of an even number of matrices in* $\mathfrak{gl}_P(n)$*.*

Taking into consideration the Lemma 3.2.1, we are now in conditions to state the next result that will be important in further developments.

Proposition 3.2.2 *Let* $P \in G_{k,n}$*,* $A \in \mathfrak{gl}_P(n)$ *and* $t \in \mathbb{R}$*. Then,*

$$(I-2P)\mathrm{e}^{tA} = \mathrm{e}^{-tA}(I-2P), \tag{3.15}$$

and, consequently,

$$\mathrm{e}^{2tA}(I-2P) = \mathrm{e}^{\mathrm{ad}_{tA}}(I-2P). \tag{3.16}$$

Proof. Taking into account the definition of matrix exponential, the first identity (3.15) is an immediate consequence of properties 6. and 7. of Lemma 3.2.1. The second identity (3.16) is obtained from the first one just by a few computations and considering the Campbell-Hausdorff Formula in Lemma 2.6.1. In fact, from (3.15) we get that

$$\mathrm{e}^{tA}(I-2P)\mathrm{e}^{tA} = (I-2P),$$

then, multiplying both terms on the left by e^{tA}, and on the right by e^{-tA}, we have

$$\mathrm{e}^{2tA}(I-2P) = \mathrm{e}^{tA}(I-2P)\mathrm{e}^{-tA},$$

which, according with the Campbell-Hausdorff Formula, proves the identity (3.16).

□

Furthermore, the Lemma 3.2.1 can be used to prove that the restriction to $\mathfrak{gl}_P(n)$ of the adjoint operator at P,

$$\begin{aligned} \mathrm{ad}_P|_{\mathfrak{gl}_P(n)} : \mathfrak{gl}_P(n) &\to \mathfrak{gl}_P(n) \\ A &\mapsto \mathrm{ad}_P(A) = [P,A], \end{aligned} \tag{3.17}$$

is an isometry. Moreover, together with the decomposition (2.38), it can be used to derive the properties present in the following proposition for the adjoint operator in the sets of matrices defined in (3.9), whose sketch of proof can be found in Batzies et al. [5]. Also, notice that, using this notation, it is possible to rewrite some of the results presented before. For instance, (3.11) of Remark 3.2.1 can now be rewritten as follows:

$$\mathrm{ad}_P\left(A^{2j}\right) = 0, \quad \text{for all } P \in G_{k,n},\ A \in \mathfrak{gl}_P(n) \text{ and } j \in \mathbb{N}. \tag{3.18}$$

Proposition 3.2.3 (Batzies et al. [5]) *Let $P \in G_{k,n}$. Then,*

1. $\mathrm{ad}_P(\mathfrak{gl}(n)) = \mathfrak{gl}_P(n)$;
2. $\mathrm{ad}_P(\mathfrak{s}(n)) = \mathrm{ad}_P(\mathfrak{s}_P(n)) = \mathfrak{so}_P(n)$;
3. $\mathrm{ad}_P(\mathfrak{so}(n)) = \mathrm{ad}_P(\mathfrak{so}_P(n)) = \mathfrak{s}_P(n)$.

Proof. Although in Batzies et al. [5] we can find an outline of the proof of this proposition, in what follows, for the sake of completeness, we prove in detail, for instance, the first identity present in property 3.. The inclusion $\mathrm{ad}_P(\mathfrak{so}_P(n)) \subset \mathrm{ad}_P(\mathfrak{so}(n))$ is immediate. In order to prove the other inclusion we need to prove that for all $X \in \mathfrak{so}(n)$, there exists $Y \in \mathfrak{so}_P(n)$ such that $[P,X] = [P,Y]$, i.e., such that $\mathrm{ad}_P(X) = \mathrm{ad}_P(Y)$. We will show that $Y = [P,[P,X]]$ solves the problem. First notice that $[P,X] \in \mathfrak{s}_P(n)$. In fact, $[P,X] \in \mathfrak{s}(n)$ and, furthermore, $[P,X]P + P[P,X] = [P,X]$, since

$$\begin{aligned} [P,X]P + P[P,X] &= (PX - XP)P + P(PX - XP) \\ &= PXP - XP + PX - PXP \\ &= [P,X]. \end{aligned}$$

Therefore, since $[P,X] \in \mathfrak{s}(n)$, it holds that $[P,[P,X]] \in \mathfrak{so}_P(n)$. Indeed, $[P,[P,X]] \in \mathfrak{so}(n)$ and

$$\begin{aligned} [P,[P,X]]P + P[P,[P,X]] &= (PX - 2PXP + XP)P + P(PX - 2PXP + XP) \\ &= PX - 2PXP + XP \\ &= [P,[P,X]], \end{aligned}$$

which means that $[P,[P,X]] \in \mathfrak{so}_P(n)$. Consequently, by Lemma 3.2.1, it is immediate that with $Y = [P,[P,X]]$, we have that

$$[P,Y] = [P,[P,[P,X]]] = [P,X].$$

$\square$

Before introduce the next subsection let us present the next results.

Proposition 3.2.4 *Let $A_1,\ A_2 \in \mathfrak{gl}_P(n)$ and $P \in G_{k,n}$. Then,*

$$[A_1,P] = [A_2,P] \Longleftrightarrow A_1 = A_2. \tag{3.19}$$

Proof. By the definition of Lie bracket we have that

$$\begin{aligned} [A_1,P] = [A_2,P] &\Longleftrightarrow A_1P - PA_1 = A_2P - PA_2 \\ &\Longleftrightarrow A_1 - PA_1 - PA_1 = A_2 - PA_2 - PA_2 \\ &\Longleftrightarrow A_1 - 2PA_1 = A_2 - 2PA_2 \\ &\Longleftrightarrow (I - 2P)A_1 = (I - 2P)A_2 \\ &\Longleftrightarrow A_1 = (I - 2P)^{-1}(I - 2P)A_2 \\ &\Longleftrightarrow A_1 = A_2. \end{aligned}$$

Since $A_1,\ A_2 \in \mathfrak{gl}_P(n)$, in the transition from the first line to the second one it was used the fact that $A_iP = A_i - PA_i$, for $i = 1,2$. Also, in the last equivalence, notice that since $P \in G_{k,n}$, we have that the matrix $I - 2P$ is orthogonal and symmetric, so its inverse is itself.

□

Remark 3.2.2 *Given $P \in G_{k,n}$, the result (3.19) of Proposition 3.2.4 is equivalent to*

$$[A,P] = 0,\ A \in \mathfrak{gl}_P(n) \Longleftrightarrow A = 0. \tag{3.20}$$

Indeed, one just needs to take into consideration the definition of Lie bracket and the fact that the matrix $I - 2P$ is invertible.

Proposition 3.2.5 *Let $P \in G_{k,n}$ and $B, X \in \mathfrak{so}_P(n)$. Then, $\mathrm{ad}_B^{2m}(X) \in \mathfrak{so}_P(n)$, $m \in \mathbb{N}$.*

Proof. Let $P \in G_{k,n}$ and $B, X \in \mathfrak{so}_P(n)$. In what follows we prove the result for $m = 1$. The proof for an arbitrary $m \in \mathbb{N}$ is an immediate consequence of the proof for $m = 1$, and of the definition of the adjoint operator at $B \in \mathfrak{so}_P(n)$.

Proof for $m = 1$:

Taking into account (2.39), and according with the definition of the adjoint operator, it is immediate that $\mathrm{ad}_B^2(X) \in \mathfrak{so}(n)$. Therefore, to prove that $\mathrm{ad}_B^2(X) \in \mathfrak{so}_P(n)$, it remains to show that $\mathrm{ad}_B^2(X)P + P\mathrm{ad}_B^2(X) = \mathrm{ad}_B^2(X)$. We have that,

$$\mathrm{ad}_B^2(X) = [B,[B,X]] = B^2X - 2BXB + XB^2.$$

On the other hand,

$$\mathrm{ad}_B^2(X)P = B^2XP - 2BXBP + XB^2P,$$

and

$$P\mathrm{ad}_B^2(X) = PB^2X - 2PBXB + PXB^2.$$

By Lemma 3.2.1, we know that $PB^2 = B^2P$, and since $X \in \mathfrak{so}_P(n)$ we obtain that

$$B^2XP + PB^2X = B^2XP + B^2PX = B^2(XP + PX) = B^2X, \tag{3.21}$$

and

$$XB^2P + PXB^2 = XPB^2 + PXB^2 = (XP + PX)B^2 = XB^2. \tag{3.22}$$

Also, since $B,\ X \in \mathfrak{so}_P(n)$, by (3.13) we know that $[BX,P] = 0$. Therefore,

$$\begin{aligned} -2BXBP - 2PBXB &= -2BX(B - PB) - 2PBXB \\ &= -2BXB + 2(BXP - PBX)B \\ &= -2BXB + 2[BX,P]B \\ &= -2BXB. \end{aligned} \tag{3.23}$$

From (3.21), (3.22) and (3.23), it holds that

$$\begin{aligned} \mathrm{ad}_B^2(X)P + P\mathrm{ad}_B^2(X) &= B^2X - 2BXB + XB^2 \\ &= \mathrm{ad}_B^2(X), \end{aligned}$$

which proves the result for $m = 1$.

$\square$

3.2.1 Tangent and Normal Spaces

In order to characterize the tangent space of $G_{k,n}$ at an arbitrary point P in $G_{k,n}$, hereafter denoted by $T_PG_{k,n}$, one has to consider any smooth curve, $t \in [0,\tau] \mapsto \alpha(t) \in G_{k,n}$ ($\tau > 0$), satisfying $\alpha(0) = P$ and must derive conditions for $\dot{\alpha}(0)$. Therefore, one can obtain the following result.

Proposition 3.2.6 *Let $P \in G_{k,n}$. Then,*

$$T_PG_{k,n} = \mathfrak{s}_P(n). \tag{3.24}$$

Proof. Let $t \in [0,\tau] \mapsto \alpha(t)$, ($\tau > 0$) be a smooth curve in $G_{k,n}$ such that $\alpha(0) = P$. Then, since $\alpha(t) \in G_{k,n}$, $t \in [0,\tau]$, we know that

$$\alpha^2(t) = \alpha(t), \quad \text{for all} \quad t \in [0,\tau].$$

Therefore, differentiating the last equality with respect to t, it yields that

$$\dot{\alpha}(t)\alpha(t) + \alpha(t)\dot{\alpha}(t) = \dot{\alpha}(t), \quad \text{for all} \quad t \in [0,\tau].$$

Then, for $t = 0$, since $\alpha(0) = P$, one has

$$\dot{\alpha}(0)P + P\dot{\alpha}(0) = \dot{\alpha}(0).$$

Since for all t, $\alpha(t) \in \mathfrak{s}(n)$, then for all t, $\dot{\alpha}(t) \in \mathfrak{s}(n)$, in particular $\dot{\alpha}(0) \in \mathfrak{s}(n)$. Consequently, from (3.2.1), denoting $\dot{\alpha}(0)$ by V, we reach the conclusion that

$$VP + PV = V,$$

which means that the vector V in $T_PG_{k,n}$ belongs to $\mathfrak{s}_P(n)$, and so, we can conclude that $T_PG_{k,n} \subset \mathfrak{s}_P(n)$. In order to prove the result, we need to show that the inclusion $\mathfrak{s}_P(n) \subset T_PG_{k,n}$ is also true. Let $A \in \mathfrak{s}_P(n)$. Then, by 2. of Proposition 3.2.3, we have that $[P,A] = \mathrm{ad}_P(A) \in \mathfrak{so}_P(n)$. Therefore, considering the curve β defined by

$$\beta(t) = \mathrm{e}^{-[P,A]t}P\mathrm{e}^{[P,A]t}, \quad t \in [0,\tau],$$

we have that $\beta(0) = P$, and for all $t \in [0,\tau]$, $\beta(t)$ leaves in $G_{k,n}$, because for all t, $\beta(t)^\top = \beta(t)$, $\beta^2(t) = \beta(t)$ and $\operatorname{rank}(\beta(t)) = \operatorname{rank}(P) = k$. Moreover,

$$\dot{\beta}(t) = -[P,A]\beta(t) + \beta(t)[P,A],$$

and so, since $\beta(0) = P$, and by 4. of Lemma 3.2.1, we get

$$\begin{aligned}\dot{\beta}(0) &= -[P,A]\beta(0) + \beta(0)[P,A] \\ &= -[P,A]P + P[P,A] \\ &= [P,[P,A]] = A,\end{aligned}$$

which means that $A \in T_P G_{k,n}$.

□

Taking into account the Lemma 3.2.1 and the relations in (2.39) it is possible to obtain the next alternative characterization of $T_P G_{k,n}$.

Proposition 3.2.7 *Let $P \in G_{k,n}$. Then,*

$$T_P G_{k,n} = \big\{[P,\Omega] \,:\, \Omega \in \mathfrak{so}_P(n)\big\}. \tag{3.25}$$

Proof. We start to prove that given an arbitrary $V \in T_P G_{k,n} = \mathfrak{s}_P(n)$, there exists an $\Omega \in \mathfrak{so}_P(n)$, such that $V = [P,\Omega]$. For that, we will show that $\Omega = [P,V]$ satisfies the requirements. Indeed, from (2.39), we get that $\Omega = [P,V] \in \mathfrak{so}(n)$ and, from 1. of Lemma 3.2.1, we obtain

$$\begin{aligned}[P,V]P + P[P,V] &= PVP - VP^2 + P^2V - PVP \\ &= -VP + PV \\ &= [P,V],\end{aligned}$$

which means that $\Omega = [P,V] \in \mathfrak{so}_P(n)$. Moreover, from 4. of Lemma 3.2.1, we have that $[P,\Omega] = [P,[P,V]] = V$. In order to prove the other inclusion, let us consider $\Omega \in \mathfrak{so}_P(n)$. Then, we know that $\Omega \in \mathfrak{so}(n)$ and $\Omega P + P\Omega = \Omega$. By (2.39), it holds that $[P,\Omega] \in \mathfrak{s}(n)$. So, in order to show that $[P,\Omega] \in \mathfrak{s}_P(n) = T_P G_{k,n}$, it is enough to prove that $[P,\Omega]P + P[P,\Omega] = [P,\Omega]$. But, similar to what was done above, from 1. of Lemma 3.2.1, we can, easily, conclude that

$$\begin{aligned}[P,\Omega]P + P[P,\Omega] &= P\Omega P - \Omega P^2 + P^2\Omega - P\Omega P \\ &= -\Omega P + P\Omega \\ &= [P,\Omega].\end{aligned}$$

□

Remark 3.2.3 *Observe that (3.25) can be rewritten as*

$$T_P G_{k,n} = \big\{\mathrm{ad}_P(\Omega) \,:\, \Omega \in \mathfrak{so}_P(n)\big\}. \tag{3.26}$$

Then, using previous considerations and results, together with some computations, leads to another alternative description of the tangent space at a point $P \in G_{k,n}$, as follows

$$T_P G_{k,n} = \left\{ \mathrm{ad}_P^2(S) \,:\, S \in \mathfrak{s}(n) \right\}, \tag{3.27}$$

which, in some future developments, will be more convenient to characterize the normal space at P.

Before proceeding, it follows that the Riemannian metric induced by the Euclidean inner product (2.37) can be defined as

$$\langle [P,\Omega_1],[P,\Omega_2] \rangle = \langle [\Omega_1,P],[\Omega_2,P] \rangle = -\mathrm{tr}\big(\Omega_1\Omega_2\big), \tag{3.28}$$

for $\Omega_1, \Omega_2 \in \mathfrak{so}_P(n)$.

Actually, since $P \in G_{k,n}$, using essentially 1. of Lemma 3.2.1, the fact that $\Omega_1 \in \mathfrak{so}_P(n)$, implies that $\Omega_1 P = \Omega_1 - P\Omega_1$ and some properties of the matrix trace we get,

$$\begin{aligned}
\langle [P,\Omega_1],[P,\Omega_2] \rangle &= \mathrm{tr}\big([P,\Omega_1]^\top [P,\Omega_2]\big) \\
&= \mathrm{tr}\big((P\Omega_1 - \Omega_1 P)^\top (P\Omega_2 - \Omega_2 P)\big) \\
&= \mathrm{tr}\big((-\Omega_1 P + P\Omega_1)(P\Omega_2 - \Omega_2 P)\big) \\
&= \mathrm{tr}\big(-\Omega_1 P\Omega_2 + \Omega_1 P\Omega_2 P + P\Omega_1 P\Omega_2 - P\Omega_1\Omega_2 P\big) \\
&= \mathrm{tr}\big(-\Omega_1 P\Omega_2 - P\Omega_1\Omega_2 P\big) \\
&= \mathrm{tr}\big(-(\Omega_1 - P\Omega_1)\Omega_2 - P\Omega_1\Omega_2 P\big) \\
&= \mathrm{tr}\big(-\Omega_1\Omega_2 + P\Omega_1\Omega_2 - P\Omega_1\Omega_2 P\big) \\
&= -\mathrm{tr}\big(\Omega_1\Omega_2\big) + \mathrm{tr}\big(P\Omega_1\Omega_2\big) - \mathrm{tr}\big(P^2\Omega_1\Omega_2\big) \\
&= -\mathrm{tr}\big(\Omega_1\Omega_2\big).
\end{aligned}$$

Consequently, from Remark 3.2.3 and by (3.28) it is immediate that the normal space at P, with respect to the Riemannian metric defined in (3.28), is defined by

$$(T_P G_{k,n})^\perp = \left\{ Z - \mathrm{ad}_P^2(Z) \,:\, Z \in \mathfrak{s}(n) \right\}. \tag{3.29}$$

We note that the descriptions of the tangent and normal spaces mentioned above are in accordance with the ones that have already appeared in Hüper and Silva Leite [31].

Proposition 3.2.8 *Let $P \in G_{k,n}$ and $\Omega \in \mathfrak{so}_P(n)$. Then,*

$$[\Omega,[\Omega,P]] \in (T_P G_{k,n})^\perp. \tag{3.30}$$

Proof. Let $P \in G_{k,n}$, $\Omega \in \mathfrak{so}_P(n)$ and $[\Omega_1,P]$, with $\Omega_1 \in \mathfrak{so}_P(n)$, be an arbitrary element of $T_P G_{k,n}$. Then, taking into account 1. and 3. of Lemma 3.2.1 and some properties of the matrix trace, we have that

$$
\begin{aligned}
\langle[\Omega_1,P],[\Omega,[\Omega,P]]\rangle &= \operatorname{tr}\big([\Omega_1,P]^\top[\Omega,[\Omega,P]]\big)\\
&= \operatorname{tr}\big((\Omega_1 P-P\Omega_1)^\top(\Omega(\Omega P-P\Omega)-(\Omega P-P\Omega)\Omega)\big)\\
&= \operatorname{tr}\big((-P\Omega_1+\Omega_1 P)(\Omega^2 P-2\Omega P\Omega+P\Omega^2)\big)\\
&= \operatorname{tr}\big((-P\Omega_1+\Omega_1 P)(2\Omega^2 P-2\Omega P\Omega)\big)\\
&= 2\operatorname{tr}\big(-P\Omega_1\Omega^2 P+P\Omega_1\Omega P\Omega+\Omega_1 P\Omega^2 P-\Omega_1\underbrace{P\Omega P}_{=0}\Omega\big)\\
&= 2\big(\operatorname{tr}(-P\Omega_1\Omega^2 P)+\operatorname{tr}(\underbrace{P\Omega P}_{=0}\Omega_1\Omega)+\operatorname{tr}(\underbrace{P\Omega_1 P}_{=0}\Omega^2)\big)\\
&= 2\operatorname{tr}\big(P\Omega^2\Omega_1 P\big)=2\operatorname{tr}\big(\Omega^2\underbrace{P\Omega_1 P}_{=0}\big)=0.
\end{aligned}
$$

Consequently, $[\Omega,[\Omega,P]]\in(T_P G_{k,n})^\perp$. □

3.2.2 Geodesics and Geodesic Distance

In this subsection we present some results in order to compute explicitly, in $G_{k,n}$, the geodesic satisfying some initial conditions, the minimizing geodesic arc joining two points and the geodesic distance between two points of the manifold.

Proposition 3.2.9 (Batzies et al. [5]) *The unique geodesic $t\mapsto\gamma(t)$ in $G_{k,n}$, satisfying the initial conditions $\gamma(0)=P$ and $\dot\gamma(0)=[\Omega,P]$, where $\Omega\in\mathfrak{so}_P(n)$, is given by*

$$\gamma(t)=\mathrm{e}^{t\Omega}P\mathrm{e}^{-t\Omega}. \tag{3.31}$$

Proposition 3.2.10 *Let $P\in G_{k,n}$ and γ be the geodesic in $G_{k,n}$, as defined in (3.31), then for all t,*

$$\Omega\in\mathfrak{so}_P(n)\Longleftrightarrow\Omega\in\mathfrak{so}_{\gamma(t)}(n). \tag{3.32}$$

Proof. It is enough to show that $\Omega\in\mathfrak{so}_P(n)$ implies that $\Omega\in\mathfrak{so}_{\gamma(t)}(n)$, for all t. The other implication is immediate, since $P=\gamma(0)$. Therefore, with the assumption that $\Omega\in\mathfrak{so}_P(n)$, we have that $\Omega\in\mathfrak{so}(n)$ and $\Omega P+P\Omega=\Omega$. Thus, for each t,

$$
\begin{aligned}
\Omega\gamma(t)+\gamma(t)\Omega &= \Omega\mathrm{e}^{t\Omega}P\mathrm{e}^{-t\Omega}+\mathrm{e}^{t\Omega}P\mathrm{e}^{-t\Omega}\Omega\\
&= \mathrm{e}^{t\Omega}\Omega P\mathrm{e}^{-t\Omega}+\mathrm{e}^{t\Omega}P\Omega\mathrm{e}^{-t\Omega}\\
&= \mathrm{e}^{t\Omega}(\Omega P+P\Omega)\mathrm{e}^{-t\Omega}\\
&= \mathrm{e}^{t\Omega}\Omega\mathrm{e}^{-t\Omega}\\
&= \Omega,
\end{aligned}
$$

which means that $\Omega\in\mathfrak{so}_{\gamma(t)}(n)$, for each t.

□

Remark 3.2.4 *With similar arguments to those used in Remark 3.1.1, we can also state that the Grassmann manifold is geodesically complete, so any two points can be joined by a geodesic. To* *obtain an explicit formula for such geodesic, we also have to impose a small restriction, as stated in the next proposition.*

The next result gives an explicit parametrization of the minimizing geodesic arc connecting two points in $G_{k,n}$. Although the expression has already appeared in Batzies et al. [5], we present bellow an easier alternative proof which can be done essentially due to the Proposition 3.2.2.

Proposition 3.2.11 *Let $P, Q \in G_{k,n}$ be such that the orthogonal matrix $(I-2Q)(I-2P)$ has no negative real eigenvalues. Then, the minimizing geodesic arc in $G_{k,n}$, with respect to the Riemannian metric (3.28), that joins P (at $t=0$) to Q (at $t=1$), is parameterized explicitly by*

$$\gamma(t) = \mathrm{e}^{t\Omega} P \mathrm{e}^{-t\Omega}, \tag{3.33}$$

with $\Omega = \frac{1}{2}\log((I-2Q)(I-2P)) \in \mathfrak{so}_P(n)$.

Proof. Let $P, Q \in G_{k,n}$ and $\gamma(t) = \mathrm{e}^{t\Omega} P \mathrm{e}^{-t\Omega},\ t \in [0,1]$ be such that $\gamma(0) = P$. In order to prove the result we need to obtain $\Omega \in \mathfrak{so}_P(n)$, such that $\gamma(1) = Q$, i.e., such that $\mathrm{e}^{\Omega} P \mathrm{e}^{-\Omega} = Q$. According with Proposition 3.2.2, and since the inverse of the matrix $I-2P$ is itself, the following holds.

$$\begin{aligned}
\mathrm{e}^{\Omega} P \mathrm{e}^{-\Omega} = Q \quad &\Longleftrightarrow \quad \mathrm{e}^{\Omega}(I-2P)\mathrm{e}^{-\Omega} = I-2Q \\
&\Longleftrightarrow \quad \mathrm{e}^{2\Omega}(I-2P) = I-2Q \\
&\Longleftrightarrow \quad \mathrm{e}^{2\Omega} = (I-2Q)(I-2P) \\
&\Longleftrightarrow \quad \Omega = \frac{1}{2}\log((I-2Q)(I-2P).
\end{aligned}$$

We have that $(I-2Q)(I-2P) \in \mathrm{SO}(n)$. Then, $\Omega = \dfrac{1}{2}\log((I-2Q)(I-2P)) \in \mathfrak{so}(n)$. Therefore, in order to prove that $\Omega \in \mathfrak{so}_P(n)$, it remains to show that $\Omega P + P\Omega = \Omega$, which is equivalent to prove that

$$2\Omega(I-2P) + (I-2P)2\Omega = 0.$$

Taking into account the properties in Lemma 2.6.1, and since $(I-2P)^2 = I$, we get

$$\begin{aligned}
&2\Omega(I-2P) + (I-2P)2\Omega \\
&\quad = (\log((I-2Q)(I-2P)))(I-2P) + (I-2P)(\log((I-2Q)(I-2P))) \\
&\quad = (I-2P)(I-2P)(\log((I-2Q)(I-2P)))(I-2P) + (I-2P)(\log((I-2Q)(I-2P))) \\
&\quad = (I-2P)\left(\log((I-2P)(I-2Q)(I-2P)^2)\right) + (I-2P)(\log((I-2Q)(I-2P))) \\
&\quad = (I-2P)\left(-\log(((I-2P)(I-2Q))^{-1})\right) + (I-2P)(\log((I-2Q)(I-2P))) \\
&\quad = (I-2P)(-\log((I-2Q)(I-2P))) + (I-2P)(\log((I-2Q)(I-2P))) \\
&\quad = 0,
\end{aligned}$$

which proves the result.

$\square$

Remark 3.2.5 *Notice that the orthogonal matrix* $(I-2Q)(I-2P)$ *belongs to* $\mathrm{SO}(n)$*, since the requirement that it has no negative real eigenvalues automatically excludes the orthogonal matrices with determinant minus one.*

To finish this subsection, taking into account the Proposition 3.2.11, we can obtain an explicit formula to compute the geodesic distance between two points in $G_{k,n}$, as follows in the next result whose proof is immediate.

Proposition 3.2.12 *Let* $P,Q \in G_{k,n}$ *be such that the orthogonal matrix* $(I-2Q)(I-2P)$ *has no negative real eigenvalues. Then, the geodesic distance between the points* P *and* Q *is given, explicitly, by*

$$d^2(P,Q) = -\frac{1}{4}\mathrm{tr}\big(\log^2((I-2Q)(I-2P))\big). \tag{3.34}$$

3.2.3 Representing Images by Points in a Grassmann Manifold

In this subsection, we explain how to associate to a set of images a point in the Grassmann manifold $G_{k,n}$, where n is the dimension of the space of features and k is related to the principal features of the images. This enlightens the importance of the Grassmann manifold in many engineering applications, in particular to solve some computer vision problems.

In the context of image processing, a feature vector is a collection of important information that describes an image, differentiating that image from others. Some examples of features are: colour, gray levels, pixel intensities, shapes, edges and gradients.

Given a set of m images of the same object, we associate to that set a point in a Grassmann manifold $G_{k,n}$ as follows:

1. Each image corresponds to a column matrix in the space of features, so that the m images can be represented by a rectangular matrix $X \in \mathbb{R}^{n\times m}$. We assume that $m<n$.

2. The matrix X is then decomposed using the Singular Value Decomposition (SVD)

$$X = U\Sigma V^\top, \tag{3.35}$$

   where $V^\top$ denotes the transpose of the matrix V, the matrices U and V are orthogonal of order n and m respectively ($UU^\top = I_n$, $VV^\top = I_m$) and Σ is a quasi-diagonal matrix containing the singular values $\sigma_1,\cdots,\sigma_m$ of X, in non-increasing order, along the main diagonal. If $\mathrm{rank}(X) = r$ and u_i and v_i denote the column vectors of U and V respectively, the SVD decomposition (3.35) can be written as

$$X = \sum_{i=1}^{r} \sigma_i u_i v_i^\top. \tag{3.36}$$

   Since $XX^\top = U(\Sigma\Sigma^\top)U^\top$, the columns of U are the eigenvectors associated to the eigenvalues λ_i of $XX^\top$, which are the non-negative square roots of the singular values and are, by convention, also descendent sorted ($\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0$). The columns of the matrix U are called the eigenvectors of the SVD decomposition and the first columns correspond to the main dominant directions in the image structure.

3. When a set of images is SVD transformed it is not compressed. Image compression deals with the problem of reducing the amount of computer memory required to represent a digital image. Since the great amount of the image information lies in the first singular values, compression of data can be achieved replacing the matrix X by a good approximation of smaller rank, say of rank $k < r$. The closest matrix of rank k is obtained by truncating the sum in (3.36) after the first k terms to obtain
$$X \approx \sum_{i=1}^{k} \sigma_i u_i v_i^\top .$$
As k increases, the image quality increases, but so does the amount of memory needed to store the image. This means that smaller rank SVD approximations are preferable, but the choice of k also depends on the dimensionality of the data. The above truncation corresponds to deleting the last $n-k$ columns of the orthogonal matrix U, to form the submatrix $S_{n\times k}$, whose columns form a k-orthonormal frame in $\mathbb{R}^n$, i.e., $S^\top S = I_k$.

4. From the previous matrix S, we compute a square matrix of order n, $P = SS^\top$, which is symmetric, idempotent ($P^2 = P$) and has rank k.

   The matrix P gives a representation of the data in the Grassmann manifold $G_{k,n}$.

3.3 Riemannian Structure of the Manifold $G_{k,n} \times \mathrm{SO}(n)$

Let n be a positive integer and let k be a positive integer smaller than n. In what follows, we will consider the Generalized Essential manifold $G_{k,n} \times \mathrm{SO}(n)$, that is the cartesian product of the real Grassmann manifold $G_{k,n}$, consisting of all k-dimensional (linear) subspaces of $\mathbb{R}^n$, and the rotation orthogonal group $\mathrm{SO}(n)$ of all orientation preserving rotational transformations in $\mathbb{R}^n$. Therefore,
$$G_{k,n} \times \mathrm{SO}(n) = \{(S,R) :\ S \in G_{k,n},\ R \in \mathrm{SO}(n)\}. \tag{3.37}$$
Taking into account the isospectral feature of $G_{k,n}$, another characterization of the real manifold $G_{k,n} \times \mathrm{SO}(n)$ can be defined as follows
$$G_{k,n} \times \mathrm{SO}(n) = \left\{P = \left(\Theta E_0 \Theta^\top, R\right) :\ \Theta, R \in \mathrm{SO}(n)\right\}, \tag{3.38}$$
with $E_0 = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$ and where I_k denotes the identity matrix of order k. It is a smooth compact connected manifold of real dimension $k(n-k) + n(n-1)/2$.

Since any element of $G_{k,n} \times \mathrm{SO}(n)$ is a pair of the form $\left(\Theta E_0 \Theta^\top, R\right)$, with $\Theta, R \in \mathrm{SO}(n)$ we have that the first component of the previous pair is a symmetric matrix and the second one is a $n \times n$ orthogonal matrix. Then, the manifold $G_{k,n} \times \mathrm{SO}(n)$ can be seen as an embedded submanifold of $\mathfrak{s}(n) \times \mathbb{R}^{n\times n}$, with metric induced by the metric of the embedding manifold which is described by the relation
$$\langle (J,K),(L,M) \rangle_{\mathfrak{s}(n)\times\mathbb{R}^{n\times n}} = \langle J,L \rangle_{\mathfrak{s}(n)} + \langle K,M \rangle_{\mathbb{R}^{n\times n}}, \tag{3.39}$$

where the metric $\langle .,. \rangle$ in the right hand side is related to the Frobenius norm for matrices, that is, $\langle A,B \rangle = \mathrm{tr}(A^\top B)$. Using the characterization (3.38), the tangent space to the manifold $G_{k,n} \times \mathrm{SO}(n)$ at an arbitrary point $P_0 = (\Theta_0 E_0 \Theta_0^\top, R_0)$ is given by

$$T_{P_0}\left(G_{k,n} \times \mathrm{SO}(n)\right) = \left\{ \left(\Theta_0 \left[\Omega, E_0\right] \Theta_0^\top, R_0 C\right) \,:\, \Omega \in \mathfrak{so}_{E_0}(n), C \in \mathfrak{so}(n) \right\} \tag{3.40}$$

or, equivalently, by

$$T_{P_0}\left(G_{k,n} \times \mathrm{SO}(n)\right) = \left\{ \left(\Theta_0 \begin{bmatrix} 0 & Z \\ Z^\top & 0 \end{bmatrix} \Theta_0^\top, R_0 C\right) \,:\, Z \in \mathbb{R}^{k\times(n-k)}, C \in \mathfrak{so}(n) \right\}. \tag{3.41}$$

Moreover, considering the fact that $G_{k,n} \times \mathrm{SO}(n)$ and $T_{P_0}\left(G_{k,n} \times \mathrm{SO}(n)\right)$ are real manifolds of dimension $k(n-k) + n(n-1)/2$ and the embedding manifold $\mathfrak{s}(n) \times \mathbb{R}^{n\times n}$ is a real manifold of dimension $(3n^2+n)/2$, the orthogonal complement, $\left(T_{P_0}\left(G_{k,n} \times \mathrm{SO}(n)\right)\right)^\perp$, of $T_{P_0}\left(G_{k,n} \times \mathrm{SO}(n)\right)$ in $\mathfrak{s}(n) \times \mathbb{R}^{n\times n}$, is given by

$$\left(T_{P_0}\left(G_{k,n} \times \mathrm{SO}(n)\right)\right)^\perp = \left\{ \left(\Theta_0 \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \Theta_0^\top, R_0 S\right) \,:\, S_1 \in \mathfrak{s}(k), S_2 \in \mathfrak{s}(n-k), S \in \mathfrak{s}(n) \right\}. \tag{3.42}$$

Observe that $\mathfrak{s}(i)$, $i = k, n-k, n$ has real dimension $i(i+1)/2$.

3.3.1 Geodesics and Geodesic Distance

Considering the results of the previous sections and the expressions of the geodesics on the manifolds $\mathrm{SO}(n)$ and $G_{k,n}$ we can conclude the following results for the manifold $G_{k,n} \times \mathrm{SO}(n)$.

Proposition 3.3.1 *Let $(P,R) \in G_{k,n} \times \mathrm{SO}(n)$, $X \in \mathfrak{so}_P(n)$ and $Y \in \mathfrak{so}(n)$. The curve $t \mapsto \gamma(t)$ in $G_{k,n} \times \mathrm{SO}(n)$, defined by*

$$\gamma(t) = (\gamma_1(t), \gamma_2(t)) = \left(e^{tX} P e^{-tX}, R e^{tY}\right), \tag{3.43}$$

is the unique geodesic in $G_{k,n} \times \mathrm{SO}(n)$, satisfying the initial conditions $\gamma(0) = (P,R)$ and $\dot{\gamma}(0) = ([X,P], RY)$.

Proposition 3.3.2 *Let $(P_1,R_1), (P_2,R_2) \in G_{k,n} \times \mathrm{SO}(n)$ be such that $\sigma(R_1^{-1}R_2) \cap \mathbb{R}^- = \varnothing$ and $\sigma((I-2P_2)(I-2P_1)) \cap \mathbb{R}^- = \varnothing$. Then, the minimizing geodesic arc connecting the point (P_1,R_1) (at $t=0$) to the point (P_2,R_2) (at $t=1$), is given explicitly by*

$$\gamma(t) = (\gamma_1(t), \gamma_2(t)) = \left(e^{tX} P_1 e^{-tX}, R_1 e^{tY}\right), \tag{3.44}$$

or, equivalently,

$$\gamma(t) = (\gamma_1(t), \gamma_2(t)) = \left(e^{tX} P_1 e^{-tX}, e^{t\overline{Y}} R_1\right), \tag{3.45}$$

where $X = \frac{1}{2}\log((I-2P_2)(I-2P_1)) \in \mathfrak{so}_{P_1}(n)$, $Y = \log(R_1^{-1}R_2) \in \mathfrak{so}(n)$ and $\overline{Y} = \log(R_2R_1^{-1}) \in \mathfrak{so}(n)$.

Proposition 3.3.3 *Let $(P_1,R_1), (P_2,R_2) \in G_{k,n} \times \mathrm{SO}(n)$ be such that $\sigma(R_1^{-1}R_2) \cap \mathbb{R}^- = \varnothing$ and $\sigma((I-2P_2)(I-2P_1)) \cap \mathbb{R}^- = \varnothing$. Then, the geodesic distance between the points (P_1,R_1) and (P_2,R_2) is* *given explicitly by*

$$d^2\left((P_1,R_1),(P_2,R_2)\right) = -\frac{1}{4}\mathrm{tr}\left(\log^2((I-2P_2)(I-2P_1))\right) - \mathrm{tr}\left(\log^2(R_1^{-1}R_2)\right),$$

or, equivalently, (3.46)

$$d^2\left((P_1,R_1),(P_2,R_2)\right) = -\frac{1}{4}\mathrm{tr}\left(\log^2((I-2P_2)(I-2P_1))\right) - \mathrm{tr}\left(\log^2(R_2R_1^{-1})\right).$$

3.4 The Normalized Essential Manifold

The Normalized Essential Manifold is a particular case of the product manifold presented in the previous section, corresponding to the choice of $k=2$ and $n=3$. The reason to single out this special case is due to its important applications in modern computer vision. As will be explained next, the Normalized Essential Manifold encodes the so called epipolar constraint between points in two projective views.

3.4.1 Geometric Formulation

It is well-known from computer vision literature that the intrinsic projective geometry between two views of the same scene is independent of the scene structure and only depends on the cameras internal parameters and relative pose (see, for instance, Hartley and Zisserman [23]). In this work we deal with calibrated cameras, that is, we assume that the camera parameters are known. We also assume for simplicity that the scene is static and that the images are taken by two identical pinhole cameras, with focal length equal to one, or equivalently by the same camera at different locations. The two cameras are denoted by C_1 and C_2. Each camera is represented by an orthonormal reference frame and can therefore be described as a change of coordinates relatively to an inertial reference frame. The corresponding images of the scene structure p are denoted by X_1 and X_2, respectively, as shown in schematic Fig. 3.1.

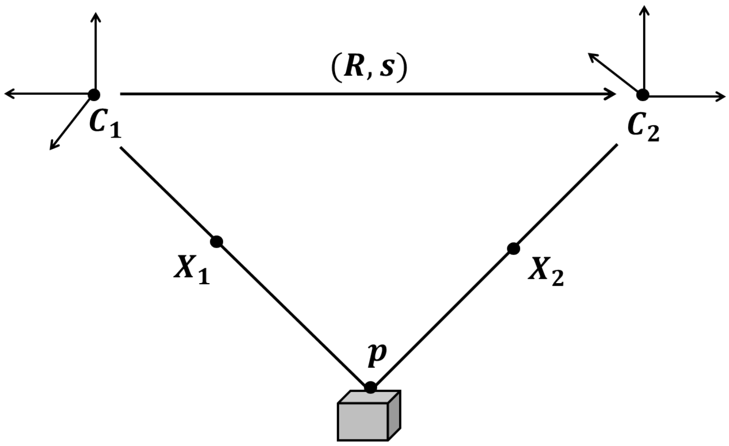

Fig. 3.1 Geometry between two views of the same scene structure.

Without loss of generality, we can assume that the inertial frame corresponds to one of the two cameras, say C_1, while the other is positioned and oriented according to an element (R,s) of the special

Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$, where R denotes a rotation and s represents a translation vector of the displacement of the first camera C_1 into the second one C_2. Let s_1, s_2 and s_3 be the coordinates of s with respect to the first camera basis ($s = [s_1\ s_2\ s_3]^\top$) and $x_1,\ x_2 \in \mathbb{R}^3$ be the homogeneous coordinates of the projection of the same point p onto the two image planes of the cameras. If we call $X_1 \in \mathbb{R}^3$ and $X_2 \in \mathbb{R}^3$ the 3D coordinates of the point p relative to the two camera frames, they are related by a rigid body motion:

$$X_2 = RX_1 + s,$$

where $X_i = \lambda_i x_i$, $i = 1,2$, can be written in terms of the image points x_i, $i = 1,2$ and the depths $\lambda_i, i = 1,2$, ($\lambda_i > 0$). So, the last equation can be written as

$$\lambda_2 x_2 = R\lambda_1 x_1 + s. \tag{3.47}$$

Consider the Lie algebra isomorphism

$$\begin{aligned} \widehat{\ }: \mathbb{R}^3 &\longrightarrow \mathfrak{so}(3) \\ s = [s_1\ s_2\ s_3]^\top &\longmapsto \widehat{S} := \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}, \end{aligned}$$

between $\mathbb{R}^3$ equipped with the cross product $\times$ and $\mathfrak{so}(3)$ equipped with the commutator. It is immediate that for any vector $v \in \mathbb{R}^3$, $\widehat{S}v = s \times v$. Also, $\mathrm{tr}\big(\widehat{S}^\top \widehat{S}\big) = 2\|s\|^2$. Multiplying (on the left) both sides of the equation (3.47) by $\widehat{S}$ we then obtain

$$\lambda_2 \widehat{S} x_2 = \lambda_1 \widehat{S} R x_1.$$

Now, by taking the inner product of both sides of the previous equation with x_2, it follows

$$x_2^\top \widehat{S} R x_1 = 0, \tag{3.48}$$

which is called the *epipolar constraint* (Longuet-Higgins, [48]). This intrinsic constraint is independent of depth information and decouples the *problem of motion recovery* from 3D structure. This problem consists in finding $(R,s) \in \mathrm{SE}(3)$ from the known image points x_1 and x_2 and the epipolar constraint. The matrix $E = \widehat{S}R$ in (3.48), which captures the relative orientation between the two cameras, is called the *essential matrix* and the set of all such matrices is the so-called Essential Manifold.

3.4.2 Riemannian Structure of the Normalized Essential Manifold

For many of the applications concerning essential matrices, it is enough to work with those where the translation vector s has norm 1, or equivalently, $\mathrm{tr}(\widehat{S}^\top \widehat{S}) = 2$. This set, referred in the literature as the *Normalized Essential Manifold*, is defined as

$$\mathscr{E} = \left\{ \widehat{S}R : \widehat{S} \in \mathfrak{so}(3), R \in \mathrm{SO}(3), \frac{1}{2}\mathrm{tr}(\widehat{S}^\top \widehat{S}) = 1 \right\}. \tag{3.49}$$

We are going to show that this manifold can be realized as a particular case of the product manifold studied in the previous section and refer to Helmke et al. [24] for details. But first we need a preliminary result.

Proposition 3.4.1 *The Normalized Essential Manifold can be defined, alternatively, by*

$$\mathscr{E} = \left\{ UE_0V^\top, \text{ where } E_0 = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } U,V \in \mathrm{SO}(3) \right\}. \tag{3.50}$$

Proof. We first show that the set defined in (3.49) is contained in the set defined in (3.50). To see this, we first prove that every normalized essential matrix $E = \widehat{S}R$ has singular values $1,1,0$. Recall that the singular values of E are the nonnegative square roots of the eigenvalues of $E^\top E$. Since $E^\top E = R^\top \widehat{S}^\top \widehat{S}R$, the eigenvalues of $E^\top E$ are the same as those of $\widehat{S}^\top \widehat{S} = -\widehat{S}^2$, which are $\|s\|^2, \|s\|^2, 0$. But $\|s\| = 1$, so $1,1,0$ are the singular values of E. As a consequence, given $E \in \mathscr{E}$, there exist matrices $U,V \in \mathrm{O}(3)$ such that $E = UE_0V^\top$, where $E_0 = \mathrm{diag}(1,1,0)$. Since when U or V have negative determinant, they may be replaced by the product of a matrix in SO(3) by $\mathrm{diag}(1,1,-1)$ without affecting the decomposition of E above, we can conclude that

$$\text{Given } E \in \mathscr{E}, \text{ there exist } U,V \in \mathrm{SO}(3) \text{ such that } E = UE_0V^\top. \tag{3.51}$$

We now show that the set defined in (3.50) is contained in the set defined in (3.49). This follows from the following observation. Let

$$A_{12} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{so that} \quad \mathrm{e}^{-\frac{\pi}{2}A_{12}} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_{12}\mathrm{e}^{-\frac{\pi}{2}A_{12}} = E_0.$$

So, we can write $E = UE_0V^\top = \left(UA_{12}U^\top\right)\left(U\mathrm{e}^{-\frac{\pi}{2}A_{12}}V^\top\right)$ and conclude that, given $U,V \in \mathrm{SO}(3)$, there exist $\widehat{S} = UA_{12}U^\top \in \mathfrak{so}(3)$ and $R = U\mathrm{e}^{-\frac{\pi}{2}A_{12}}V^\top \in \mathrm{SO}(3)$ such that $E = \widehat{S}R$. Moreover, $\frac{1}{2}\mathrm{tr}(\widehat{S}^\top \widehat{S}) = 1$, which concludes the proof.

□

Proposition 3.4.2 *The Normalized Essential Manifold $\mathscr{E}$ is diffeomorphic to $G_{2,3} \times \mathrm{SO}(3)$.*

Proof. It is enough to show that the mapping

$$\begin{array}{rccc} \varphi: & \mathscr{E} & \to & G_{2,3} \times \mathrm{SO}(3) \\ & UE_0V^\top & \mapsto & \left(UE_0U^\top, UV^\top\right) \end{array} \tag{3.52}$$

is a diffeomorphism. It is easily seen that φ is smooth and bijective and its inverse $\varphi^{-1}(X,Y) = XY$ is also smooth. So, $\mathscr{E}$ is diffeomorphic to $G_{2,3} \times \mathrm{SO}(3)$.

□

As a consequence, we may look at the Normalized Essential Manifold as being the 5-dimensional smooth manifold defined as

$$\mathscr{E} := \left\{ \left(U E_0 U^\top, R \right),\ \text{such that}\ U, R \in \mathrm{SO}(3)\ \text{and}\ E_0 = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} \right\}. \tag{3.53}$$

As such it can be considered embedded in the Euclidean space $\mathfrak{s}(3) \times \mathbb{R}^{3\times 3}$ with the natural metric defined as

$$\langle (J,K), (L,M) \rangle_{\mathfrak{s}(3)\times\mathbb{R}^{3\times 3}} = \langle J, L \rangle_{\mathfrak{s}(3)} + \langle K, M \rangle_{\mathbb{R}^{3\times 3}}. \tag{3.54}$$

With the above parametrization, the tangent space at a point $P_0 = (\Theta_0 E_0 \Theta_0^\top, R_0)$ of the Normalized Essential Manifold, and the corresponding orthogonal space are, respectively, given by:

$$T_{P_0}\mathscr{E} = \left\{ \left(\Theta_0 \left[\Omega, E_0 \right] \Theta_0^\top, R_0 C \right) : \Omega \in \mathfrak{so}_{E_0}(3), C \in \mathfrak{so}(3) \right\} \tag{3.55}$$

or, equivalently,

$$T_{P_0}\mathscr{E} = \left\{ \left(\Theta_0 \begin{bmatrix} 0 & \Lambda \\ \Lambda^\top & 0 \end{bmatrix} \Theta_0^\top, R_0 C \right) : \Lambda \in \mathbb{R}^{2\times 1}, C \in \mathfrak{so}(3) \right\} \tag{3.56}$$

and

$$(T_{P_0}\mathscr{E})^\perp = \left\{ \left(\Theta_0 \begin{bmatrix} B & 0 \\ 0 & b \end{bmatrix} \Theta_0^\top, R_0 S \right) : B \in \mathfrak{s}(2), b \in \mathbb{R}, S \in \mathfrak{s}(3) \right\}. \tag{3.57}$$

Note that, as expected, the dimensions of the above spaces match with the dimension of the embedding space, which is fifteen. Indeed, $\dim(T_{P_0}\mathscr{E}) = 5$ and $\dim\left((T_{P_0}\mathscr{E})^\perp\right) = 10$.

Chapter 4

Polynomial Interpolation using the De Casteljau Algorithm

4.1 Introduction

Interpolation problems of data arise in different areas ranging from robotics, engineering, computer vision to industrial and medical applications (see, e.g., Bressan [8]). This kind of problems on manifolds are needed in a growing number of applications and have sparked the research interest of the mathematical community. A well-known recursive procedure to generate interpolating polynomial curves in Euclidean spaces is the classical De Casteljau algorithm which was introduced, independently, by De Casteljau [17] and Bézier [9]. It is a simple and powerful tool widely used in the field of Computer Aided Geometric Design (CAGD), particularly due to the fact that it is, essentially, geometrically based.

In this chapter, using the classical De Casteljau algorithm and its generalization to geodesically complete Riemannian manifolds, we present solutions of an interpolating problem that generalizes cubic splines in Euclidean spaces. It is well-known that these curves minimize the acceleration, which makes them particularly useful in many engineering applications. Also having this in mind, we pay particular attention to problems with two different types of boundary conditions. One are the well-known Hermite boundary conditions, and the other consists of initial and final points, initial velocity and initial covariant acceleration.

Based on the work of Crouch et al. [13], concerning the reinterpretation of the De Casteljau algorithm for connected and compact Lie groups and for spheres, we review the algorithm to generate cubic polynomials in the Euclidean space $\mathbb{R}^m$, followed by a detailed description and implementation of that algorithm for the Grassmann manifold $G_{k,n}$. The main feature of the algorithm is based on recursive geodesic interpolation. The generation of a cubic spline from cubic polynomials is immediate for the second type of boundary conditions but not for those of Hermite type, which are more natural from the point of view of applications. To overcome this difficulty, we show how each type of boundary conditions are related to the other for the interpolation problem in $G_{k,n}$. We also briefly review the interpolation problem in $\mathrm{SO}(n)$, which has been studied in Crouch et al. [13], in order to extend the results to $G_{k,n} \times \mathrm{SO}(n)$ and, in particular, to the Normalized Essential manifold.

4.1.1 Literature Review

The De Casteljau algorithm and the Bézier curves have been used for a long time and on a widely variety of different areas of knowledge. For a general presentation of the De Casteljau algorithm and an overview of the historical evolution and existing works dealing with this concept we mention, for instance, Farin [20] and literature cited therein.

Most of the reality problems require the application of the De Casteljau algorithm in manifolds different from the Euclidean space $\mathbb{R}^m$. In this sense, there are known in the literature many references of works relating to the study of the algorithm, in other manifolds, and with different approaches. We start by referring, for instance, the early work of Shoemake [70] and the work of Altafini [2].

Furthermore, interpolation problems on manifolds are needed in a growing number of applications and have been studied by several authors, starting with the pioneer work of Noakes, Heinzinger and Paden in [59]. Following this, other authors further developed the theory of geometric splines on manifolds using a variational approach (see, for instance, Camarinha [10], Crouch and Silva Leite [14] and [15], and Zhang and Noakes [74]). A more general variational problem, that of fitting a curve to data points on a Riemannian manifold, was presented and studied in Machado et al. [52]. The main drawback of this approach is that the solution curves are not given explicitly. Instead, one obtains a highly nonlinear third order differential equation for the velocity vector field. To overcome the complexity of the previous approach, other procedures have been developed, namely the generalizations of the classical De Casteljau algorithm for Riemannian manifolds, which can be found, for instance, in Park and Ravani [62], Crouch et al. [13], J. Jakubiak [32] and Nava-Yazdani and Polthier [58].

Recently, a modified Casteljau algorithm to solve interpolation problems on Stiefel manifolds was presented in Krakowski et al. [42] and used in Batista et al. [4] to address multiple source domain adaptation, by taking advantage of the smoothness of the interpolating curves on the Stiefel manifold to walk along a set of multiple domains.

The inclusion of the approach of this theme in this work is motivated by the existence in the literature of an increasing of a recent emerging need for spline interpolation on fields where the Normalized Essential manifold plays an important role.

4.1.2 Main Contributions

After bringing forward the interpolation problem that will be under study we review the De Casteljau algorithm to generate cubic polynomials in geodesically complete Riemannian manifolds and its implementation when the manifold is the Euclidean space $\mathbb{R}^m$.

Following this approach and the work of Crouch et al. [13] for connected and compact Lie groups and for spheres, one of our main contribution is the implementation of the De Casteljau algorithm for the generation of cubic polynomials in the Grassmann manifold $G_{k,n}$ and its application to generating cubic splines to solve the interpolation problem in the same manifold. In order to generate cubic splines from cubic polynomials in the Grassmann manifold, we first relate the two types of boundary conditions mentioned in Section 4.1 and then describe the main steps to obtain the interpolating curve.

Furthermore, we develop expressions for the derivative of the geometric cubic polynomial generated by the generalized De Casteljau algorithm in $G_{k,n}$, and for the covariant derivative of the velocity vector field along this curve, that to the best of our knowledge are new results. These results have been published recently in Pina and Silva Leite [65] and a shorter version, Pina and Silva Leite [64], has just been submitted.

The implementation of the De Casteljau algorithm on the special orthogonal group $\mathrm{SO}(n)$ and on the Grassmann manifold $G_{k,n}$ provides the groundwork to obtain the solution of the interpolation problem for the manifold $G_{k,n} \times \mathrm{SO}(n)$ and, in particular, for the Normalized Essential manifold.

4.2 Formulation of the Interpolation Problem on Manifolds

Let M be a m-dimensional connected Riemannian manifold, which is also geodesically complete. Given a set of $\ell+1$ distinct points $p_i \in M$, with $i = 0, 1, \ldots, \ell$, a discrete sequence of $\ell+1$ fixed times t_i, where

$$0 = t_0 < t_1 < \cdots < t_{\ell-1} < t_\ell = \tau,$$

and the vectors ξ_0 tangent to M at p_0 and ξ_ℓ tangent to M at p_ℓ (or, alternatively, the vectors ξ_0 and η_0 tangent to M at p_0), we intend to solve the following problem.

Problem 4.2.1 *Find a $\mathscr{C}^2$-smooth curve*

$$\gamma : [0, \tau] \to M$$

satisfying the interpolation conditions:

$$\gamma(t_i) = p_i, \qquad 1 \leq i \leq \ell - 1, \tag{4.1}$$

and the boundary conditions:

$$\begin{aligned} \gamma(0) = p_0, \quad \gamma(\tau) = p_\ell, \\ \dot{\gamma}(0) = \xi_0 \in T_{p_0}M, \quad \dot{\gamma}(\tau) = \xi_\ell \in T_{p_\ell}M \end{aligned} \tag{4.2}$$

or, alternatively, the boundary conditions:

$$\begin{aligned} \gamma(0) = p_0, \quad \gamma(\tau) = p_\ell, \\ \dot{\gamma}(0) = \xi_0 \in T_{p_0}M, \quad \frac{D\dot{\gamma}}{dt}(0) = \eta_0 \in T_{p_0}M. \end{aligned} \tag{4.3}$$

During this chapter we will consider $\tau = 1$. Notice that there's no loss of generality, since the interval $[0,1]$ can be replaced by any other interval $[a,b]$, $a < b$, just by choosing the reparametrization $(t \to s)$ defined by $t = (s-a)/(b-a)$.

Solutions of these problems will be called geometric cubic splines, since they are natural extensions to Riemannian manifolds of Euclidean cubic splines, the curves that minimize acceleration. If no interpolation conditions (4.1) are required, the solutions of these problems are geometric cubic polynomials, which happen to be $\mathscr{C}^\infty$-smooth. Generalizations to higher degree polynomials would require more boundary conditions, higher order of smoothness and more elaborate computations. The cases presented here will give enough insight about the theoretical aspects and the complexity of the corresponding implementations.

The boundary conditions (4.2) are symmetrically defined and are known, in the literature, as *Hermite boundary conditions*. The boundary conditions (4.3), which are not symmetrically specified, have computational advantages whenever instead of polynomial interpolating curves one is interested in interpolating polynomial spline curves, as we will see.

Problem 4.2.1 can be solved using cubic polynomials generated by the generalized De Casteljau algorithm on manifolds. The classical version of this algorithm in $\mathbb{R}^m$ was developed at 1959 by Paul De Casteljau [17], and it is a recursive process to generate interpolating polynomial curves of arbitrary degree in the Euclidean space $\mathbb{R}^m$. These curves are known in the literature as *Bernstein-Bézier curves*, or simply, *Bézier curves* (see, e.g., Bézier [9] and Farin [20]). This process is essentially based on a geometric and on an algebraic construction in which any two points of $\mathbb{R}^m$ are joined by a polynomial curve, and where each step is described in terms of successive linear interpolation. For an arbitrary manifold, the linear interpolation is replaced by geodesic interpolation, this being the reason why we assume that the manifold M is geodesically complete, at least in a sufficiently big neighbourhood of the given data points. The general De Casteljau algorithm is presented in the next section.

4.3 Cubic Polynomials in Manifolds, using De Casteljau Algorithm

Given a set of four distinct points $\{x_0, x_1, x_2, x_3\}$ in M, a smooth curve

$$t \in [0,1] \mapsto \beta_3(t) := \beta_3(t, x_0, x_1, x_2, x_3)$$

in M, joining x_0 (at $t = 0$) and x_3 (at $t = 1$), can be constructed by three successive geodesic interpolation steps as follows.

Algorithm 4.3.1 (Generalized De Casteljau Algorithm)

Given four distinct points x_0, x_1, x_2 and x_3 in M:

Step 1. *Construct three geodesic arcs $\beta_1(t, x_i, x_{i+1})$, $t \in [0,1]$ joining, for $i = 0, 1, 2$, the points x_i (at $t = 0$) and x_{i+1} (at $t = 1$).*

Step 2. *Construct two families of geodesic arcs*

$$\beta_2(t, x_0, x_1, x_2) = \beta_1(t, \beta_1(t, x_0, x_1), \beta_1(t, x_1, x_2)),$$
$$\beta_2(t, x_1, x_2, x_3) = \beta_1(t, \beta_1(t, x_1, x_2), \beta_1(t, x_2, x_3)),$$

joining, for $i = 0, 1$ and $t \in [0,1]$, the point $\beta_1(t, x_i, x_{i+1})$ (at $t = 0$) with the point $\beta_1(t, x_{i+1}, x_{i+2})$ (at $t = 1$).

Step 3. *Construct the family of geodesic arcs*

$$\beta_3(t, x_0, x_1, x_2, x_3) = \beta_1(t, \beta_2(t, x_0, x_1, x_2), \beta_2(t, x_1, x_2, x_3)),$$

joining, for each $t \in [0,1]$, the points $\beta_2(t, x_0, x_1, x_2)$ (at $t = 0$) and $\beta_2(t, x_1, x_2, x_3)$ (at $t = 1$).

The curve $t \in [0,1] \mapsto \beta_3(t) := \beta_3(t, x_0, x_1, x_2, x_3)$ obtained in Step 3. of Algorithm 4.3.1 is called *geometric cubic polynomial* in M. It is important to observe that this curve joins the points x_0 (at

$t = 0$) and x_3 (at $t = 1$), but does not pass through the other two points x_1 and x_2. These points are usually called by *control points*, since they influence the shape of the curve. This algorithm can also be used to generate $\mathscr{C}^2$-smooth *geometric cubic polynomial splines* by piecing together, in a smooth manner, several geometric cubic polynomials and which are interpolating curves of the data.

A few remarks should be made concerning the general applicability of this construction. In fact, although the geometry of a Riemannian manifold possesses enough structure to formulate the construction, the basic ingredients used, the geodesic arcs, are implicitly defined by a set of nonlinear differential equations. Therefore, the Algorithm 4.3.1 can be only practically implemented when we can reduce the calculation of these geodesic arcs to a manageable form.

For this reason, in what follows we will present the implementation of the Algorithm 4.3.1, when the manifold M is one of the three particular manifolds, $\mathbb{R}^m$, $\mathrm{SO}(n)$ and $G_{k,n}$. The result for the Normalized Essential Manifold $\mathscr{E} = G_{k,n} \times \mathrm{SO}(n)$, with $n = 3$ and $k = 2$, is an immediate consequence of the implementation on the manifolds $\mathrm{SO}(n)$ and $G_{k,n}$.

4.3.1 Cubic Polynomials in $\mathbb{R}^m$

For the particular situation when the geodesically complete Riemannian manifold is $M = \mathbb{R}^m$ we recover the classical De Casteljau Algorithm.

Algorithm 4.3.2 (Classical De Casteljau Algorithm)

Given four distinct points x_0, x_1, x_2 and x_3 in $M = \mathbb{R}^m$:

Step 1. *Construct three straight line arcs joining, for $i = 0,1,2$, the points x_i (at $t = 0$) and x_{i+1} (at $t = 1$):*

$$\begin{aligned}
\beta_1(t,x_0,x_1) &= tx_1 + (1-t)x_0, \\
\beta_1(t,x_1,x_2) &= tx_2 + (1-t)x_1, \\
\beta_1(t,x_2,x_3) &= tx_3 + (1-t)x_2, \quad t \in [0,1].
\end{aligned}$$

Step 2. *Construct two straight line arcs*

$$\begin{aligned}
\beta_2(t,x_0,x_1,x_2) &= t^2x_2 + 2t(1-t)x_1 + (1-t)^2x_0, \\
\beta_2(t,x_1,x_2,x_3) &= t^2x_3 + 2t(1-t)x_2 + (1-t)^2x_1.
\end{aligned}$$

Step 3. *Construct the straight line arc*

$$\beta_3(t,x_0,x_1,x_2,x_3) = t^3x_3 + 3t^2(1-t)x_2 + 3t(1-t)^2x_1 + (1-t)^3x_0. \quad (4.4)$$

The curve $t \in [0,1] \mapsto \beta_3(t) := \beta_3(t,x_0,x_1,x_2,x_3)$ is the cubic polynomial that interpolates the points x_0 (at $t = 0$) and x_3 (at $t = 1$), but does not pass through the control points x_1 and x_2. We illustrate this situation in Figure 4.1.

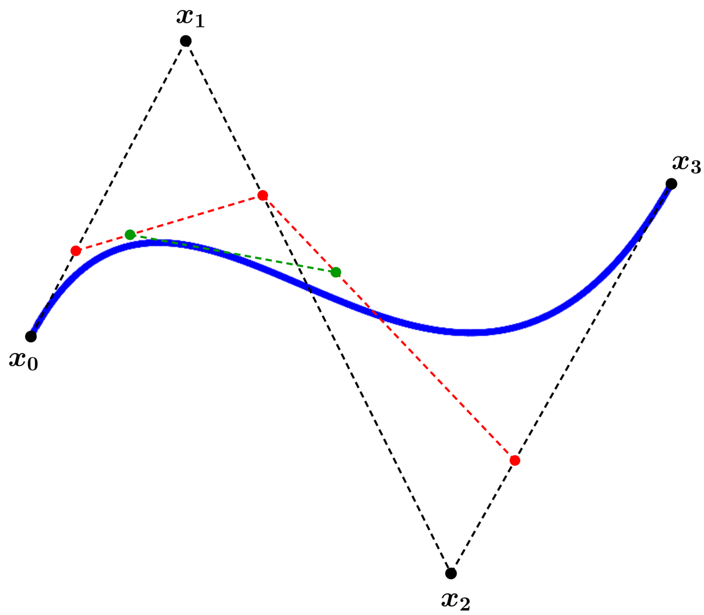

Fig. 4.1 Cubic polynomial defined by the De Casteljau algorithm in $\mathbb{R}^m$.

Obtaining the Control Points from the Boundary Conditions

Previously, we introduce the problem of finding a interpolating cubic polynomial curve in M, given a sequence of four points in the manifold M. However, in most of the problems that arise from applications we are faced with some boundary constraints, instead of the four points. Therefore, it is important to know how we can obtain the control points x_1 and x_2 from the boundary data and, then, be able to carry out the implementation of the De Casteljau algorithm to get the required cubic interpolating polynomial. This is what we are going to present in this subsection, when the manifold M is $\mathbb{R}^m$ and when the boundary conditions are of the type (4.2) and of the type (4.3). In coming after sections, similar work will be done for other manifolds, such as the manifolds $G_{k,n}$ and $\mathrm{SO}(n)$.

- **Case 1 - The Boundary Conditions are of the Type (4.2)**

Problem 4.3.1 *Given two points x_0, x_3 in $\mathbb{R}^m$ and two vectors $v_0 \in T_{x_0}\mathbb{R}^m \cong \mathbb{R}^m$ and $v_3 \in T_{x_3}\mathbb{R}^m \cong \mathbb{R}^m$, find a cubic polynomial $t \in [0,1] \mapsto \beta_3(t)$ in $\mathbb{R}^m$ satisfying the boundary conditions:*

$$\beta_3(0) = x_0, \qquad \beta_3(1) = x_3, \qquad \dot{\beta}_3(0) = v_0 \qquad \textit{and} \qquad \dot{\beta}_3(1) = v_3. \tag{4.5}$$

Solution to Problem 4.3.1: From the relation (4.4), we have that

$$\begin{aligned} &\bullet\ \dot{\beta}_3(0) = 3(x_1 - x_0), \\ &\bullet\ \dot{\beta}_3(1) = 3(x_3 - x_2). \end{aligned} \tag{4.6}$$

Then, we just need to choose the control points x_1 and x_2 necessary to apply the De Casteljau Algorithm 4.3.2 as

$$\begin{aligned} &\bullet\ x_1 = \frac{1}{3}v_0 + x_0, \\ \text{and} & \\ &\bullet\ x_2 = -\frac{1}{3}v_3 + x_3. \end{aligned} \tag{4.7}$$

Therefore, the curve $t \in [0,1] \mapsto \beta_3(t) = \beta_3(t,x_0,x_1,x_2,x_3)$, given by the Step 3. of the mentioned algorithm is the required cubic polynomial.

• Case 2 - The Boundary Conditions are of the Type (4.3)

Problem 4.3.2 *Given two points x_0, x_3 in $\mathbb{R}^m$ and two vectors v_0, w_0 in $T_{x_0}\mathbb{R}^m \cong \mathbb{R}^m$, find a cubic polynomial $t \in [0,1] \mapsto \beta_3(t)$ in $\mathbb{R}^m$ satisfying the boundary conditions:*

$$\beta_3(0) = x_0, \qquad \beta_3(1) = x_3, \qquad \dot{\beta}_3(0) = v_0 \qquad \textit{and} \qquad \ddot{\beta}_3(0) = w_0. \tag{4.8}$$

Solution to Problem 4.3.2: From the relation (4.4), we have that

$$\begin{aligned} &\bullet\ \dot{\beta}_3(0) = 3(x_1 - x_0), \\ &\bullet\ \ddot{\beta}_3(0) = 6((x_2 - x_1) - (x_1 - x_0)). \end{aligned} \tag{4.9}$$

Then, the control points x_1 and x_2 are given by

$$\begin{aligned} &\bullet\ x_1 = \frac{1}{3}v_0 + x_0, \\ \text{and} & \\ &\bullet\ x_2 = \frac{1}{6}w_0 + \frac{2}{3}v_0 + x_0. \end{aligned} \tag{4.10}$$

Consequently, applying the De Casteljau Algorithm, we obtain the required cubic polynomial given by $\beta_3(t) = \beta_3(t,x_0,x_1,x_2,x_3)$, $t \in [0,1]$.

Remark 4.3.1

1. *When the manifold M is the Euclidean space $\mathbb{R}^m$, the Problem 4.3.1 and the Problem 4.3.2 are equivalent. In fact, given the boundary conditions of the type (4.2), we can obtain the vector w_0 by:*

$$w_0 = 6\left(-\frac{1}{3}v_3 - \frac{2}{3}v_0 + x_3 - x_0\right).$$

   Similarly, given the boundary conditions of the type (4.3), we can obtain the vector v_3 by:

$$v_3 = 3\left(-\frac{1}{6}w_0 - \frac{2}{3}v_0 + x_3 - x_0\right).$$

2. *The Algorithm 4.3.2 can be generalized for polynomials of arbitrary degree r. In this sense, given a set $\{x_0, x_1, \ldots, x_r\}$ of $r+1$ distinct points in the Euclidean space $\mathbb{R}^m$, a smooth curve*

$t \in [0,1] \mapsto \beta_r(t) := \beta_r(t, x_0, x_1, \ldots, x_r)$ in $\mathbb{R}^m$ joining x_0 (at $t = 0$) and x_r (at $t = 1$), can be constructed by successive linear interpolation in the following way:

- $\beta_0(t, x_i) = x_i, \qquad \forall i = 0, \ldots, r,$
- $$\beta_j(t, x_i, \ldots, x_{i+j}) = t\beta_{j-1}(t, x_{i+1}, \ldots, x_{i+j}) + (1-t)\beta_{j-1}(t, x_i, \ldots, x_{i+j-1}), \tag{4.11}$$
  $j = 1, \ldots, r, \;\; i = 0, \ldots, r-j.$

4.4 Implementation of the De Casteljau Algorithm in $G_{k,n}$

Although the Grassmann manifold is geodesically complete, we have seen that an explicit formula for the geodesic that joins two points may be unknown in some particular situations. So, in this case the implementation of the De Casteljau algorithm is restricted to a convex open subset of the manifold where the expression to compute the geodesic arc joining two points is well-defined. Having this in mind, in what follows we begin with a theoretical detailed description of the generalized De Casteljau algorithm for this particular manifold.

Given four distinct points x_0, x_1, x_2 and x_3 in $M = G_{k,n}$, the three steps of the Algorithm 4.3.1, which allows to generate a cubic polynomial $t \in [0,1] \mapsto \beta_3(t)$ in M, such that $\beta_3(0) = x_0$ and $\beta_3(1) = x_3$, i.e., joining x_0 (at $t = 0$) and x_3 (at $t = 1$), for this particular manifold, can be described as follows.

Step 1. For all $i = 0, 1, 2$, construct the geodesic arc joining the points x_i and x_{i+1} and given by

$$\beta_1(t, x_i, x_{i+1}) = \mathrm{e}^{t\Omega_i^1} x_i \mathrm{e}^{-t\Omega_i^1} = \mathrm{e}^{t\,\mathrm{ad}_{\Omega_i^1}} x_i, \tag{4.12}$$

with $\Omega_i^1 = \dfrac{1}{2}\log((I - 2x_{i+1})(I - 2x_i)) \in \mathfrak{so}_{x_i}(n)$.

Step 2. Construct, for each $t \in [0,1]$, two geodesic arcs, the first

$$\beta_2(t, x_0, x_1, x_2) = \mathrm{e}^{t\Omega_0^2(t)} \beta_1(t, x_0, x_1) \mathrm{e}^{-t\Omega_0^2(t)} = \mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^2(t)}} \beta_1(t, x_0, x_1), \tag{4.13}$$

with $\Omega_0^2(t) = \dfrac{1}{2}\log((I - 2\beta_1(t, x_1, x_2))(I - 2\beta_1(t, x_0, x_1))) \in \mathfrak{so}_{\beta_1(t, x_0, x_1)}(n)$, joining the point $\beta_1(t, x_0, x_1)$, with the point $\beta_1(t, x_1, x_2)$. The second one

$$\beta_2(t, x_1, x_2, x_3) = \mathrm{e}^{t\Omega_1^2(t)} \beta_1(t, x_1, x_2) \mathrm{e}^{-t\Omega_1^2(t)} = \mathrm{e}^{t\,\mathrm{ad}_{\Omega_1^2(t)}} \beta_1(t, x_1, x_2), \tag{4.14}$$

with $\Omega_1^2(t) = \dfrac{1}{2}\log((I - 2\beta_1(t, x_2, x_3))(I - 2\beta_1(t, x_1, x_2))) \in \mathfrak{so}_{\beta_1(t, x_1, x_2)}(n)$, joining the point $\beta_1(t, x_1, x_2)$, with the point $\beta_1(t, x_2, x_3)$.

Step 3. Construct, for each $t \in [0,1]$, the geodesic arc

$$\beta_3(t, x_0, x_1, x_2, x_3) = \mathrm{e}^{t\Omega_0^3(t)} \beta_2(t, x_0, x_1, x_2) \mathrm{e}^{-t\Omega_0^3(t)} = \mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^3(t)}} \beta_2(t, x_0, x_1, x_2), \tag{4.15}$$

with $\Omega_0^3(t) = \dfrac{1}{2}\log((I-2\beta_2(t,x_1,x_2,x_3))(I-2\beta_2(t,x_0,x_1,x_2))) \in \mathfrak{so}_{\beta_2(t,x_0,x_1,x_2)}(n)$ connecting the point $\beta_2(t,x_0,x_1,x_2)$ with the point $\beta_2(t,x_1,x_2,x_3)$.

Therefore, as a result of applying the De Casteljau Algorithm to the given four points in $M = G_{k,n}$, we obtain the next definition of a geometric cubic polynomial in the Grassmann manifold.

Definition 4.4.1 *The curve $t \in [0,1] \mapsto \beta_3(t) := \beta_3(t,x_0,x_1,x_2,x_3)$ in $M = G_{k,n}$ defined by*

$$\begin{aligned} \beta_3(t) &= \mathrm{e}^{t\Omega_0^3(t)}\mathrm{e}^{t\Omega_0^2(t)}\mathrm{e}^{t\Omega_0^1}x_0\mathrm{e}^{-t\Omega_0^1}\mathrm{e}^{-t\Omega_0^2(t)}\mathrm{e}^{-t\Omega_0^3(t)} \\ &= \mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^3(t)}}\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^2(t)}}\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^1}}x_0, \end{aligned} \tag{4.16}$$

with Ω_0^1, Ω_0^2 and Ω_0^3 given as mentioned above is called a geometric cubic polynomial *in the Grassmann manifold, associated to the points x_i, $i = 0,1,2,3$.*

Remark 4.4.1 *Notice that, as expected, the curve just defined joins the points x_0 (at $t = 0$) and x_3 (at $t = 1$). It is obvious that $\beta_3(0) = x_0$. To see that $\beta_3(1) = x_3$, we start to observe that, from the definition of Ω_j^i, $i = 1,2,3$, $j = 0,1,2$, it can be easily derived the following boundary conditions:*

- $\Omega_0^2(0) = \Omega_0^3(0) = \Omega_0^1$ *and* $\Omega_1^2(0) = \Omega_1^1$*;*
- $\Omega_1^2(1) = \Omega_0^3(1) = \Omega_2^1$ *and* $\Omega_0^2(1) = \Omega_1^1$.

Thus, considering these boundary conditions together with the definition of the geodesic arcs (4.12), we obtain that

$$\beta_3(1) = \mathrm{e}^{\mathrm{ad}_{\Omega_0^3(1)}}\mathrm{e}^{\mathrm{ad}_{\Omega_0^2(1)}}\mathrm{e}^{\mathrm{ad}_{\Omega_0^1}}x_0 = \mathrm{e}^{\mathrm{ad}_{\Omega_2^1}}\mathrm{e}^{\mathrm{ad}_{\Omega_1^1}}\mathrm{e}^{\mathrm{ad}_{\Omega_0^1}}x_0 = x_3. \tag{4.17}$$

Next, we will present a few results, related with Ω_j^i, $i = 1,2,3$, $j = 0,1,2$, that will be used later on.

Lemma 4.4.1 *Let Ω_j^i, $i = 1,2,3$, $j = 0,1,2$ be defined as at the beginning of Section 4.4. Then, the following identities hold:*

$$\begin{array}{lrcl} (i) & \mathrm{e}^{2\Omega_0^2(t)} &=& \mathrm{e}^{2t\Omega_1^1}\mathrm{e}^{2(1-t)\Omega_0^1} \\ (ii) & \mathrm{e}^{2\Omega_1^2(t)} &=& \mathrm{e}^{2t\Omega_2^1}\mathrm{e}^{2(1-t)\Omega_1^1} \\ (iii) & \mathrm{e}^{2\Omega_0^3(t)} &=& \mathrm{e}^{2t\Omega_1^2(t)}\mathrm{e}^{2(1-t)\Omega_0^2(t)}. \end{array} \tag{4.18}$$

Proof. The proof of all these identities uses the definition of Ω_j^i, $i = 1,2,3$, $j = 0,1,2$ and the Proposition 3.2.2. We prove the last one in detail, the others can be proved using similar arguments, but have an easier proof. Proof of (iii): From the definition of Ω_0^3, we know that

$$\Omega_0^3(t) = \frac{1}{2}\log((I-2\beta_2(t,x_1,x_2,x_3))(I-2\beta_2(t,x_0,x_1,x_2))).$$

Then, using the relations (4.13) and (4.14), we have that

$$\begin{aligned}
e^{2\Omega_0^3(t)} &= (I-2\beta_2(t,x_1,x_2,x_3))(I-2\beta_2(t,x_0,x_1,x_2))\\
&= (I-2e^{t\,\mathrm{ad}_{\Omega_1^2(t)}}\beta_1(t,x_1,x_2))(I-2e^{t\,\mathrm{ad}_{\Omega_0^2(t)}}\beta_1(t,x_0,x_1))\\
&= (I-2e^{t\Omega_1^2(t)}\beta_1(t,x_1,x_2)e^{-t\Omega_1^2(t)})(I-2e^{t\Omega_0^2(t)}\beta_1(t,x_0,x_1)e^{-t\Omega_0^2(t)}).
\end{aligned}$$

Therefore, since $\Omega_1^2(t) \in \mathfrak{so}_{\beta_1(t,x_1,x_2)}(n)$ and $\Omega_0^2(t) \in \mathfrak{so}_{\beta_1(t,x_0,x_1)}(n)$, using Proposition 3.2.2 and the definition of Ω_0^2, we obtain that

$$\begin{aligned}
e^{2\Omega_0^3(t)} &= e^{t\Omega_1^2(t)}(I-2\beta_1(t,x_1,x_2))e^{-t\Omega_1^2(t)}e^{t\Omega_0^2(t)}(I-2\beta_1(t,x_0,x_1))e^{-t\Omega_0^2(t)}\\
&= e^{2t\Omega_1^2(t)}(I-2\beta_1(t,x_1,x_2))(I-2\beta_1(t,x_0,x_1))e^{-2t\Omega_0^2(t)}\\
&= e^{2t\Omega_1^2(t)}e^{2\Omega_0^2(t)}e^{-2t\Omega_0^2(t)}\\
&= e^{2t\Omega_1^2(t)}e^{2(1-t)\Omega_0^2(t)}.
\end{aligned}$$

□

Lemma 4.4.2 *Let Ω_j^i, $i=1,2,3,\ j=0,1,2$, be defined as in the steps at the beginning of Section 4.4. Then, the following identities hold:*

$$\begin{array}{lrcl}
(i) & e^{(t-1)\mathrm{ad}_{\Omega_0^2(t)}}e^{t\,\mathrm{ad}_{\Omega_1^1}} &=& e^{t\,\mathrm{ad}_{\Omega_0^2(t)}}e^{(t-1)\mathrm{ad}_{\Omega_0^1}}\\
(ii) & e^{(t-1)\mathrm{ad}_{\Omega_1^2(t)}}e^{t\,\mathrm{ad}_{\Omega_2^1}} &=& e^{t\,\mathrm{ad}_{\Omega_1^2(t)}}e^{(t-1)\mathrm{ad}_{\Omega_1^1}}\\
(iii) & e^{(t-1)\mathrm{ad}_{\Omega_0^3(t)}}e^{t\,\mathrm{ad}_{\Omega_1^2(t)}} &=& e^{t\,\mathrm{ad}_{\Omega_0^3(t)}}e^{(t-1)\mathrm{ad}_{\Omega_0^2(t)}}.
\end{array} \tag{4.19}$$

Proof. We will prove the identities (i) and (iii). The proof of the identity (ii) is similar to the proof of the identity (i). All the proves uses the Lemma 4.4.1, the Proposition 3.2.2, the definition of Ω_j^i, $i=1,2,3,\ j=0,1,2$ and the definition of the geodesic arcs β_i, $i=1,2$.

Proof of (i):

Since $\Omega_0^2(t) \in \mathfrak{so}_{\beta_1(t,x_0,x_1)}(n)$ and $\beta_1(t,x_0,x_1) \in G_{k,n}$, by the Proposition 3.2.2, we have that

$$e^{2\Omega_0^2(t)}\,(I-2\beta_1(t,x_0,x_1)) = e^{\mathrm{ad}_{\Omega_0^2(t)}}\,(I-2\beta_1(t,x_0,x_1)).$$

But, from (4.12), we know that $\beta_1(t,x_0,x_1) = e^{t\,\mathrm{ad}_{\Omega_0^1}}x_0$, and thus we get

$$e^{2\Omega_0^2(t)}e^{t\,\mathrm{ad}_{\Omega_0^1}}\,(I-2x_0) = e^{\mathrm{ad}_{\Omega_0^2(t)}}e^{t\,\mathrm{ad}_{\Omega_0^1}}\,(I-2x_0). \tag{4.20}$$

Then, taking into account the identity (i) of the Lemma 4.4.1 and the Proposition 3.2.2, the left-hand side of (4.20) can be rewritten as

$$\begin{aligned} \mathrm{e}^{2\Omega_0^2(t)}\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^1}}(I-2x_0) &= \mathrm{e}^{2\Omega_0^2(t)}\mathrm{e}^{t\,\Omega_0^1}(I-2x_0)\,\mathrm{e}^{-t\,\Omega_0^1} \\ &= \mathrm{e}^{2\Omega_0^2(t)}\mathrm{e}^{2t\,\Omega_0^1}(I-2x_0) \\ &= \mathrm{e}^{2t\Omega_1^1}\mathrm{e}^{2(1-t)\Omega_0^1}\mathrm{e}^{2t\,\Omega_0^1}(I-2x_0) \\ &= \mathrm{e}^{2t\Omega_1^1}\mathrm{e}^{2\,\Omega_0^1}(I-2x_0) \\ &= \mathrm{e}^{2t\Omega_1^1}\mathrm{e}^{\mathrm{ad}_{\Omega_0^1}}(I-2x_0) \\ &= \mathrm{e}^{2t\Omega_1^1}(I-2x_1) \\ &= \mathrm{e}^{t\mathrm{ad}_{\Omega_1^1}}\mathrm{e}^{\mathrm{ad}_{\Omega_0^1}}(I-2x_0)\,. \end{aligned} \tag{4.21}$$

Therefore, comparing the last right-hand side of (4.21) with the right-hand side of (4.20), we obtain that $\mathrm{e}^{t\mathrm{ad}_{\Omega_1^1}}\mathrm{e}^{\mathrm{ad}_{\Omega_0^1}}(I-2x_0)=\mathrm{e}^{\mathrm{ad}_{\Omega_0^2(t)}}\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^1}}(I-2x_0)$. Since $(I-2x_0)^{-1}=(I-2x_0)$, it holds that $\mathrm{e}^{-\mathrm{ad}_{\Omega_0^2(t)}}\mathrm{e}^{t\mathrm{ad}_{\Omega_1^1}}\mathrm{e}^{\mathrm{ad}_{\Omega_0^1}}=\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^1}}$, which is equivalent to

$$\mathrm{e}^{-\mathrm{ad}_{\Omega_0^2(t)}}\mathrm{e}^{t\mathrm{ad}_{\Omega_1^1}}=\mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_0^1}}. \tag{4.22}$$

Consequently, multiplying the both sides of the last equality (4.22) by $\mathrm{e}^{t\mathrm{ad}_{\Omega_0^2(t)}}$, we get

$$\mathrm{e}^{(t-1)\mathrm{ad}_{\Omega_0^2(t)}}\mathrm{e}^{t\mathrm{ad}_{\Omega_1^1}}=\mathrm{e}^{t\mathrm{ad}_{\Omega_0^2(t)}}\mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_0^1}},$$

which proves the result.

Proof of (iii):

We will start to show that

$$\mathrm{e}^{\mathrm{ad}_{\Omega_0^2(t)}}\beta_1(t,x_0,x_1)=\beta_1(t,x_1,x_2). \tag{4.23}$$

From the relation (4.22), it yields that

$$\mathrm{e}^{\mathrm{ad}_{\Omega_0^2(t)}}=\mathrm{e}^{t\mathrm{ad}_{\Omega_1^1}}\mathrm{e}^{(1-t)\,\mathrm{ad}_{\Omega_0^1}}. \tag{4.24}$$

Then, according with the relations (4.12) and (4.24), we have that

$$\begin{aligned} \mathrm{e}^{\mathrm{ad}_{\Omega_0^2(t)}}\beta_1(t,x_0,x_1) &= \mathrm{e}^{t\mathrm{ad}_{\Omega_1^1}}\mathrm{e}^{(1-t)\,\mathrm{ad}_{\Omega_0^1}}\beta_1(t,x_0,x_1) \\ &= \mathrm{e}^{t\mathrm{ad}_{\Omega_1^1}}\mathrm{e}^{(1-t)\,\mathrm{ad}_{\Omega_0^1}}\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^1}}x_0 \\ &= \mathrm{e}^{t\mathrm{ad}_{\Omega_1^1}}\mathrm{e}^{\mathrm{ad}_{\Omega_0^1}}x_0 \\ &= \mathrm{e}^{t\mathrm{ad}_{\Omega_1^1}}x_1 \\ &= \beta_1(t,x_1,x_2), \end{aligned}$$

which proves the identity (4.23). Since $\Omega_0^3(t) \in \mathfrak{so}_{\beta_2(t,x_0,x_1,x_2)}(n)$ and $\beta_2(t,x_0,x_1,x_2) \in G_{k,n}$, by the Proposition 3.2.2 and by (4.13), we have that

$$\begin{aligned} e^{2\Omega_0^3(t)}\left(I-2\beta_2(t,x_0,x_1,x_2)\right) &= e^{\mathrm{ad}_{\Omega_0^3(t)}}\left(I-2\beta_2(t,x_0,x_1,x_2)\right) \\ &= e^{\mathrm{ad}_{\Omega_0^3(t)}} e^{t\mathrm{ad}_{\Omega_0^2(t)}}\left(I-2\beta_1(t,x_0,x_1)\right). \end{aligned} \tag{4.25}$$

Then, using the Proposition 3.2.2, together with the identity (iii) of the Lemma 4.4.1 and the equality (4.23), we obtain that the left-hand side of (4.25) can be rewritten as

$$\begin{aligned} e^{2\Omega_0^3(t)}\left(I-2\beta_2(t,x_0,x_1,x_2)\right) &= e^{2\Omega_0^3(t)} e^{2t\Omega_0^2(t)}\left(I-2\beta_1(t,x_0,x_1)\right) \\ &= e^{2t\Omega_1^2(t)} e^{2(1-t)\Omega_0^2(t)} e^{2t\Omega_0^2(t)}\left(I-2\beta_1(t,x_0,x_1)\right) \\ &= e^{2t\Omega_1^2(t)} e^{2\Omega_0^2(t)}\left(I-2\beta_1(t,x_0,x_1)\right) \\ &= e^{2t\Omega_1^2(t)} e^{\mathrm{ad}_{\Omega_0^2(t)}}\left(I-2\beta_1(t,x_0,x_1)\right) \\ &= e^{2t\Omega_1^2(t)}\left(I-2e^{\mathrm{ad}_{\Omega_0^2(t)}}\beta_1(t,x_0,x_1)\right) \\ &= e^{2t\Omega_1^2(t)}\left(I-2\beta_1(t,x_1,x_2)\right) \\ &= e^{t\mathrm{ad}_{\Omega_1^2(t)}}\left(I-2\beta_1(t,x_1,x_2)\right) \\ &= e^{t\mathrm{ad}_{\Omega_1^2(t)}} e^{\mathrm{ad}_{\Omega_0^2(t)}}\left(I-2\beta_1(t,x_0,x_1)\right). \end{aligned} \tag{4.26}$$

Therefore, from (4.25) and (4.26), we get that

$$e^{t\mathrm{ad}_{\Omega_1^2(t)}} e^{\mathrm{ad}_{\Omega_0^2(t)}}\left(I-2\beta_1(t,x_0,x_1)\right) = e^{\mathrm{ad}_{\Omega_0^3(t)}} e^{t\mathrm{ad}_{\Omega_0^2(t)}}\left(I-2\beta_1(t,x_0,x_1)\right).$$

Also, since $\beta_1(t,x_0,x_1) \in G_{k,n}$, we know that $\left(I-2\beta_1(t,x_0,x_1)\right)\left(I-2\beta_1(t,x_0,x_1)\right) = I$, and thus

$$e^{-\mathrm{ad}_{\Omega_0^3(t)}} e^{t\mathrm{ad}_{\Omega_1^2(t)}} e^{\mathrm{ad}_{\Omega_0^2(t)}} = e^{t\mathrm{ad}_{\Omega_0^2(t)}}.$$

Consequently,

$$e^{-\mathrm{ad}_{\Omega_0^3(t)}} e^{t\mathrm{ad}_{\Omega_1^2(t)}} = e^{(t-1)\mathrm{ad}_{\Omega_0^2(t)}},$$

and then, multiplying the both sides of this identity by $e^{t\mathrm{ad}_{\Omega_0^3(t)}}$, allows to obtain (iii).

□

We are now in conditions to state the following result which contains an alternative way of defining the geometric cubic polynomial β_3 in $M = G_{k,n}$. The importance of this result lies in the fact that this alternative expression will be particularly useful in the computation of the derivatives of the cubic polynomial at the endpoint ($t = 1$).

Theorem 4.4.1 *Let $t \in [0,1] \mapsto \beta_3(t)$ be the geometric cubic polynomial in $M = G_{k,n}$ that results from the De Casteljau algorithm, which joins the point x_0 (at $t = 0$) to the point x_3 (at $t = 1$) and having control points x_1 and x_2. So, according to* (4.16)*,*

$$\beta_3(t) = e^{t\,\mathrm{ad}_{\Omega_0^3(t)}} e^{t\,\mathrm{ad}_{\Omega_0^2(t)}} e^{t\,\mathrm{ad}_{\Omega_0^1}} x_0,$$

where Ω_0^3, Ω_0^2 and Ω_0^1 are as defined at the beginning of Section 4.4. Define another curve $t \in [0,1] \mapsto \gamma(t)$ in $M = G_{k,n}$, by

$$\gamma(t) = \mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_0^3(t)}} \mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_1^2(t)}} \mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_2^1}} x_3, \tag{4.27}$$

where Ω_1^2 and Ω_2^1 are also as defined at the beginning of Section 4.4. Then,

$$\beta_3(t) = \gamma(t), \quad t \in [0,1].$$

Proof. Taking into consideration the relation (4.17) and applying the identities (i), (ii) and (iii) of Lemma 4.4.2, by a few simplifications it holds that, for all $t \in [0,1]$:

$$\begin{aligned}
\gamma(t) &= \mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_0^3(t)}} \mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_1^2(t)}} \mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_2^1}} x_3 \\
&\overset{(4.17)}{=} \mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_0^3(t)}} \mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_1^2(t)}} \mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_2^1}} \mathrm{e}^{\mathrm{ad}_{\Omega_2^1}} \mathrm{e}^{\mathrm{ad}_{\Omega_1^1}} \mathrm{e}^{\mathrm{ad}_{\Omega_0^1}} x_0 \\
&= \mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_0^3(t)}} \mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_1^2(t)}} \mathrm{e}^{t\,\mathrm{ad}_{\Omega_2^1}} \mathrm{e}^{\mathrm{ad}_{\Omega_1^1}} \mathrm{e}^{\mathrm{ad}_{\Omega_0^1}} x_0 \\
&\overset{(ii)}{=} \mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_0^3(t)}} \mathrm{e}^{t\,\mathrm{ad}_{\Omega_1^2(t)}} \mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_1^1}} \mathrm{e}^{\mathrm{ad}_{\Omega_1^1}} \mathrm{e}^{\mathrm{ad}_{\Omega_0^1}} x_0 \\
&\overset{(iii)}{=} \mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^3(t)}} \mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_0^2(t)}} \mathrm{e}^{t\,\mathrm{ad}_{\Omega_1^1}} \mathrm{e}^{\mathrm{ad}_{\Omega_0^1}} x_0 \\
&\overset{(i)}{=} \mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^3(t)}} \mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^2(t)}} \mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_0^1}} \mathrm{e}^{\mathrm{ad}_{\Omega_0^1}} x_0 \\
&= \mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^3(t)}} \mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^2(t)}} \mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^1}} x_0 \\
&= \beta_3(t).
\end{aligned}$$

□

We now state some results about derivatives involving Ω_j^i, $i = 1,2,3,\ j = 0,1,2$ that will be necessary to fully understand other important developments in this chapter.

Lemma 4.4.3 *For $j = 2,3$, let $i = 3 - j$. Then*

1. $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{2\Omega_0^j(t)}\right)\Big|_{t=0} = 2\chi_0\left(\dot{\Omega}_0^j(0)\right)\mathrm{e}^{2\Omega_0^1}, \quad \textit{with} \quad \chi_0 := \int_0^1 \mathrm{e}^{u\,\mathrm{ad}_{2\Omega_0^1}} du. \tag{4.28}$$

2. $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{2\Omega_i^j(t)}\right)\Big|_{t=1} = 2\chi_1\left(\dot{\Omega}_i^j(1)\right)\mathrm{e}^{2\Omega_2^1}, \quad \textit{with} \quad \chi_1 := \int_0^1 \mathrm{e}^{u\,\mathrm{ad}_{2\Omega_2^1}} du. \tag{4.29}$$

Proof.

1. From Lemma 2.6.3, we have that, for $j = 2,3$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{2\Omega_0^j(t)}\right) = \int_0^1 \mathrm{e}^{u\,\mathrm{ad}_{2\Omega_0^j(t)}}(2\dot{\Omega}_0^j(t))du\,\mathrm{e}^{2\Omega_0^j(t)}.$$

Then, evaluating at $t=0$, and since $\Omega_0^j(0)=\Omega_0^1$ for $j=2,3$, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{2\Omega_0^j(t)}\right)\Big|_{t=0} &= 2\int_0^1 \mathrm{e}^{u\,\mathrm{ad}_{2\Omega_0^1}}(\dot\Omega_0^j(0))du\,\mathrm{e}^{2\Omega_0^1}\\
&= 2\int_0^1 \mathrm{e}^{u\,\mathrm{ad}_{2\Omega_0^1}}du\,(\dot\Omega_0^j(0))\,\mathrm{e}^{2\Omega_0^1}\\
&= 2\chi_0\left(\dot\Omega_0^j(0)\right)\mathrm{e}^{2\Omega_0^1},
\end{aligned}$$

with $\chi_0=\int_0^1 \mathrm{e}^{u\,\mathrm{ad}_{2\Omega_0^1}}du.$

2. The result can be proved with similar computations to those made in the proof of statement 1., just taking into account Lemma 2.6.3, and the fact that, for $j=2,3$ and $i=3-j$, we have that $\Omega_i^j(1)=\Omega_2^1$ and

$$\left(\Delta^L_{2\Omega_i^j(t)}(t)\right)\Big|_{t=1}=2\chi_1\left(\dot\Omega_i^j(1)\right), \tag{4.30}$$

with $\chi_1=\int_0^1 \mathrm{e}^{u\,\mathrm{ad}_{2\Omega_2^1}}du.$

□

Remark 4.4.2 *Notice that χ_0 and χ_1 defined in Lemma 4.4.3 can be, alternatively, rewritten as*

$$\chi_0:=f\left(\mathrm{ad}_{2\Omega_0^1}\right),\qquad \chi_1:=f\left(\mathrm{ad}_{2\Omega_2^1}\right), \tag{4.31}$$

where f is given as in (2.40).

Lemma 4.4.4 *For $j=2,3$, let $i=3-j$. Then, from Lemma 2.6.3 replacing $X(t)$ by, respectively, $2(1-t)\Omega_0^j(t)$ and $2t\Omega_i^j(t)$, we have that*

$$\left(\Delta^L_{2(1-t)\Omega_0^j(t)}(t)\right)\Big|_{t=0}=-2\Omega_0^1+2\chi_0\left(\dot\Omega_0^j(0)\right)\quad\textit{and}\quad\left(\Delta^L_{2t\Omega_i^j(t)}(t)\right)\Big|_{t=1}=2\Omega_2^1+2\chi_1\left(\dot\Omega_i^j(1)\right), \tag{4.32}$$

with χ_0 and χ_1 defined as in Lemma 4.4.3.

Proof. To show the first identity, we have that, for $j=2,3$,

$$\begin{aligned}
\left(\Delta^L_{2(1-t)\Omega_0^j(t)}(t)\right)\Big|_{t=0} &= \left(\int_0^1 \mathrm{e}^{2(1-t)u\,\mathrm{ad}_{\Omega_0^j(t)}}(-2\Omega_0^j(t)+2(1-t)\dot\Omega_0^j(t))du\right)\Big|_{t=0}\\
&= \left(\int_0^1 \mathrm{e}^{2(1-t)u\,\mathrm{ad}_{\Omega_0^j(t)}}(-2\Omega_0^j(t))du\right)\Big|_{t=0}\\
&\quad+\left(\int_0^1 \mathrm{e}^{2(1-t)u\,\mathrm{ad}_{\Omega_0^j(t)}}(2(1-t)\dot\Omega_0^j(t))du\right)\Big|_{t=0}\\
&= -\int_0^1 \mathrm{e}^{2u\,\mathrm{ad}_{\Omega_0^j(0)}}(2\Omega_0^j(0))du+\int_0^1 \mathrm{e}^{2u\,\mathrm{ad}_{\Omega_0^j(0)}}(2\dot\Omega_0^j(0))du\\
&= -\int_0^1 \mathrm{e}^{u\,\mathrm{ad}_{2\Omega_0^1}}(2\Omega_0^1)du+2\int_0^1 \mathrm{e}^{u\,\mathrm{ad}_{2\Omega_0^1}}du(\dot\Omega_0^j(0))\\
&= -2\Omega_0^1+2\chi_0\left(\dot\Omega_0^j(0)\right).
\end{aligned}$$

The proof of the second identity can be achieved with analogous computations, and taking in consideration that, for $j = 2,3$ and $i = 3 - j$, it holds that $\Omega_i^j(1) = \Omega_2^1$.

□

Before proceeding it is important to make the next remark.

Remark 4.4.3 *In what follows, we must guarantee that the operators χ_0 and χ_1 have inverse. From the definition of f and g, respectively, in (2.40) and (2.41), we know that $f(A)g(e^A) = I$, for $\|e^A - I\| < 1$. So, if this restriction holds for $A = \mathrm{ad}_{2\Omega_0^1}$ and for $A = \mathrm{ad}_{2\Omega_2^1}$, taking into account the definition of χ_0 and χ_1 in Remark 4.4.2, we immediately obtain*

$$\chi_0^{-1} := g\left(\mathrm{e}^{\mathrm{ad}_{2\Omega_0^1}}\right) \quad \textit{and} \quad \chi_1^{-1} := g\left(\mathrm{e}^{\mathrm{ad}_{2\Omega_2^1}}\right). \tag{4.33}$$

Lemma 4.4.5 *For $j = 2,3$, let $i = 3 - j$. Then*

1.

$$\dot{\Omega}_0^j(0) = (j-1)\chi_0^{-1}\left(\Omega_1^1 - \Omega_0^1\right), \tag{4.34}$$

with χ_0^{-1} the inverse of the operator χ_0.

2.

$$\dot{\Omega}_i^j(1) = (j-1)\chi_1^{-1}\left(\Omega_2^1 - \mathrm{e}^{2\Omega_2^1}\Omega_1^1\mathrm{e}^{-2\Omega_2^1}\right), \tag{4.35}$$

with χ_1^{-1} the inverse of the operator χ_1.

Proof.

1. We will start to show that the identity (4.34) holds for $j = 2$. Differentiating with respect to t, both sides of the identity (i) of Lemma 4.4.1, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{2\Omega_0^2(t)}\right) = 2\Omega_1^1\mathrm{e}^{2\Omega_0^2(t)} + \mathrm{e}^{2t\Omega_1^1}(-2\Omega_0^1)\mathrm{e}^{2(1-t)\Omega_0^1},$$

and, since $\Omega_0^2(0) = \Omega_0^1$, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{2\Omega_0^2(t)}\right)\right|_{t=0} = 2\Omega_1^1\mathrm{e}^{2\Omega_0^2(0)} - 2\Omega_0^1\mathrm{e}^{2\Omega_0^1} = 2(\Omega_1^1 - \Omega_0^1)\mathrm{e}^{2\Omega_0^1}. \tag{4.36}$$

But, from Lemma 4.4.3, considering $j = 2$, we also have that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{2\Omega_0^2(t)}\right)\right|_{t=0} = 2\chi_0\left(\dot{\Omega}_0^2(0)\right)\mathrm{e}^{2\Omega_0^1}, \tag{4.37}$$

with $\chi_0 = \displaystyle\int_0^1 \mathrm{e}^{u\,\mathrm{ad}_{2\Omega_0^1}}du$.

Then, comparing the expressions (4.36) and (4.37) we get

$$2\chi_0\left(\dot{\Omega}_0^2(0)\right)\mathrm{e}^{2\Omega_0^1} = 2(\Omega_1^1 - \Omega_0^1)\mathrm{e}^{2\Omega_0^1} \quad \Leftrightarrow \quad \dot{\Omega}_0^2(0) = \chi_0^{-1}\left(\Omega_1^1 - \Omega_0^1\right), \tag{4.38}$$

which proves the result, for $j = 2$.

Now, we will show that the identity (4.34) also holds for $j=3$. Similarly, differentiating with respect to t, both sides of the identity (iii) of Lemma 4.4.1, and evaluating at $t=0$, it yields that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{2\Omega_0^3(t)}\right)\Big|_{t=0} &= \left(\Delta^L_{2t\Omega_1^2(t)}(t)\mathrm{e}^{2\Omega_0^3(t)}+\mathrm{e}^{2t\Omega_1^2(t)}\Delta^L_{2(1-t)\Omega_0^2(t)}(t)\mathrm{e}^{2(1-t)\Omega_0^2(t)}\right)\Big|_{t=0} \\ &= \left(\Delta^L_{2t\Omega_1^2(t)}(t)\right)\Big|_{t=0}\mathrm{e}^{2\Omega_0^3(0)}+\left(\Delta^L_{2(1-t)\Omega_0^2(t)}(t)\right)\Big|_{t=0}\mathrm{e}^{2\Omega_0^2(0)}.\end{aligned} \tag{4.39}$$

Since $\Omega_1^2(0)=\Omega_1^1$, by Proposition 2.6.2, we have that $\left(\Delta^L_{2t\Omega_1^2(t)}(t)\right)\Big|_{t=0}=2\Omega_1^2(0)=2\Omega_1^1$. Also, by Lemma 4.4.4 and, taking into account the relation (4.38), we have

$$\begin{aligned}\left(\Delta^L_{2(1-t)\Omega_0^2(t)}(t)\right)\Big|_{t=0} &= -2\Omega_0^1+2\chi_{\scriptscriptstyle 0}\left(\dot\Omega_0^2(0)\right) \\ &= -2\Omega_0^1+2\chi_{\scriptscriptstyle 0}\chi_{\scriptscriptstyle 0}^{-1}\left(\Omega_1^1-\Omega_0^1\right) \\ &= -4\Omega_0^1+2\Omega_1^1.\end{aligned}$$

Consequently, since $\Omega_0^2(0)=\Omega_0^3(0)=\Omega_0^1$, the relation (4.39) can be rewritten as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{2\Omega_0^3(t)}\right)\Big|_{t=0} &= 2\Omega_1^1\mathrm{e}^{2\Omega_0^1}+\left(-4\Omega_0^1+2\Omega_1^1\right)\mathrm{e}^{2\Omega_0^1} \\ &= 4\left(\Omega_1^1-\Omega_0^1\right)\mathrm{e}^{2\Omega_0^1}.\end{aligned} \tag{4.40}$$

Therefore, from (4.40) and Lemma 4.4.3, with $j=3$, we get

$$2\chi_{\scriptscriptstyle 0}\left(\dot\Omega_0^3(0)\right)\mathrm{e}^{2\Omega_0^1}=4\left(\Omega_1^1-\Omega_0^1\right)\mathrm{e}^{2\Omega_0^1}\quad\Leftrightarrow\quad\dot\Omega_0^3(0)=2\chi_{\scriptscriptstyle 0}^{-1}\left(\Omega_1^1-\Omega_0^1\right), \tag{4.41}$$

which also proves the result, for $j=3$.

2. The proof uses identical arguments to those applied to show statement 1., and we present next a sketch of it. Similar to what was made in the proof of the statement 1., we begin with the proof of the result for $j=2$, since it will be needed to complete the proof of the result when $j=3$.

For $j=2$:

Start to differentiate with respect to t, both sides of the identity (ii) of Lemma 4.4.1 and evaluate them at $t=1$. The result is then achieved, with a few calculations, considering Lemma 4.4.3, the identity $\left(\Delta^L_{2(1-t)\Omega_1^1}(t)\right)\Big|_{t=1}=-2\Omega_1^1$, the fact that $\Omega_1^2(1)=\Omega_2^1$ and the relation (4.30), with $j=2$ and $i=1$.

For $j=3$:

Start differentiating with respect to t, both sides of the identity (iii) of Lemma 4.4.1 and evaluating them at $t=1$. Then, with some computations, taking into account Lemma 4.4.3, Lemma 4.4.4, the relation (4.30), with $j=3$ and $i=0$, the fact that $\Omega_0^3(1)=\Omega_1^2(1)=\Omega_2^1$ and that, by Proposition 2.6.2, we have $\left(\Delta^L_{2(1-t)\Omega_0^2(t)}(t)\right)\Big|_{t=1}=-2\Omega_0^2(1)=-2\Omega_1^1$, it holds that

$$\dot\Omega_0^3(1)=\Omega_2^1+\dot\Omega_1^2(1)-\chi_{\scriptscriptstyle 1}^{-1}\left(\mathrm{e}^{2\Omega_2^1}\Omega_1^1\mathrm{e}^{-2\Omega_2^1}\right).$$

Therefore, making a few calculations, the result is obtained replacing $\dot{\Omega}_1^2(1)$ by the identity (4.35), previously proved for $j=2$, and attending to the fact that $\chi_1\left(\Omega_2^1\right)=\Omega_2^1$.

$\square$

We are now in conditions to prove the following result and its corollaries.

Theorem 4.4.2 *The polynomial curve $t\in[0,1]\mapsto\beta_3(t)$ in $M=G_{k,n}$ defined in (4.16) satisfies the boundary conditions $\beta_3(0)=x_0$, $\beta_3(1)=x_3$ and*

$$\dot{\beta}_3(t)=[\Omega(t),\beta_3(t)], \tag{4.42}$$

with $\Omega(t)=\Delta^L_{t\Omega_0^3(t)}(t)+\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^3(t)}}(\Delta^L_{t\Omega_0^2(t)}(t))+\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^3(t)}}\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^2(t)}}\Omega_0^1\in\mathfrak{so}_{\beta_3(t)}(n)$.

Proof. We have already pointed out in Remark 4.4.1 that $\beta_3(0)=x_0$ and $\beta_3(1)=x_3$. Differentiating (4.16) with respect to t, and since $\mathrm{e}^{t\Omega_0^1}\Omega_0^1\mathrm{e}^{-t\Omega_0^1}=\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^1}}\Omega_0^1=\Omega_0^1$, we obtain

$$\begin{aligned}\dot{\beta}_3(t)=\;&\Delta^L_{t\Omega_0^3(t)}(t)\beta_3(t)+\mathrm{e}^{t\Omega_0^3(t)}\Delta^L_{t\Omega_0^2(t)}(t)\mathrm{e}^{-t\Omega_0^3(t)}\beta_3(t)+\left[\mathrm{e}^{t\Omega_0^3(t)}\mathrm{e}^{t\Omega_0^2(t)}\Omega_0^1\mathrm{e}^{-t\Omega_0^2(t)}\mathrm{e}^{-t\Omega_0^3(t)},\beta_3(t)\right]\\&+\beta_3(t)\mathrm{e}^{t\Omega_0^3(t)}\mathrm{e}^{t\Omega_0^2(t)}\Delta^L_{-t\Omega_0^2(t)}(t)\mathrm{e}^{-t\Omega_0^2(t)}\mathrm{e}^{-t\Omega_0^3(t)}+\beta_3(t)\mathrm{e}^{t\Omega_0^3(t)}\Delta^L_{-t\Omega_0^3(t)}(t)\mathrm{e}^{-t\Omega_0^3(t)}.\end{aligned} \tag{4.43}$$

Using Proposition 2.6.1, with $A(t)$ replaced by $t\Omega_0^j(t)$, for $j=2,3$, the fourth and fifth terms in (4.43) can be rewritten, respectively, as

$$-\beta_3(t)\mathrm{e}^{t\Omega_0^3(t)}\Delta^L_{t\Omega_0^2(t)}(t)\mathrm{e}^{-t\Omega_0^3(t)}\quad\text{and}\quad-\beta_3(t)\Delta^L_{t\Omega_0^3(t)}(t).$$

Then, we get

$$\begin{aligned}\dot{\beta}_3(t)\;&=\;\left[\Delta^L_{t\Omega_0^3(t)}(t),\beta_3(t)\right]+\left[\mathrm{e}^{t\Omega_0^3(t)}\Delta^L_{t\Omega_0^2(t)}(t)\mathrm{e}^{-t\Omega_0^3(t)},\beta_3(t)\right]\\&\quad+\left[\mathrm{e}^{t\Omega_0^3(t)}\mathrm{e}^{t\Omega_0^2(t)}\Omega_0^1\mathrm{e}^{-t\Omega_0^2(t)}\mathrm{e}^{-t\Omega_0^3(t)},\beta_3(t)\right]\\&=\;[\Omega(t),\beta_3(t)],\end{aligned}$$

with

$$\begin{aligned}\Omega(t)\;&=\;\Delta^L_{t\Omega_0^3(t)}(t)+\mathrm{e}^{t\Omega_0^3(t)}\Delta^L_{t\Omega_0^2(t)}(t)\mathrm{e}^{-t\Omega_0^3(t)}+\mathrm{e}^{t\Omega_0^3(t)}\mathrm{e}^{t\Omega_0^2(t)}\Omega_0^1\mathrm{e}^{-t\Omega_0^2(t)}\mathrm{e}^{-t\Omega_0^3(t)}\\&=\;\Delta^L_{t\Omega_0^3(t)}(t)+\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^3(t)}}(\Delta^L_{t\Omega_0^2(t)}(t))+\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^3(t)}}\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^2(t)}}\Omega_0^1\in\mathfrak{so}_{\beta_3(t)}(n),\end{aligned}$$

which proves the result. $\square$

Corollary 4.4.1 *Let $t\in[0,1]\mapsto\beta_3(t)$ be the geometric cubic polynomial in $M=G_{k,n}$ defined in (4.16) and $\Omega(t)\in\mathfrak{so}_{\beta_3(t)}(n)$ as defined in Theorem 4.4.2. Then, $\dfrac{D\dot{\beta}_3}{dt}(t)=\left[\dot{\Omega}(t),\beta_3(t)\right]$.*

Proof. Differentiating the relation (4.42) of Theorem 4.4.2 with respect to t, we get

$$\begin{aligned}\ddot{\beta}_3(t)\;&=\;[\dot{\Omega}(t),\beta_3(t)]+[\Omega(t),\dot{\beta}_3(t)]\\&=\;[\dot{\Omega}(t),\beta_3(t)]+[\Omega(t),[\Omega(t),\beta_3(t)]].\end{aligned}$$

By Proposition 3.2.8, we have that $[\Omega(t),[\Omega(t),\beta_3(t)]] \in (T_{\beta_3(t)}G_{k,n})^{\perp}$. Therefore, since $[\dot{\Omega}(t),\beta_3(t)] \in T_{\beta_3(t)}G_{k,n}$, we obtain that

$$\frac{D\dot{\beta}_3}{dt}(t) = \left[\dot{\Omega}(t),\beta_3(t)\right].$$

□

Corollary 4.4.2 *Let $t \in [0,1] \mapsto \beta_3(t)$ be the geometric cubic polynomial in $M = G_{k,n}$ defined in (4.16) and $\Omega(t) \in \mathfrak{so}_{\beta_3(t)}(n)$ as defined in Theorem 4.4.2. Then,*

$$\dot{\beta}_3(0) = [3\Omega_0^1, x_0] \quad \textit{and} \quad \frac{D\dot{\beta}_3}{dt}(0) = 6\left[\chi_0^{-1}\left(\Omega_1^1 - \Omega_0^1\right), x_0\right].$$

Proof. From Theorem 4.4.2, we have that $\beta_3(0) = x_0$ and that

$$\dot{\beta}_3(0) = [\Omega(0),\beta_3(0)],$$

with $\Omega(0) = \left(\Delta^L_{t\Omega_0^3(t)}(t)\right)\Big|_{t=0} + \left(\Delta^L_{t\Omega_0^2(t)}(t)\right)\Big|_{t=0} + \Omega_0^1 \in \mathfrak{so}_{\beta_3(0)}(n)$. But, since $\Omega_0^2(0) = \Omega_0^3(0) = \Omega_0^1$, from Proposition 2.6.2, we obtain that $\Omega(0) = 3\Omega_0^1$. Therefore, $\dot{\beta}_3(0) = [3\Omega_0^1, x_0]$. From Corollary 4.4.1, and since $\beta_3(0) = x_0$, we know that

$$\frac{D\dot{\beta}_3}{dt}(0) = \left[\dot{\Omega}(0), x_0\right]. \tag{4.44}$$

In order to compute $\dot{\Omega}(0)$, let us consider $\omega_1(t) := e^{t\,\mathrm{ad}_{\Omega_0^3(t)}}(\Delta^L_{t\Omega_0^2(t)}(t))$ and $\omega_2(t) := e^{t\,\mathrm{ad}_{\Omega_0^3(t)}} e^{t\,\mathrm{ad}_{\Omega_0^2(t)}} \Omega_0^1$. Then, $\Omega(t) = \Delta^L_{t\Omega_0^3(t)}(t) + \omega_1(t) + \omega_2(t) \in \mathfrak{so}_{\beta_3(t)}(n)$ and, differentiating with respect to t, we have that

$$\dot{\Omega}(t) = \dot{\Delta}^L_{t\Omega_0^3(t)}(t) + \dot{\omega}_1(t) + \dot{\omega}_2(t),$$

with

$$\begin{aligned}
\dot{\omega}_1(t) &= \Delta^L_{t\Omega_0^3(t)}(t)\omega_1(t) + e^{t\,\mathrm{ad}_{\Omega_0^3(t)}}(\dot{\Delta}^L_{t\Omega_0^2(t)}(t)) + e^{t\,\mathrm{ad}_{\Omega_0^3(t)}}\left(\Delta^L_{t\Omega_0^2(t)}(t)\Delta^L_{-t\Omega_0^3(t)}(t)\right)\\
&= \left[\Delta^L_{t\Omega_0^3(t)}(t), \omega_1(t)\right] + e^{t\,\mathrm{ad}_{\Omega_0^3(t)}}(\dot{\Delta}^L_{t\Omega_0^2(t)}(t))
\end{aligned}$$

and

$$\begin{aligned}
\dot{\omega}_2(t) &= \Delta^L_{t\Omega_0^3(t)}(t)\omega_2(t) + e^{t\,\mathrm{ad}_{\Omega_0^3(t)}}\left(\Delta^L_{t\Omega_0^2(t)}(t) e^{t\,\mathrm{ad}_{\Omega_0^2(t)}}\Omega_0^1\right) + e^{t\,\mathrm{ad}_{\Omega_0^3(t)}} e^{t\,\mathrm{ad}_{\Omega_0^2(t)}}\left(\Omega_0^1\Delta^L_{-t\Omega_0^2(t)}(t)\right)\\
&\quad + e^{t\,\mathrm{ad}_{\Omega_0^3(t)}}\left(e^{t\,\mathrm{ad}_{\Omega_0^2(t)}}(\Omega_0^1)\Delta^L_{-t\Omega_0^3(t)}(t)\right)\\
&= \left[\Delta^L_{t\Omega_0^3(t)}(t) + e^{t\,\mathrm{ad}_{\Omega_0^3(t)}}\left(\Delta^L_{t\Omega_0^2(t)}(t)\right), \omega_2(t)\right].
\end{aligned}$$

Therefore, evaluating at $t = 0$, and according with Proposition 2.6.2 and Proposition 2.6.3, we get that

$$\dot{\Omega}(0) = 2\dot{\Omega}_0^3(0) + \dot{\omega}_1(0) + \dot{\omega}_2(0),$$

with

$$\begin{aligned}\dot{\omega}_1(0) &= \Omega_0^3(0)\Omega_0^2(0)+2\dot{\Omega}_0^2(0)-\Omega_0^2(0)\Omega_0^3(0)\\ &= \left[\Omega_0^3(0),\Omega_0^2(0)\right]+2\dot{\Omega}_0^2(0)\end{aligned}$$

and

$$\begin{aligned}\dot{\omega}_2(0) &= \Omega_0^3(0)\Omega_0^1+\Omega_0^2(0)\Omega_0^1-\Omega_0^1\Omega_0^2(0)-\Omega_0^1\Omega_0^3(0)\\ &= \left[\Omega_0^3(0)+\Omega_0^2(0),\Omega_0^1\right].\end{aligned}$$

Due to the fact that $\Omega_0^2(0)=\Omega_0^3(0)=\Omega_0^1$, it holds that $\dot{\omega}_1(0)=2\dot{\Omega}_0^2(0)$ and $\dot{\omega}_2(0)=0$. Then, by Lemma 4.4.5, we can conclude that

$$\begin{aligned}\dot{\Omega}(0) &= 2\dot{\Omega}_0^3(0)+2\dot{\Omega}_0^2(0)\\ &= 6\chi_0^{-1}\left(\Omega_1^1-\Omega_0^1\right).\end{aligned} \tag{4.45}$$

Consequently, assuming (4.45), we have that the relation (4.44) can be rewritten as

$$\frac{D\dot{\beta}_3}{dt}(0)=6\left[\chi_0^{-1}\left(\Omega_1^1-\Omega_0^1\right),x_0\right].$$

□

On the next result we derive an expression for the derivative of the geometric cubic polynomial β_3, and for the covariant derivative of the velocity vector field along the curve β_3, at the endpoint $t=1$. For that, it was fundamental the use of the alternative expression of β_3 established in the Theorem 4.4.1.

Theorem 4.4.3 *Let $t\in[0,1]\mapsto\beta_3(t)$ be the geometric cubic polynomial in $M=G_{k,n}$ given by the alternative formula present in Theorem 4.4.1. Then,*

$$\dot{\beta}_3(1)=[3\Omega_2^1,x_3] \qquad \textit{and} \qquad \frac{D\dot{\beta}_3}{dt}(1)=6\left[\chi_1^{-1}\left(\Omega_2^1-\mathrm{e}^{2\Omega_2^1}\Omega_1^1\mathrm{e}^{-2\Omega_2^1}\right),x_3\right]. \tag{4.46}$$

Proof. The alternative formula for β_3 present in Theorem 4.4.1 is

$$\beta_3(t)=\mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_0^3(t)}}\mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_1^2(t)}}\mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_2^1}}x_3. \tag{4.47}$$

Making a few calculations similar to those that where done in the proof of the Theorem 4.4.2, it is possible to show that, differentiating with respect to t the alternative expression (4.47), of the geometric cubic polynomial β_3 we obtain that

$$\dot{\beta}_3(t)=[\overline{\Omega}(t),\beta_3(t)], \tag{4.48}$$

with $\overline{\Omega}(t)=\Delta^L_{(t-1)\Omega_0^3(t)}(t)+\mathrm{e}^{(t-1)\mathrm{ad}_{\Omega_0^3(t)}}(\Delta^L_{(t-1)\Omega_1^2(t)}(t))+\mathrm{e}^{(t-1)\mathrm{ad}_{\Omega_0^3(t)}}\mathrm{e}^{(t-1)\mathrm{ad}_{\Omega_1^2(t)}}\Omega_2^1\in\mathfrak{so}_{\beta_3(t)}(n)$ and where $\beta_3(t)$ is given by (4.47).

The ingredients to prove the previous identity (4.48) are, essentially, the formula for the derivative of the exponential map given by Lemma 2.6.3 and the relation in Proposition 2.6.1.

Therefore, to obtain the first identity in (4.46), observe that, for $j=2,3$ and $i=3-j$, we have that $\left.\mathrm{e}^{(t-1)\mathrm{ad}_{\Omega_i^j(t)}}\right|_{t=1}=I$. Furthermore, consider the Proposition 2.6.2 with $k=1$, and the Remark 4.4.1, namely, the fact that $\Omega_1^2(1)=\Omega_0^3(1)=\Omega_2^1$. All the rest are simple computations.

In order to prove the second identity in (4.46), notice that differentiating (4.48) with respect to t, using similar arguments to those in the proof of the Corollary 4.4.1, and taking into consideration the Lemma 2.6.3 and the Proposition 2.6.1, with a few calculations, we get that

$$\frac{D\dot{\beta}_3}{dt}(t)=\left[\dot{\overline{\Omega}}(t),\beta_3(t)\right],$$

with

$$\begin{aligned}\dot{\overline{\Omega}}(t)= &\ \dot{\Delta}^L_{(t-1)\Omega_0^3(t)}(t)+\left[\Delta^L_{(t-1)\Omega_0^3(t)}(t),\overline{\omega}_1(t)\right]+\mathrm{e}^{(t-1)\mathrm{ad}_{\Omega_0^3(t)}}\left(\dot{\Delta}^L_{(t-1)\Omega_1^2(t)}(t)\right)\\ &+\left[\Delta^L_{(t-1)\Omega_0^3(t)}(t)+\mathrm{e}^{(t-1)\mathrm{ad}_{\Omega_0^3(t)}}\left(\Delta^L_{(t-1)\Omega_1^2(t)}(t)\right),\overline{\omega}_2(t)\right],\end{aligned} \tag{4.49}$$

and, where $\overline{\omega}_1(t)=\mathrm{e}^{(t-1)\mathrm{ad}_{\Omega_0^3(t)}}(\Delta^L_{(t-1)\Omega_1^2(t)}(t))$ and $\overline{\omega}_2(t)=\mathrm{e}^{(t-1)\mathrm{ad}_{\Omega_0^3(t)}}\mathrm{e}^{(t-1)\mathrm{ad}_{\Omega_1^2(t)}}\Omega_2^1$.

Consequently, evaluating at $t=1$ and, essentially, due to Proposition 2.6.3, Lemma 4.4.5, Proposition 2.6.2 with $k=1$, and Remark 4.4.1, we obtain that

$$\frac{D\dot{\beta}_3}{dt}(1)=6\left[\chi_1^{-1}\left(\Omega_2^1-\mathrm{e}^{2\Omega_2^1}\Omega_1^1\mathrm{e}^{-2\Omega_2^1}\right),x_3\right],$$

as required. □

Obtaining the Control Points from the Boundary Conditions

In this subsection similar to what was done for $M=\mathbb{R}^m$, we will show how to get the control points from the boundary conditions in order to, then, implement the De Casteljau algorithm to solve interpolation data problems that arise from different areas involving the manifold $M=G_{k,n}$.

- **Case 1 - The Boundary Conditions are of the Type (4.2)**

Problem 4.4.1 *Given two points x_0, x_3 in $M=G_{k,n}$ and two tangent vectors $[V_0,x_0]\in T_{x_0}G_{k,n}$ and $[V_3,x_3]\in T_{x_3}G_{k,n}$, find a geometric cubic polynomial $t\in[0,1]\mapsto\beta_3(t)$ in M satisfying the boundary conditions:*

$$\beta_3(0)=x_0,\qquad \beta_3(1)=x_3,\qquad \dot{\beta}_3(0)=[V_0,x_0]\qquad \textit{and}\qquad \dot{\beta}_3(1)=[V_3,x_3], \tag{4.50}$$

where $V_0\in\mathfrak{so}_{x_0}(n)$ and $V_3\in\mathfrak{so}_{x_3}(n)$.

According to the implementation of the algorithm, in order to generate the cubic polynomial that satisfies (4.50), we must be able to choose the control points x_1 and x_2 from those boundary conditions. The following theorem answers this question.

Theorem 4.4.4 *The control points x_1 and x_2, used in the De Casteljau algorithm to generate the geometric cubic polynomial β_3 in $G_{k,n}$ that satisfies the boundary conditions (4.50), are given by:*

$$x_1 = \frac{1}{2}\left(I - \mathrm{e}^{\frac{2}{3}V_0}(I-2x_0)\right) \quad \textit{and} \quad x_2 = \frac{1}{2}\left(I - (I-2x_3)\mathrm{e}^{\frac{2}{3}V_3}\right). \tag{4.51}$$

Proof. From Corollary 4.4.2, we know that $\dot{\beta}_3(0) = [3\Omega_0^1, x_0]$, with $\Omega_0^1 = \dfrac{1}{2}\log((I-2x_1)(I-2x_0)) \in \mathfrak{so}_{x_0}(n)$. Then, considering the expression of $\dot{\beta}_3(0)$ in (4.50) and the Proposition 3.2.4, it follows that $3\Omega_0^1 = V_0$ and thus,

$$\Omega_0^1 = \frac{1}{3}V_0. \tag{4.52}$$

According with the definition of Ω_0^1, we have that $\mathrm{e}^{2\Omega_0^1} = (I-2x_1)(I-2x_0)$, which is equivalent to $I - 2x_1 = \mathrm{e}^{2\Omega_0^1}(I-2x_0)$. Therefore, solving the last equality in order to get x_1, it yields that

$$x_1 = \frac{1}{2}\left(I - \mathrm{e}^{2\Omega_0^1}(I-2x_0)\right). \tag{4.53}$$

Furthermore, taking into account the identity (4.53), together with the relation in (4.52), we can conclude that the control point x_1 can be obtained, explicitly, from x_0 and V_0 as

$$x_1 = \frac{1}{2}\left(I - \mathrm{e}^{\frac{2}{3}V_0}(I-2x_0)\right). \tag{4.54}$$

To reach, from the boundary conditions, an explicit expression to compute the control point x_2 notice that, in the Theorem 4.4.3, we proved that $\dot{\beta}_3(1) = [3\Omega_2^1, x_3]$. On the other hand, from (4.50), $\dot{\beta}_3(1) = [V_3, x_3]$. So, by the Proposition 3.2.4 we obtain that $V_3 = 3\Omega_2^1$, which implies that

$$\Omega_2^1 = \frac{1}{3}V_3. \tag{4.55}$$

Therefore, from the definition of Ω_2^1, we have that $\mathrm{e}^{2\Omega_2^1} = (I-2x_3)(I-2x_2)$, and, making a few calculations in order to obtain x_2, it holds that $x_2 = \dfrac{1}{2}\left(I - (I-2x_3)\mathrm{e}^{2\Omega_2^1}\right)$. Thus, replacing in the last equality Ω_2^1 by (4.55), it allows to conclude that the control point x_2 can be obtained, explicitly, from x_3 and V_3 as

$$x_2 = \frac{1}{2}\left(I - (I-2x_3)\mathrm{e}^{\frac{2}{3}V_3}\right). \tag{4.56}$$

Consequently, we can conclude that the two control points x_1 and x_2 can be obtained, explicitly, from the boundary conditions, as required.

$\square$

We are now able to state that the geometric cubic polynomial β_3 in the Grassmann manifold, satisfying the boundary conditions in (4.50), and which solves the Problem 4.4.1, can be obtained by the implementation, in $G_{k,n}$, of the De Casteljau algorithm described at the beginning of Section 4.4, and using the four points $x_0, x_1, x_2, x_3 \in G_{k,n}$, with the control points x_1 and x_2 given by (4.51).

• Case 2 - The Boundary Conditions are of the Type (4.3)

Problem 4.4.2 *Given two points x_0, x_3 in $M = G_{k,n}$ and two vectors $[V_0,x_0]$ and $[W_0,x_0]$ in $T_{x_0}G_{k,n}$, find a geometric cubic polynomial $t \in [0,1] \mapsto \beta_3(t)$ in M satisfying the boundary conditions:*

$$\beta_3(0) = x_0, \qquad \beta_3(1) = x_3, \qquad \dot{\beta}_3(0) = [V_0,x_0] \qquad \textit{and} \qquad \frac{D\dot{\beta}_3}{dt}(0) = [W_0,x_0], \tag{4.57}$$

where $V_0, W_0 \in \mathfrak{so}_{x_0}(n)$.

Similar to what was done in Case 1, we can state the forthcoming result.

Theorem 4.4.5 *The control points x_1 and x_2, used in the De Casteljau algorithm to generate the geometric cubic polynomial β_3 in $G_{k,n}$ that satisfies the boundary conditions (4.57), are given by:*

$$x_1 = \frac{1}{2}\left(I - \mathrm{e}^{\frac{2}{3}V_0}(I-2x_0)\right) \quad \textit{and} \quad x_2 = \frac{1}{2}\left(I - \mathrm{e}^{\frac{1}{3}\chi_0(W_0)+\frac{2}{3}V_0}\,\mathrm{e}^{\frac{2}{3}V_0}(I-2x_0)\right). \tag{4.58}$$

Proof. According with the computations done in Case 1, we know that Ω_0^1 is given by (4.52). Consequently, the control point x_1 can also be obtained, explicitly, from the boundary conditions using the expression provided in (4.54).

In order to obtain the control point x_2, notice that taking into account the expressions of $\frac{D\dot{\beta}_3}{dt}(0)$ in (4.57) and in the Corollary 4.4.2, from the Proposition 3.2.4, it holds that $6\chi_0^{-1}\left(\Omega_1^1 - \Omega_0^1\right) = W_0$. Thus, $\Omega_1^1 = \frac{1}{6}\chi_0(W_0) + \Omega_0^1$. Then, considering (4.52), we obtain that

$$\Omega_1^1 = \frac{1}{6}\chi_0(W_0) + \frac{1}{3}V_0. \tag{4.59}$$

From the definition of Ω_1^1, we have that $\mathrm{e}^{2\Omega_1^1} = (I-2x_2)(I-2x_1)$, and a few calculations leads to,

$$x_2 = \frac{1}{2}\left(I - \mathrm{e}^{2\Omega_1^1}(I-2x_1)\right). \tag{4.60}$$

Therefore, using the relations (4.54) and (4.59), the expression (4.60) to obtain the control point x_2 can be rewritten as

$$x_2 = \frac{1}{2}\left(I - \mathrm{e}^{\frac{1}{3}\chi_0(W_0)+\frac{2}{3}V_0}\mathrm{e}^{\frac{2}{3}V_0}(I-2x_0)\right),$$

which ends the proof of the result.

□

Thus, now we have all the ingredients to perform the computation, using the De Casteljau algorithm, of the required geometric cubic polynomial given by $\beta_3(t) = \beta_3(t,x_0,x_1,x_2,x_3)$, $t \in [0,1]$, that solves the Problem 4.4.2.

The Case 1, corresponding to the Hermite boundary conditions, can be considered simpler than the Case 2, since it doesn't involves the computation of covariant derivatives. However, the Case 2, where the data is not symmetrically specified, in some applications, has computational advantages over the Case 1, namely whenever the goal is to generated cubic splines, i.e., to piecing together several geometric cubic polynomials so that the overall curve is $\mathscr{C}^2$-smooth.

Therefore, before finishing this section it is important to observe that if we are faced with a problem involving the boundary conditions of type (4.2), we can transform it into a problem with boundary conditions of the type (4.3). More precisely, if we have a problem of the type of the Problem 4.4.1, then we can convert it into a problem of the kind of the Problem 4.4.2. The reverse situation is also true. This is what we state and prove in the next two results, which can be regarded as immediate consequences of the last two theorems.

Proposition 4.4.1 *Every Problem 4.4.1 can be transformed in a Problem 4.4.2, considering* $W_0 \in \mathfrak{so}_{x_0}(n)$ *given by*

$$W_0 = 3\chi_0^{-1}\left(\log\left((I-2x_3)\mathrm{e}^{\frac{2}{3}V_3}\mathrm{e}^{\frac{2}{3}V_0}(I-2x_0)\right) - \frac{2}{3}V_0\right), \tag{4.61}$$

where $\chi_0^{-1} = g\left(\mathrm{e}^{\mathrm{ad}_{\frac{2}{3}V_0}}\right)$*, with* g *defined as in (2.41).*

Proof. Given two points x_0, x_3 in $M = G_{k,n}$ and two vectors $[V_0, x_0] \in T_{x_0}G_{k,n}$ and $[V_3, x_3] \in T_{x_3}G_{k,n}$ satisfying (4.50), it is enough to prove how we can construct the vector $[W_0, x_0] \in T_{x_0}G_{k,n}$ satisfying (4.57). From (4.59), we have that

$$2\Omega_1^1 = \frac{1}{3}\chi_0(W_0) + \frac{2}{3}V_0. \tag{4.62}$$

Furthermore, from the definition of Ω_1^1, we know that $\mathrm{e}^{2\Omega_1^1} = (I-2x_2)(I-2x_1)$ and, from (4.51), it holds that

$$(I-2x_2) = (I-2x_3)\mathrm{e}^{\frac{2}{3}V_3} \quad \text{and} \quad (I-2x_1) = \mathrm{e}^{\frac{2}{3}V_0}(I-2x_0).$$

Therefore, we get that $\mathrm{e}^{2\Omega_1^1} = (I-2x_3)\mathrm{e}^{\frac{2}{3}V_3}\mathrm{e}^{\frac{2}{3}V_0}(I-2x_0)$, and, thus

$$2\Omega_1^1 = \log\left((I-2x_3)\mathrm{e}^{\frac{2}{3}V_3}\mathrm{e}^{\frac{2}{3}V_0}(I-2x_0)\right). \tag{4.63}$$

Consequently, comparing the expressions for $2\Omega_1^1$ described in (4.62) and in (4.63), we obtain (4.61). □

Proposition 4.4.2 *Every Problem 4.4.2 can be transformed in a Problem 4.4.1, considering* $V_3 \in \mathfrak{so}_{x_3}(n)$ *given by*

$$V_3 = \frac{3}{2}\log\left((I-2x_3)\mathrm{e}^{\frac{1}{3}\chi_0(W_0)+\frac{2}{3}V_0}(I-2x_0)\mathrm{e}^{-\frac{2}{3}V_0}\right), \tag{4.64}$$

where $\chi_0 = f\left(\mathrm{ad}_{\frac{2}{3}V_0}\right)$*, with* f *given as in (2.40).*

Proof. Let x_0, x_3 in $M = G_{k,n}$ and two vectors $[V_0, x_0]$, $[W_0, x_0] \in T_{x_0}G_{k,n}$ satisfying (4.57). In order to prove how we can construct the vector $[V_3, x_3] \in T_{x_3}G_{k,n}$ satisfying (4.50), it is enough to show that we can obtain $V_3 \in \mathfrak{so}_{x_3}(n)$ from the conditions of Problem 4.4.2. Indeed, from (4.61) we obtain that

$$e^{\frac{1}{3}\chi_0(W_0)+\frac{2}{3}V_0} = (I-2x_3)\, e^{\frac{2}{3}V_3} e^{\frac{2}{3}V_0}\,(I-2x_0)\,,$$

which is equivalent to $(I-2x_3)\, e^{\frac{1}{3}\chi_0(W_0)+\frac{2}{3}V_0}\,(I-2x_0)\, e^{-\frac{2}{3}V_0} = e^{\frac{2}{3}V_3}$. Therefore, the identity (4.64) comes out.

□

4.5 Generating Cubic Splines in $G_{k,n}$

According to the relationship between the two types of boundary conditions, we can concentrate now in solving the interpolation Problem 4.2.1 in $G_{k,n}$ with boundary conditions of type (4.3). The crucial procedure is the generation of the first cubic polynomial, denoted by γ_1, joining p_0 to p_1 and having prescribed initial velocity $[V_0, p_0]$ and initial covariant acceleration $[W_0, p_0]$. Although this has already been described in the previous section, we summarize the results here for the sake of completeness. We also adapt the notations so that the curve is given in terms of the data. The interpolation curve γ of Problem 4.2.1 may be generated by piecing together cubic polynomials defined on each subinterval $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, \ell$. Without loss of generality, we assume that all spline segments are parameterized in the $[0,1]$ time interval.

4.5.1 Generating the First Spline Segment γ_1

Apply the De Casteljau algorithm to obtain

$$\gamma_1(t) \quad = \quad e^{t\,\mathrm{ad}_{\Omega_0^3(t)}} e^{t\,\mathrm{ad}_{\Omega_0^2(t)}} e^{t\,\mathrm{ad}_{\Omega_0^1}} p_0, \tag{4.65}$$

where

$$\begin{aligned}
\Omega_0^1 &= \frac{1}{2}\log((I-2x_1)(I-2p_0));\\
\Omega_1^1 &= \frac{1}{2}\log((I-2x_2)(I-2x_1));\\
\Omega_2^1 &= \frac{1}{2}\log((I-2p_1)(I-2x_2));\\
\Omega_0^2(t) &= \frac{1}{2}\log((I-2e^{t\,\mathrm{ad}_{\Omega_1^1}}x_1)(I-2e^{t\,\mathrm{ad}_{\Omega_0^1}}p_0));\\
\Omega_1^2(t) &= \frac{1}{2}\log((I-2e^{t\,\mathrm{ad}_{\Omega_2^1}}x_2)(I-2e^{t\,\mathrm{ad}_{\Omega_1^1}}x_1));\\
\Omega_0^3(t) &= \frac{1}{2}\log((I-2e^{t\,\mathrm{ad}_{\Omega_1^2(t)}}e^{t\,\mathrm{ad}_{\Omega_1^1}}x_1)(I-2e^{t\,\mathrm{ad}_{\Omega_0^2}(t)}e^{t\,\mathrm{ad}_{\Omega_0^1}}p_0)),
\end{aligned}$$

and the control points are given by

$$\begin{aligned}
x_1 &= \frac{1}{2}(I-e^{\frac{2}{3}V_0}(I-2p_0));\\
x_2 &= \frac{1}{2}(I-e^{\frac{1}{3}\chi_0(W_0)+\frac{2}{3}V_0}e^{\frac{2}{3}V_0}(I-2p_0)).
\end{aligned}$$

4.5.2 Generating Consecutive Spline Segments

After having generated the first spline segment, one continues in a similar way for the second spline segment. Since the cubic spline is required to be $\mathscr{C}^2$-smooth, the initial velocity and initial covariant acceleration for this second spline segment must equal the end velocity and the end covariant acceleration of the previous spline segment, which are given by the formulas in Theorem 4.4.3. The other $\ell-2$ consecutive segments are generated similarly. The solution of Problem 4.2.1 is the cubic spline curve resulting from the concatenation of the ℓ consecutive segments.

Remark 4.5.1 *Although the De Casteljau algorithm is theoretically very interesting, it becomes clear from the previous formulas that its implementation is not straightforward. This is the main reason for searching alternatives to generate interpolating curves. The method that will be presented in Chapter 5.*

4.6 Cubic Polynomials and Cubic Splines in SO(n)

For the sake of completeness, we include here the formulas that can be derived from the results in Crouch et al. [13] for cubic polynomials obtained by the De Casteljau algorithm when $M = \mathrm{SO}(n)$. Similar to the Grassmann manifold it is necessary to restricted the implementation to a sufficiently big neighbourhood of the given points where the explicit formula to compute the geodesic arc joining two arbitrary points that depends only on these points is well-defined.

Given four distinct points x_0, x_1, x_2 and x_3 in SO(n), the three steps of the Algorithm 4.3.1, which allows to generate a cubic polynomial $t \in [0,1] \mapsto \beta_3(t)$ such that $\beta_3(0) = x_0$ and $\beta_3(1) = x_3$, is briefly described as follows.

Step 1. For all $i = 0,1,2$, construct the geodesic arc joining the points x_i and x_{i+1} and given by

$$\beta_1(t,x_i,x_{i+1}) = x_i \mathrm{e}^{tV_i^1}, \tag{4.66}$$

with $V_i^1 = \log(x_i^{-1}x_{i+1}) \in \mathfrak{so}(n)$.

Step 2. Construct, for each $t \in [0,1]$, two geodesic arcs, the first

$$\beta_2(t,x_0,x_1,x_2) = \beta_1(t,x_0,x_1)\mathrm{e}^{tV_0^2(t)} = x_0 \mathrm{e}^{tV_0^1} \mathrm{e}^{tV_0^2(t)}, \tag{4.67}$$

with $V_0^2(t) = \log((\beta_1(t,x_0,x_1))^{-1}\beta_1(t,x_1,x_2)) = \log(\mathrm{e}^{(1-t)V_0^1}\mathrm{e}^{tV_1^1}) \in \mathfrak{so}(n)$, joining the point $\beta_1(t,x_0,x_1)$, with the point $\beta_1(t,x_1,x_2)$. The second one

$$\beta_2(t,x_1,x_2,x_3) = \beta_1(t,x_1,x_2)\mathrm{e}^{tV_1^2(t)} = x_1 \mathrm{e}^{tV_1^1} \mathrm{e}^{tV_1^2(t)}, \tag{4.68}$$

with $V_1^2(t) = \log((\beta_1(t,x_1,x_2))^{-1}\beta_1(t,x_2,x_3)) = \log(\mathrm{e}^{(1-t)V_1^1}\mathrm{e}^{tV_2^1}) \in \mathfrak{so}(n)$, joining the point $\beta_1(t,x_1,x_2)$, with the point $\beta_1(t,x_2,x_3)$.

Step 3. Construct, for each $t \in [0,1]$, the geodesic arc

$$\beta_3(t,x_0,x_1,x_2,x_3) = \beta_2(t,x_0,x_1,x_2)\mathrm{e}^{tV_0^3(t)} = x_0 \mathrm{e}^{tV_0^1} \mathrm{e}^{tV_0^2(t)} \mathrm{e}^{tV_0^3(t)} \tag{4.69}$$

with $V_0^3(t) = \log((\beta_2(t,x_0,x_1,x_2))^{-1}\beta_2(t,x_1,x_2,x_3)) = \log(\mathrm{e}^{(1-t)V_0^2}\mathrm{e}^{tV_1^2}) \in \mathfrak{so}(n)$ connecting the point $\beta_2(t,x_0,x_1,x_2)$ with the point $\beta_2(t,x_1,x_2,x_3)$.

The formulas for the generation of the cubic splines and the solution of Problem 4.2.1 when $M = \mathrm{SO}(n)$ can then be obtained from the results in Crouch et al. [13] for general connected and compact Lie groups.

4.7 Solving the Interpolation Problem for the Normalized Essential Manifold

It is clear from the last two sections that the implementation of the De Casteljau algorithm on both the Grassmann manifold and the special orthogonal group heavily relies on computing matrix exponentials and matrix logarithms. The same happens if one is interested in extending that procedure to $G_{k,n} \times \mathrm{SO}(n)$. In general, there are no explicit formulas to compute those matrix functions and one would have to use stable numerical algorithms that are already available, for instance in Higham [26]. However, for the Normalized Essential Manifold $\mathscr{E} = G_{2,3} \times \mathrm{SO}(3)$ we can use the following explicit formulas to compute exponentials of matrices in $\mathfrak{so}(3)$ and logarithms of matrices in $\mathrm{SO}(3)$ that are available, for instance, in Murray et al. [56]:

- For any skew-symmetric matrix $A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \in \mathfrak{so}(3)$, we have the following formula, known as Rodrigues's Formula:

$$\mathrm{e}^A = \begin{cases} I + \dfrac{\sin\|a\|}{\|a\|}A + \dfrac{1-\cos\|a\|}{\|a\|^2}A^2, & \|a\| \neq 0 \\ I, & \|a\| = 0 \end{cases}, \tag{4.70}$$

  where $\|a\| := \sqrt{a_1^2 + a_2^2 + a_3^2}$.

- And for any orthogonal matrix $R \in \mathrm{SO}(3)$, such that $\mathrm{tr}(R) \neq -1$, we have

$$\log(R) = \begin{cases} \dfrac{\theta}{2\sin\theta}\left(R - R^\top\right), & R \neq I \\ 0, & R = I \end{cases}, \tag{4.71}$$

  where $\theta = \arccos\left(\frac{\mathrm{tr}(R)-1}{2}\right)$.

We note that, in the SO(3) case, the condition $\mathrm{tr}(R) \neq -1$ is equivalent to $\sigma(R) \cap \mathbb{R}_0^- = \varnothing$, where $\sigma(R)$ denotes the spectrum of the matrix R. So, under our assumption that $\mathrm{tr}(R) \neq -1$, the formula (4.71) to compute the principal logarithm is in accordance with a remark in Section 2.6 about the existence and uniqueness of the principal logarithm for matrices whose spectrum doesn't intersect $\mathbb{R}_0^-$.

Chapter 5

Rolling Riemannian Manifolds

5.1 Introduction

Rolling maps describe how two connected and oriented manifolds M_0 and M_1 of the same dimension n, both isometrically embedded in the same Riemannian complete m-dimensional manifold $\overline{M}$, $(1 \leq n < m)$, roll on each other without slipping and without twisting.

Rolling motions are rigid-like motions that result from the action of the group of isometries of the embedding manifold, subject to holonomic constraints (the rolling conditions) and nonholonomic constraints (the non-slip and non-twist conditions). Throughout the thesis we assume that M_0 is static and M_1 is rolling over M_0. M_1 will be called the rolling manifold.

This chapter starts with the definition of rolling map for manifolds embedded in a general Riemannian manifold. A geometric interpretation of the rolling conditions and of the non-slip and non-twist conditions is also presented. Using the geometric interpretation of the non-twist conditions we will make a clear connection between rolling and parallel transport. We then prove some interesting properties that allow to reduce the study of rolling motions to the case where the static manifold is the affine tangent space at a point of the rolling manifold. We then proceed with the particular situation when the embedding manifold is an Euclidean space. In this case, the holonomic and nonholonomic constraints can be simplified in order to derive the general structure for the kinematic equations of the rolling motions. This is then specialized to the case when M_1 is the product manifold $G_{k,n} \times \mathrm{SO}(n)$. In particular, when $n = 3$ and $k = 2$, this manifold is the Normalized Essential manifold that plays an important role in the area of Vision, as already mentioned.

The chapter ends with the study of the controllability of the kinematic equations describing the rolling motion of Grassmann Manifolds over the affine tangent space at a point. These kinematic equations, which can be seen as a nonlinear nonholonomic control system evolving on a certain Lie group, are proved to be controllable in some subgroup of the group of isometries of the embedding space of the two manifolds. Moreover, we also present a constructive proof of controllability of the rolling motion by showing that the all admissible configurations can be recovered by motions that do not violate the non-slip and non-twist constraints.

5.1.1 Literature Review

The operation of rolling a surface over another surface appeared in the literature as early as 1919, in the work of Paul Appell [3]. The rolling sphere problem, consisting on a 2-sphere rolling on a tangent plane, is the most classical of all nonholonomic systems and has been studied for a long time due to its interest in Physics and Engineering. But the first modern treatment of rolling without slip and twist didn't appear until 1978, with the work of Nomizu [60]. And a mathematical definition of rolling map for manifolds embedded in an Euclidean space was presented by Sharpe [68] only in 1997. In this case, the group of orientation preserving isometries of the ambient space is the special Euclidean group. Although, according to Nash Theorem [57], every finite dimensional Riemannian manifold can be smoothly isometrically embedded in a sufficiently high-dimensional Euclidean space, finding an appropriate embedding is not necessarily an easy task. For that reason, the concept of rolling has been extended to a more general Riemannian framework in Hüper et al. [29]. This will be our starting point. The study of rolling motions and the derivation of the corresponding kinematic equations have also appeared in the literature before. In some cases the study was motivated by real applications, but quite often it served as an inspiration to reach a generalized theory. Examples of that include Hüper and Silva Leite [31], Zimmerman [75], Jurdjevic and Zimmerman [38], Hüper et al. [28], and Krakowski and Silva Leite [44]. For a survey about rolling motions we also refer to Chapter 21 in the upcoming book edited by Grohs et al. [22].

In which concerns controllability, the existing literature is quite sparse, but we can find partial results, for instance, in Biscolla [6], Jurdjevic [35], Zimmerman [75], Kleinsteuber et al. [41] and Marques [53].

5.1.2 Main Contributions

After revisiting the definition of rolling map, which is the key mathematical concept for all the rolling problems studied here, we give a geometric interpretation of all the conditions in that definition. Our main contribution is a refinement of the geometric interpretation of the non-twist conditions which allows to prove other interesting properties of rolling and, consequently, simplifies the study of rolling motions. In particular, when the non-twist conditions are rewritten in terms of parallel vector fields, a clear connection between rolling and parallel transport will pop up.

When specializing to the rolling manifold $G_{k,n} \times \mathrm{SO}(n)$ we follow the approach in Hüper and Silva Leite [31], also used successfully in Crouch and Silva Leite [16] and Marques [53] for certain pseudo-Euclidean manifolds, taking advantage of its matrix representation. The definition of rolling map is adjusted in order to avoid destroying the matrix structure of that manifold, what otherwise would bring lots of complicated computations. These results for the case when $k = 2$ and $n = 3$, i.e., rolling motions of the Normalized Essential Manifold, have already been published in Machado et al. [51].

Following the approach of Jurdjevic [35], Jurdjevic and Sussmann [37] and Zimmerman [75], we prove that the kinematic equations that describe the pure rolling motions of the Grassmann manifold $G_{k,n}$ over the affine tangent space at a point, is controllable in $G = \mathrm{SO}(n) \times T_{P_0}G_{k,n}$, whenever $k(n-k) \neq 1$. Furthermore, inspired by the works of Kleinsteuber et al. [41] and of Biscolla [6], a constructive proof of controllability of the rolling motion is presented by showing how the forbidden motions of twisting and slipping can be accomplished by rolling without breaking the nonholonomic constraints of non-slip and non-twist. These results have already been published in Pina and Silva Leite [63].

5.2 General Definition of Rolling Map

Let M_0 and M_1 be two connected and oriented manifolds of the same dimension n, both isometrically embedded in the same Riemannian complete m-dimensional manifold $(\overline{M},\overline{g})$ $(1 \leq n < m)$. Denote by $\overline{\nabla}$ the Riemannian connection associated with the metric $\overline{g}$ and by $\overline{G}$ the connected component of $\mathsf{Isom}(\overline{\mathsf{M}})$ containing the identity $e_{\overline{G}}$ and preserving orientation.

A rolling motion of M_1 over M_0, without slipping and twisting, will be described by a rolling map which is a curve in $\overline{G}$ that satisfies certain constraints. Assume that $J = [0, \tau]$, $(\tau > 0)$, is a real interval and consider a curve in $\overline{G}$

$$\begin{aligned} h : J &\to \overline{G} \\ t &\mapsto h(t) \end{aligned} \qquad \text{where} \qquad \begin{aligned} h(t) : \overline{M} &\to \overline{M} \\ p &\mapsto q = h(t)(p) \end{aligned}. \tag{5.1}$$

Before defining the concept of a rolling map, we introduce some helpful notations.

We denote by $\dot{h}(t) := \frac{\mathrm{d}}{\mathrm{d}\sigma}(h(\sigma))\big|_{\sigma=t}$ the vector tangent to the curve $\sigma \mapsto h(\sigma)$ at t.

The action of $\overline{G}$ on $\overline{M}$ induces a natural action on tangent vectors to $\overline{M}$, which is the pushforward or differential of $h(t)$, defined by

$$\begin{aligned} d_p h(t) : T_p\overline{M} &\to T_{h(t)(p)}\overline{M} \\ \eta &\mapsto \frac{\mathrm{d}}{\mathrm{d}\sigma}\left[(h(t)(y(\sigma))\right]\Big|_{\sigma=0} \end{aligned}, \tag{5.2}$$

where $y :]-\varepsilon, \varepsilon[\to \overline{M}$ is a smooth curve in $\overline{M}$, such that $y(0) = p$ and $\dot{y}(0) = \eta$.

The action of $\overline{G}$ on $\overline{M}$ also induces the following actions that will be used in the definition of rolling map (Definition 5.2.1):

$$\begin{aligned} \dot{h}(t) : \overline{M} &\to T\overline{M} \\ p &\mapsto \dot{h}(t)(p) := \frac{\mathrm{d}}{\mathrm{d}\sigma}\left[h(\sigma)(p)\right]\Big|_{\sigma=t} \in T_{h(t)(p)}\overline{M}, \end{aligned} \tag{5.3}$$

which is the action of the vector tangent to the curve $\sigma \mapsto h(\sigma)$ at t on the point $p \in \overline{M}$;

$$\begin{aligned} \dot{h}(t) \circ h(t)^{-1} : \overline{M} &\to T\overline{M} \\ p &\mapsto \left(\dot{h}(t) \circ h(t)^{-1}\right)(p) := \frac{\mathrm{d}}{\mathrm{d}\sigma}\left[\left(h(\sigma)\ \left(h(t)^{-1}\right)(p)\right)\right]\Big|_{\sigma=t} \in T_p\overline{M}; \end{aligned} \tag{5.4}$$

and

$$
\begin{aligned}
d_p\left(\dot{h}(t)\circ h(t)^{-1}\right) : T_p\overline{M} &\to T_{\dot{h}(t)\circ h(t)^{-1}(p)}\left(T\overline{M}\right) \\
\eta \mapsto d_p\left(\dot{h}(t)\circ h(t)^{-1}\right)(\eta) &:= \frac{\mathrm{d}}{\mathrm{d}\sigma}\left[\left(\dot{h}(t)\circ h(t)^{-1}\right)(y(\sigma))\right]\Big|_{\sigma=0},
\end{aligned}
\tag{5.5}
$$

where y and η are as above.

Definition 5.2.1 *Let M_0 and M_1 be two n-dimensional connected and oriented manifolds isometrically embedded in an m-dimensional complete Riemannian manifold $\overline{M}$ and let $\overline{G}$ be the connected component of the group of isometries of $\overline{M}$ that contains the identity and preserves orientation. A* rolling map of M_1 over M_0, without slipping and twisting, *is a smooth curve $h : J \to \overline{G}$, satisfying, for all $t \in J$, the following constraints:*

1. *Rolling conditions: There exists a smooth curve $\alpha_1 : J \to M_1$, such that*

   (a) $h(t)(\alpha_1(t)) \in M_0$;

   (b) $T_{h(t)(\alpha_1(t))}(h(t)(M_1)) = T_{h(t)(\alpha_1(t))}M_0$.

   The curve α_1 is called the rolling curve *and the curve $\alpha_0 : J \to M_0$, defined by*

   $$\alpha_0(t) := h(t)(\alpha_1(t)), \tag{5.6}$$

   is called the development of α_1 on M_0.

2. *No-slip condition:*

   $$\left(\dot{h}(t)\circ h(t)^{-1}\right)(\alpha_0(t)) = 0. \tag{5.7}$$

3. *No-twist conditions:*

   (a) (Tangential part)

   $$d_{\alpha_0(t)}\left(\dot{h}(t)\circ h(t)^{-1}\right)\left(T_{\alpha_0(t)}M_0\right) \subset \left(T_{\alpha_0(t)}M_0\right)^{\perp}, \tag{5.8}$$

   (b) (Normal part)

   $$d_{\alpha_0(t)}\left(\dot{h}(t)\circ h(t)^{-1}\right)\left(T_{\alpha_0(t)}M_0\right)^{\perp} \subset T_{\alpha_0(t)}M_0. \tag{5.9}$$

Remark 5.2.1

1. *In Sharpe [68], it has been proven that given any smooth curve in M_1, there exists a unique rolling map along that curve. This existence and uniqueness property of rolling maps has been generalized to general Riemannian submanifolds in Hüper et al. [29].*

2. *Rolling along piecewise-smooth curves doesn't bring any difficulty. One only has to make a minor adjustment in the conditions of the previous definition that involve derivatives, namely replacing "for all t" by "for almost all t".*

3. *This definition also makes sense when $M_1 = M_0$. In this case $h(t) : \overline{M} \to \overline{M}$ is the identity map and the rolling curve coincides with its development.*

Throughout this thesis the term "pure rolling" or simply "rolling" will always mean rolling, without slipping and without twisting.

5.2.1 Geometric Interpretation of the Rolling Map Conditions

The rolling conditions in the Definition 5.2.1 are holonomic constraints since they are restrictions on the configurations of the two manifolds. We call admissible configurations the configurations imposed by the rolling conditions. Contrary to that, the no-slip and no-twist conditions being nonintegrable constraints on velocities are nonholonomic constants. This terminology was introduced by Heinrich Hertz in the nineteenth century, and details about these restrictions can be found, for instance, in Lützen [49].

In this section we will give geometric interpretation of the three rolling map conditions of Definition 5.2.1. Before, we start to observe that the embeddings of M_1 and M_0 into $\overline{M}$ split the tangent space of $\overline{M}$ into direct sums:

$$\begin{aligned} T_p\overline{M} &= T_pM_1 \oplus (T_pM_1)^{\perp}, \quad p \in M_1, \\ T_q\overline{M} &= T_qM_0 \oplus (T_qM_0)^{\perp}, \quad q \in M_0, \end{aligned} \tag{5.10}$$

According to these splittings, if $p \in M_1$ any vector $V \in T_p\overline{M}$ can be written uniquely as the sum $V = V^{\top} + V^{\perp}$, where $V^{\top} \in T_pM_1$ is tangent to M_1 at p, while $V^{\perp} \in \left(T_pM_1\right)^{\perp}$ is normal. Analogous projections can be defined for M_0.

• Geometric Interpretation of the Rolling Conditions

- The first rolling condition means that, during the motion, the development curve α_0 is being drawn on M_0 by the point of contact between the moving manifold, which at any instant t is $h(t)M_1$, and the static manifold M_0.

- The second rolling condition means that, at each time $t \in J$, both manifolds $h(t)M_1$ and M_0 have the same tangent space.

Note that the second rolling condition is also equivalent to

$$d_{\alpha_1(t)}h(t)\left(T_{\alpha_1(t)}M_1\right) = T_{\alpha_0(t)}M_0. \tag{5.11}$$

This is easily seen from the following identities:

$$d_{\alpha_1(t)}h(t)\left(T_{\alpha_1(t)}M_1\right) \overset{\text{by (2.24)}}{=} T_{h(t)(\alpha_1(t))}(h(t)M_1) \overset{\text{by (1b) of Definition 5.2.1}}{=} T_{\alpha_0(t)}M_0.$$

Also, due to the splitting (5.10) and the fact that the differential of $h(t)$ is a linear isomorphism, the rolling condition (5.11) also implies that

$$d_{\alpha_1(t)}h(t)\left(T_{\alpha_1(t)}M_1\right)^{\perp} = \left(T_{\alpha_0(t)}M_0\right)^{\perp}. \tag{5.12}$$

Figure 5.1 illustrates the rolling of the Euclidean 2-sphere over its affine tangent space at the south pole p_0.

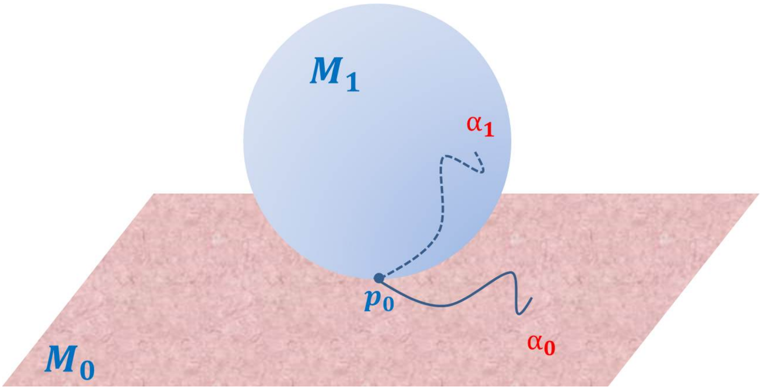

Fig. 5.1 Illustration of the rolling conditions for the rolling of the 2-sphere embedded in $\mathbb{R}^3$.

• Geometric Interpretation of the No-Slip Condition

The no-slip condition (5.7) is equivalent to

$$\dot{\alpha}_0(t) = d_{\alpha_1(t)}h(t)(\dot{\alpha}_1(t)). \tag{5.13}$$

Therefore, this condition has the interpretation that during the rolling motion the velocities of the rolling curve and of its development at each point of contact are the same.

To show this, first notice that differentiating the expression (5.6) with respect to t yields

$$\dot{\alpha}_0(t) = \dot{h}(t)(\alpha_1(t)) + d_{\alpha_1(t)}h(t)(\dot{\alpha}_1(t)). \tag{5.14}$$

Assuming that (5.13) holds, we get that

$$\dot{h}(t)(\alpha_1(t)) = 0 \Leftrightarrow \left(\dot{h}(t)\circ h(t)^{-1}\right)(\alpha_0(t)) = 0.$$

On the other hand, if the condition (5.7) is satisfied, then according with (5.6), we have that $\dot{h}(t)(\alpha_1(t)) = 0$. Consequently, taking into account (5.14), one gets the relation (5.13).

• Geometric Interpretation of the No-Twist Conditions

A geometric interpretation for the no-twist conditions will be crucial to prove other important results. However, that is not so easy to obtain and requires some preliminary considerations.
From relation (5.5), we know that the map $d_{\alpha_0(t)}\left(\dot{h}(t)\circ h(t)^{-1}\right)$ maps $T_{\alpha_0(t)}\overline{M}$ to $T_{\dot{h}(t)\circ h(t)^{-1}(\alpha_0(t))}\left(T\overline{M}\right)$. Therefore, according with (5.10), it makes sense to apply $d_{\alpha_0(t)}\left(\dot{h}(t)\circ h(t)^{-1}\right)$ to elements of $T_{\alpha_0(t)}M_0$ or to elements of $\left(T_{\alpha_0(t)}M_0\right)^{\perp}$.
A vector field V along a curve $\gamma(t)$ is usually denoted by $V_{\gamma(t)}$. However, to simplify notations, we will denote this vector field simply by $V(t)$, as long as it is clear what is the curve. Let $V_0(t)\in T_{\alpha_0(t)}\overline{M}$. By the chain rule, we have that

$$d_{\alpha_0(t)}\left(\dot{h}(t)\circ h(t)^{-1}\right)(V_0(t)) = \left(d_{h(t)^{-1}\alpha_0(t)}\dot{h}(t)\circ d_{\alpha_0(t)}h(t)^{-1}\right)(V_0(t)). \tag{5.15}$$

Then, since $h(t)^{-1}(\alpha_0(t)) = \alpha_1(t)$ and $d_{\alpha_0(t)}h(t)^{-1} = (d_{\alpha_1(t)}h(t))^{-1}$, we can rewritten the previous relation (5.15) as

$$d_{\alpha_0(t)}\left(\dot{h}(t)\circ h(t)^{-1}\right)(V_0(t)) = d_{\alpha_1(t)}\dot{h}(t)\,(d_{\alpha_1(t)}h(t))^{-1}(V_0(t)). \tag{5.16}$$

We are now in conditions to prove the following result.

Proposition 5.2.1 *The no-twist conditions of Definition 5.2.1 are, respectively, equivalent to:*

1. *Tangential part:*

$$(d_{\alpha_1(t)}\dot{h}(t)(V_1(t)))^{\top} = 0, \quad \textit{for all vector fields } V_1(t)\in T_{\alpha_1(t)}M_1, \textit{ and all } t\in J. \tag{5.17}$$

2. *Normal part:*

$$(d_{\alpha_1(t)}\dot{h}(t)(Y_1(t)))^{\perp} = 0, \quad \textit{for all vector fields } Y_1(t)\in (T_{\alpha_1(t)}M_1)^{\perp}, \textit{ and all } t\in J. \tag{5.18}$$

Proof.

1. Let $V_0(t)\in T_{\alpha_0(t)}M_0\subset T_{\alpha_0(t)}\overline{M}$. Since $(d_{\alpha_1(t)}h(t))^{-1}$ maps $T_{\alpha_0(t)}M_0$ to $T_{\alpha_1(t)}M_1$, we get that $(d_{\alpha_1(t)}h(t))^{-1}(V_0(t)) =: V_1(t)\in T_{\alpha_1(t)}M_1$. Then, taking into account (5.16), we obtain that

$$d_{\alpha_0(t)}\left(\dot{h}(t)\circ h(t)^{-1}\right)(V_0(t)) = d_{\alpha_1(t)}\dot{h}(t)(V_1(t)).$$

Consequently, the condition (5.8) of the tangential part of the no-twist conditions is equivalent to

$$(d_{\alpha_1(t)}\dot{h}(t)(V_1(t)))^{\top} = 0, \quad \text{with} \quad V_1(t)\in T_{\alpha_1(t)}M_1.$$

2. The proof of the normal part is similar to the proof of the first part. In fact, if $Y_0(t) \in (T_{\alpha_0(t)}M_0)^\perp \subset T_{\alpha_0(t)}\overline{M}$, using (5.16) and the fact that $(d_{\alpha_1(t)}h(t))^{-1}$ maps $(T_{\alpha_0(t)}M_0)^\perp$ to $(T_{\alpha_1(t)}M_1)^\perp$ we obtain that

$$d_{\alpha_0(t)}\left(\dot{h}(t)\circ h(t)^{-1}\right)(Y_0(t)) = d_{\alpha_1(t)}\dot{h}(t)(Y_1(t)), \quad Y_1(t)\in(T_{\alpha_1(t)}M_1)^\perp.$$

Therefore, the condition (5.9) of the normal part of the no-twist conditions is equivalent to

$$(d_{\alpha_1(t)}\dot{h}(t)(Y_1(t)))^\perp = 0, \quad \text{with} \quad Y_1(t)\in(T_{\alpha_1(t)}M_1)^\perp.$$

□

Proposition 5.2.1 gives an interpretation of the no-twist conditions involving the notions of tangent and normal projections and allows us to prove the next result, which gives an interpretation of the same conditions using the concepts of covariant derivative and normal covariant derivative along the rolling curve.

Notations: Let $\overline{\nabla}$ be the connection on $\overline{M}$ compatible with the Riemannian metric $\overline{g}$, ∇^1 and ∇^0 the connections on M_1 and M_0 respectively, compatible with the corresponding induced metric. While $\frac{\overline{D}}{dt}$ denotes the ambient covariant derivative, in what follows, and to simplify notations, we write $\frac{D}{dt}$ indistinctly for the covariant derivative of vector fields along curves in M_1 and M_0. But it should be clear from the context that for a vector field $V_1(t)$ along the curve $t\mapsto\alpha_1(t)\in M_1$, $\frac{D}{dt}V_1(t) := {\nabla^1}_{\dot{\alpha}_1(t)}V_1(t)$, and for a vector field $V_0(t)$ along the curve $t\mapsto\alpha_0(t)\in M_0$, $\frac{D}{dt}V_0(t) := {\nabla^0}_{\dot{\alpha}_0(t)}V_0(t)$. We use the same abuse of notations for the normal covariant derivative $\frac{D^\perp}{dt}$.

Proposition 5.2.2 *The no-twist conditions of Definition 5.2.1 are, respectively, equivalent to:*

1. *Tangential part:*
$$d_{\alpha_1(t)}h(t)\frac{D}{dt}V_1(t) = \frac{D}{dt}d_{\alpha_1(t)}h(t)V_1(t), \tag{5.19}$$
for any tangent vector field $V_1(t)$ along $\alpha_1(t)$, and all $t\in J$.

2. *Normal part:*
$$d_{\alpha_1(t)}h(t)\frac{D^\perp}{dt}Y_1(t) = \frac{D^\perp}{dt}d_{\alpha_1(t)}h(t)Y_1(t), \tag{5.20}$$
for any normal vector field $Y_1(t)$ along $\alpha_1(t)$, and all $t\in J$.

Proof.

1. Let $V_1(t)\in T_{\alpha_1(t)}M_1\subset T_{\alpha_1(t)}\overline{M}$ be a tangent vector field along $\alpha_1(t)$. By the product derivative rule for covariant derivatives we have
$$\frac{\overline{D}}{dt}\left(d_{\alpha_1(t)}h(t)(V_1(t))\right) = d_{\alpha_1(t)}\dot{h}(t)V_1(t) + d_{\alpha_1(t)}h(t)\left(\frac{\overline{D}}{dt}V_1(t)\right). \tag{5.21}$$
According to the Proposition 5.2.1, we know that the tangential part of the no-twist conditions is equivalent to (5.17). So, using this fact together with (5.21) and the Gauss formula along a

curve, we obtain

$$
\begin{aligned}
\left(d_{\alpha_1(t)}\dot{h}(t)V_1(t)\right)^{\top}=0 \quad &\Leftrightarrow \quad \left(\frac{\overline{D}}{dt}\left(d_{\alpha_1(t)}h(t)V_1(t)\right)-d_{\alpha_1(t)}h(t)\left(\frac{\overline{D}}{dt}V_1(t)\right)\right)^{\top}=0\\
&\Leftrightarrow \quad \frac{D}{dt}\left(d_{\alpha_1(t)}h(t)V_1(t)\right)-d_{\alpha_1(t)}h(t)\frac{D}{dt}V_1(t)=0\\
&\Leftrightarrow \quad d_{\alpha_1(t)}h(t)\frac{D}{dt}V_1(t)=\frac{D}{dt}\left(d_{\alpha_1(t)}h(t)V_1(t)\right),
\end{aligned}
$$

which proves the result.

2. Let $Y_1(t)\in(T_{\alpha_1(t)}M_1)^{\perp}\subset T_{\alpha_1(t)}\overline{M}$ be a normal vector field along $\alpha_1(t)$ on M. Similarly to the tangential part, and taking into account Proposition 5.2.1, we know that the normal no-twist condition is equivalent to (5.18). Therefore,

$$
\begin{aligned}
\left(d_{\alpha_1(t)}\dot{h}(t)Y_1(t)\right)^{\perp}=0 \quad &\Leftrightarrow \quad \left(\frac{\overline{D}}{dt}\left(d_{\alpha_1(t)}h(t)Y_1(t)\right)-d_{\alpha_1(t)}h(t)\left(\frac{\overline{D}}{dt}Y_1(t)\right)\right)^{\perp}=0\\
&\Leftrightarrow \quad \frac{D^{\perp}}{dt}\left(d_{\alpha_1(t)}h(t)Y_1(t)\right)-d_{\alpha_1(t)}h(t)\frac{D^{\perp}}{dt}Y_1(t)=0\\
&\Leftrightarrow \quad d_{\alpha_1(t)}h(t)\frac{D^{\perp}}{dt}Y_1(t)=\frac{D^{\perp}}{dt}\left(d_{\alpha_1(t)}h(t)Y_1(t)\right).
\end{aligned}
$$

□

The outcomes of the last two propositions are crucial to prove an interesting geometric interpretation of the no-twist conditions involving the notion of parallel vectors fields, as shown in the following result.

Theorem 5.2.1 *The no-twist conditions of Definition 5.2.1 are, respectively, equivalent to:*

1. *Tangential part:*

   (i) A vector field $V_1(t)$ is tangent parallel along the curve $\alpha_1(t)$ if, and only if, $V_0(t)=d_{\alpha_1(t)}h(t)(V_1(t))$ is tangent parallel along $\alpha_0(t)$.

2. *Normal part:*

   (ii) A vector field $Y_1(t)$ is normal parallel along the curve $\alpha_1(t)$ if, and only if, $Y_0(t)=d_{\alpha_1(t)}h(t)(Y_1(t))$ is normal parallel along $\alpha_0(t)$.

Proof. We will prove the equivalence of the tangential part. The proof of the equivalence of the normal part can be done similarly.

We first assume that (5.8) of Definition 5.2.1 holds. Then, by Proposition 5.2.2, (5.8) is equivalent to the identity (5.19). So, it is clear that $\frac{D}{dt}V_1(t)=0$, if and only if, $\frac{D}{dt}\left(d_{\alpha_1(t)}h(t)(V_1(t))\right)=0$. Consequently, condition (5.8) implies condition (i) of Theorem 5.2.1.

To prove that condition (i) of Theorem 5.2.1 implies (5.8) of Definition 5.2.1, let $V_1(t)$ be an arbitrary tangent vector field along the curve $\alpha_1(t)$ and $\{E_1(t),\ldots,E_n(t)\}$ (where n is the dimension of M_0 and M_1) be a parallel tangent frame field along the rolling curve $\alpha_1(t)$, so that

$$V_1(t) = \sum_{i=1}^{n} v_i(t)E_i(t) \quad \text{and} \quad \frac{D}{dt}V_1(t) = \sum_{i=1}^{n} \dot{v}_i(t)E_i(t).$$

Considering $\hat{E}_i(t) := d_{\alpha_1(t)}h(t)(E_i(t))$, and taking into account (i) of Theorem 5.2.1, we can guarantee that $\{\hat{E}_1(t),\ldots,\hat{E}_n(t)\}$ is a parallel tangent frame field along the development curve $\alpha_0(t)$. Then, since $d_{\alpha_1(t)}h(t)$ is a linear map (isomorphism) and according to properties of the covariant derivative we obtain that

$$d_{\alpha_1(t)}h(t)\left(\frac{D}{dt}V_1(t)\right) = \sum_{i=1}^{n} \dot{v}_i(t)d_{\alpha_1(t)}h(t)(E_i(t)) = \sum_{i=1}^{n} \dot{v}_i(t)\hat{E}_i(t)$$

and

$$\frac{D}{dt}\left(d_{\alpha_1(t)}h(t)(V_1(t))\right) = \frac{D}{dt}\left(\sum_{i=1}^{n} v_i(t)\hat{E}_i(t)\right) = \sum_{i=1}^{n} \dot{v}_i(t)\hat{E}_i(t).$$

Therefore,

$$d_{\alpha_1(t)}h(t)\left(\frac{D}{dt}V_1(t)\right) = \frac{D}{dt}\left(d_{\alpha_1(t)}h(t)V_1(t)\right).$$

Consequently, by Proposition 5.2.2, we can conclude that (i) of Theorem 5.2.1 implies (5.8) of Definition 5.2.1.

□

Remark 5.2.2

1. *Godoy et al. [21] proved the equivalent formulation of the no-twist conditions present in Theorem 5.2.1 when the manifolds are embedded in an Euclidean space and whenever the no-slip condition holds. Our proof doesn't need this assumption what makes the last theorem even more interesting. This is due to the fact that in some engineering problems the constraint of no-slip is overlooked.*

2. *Note that the normal part of the no-twist conditions is always satisfied for manifolds of co-dimension* 1 *and the tangential part of the no-twist conditions is always satisfied for manifolds of dimension* 1.

5.2.2 Rolling and Parallel Transport

The non-twist conditions allow to make a connection between rolling motions and parallel transport. This is better seen from the geometric interpretation of those conditions. To see this, let $V_0 \in T_{p_0}M_1 = T_{p_0}M_0$ and $V_0(t)$ be the parallel transport of V_0 along the development curve α_0. Define $V_1(t) := d_{\alpha_0(t)}h(t)^{-1}(V_0(t)) \in T_{\alpha_1(t)}M_1$. According to Theorem 5.2.1, $V_1(t)$ is parallel along α_1 and $V_1(0) = V_0$. This is the only parallel vector field along the rolling curve that satisfies the given initial condition.

Since the parallel vector fields on Euclidean spaces are the constant vector fields, this explains the formulas derived in Hüper and Silva Leite [31] for the parallel transport along curves of some particular manifolds rolling on the affine tangent space at a point. We will come back to this point later in this chapter.

If one chooses $V_0 = \dot{\alpha}_1(0)$, it also follows immediately from the above considerations that rolling along a geodesic always generates a geodesic on the static manifold. This important property has already been stated in Sharpe [68], but is also included here for the sake of completeness.

Proposition 5.2.3 *Let $h : J \to \overline{G}$ be a rolling map of M_1 over M_0, with rolling curve α_1 and development curve α_0. The rolling curve α_1 is a geodesic on M_1, if and only if, the development curve α_0 is a geodesic on M_0.*

5.2.3 Properties of Rolling Motions

In this section we show two important properties of rolling motions, without slip and twist, namely, the symmetry and the transitivity. It is important to notice that Proposition 2.3.1, will be useful to prove these rolling properties. Note also, that given M_1 and M_2 two differential manifolds of the same dimension, if $f : M_1 \to M_2$ is a diffeomorphism, then $d_p f : T_pM_1 \to T_{f(p)}M_2$ is an isomorphism between the tangent spaces. So, it follows that any tangent vector $V_2 \in T_{f(p)}M_2$ is always the image of a tangent vector $V_1 \in T_pM_1$ and vice versa.

Theorem 5.2.2 (Symmetry) *Let M_1 and M_2 be two connected submanifolds, of the same dimension, of a complete Riemannian manifold $\overline{M}$ tangent to each other at some point $p \in M_1 \cap M_2$. Suppose that:*

(i) $h : J \to \overline{G}$ is a rolling map of M_1 over M_2 with rolling curve $\alpha_1 : J \to M_1$ and development curve $\alpha_2 : J \to M_2$.

Then, it follows that:

(ii) $h^{-1} : J \to \overline{G}$ defined by $h^{-1}(t) := h(t)^{-1}$ is a rolling map of M_2 over M_1, with rolling curve α_2 and development curve α_1.

Proof. To prove the result we need to show that, for all $t \in J$, the map h^{-1} fulfills the properties 1, 2 and 3 of Definition 5.2.1 or, equivalently, properties derived in Section 5.2.1.

1. Proof of the Rolling Conditions:

   Since $h^{-1}(t)(\alpha_2(t)) = h(t)^{-1}(h(t)(\alpha_1(t))) = \alpha_1(t)$, it is immediately that $h^{-1}(t)(\alpha_2(t)) \in M_1$ and that α_1 is the development curve for h^{-1}. Moreover, by (i), we know that

$$T_{\alpha_2(t)}(h(t)(M_1)) = T_{\alpha_2(t)}M_2. \tag{5.22}$$

   Therefore,

$$\begin{aligned} T_{\alpha_1(t)}(h^{-1}(t)(M_2)) &= T_{h^{-1}(t)(\alpha_2(t))}(h^{-1}(t)(M_2)) \\ &\overset{(2.24)}{=} d_{\alpha_2(t)}h^{-1}(t)(T_{\alpha_2(t)}M_2) \\ &\overset{(5.22)}{=} d_{\alpha_2(t)}h^{-1}(t)(T_{\alpha_2(t)}(h(t)(M_1))) \\ &\overset{(2.24)}{=} T_{h^{-1}(t)(\alpha_2(t))}h^{-1}(t)(h(t)(M_1)) \\ &= T_{h^{-1}(t)(\alpha_2(t))}M_1 \\ &= T_{\alpha_1(t)}M_1. \end{aligned}$$

2. Proof of the No-Slip Condition:

   Considering the no-slip condition (5.13), we need to prove that $\dot{\alpha}_1(t) = d_{\alpha_2(t)}h^{-1}(t)(\dot{\alpha}_2(t))$. By (i), we know that $\dot{\alpha}_2(t) = d_{\alpha_1(t)}h(t)(\dot{\alpha}_1(t))$. Then, $(d_{\alpha_1(t)}h(t))^{-1}(\dot{\alpha}_2(t)) = \dot{\alpha}_1(t)$, and since $d_{\alpha_2(t)}h^{-1}(t) = (d_{\alpha_1(t)}h(t))^{-1}$, we get

   $$\dot{\alpha}_1(t) = d_{\alpha_2(t)}h^{-1}(t)(\dot{\alpha}_2(t)).$$

3. Proof of the No-Twist Conditions:

   In what follows we will show the tangential part of the no-twist conditions. The proof of the normal part is similar. According with Theorem 5.2.1 we just need to prove that a vector field $V_2(t)$ is tangent parallel to M_2 along the curve $\alpha_2(t)$, if and only if, $V_1(t) = d_{\alpha_2(t)}h^{-1}(t)(V_2(t))$ is tangent parallel along the curve $\alpha_1(t)$. By (i), we have that $V_1(t)$ is tangent parallel to M_1 along the curve $\alpha_1(t)$, if and only if, $V_2(t) = d_{\alpha_1(t)}h(t)(V_1(t))$ is tangent parallel to M_2 along the curve $\alpha_2(t)$. From the last equality, we obtain that:

   $$(d_{\alpha_1(t)}h(t))^{-1}(V_2(t)) = V_1(t),$$

   which, since $h(t)^{-1}(\alpha_2(t)) = \alpha_1(t)$ and $d_{\alpha_2(t)}h^{-1}(t) = (d_{\alpha_1(t)}h(t))^{-1}$, is equivalent to $V_1(t) = d_{\alpha_2(t)}h^{-1}(t)(V_2(t))$. Therefore, we conclude the result.

$\square$

Theorem 5.2.3 (Transitivity) *Let M_1, M_2 and M_3 be three connected submanifolds, of the same dimension, of a complete Riemannian manifold $\overline{M}$ tangent to each other at some point $p \in M_1 \cap M_2 \cap M_3$. Suppose that the following two conditions hold:*

(i) $h_1 : J \to \overline{G}$ is a rolling map of M_1 over M_2 with rolling curve $\alpha_1 : J \to M_1$ and development curve $\alpha_2 : J \to M_2$.

(ii) $h_2 : J \to \overline{G}$ is a rolling map of M_2 over M_3 with rolling curve $\alpha_2 : J \to M_2$ and development curve $\alpha_3 : J \to M_3$.

Then, it follows that:

(iii) $h_2 \circ h_1 : J \to \overline{G}$ defined by $(h_2 \circ h_1)(t) := h_2(t)h_1(t)$ is a rolling map of M_1 over M_3, with rolling curve α_1 and development curve α_3.

Proof. We will prove that, for all $t \in J$, the map $h_2 \circ h_1$ satisfies the properties 1, 2 and 3 of Definition 5.2.1.

1. Proof of the Rolling Conditions:

   By (i),

   $$\alpha_2(t) = h_1(t)(\alpha_1(t)) \in M_2$$

   and

   $$T_{\alpha_2(t)}(h_1(t)(M_1)) = T_{\alpha_2(t)}M_2. \tag{5.23}$$

By (ii),

$$\alpha_3(t) = h_2(t)(\alpha_2(t)) \in M_3$$

and

$$T_{\alpha_3(t)}(h_2(t)(M_2)) = T_{\alpha_3(t)}M_3. \tag{5.24}$$

Then,

$$\begin{array}{rcl}(h_2 \circ h_1)(t)(\alpha_1(t)) & = & h_2(t)h_1(t)(\alpha_1(t)) \\ & = & h_2(t)(h_1(t)(\alpha_1(t))) \\ & = & h_2(t)(\alpha_2(t)) \\ & = & \alpha_3(t) \in M_3,\end{array}$$

and

$$\begin{array}{rcl}T_{\alpha_3(t)}((h_2 \circ h_1)(t)(M_1)) & = & T_{h_2(t)(\alpha_2(t))}(h_2(t)(h_1(t)(M_1))) \\ & \overset{(2.24)}{=} & d_{\alpha_2(t)}h_2(t)(T_{\alpha_2(t)}(h_1(t)(M_1))) \\ & \overset{(5.23)}{=} & d_{\alpha_2(t)}h_2(t)(T_{\alpha_2(t)}M_2) \\ & \overset{(2.24)}{=} & T_{\alpha_3(t)}(h_2(t)(M_2)) \\ & \overset{(5.24)}{=} & T_{\alpha_3(t)}M_3.\end{array}$$

2. Proof of the No-Slip Condition:

Taking into consideration the geometric interpretation of the no-slip condition (5.13), it is enough to prove that $\dot{\alpha}_3(t) = d_{\alpha_1(t)}(h_2 \circ h_1)(t)(\dot{\alpha}_1(t))$. By (i), $\dot{\alpha}_2(t) = d_{\alpha_1(t)}h_1(t)(\dot{\alpha}_1(t))$, and by (ii), $\dot{\alpha}_3(t) = d_{\alpha_2(t)}h_2(t)(\dot{\alpha}_2(t))$. Then,

$$\begin{array}{rcl}\dot{\alpha}_3(t) & = & d_{\alpha_2(t)}h_2(t)(\dot{\alpha}_2(t)) \\ & = & d_{\alpha_2(t)}h_2(t)(d_{\alpha_1(t)}h_1(t)(\dot{\alpha}_1(t))) \\ & = & d_{\alpha_2(t)}h_2(t)d_{\alpha_1(t)}h_1(t)(\dot{\alpha}_1(t)) \\ & = & d_{h_1(t)(\alpha_1(t))}h_2(t)d_{\alpha_1(t)}h_1(t)(\dot{\alpha}_1(t)) \\ & = & d_{\alpha_1(t)}(h_2(t)h_1(t))(\dot{\alpha}_1(t)) \\ & = & d_{\alpha_1(t)}(h_2 \circ h_1)(t)(\dot{\alpha}_1(t)).\end{array}$$

3. Proof of the No-Twist Conditions:

In what follows we will show the tangential part of the no-twist conditions. The proof of the normal part is similar. According with Theorem 5.2.1 we just need to prove that a vector field $V_1(t)$ is tangent parallel to M_1 along the curve $\alpha_1(t)$, if and only if, $V_3(t) = d_{\alpha_1(t)}(h_2 \circ h_1)(t)(V_1(t))$ is tangent parallel along the curve $\alpha_3(t)$. By (i), we have that $V_1(t)$ is tangent parallel to M_1 along the curve $\alpha_1(t)$, if and only if, $V_2(t) = d_{\alpha_1(t)}h_1(t)(V_1(t))$ is tangent parallel to M_2 along the curve $\alpha_2(t)$. Moreover, by (ii), we have that $V_2(t) = d_{\alpha_1(t)}h_1(t)(V_1(t))$ is tangent parallel to M_2 along the curve $\alpha_2(t)$, if and only if, $V_3(t) = d_{\alpha_2(t)}h_2(t)(V_2(t))$ is tangent parallel to M_3 along the curve $\alpha_3(t)$. Therefore, since

$$\begin{aligned} V_3(t) &= d_{\alpha_2(t)}h_2(t)(V_2(t)) \\ &= d_{\alpha_2(t)}h_2(t)(d_{\alpha_1(t)}h_1(t)(V_1(t))) \\ &= d_{h_1(t)(\alpha_1(t))}h_2(t)(d_{\alpha_1(t)}h_1(t)(V_1(t))) \\ &= d_{\alpha_1(t)}(h_2\circ h_1)(t)(V_1(t)) \end{aligned}$$

we conclude that $\frac{D}{dt}V_3(t) = d_{\alpha_1(t)}(h_2\circ h_1)(t)\frac{D}{dt}V_1(t)$, and the result follows.

□

Remark 5.2.3 *These rolling properties will prove to be important because they allow to reduce the study of rolling maps to the rolling of a manifold on the affine tangent space at an arbitrary point. It turns out that such rolling motions play an important role in finding simple solutions for certain interpolation problems on manifolds, as we will see on Chapter 6.*

5.3 Rolling Euclidean Submanifolds

In what follows we particularize the general case to the special situation when M_0 and M_1 are manifolds isometrically embedded in the Euclidean space $\overline{M} = \mathbb{R}^m$ and, consequently, the isometry group $\overline{G}$ reduces to the special Euclidean group $\mathrm{SE}(m) = \mathrm{SO}(m) \ltimes \mathbb{R}^m$. This is the group of rigid motions in $\mathbb{R}^m$ that preserve orientation, and can be described by rotations and translations. We represent elements of $\overline{G} = \mathrm{SE}(m)$ as pairs (R,s), where $R \in \mathrm{SO}(m)$, $s \in \mathbb{R}^m$, so that the group operations are defined by:

$$(R_2,s_2)(R_1,s_1) := (R_2R_1, R_2s_1+s_2) \qquad \text{(product rule)},$$

$$(R,s)^{-1} := (R^{-1}, -R^{-1}s) \qquad \text{(inverse rule)},$$

and $(I_m,0)$ denotes the identity element. The group $\overline{G} = \mathrm{SE}(m)$ acts on points of $\mathbb{R}^m$ in the usual way via:

$$\begin{array}{rccc} \sigma: & \overline{G}\times\mathbb{R}^m & \longrightarrow & \mathbb{R}^m \\ & ((R,s),p) & \longmapsto & Rp+s \end{array}, \tag{5.25}$$

and, for each $(R,s) \in \overline{G}$, this action defines a mapping

$$\begin{array}{rccc} \sigma_{(R,s)}: & \mathbb{R}^m & \longrightarrow & \mathbb{R}^m \\ & p & \longmapsto & Rp+s \end{array},$$

which induces the differential mapping between $T_p\mathbb{R}^m \cong \mathbb{R}^m$ and $T_{Rp+s}\mathbb{R}^m \cong \mathbb{R}^m$, sending every vector $\xi \in T_p\mathbb{R}^m$ to the vector $R\xi$ in $T_{Rp+s}\mathbb{R}^m$.

For this particular case, a smooth curve

$$\begin{array}{rccc} h: & [0,\tau] & \longrightarrow & \mathrm{SE}(m) \\ & t & \longmapsto & h(t) = (R(t),s(t)) \end{array},$$

is a rolling map of M_1 over M_0, without slipping and twisting, along a smooth curve α_1 on M_1, if it satisfies the properties 1, 2 and 3 of the Definition 5.2.1, with the natural action of the group

$\mathrm{SE}(m)$ on $\mathbb{R}^m$ given by (5.25). In order to specialize the Definition 5.2.1 for the rolling of Euclidean submanifolds we first rewrite the expressions (5.3) - (5.5) adapting them to our specific situation.

Let $p \in \mathbb{R}^m$ be a point and $\eta \in T_p\mathbb{R}^m \cong \mathbb{R}^m$ be a tangent vector at p. This means that there exists a smooth curve $y :]-\varepsilon, \varepsilon[\to \mathbb{R}^m$ such that $y(0) = p$ and $\dot{y}(0) = \eta$. Then, the expression (5.3) can be rewritten as follows:

$$\begin{aligned} \dot{h}(t)(p) &= \frac{\mathrm{d}}{\mathrm{d}\sigma}[h(\sigma)(p)]\bigg|_{\sigma=t} \\ &= \frac{\mathrm{d}}{\mathrm{d}\sigma}[R(\sigma)p + s(\sigma)]\bigg|_{\sigma=t} \\ &= \dot{R}(t)p + \dot{s}(t). \end{aligned}$$

Taking in consideration the definition of the group operations, we have

$$h(\sigma)h(t)^{-1} = (R(\sigma)R(t)^{-1}, -R(\sigma)R(t)^{-1}s(t) + s(\sigma)) \;\in\; \mathrm{SE}(m).$$

Consequently, the expression (5.4) can be rewritten as

$$\begin{aligned} \left(\dot{h}(t) \circ h(t)^{-1}\right)(p) &= \frac{\mathrm{d}}{\mathrm{d}\sigma}\left[\left(h(\sigma)\,h(t)^{-1}\right)(p)\right]\bigg|_{\sigma=t} \\ &= \frac{\mathrm{d}}{\mathrm{d}\sigma}\left[R(\sigma)R(t)^{-1}p - R(\sigma)R(t)^{-1}s(t) + s(\sigma)\right]\bigg|_{\sigma=t} \\ &= \frac{\mathrm{d}}{\mathrm{d}\sigma}\left[R(\sigma)R(t)^{-1}(p - s(t)) + s(\sigma)\right]\bigg|_{\sigma=t} \\ &= \dot{R}(t)R(t)^{-1}(p - s(t)) + \dot{s}(t), \end{aligned} \tag{5.26}$$

and the expression (5.5) simplifies to

$$\begin{aligned} d_p\left(\dot{h}(t) \circ h(t)^{-1}\right)(\eta) &= \frac{\mathrm{d}}{\mathrm{d}\sigma}\left[\left(\dot{h}(t) \circ h(t)^{-1}\right)(y(\sigma))\right]\bigg|_{\sigma=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\sigma}\left[\dot{R}(t)R(t)^{-1}(y(\sigma) - s(t)) + \dot{s}(t)\right]\bigg|_{\sigma=0} \\ &= \dot{R}(t)R(t)^{-1}\eta. \end{aligned} \tag{5.27}$$

So, for this particular situation, we can rewrite the definition of a rolling map as follows.

Definition 5.3.1 *Let M_0 and M_1 be two n-dimensional connected manifolds isometrically embedded in the Euclidean space $\mathbb{R}^m$. A* rolling map of M_1 over M_0, without slipping and twisting, *is a smooth curve $t \mapsto (R(t), s(t))$ in $\mathrm{SE}(m)$, satisfying, for all $t \in J$, the following three properties:*

1. *Rolling conditions: There exists a smooth curve $\alpha_1 : J \to M_1$, such that*

   (a) $R(t)\alpha_1(t) + s(t) =: \alpha_0(t) \in M_0$;

   (b) $T_{\alpha_0(t)}(R(t)M_1 + s(t)) = T_{\alpha_0(t)}M_0$.

2. *No-slip condition:*

$$\dot{R}(t)R(t)^{-1}\left(\alpha_0(t) - s(t)\right) + \dot{s}(t) = 0. \tag{5.28}$$

3. *No-twist conditions:*

   (a) *(Tangential part)*
$$\dot{R}(t)R(t)^{-1}T_{\alpha_0(t)}M_0 \subset (T_{\alpha_0(t)}M_0)^{\perp}. \tag{5.29}$$

   (b) *(Normal part)*
$$\dot{R}(t)R(t)^{-1}(T_{\alpha_0(t)}M_0)^{\perp} \subset T_{\alpha_0(t)}M_0. \tag{5.30}$$

5.3.1 Structure of the Kinematic Equations of Rolling

The kinematic equations describe the velocity vector of the rolling map, that is the rotational and translational velocities of the rolling motion. In the situation when the manifolds M_1 and M_0 are isometrically embedded in the Euclidean space $\mathbb{R}^m$ and, consequently, rolling maps are of the form $t \mapsto h(t) = (R(t), s(t))$, the kinematic equations can be written as:

$$\begin{cases} \dot{R}(t) = A(t)R(t) \\ \dot{s}(t) = u(t) \end{cases},$$

for some matrix function A taking values in the Lie algebra $\mathfrak{so}(m)$ and some vector valued function u taking values in $\mathbb{R}^m$. Conditions on these functions are determined from the holonomic and nonholonomic constraints.

Assumption: Assume that the orthogonal group $\mathrm{SO}(m)$ acts transitively on the rolling manifold M_1 and that $M_0 = T^{\mathrm{aff}}_{p_0}M_1 := p_0 + T_{p_0}M_1$, for some $p_0 \in M_1$. This is not a significant restriction since, due to the transitivity property of rolling maps, we can always reduce to this situation.

Since $\mathrm{SO}(m)$ acts transitively on M_1, the rolling curve is always generated by the action of this group on a point, that is, $\alpha_1(t) = Q(t)p_0$, for some $Q(t) \in \mathrm{SO}(m)$, with $Q(0) = I_m$. The corresponding rolling map $h(t) = (R(t), s(t))$ must satisfy the first rolling condition, so $\alpha_0(t) = R(t)Q(t)p_0 + s(t) \in p_0 + T_{p_0}M_1$. Choose $Q(t) = R(t)^{-1}$ and $s(t) \in T_{p_0}M_1$, so that $s(0) = 0$, $\alpha_1(t) = R(t)^{-1}p_0$ and $\alpha_0(t) = p_0 + s(t)$. For this choice, the first rolling condition is clearly satisfied. We will show that $h(t) = (R(t), s(t))$ and $\alpha_1(t) = R(t)^{-1}p_0$ also satisfy all the other conditions of Definition 5.3.1. So, due to the uniqueness of the rolling map, $t \mapsto (R(t), s(t)) \in \mathrm{SE}(m)$ is a rolling map with rolling curve $t \mapsto R(t)^{-1}p_0 \in M_1$.
Since in this case $T_{\alpha_0(t)}M_0 = T_{p_0}M_0 = T_{p_0}M_1$, the second rolling condition is equivalent to $T_{\alpha_0(t)}h(t)M_1 = T_{p_0}M_1$. We show that this equality holds.
First, we prove that $T_{\alpha_0(t)}h(t)M_1 \subset T_{p_0}M_1$. Let $v(t) \in T_{\alpha_0(t)}h(t)M_1$. So, there exists a curve $\sigma \mapsto \gamma_t(\sigma) \in h(t)M_1$ such that $\gamma_t(0) = \alpha_0(t)$ and $\dot{\gamma}_t(0) = v(t)$. Define a curve $\sigma \mapsto \beta_t(\sigma) = \gamma_t(\sigma) - s(t)$. This is a curve in M_1 due to the transitive action of $\mathrm{SO}(m)$ on M_1. It happens that $\beta_t(0) = \gamma_t(0) - s(0) = \alpha_0(t)$ and $\dot{\beta}_t(0) = v(t)$. So, $v(t) \in T_{p_0}M_1$. The prove that $T_{p_0}M_1 \subset T_{\alpha_0(t)}h(t)M_1$ is similar.
Now, since $s(t) = \alpha_0(t) - p_0 \in T^{\mathrm{aff}}_{p_0}M_1$, the no-slip condition reduces to

$$\dot{s}(t) = -A(t)p_0, \tag{5.31}$$

with $A(t) = \dot{R}(t)R(t)^{-1} \in \mathfrak{so}(m)$. The structure of $A(t)$, $t \in [0, \tau]$ is determined from the no-twist conditions

$$A(t)T_{\alpha_0(t)}M_0 \subset (T_{\alpha_0(t)}M_0)^{\perp},$$
$$A(t)(T_{\alpha_0(t)}M_0)^{\perp} \subset T_{\alpha_0(t)}M_0.$$

So, for an appropriate choice of coordinates, the matrix function A has the following structure

$$A(t) = \left[\begin{array}{c|c} 0 & A_1(t) \\ \hline -A_1^{\top}(t) & 0 \end{array}\right], \tag{5.32}$$

with $A_1(t) \in \mathbb{R}^{n\times(m-n)}$, where n is the dimension of M_1. Finally, from (5.31) and (5.32), we can conclude that the kinematic equations for rolling the manifold M_1 over $M_0 = T^{\mathrm{aff}}_{p_0}M_1$ are:

$$\begin{cases} \dot{R}(t) = A(t)R(t) \\ \dot{s}(t) = -A(t)p_0 \end{cases}, \tag{5.33}$$

where the matrix function A has the structure (5.32).

Remark 5.3.1 *When M_1 is the $(m-1)$-dimensional sphere centered at the origin, with radius r, and p_0 is its south pole, then*

$$A_1(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_{m-1}(t) \end{bmatrix},$$

for some scalar functions $u_1, \dots, u_{m-1}$. Hence, if $A_{i,j} = e_i e_j^{\top} - e_j e_i^{\top}$, where e_ℓ, $\ell = 1, \dots, m$, denote the unitary canonical vectors of $\mathbb{R}^m$, are the elementary skewsymmetric matrices, we can write that $A(t) = \sum_{i=1}^{m-1} u_i(t)A_{i,m}$, and then, the equations (5.33) for rolling the sphere on its affine tangent space at the south pole reduce to the well-known kinematic equations

$$\begin{cases} \dot{R}(t) = \sum_{i=1}^{m-1} u_i(t)A_{i,m}R(t) \\ \dot{s}(t) = ru(t) \end{cases}.$$

5.3.2 Parallel Transport

Under the assumption that $M_0 = T^{\mathrm{aff}}_{p_0}M_1$, if $t \mapsto h(t) = (R(t), s(t)) \in \mathrm{SE}(m)$ is a rolling map with rolling curve $t \mapsto \alpha_1(t) = R(t)^{-1}p_0 \in M_1$ and $V_0 \in T_{p_0}M_1 = T_{p_0}M_0$, then the parallel transport of V_0 along α_0 is the constant vector field $V_0(t) = V_0$. So, according to Subsection 5.2.2,

$$V_1(t) := d_{\alpha_0(t)}h(t)^{-1}(V_0(t)) = R(t)^{-1}V_0$$

is the parallel transport of V_0 along the rolling curve $\alpha_1(t)$.

5.4 Rolling the Riemannian Manifold $G_{k,n} \times \mathrm{SO}(n)$

In this section we specify the rolling maps to the particular situation when M_1 is the manifold $G_{k,n} \times \mathrm{SO}(n)$ and M_0 its affine tangent space at a point P_0. Both manifolds are isometrically embedded in the vector space of matrices $\overline{M} = \mathfrak{s}(n) \times \mathbb{R}^{n\times n}$, endowed with the Euclidean Riemannian metric defined in (3.39). This approach follows that of Hüper and Silva Leite [31], where the rolling of Grassmann manifolds and of rotation groups has been studied. We recall that, according to our representation of the manifold $G_{k,n} \times \mathrm{SO}(n)$ given in (3.38), its elements are represented by pairs. Take $G = \mathrm{SO}(n) \times \mathrm{SO}(n) \times \mathrm{SO}(n)$. G acts on $M_1 = G_{k,n} \times \mathrm{SO}(n)$ via

$$\begin{array}{rccc} \phi: & G \times M_1 & \to & M_1 \\ & ((U,V,W),(A,B)) & \mapsto & (UAU^\top, VBW^\top) \end{array},$$

and this action is transitive. Indeed, given $(\Theta_1 E_0 \Theta_1^\top, R_1)$ and $(\Theta_2 E_0 \Theta_2^\top, R_2)$ in $M_1 = G_{k,n} \times \mathrm{SO}(n)$, there exists $(U,V,W) \in G$ such that $\phi((U,V,W),(\Theta_1 E_0 \Theta_1^\top, R_1)) = (\Theta_2 E_0 \Theta_2^\top, R_2)$. It is easily checked that $U = \Theta_2 \Theta_1^\top$, $V = R_2$ and $W = R_1$ comply to the requirements.

Now, define $\overline{G} := G \ltimes \overline{M}$, which is a connected Lie group with product rule

$$\Big(U_1,V_1,W_1,X_1,Y_1\Big)\Big(U_2,V_2,W_2,X_2,Y_2\Big) = \Big(U_1U_2, V_1V_2, W_1W_2, U_1X_2U_1^\top + X_1, V_1Y_2W_1^\top + Y_1\Big),$$

identity element $e_{\overline{G}} = (I,I,I,0,0)$, and inverse rule

$$\Big(U,V,W,X,Y\Big)^{-1} = \Big(U^\top, V^\top, W^\top, -U^\top XU, -V^\top YW\Big). \tag{5.34}$$

- The action of G on M_1, extends to the following action of $\overline{G}$ on the vector space $\overline{M} = \mathfrak{s}(n) \times \mathbb{R}^{n\times n}$:

$$\begin{array}{rccc} \phi: & \overline{G} \times (\mathfrak{s}(n) \times \mathbb{R}^{n\times n}) & \longrightarrow & \mathfrak{s}(n) \times \mathbb{R}^{n\times n} \\ & \Big((U,V,W,X,Y),(A,B)\Big) & \longmapsto & \Big(UAU^\top + X, VBW^\top + Y\Big) \end{array}. \tag{5.35}$$

  Indeed, for every E and F in $\overline{G}$ and (A,B) in $\mathfrak{s}(n) \times \mathbb{R}^{n\times n}$, the following easily checked identities hold:

$$\begin{aligned} \phi\left(e_{\overline{G}},(A,B)\right) &= (A,B); \\ \phi\left(EF,(A,B)\right) &= \phi\left(E,\phi(F,(A,B))\right). \end{aligned}$$

- For each fixed element $E = (U,V,W,X,Y) \in \overline{G}$, the mapping

$$\begin{array}{rccc} \phi_E: & \mathfrak{s}(n) \times \mathbb{R}^{n\times n} & \longrightarrow & \mathfrak{s}(n) \times \mathbb{R}^{n\times n} \\ & (A,B) & \longmapsto & \Big(UAU^\top + X, VBW^\top + Y\Big) \end{array}$$

  is an isometry of $\overline{M} = \mathfrak{s}(n) \times \mathbb{R}^{n\times n}$, that is, ϕ_E is a diffeomorphism and

$$\begin{array}{rccc} d_{(A,B)}\phi_E: & \mathfrak{s}(n) \times \mathbb{R}^{n\times n} & \longrightarrow & \mathfrak{s}(n) \times \mathbb{R}^{n\times n} \\ & (\xi,\eta) & \longmapsto & \Big(U\xi U^\top, V\eta W^\top\Big) \end{array}$$

is a linear isometry. This can be easily checked by showing that for every (ξ_1, η_1) and (ξ_2, η_2) belonging to $\mathfrak{s}(n) \times \mathbb{R}^{n \times n}$, the following holds:

$$\langle d\phi_E(\xi_1, \eta_1), d\phi_E(\xi_2, \eta_2) \rangle = \langle (\xi_1, \eta_1), (\xi_2, \eta_2) \rangle .$$

- For each $(U, V, W) \in G = \mathrm{SO}(n) \times \mathrm{SO}(n) \times \mathrm{SO}(n)$, the action of G on $M_1 = G_{k,n} \times \mathrm{SO}(n)$ induces the mapping

$$\begin{array}{rccc} \psi_{U,V,W} : & G_{k,n} \times \mathrm{SO}(n) & \longrightarrow & G_{k,n} \times \mathrm{SO}(n) \\ & (\Theta E_0 \Theta^\top, R) & \longmapsto & \big(U\Theta E_0 \Theta^\top U^\top, VRW^\top\big) \end{array} . \tag{5.36}$$

Remark 5.4.1 *It is not clear if $\overline{G}$ is indeed the whole connected component of the isometry group of $\mathfrak{s}(n) \times \mathbb{R}^{n \times n}$ which contains the identity and preserves orientation. However, as will be clear soon, these isometries are enough to perform the rolling motion.*

The transitive action of $G = \mathrm{SO}(n) \times \mathrm{SO}(n) \times \mathrm{SO}(n)$ on $M_1 = G_{k,n} \times \mathrm{SO}(n)$ ensures that any curve $t \mapsto \alpha_1(t) \in M_1$, satisfying $\alpha_1(0) = (\Theta_0 E_0 \Theta_0^\top, R_0)$ is defined by

$$\alpha_1(t) = \big(U(t)\Theta_0 E_0 \Theta_0^\top U(t)^\top, V(t) R_0 W(t)^\top\big),$$

with $t \mapsto U(t)$, $t \mapsto V(t)$ and $t \mapsto W(t)$ curves in $SO(n)$ that satisfy $U(0) = V(0) = W(0) = I$.

In what follows, we will show under which conditions the map

$$\begin{array}{rccl} h : & [0, \tau] & \longrightarrow & \overline{G} \\ & t & \longmapsto & h(t) = (U(t)^\top, V(t)^\top, W(t)^\top, X(t), Y(t)) \end{array} , \tag{5.37}$$

is a rolling map of the manifold $G_{k,n} \times \mathrm{SO}(n)$ over its affine tangent space $T^{\mathrm{aff}}_{P_0}(G_{k,n} \times \mathrm{SO}(n))$, along

$$\alpha_1(t) = \big(U(t)\Theta_0 E_0 \Theta_0^\top U(t)^\top, V(t) R_0 W(t)^\top\big),$$

with development curve

$$\alpha_0(t) = h(t)(\alpha_1(t)) = \big(\Theta_0 E_0 \Theta_0^\top + X(t), R_0 + Y(t)\big) = P_0 + Z(t) \in M_0, \tag{5.38}$$

where $Z(t) = (X(t), Y(t)) \in \mathfrak{s}(n) \times \mathbb{R}^{n \times n}$.

In order to do that, we must adapt the notations in Section 5.3 to the present matrix representation. First, we rewrite the expressions (5.3) - (5.5) for this new situation.

Let (A, B) be a point in $\mathfrak{s}(n) \times \mathbb{R}^{n \times n}$ and $(\xi, \eta) \in \mathfrak{s}(n) \times \mathbb{R}^{n \times n}$ be a tangent vector to a smooth curve $y : t \in]-\varepsilon, \varepsilon[\longmapsto y(t) = \big(A(t), B(t)\big) \in \mathfrak{s}(n) \times \mathbb{R}^{n \times n}$ that satisfies $y(0) = \big(A(0), B(0)\big) = (A, B)$ and $\dot{y}(0) = \big(\dot{A}(0), \dot{B}(0)\big) = (\xi, \eta)$. Then, since

$$h(t)((A, B)) = (U(t)^\top A U(t) + X(t), V(t)^\top B W(t) + Y(t)),$$

one gets

$$\begin{aligned}
\dot{h}(t)((A,B)) &= \frac{\mathrm{d}}{\mathrm{d}\sigma}\left[h(\sigma)((A,B))\right]\Big|_{\sigma=t} \\
&= \frac{\mathrm{d}}{\mathrm{d}\sigma}\left[U(\sigma)^\top A U(\sigma)+X(\sigma), V(\sigma)^\top B W(\sigma)+Y(\sigma)\right]\Big|_{\sigma=t} \\
&= \big(\dot{U}(t)^\top A U(t)+U(t)^\top A\dot{U}(t)+\dot{X}(t), \dot{V}(t)^\top B W(t)+V(t)^\top B\dot{W}(t)+\dot{Y}(t)\big).
\end{aligned}$$

This is the counterpart of (5.3). Now, taking into account the group operations we have that

$$\begin{aligned}
h(\sigma)\,h(t)^{-1} &= \big(U(\sigma)^\top U(t), V(\sigma)^\top V(t), W(\sigma)^\top W(t), -U(\sigma)^\top U(t)X(t)U(t)^\top U(\sigma)+X(\sigma), \\
&\qquad -V(\sigma)^\top V(t)Y(t)W(t)^\top W(\sigma)+Y(\sigma)\big) \in \overline{G}.
\end{aligned}$$

Hence, the counterpart of (5.4) is

$$\begin{aligned}
&\big(\dot{h}(t)\circ h(t)^{-1}\big)((A,B)) \\
&= \frac{\mathrm{d}}{\mathrm{d}\sigma}\left[\big(h(\sigma)\,h(t)^{-1}\big)((A,B))\right]\Big|_{\sigma=t} \\
&= \frac{\mathrm{d}}{\mathrm{d}\sigma}\Big|_{\sigma=t}\Big[\big(U^\top(\sigma)U(t)AU(t)^\top U(\sigma)-U^\top(\sigma)U(t)X(t)U^\top(t)U(\sigma)+X(\sigma), \\
&\qquad V^\top(\sigma)V(t)BW(t)^\top W(\sigma)-V(\sigma)^\top V(t)Y(t)W(t)^\top W(\sigma)+Y(\sigma)\big)\Big] \\
&= \big(\dot{U}(t)^\top U(t)A+AU(t)^\top\dot{U}(t)-\dot{U}(t)^\top U(t)X(t)-X(t)U(t)^\top\dot{U}(t)+\dot{X}(t), \\
&\qquad \dot{V}(t)^\top V(t)B+BW(t)^\top\dot{W}(t)-\dot{V}(t)^\top V(t)Y(t)-Y(t)W(t)^\top\dot{W}(t)+\dot{Y}(t)\big).
\end{aligned} \tag{5.39}$$

Finally, the counterpart of (5.5) is written as

$$\begin{aligned}
d_{(A,B)}\big(\dot{h}(t)\circ h(t)^{-1}\big)((\xi,\eta)) &= \frac{\mathrm{d}}{\mathrm{d}\sigma}\left[\big(\dot{h}(t)\circ h(t)^{-1}\big)\,(A(\sigma),B(\sigma))\right]\Big|_{\sigma=0} \\
&= \big(\dot{U}(t)^\top U(t)\xi+\xi U(t)^\top\dot{U}(t), \dot{V}(t)^\top V(t)\eta+\eta W(t)^\top\dot{W}(t)\big).
\end{aligned} \tag{5.40}$$

5.4.1 The Kinematic Equations of Rolling

In this section we derive the kinematic equations for the rolling motion by imposing the no-slip and no-twist conditions on $h(t)$ given by (5.37). Taking into account (5.39) and the expression for α_0 given by (5.38), the no-slip condition (5.7), for all $t\in J$, can be rewritten as

$$\begin{cases}
\dot{U}(t)^\top U(t)\Theta_0E_0\Theta_0^\top+\Theta_0E_0\Theta_0^\top U(t)^\top\dot{U}(t)+\dot{X}(t)=0 \\
\dot{V}(t)^\top V(t)R_0+R_0W(t)^\top\dot{W}(t)+\dot{Y}(t)=0
\end{cases}.$$

In what follows, for the sake of simplicity, whenever convenient we will omit the dependency on the parameter t.

If we define the skew-symmetric matrices Ω_U, Ω_V and Ω_W of $\mathfrak{so}(n)$ by

$$\Omega_U := \Theta_0^\top \dot{U}^\top U \Theta_0, \quad \Omega_V := R_0^\top \dot{V}^\top V R_0 \quad \text{and} \quad \Omega_W := R_0 \dot{W}^\top W R_0^\top, \tag{5.41}$$

the no-slip condition takes the form

$$\begin{cases} \dot{X}(t) = -\Theta_0 \big[\Omega_U(t), E_0\big] \Theta_0^\top \\ \dot{Y}(t) = \Omega_W(t) R_0 - R_0 \Omega_V(t) \end{cases}, \qquad \text{for all } t \in J. \tag{5.42}$$

Now, using (5.40), the tangential part of the no-twist conditions is equivalent to showing that, for all $(\xi, \eta) \in T_{\alpha_0(t)} M_0$,

$$\big(\dot{U}^\top U \xi + \xi U^\top \dot{U}, \dot{V}^\top V \eta + \eta W^\top \dot{W}\big) \in (T_{\alpha_0(t)} M_0)^\perp, \tag{5.43}$$

and similarly, the normal part of the no-twist conditions is equivalent to showing that, for all $(\xi, \eta) \in (T_{\alpha_0(t)} M_0)^\perp$,

$$\big(\dot{U}^\top U \xi + \xi U^\top \dot{U}, \dot{V}^\top V \eta + \eta W^\top \dot{W}\big) \in T_{\alpha_0(t)} M_0. \tag{5.44}$$

But, $T_{\alpha_0(t)} M_0 = T_{P_0}(G_{k,n} \times \mathrm{SO}(n))$ (and, similarly, for the normal space). So, taking into account the notations (5.41), we have that

$$\dot{U}^\top U = \Theta_0 \Omega_U \Theta_0^\top, \quad \dot{V}^\top V = R_0 \Omega_V R_0^\top, \quad W^\top \dot{W} = -R_0^\top \Omega_W R_0. \tag{5.45}$$

Consequently, the normal part of the no-twist conditions (5.44) is equivalent to

$$\Big(\Big[\Theta_0 \Omega_U \Theta_0^\top, \xi\Big], R_0 \Omega_V R_0^\top \eta - \eta R_0^\top \Omega_W R_0\Big) \in T_{P_0}(G_{k,n} \times \mathrm{SO}(n)), \tag{5.46}$$

for all $(\xi, \eta) \in (T_{P_0}(G_{k,n} \times \mathrm{SO}(n)))^\perp$. But, according to (3.42), for $(\xi, \eta) \in (T_{P_0}(G_{k,n} \times \mathrm{SO}(n)))^\perp$, we have that

$$\xi = \Theta_0 \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \Theta_0^\top, \quad S_1 \in \mathfrak{s}(k),\ S_2 \in \mathfrak{s}(n-k) \qquad \text{and} \qquad \eta = R_0 S, \quad S \in \mathfrak{s}(n).$$

Hence, writing the skew-symmetric matrix Ω_U as

$$\Omega_U = \begin{bmatrix} \Omega_1 & \Omega_2 \\ -\Omega_2^\top & \Omega_3 \end{bmatrix},$$

where $\Omega_1 \in \mathfrak{so}(k)$, $\Omega_2 \in \mathbb{R}^{k \times (n-k)}$, $\Omega_3 \in \mathfrak{so}(n-k)$, and taking into account that

$$\Big[\Theta_0 \Omega_U \Theta_0^\top, \xi\Big] = \Theta_0 \begin{bmatrix} [\Omega_1, S_1] & \Omega_2 S_2 - S_1 \Omega_2 \\ -\Omega_2^\top S_1 + S_2 \Omega_2^\top & [\Omega_3, S_2] \end{bmatrix} \Theta_0^\top,$$

the characterization of the tangent space (3.41) enables us to conclude that

$$[\Omega_1, S_1] = 0, \ \text{ for all } S_1 \in \mathfrak{s}(k) \tag{5.47}$$

and

$$[\Omega_3, S_2] = 0, \text{ for all } S_2 \in \mathfrak{s}(n-k). \tag{5.48}$$

The relations (5.47) and (5.48) imply, respectively, that $\Omega_1 = 0$ and $\Omega_3 = 0$. Therefore, we obtain that the skew-symmetric matrix Ω_U must have the constrained structure

$$\Omega_U = \begin{bmatrix} 0 & \Omega_2 \\ -\Omega_2^\top & 0 \end{bmatrix}.$$

Additionally, the second component present in relation (5.46), with $\eta = R_0 S$, $S \in \mathfrak{s}(n)$, should be of the form $R_0 C$, with $C \in \mathfrak{so}(n)$. This requires that the matrix $\left(\Omega_V S - S R_0^\top \Omega_W R_0\right)$ should be skew-symmetric, for all $S \in \mathfrak{s}(n)$. Using this requirement, and after some simple calculations, one concludes that this is equivalent to

$$\left[\Omega_V, S\right] + \left[R_0^\top \Omega_W R_0, S\right] = 0, \text{ for all } S \in \mathfrak{s}(n),$$

which means that

$$\left[\Omega_V + R_0^\top \Omega_W R_0, S\right] = 0, \text{ for all } S \in \mathfrak{s}(n).$$

Hence, $\Omega_V + R_0^\top \Omega_W R_0 = 0$, that is

$$\Omega_V = -R_0^\top \Omega_W R_0.$$

Therefore, the *normal part of the no-twist conditions* for the manifold $G_{k,n} \times \mathrm{SO}(n)$ is equivalent to requiring that

$$\Omega_U = \begin{bmatrix} 0 & \Omega_2 \\ -\Omega_2^\top & 0 \end{bmatrix} \quad \text{and} \quad \Omega_V = -R_0^\top \Omega_W R_0. \tag{5.49}$$

Finally, we must impose the tangential part of the no-twist conditions. But, it turns out that if the above conditions (5.49) hold, the tangential part of the no-twist conditions holds as well. Indeed, by (5.45), the relation (5.43), for all $(\xi, \eta) \in T_{\alpha_0(t)} M_0$, is equivalent to

$$\left(\left[\Theta_0 \Omega_U \Theta_0^\top, \xi\right], R_0 \Omega_V R_0^\top \eta - \eta R_0^\top \Omega_W R_0\right) \in (T_{\alpha_0(t)} M_0)^\perp. \tag{5.50}$$

So, since $(\xi, \eta) \in T_{\alpha_0(t)} M_0 = T_{P_0}(G_{k,n} \times \mathrm{SO}(n))$, we must have

$$\xi = \Theta_0 \begin{bmatrix} 0 & Z \\ Z^\top & 0 \end{bmatrix} \Theta_0^\top, \quad Z \in \mathbb{R}^{k \times (n-k)} \qquad \text{and} \qquad \eta = R_0 C, \quad C \in \mathfrak{so}(n).$$

Hence, using (5.49), after some calculations we obtain that

$$\left[\Theta_0 \Omega_U \Theta_0^\top, \xi\right] = \Theta_0 \begin{bmatrix} \Omega_2 Z^\top + Z \Omega_2^\top & 0 \\ 0 & -\Omega_2^\top Z - Z \Omega_2 \end{bmatrix} \Theta_0^\top,$$

which is in accordance with the characterization of the orthogonal complement (3.42). Moreover, the second component present in relation (5.50) should be of the form $R_0 S$, with $S \in \mathfrak{s}(n)$ and, taking into account (5.49), this requires that the matrix $(\Omega_V C + C \Omega_V)$ should be symmetric, for all $C \in \mathfrak{so}(n)$. A few computations show that this requirement is verified. Thus, *the no-twist conditions reduce to the equations (5.49).*

Now, if the second condition in (5.49) is used in (5.42), one obtains

$$\begin{cases} \dot{X}(t) = -\Theta_0 \big[\Omega_U(t), E_0\big] \Theta_0^\top \\ \dot{Y}(t) = -2R_0 \Omega_V(t) \end{cases}, \qquad \text{for all } t \in J. \tag{5.51}$$

The no-slip condition reduces to equations (5.51).

We can now state the main theorem of this section.

Theorem 5.4.1 *For $t \in J$, let $\Omega_U(t)$, $\Omega_V(t) \in \mathfrak{so}(n)$ with $\Omega_U(t) = \begin{bmatrix} 0 & \Omega_2(t) \\ -\Omega_2^\top(t) & 0 \end{bmatrix}$, and where $t \mapsto \Omega_2(t) \in \mathbb{R}^{k\times(n-k)}$. If (U,V,W,X,Y) is the solution of the following system of differential equations, evolving on $\overline{G}$,*

$$\begin{cases} \dot{U}(t) = -U(t)\Theta_0\Omega_U(t)\Theta_0^\top \\ \dot{V}(t) = -V(t)R_0\Omega_V(t)R_0^\top \\ \dot{W}(t) = W(t)\Omega_V(t) \\ \dot{X}(t) = -\Theta_0\big[\Omega_U(t), E_0\big]\Theta_0^\top \\ \dot{Y}(t) = -2R_0\Omega_V(t) \end{cases}, \tag{5.52}$$

with initial condition $(U(0),V(0),W(0),X(0),Y(0)) = (I,I,I,0,0)$, then

$$t \mapsto h(t) = (U(t)^\top, V(t)^\top, W(t)^\top, X(t), Y(t)) \in \overline{G}$$

is a rolling map (in the sense of Definition 5.2.1) of the manifold $G_{k,n} \times \mathrm{SO}(n)$ over the affine tangent space at the point $P_0 = (\Theta_0 E_0 {\Theta_0}^\top, R_0)$, along the rolling curve

$$t \mapsto \alpha_1(t) = \big(U(t)\Theta_0E_0\Theta_0^\top U(t)^\top, V(t)R_0W(t)^\top\big),$$

with development curve

$$t \mapsto \alpha_0(t) = \big(\Theta_0E_0\Theta_0^\top + X(t), R_0 + Y(t)\big).$$

Proof. We have already proved, before the statement of the theorem, that equations (5.52) encode the no-slip and the no-twist conditions. Since the curve α_1 clearly lives in the manifold $G_{k,n} \times \mathrm{SO}(n)$ and $\alpha_0(t) = h(t)(\alpha_1(t)) = P_0 + Z(t)$, with $Z(t) = (X(t), Y(t)) \in \mathfrak{s}(n) \times \mathbb{R}^{n\times n}$, to complete the proof it is enough to show that $Z(t) \in T_{P_0}\big(G_{k,n} \times \mathrm{SO}(n)\big)$. But since $\Omega_U(t)$ and $\Omega_V(t)$ are skew-symmetric, it follows from the last two equations of (5.52) that $\dot{Z}(t) \in T_{P_0}\big(G_{k,n} \times \mathrm{SO}(n)\big)$. This, together with the initial condition $Z(0) = (0,0)$, implies that $Z(t) \in T_{P_0}\big(G_{k,n} \times \mathrm{SO}(n)\big)$, that is, $\alpha_0(t) \in T_{P_0}^{\mathrm{aff}}\big(G_{k,n} \times \mathrm{SO}(n)\big)$.

$\square$

Remark 5.4.2 *Equations (5.52), which encode the non-holonomic constraints of no-slip and no-twist are called the* ***kinematic equations*** *for rolling $G_{k,n} \times \mathrm{SO}(n)$ over the affine tangent space at the point P_0. The choice of Ω_U and Ω_V completely determines the solutions of the kinematic equations and, consequently, the rolling curve (and its development). For that reason, we say that these two functions are the "control functions" of the motion.*

Remark 5.4.3 *Notice that we have started with a general curve in M_1 and ended up with restrictions in order that it is a rolling curve. This seems to contradict the statement made in Sharpe [68] that for each curve there is a rolling map that rolls the manifold on that curve. The next subsection is introduced here to show that there is no contradiction.*

5.4.2 An Important Observation

We consider the two manifolds separately, starting with the Grassmann manifold, and will show that in each case the rolling curve can always be expressed in a more simplified form.

First case: $M_1 = G_{k,n}$

In this case, $G = \mathrm{SO}(n)$ is the connected Lie group that acts transitively on M_1. Fix the point E_0 in $G_{k,n}$ and let K be the isotropy subgroup at E_0, i.e., $K := \{\theta \in \mathrm{SO}(n) : \theta E_0 \theta^\top = E_0\}$. Then

$$K = \left\{ \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix} : \quad \theta_1 \in \mathrm{SO}(k),\, \theta_2 \in \mathrm{SO}(n-k) \right\} \cong \mathrm{SO}(k) \times \mathrm{SO}(n-k). \tag{5.53}$$

The Lie algebra $\mathfrak{so}(n)$ admits a direct sum decomposition (Cartan decomposition)

$$\mathfrak{so}(n) = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is a vector space of $\mathfrak{so}(n)$. More precisely,

$$\mathfrak{k} = \left\{ \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \quad A_1 \in \mathfrak{so}(k), A_2 \in \mathfrak{so}(n-k) \right\} \cong \mathfrak{so}(k) \times \mathfrak{so}(n-k);$$

$$\mathfrak{p} = \left\{ \begin{bmatrix} 0 & B \\ -B^\top & 0 \end{bmatrix} : \quad B \in \mathbb{R}^{k \times (n-k)} \right\}.$$

The following commutation relations hold:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}. \tag{5.54}$$

Notice that $\mathfrak{p}$ is Ad_K-invariant, that is, $Ad_K(\mathfrak{p}) \subset \mathfrak{p}$. Indeed, for every $h \in K$ and $B \in \mathfrak{p}$, we have $Ad_h(B) = hBh^{-1}$. Since the exponential map is surjective for special orthogonal groups, $h = e^A$, for some $A \in \mathfrak{k}$. But then,

$$Ad_h(B) = e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \cdots \in \mathfrak{p}, \text{ since } [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}.$$

Now, let α_1 be any curve in $G_{k,n}$, starting at the point E_0. Due to the transitive action of G on $G_{k,n}$, $\alpha_1(t) = \theta(t)E_0\theta(t)^{-1}$, for some $\theta(t) \in \mathrm{SO}(n)$. We will show that there exists $\theta_1 \in \mathrm{SO}(n)$, such that

1. $\theta_1(t)^{-1}\,\dot{\theta}_1(t) \in \mathfrak{p}$;
2. $\alpha_1(t) = \theta_1(t)E_0\theta_1(t)^{-1}$.

Indeed, since $\theta \in \mathrm{SO}(n)$, $\dot{\theta}(t) = \theta(t)\Omega(t)$, for some $\Omega(t) \in \mathfrak{so}(n)$. Due to the Cartan decomposition above, $\Omega(t)$ decomposes uniquely as $\Omega(t) = A(t) + B(t)$, with $A(t) \in \mathfrak{k}$ and $B(t) \in \mathfrak{p}$. Now define a new curve in $\mathrm{SO}(n)$ by $\theta_1(t) := \theta(t)H(t)$, where $H(t) \in K$ is the unique solution of the initial value problem $\dot{H}(t) = -A(t)H(t)$, $H(0) = I_n$. It turns out that, due to the fact that K keeps E_0 invariant,

$$\theta_1(t)E_0\theta_1(t)^{-1} = \theta(t)H(t)E_0H(t)^{-1}\theta(t)^{-1} = \theta(t)E_0\theta(t)^{-1} =: \alpha_1(t),$$

and, moreover, using the above assumptions we have

$$\begin{aligned}
\dot{\theta}_1(t) &= \dot{\theta}(t)H(t) + \theta(t)\dot{H}(t) = \theta(t)\Omega(t)H(t) - \theta(t)A(t)H(t) \\
&= \theta(t)B(t)H(t) = \theta(t)H(t)H(t)^{-1}B(t)H(t) = \theta_1(t)(H(t)^{-1}B(t)H(t)).
\end{aligned}$$

Finally, since $H(t) \in K$ and $B(t) \in \mathfrak{p}$, the Ad_K-invariance of $\mathfrak{p}$ guarantees that $H(t)^{-1}B(t)H(t) \in \mathfrak{p}$, concluding our proof.

Second case: $M_1 = \mathrm{SO}(n)$

The same arguments can be used when M_1 is the rotation group, but the calculations get more involved because the group G has an extra component in this case.

Indeed, when $M_1 = \mathrm{SO}(n)$, $G = \mathrm{SO}(n) \times \mathrm{SO}(n)$ is a Lie group that acts transitively on M_1 via

$$\begin{array}{rccc}
\phi: & G \times M_1 & \to & M_1 \\
 & ((V,W),R) & \mapsto & VRW^\top
\end{array}.$$

Without loss of generality fix the point $I_n \in \mathrm{SO}(n)$, so that the isotropy subgroup of G at I_n is given by $K := \{(V,V) \in G : \ V \in \mathrm{SO}(n)\}$. The Lie algebra of G, $\mathfrak{so}(n) \oplus \mathfrak{so}(n)$, admits a direct sum decomposition (Cartan type decomposition)

$$\mathfrak{so}(n) \oplus \mathfrak{so}(n) = \mathfrak{k} \oplus \mathfrak{p}, \tag{5.55}$$

where $\mathfrak{k} = \{(\Omega,\Omega) : \ \Omega \in \mathfrak{so}(n)\}$ is the Lie algebra of K and $\mathfrak{p} = \{(\psi,-\psi) : \ \psi \in \mathfrak{so}(n)\}$ is a vector space of $\mathfrak{so}(n) \oplus \mathfrak{so}(n)$. It is easy to see that given $(A,B) \in \mathfrak{so}(n) \oplus \mathfrak{so}(n)$, there exist unique elements in $\mathfrak{k}$ and $\mathfrak{p}$ that confirm (5.55). Indeed,

$$(A,B) = \underbrace{\left(\frac{1}{2}(A+B), \frac{1}{2}(A+B)\right)}_{\in\mathfrak{k}} + \underbrace{\left(\frac{1}{2}(A-B), -\frac{1}{2}(A-B)\right)}_{\in\mathfrak{p}}. \tag{5.56}$$

It is also easy to check that

$$[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}, \quad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p},$$

and, moreover, $\mathfrak{p}$ is Ad_K-invariant. This last statement also uses the fact that the exponential map in $\mathrm{SO}(n)$ is surjective, so that every element of $\mathrm{SO}(n)$ is of the form e^A, for some $A \in \mathfrak{so}(n)$. To check the invariance, let $H = (e^A, e^A) \in K$ and $(\psi, -\psi) \in \mathfrak{p}$. Then, it holds that,

$$Ad_H(\psi,-\psi) = H(\psi,-\psi)H^{-1} = (e^A,e^A)(\psi,-\psi)(e^{-A},e^{-A}) = (e^A\psi e^{-A}, -e^A\psi e^{-A}) \in \mathfrak{p},$$

as required.

Now, let $\alpha_1(t) = V(t)I_nW(t)^\top$, for some $V(t), W(t) \in \mathrm{SO}(n)$, be a curve in $\mathrm{SO}(n)$, satisfying $\alpha_1(0) = I_n$. Our goal now is to show that there exists $(V_1(t), W_1(t)) \in G = \mathrm{SO}(n) \times \mathrm{SO}(n)$, such that, for all t,

1. $(V_1(t)^{-1}\dot{V}_1(t), W_1(t)^{-1}\dot{W}_1(t)) \in \mathfrak{p}$;
2. $\alpha_1(t) = V_1(t)I_nW_1(t)^\top$.

To prove 1., first note that

$$\dot{V}(t) = V(t)\Omega(t), \quad \dot{W}(t) = W(t)\psi(t), \tag{5.57}$$

for some $\Omega(t), \psi(t) \in \mathfrak{so}(n)$, and, according to (5.56),

$$(\Omega(t),\psi(t)) = \left(\frac{1}{2}(\Omega(t)+\psi(t)), \frac{1}{2}(\Omega(t)+\psi(t))\right) + \left(\frac{1}{2}(\Omega(t)-\psi(t)), -\frac{1}{2}(\Omega(t)-\psi(t))\right),$$

with the first term of the second member of this equality belonging to $\mathfrak{k}$ and the second one belonging to $\mathfrak{p}$. Secondly, define a new curve in $G = \mathrm{SO}(n) \times \mathrm{SO}(n)$ by:

$$(V_1(t), W_1(t)) := (V(t), W(t))(Y(t), Y(t)) = (V(t)Y(t), W(t)Y(t)), \tag{5.58}$$

where $Y(t)$ is the unique solution of the initial value problem in $\mathrm{SO}(n)$ given by

$$\begin{cases} \dot{Y}(t) = -\dfrac{1}{2}(\Omega(t)+\psi(t))Y(t) \\ Y(0) = I_n \end{cases}. \tag{5.59}$$

Then, using the identities in (5.57), (5.58) and (5.59), we can write the following where, for the sake of simplicity, we don't specify the dependency on t:

$$\begin{aligned}
(\dot{V}_1, \dot{W}_1) &\overset{(5.58)}{=} (\dot{V}Y + V\dot{Y}, \dot{W}Y + W\dot{Y}) \\
&\underset{(5.59)}{\overset{(5.57)}{=}} \left(V\Omega Y - \frac{1}{2}V\Omega Y - \frac{1}{2}V\psi Y, W\psi Y - \frac{1}{2}W\Omega Y - \frac{1}{2}W\psi Y\right) \\
&= \left(V\frac{1}{2}(\Omega-\psi)Y, W\left(-\frac{1}{2}(\Omega-\psi)\right)Y\right) \\
&= \left(VYY^{-1}\frac{1}{2}(\Omega-\psi)Y, WYY^{-1}\left(-\frac{1}{2}(\Omega-\psi)\right)Y\right).
\end{aligned}$$

Defining $B(t) := \frac{1}{2}(\Omega(t) - \psi(t))$ and $Y(t) = e^{-C(t)}$, $C(t) \in \mathfrak{so}(n)$, we get

$$\left(\dot V_1(t), \dot W_1(t)\right) = \left(V_1(t)e^{C(t)}B(t)e^{-C(t)}, W_1(t)e^{C(t)}(-B(t))e^{-C(t)}\right),$$

that is, $V_1(t)^{-1}\dot V_1(t) = e^{C(t)}B(t)e^{-C(t)} \in \mathfrak{so}(n)$ and $W_1(t)^{-1}\dot W_1(t) = e^{C(t)}(-B(t))e^{-C(t)} \in \mathfrak{so}(n)$. So,

$$\left(V_1(t)^{-1}\dot V_1(t), W_1(t)^{-1}\dot W_1(t)\right) \in \mathfrak{p},$$

as required in 1.. To check 2., just note that

$$V_1(t)I_nW_1(t)^\top \overset{(5.59)}{=} V(t)Y(t)I_nY(t)^\top W(t)^\top = V(t)I_nW(t)^\top = \alpha_1(t).$$

5.4.3 Rolling $G_{k,n} \times \mathrm{SO}(n)$ along Geodesics

For the special situation where the control functions are constant $(n \times n)$ skew-symmetric matrices, let us say, $\Omega_U(t) = \Omega_U = \begin{bmatrix} 0 & \Omega_2 \\ -\Omega_2^\top & 0 \end{bmatrix}$, $\Omega_2 \in \mathbb{R}^{k\times(n-k)}$ and $\Omega_V(t) = \Omega_V$, the kinematic equations (5.52) can be solved explicitly and

$$\begin{cases} U(t) = \Theta_0 e^{-t\Omega_U}\Theta_0^\top \\ V(t) = R_0 e^{-t\Omega_V}R_0^\top \\ W(t) = e^{t\Omega_V} \\ X(t) = -t\Theta_0\left[\Omega_U, E_0\right]\Theta_0^\top \\ Y(t) = -2tR_0\Omega_V \end{cases}. \tag{5.60}$$

In this case, the rolling curve

$$t \mapsto \alpha_1(t) = \left(\Theta_0 e^{-t\Omega_U}E_0e^{t\Omega_U}\Theta_0^\top, R_0e^{-2t\Omega_V}\right) \tag{5.61}$$

is a geodesic on $G_{k,n} \times \mathrm{SO}(n)$, passing through P_0 at $t = 0$ with velocity

$$\dot\alpha_1(0) = \left(\Theta_0\left[-\Omega_U, E_0\right]\Theta_0^\top, -2R_0\Omega_V\right) = \left(\Theta_0\left[E_0, \Omega_U\right]\Theta_0^\top, 2R_0\Omega_V^\top\right)$$

and, consequently,

$$t \mapsto \alpha_0(t) = P_0 + Z(t) = P_0 + (X(t), Y(t)) = P_0 + t\left(\Theta_0\left[E_0, \Omega_U\right]\Theta_0^\top, 2R_0\Omega_V^\top\right)$$

is also a geodesic in the affine tangent space $T_{P_0}^{\mathrm{aff}}\left(G_{k,n} \times \mathrm{SO}(n)\right)$ satisfying $\alpha_0(0) = P_0$. The second statement is obvious since a geodesic in the affine space is a straight line. The first statement can be checked differentiating α_1 twice and noticing that $\ddot\alpha_1(t)$ belongs to $\left(T_{\alpha_1(t)}\left(G_{k,n} \times \mathrm{SO}(n)\right)\right)^\perp$. Indeed, differentiating (5.61), with respect to t, we obtain that

$$\dot{\alpha}_1(t) = \left(\Theta_0 e^{-t\Omega_U}\left[E_0, e^{t\Omega_U}\Omega_U e^{-t\Omega_U}\right] e^{t\Omega_U}\Theta_0^\top, R_0 e^{-2t\Omega_V}(-2\Omega_V)\right). \tag{5.62}$$

Then, using the fact that $e^{t\Omega_U}\Omega_U e^{-t\Omega_U} = \Omega_U$ and the anticommutativity of the matrix commutator, we can write

$$\dot{\alpha}_1(t) = \left(\Theta_0 e^{-t\Omega_U}\left[-\Omega_U, E_0\right] e^{t\Omega_U}\Theta_0^\top, R_0 e^{-2t\Omega_V}(-2\Omega_V)\right) \in T_{\alpha_1(t)}\left(G_{k,n}\times \mathrm{SO}(n)\right) \tag{5.63}$$

which, is in accordance with (3.40). Therefore, differentiating (5.63), with respect to t, a few computations leads to

$$\ddot{\alpha}_1(t) = \left(\Theta_0 e^{-t\Omega_U}\left[\left[E_0, \Omega_U\right], \Omega_U\right] e^{t\Omega_U}\Theta_0^\top, R_0 e^{-2t\Omega_V}(4\Omega_V^2)\right) \tag{5.64}$$

Hence, taking into account that

$$\left[\left[E_0, \Omega_U\right], \Omega_U\right] = \begin{bmatrix} -2\Omega_2\Omega_2^\top & 0 \\ 0 & 2\Omega_2^\top\Omega_2 \end{bmatrix},$$

with $-2\Omega_2\Omega_2^\top \in \mathfrak{s}(k)$ and $2\Omega_2^\top\Omega_2 \in \mathfrak{s}(n-k)$, we have that

$$\ddot{\alpha}_1(t) = \left(\Theta_0 e^{-t\Omega_U}\begin{bmatrix} -2\Omega_2\Omega_2^\top & 0 \\ 0 & 2\Omega_2^\top\Omega_2 \end{bmatrix} e^{t\Omega_U}\Theta_0^\top, R_0 e^{-2t\Omega_V}(4\Omega_V^2)\right) \tag{5.65}$$

which, is in accordance with (3.42). So, $\ddot{\alpha}_1(t)$ belongs to $\left(T_{\alpha_1(t)}\left(G_{k,n}\times \mathrm{SO}(n)\right)\right)^\perp$. Therefore, the covariant derivative of $\dot{\alpha}_1$, for all t, is identically zero and, thus, (5.61) is a geodesic on $G_{k,n}\times \mathrm{SO}(n)$, passing through P_0 at $t=0$. We summarize the previous in the following corollary of Theorem 5.4.1.

Corollary 5.4.1 *If the control functions Ω_U and Ω_V are constant skew-symmetric matrices, then*

$$h(t) = \left(\Theta_0 e^{t\Omega_U}\Theta_0^\top, R_0 e^{t\Omega_V} R_0^\top, e^{-t\Omega_V}, -t\Theta_0\left[\Omega_U, E_0\right]\Theta_0^\top, -2tR_0\Omega_V\right)$$

is the rolling map of $G_{k,n}\times \mathrm{SO}(n)$, without slipping and twisting, along the geodesic

$$t \mapsto \alpha_1(t) = \left(\Theta_0 e^{-t\Omega_U} E_0 e^{t\Omega_U}\Theta_0^\top, R_0 e^{-2t\Omega_V}\right) \in G_{k,n}\times \mathrm{SO}(n) \tag{5.66}$$

with development curve

$$t \mapsto \alpha_0(t) = P_0 + Z(t) = P_0 + (X(t), Y(t)) = P_0 + t\left(\Theta_0\left[E_0, \Omega_U\right]\Theta_0^\top, 2R_0\Omega_V^\top\right), \tag{5.67}$$

also a geodesic in the affine tangent space $T_{P_0}^{aff}\left(G_{k,n}\times \mathrm{SO}(n)\right)$, satisfying $\alpha_0(0) = \alpha_1(0) = P_0$.

5.5 Controllability Aspects of Rolling Motions

Controllability deals with the ability of a system with noholonomic constraints to generate all the forbidden motions. In this section we deal with controllability aspects for the pure rolling of the Grassmann manifold $G_{k,n}$ over its affine tangent space at a point, and consider two aspects of the same problem. The first deals with the controllability of the kinematic equations and the second is a constructive proof of controllability.

The controllability of the kinematic equations for the rolling of $\mathrm{SO}(n)$ has already been proved in Marques [53]. So, we concentrate here on the Grassmann manifold only.

5.5.1 Controllability of the Kinematic Equations for the Pure Rolling of $G_{k,n}$

In which concerns the Grassmann manifold rolling over the affine tangent space at a point P_0,

$$M_1 = G_{k,n}, \;\; M_0 = T_{P_0}^{\mathrm{aff}} G_{k,n} := P_0 + T_{P_0} G_{k,n}, \;\; \overline{M} = \mathfrak{s}(n),$$

and the isometry group of $\overline{M}$ is the semi-direct product

$$\overline{G} = \mathrm{SO}(n) \ltimes \mathfrak{s}(n).$$

Elements in $\overline{G}$ are represented by pairs (U,X), and the action of $\overline{G}$ on $\mathfrak{s}(n)$ is defined by $(U,X)S = USU^\top + X$.

Assume, without loss of generality, that $P_0 = E_0 = \left[\begin{smallmatrix} I_k & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Notice that $G_{k,n} \cap T_{P_0}^{\mathrm{aff}} G_{k,n} = \{P_0\}$. A rolling map consists of a curve in $\overline{G}$ whose velocity vector field is restricted to a certain distribution due to the nonholonomic constraints of no-slip and no-twist. This distribution characterizes the kinematic equations of the rolling motion.

- **Kinematic Equations**

From (5.52) and defining the matrices

$$A(t) := -\Omega_U(t) = \begin{bmatrix} 0 & -\Omega_2(t) \\ \Omega_2^\top(t) & 0 \end{bmatrix} \quad \text{and} \quad B(t) := -[\Omega_U(t), P_0] = \begin{bmatrix} 0 & \Omega_2(t) \\ \Omega_2^\top(t) & 0 \end{bmatrix},$$

with $t \mapsto \Omega_2(t) \in \mathbb{R}^{k\times(n-k)}$, it is possible to conclude the following result that has already been derived in Hüper and Silva Leite [31], and that will be the starting point for the main results of this section.

Theorem 5.5.1 *If (U,X) is the solution of the following coupled system of differential equations (the kinematic equations)*

$$\begin{cases} \dot{U}(t) = U(t)A(t) \\ \dot{X}(t) = B(t) \end{cases} \tag{5.68}$$

and satisfying $(U(0), X(0)) = (I_n, 0_n)$, *then* $t \mapsto h(t) = (U^\top(t), X(t)) \in \overline{G}$ *is a rolling map along the curve* $t \mapsto \alpha_1(t) = UP_0U^\top(t) \in G_{k,n}$, *with development curve* $t \mapsto \alpha_0(t) = P_0 + X(t) \in T^{aff}_{P_0}G_{k,n}$.

Remark 5.5.1 *When* Ω_2 *is constant, the kinematic equations can be solved explicitly to obtain*

$$\begin{cases} U(t) = e^{tA} \\ X(t) = tB \end{cases}, \tag{5.69}$$

$$\text{with } A = \begin{bmatrix} 0 & -\Omega_2 \\ \Omega_2^\top & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & \Omega_2 \\ \Omega_2^\top & 0 \end{bmatrix} \text{ constant matrices.}$$

In this case the rolling curve $\alpha_1(t) = e^{tA}P_0e^{-tA}$ *is a geodesic in* $G_{k,n}$ *and its development* $\alpha_0(t) = P_0 + tB$ *is a geodesic in* $T^{aff}_{P_0}G_{k,n}$.

It is clear that the choice of the function Ω_2 completely determines the rolling curve (or equivalently its development, since $\alpha_0(t) = h(t)\alpha_1(t)$). So, the kinematic equations (5.68) may be seen as a control system, with unrestricted controls given by the entries of the matrix function Ω_2, evolving on the Lie group $G = \mathrm{SO}(n) \times \mathbb{V}$, where $\mathbb{V} = T_{P_0}G_{k,n}$ is an additive Lie group. A natural question to ask is whether or not this system is controllable. This issue will be addressed bellow, following a procedure that has been used to prove controllability of the rolling sphere in Jurdjevic [35] and Zimmerman [75].

• Controllability of the Kinematic Equations

In the language of geometric control theory, the kinematic equations (5.68) form a sub-Riemannian control system without drift, evolving on the connected Lie group $G = \mathrm{SO}(n) \times \mathbb{V}$, where $\mathbb{V} = T_{P_0}G_{k,n}$. The Lie algebra of this group is $\mathscr{L}(G) = \mathfrak{so}(n) \oplus \mathbb{V}$, equipped with the Lie bracket which is the commutator in the first component and the trivial bracket in the second component of this direct sum. Controllability means that every two points in G can be joined by trajectories of the system in finite time.

In which concerns proving controllability of the kinematic equations, the first thing that needs to be checked is the algebraic property, known in the literature as *bracket generating property*. This is a necessary condition for controllability but, in general, it is not sufficient. We refer to Jurdjevic and Sussmann [37] and Jurdjevic [35] for details concerning controllability of systems evolving on Lie groups. However, in the present situation, we can use a pioneer result about control systems on Lie groups which guarantees that, under some conditions, the bracket generating property is equivalent to controllability.

Theorem 5.5.2 below is a paraphrase of Theorem 7.1 in Jurdjevic and Sussmann [37]. Although the statement of the theorem in Jurdjevic and Sussmann [37] is for right-invariant control systems, it is also true for left-invariant control systems, since these two classes of control systems are equivalent via inversion. More precisely,

$$\dot{Z} = ZW(t) \Longleftrightarrow \dot{Y} = -W(t)Y, \quad \text{where } Y = Z^{-1}.$$

Theorem 5.5.2 *A left-invariant control system without drift and unrestricted controls, evolving on a connected Lie group G, is controllable if and only if the control vector fields generate the Lie algebra of G, i.e., satisfy the bracket generating property.*

It is not clear from the kinematic equations (5.68) that we are in the presence of a control system that fits the conditions of this theorem, namely that it is a left-invariant control system. To convince the readers that this is the case, we rewrite (5.68) in the form

$$\dot{Z}(t) = Z(t)\underbrace{\left(\sum_{i=1}^{r} u_i(t)W_i\right)}_{W(t)}, \quad Z \in G, \quad W_i \in \mathscr{L}(G),$$

where $r = k(n-k)$. For that, define Z in terms of U and X, and W in terms of the matrices A and B, as block diagonal matrices:

$$\begin{aligned} Z &:= \operatorname{diag}\left(U, \left[\begin{array}{c|c} I_n & X \\ \hline 0_n & I_n \end{array}\right]\right), \\ W &:= \operatorname{diag}\left(A, \left[\begin{array}{c|c} 0_n & B \\ \hline 0_n & 0_n \end{array}\right]\right). \end{aligned} \tag{5.70}$$

With these identifications, a simple calculation shows that the kinematic equations (5.68) can be written as

$$\dot{Z}(t) = Z(t)W(t), \tag{5.71}$$

which is a left-invariant control system without drift, evolving on the connected Lie group $G = \mathrm{SO}(n) \times \mathbb{V}$. So, according to Theorem 5.5.2, all we need to prove is that the control vector fields generate the Lie algebra of G. To do this without getting too weird notations, we work with the first representation of the kinematic equations, i.e., with equations (5.68).

Let $E_{i,j}$ denote the square matrix with entry (i,j) equal to 1 and all other entries equal to 0. Define the elementary skewsymmetric matrices $A_{i,j} := E_{i,j} - E_{j,i}$, and the elementary symmetric matrices $B_{i,j} := E_{i,j} + E_{j,i}$. A canonical basis for the Lie algebra $\mathscr{L}(G)$ is defined as:

$$\{(A_{i,j},0),\ 1 \le i < j \le n\} \cup \{(0,B_{i,k+j}),\ i = 1,\cdots,k;\ j = 1,\cdots,n-k\}. \tag{5.72}$$

The left-invariant control vector fields in (5.68) can be identified with the following elements in $\mathscr{L}(G)$:

$$\{(A_{i,k+j},B_{i,k+j}),\ i = 1,\cdots,k;\ j = 1,\cdots,n-k\}. \tag{5.73}$$

Note that $\dim(\mathscr{L}(G)) = k(n-k) + n(n-1)/2$, while the system has only $k(n-k)$ control functions, which are the entries of the matrix function U. So, we are in the presence of an underactuated control system. The following commutator properties, where δ_{ij} denotes the Kronecker delta (which is 1 if $i=j$ and 0 if $i\neq j$), will be important:

$$\left[A_{i,j},A_{f,l}\right]=\delta_{il}A_{j,f}+\delta_{jf}A_{i,l}-\delta_{if}A_{j,l}-\delta_{jl}A_{i,f}. \tag{5.74}$$

In our situation, to prove that the kinematic equations (5.68) are controllable reduces to showing that every element in the canonical basis (5.72) can be written as linear combinations of the $k(n-k)$ control vector fields in (5.73) and their Lie brackets.

Theorem 5.5.3 *The control vector fields in (5.68) are bracket generating, except when $k(n-k)=1$.*

Proof. It is enough to show that every element in the canonical basis (5.72) can be written as linear combinations of the $k\,(n-k)$ elements in (5.73) and their Lie brackets. Recall that if (Y_1,Z_1) and (Y_2,Z_2) belong to $\mathscr{L}(G)=\mathfrak{so}(n)\oplus\mathbb{V}$, then

$$[(Y_1,Z_1),(Y_2,Z_2)]_{\mathscr{L}(G)}=([Y_1,Y_2]_{\mathfrak{so}(n)},0). \tag{5.75}$$

Using (5.74) and (5.75), we show that all the basis elements can be obtained by, at most, second order brackets of control vector fields in (5.73).
First, we generate basis elements of the form $(A_{i,j},0)$ and $(A_{i,k+j},0)$:

$$\begin{aligned}&\text{For } 1\leq i<j\leq k, \text{ and any } l\in\{1,\cdots,n-k\},\\ &(A_{i,j},0)=-\left[(A_{i,k+l},B_{i,k+l}),(A_{j,k+l},B_{j,k+l})\right];\end{aligned} \tag{5.76}$$

$$\begin{aligned}&\text{For } 1\leq i,j\leq n-k, \text{ and any } m\in\{1,\cdots,k\},\\ &(A_{k+i,k+j},0)=\left[(A_{m,k+j},B_{m,k+j}),(A_{m,k+i},B_{m,k+i})\right].\end{aligned} \tag{5.77}$$

Second, we generate basis elements of the form $(A_{i,k+j},0)$ using elements from (5.73) and from (5.76):

$$\begin{aligned}&\text{For } i=1,\cdots,k;\ j=1,\cdots,n-k,\\ &(A_{i,k+j},0)=\left[(A_{i,l},0),(A_{l,k+j},B_{l,k+j})\right].\end{aligned} \tag{5.78}$$

Finally, we generate elements of the form $(0,B_{i,k+j})$ using elements from (5.73) and from (5.78):

$$\begin{aligned}&\text{For } i=1,\cdots,k; j=1,\cdots,n-k,\\ &(0,B_{i,k+j})=(A_{i,k+j},B_{i,k+j})-(A_{i,k+j},0).\end{aligned} \tag{5.79}$$

This completes the decoupling and proves the statement.
Notice that we have to exclude the situation when $k(n-k)=1$, or equivalently $k=1$ and $n=2$, since in this case $\dim(\mathscr{L}(G))=2$ and there is only one control vector field that can't generate a 2-dimensional Lie algebra.

□

Corollary 5.5.1 *Whenever $k(n-k) \neq 1$, the control system (5.68), describing the pure rolling motions of the Grassmann manifold $G_{k,n}$ over the affine tangent space at the point P_0, is controllable in $G = \mathrm{SO}(n) \times T_{P_0}G_{k,n}$.*

This is the counterpart of a result about controllability of the rolling n-sphere that can be found, for instance, in Jurdjevic [35] and Zimmerman [75].

5.5.2 Constructive Proof of Controllability for the Grassmann Manifold

Due to the nonholonomic constraints of the rolling, there are some motions resulting from the action of $\overline{G}$ on $G_{k,n}$ that are forbidden. These are the slips and the twists. A constructive proof of controllability for this rolling system consists in showing how the forbidden motions can be generated from admissible motions, that is by rolling without twisting and without slipping. We first give precise definitions of the forbidden motions.

Definition 5.5.1 *A slip is any motion of $G_{k,n}$ that results from the action of elements in $\overline{G}$ of the form (I_n, X), where $X \in T_{P_0}G_{k,n}$. That is, a slip is a pure translation (in the embedding space) by the vector X.*

Definition 5.5.2 *A twist is any motion that results from the action of elements in $\overline{G}$ that keep P_0 invariant.*

Remark 5.5.2 *Notice that for (U,X) to keep P_0 invariant we must have $UP_0U^\top + X = P_0$, and this identity implies that $P_0 - X \in G_{k,n} \cap T_{P_0}^{aff}G_{k,n}$. But the only point that belongs to this intersection is P_0, so $X = 0$. Moreover, we must have $UP_0U^\top = P_0$, that is, U belongs to the isotropy subgroup of $SO(n)$ at P_0, which is defined as*

$$K := \{U \in \mathrm{SO}(n), \text{ such that } UP_0U^\top = P_0\}.$$

We conclude that twists are generated by elements in G of the form $(U,0)$, with $U \in K$. Also notice that elements in K have the following structure:

$$K = \left\{ U = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}, U_1 \in \mathrm{SO}(k), U_2 \in \mathrm{SO}(n-k) \right\}.$$

A natural question to ask at this point is: how can we generate forbidden motions? The generation of the forbidden motions using only admissible motions is similar to what happens for the rolling sphere. Cartan decompositions of the Lie algebra $\mathfrak{so}(n)$ and corresponding decompositions of $\mathrm{SO}(n)$ play a crucial role to address this problem, but for a particular case, such as the Grassmann manifold, those decompositions only appear implicitly. We are inspired by the work of Kleinsteuber et al. [41] to generate twists and by the work of Biscolla [6] to generate slips.

How to Generate Forbidden Motions

• Generating Twists

It is known that any element in $\mathrm{SO}(n)$ can be written as a finite product of Givens rotations, which are elements of the form $e^{\tau A_{i,j}}$, $\tau \in \mathbb{R}$ (see, for instance, Kleinsteuber et al. [41] for some details). So, every twist can also be decomposed as a finite product of elements of the form

$$\left[\begin{array}{c|c} e^{\tau_1 A_{i,j}} & 0 \\ \hline 0 & e^{\tau_2 A_{k+l,k+m}} \end{array}\right], \tag{5.80}$$

where $1 \leq i < j \leq k$ and $1 \leq l < m \leq n-k$.

In order to generate a twist out of admissible motions it is enough to show that each one of the block diagonal elements in (5.80) can be decomposed into products of Givens rotations generated by elements of the form $A_{r,k+s}$, for $r = 1, \cdots, k$ and $s = 1, \cdots, n-k$, so that the sum of all angles of rotation adds up to zero. Note that these are the elements in the Lie algebra of $\mathrm{SO}(n)$ related to the control vector fields.

In order to show that this is indeed possible, we first prove the following result.

Proposition 5.5.1 *Let A, B and C be any three square matrices of the same arbitrary order that satisfy the following commuting relations:*

$$[A,B] = C \;\; and \;\; [A,C] = -B. \tag{5.81}$$

Then, for any real parameter τ,

$$\begin{aligned} e^{\tau C} &= e^{(\pi/2)A} e^{\tau B} e^{-(\pi/2)A} \\ &= e^{-(\pi/2)A} \, e^{-\tau B} \, e^{(\pi/2)A}, \end{aligned} \tag{5.82}$$

and, consequently,

$$e^{\tau C} = e^{(\pi/2)A} \, e^{(\tau/2)B} \, e^{-\pi A} \, e^{-(\tau/2)B} \, e^{(\pi/2)A}. \tag{5.83}$$

Proof. The proof of the identity (5.83) is based on the commuting relations above and on properties of the exponential mapping, including the Campbell-Hausdorff formula

$$e^{tA} B e^{-tA} = e^{t\,\mathrm{ad}_A} B = \sum_{i=0}^{+\infty} \frac{t^i}{i!} \mathrm{ad}_A^i B,$$

where ad_A is the adjoint operator defined by $\mathrm{ad}_A B := [A,B]$, and $\mathrm{ad}_A^i B := \mathrm{ad}_A^{i-1} (\mathrm{ad}_A B)$, for $i = 2, 3, \cdots$. Indeed, when the commuting relations (5.81) hold, then

$$e^{tA} B e^{-tA} = B \cos t + C \sin t,$$

and so, choosing $t = \pm\pi/2$, we get

$$C = e^{\pm(\pi/2)A}\,(\pm B)\,e^{\mp(\pi/2)A},$$

which implies

$$e^{\tau C} = e^{\pm(\pi/2)A}\,e^{\pm\tau B}\,e^{\mp(\pi/2)A}.$$

The identity (5.83) is obtained from the previous, with a convenient choice of the signals, as follows.

$$\begin{aligned} e^{\tau C} &= e^{(\tau/2)C}\,e^{(\tau/2)C} \\ &= e^{(\pi/2)A}\,e^{(\tau/2)B}\,e^{-\pi A}\,e^{-(\tau/2)B}\,e^{(\pi/2)A}. \end{aligned}$$

□

When A, B and C are skewsymmetric matrices of the form $A_{i,j}$, each term in (5.83) is a plane rotation (in the plane spanned by e_i and e_j) and Figure 5.2 illustrates the rectangle, in that plane, that is the development curve of the rolling motion that performs the twist $e^{\tau C}$.

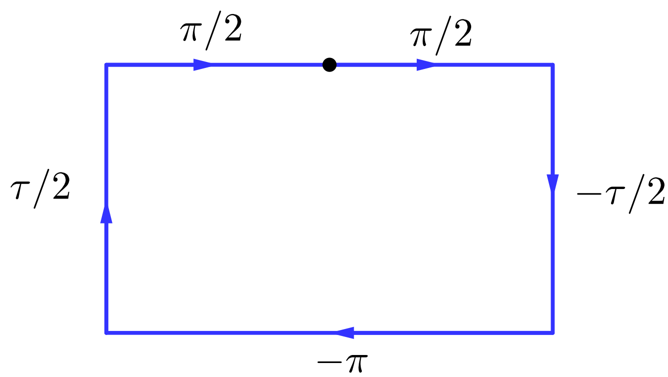

Fig. 5.2 Development curve that performs a twist.

Corollary 5.5.2 *The following identities hold:*

a) *For* $1 \leq i < j \leq k$, $l \in \{1, \cdots, n-k\}$ *and* $\tau_1 \in \mathbb{R}$,

$$\begin{aligned} e^{\tau_1 A_{i,j}} &= e^{(\pi/2)A_{j,k+l}}\,e^{(\tau_1/2)A_{i,k+l}}\,e^{-\pi A_{j,k+l}} \\ &\quad e^{-(\tau_1/2)A_{i,k+l}}\,e^{(\pi/2)A_{j,k+l}}. \end{aligned}$$

b) *For* $1 \leq l < m \leq n-k$, $i \in \{1, \cdots, k\}$ *and* $\tau_2 \in \mathbb{R}$,

$$\begin{aligned} e^{\tau_2 A_{k+l,k+m}} &= e^{(\pi/2)A_{i,k+m}}\,e^{(\tau_2/2)A_{i,k+l}}\,e^{-\pi A_{i,k+m}} \\ &\quad e^{-(\tau_2/2)A_{i,k+l}}\,e^{(\pi/2)A_{i,k+m}}. \end{aligned}$$

Proof. This is an immediate consequence of the fact that the triples

$$\{A_{j,k+l}, A_{i,k+l}, A_{i,j}\} \text{ and } \{A_{i,k+m}, A_{i,k+l}, A_{k+l,k+m}\}$$

satisfy the commuting relations (5.81), as the triple $\{A,B,C\}$ does.

□

• Generating Slips

Without loss of generality, we may restrict to the situation of a slip from the point P_0 to the point $Q_1 = P_0 + \tau B_{1,k+1}$, for some $\tau > 0$. This is a pure translation in the ambient space by the vector $\tau B_{1,k+1} \in T_{P_0}G_{k,n}$. The distance from P_0 to Q_1 is easily computed, since we know that $t \mapsto \beta(t) = P_0 + tB_{1,k+1}$ is the geodesic that satisfies $\beta(0) = P_0$ and $\beta(\tau) = Q_1$, and

$$d(P_0,Q_1) = \int_0^\tau \parallel \dot{\beta}(t) \parallel \, dt = \tau\sqrt{\operatorname{tr}(B_{1,k+1}^2)} = \tau\sqrt{2}.$$

There are two situations to consider, the first one when τ is a multiple of 2π, and the second one otherwise. The corresponding development curves are shown in Figure 5.3, and details about this construction are given in what follows.

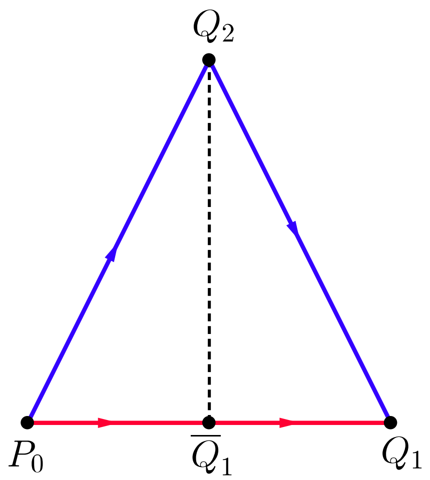

Fig. 5.3 Triangle showing the development curve for the rolling map that generates a slip from P_0 to Q_1.

1. τ is a multiple of 2π, say $\tau = 2\pi l$.

In this case, the slip can be generated by rolling (without slip or twist) $G_{k,n}$ along a geodesic arc so that its development curve is the geodesic arc in $T_{P_0}^{\text{aff}}G_{k,n}$ that joins P_0 to Q_1. This geodesic arc is represented in the previous figure by the red line. The corresponding rolling map is

$$h(t) = \left(e^{tA_{1,k+1}}, t\,B_{1,k+1}\right).$$

At $t = 2\pi l$, we have

$$h(2\pi l) = \left(e^{2\pi l A_{1,k+1}}, 2\pi l\,B_{1,k+1}\right) = (I_n, 2\pi l\,B_{1,k+1}).$$

So, we have generated the slip from P_0 to Q_1 by rolling without twist or slip.

2. τ is not a multiple of 2π.

In this case, we can generate the slip by rolling (without slip or twist) along a broken geodesic composed of two geodesic arcs of equal length. These arcs, which are represented in the previous figure by the blue lines, form an isosceles triangle together with the geodesic arc joining P_0 to Q_1. There are many possible choices for the plane where such triangle lives, and one possible choice for the third vertex Q_2 is

$$Q_2 = \overline{Q}_1 + \tau_1 B_{1,k+2} = P_0 + \frac{\tau}{2} B_{1,k+1} + \tau_1 B_{1,k+2},$$

where $\overline{Q}_1$ is the midpoint between P_0 and Q_1 and τ_1 must be chosen so that

$$d(P_0, Q_2) = d(Q_1, Q_2) = 2\sqrt{2}\pi r, \text{ for some } r \in \mathbb{N}.$$

The first equality follows from the requirement that the triangle is isosceles, and the second equality ensures that rolling along each one of the equal sides of the triangle generates a slip. A simple calculation shows that τ_1 must satisfy

$$\tau_1^2 = 4\pi^2 r^2 - \tau^2/4, \quad \text{for some } r \in \mathbb{N}.$$

Rolling $G_{k,n}$ along that broken geodesic arc will not affect its orientation and generates the slip from P_0 to Q_1, as required.

Therefore, we have shown how the forbidden motions of twist and slip can be generated just by rolling without twisting and without slipping.

Chapter 6

Solving Interpolation Problems using Rolling Motions

6.1 Introduction

We return to interpolation problems on manifolds in order to present a simpler approach based on rolling motions that somehow overcomes the drawbacks of other interpolating techniques, such as the De Casteljau Algorithm previously presented. In this chapter, the basic idea to solve an interpolation problem on a non-Euclidean manifold is to project the data from that manifold to a simpler manifold, where the problem may be easily solved, and then reversing the process by projecting the resulting interpolation curve to the original manifold. As we will explain, this can be done successfully using rolling motions of the given manifold over its affine tangent space at a point. Since there are several useful algorithms to solve interpolation problems on vector spaces, and consequently also on affine spaces, what needs to be explained is how to project data from one manifold to another one using rolling motions. The interesting observation is that this technique is coordinate-free and it produces an interpolating curve that is given in closed form. This is particularly useful for solving real problems, such as motion planning for robots or other mechanical systems whose configuration spaces contain components which are Lie groups or symmetric spaces, or to reconstruct a video from several images of a scene, using the Normalized Essential manifold.

For the sake of simplicity, in this chapter we only give details for a $\mathscr{C}^2$-smooth interpolating curve that solves a two-boundary value problem of Hermite type, but more general problems can be solved in a similar way, although computationally more expensive.

6.1.1 Literature Review

Interpolation schemes for manifolds with a complicated geometry, based on the idea of projecting the interpolating data to a simpler manifold using local diffeomorphisms, have already appeared in the literature before. A scheme that combines rolling with other local diffeomorphisms was proposed for the first time in Jupp and Kent [34] for the S^2 sphere and in Hüper and Silva Leite [30] to solve spherical interpolating problems on any dimension. In this context, the term unwrapping was used to describe how the diffeomorphism mapped data on the n-sphere to data on the tangent hyperplane at a point and the term wrapping to describe the inverse diffeomorphism. These unwrapping/wrapping techniques combined with rolling/unrolling have been applied to SO(3) in Shen et al. [69], to other Euclidean submanifolds in Hüper and Silva Leite [31], to some pseudo-Euclidean manifolds in Marques [53] and to ellipsoids equipped with a left-invariant metric in Krakowski and Silva Leite [43]. This combination of techniques reduces distortions introduced by flattening the manifold (see, for instance, Vemulapalli and Chellapa [72]).

6.1.2 Main Contributions

After formulating the interpolation problem that will be under study, we first unify existing solutions of the problem for manifolds embedded in Euclidean spaces with the additional condition that the first component of the isometry group, acts transitively on the rolling manifold and keeps this manifold invariant. This condition was seemingly overlooked in Hüper and Silva Leite [31], the first work to propose an interpolation algorithm on manifolds. The main advantage of the solution presented here is that the interpolating curve is given in closed form. We also confirm that the proposed algorithm works for the manifold $G_{k,n} \times \mathrm{SO}(n)$, which is isometrically embedded in the matrix space $\overline{M} = \mathfrak{s}(n) \times \mathbb{R}^{n\times n}$, endowed with the Euclidean Riemannian metric defined in (3.39). The main result concerning interpolating curves for $\mathrm{SO}(n)$ and for the Grassmann manifolds has already appeared in Hüper and Silva Leite [31], but its proof was only given for the first case. So, for the sake of completeness, we include here details of the proof for the Grassmann manifold. The proposed algorithm was implemented in Hüper and Silva Leite [31] for spheres only, using rolling maps and stereographic projection on the affine tangent space at the south pole. Here we implement the algorithm on the rotation group and on the Grassmann manifold, preparing the ground for applications in the Normalized Essential manifold, and the unwrapping/wrapping techniques are performed with rolling maps. So, contrary to previous works, the implementation of the interpolation algorithm on manifolds presented here is based on rolling motions only.

6.2 Formulation of the Interpolation Problem on Manifolds

Let M_1 be a connected and oriented smooth n-dimensional manifold isometrically embedded in a Riemannian complete m-dimensional manifold $(\overline{M}, \overline{g})$, with $1 \leq n < m$. Given a set of $\ell+1$ distinct points $p_i \in M_1$, with $i = 0, 1, \dots, \ell$, a discrete sequence of $\ell+1$ fixed times t_i, where

$$0 = t_0 < t_1 < \cdots < t_{\ell-1} < t_\ell = \tau, \tag{6.1}$$

and two tangent vectors ξ_0 and ξ_ℓ to M_1 at p_0 and p_ℓ respectively, we intend to solve the interpolation problem stated at the beginning of Chapter 4, with boundary data of Hermite type. We repeat here its formulation:

Problem 6.2.1 *Find a $\mathscr{C}^2$-smooth curve*

$$\gamma\colon [0, \tau] \to M_1 \tag{6.2}$$

satisfying the interpolation conditions:

$$\gamma(t_i) = p_i, \qquad 1 \leq i \leq \ell - 1, \tag{6.3}$$

and the boundary conditions:

$$\begin{aligned} \gamma(0) = p_0, &\quad \gamma(\tau) = p_\ell, \\ \dot{\gamma}(0) = \xi_0 \in T_{p_0}M_1, &\quad \dot{\gamma}(\tau) = \xi_\ell \in T_{p_\ell}M_1. \end{aligned} \tag{6.4}$$

6.3 Solving the Interpolation Problem on Manifolds Embedded in Euclidean Spaces

In order to be able to apply rolling techniques to solve the interpolation problem, we have to make some assumptions.

Assumptions:

1. Assume that the given Riemannian manifolds M_1 and $M_0 = T_{p_0}^{\mathrm{aff}} M_1$ are both isometrically embedded in some Euclidean space $\overline{M}$.

   This is not very restrictive since Whitney's Theorem (Whitney [73]) guarantees that such embedding is always possible.

2. Assume that $\overline{G} = G \ltimes \overline{M}$ is the isometry group of the embedding space.

   In case $\overline{M} = \mathbb{R}^m$, $\overline{G} = \mathrm{SE}(m) = \mathrm{SO}(m) \ltimes \mathbb{R}^m$.

3. Assume that the action of G keeps M_1 invariant and is transitive. Assume further that if $t \to h(t) = (g(t), s(t)) \in \overline{G}$ is a rolling map of M_1 on M_0, along a curve α_1 satisfying $\alpha_1(0) = p_0$, then $s(t) = \alpha_0(t) - p_0$, where α_0 is the development curve.

6.3.1 Algorithm to Solve the Interpolation Problem when $\overline{M} = \mathbb{R}^m$

The algorithm that generates a solution for the interpolation Problem 6.2.1 is based on a mixed technique of rolling/unrolling and unwrapping/wrapping and is described in the following five steps.

Algorithm 6.3.1 (Interpolation Algorithm)

Step 1. *Choose an arbitrary smooth rolling curve $\alpha_1 : [0, \tau] \longrightarrow M_1$, connecting p_0 with p_ℓ, i.e., such that $\alpha_1(0) = p_0$ and $\alpha_1(\tau) = p_\ell$.*

Step 2. *Unwrap the boundary data from M_1 to M_0 by rolling M_1 along α_1, with rolling map $h(t) = (R(t), s(t))$, so that:*

$$\begin{aligned} p_0 = \alpha_1(0) \in M_1 &\mapsto q_0 := \alpha_0(0) = p_0 \in M_0 \\ p_\ell = \alpha_1(\tau) \in M_1 &\mapsto q_\ell := \alpha_0(\tau) \in M_0 \\ \xi_0 \in T_{p_0}M_1 &\mapsto \eta_0 := d_{\alpha_1(0)}h(0)(\xi_0) = \xi_0 \in T_{q_0}(M_0) \\ \xi_\ell \in T_{p_\ell}M_1 &\mapsto \eta_\ell := d_{\alpha_1(\tau)}h(\tau)(\xi_\ell) \in T_{q_\ell}(M_0) \end{aligned}$$

Step 3. *While performing Step 2. in the* $[0,\tau]$ *interval, unwrap the remaining interpolating points* p_i *at* t_i*,* $i=1,\dots,\ell-1$*, from* M_1 *to* M_0*, using a suitable local diffeomorphism* $\phi:\mathscr{V}\subset M_1\longrightarrow\phi(\mathscr{V})\subset M_0$*, with* $\mathscr{V}$ *an open neighbourhood of* p_0*, satisfying the conditions* $\phi(p_0)=p_0$ *and* $d_{p_0}\phi=\mathrm{id}$*, and indexed by the rolling map* h*, so that*

$$\begin{aligned} p_i\mapsto q_i &:= \phi\big(h(t_i)p_i-\alpha_0(t_i)+p_0\big)+\alpha_0(t_i)-p_0 \\ &= \phi\big(R(t_i)p_i\big)+s(t_i). \end{aligned} \tag{6.5}$$

Step 4. *Solve the interpolating problem on* M_0 *for the projected data* $\{q_0,\dots,q_\ell;\eta_0,\eta_\ell\}$*, to generate a* $\mathscr{C}^2$*-smooth curve* $\beta:[0,\tau]\longrightarrow M_0$ *satisfying*

$$\beta(t_i)=q_i,\ i=0,\dots,\ell,\quad \dot\beta(0)=\eta_0 \quad \text{and} \quad \dot\beta(\tau)=\eta_\ell.$$

Step 5. *Wrap* $\beta([0,\tau])$ *onto the manifold* M_1 *using* ϕ^{-1}*, while unrolling* M_1 *along* α_1 *(in reverse time), to produce a curve* γ*, defined by the following explicit formula.*

$$\begin{aligned} \gamma(t) &:= h(t)^{-1}\Big(\phi^{-1}\big(\beta(t)-\alpha_0(t)+p_0\big)+\alpha_0(t)-p_0\Big) \\ &= h(t)^{-1}\Big(\phi^{-1}\big(\beta(t)-s(t)\big)+s(t)\Big). \end{aligned} \tag{6.6}$$

The curve β will be a cubic spline that can be easily obtained, for instance using the classical De Casteljau algorithm presented in Chapter 4. Figure 6.1 illustrates the formula (6.5) when M_1 is the Euclidean 2-sphere.

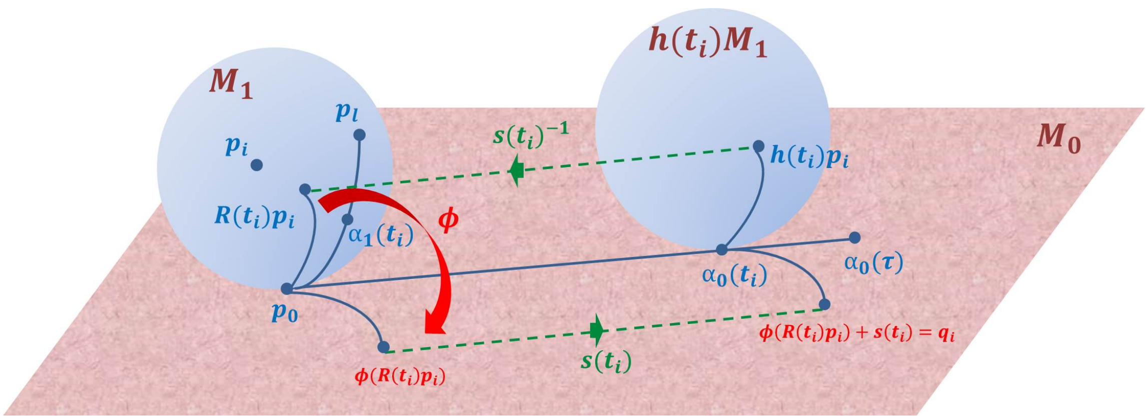

Fig. 6.1 Illustration of (6.5) for the rolling 2-sphere.

Before we prove that this algorithm produces the solution of the formulated problem, we make some remarks.

Remark 6.3.1

1. *The expression (6.5) for the points* q_i*,* $i=1,\dots,\ell-1$*, is well defined. In fact, with the assumption that the orthogonal group* $\mathrm{SO}(m)$ *acts transitively on the rolling manifold* M_1*,* $R(t_i)p_i$ *belongs to* M_1 *and, consequently,* $\phi\big(R(t_i)p_i\big)$ *and* $q_i=\phi\big(R(t_i)p_i\big)+s(t_i)$*,* $i=1,\dots,\ell-1$*, belong to* M_0*.*

2. *The expression (6.5) is also valid for $i = 0$ and for $i = \tau$, as can be easily showed using the conditions on the diffeomorphism ϕ and on the rolling map h. Indeed, for $i = 0$ we have*

$$p_0 \quad \mapsto \quad \phi\big(R(0)p_0\big) + s(0) = \phi\big(p_0\big) + s(0) = \phi\big(p_0\big) = p_0,$$

and for $i = \tau$ we get

$$\begin{aligned} p_l \quad \mapsto \quad & \phi\big(R(\tau)p_\ell\big) + s(\tau) = \phi\big(R(\tau)\alpha_1(\tau)\big) + s(\tau) \\ & = \phi\big(\alpha_0(\tau) - s(\tau)\big) + s(\tau) = \phi(p_0) + s(\tau) = p_0 + s(\tau) = \alpha_0(\tau) = q_l. \end{aligned}$$

3. *It is not necessary that all points p_i are in the neighborhood $\mathscr{V}$. What we have to guarantee is that the diffeomorphism ϕ and the interpolating curve β on M_0 are chosen so that, for all $i = 0,\ldots,\ell$, $R(t_i)p_i$ belongs to $\mathscr{V}$ and, for all $t \in [0,\tau]$, $\beta(t) - \alpha_0(t) + p_0$ belongs to $\phi\big(\mathscr{V}\big)$.*

We are now in conditions to prove the main result of this section.

Theorem 6.3.1 *Under the assumptions on M_1 given at the beginning of this section, the curve $\gamma : [0,\tau] \longrightarrow M_1$ defined by (6.6) solves the Problem 6.2.1.*

Proof. Since $h(t) = (R(t), s(t)) \in \overline{G} = \mathrm{SE}(m)$ is a rolling map of M_1 on M_0 along the smooth rolling curve $\alpha_1(t) = R(t)^{-1}p_0$, joining p_0 (at $t = 0$) to p_ℓ (at $t = \tau$), with development curve $\alpha_0(t) = s(t) + p_0$, then $h(t)^{-1} = (R(t)^{-1}, -R(t)^{-1}s(t))$. Then a simple calculation allows to rewrite the relation (6.6) as

$$\begin{aligned} \gamma(t) &= R(t)^{-1}\left(\phi^{-1}(\beta(t) - s(t)) + s(t)\right) - R(t)^{-1}s(t) \\ &= R(t)^{-1}\phi^{-1}(\beta(t) - s(t)). \end{aligned} \tag{6.7}$$

Therefore, since $\beta(t_i) = q_i$ and (6.5) holds, we have $\beta(t_i) - s(t_i) = \phi\big(R(t_i)p_i\big)$ and, consequently,

$$\begin{aligned} \gamma(t_i) &= R(t_i)^{-1}\phi^{-1}\big(\beta(t_i) - s(t_i)\big) \\ &= R(t_i)^{-1}\phi^{-1}\left(\phi\left(R(t_i)p_i\right)\right) \\ &= p_i, \quad \text{for} \quad i = 0,\ldots,\ell, \end{aligned}$$

that is, the curve γ interpolates the points p_i at time t_i, for all $i = 0,\ldots,\ell$.

To show that γ also satisfies the boundary velocities, take derivatives on both sides of (6.7) to obtain

$$\dot\gamma(t) = \dot R(t)^{-1}(\phi^{-1}\big(\beta(t) - s(t)\big)) + R(t)^{-1}d_{\big(\beta(t)-s(t)\big)}\phi^{-1}\big(\dot\beta(t) - \dot s(t)\big).$$

Now, the no-slip condition (5.31) enables to conclude that $\dot s(t) = R(t)\dot\alpha_1(t)$, and taking into consideration that $\beta(0) = p_0$, $\dot\beta(0) = \eta_0$, $\phi(p_0) = p_0$ and $d_{p_0}\phi = \mathrm{id}$, we obtain

$$\begin{aligned} \dot\gamma(0) &= \dot R(0)^{-1}\phi^{-1}\big(\beta(0) - s(0)\big) + R(0)^{-1}d_{\big(\beta(0)-s(0)\big)}\phi^{-1}\big(\dot\beta(0) - \dot s(0)\big) \\ &= \dot R(0)^{-1}\phi^{-1}(p_0) + \dot\beta(0) - \dot s(0) \\ &= \dot R(0)^{-1}p_0 + \eta_0 - \dot\alpha_1(0) \\ &= \eta_0 = \xi_0. \end{aligned}$$

Furthermore, since $\beta(\tau)=\alpha_0(\tau)=s(\tau)-p_0$, we have that $\beta(\tau)-s(\tau)=p_0$. Therefore, taking also into account that $\dot{\beta}(\tau)=\eta_\ell$, we get

$$\begin{aligned}
\dot{\gamma}(\tau) &= \dot{R}(\tau)^{-1}\phi^{-1}\big(\beta(\tau)-s(\tau)\big)+R(\tau)^{-1}d_{\big(\beta(\tau)-s(\tau)\big)}\phi^{-1}\big(\dot{\beta}(\tau)-\dot{s}(\tau)\big)\\
&= \dot{R}(\tau)^{-1}\phi^{-1}(p_0)+R(\tau)^{-1}\big(\dot{\beta}(\tau)-\dot{s}(\tau)\big)\\
&= \dot{R}(\tau)^{-1}p_0+R(\tau)^{-1}\eta_\ell-\dot{\alpha}_1(\tau)\\
&= R(\tau)^{-1}\eta_\ell=R(\tau)^{-1}R(\tau)\xi_\ell=\xi_\ell.
\end{aligned}$$

Finally, the resulting curve γ is $\mathscr{C}^2$-smooth by construction, since h and ϕ are smooth and β is $\mathscr{C}^2$-smooth. This concludes the proof. □

Remark 6.3.2 *It is clear that the degree of smoothness of the interpolating curve only depends on the degree of smoothness of the curve β. So, this algorithm can be easily extended to generate $\mathscr{C}^k$-smooth interpolating curves, for $k>2$, as long as we are given enough boundary conditions.*

6.3.2 Solving the Interpolation Problem when $M_1=G_{k,n}\times\mathrm{SO}(n)$

For this manifold, the embedding space is $\overline{M}=\mathfrak{s}(n)\times\mathbb{R}^{n\times n}$, endowed with the Euclidean Riemannian metric defined in (3.39) and the first component of the isometry group $\overline{G}:=G\ltimes\overline{M}$ which is $G=\mathrm{SO}(n)\times\mathrm{SO}(n)\times\mathrm{SO}(n)$, acts transitively on $G_{k,n}\times\mathrm{SO}(n)$ and keeps this manifold invariant. So, this manifold fits in the general framework of Euclidean submanifolds that satisfy the assumptions at the beginning of Section 6.3, simply replacing $\mathrm{SO}(n)$ by $\mathrm{SO}(n)\times\mathrm{SO}(n)\times\mathrm{SO}(n)$ and $\mathbb{R}^m$ by $\mathfrak{s}(n)\times\mathbb{R}^{n\times n}$. For the sake of completeness, we give an independent proof that the proposed algorithm indeed solves Problem 6.2.1 for this particular manifold.

In order to achieve this goal, we start with $M_1=G_{k,n}$ followed by $M_1=\mathrm{SO}(n)$. The interpolating points p_i of Problem 6.2.1 will be denoted by S_i when $M_1=G_{k,n}$, and by R_i when $M_1=\mathrm{SO}(n)$.

• Solving in the Grassmann Manifold $G_{k,n}$

Before proving that Algorithm 6.3.1 solves the smooth interpolation problem in $G_{k,n}$, we rewrite some of the formulas derived in the previous Chapter 5 in a more convenient form.

Let $h(t)=(U(t)^\top,X(t))\in\overline{G}=\mathrm{SO}(n)\ltimes\mathfrak{s}(n)$ be a rolling map of $M_1=G_{k,n}$ rolling over $M_0=T^{\text{aff}}_{S_0}G_{k,n}$, with $S_0=\Theta_0E_0\Theta_0^\top\in G_{k,n}$. Then, we can write the following.

- The rolling curve is given by $\alpha_1(t)=U(t)S_0U(t)^\top$ and the development curve is given by $\alpha_0(t)=S_0+X(t)$, with $X(t)\in\mathfrak{s}(n)$.

- The no-slip condition (first equation in (5.51)) can be rewritten as

$$\begin{aligned}
\dot{X}(t) &= -\Theta_0\left[\Omega_U(t), E_0\right]\Theta_0^\top \\
&= -\Theta_0\left[\Theta_0^\top \dot{U}(t)^\top U(t)\Theta_0, E_0\right]\Theta_0^\top \\
&= -\left[\dot{U}(t)^\top U(t), \Theta_0 E_0 \Theta_0^\top\right] \\
&= -\left[\dot{U}(t)^\top U(t), S_0\right] \\
&= \left[S_0, \dot{U}(t)^\top U(t)\right].
\end{aligned}$$

Then, since $\dot{\alpha}_1(t) = \dot{U}(t) S_0 U(t)^\top + U(t) S_0 \dot{U}(t)^\top$, we have that

$$\begin{aligned}
U(t)^\top \dot{\alpha}_1(t) U(t) &= U(t)^\top \left(\dot{U}(t) S_0 U(t)^\top + U(t) S_0 \dot{U}(t)^\top\right) U(t) \\
&= U(t)^\top \dot{U}(t) S_0 + S_0 \dot{U}(t)^\top U(t) \\
&= -\dot{U}(t)^\top U(t) S_0 + S_0 \dot{U}(t)^\top U(t) \\
&= \left[S_0, \dot{U}(t)^\top U(t)\right] = \dot{X}(t).
\end{aligned}$$

- From Section 5.4.3, the rolling curve

$$\begin{aligned}
\alpha_1(t) &= \Theta_0 e^{-t\Omega_U} E_0 e^{t\Omega_U} \Theta_0^\top \\
&= \Theta_0 e^{-t\Omega_U} \Theta_0^\top \Theta_0 E_0 \Theta_0^\top \Theta_0 e^{t\Omega_U} \Theta_0^\top \\
&= e^{-t(\Theta_0 \Omega_U \Theta_0^\top)} \Theta_0 E_0 \Theta_0^\top e^{t(\Theta_0 \Omega_U \Theta_0^\top)} \\
&= e^{t\overline{\Omega}_1} S_0 e^{-t\overline{\Omega}_1},
\end{aligned}$$

with $\overline{\Omega}_1 = -\Theta_0 \Omega_U \Theta_0^\top \in \mathfrak{so}_{S_0}(n)$, is a geodesic on $M_1 = G_{k,n}$, passing through S_0 at $t = 0$ with velocity $\left[\,\overline{\Omega}_1, S_0\right]$. Also, the development curve α_0 on $M_0 = T_{S_0}^{\text{aff}} G_{k,n}$ can be rewritten as

$$\alpha_0(t) = S_0 + t\left[\,\overline{\Omega}_1, S_0\right],$$

and is a geodesic on M_0, satisfying $\alpha_0(0) = S_0$.

- In this case $d_{\alpha_1(\tau)} h(\tau)(\xi_\ell) = U(\tau)^\top \xi_\ell U(\tau)$ and so, $\eta_\ell := U(\tau)^\top \xi_\ell U(\tau)$.

Using the previous, if the Algorithm 6.3.1 is applied to the manifold $M_1 = G_{k,n}$, we obtain the counterpart of the curve in (6.6):

$$\gamma(t) = U(t)\,\phi^{-1}(\beta(t) - X(t))\,U(t)^\top. \tag{6.8}$$

We are, now, in conditions to prove the following result.

Theorem 6.3.2 *The curve $\gamma : [0,\tau] \to G_{k,n}$ given by (6.8), where $t \mapsto h(t) = (U(t)^\top, X(t))$ is the rolling map along a smooth rolling curve α_1 that joins S_0 (at $t=0$) to S_ℓ (at $t=\tau$), and β is the curve in $T^{aff}_{S_0}G_{k,n}$ obtained in Step 4. of Algorithm 6.3.1, solves the interpolation Problem 6.2.1 for $G_{k,n}$.*

Proof. Let $h(t) = (U(t)^\top, X(t)) \in \overline{G} = \mathrm{SO}(n) \ltimes \mathfrak{s}(n)$ be the rolling map of $M_1 = G_{k,n}$ on $M_0 = T^{\mathrm{aff}}_{S_0}G_{k,n}$ along a smooth rolling curve α_1 that joins S_0 (at $t=0$) to S_ℓ (at $t=\tau$), with development curve α_0 and such that $h(0) = e_{\overline{G}}$. According with the formulas derived above and Section 5.4, we have that $X(t) = \alpha_0(t) - S_0$, and the rolling curve can be defined by $\alpha_1(t) = U(t)S_0U(t)^\top$. Furthermore, $h(t)^{-1} = (U(t), -U(t)X(t)U(t)^\top)$ and a simple calculation allows to rewrite the curve (6.6) as

$$\begin{aligned}\gamma(t) &= U(t)\left(\phi^{-1}(\beta(t)-X(t))+X(t)\right)U(t)^\top - U(t)X(t)U(t)^\top \\ &= U(t)\phi^{-1}(\beta(t)-X(t))U(t)^\top.\end{aligned} \tag{6.9}$$

Since $h(t_i)S_i = (U(t_i)^\top, X(t_i))S_i = U(t_i)^\top S_i U(t_i) + X(t_i)$, we have that $h(t_i)S_i - X(t_i) = U(t_i)^\top S_i U(t_i)$. Then, from $\beta(t_i) = \phi\big(h(t_i)S_i - X(t_i)\big) + X(t_i) = \phi\big(U(t_i)^\top S_i U(t_i)\big) + X(t_i)$, we obtain that $\beta(t_i) - X(t_i) = \phi\big(U(t_i)^\top S_i U(t_i)\big)$. Consequently, for all $i = 0,1,\ldots,\ell$ we have that

$$\begin{aligned}\gamma(t_i) &= U(t_i)\phi^{-1}(\beta(t_i)-X(t_i))U(t_i)^\top \\ &= U(t_i)\phi^{-1}\left(\phi(U(t_i)^\top S_i U(t_i))\right)U(t_i)^\top \\ &= S_i.\end{aligned}$$

Moreover, from (6.9) we get that

$$\begin{aligned}\dot{\gamma}(t) = \dot{U}(t)\phi^{-1}\left(\beta(t)-X(t)\right)U(t)^\top &+ U(t)d_{(\beta(t)-X(t))}\phi^{-1}\big(\dot{\beta}(t)-\dot{X}(t)\big)U(t)^\top \\ &+ U(t)\phi^{-1}\big(\beta(t)-X(t)\big)\dot{U}(t)^\top.\end{aligned}$$

To prove that γ satisfies the boundary conditions, note that from the no-slip condition we conclude that $\dot{X}(t) = U(t)^\top\dot{\alpha}_1(t)U(t)$. Also, from $h(0) = e_{\overline{G}}$, we have that $U(0) = I_n$ and $X(0) = 0$. Therefore, taking into consideration that $\beta(0) = S_0$, $\dot{\beta}(0) = \eta_0$, $\phi(S_0) = S_0$ and $d_{S_0}\phi = \mathrm{id}$, we have that

$$\begin{aligned}\dot{\gamma}(0) &= \dot{U}(0)\phi^{-1}\left(\beta(0)-X(0)\right)U(0)^\top + U(0)d_{(\beta(0)-X(0))}\phi^{-1}\big(\dot{\beta}(0)-\dot{X}(0)\big)U(0)^\top \\ &\quad + U(0)\phi^{-1}\big(\beta(0)-X(0)\big)\dot{U}(0)^\top \\ &= \dot{U}(0)\phi^{-1}\left(S_0\right) + d_{S_0}\phi^{-1}\big(\dot{\beta}(0)-\dot{X}(0)\big) + \phi^{-1}\big(S_0\big)\dot{U}(0)^\top \\ &= \dot{U}(0)S_0 + \dot{\beta}(0) - \dot{X}(0) + S_0\dot{U}(0)^\top \\ &= \dot{\alpha}_1(0) + \eta_0 - \dot{\alpha}_1(0) \\ &= \eta_0 = \xi_0.\end{aligned}$$

Furthermore, since $\beta(\tau) = \alpha_0(\tau) = X(\tau) - S_0$, we have that $\beta(\tau) - X(\tau) = S_0$. Therefore, taking also into account that $\dot{\beta}(\tau) = \eta_\ell$ and $U(\tau)^\top \xi_\ell U(\tau) = \eta_\ell$, we get that

$$\begin{aligned}
\dot{\gamma}(\tau) &= \dot{U}(\tau)\phi^{-1}\left(\beta(\tau)-X(\tau)\right)U(\tau)^\top + U(\tau)d_{(\beta(\tau)-X(\tau))}\phi^{-1}\big(\dot{\beta}(\tau)-\dot{X}(\tau)\big)U(\tau)^\top \\
&\quad + U(\tau)\phi^{-1}\big(\beta(\tau)-X(\tau)\big)\dot{U}(\tau)^\top \\
&= \dot{U}(\tau)\phi^{-1}\left(S_0\right)U(\tau)^\top + U(\tau)d_{S_0}\phi^{-1}\big(\dot{\beta}(\tau)-\dot{X}(\tau)\big)U(\tau)^\top + U(\tau)\phi^{-1}\big(S_0\big)\dot{U}(\tau)^\top \\
&= \dot{U}(\tau)S_0U(\tau)^\top + U(\tau)\big(\dot{\beta}(\tau)-\dot{X}(\tau)\big)U(\tau)^\top + U(\tau)S_0\dot{U}(\tau)^\top \\
&= \dot{\alpha}_1(\tau) + U(\tau)\dot{\beta}(\tau)U(\tau)^\top - U(\tau)\dot{X}(\tau)U(\tau)^\top \\
&= \dot{\alpha}_1(\tau) + U(\tau)\eta_\ell U(\tau)^\top - \dot{\alpha}_1(\tau) \\
&= U(\tau)\eta_\ell U(\tau)^\top \\
&= \xi_\ell.
\end{aligned}$$

Finally, the $\mathscr{C}^2$-smoothness of the resulting curve γ is obtained by construction, since h and ϕ are smooth and β is $\mathscr{C}^2$-smooth. This concludes the proof.

□

• Solving in the Rotation Group $\mathrm{SO}(n)$

The counterpart of the last theorem when $M_1 = \mathrm{SO}(n)$ was proved in Hüper and Silva Leite [31] and we only include its statement below. However, as a preparation for the implementation of the algorithm in the next section, we rewrite next some of the formulas that will be needed. Considering $h(t) = (V(t)^\top, W(t)^\top, Y(t)) \in \overline{G} = \mathrm{SO}(n)\times\mathrm{SO}(n)\ltimes\mathbb{R}^{n\times n}$ a rolling map of $M_1 = \mathrm{SO}(n)$ rolling over $M_0 = T^{\mathrm{aff}}_{R_0}\mathrm{SO}(n)$, with $R_0 \in \mathrm{SO}(n)$, we have that:

- The rolling curve is given by $\alpha_1(t) = V(t)R_0W(t)^\top$ and the development curve is defined by $\alpha_0(t) = R_0 + Y(t)$, with $Y(t)\in\mathbb{R}^{n\times n}$.
- From Section 5.4.3, the rolling curve
$$\alpha_1(t) = R_0e^{-2t\Omega_V} = R_0e^{t\overline{\Omega}_2},$$
with $\overline{\Omega}_2 := -2\Omega_V \in \mathfrak{so}(n)$, is a geodesic on $M_1 = \mathrm{SO}(n)$, passing through R_0 at $t=0$ with velocity $\overline{\Omega}_2$. Also, the development curve α_0 on $M_0 = T^{\mathrm{aff}}_{R_0}\mathrm{SO}(n)$ can be rewrite as
$$\alpha_0(t) = R_0 + tR_0\overline{\Omega}_2,$$
and is a geodesic on M_0, satisfying $\alpha_0(0) = R_0$.
- In this case, $d_{\alpha_1(\tau)}h(\tau)(\xi_\ell) = V(\tau)^\top\xi_\ell W(\tau)$ and so $\eta_\ell := V(\tau)^\top\xi_\ell W(\tau)$.

Then, the curve (6.6) obtained at the last step of the Algorithm 6.3.1 when $M_1 = \mathrm{SO}(n)$ simplifies to

$$\gamma(t) = V(t)\phi^{-1}(\beta(t)-Y(t))W(t)^\top, \tag{6.10}$$

and the counterpart of Theorem 6.3.2, whose proof appears in Hüper and Silva Leite [31], can be state in the following way:

Theorem 6.3.3 *The curve $\gamma : [0,\tau] \to \mathrm{SO}(n)$ given by (6.10), where $t \mapsto h(t) = (V(t)^\top, W(t)^\top, Y(t))$ is the rolling map along a smooth rolling curve α_1 that joins R_0 (at $t=0$) to R_ℓ (at $t=\tau$), and β is the curve in $T_{R_0}^{aff}\mathrm{SO}(n)$ obtained in Step 4. of Algorithm 6.3.1, solves the interpolation Problem 6.2.1 for $\mathrm{SO}(n)$.*

To conclude this section we write down the explicit formula for the interpolating curve on the product manifold $G_{k,n} \times \mathrm{SO}(n)$. Here the local diffeomorphism ϕ that is used to perform the unwrapping/wrapping will be denoted by ϕ_1 when $M_1 = G_{k,n}$ and by ϕ_2 when $M_1 = \mathrm{SO}(n)$. Similarly, the cubic spline β that solves the interpolation problem on M_0, will be denoted by β_1 when $M_0 = T_{S_0}^{\mathrm{aff}} G_{k,n}$ and by β_2 when $M_0 = T_{R_0}^{\mathrm{aff}}\mathrm{SO}(n)$.

Corollary 6.3.1 *The curve $\gamma : [0,\tau] \longrightarrow M_1$ defined by*

$$\begin{aligned} \gamma(t) &= (\gamma_1(t), \gamma_2(t)) \\ &= \big(U(t)\phi_1^{-1}(\beta_1(t) - X(t))U(t)^\top, V(t)\phi_2^{-1}(\beta_2(t) - Y(t))W(t)^\top\big) \end{aligned} \tag{6.11}$$

solves the Problem 6.2.1 when $M_1 = G_{k,n} \times \mathrm{SO}(n)$.

Remark 6.3.3 *Although Corollary 6.3.1 guarantees that an explicitly solution to the smooth interpolation problem on $G_{k,n} \times \mathrm{SO}(n)$ exists, it says nothing about its uniqueness. It is clear that the interpolating curve depends on the choice of a rolling curve and a local diffeomorphism.*

6.4 Implementation of the Interpolation Algorithm in $G_{k,n} \times \mathrm{SO}(n)$

Theoretically, the proposed algorithm always generates an interpolating curve which depends on the choice of a rolling map and of a local diffeomorphism. In practice, it is important that the algorithm can be implemented. In which concerns the rolling/unrolling, the rolling curve should be chosen so that the kinematic equations of rolling can be solved explicitly. The obvious choice is the geodesic joining the first point (at $t=0$) to the last point (at $t=\tau$). In which concerns the wrapping/unwrapping, our choice for the local diffeomorphism are the so called normal coordinates, which always exist for the Levi-Civita connection of a Riemannian manifold and can also be described by rolling along geodesics. So, our implementation of the algorithm depends on rolling along geodesics only. This will be made more precise for the manifolds $G_{k,n}$ and $\mathrm{SO}(n)$, where we use h_1 and ϕ_1 for the rolling map and local diffeomorphism in the first manifold and h_2 and ϕ_2 for the rolling map and local diffeomorphism in the second. The result for the manifold $G_{k,n} \times \mathrm{SO}(n)$ is an immediate consequence. In the sequel, we keep the notations introduced at the end of the previous section.

6.4.1 Implementation in $\mathrm{SO}(n)$

Choosing the Rolling Curve:

Consider as rolling curve the geodesic arc $\alpha_1 : [0,\tau] \to \mathrm{SO}(n)$ joining R_0 (at $t=0$) to R_ℓ (at $t=\tau$), which is defined as

$$\alpha_1(t) = R_0 \mathrm{e}^{t\overline{\Omega}_2}, \tag{6.12}$$

with $\overline{\Omega}_2 = \dfrac{1}{\tau}\log(R_0^\top R_\ell) \in \mathfrak{so}(n)$. Then, according to the developments in Chapter 5, the corresponding rolling map to unwrap the boundary data on $T^{\mathrm{aff}}_{R_0}\mathrm{SO}(n)$ is given explicitly by

$$h_2(t) = (R_0 \mathrm{e}^{-\frac{1}{2}t\overline{\Omega}_2} R_0^\top, \mathrm{e}^{\frac{1}{2}t\overline{\Omega}_2}, tR_0\overline{\Omega}_2), \tag{6.13}$$

and, consequently,

$$h_2(t)^{-1} = (R_0 \mathrm{e}^{\frac{1}{2}t\overline{\Omega}_2} R_0^\top, \mathrm{e}^{-\frac{1}{2}t\overline{\Omega}_2}, -tR_0\mathrm{e}^{\frac{1}{2}t\overline{\Omega}_2}\overline{\Omega}_2\mathrm{e}^{\frac{1}{2}t\overline{\Omega}_2}) = (R_0 \mathrm{e}^{\frac{1}{2}t\overline{\Omega}_2} R_0^\top, \mathrm{e}^{-\frac{1}{2}t\overline{\Omega}_2}, -tR_0\overline{\Omega}_2\mathrm{e}^{t\overline{\Omega}_2}). \tag{6.14}$$

So, we already have the following data that enters in the expression of the interpolating curve (6.10):

$$Y(t) = tR_0\overline{\Omega}_2, \quad V(t) = R_0\mathrm{e}^{\frac{1}{2}t\overline{\Omega}_2}R_0^\top \quad \text{and} \quad W(t)^\top = \mathrm{e}^{\frac{1}{2}t\overline{\Omega}_2}.$$

Choosing the Local Diffeomorphism:

The local diffeomorphism ϕ_2 used to unwrap the intermediate interpolation points, is chosen so that the image of a point R in a neighbourhood $\mathcal{V}_2$ of R_0 is the point in $T^{\mathrm{aff}}_{R_0}\mathrm{SO}(n)$ that results from rolling $\mathrm{SO}(n)$ over the affine tangent space at R_0, along the geodesic that joins R_0 to R.

Therefore,

$$\begin{array}{rccl} \phi_2 : & \mathcal{V}_2 \subset \mathrm{SO}(n) & \longrightarrow & \phi_2(\mathcal{V}_2) \subset T^{\mathrm{aff}}_{R_0}\mathrm{SO}(n) \\ & R & \longmapsto & \phi_2(R) = R_0 + R_0\overline{Y}, \end{array} \tag{6.15}$$

with $\overline{Y} = \log(R_0^\top R) \in \mathfrak{so}(n)$, and

$$\begin{array}{rccl} \phi_2^{-1} : & \phi_2(\mathcal{V}_2) \subset T^{\mathrm{aff}}_{R_0}\mathrm{SO}(n) & \longrightarrow & \mathcal{V}_2 \subset \mathrm{SO}(n) \\ & \overline{R} = R_0 + R_0\overline{Y} & \longmapsto & \phi_2^{-1}(\overline{R}) = R_0\mathrm{e}^{R_0^\top(\overline{R}-R_0)} = R_0\mathrm{e}^{\overline{Y}}. \end{array} \tag{6.16}$$

But in order to prove that this diffeomorphism is appropriate, we need to show that it satisfies the two requirements: $\phi_2(R_0) = R_0$ and $d_{R_0}\phi_2 = \mathrm{id}$. This is what we do next.

Since $\phi_2(R_0) = R_0 + R_0\overline{Y}$, with $\overline{Y} = \log(R_0^\top R_0) = \log(I) = 0$, we immediately have $\phi_2(R_0) = R_0$. Furthermore,

$$\begin{array}{rccl} d_{R_0}\phi_2 : & T_{R_0}\mathrm{SO}(n) & \longrightarrow & T_{R_0}\left(T^{\mathrm{aff}}_{R_0}\mathrm{SO}(n)\right) \equiv T_{R_0}\mathrm{SO}(n) \\ & R_0A & \longmapsto & \dfrac{\mathrm{d}}{\mathrm{d}t}[\phi_2(R(t))]\Big|_{t=0} = \dfrac{\mathrm{d}}{\mathrm{d}t}[R_0 + R_0\overline{Y}(t)]\Big|_{t=0} = R_0\dot{\overline{Y}}(0), \end{array}$$

where $t \mapsto R(t)$ is a curve in $\mathrm{SO}(n)$, satisfying $R(0) = R_0$ and $\dot{R}(0) = R_0 A$, with $A \in \mathfrak{so}(n)$. Since, $\overline{Y}(t) = \log(R_0^\top R(t))$, from Lemma 2.6.4, we obtain that

$$\dot{\overline{Y}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\log(R_0^\top R(t))\Big) = \frac{u}{\mathrm{e}^u - 1}\bigg|_{u=\mathrm{ad}_{\log(R_0^\top R(t))}} \Big(R_0^\top \dot{R}(t) R^{-1}(t) R_0\Big).$$

Evaluating at $t = 0$, we get

$$\dot{\overline{Y}}(0) = \frac{\mathrm{d}}{\mathrm{d}t}\Big[\log(R_0^\top R(t))\Big]\bigg|_{t=0} = R_0^\top R_0 A R_0^{-1} R_0 = A.$$

Therefore, $d_{R_0}\phi_2(R_0 A) = R_0 \dot{\overline{Y}}(0) = R_0 A$, which proves that $d_{R_0}\phi_2 = \mathrm{id}$.

6.4.2 Implementation in $G_{k,n}$

Similarly to what was done for the manifold $\mathrm{SO}(n)$, we need to define explicitly the rolling and the local diffeomorphism.

Choosing the Rolling Curve:

Consider the rolling curve the geodesic arc $\alpha_1 : [0,\tau] \longrightarrow G_{k,n}$ that joins the point S_0 (at $t = 0$) with the point S_ℓ (at $t = \tau$), and which is defined, explicitly, as

$$\alpha_1(t) = \mathrm{e}^{t\overline{\Omega}_1} S_0 \mathrm{e}^{-t\overline{\Omega}_1}, \tag{6.17}$$

with $\overline{\Omega}_1 = \dfrac{1}{2\tau}\log((I - 2S_\ell)(I - 2S_0)) \in \mathfrak{so}_{S_0}(n)$.

From Chapter 5, the corresponding rolling map h_1 to unwrap the boundary data on $T_{S_0}^{\mathrm{aff}} G_{k,n}$ by rolling $G_{k,n}$ along the curve α_1 is defined, explicitly, by

$$h_1(t) = (\mathrm{e}^{-t\overline{\Omega}_1}, t\,[\,\overline{\Omega}_1, S_0]), \tag{6.18}$$

and, then

$$h_1(t)^{-1} = \Big(\mathrm{e}^{t\overline{\Omega}_1}, -t\mathrm{e}^{t\overline{\Omega}_1}\,[\,\overline{\Omega}_1, S_0]\,\mathrm{e}^{-t\overline{\Omega}_1}\Big) = \Big(\mathrm{e}^{t\overline{\Omega}_1}, -t\Big[\,\overline{\Omega}_1, \mathrm{e}^{t\,\mathrm{ad}_{\overline{\Omega}_1}}(S_0)\Big]\Big). \tag{6.19}$$

Consequently, in the expression of the interpolating curve (6.8), we get

$$U(t) = \mathrm{e}^{t\overline{\Omega}_1} \quad \text{and} \quad X(t) = t\,[\,\overline{\Omega}_1, S_0]\,.$$

Choosing the Local Diffeomorphism:

Set up the local diffeomorphism ϕ_1, to perform the unwrapping technique, between an open neighbourhood $\mathscr{V}_1 \subset G_{k,n}$ of S_0 and $\phi_1(\mathscr{V}_1) \subset T_{S_0}^{\mathrm{aff}} G_{k,n}$, by using rolling motions along geodesic arcs joining the point S_0 with a point $S \in \mathscr{V}_1$, as follows

$$\begin{array}{rccc} \phi_1 : & \mathscr{V}_1 \subset G_{k,n} & \longrightarrow & \phi_1(\mathscr{V}_1) \subset T_{S_0}^{\mathrm{aff}} G_{k,n} \\ & S & \longmapsto & \phi_1(S) = S_0 + [\Lambda, S_0]\,, \end{array} \tag{6.20}$$

where $\Lambda = \frac{1}{2}\log((I-2S)(I-2S_0)) \in \mathfrak{so}_{S_0}(n)$ and $[\Lambda, S_0]$ denotes the velocity of the geodesic arc connecting S_0 to S.

To compute the inverse of this diffeomorphism, notice that from $\Lambda = \frac{1}{2}\log((I-2S)(I-2S_0))$ we can solve for S in the following way:

$$\Lambda = \frac{1}{2}\log((I-2S)(I-2S_0)) \Longleftrightarrow \mathrm{e}^{2\Lambda} = (I-2S)(I-2S_0) \Longleftrightarrow S = \frac{1}{2}\big(I-\mathrm{e}^{2\Lambda}(I-2S_0)\big),$$

and applying Proposition 3.2.2, we have

$$S = \frac{1}{2}\big(I-\mathrm{e}^{\Lambda}(I-2S_0)\mathrm{e}^{-\Lambda}\big) = \mathrm{e}^{\Lambda}S_0\mathrm{e}^{-\Lambda}.$$

Consequently,

$$\begin{array}{rlcl} \phi_1^{-1}: & \phi_1(\mathcal{V}_1)\subset T^{\mathrm{aff}}_{S_0}G_{k,n} & \longrightarrow & \mathcal{V}_1\subset G_{k,n} \\ & \overline{S} = S_0+[\Lambda,S_0] & \longmapsto & \phi_1^{-1}(\overline{S}) = \mathrm{e}^{\Lambda}S_0\mathrm{e}^{-\Lambda}. \end{array} \tag{6.21}$$

Now, to show that the diffeomorphism ϕ_1, thus defined, satisfies $\phi_1(S_0)=S_0$ and $d_{S_0}\phi_1 = \mathrm{id}$, observe that, for $S=S_0\in\mathcal{V}_1$ we obtain

$$\Lambda = \frac{1}{2}\log((I-2S_0)(I-2S_0)) = \frac{1}{2}\log(I) = 0.$$

Consequently, $\phi_1(S_0) = S_0+[0,S_0] = S_0$. In order to show that $d_{S_0}\phi_1 = \mathrm{id}$, note that

$$\begin{array}{rlcl} d_{S_0}\phi_1: & T_{S_0}G_{k,n} & \longrightarrow & T_{S_0}\left(T^{\mathrm{aff}}_{S_0}G_{k,n}\right)\equiv T_{S_0}G_{k,n} \\ & [\psi,S_0] & \longmapsto & \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\phi_1(S(t))\right)\Big|_{t=0} = \dfrac{\mathrm{d}}{\mathrm{d}t}\left(S_0+[\overline{\psi}(t),S_0]\right)\Big|_{t=0} = \left[\dot{\overline{\psi}}(0),S_0\right], \end{array}$$

where $t\mapsto S(t)$ is a curve in $G_{k,n}$, satisfying $S(0)=S_0$ and $\dot{S}(0)=[\psi,S_0]$, with $\overline{\psi}(t) = \frac{1}{2}\log((I-2S(t))(I-2S_0)) \in \mathfrak{so}_{S_0}(n)$. Let us consider the matrix valued function L defined by $L(t) = (I-2S(t))(I-2S_0)$. Then, $L(0)=I$ and, with this notation, we have that $\overline{\psi}(t) = \frac{1}{2}\log(L(t))$. Therefore, from Lemma 2.6.4, we obtain that

$$\begin{aligned} \dot{\overline{\psi}}(t) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\log(L(t)) \\ &= \frac{1}{2}\left.\frac{u}{\mathrm{e}^u-1}\right|_{u=\mathrm{ad}_{\log(L(t))}}\left(-2\dot{S}(t)(I-2S_0)(I-2S_0)^{-1}(I-2S(t))^{-1}\right) \\ &= \frac{1}{2}\left.\frac{u}{\mathrm{e}^u-1}\right|_{u=\mathrm{ad}_{\log(L(t))}}\left(-2\dot{S}(t)(I-2S(t))^{-1}\right). \end{aligned}$$

Consequently, at $t=0$,

$$\begin{aligned} \dot{\overline{\psi}}(0) &= \frac{1}{2}\left.\frac{u}{\mathrm{e}^u-1}\right|_{u=\mathrm{ad}_{\log(L(0))}}\left(-2\dot{S}(0)(I-2S(0))^{-1}\right) \\ &= \frac{1}{2}(-2)\dot{S}(0)(I-2S_0) \\ &= -[\psi,S_0](I-2S_0) = -[\psi,S_0]+2[\psi,S_0]S_0. \end{aligned}$$

Furthermore, from Lemma 3.2.1, and since $S_0^2 = S_0$, we have that

$$2\,[\psi, S_0]\, S_0 = 2(\psi S_0^2 - S_0\psi S_0) = 2\psi S_0.$$

Therefore, since $\psi \in \mathfrak{so}_{S_0}(n)$, we obtain $\dot{\overline{\psi}}(0) = -[\psi, S_0] + 2\psi S_0 = \psi S_0 + S_0\psi = \psi$. Consequently,

$$d_{S_0}\phi_1([\psi, S_0]) = \left[\dot{\overline{\psi}}(0), S_0\right] = [\psi, S_0],$$

which proves that $d_{S_0}\phi_1 = \mathrm{id}$.

Remark 6.4.1

The $\mathscr{C}^2$-smooth interpolating curves β_2 in $T^{aff}_{R_0}\mathrm{SO}(n)$ and β_1 in $T^{aff}_{S_0}G_{k,n}$ can be obtained using the classical De Casteljau Algorithm (Chapter 4), which works well for these affine spaces.

To conclude, from the previous two subsections, and taking into consideration Corollary 6.3.1, the implementation of the algorithm to obtain an explicitly solution to the smooth interpolation Problem 6.2.1 in $G_{k,n} \times \mathrm{SO}(n)$ is clear, and the interpolating curve (6.11) is given by,

$$\gamma(t) = \left(\mathrm{e}^{t\overline{\Omega}_1}\phi_1^{-1}\left(\beta_1(t) - t\left[\,\overline{\Omega}_1, S_0\right]\right)\mathrm{e}^{-t\overline{\Omega}_1},\ R_0\mathrm{e}^{\frac{1}{2}t\overline{\Omega}_2}R_0^\top\phi_2^{-1}\ \left(\beta_2(t) - tR_0\overline{\Omega}_2\right)\mathrm{e}^{\frac{1}{2}t\overline{\Omega}_2}\right)\ , \tag{6.22}$$

with $\overline{\Omega}_1 = \dfrac{1}{2\tau}\log((I-2S_\ell)(I-2S_0)) \in \mathfrak{so}_{S_0}(n)$ and $\overline{\Omega}_2 = \dfrac{1}{\tau}\log(R_0^\top R_\ell) \in \mathfrak{so}(n)$.

Before finishing this section it is important to notice that the implementation of the algorithm in $G_{k,n}$ and in $\mathrm{SO}(n)$ (and, consequently, in $G_{k,n} \times \mathrm{SO}(n)$), requires that one knows how to compute matrix exponentials of skewsymmetric matrices and logarithms of orthogonal matrices. This is a major problem that for $n \neq 3$ can only be solved numerically. Fortunately, there are numerical methods to compute these matrix functions that are stable in the sense that each step of the algorithm produces a matrix that leaves in the right space. The interested reader can see, for instance, Cardoso and Silva Leite [11] and Higham [26]. When $n = 3$, there are explicit formulas to compute exponentials of matrices in $\mathfrak{so}(3)$ and logarithms of matrices in $\mathrm{SO}(3)$ and, for that reason, the implementation of the interpolating algorithm on the Normalized Essential manifold can be performed with exact formulas, as will be seen in the next section.

6.5 Implementation of the Interpolation Algorithm in the Normalized Essential Manifold

The interpolation algorithm in the Normalized Essential manifold $\mathscr{E} = G_{2,3} \times \mathrm{SO}(3)$ may be regarded as a particular case of the previous Section 6.4. Although, using the results obtained in that section, in this section we will give other details for the curve in $\mathscr{E}$, that interpolates the points $P_i = (S_i, R_i)$ in $\mathscr{E}$ (at $t = t_i$) , for $i = 0, \ldots, \ell$, and which has initial and final velocity prescribed. From Section 6.4, we know that this interpolating curve is defined, in closed form by (6.22), with $\overline{\Omega}_1 \in \mathfrak{so}_{S_0}(3)$ and $\overline{\Omega}_2 \in \mathfrak{so}(3)$.

Furthermore, for all $i = 0, \ldots, \ell$, we have that

$$\begin{aligned}\beta_1(t_i) &= \phi_1\left(\mathrm{e}^{-t_i\overline{\Omega}_1} S_i \mathrm{e}^{t_i\overline{\Omega}_1}\right) + t_i \left[\,\overline{\Omega}_1, S_0\right] \\ &= S_0 + [\Lambda_i, S_0] + \left[t_i\,\overline{\Omega}_1, S_0\right] = S_0 + \left[\Lambda_i + t_i\overline{\Omega}_1, S_0\right],\end{aligned}$$

with $\Lambda_i = \dfrac{1}{2}\log\left((I - 2\mathrm{e}^{-t_i\overline{\Omega}_1} S_i \mathrm{e}^{t_i\overline{\Omega}_1})(I - 2S_0)\right) \in \mathfrak{so}_{S_0}(3)$, and

$$\begin{aligned}\beta_2(t_i) &= \phi_2\left(R_0 \mathrm{e}^{-\frac{1}{2}t_i\overline{\Omega}_2} R_0^\top R_i \mathrm{e}^{-\frac{1}{2}t_i\overline{\Omega}_2}\right) + t_i R_0 \overline{\Omega}_2 \\ &= R_0 + R_0 \overline{Y}_i + t_i R_0 \overline{\Omega}_2 = R_0 + R_0\left(\overline{Y}_i + t_i\overline{\Omega}_2\right),\end{aligned}$$

with $\overline{Y}_i = \log\left(\mathrm{e}^{-\frac{1}{2}t_i\overline{\Omega}_2} R_0^\top R_i \mathrm{e}^{-\frac{1}{2}t_i\overline{\Omega}_2}\right) \in \mathfrak{so}(3)$.

Consequently, applying the classical De Casteljau Algorithm, respectively, on $T^{\mathrm{aff}}_{S_0} G_{2,3}$ $(j = 1)$ and on $T^{\mathrm{aff}}_{R_0}\mathrm{SO}(3)$ $(j = 2)$, it is possible to obtain the $\mathscr{C}^2$-smooth interpolating curve β_j, $j = 1, 2$ that interpolates the points $\beta_j(t_i)$, for all $i = 0, \ldots, \ell$ and $j = 1, 2$.

For all $t \in [0, \tau]$, $\beta_1(t) \in T^{\mathrm{aff}}_{S_0} G_{2,3}$, and thus $\beta_1(t) = S_0 + [\upsilon(t), S_0]$, with $\upsilon(t) \in \mathfrak{so}_{S_0}(3)$. Then, we have that $\upsilon(t) = [\beta_1(t), S_0]$ and, since $X(t) = t\left[\,\overline{\Omega}_1, S_0\right]$, we get

$$\begin{aligned}\phi_1^{-1}(\beta_1(t) - X(t)) &= \phi_1^{-1}\left(S_0 + \left[\upsilon(t) -\, t\overline{\Omega}_1, S_0\right]\right) \\ &= \mathrm{e}^{\upsilon(t) -\, t\overline{\Omega}_1} S_0 \mathrm{e}^{-(\upsilon(t) -\, t\overline{\Omega}_1)}.\end{aligned}$$

Similarly, for all $t \in [0, \tau]$, $\beta_2(t) \in T^{\mathrm{aff}}_{R_0}\mathrm{SO}(3)$, and thus $\beta_2(t) = R_0 + R_0 A(t)$, with $A(t) \in \mathfrak{so}(3)$. Then, we obtain that $A(t) = R_0^\top(\beta_2(t) - R_0) = R_0^\top\beta_2(t) - I_3$ and, since $Y(t) = tR_0\overline{\Omega}_2$, we get

$$\begin{aligned}\phi_2^{-1}(\beta_2(t) - Y(t)) &= \phi_2^{-1}\left(R_0 + R_0\left(A(t) -\, t\overline{\Omega}_2\right)\right) \\ &= R_0 \mathrm{e}^{A(t) -\, t\overline{\Omega}_2}.\end{aligned}$$

Therefore, after knowing how ϕ_j^{-1} and β_j, $j = 1, 2$ are defined in terms of the data, we can conclude that the $\mathscr{C}^2$-smooth interpolating curve (6.22) in the Normalized Essential manifold $\mathscr{E}$, can be rewritten just in terms of the data by

$$\gamma(t) = \left(\mathrm{e}^{t\overline{\Omega}_1}\mathrm{e}^{\upsilon(t) -\, t\overline{\Omega}_1} S_0 \mathrm{e}^{-(\upsilon(t) -\, t\overline{\Omega}_1)}\mathrm{e}^{-t\overline{\Omega}_1},\ R_0 \mathrm{e}^{\frac{1}{2}t\overline{\Omega}_2}\mathrm{e}^{A(t) -\, t\overline{\Omega}_2}\mathrm{e}^{\frac{1}{2}t\overline{\Omega}_2}\right). \tag{6.23}$$

To finish, it is important to notice that, contrary to all other situations, for the Normalized Essential manifold $\mathscr{E}$ we can go a step further in the computation of the interpolating curve, because in this case there are explicit formulas (4.70) and (4.71) to compute, respectively, exponentials of matrices in $\mathfrak{so}(3)$ and logarithms of matrices in $\mathrm{SO}(3)$.

Chapter 7

Final Remarks: Related Work and Future Research Directions

We end with some final comments about work that is related to the topics of this thesis and also point out some directions for future research.

7.1 Variational Approach to Solve Interpolation Problems

Although this thesis is about solving interpolation problems on manifolds, it doesn't cover all possible methods available in the literature. In particular, the variational approach that produces Riemannian cubic polynomials, curves that minimize a certain energy functional, was only briefly mentioned in Subsection 4.1.1. These polynomial curves have been defined as the extremal curves for the problem

$$\min_{\gamma\in\Omega} L(\gamma) = \frac{1}{2}\int_0^1 \left\langle \frac{D^2\gamma}{dt^2}, \frac{D^2\gamma}{dt^2} \right\rangle dt$$

over the family Ω of smooth paths $\gamma\colon [0,1] \to M$ that satisfy the boundary conditions:

$$\gamma(0) = p_0, \qquad \gamma(1) = p_1,$$

$$\frac{d\gamma}{dt}(0) = v_0, \quad \frac{d\gamma}{dt}(1) = v_1,$$

where p_0 and p_1 are given points in M, $v_0 \in T_{p_0}M$ and $v_1 \in T_{p_1}M$ are prescribed initial and final velocities.

These problems, and even higher-order variations of them, have been extensively studied by several authors, starting with the pioneer work of Noakes, Heinzinger and Paden in [59], followed by Crouch and Silva Leite [14, 15], Camarinha [10], and Zhang and Noakes [74], just to name a few.

It is known that Riemannian cubic polynomials are solutions of the Euler-Lagrange equations associated to the optimization problem above. These differential equations evolving in M are 4th-order, highly nonlinear, with complicated coupled boundary conditions. They are of the form:

$$\frac{D^4\gamma}{dt^4} + R\left(\frac{D^2\gamma}{dt^2}, \frac{d\gamma}{dt}\right)\frac{d\gamma}{dt} = 0,$$

where R denotes the curvature tensor. In $\mathbb{R}^m$ this reduces to $\dfrac{d^4\gamma}{dt^4}$, giving rise to Euclidean cubic polynomials. On other Riemannian manifolds, however, solving the Euler-Lagrange equations is a big challenge and in spite of some progress for particular cases, explicit formulas for Riemannian cubic polynomials have not been found yet. This emphasizes the importance of alternative methods, such as those studied throughout this thesis.

Since the solution curves of this problem minimize the average covariant acceleration, which is an important property in many real applications, knowing approximated solutions might be particularly useful. Such solutions could be used to compare cubic polynomial curves on Riemannian manifolds produced by the variational approach with those generated by the De Casteljau geometric construction, an issue that, to the best of our knowledge, has not yet been addressed properly. It may also inspire appropriate modifications in the later algorithm in order that the resulting curve is closer to the optimal one obtained with the variational approach. However, solving the Euler-Lagrange equations numerically requires the use of numeric integrators on manifolds, a subject that is out of our objective at the moment.

7.2 Rolling Motions of Riemannian Symmetric Spaces

In this section we briefly recall the essentials of Riemannian symmetric spaces and present a simple description of the kinematic equations of the rolling motion of such Riemannian manifolds.

We assume that a Lie group G acts transitively on a Riemannian manifold M and use the notations defined in Section 2.5 for this action. In particular, we write $g(p)$ for the image of a point $p \in M$ under the diffeomorphism $\phi_g : M \to M$ associated to the action $\phi : G \times M \to M$.

For a fixed point $p_0 \in M$, let K be the isotropy subgroup at p_0, and assume that the Lie algebra $\mathfrak{g}$ of G admits a direct sum decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is a vector space of $\mathfrak{g}$, satisfying the following commutator relations:

$$[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}, \quad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}. \tag{7.1}$$

Notice that the last of the previous relations implies that $\mathfrak{p}$ is Ad_K-invariant, i.e., $Ad_K(\mathfrak{p}) \subset \mathfrak{p}$. Also, by differentiating the action with respect to p_0, one obtains an isomorphism between $\mathfrak{p}$ and $T_{p_0}M$.

Under the above assumptions, M is a Riemannian symmetric space (symmetric space for short) that can be identified with the quotient G/K.

It happens that every curve in M is the projection (under the natural projection $\pi : G \to M = G/K$ given by $\pi(g) = \phi_g(p_0)$) of a horizontal curve in G. A curve $t \mapsto g(t)$ in G is said to be horizontal if $\dot{g}(t) = g(t)\Omega(t)$ with $\Omega(t) \in \mathfrak{p}$. For more details about these statements, we refer to Jurdjevic [36].

The Riemannian manifolds that appear in this thesis are particular cases of symmetric spaces. In Subsection 5.4.2 we have already described the transitive action of a certain Lie group on each of those manifolds and all the other ingredients that make them symmetric spaces. So, it is natural to look at the rolling motions of the Grassmann manifold and of $\mathrm{SO}(n)$ as particular cases of rolling motions of general symmetric spaces. This has not been the objective of our thesis, but at this point it is important to have a broader picture of the results already presented, also having in mind the work that has appeared, in the meanwhile, in Krakowski et al. [46] and Krakowski and Silva Leite [45]. In these papers, the authors have shown that the natural decomposition of the Lie algebra associated to a symmetric space provides the structure of the kinematic equations that describe the rolling motion of that space upon its affine tangent space at a point. This is clearly observed in our particular cases.

The following statement gives a clear and simple description of the kinematic equations of rolling a symmetric space on the affine tangent space at a point, along a curve starting at p_0, under the assumption that it has no flat points.

Theorem 7.2.1 *Let $t \mapsto \alpha(t) = \pi(g(t))$ be a curve in M starting at the point p_0, where $t \mapsto g(t)$ is a horizontal curve in G starting at e_G, i.e., $\dot{g}(t) = g(t)\Omega(t)$ with $\Omega(t) \in \mathfrak{p}$. Then, the kinematic equations for the rolling motion of the symmetric space $M = G/K$ along the curve $t \mapsto \alpha(t)$ are given by:*

$$\left\{ \begin{array}{rcl} \dot{g}(t) & = & g(t)\,\Omega(t) \\ \dot{s}(t) & = & d_{e_G}\pi\left(\Omega(t)\right) \end{array} \right. , \tag{7.2}$$

and $t \mapsto (g(t)^{\top}, s(t))$ is the corresponding rolling map.

7.3 Modified De Casteljau Algorithm for Other Riemannian Manifolds

On a recent article, Modin et al. [55] presented a numerical algorithm for $\mathscr{C}^2$-smooth splines on symmetric spaces based on the Bézier curves. In terms of computational efficiency and numerical tractability, the new method surpasses those based on Riemannian cubics. This is not a surprise since the later are solutions of a highly nonlinear boundary value problem, while the curves obtained by the former approach are given in closed form. Nevertheless, since the $\mathscr{C}^2$-smoothness conditions are significantly more complicated in the presence of Hermite boundary conditions, the numerical algorithm presented in Modin et al. [55] may help in the comparison between the interpolating curves produced by distinct approaches, in particular those already mentioned in the previous section.

The classical De Casteljau algorithm and its generalizations, such as the one presented in this thesis, is based on successive geodesic interpolation and, so, can be implemented whenever there is an explicit formula for geodesics joining two points. But contrary to most symmetric spaces, such as Grassmann manifolds and special orthogonal groups, such explicit formulas do not exist for certain manifolds that are particularly important in many applications. This is the case, for instance, in the Stiefel manifold, consisting of k-orthonormal frames in $\mathbb{R}^m$. However, a modification of the De Casteljau algorithm, presented in Krakowski et al. [42], where geodesic interpolation was replaced by quasi-geodesic interpolation turned out to be successful to generate $\mathscr{C}^1$-smooth interpolating curves defined in closed form. An interesting open problem is to solve the $\mathscr{C}^2$-smoothness condition on the Stiefel manifold, which is significantly more complicated than the $\mathscr{C}^1$-smooth condition. Stiefel manifolds are particularly important in computer vision, and the $\mathscr{C}^2$-smooth condition is often required in applications to avoid discontinuities in accelerations. We hope to be able to address this issue in the future, inspired by the modifications of the De Casteljau algorithm in Krakowski et al. [42].

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