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UNIVERSIDADE D
COIMBRA

Maria Elisa Barbosa Silveira

DRUG TRANSPORT ENHANCED BY TEMPERATURE: MATHEMATICAL ANALYSIS AND NUMERICAL SIMULATION

Tese no âmbito do Programa Interuniversitário de Doutoramento em Matemática, orientada pelos Professores Doutores José Augusto Ferreira e Maria Paula de Oliveira e apresentada ao Departamento de Matemática da Faculdade de Ciências e Tecnologia da Universidade de Coimbra.

Novembro de 2020

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Maria Elisa Barbosa Silveira

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U.PORTO

UC|UP Joint PhD Program in Mathematics

Programa Interuniversitário de Doutoramento em Matemática

PhD Thesis | Tese de Doutoramento

November 2020

Acknowledgements

I would like to thank my supervisor Professor José Augusto Ferreira for his guidance, patience and huge support throughout this process. His optimistic and passionate way of seeing things encouraged me and pushed me through these three years. His ability to share knowledge created in me a desire to be part and always learn more about mathematics.

I would like to thank my supervisor Maria Paula de Oliveira for her support, for reading my work and for the inspiring enthusiastic way in which she shares mathematics.

I would like to thank Professor Gonçalo Pena for his collaboration and openness to help me develop my work. I learnt how to think when we are programming.

To my PhD colleagues from Porto and from Coimbra I sincerely thank for your time and chats during mealtime and breaks. I have learnt a lot from you. In particular, I am thankful to Jorge and Willian who are my friends and have always listened to me. We have both laughed and cried together.

I want to thank all my friends for being who they are and being there when I needed them.

To my home family, mum, dad and Helena, I am finally finishing my studies, at least at the university. None of this would be possible without your example, values, support and encouragement during the past ten years. Thank you for being always there, for listening by stories, for giving me the right advices, for teaching me to handle my frustrations and step backs and always overcome difficulties trying to reach success. I love you.

To my big family, I am very proud of being the first person in my family to finish a PhD Thesis (and hopefully, a PhD), but at the same time I feel that it is a big responsibility. I hope that some of you will be inspired by my achievements.

I want to thank Tete and Lulu for their smiles, kisses and hugs, even though you still do not realise, you were one of my sources of strength and inspiration.

To my love, Paulo, thank you for being the perfect partner, I am blessed to have met you during this process. You are my rock and my inspiration. I love you.

I would like to thank Fundação para a Ciência e Tecnologia for the support given by the PhD grants PD/BI/113690/2015 and PD/BD/128058/2016 funded by the Portuguese Government through FCT/MCTES. I would also like to thank Centro de Matemática da Universidade de Coimbra that welcomed me and allowed to be a part of the project UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MCTES and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

Abstract

The motivation of the studies presented in this thesis is the modelling of drug delivery, from polymeric matrices, enhanced by external stimuli, namely by heat. Drug delivery is a large domain of active research on the development of new materials and transport systems for efficient therapeutic release of drugs. Many new drug delivery systems are at an experimental stage. Therefore, mathematical modelling and numerical simulation of drug release, appears as an important coadjuvant in such pioneering experimental studies.

In this thesis, we study numerical methods for systems of nonlinear parabolic equations and systems composed by a nonlinear elliptic equation coupled with two nonlinear parabolic equations. The first systems can be used to describe drug transport enhanced by heat, while the second one can be used to describe iontophoresis drug delivery. The numerical methods proposed may be viewed as finite difference methods as well as fully discrete piecewise linear finite element methods.

Concerning the first class of systems, systems of nonlinear parabolic equations, we analyse their stability and convergence when the solutions are in H^3. We prove that the approximations of the solutions and their gradients are second order convergent with respect to discrete L^2-norms.

Regarding the numerical methods proposed for the second class of systems, a nonlinear elliptic equation coupled with a system of nonlinear parabolic equations, we prove second order convergence, with respect to a discrete L^2-norm, for the solutions of the parabolic equations. We propose a numerical method for the elliptic equation, whose solutions converge, to the corresponding continuous solutions, with a second order convergence rate with respect to a discrete version of the usual H^1-norm. Concerning these systems, the main difficulty in the design of efficient and accurate numerical methods is the dependence of the convective velocity, of one of the time dependent equations, on the gradient of the solution of the nonlinear elliptic equation. The convergence results established can be viewed as supraconvergence results, in the framework of finite difference convergence theory, and supercloseness results, within finite element convergence theory. The stability of the numerical methods proposed is also addressed. As we are dealing with nonlinear problems, to get local stability, a usual required assumption is the boundedness of the sequence of numerical solutions. We prove that this assumption is a consequence of the error properties and therefore we conclude the stability of the proposed methods. Numerical experiments illustrating the convergence results obtained are included. These experiments show the sharpness of the smoothness assumptions on the solutions of the differential problems. From the perspective of drug delivery applications, the qualitative behaviour of the previous systems is numerically studied in different scenarios: drug transport enhanced by heat and iontophoresis drug delivery. Finally, we remark that the mathematical models studied before were established using Fick's law for the drug flux, which does not take into account the viscoelastic properties of the matrices where the drug is dispersed. In the last chapter we present an exploratory study of pulsatile drug delivery from thermoresponsive polymeric matrices. The model takes into account the mechanistic properties of the polymeric matrix under the effect of temperature and is represented by a moving boundary initial value problem. A semi-analytic approach is considered and the solution of the initial boundary value problem is obtained using Fourier analysis.

Keywords: Stimuli drug delivery - Heat - Finite difference methods - Piecewise finite element methods - Convergence - Stability

Resumo

A motivação para o estudo apresentado nesta dissertação é a libertação de fármacos de matrizes poliméricas, estimulada por estímulos externos, nomeadamente pela temperatura. A libertação de fármacos faz parte de um domínio alargado da investigação ativa do desenvolvimento de novos materiais e sistemas de transporte para uma eficiente libertação terapêutica de fármacos. Muitos sistemas de libertação de fármacos mais recentes encontram-se numa etapa experimental. Por esta razão, a modelação matemática e a simulação numérica de libertação de fármacos surgem como um coadjuvante importante em tais estudos experimentais pioneiros. Nesta dissertação, estudamos os métodos numéricos para os sistemas de equações parabólicas não lineares e sistemas de uma equação elíptica não linear associada a duas equações parabólicas não lineares. Os primeiros sistemas podem ser utilizados para descrever o transporte de fármacos estimulado pela temperatura, ao passo que o segundo pode ser utilizado para descrever a libertação de fármacos através da iontoforese. Os métodos numéricos propostos podem ser vistos como métodos de diferenças finitas, bem como método de elementos finitos segmentado linear completamente discreto. A respeito dos sistemas de primeira classe, sistemas de equações parabólicas não lineares, analisamos a sua estabilidade e convergência quando as soluções estão em H^3. Provamos que as aproximações das soluções e os seus gradientes são convergentes de segunda ordem relativamente às normas-L^2 discretas. Quanto aos métodos numéricos propostos para a segunda classe de sistemas, uma equação elíptica não linear associada a um sistema de equações parabólicas não lineares, provamos a convergência de segunda ordem, relativamente à norma-L^2 discreta, para as soluções das equações parabólicas. Propomos um método numérico para a equação elíptica, cujas soluções convergem para as correspondentes derivadas da solução contínua, com uma taxa de convergência de segunda ordem, relativamente a uma versão discreta da norma-H^1 usual. A respeito destes sistemas, a principal dificuldade na conceção de métodos numéricos eficientes e exatos é a dependência da velocidade convectiva, de uma das equações dependentes do tempo, no gradiente da solução da equação elíptica não linear.

Os resultados da convergência estabelecida podem ser vistos como resultados de supraconvergência, no âmbito da teoria da convergência de diferenças finitas e como resultados de superaproximação, dentro da teoria da convergência de elementos finitos. Também abordamos a estabilidade de métodos numéricos propostos. Visto tratar-se de problemas não lineares, de forma a obter estabilidade local, uma das imposições comuns necessárias é a limitação da sequência de soluções numéricas. Provamos que este imposição é uma consequência das propriedades do erro e, portanto, concluímos a estabilidade dos métodos propostos. Estão incluídas as experimentações numéricas ilustrativas dos resultados de convergência obtidos. Estas experimentações comprovam a precisão das hipóteses de regularidade das soluções dos problemas diferenciais. Do ponto de vista das aplicações da libertação de fármacos, o comportamento qualitativo dos sistemas anteriores é estudado numericamente em cenários diferentes: transporte de fármacos estimulado pela temperatura ou libertação de fármacos através da iontoforese. Por fim, observamos que os modelos matemáticos estudados anteriormente foram definidos utilizando a Lei de Fick para fluxo de fármacos que não consideram as propriedades viscoelásticas das matrizes em que os fármacos são distribuídos. Neste último capítulo, apresentamos um estudo exploratório da libertação pulsátil de fármacos para matrizes poliméricas termoresponsivas. O modelo tem em consideração as propriedades mecânicas da matriz polimérica sob o efeito da temperatura e é representado através de um problema de condições iniciais e de fronteira móvel. é considerada uma abordagem semi-analítica e a solução do problema de condições iniciais e de fronteira é obtida utilizando a análise de Fourier.

Palavras-chave: Libertação estimulada de fármacos - Temperatura - Método de diferenças Finitas - Método de Elementos Finitos Segmentado Linear - Convergência - Estabilidade

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Chapter I

Introduction

1 General framework

Drug delivery is a large domain of research on the development of new materials or transport systems, used for efficient therapeutic release of drugs. It plays a crucial role in disease treatment and it represents an important tool in the advancement of precision medicine. The most challenging problems faced by researchers in the area are the development of systems for targeted release, controlled release or enhanced release. Targeted release refers to systems that directly deliver drugs to specific parts of the body, avoiding global systemic absorption and degradation within the gastrointestinal tract, if taken orally. A therapeutic concentration of the drug is achieved at a specific site- the target organ or tissue- for a desired period without causing undesirable side effects. Examples of targeted delivery systems are cardiovascular drug delivery stents or intravitreal implants, where the drug is dispersed in a polymeric platform. In addition, the use of nanoparticles, as drug carriers, has revolutionized how drugs are delivered. After the drug targets the tissue or organ the release can be sustained or enhanced. The release is controlled or sustained when it is extended over a period, to keep concentration levels within a therapeutic window (see Figure I.1). In the case of polymeric implants the release can be controlled by tuning the properties of the polymer and of the drug-polymer interactions. The enhancement of drug release and of drug transport through the target tissue or organ is achieved with different stimuli as chemical enhancers or physical enhancers as electric fields, magnetic fields, ultrasounds or heat sources. These stimuli can be applied separately or coupled. They are used in different areas, but oncology is one of the most promising and challenging (see [10], [11], [24], [36] and [40])). In oncologic diseases, the transport of the chemotherapy cocktails can be made by specific nanoparticles and the stimuli act to enhance the drug release from the transporter ([19], [29], [35], [36], [39], [41]).

Another area of application is transdermal delivery where it is crucial to enhance the permeability of the stratum corneum, the outermost layer of the epidermis. In this case external stimuli as heat, electric fields or ultrasounds have been used with great success (see for instance [6], [31], [27], [37], [41], [49] and [56]). Ultrasounds play also a very important role in drug delivery to the brain where the stimulus act as a disruptor of the blood brain barrier ([34], [55]).

In the present dissertation we are mainly interested on drug delivery systems where the drug release is enhanced by the temperature. Heat has been used as enhancer in different situations as for

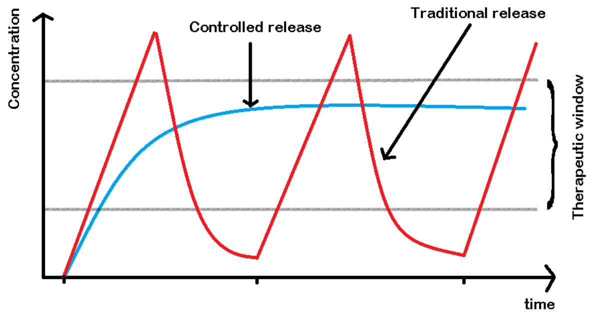

Fig. I.1 Different profiles of release

instance, in transdermal drug delivery. An increasing body of evidence suggests that temperature largely influences drug distribution, altering rate profile ([31]).

One popular application of heat in transdermal drug delivery are patches. We mention, for example patches where dispersed iron powder represent a heat source. Oxidation of the iron powder generates an increase of the temperature that lead to an increase of permeability of the skin as well as a decrease in its Young modulus ([56]). Consequently, an augmented the drug flux through the skin is observed, and due to the increase of the superficial blood perfusion, an intensification of the drug absorption occurs (see [41], [49]). Heat can be also generated by the application of other stimuli as ultrasounds [27] or electric fields [6].

When temperature increases, the pattern of Brownian motion is altered. In fact the, *rate* of diffusion defined by the diffusion coefficient strongly depends on temperature. In the case of spherical particles, diffusing in a liquid with low Reynolds number, the Stokes-Einstein equation postulates that the diffusion coefficient D is defined by

$$D = \frac{K_B T}{6\pi\eta r},$$

where T denotes the temperature, K_B is the Boltzman constant, r represents the radius of a spherical drug molecule and η the viscosity.

The diffusion coefficient in solids at a specific temperature T is given by the Arrhenius equation

$$D = D_0 \exp\left(-\frac{E_{\mathrm{A}}}{RT}\right), \tag{I.1}$$

where D_0 is the maximal diffusion coefficient (at infinite temperature), E_A is the activation energy for diffusion, and R denotes the universal gas constant.

In Chapter III we describe the drug transport in a tissue or organ Ω, by the convection-diffusion-reaction equation

$$\frac{\partial c}{\partial t} + \nabla.(v(T)c) = \nabla.(D_d(T)\nabla c) + Q(c) \text{ in } \Omega \times (0, t_f] \tag{I.2}$$

where c denotes the concentration and the temperature T is governed by

$$\frac{\partial T}{\partial t} = \nabla.(D_T(T)\nabla T) + G(T) \text{ in } \Omega \times (0, t_f]. \tag{I.3}$$

In (I.2), $v(T)$ denotes the drug release velocity, D_d is the diffusion coefficient and t_f denotes the final time. To describe the dependence of drug distribution on temperature, we assume that D_d in (I.2) is a function of the temperature T. We observe that equations (I.2) and (I.3) describe the drug evolution in two different situations: when heat is generated by a source term, like in heat patches applications, or when heat is generated as a secondary stimulus. Equations (I.2) and (I.3) should be coupled with drug release equations from a reservoir (nanoparticles loaded with the drug) as well as the equations for the reservoir transport. Moreover, in equation (I.2), the convective velocity depends explicitly in the temperature. To complete the model an equation for the velocity can be added [2].

The concentration and temperature equations, (I.2) and (I.3), respectively, are complemented with homogeneous Dirichlet boundary conditions

$$c(t) = 0 \text{ on } \partial\Omega \times (0,t_f], T(t) = 0 \text{ on } \partial\Omega \times (0,t_f], \tag{I.4}$$

and initial conditions

$$c(0) = c_0 \text{ in } \overline{\Omega}, T(0) = T_0 \text{ in } \overline{\Omega}. \tag{I.5}$$

However, heat is also generated as a consequence of the application of other physical enhancers as electric fields or ultrasounds. We remark that electric fields are currently used to enhance drug transport through the skin namely for electric charged drug molecules. In this case a convective drug transport arises induced by the electric field defined by the gradient of the electric potential ([6]). Moreover, nowadays we can assist to the use of electric fields to enhance drug transport and drug absorption in other contexts like pancreatic cancer ([8], [9]), breast cancer ([33]), ophthalmic applications (see [40], [28] and the references therein).

Several authors have addressed the problem of drug transport electrically enhanced. We mention without being exhaustive [6], [16], [42] and [47]. Thereby, in chapter IV is introduced the following model that includes a nonlinear elliptic equation for the electric field and the two parabolic equations from the previous model with the necessary adaptations.

$$-\nabla.(\sigma(|\nabla\phi|)\nabla\phi) = f \text{ in}\,\Omega, \tag{I.6}$$

$$\frac{\partial T}{\partial t} = \nabla.(D_T(T)\nabla T) + G(T) + F(\nabla\phi) \text{ in}\,\Omega \times (0,t_f] \tag{I.7}$$

$$\frac{\partial c}{\partial t} + \nabla.(v(T,\nabla\phi)c) = \nabla.(D_d(T)\nabla c) + Q(c) \text{ in}\,\Omega \times (0,t_f]. \tag{I.8}$$

that is completed by the following boundary and initial conditions

$$\phi = 0, T = 0, c = 0 \text{ on}\,\partial\Omega \times (0,t_f] \tag{I.9}$$

and

$$T(0) = T_0,\ c(0) = c_0 \text{ in}\,\overline{\Omega}. \tag{I.10}$$

In (I.6), $f = 0$, ϕ represents the electric potential, σ denotes the electrical conductivity coefficient. In [6] while studying the transdermal electroporation the authors defined σ by

$$\sigma(y) = \sigma_1 + (\sigma_0 - \sigma_1)\frac{e^{\frac{1}{B}(y-y_1)} - 1}{e^{\frac{1}{B}(y-y_0)} - 1}, \tag{I.11}$$

where $\sigma_i, y_i, i = 0, 1$, and B are convenient constants that can be computed from laboratorial data. In the model proposed in this chapter we give equation (I.7), that describes the generation of heat by an electric field, the form of a modified Pennes' bioheat equation. Penne's equation reads

$$\rho k_s \frac{\partial T}{\partial t} = \nabla.(D_T \nabla T) - \omega_m c_b (T - T_a) + q + Q_J, \tag{I.12}$$

where T denotes the temperature, ρ represents the tissue density, k_s is the specific heat of the tissue, D_T is the thermal conductivity, T_a is the arterial blood temperature, q is the metabolic volumetric heat generation, ω_m is the nondirectional blood flow associated with perfusion, c_b is the specific heat of blood and in (I.13), Q_J that denotes the heat generated by the applied potential ϕ, is given by $Q_J = \sigma|\nabla\phi|^2$, where σ is the electrical conductivity and $|.|$ represents the Euclidian norm (see for instance [6]). The modified form of Pennes' equation adopted in our model, assumes the form

$$\rho k_s \frac{\partial T}{\partial t} = \nabla.(D_T \nabla T) - \omega_m c_b (T - T_a) + q + F(\nabla\phi), \tag{I.13}$$

where F is given by

$$F(\nabla\phi) = \sigma(|\nabla\phi|)|\nabla\phi|^2, \tag{I.14}$$

In equation (I.8), D_d is the diffusion coefficient, v is the convective transport, that is related with the diffusion coefficient via Einstein-Smoluchowski relation, is given by the modified Nernst-Planck equation

$$v(T, \nabla\phi) = D_d \frac{zF_r}{TR} \nabla\phi + v_b, \tag{I.15}$$

where z is the drug valence, F_r Faraday's constant, R represents the universal gas constant and v_b denotes the electro-osmotic convective velocity.

The main numerical question that arises in the solution of the initial boundary value problem (IBVP) (I.6)-(I.8), (I.9), (I.10) is the dependence of the concentration variable on the gradient of the solution of the nonlinear elliptic equation (I.6). If the numerical approximation of the elliptic equation (I.6) is such that the numerical gradient does not converge to the corresponding continuous gradient, or if it converges with a lower convergence order, then the numerical approximation for the concentration does not converge to the corresponding continuous concentration or it converges with lower convergence order, respectively. This problem was previously studied in [4] for the non Fickian transport in a porous medium when Darcy's law is replaced by an elliptic linear equation for the pressure. Fickian transport in porous media were considered in [25].

In chapter V we study the effect of temperature on drug delivery, when the drug is dispersed in a thermoresponsive polymer. Thermoresponsive polymers are stimuli-responsive materials that exhibit a change of their physical properties with temperature. Ideally a thermoresponsive polymer, where a drug is dispersed, should retain it at body temperature (37 °C), and release the drug when the temperature is higher, signalizing an infectious process. As temperature increases, the force that keeps the atoms together decreases, the polymer collapses and the drug is released. More precisely thermoresponsive polymers have a phase transition temperature, the so-called Critical Solution Temperature (CST). A large class of polymers has a Lower Critical Solution Temperature (LCST), that is the polymers swell for temperatures below the LCST and they shrink when the temperature is above the LCST. In the case a drug is dispersed in a thermosensitive polymeric matrix, the drug is released as the polymer shrinks. In the chapters before V, we take into account the influence of the temperature on the diffusion coefficient and on the convection rate of the drug, but we assume that there was no effect of the temperature on the properties of the platform where the drug is dispersed. The model presented in chapter V is based on a different approach. In fact, in thermoresponsive materials, the phenomena underlying a pulsatile delivery are essentially related with changes in the properties of the platform.

In the previous chapters, we studied Fickian models that can be used to describe drug transport enhanced by heat. This means that the parabolic equations for the temperature and for the concentration are established from Fick's law for the flux

$$J(x,t) = -D\nabla \ell(x,t) \tag{I.16}$$

with the mass conservation equation

$$\frac{\partial \ell}{\partial t}(x,t) + \nabla J(x,t) = R(\ell(x,t)), x \in \Omega, t \in (0,t_f], \tag{I.17}$$

where $\ell = T, c$, and R represents the reaction terms.

2 Thesis description

One of the main objectives of the present work is the convergence analysis of finite difference methods for systems of nonlinear partial differential equations using lower smoothness assumptions for the solution than those usually considered. Classically, for linear initial value problems, the Lax-Richtmeyer equivalence theorem states that a consistent finite difference method is convergent if and only if is stable ([36]). From this result, a practical strategy used to study convergence of finite difference methods for linear initial value problems, is defined by

$$\text{Stability} + \text{Consistency} \Longrightarrow \text{Convergence},$$

where the convergence order is at least equal to the consistency order.

When the finite difference methods are defined with nonuniform meshes, the consistency order can be less than the order of the corresponding finite difference methods defined with uniform meshes. Consequently, based on the Lax-Richtmeyer theorem, we cannot conclude that the convergence order on nonuniform meshes is equal to the convergence order on uniform meshes.

There exists a long list of contributions showing that the convergence order of several linear finite difference methods defined on nonuniform grids is equal to the convergence order of the correspondent finite difference methods, defined on uniform grids. Without being exhaustive we mention the classical papers [26], [38], [18], where the analysis requires smoothness of the solutions of the continuous problem, and [5], [25], [19] where the convergence analysis requires less smoothness than in the first group of papers. We notice that only the work [25] deals with a system of partial differential equations defined by an elliptic equation and a parabolic equation.

We begin in Chapter II by studying a coupled abstract system composed by two nonlinear diffusion equations - for concentration and temperature. We propose a method that can be viewed as a fully discrete piecewise linear finite element method and we prove its supraconvergence.

In Chapter III we consider a more general coupled system - equations (I.3) and (I.2)- and we propose a stable and accurate method to compute numerical approximations for the temperature and concentration. The method is based on a piecewise linear finite element method combined with particular integration formulas. Such fully discrete method can be seen as a finite difference method defined on a nonuniform grid. We prove that the approximations for the temperature and concentration and their gradients are second order convergent with respect to discrete L^2-norms. It is well known that the piecewise linear finite element method (FEM) leads to first order approximations for the gradient. Such result shows the supercloseness of the gradient approximations presented in this work. As the fully discrete FEM is equivalent to a finite difference method defined on a nonuniform grid with first order truncation error with respect to the norm $\|.\|_{h,\infty}$. The supraconvergence of the new method is established. The results in Chapter III are published online in the journal *Computers & Mathematics with Applications*.

The system of PDE's studied in the last chapter is coupled with a nonlinear elliptic equation that will be presented in the next chapter.

The main objective of chapter IV is to design a numerical method for the IBVP (I.6)-(I.8), (I.9), (I.10) that leads to second order approximations for the numerical gradient of ϕ and for the temperature T, leading consequently to a second order approximation for the approximation of c. The main ingredient in our study is the extension of the results presented in [5] to the nonlinear elliptic equation (I.6) and the approach followed in [4] and [25]. The numerical method proposed belongs to the class of finite difference methods (FDM) but it is equivalent to a fully discrete in space piecewise linear finite element method (PLFEM). We will show that the FD approximation for ϕ is second order convergent to ϕ with respect a discrete H^1-norm while the numerical approximations for T and c are second order convergent to T and c, respectively, with respect to a discrete L^2-norm. These results are unexpected in the scope of FDM because the truncations errors associated with the FDM are only of first order with respect to the norm $\|.\|_\infty$ as well as in the scope of FEM because their are based in the piecewise linear FEM.

Until this point, we studied Fickian models to describe drug transport enhanced by temperature, in the remain chapter is introduced a hybrid Non Fickian model.

In chapter V, we construct an hybric Non Fickian mathematical model for the drug transport through the manipulation of an integro-differential equation. For the IBVP's defined in special spatial and discrete time domains, are computed the solution.

In chapter VI are addressed some conclusions about the work and referenced a variety of problems that remain open and that will be subject of study in the future.

To complete this work and cover the results needed in the proofs we attached the Appendix with relevant definitions and results.

Chapter II

Convergence analysis for an abstract coupled problem

1 Introduction

In this chapter our objective is to analyse a finite difference scheme for the following system, assuming that $\Omega = (a,b)$, t_f denotes a final time and D_T is constant:

$$\frac{\partial T}{\partial t}(x,t) = D_T \frac{\partial^2 T}{\partial x^2}(x,t), (x,t) \in \Omega \times (0,t_f] \tag{II.1}$$

and

$$\frac{\partial c}{\partial t}(x,t) = \frac{\partial}{\partial x}\left(D_d(T(x,t)) \frac{\partial c}{\partial x}(x,t) \right), (x,t) \in \Omega \times (0,t_f]\,. \tag{II.2}$$

The system (II.1), (II.2) is completed by the boundary conditions

$$T(t) = 0 \text{ and } c(t) = 0 \text{ on } \partial\Omega \times (0,t_f] \tag{II.3}$$

and the initial conditions

$$T(0) = T_0 \text{ and } c(0) = c_0 \text{ in } \overline{\Omega}. \tag{II.4}$$

To simplify the presentation the following notation are used: if $w : \overline{\Omega} \times [0,t_f] \to \mathbb{R}$, we represent by $w(t)$ the function $w(t) : \overline{\Omega} \to \mathbb{R}$ such that $w(t)(x) = w(x,t), x \in \overline{\Omega}$.

The finite difference method that will be studied is defined on nonuniform grids and it can be seen as a fully discrete piecewise linear finite element method. The convergence analysis will be performed assuming that $T(t), c(t) \in C^4(\overline{\Omega}),\ t \in (0,t_f]$. We show that the numerical approximations for $T(t)$ and $c(t)$, $T_h(t)$ and $c_h(t)$, respectively, are second order accurate approximations.

This chapter is composed by five sections that we describe briefly. In Section 2 we present some preliminary definitions and results. The first convergence results are presented in Section 3 where we establish first order estimates for the errors of $T_h(t)$ and $c_h(t)$, assuming that $T(t), c(t) \in C^3(\overline{\Omega})$. The convergence orders established in the previous section are improved in Section 4 assuming that $T(t), c(t) \in C^4(\overline{\Omega})$.

2 Preliminary definitions and results

Let Λ be a sequence of vectors h of positive entries $(h_1,\cdots,h_N)$ such that $\sum_{i=1}^{N} h_i = b-a$ and $h_{max} = \max_i h_i$, such that $h_{max} \to 0$. For $h \in \Lambda$ we introduce in $\overline{\Omega}$ the nonuniform grid

$$\overline{\Omega}_h = \{x_i, i=0,\cdots,N, x_i - x_{i-1} = h_i, i=1,\ldots,N, x_0 = a, x_N = b\}.$$

We denote by Ω_h and $\partial\Omega_h$ the set of interior nodes $\Omega \cap \overline{\Omega}_h$ and the boundary points $\partial\Omega \cap \overline{\Omega}_h$, respectively.

By W_h we represent the space of grid functions defined in $\overline{\Omega}_h$ and the space of grid functions in W_h that are null at the boundary points is denoted by $W_{h,0}$.

We introduce in the following lines, the inner products and norms in W_h and $W_{h,0}$, as well as the finite difference operators that we need at this work. In $W_{h,0}$ we define the inner product

$$(u_h, v_h)_h = \sum_{i=1}^{N-1} h_{i+1/2} u_h(x_i) v_h(x_i), u_h, v_h \in W_{h,0},$$

where $h_{i+1/2} = \dfrac{h_i + h_{i+1}}{2}$. The corresponding norm is denoted by $\|.\|_h$. We also use the related notations

$$(u_h, v_h)_+ = \sum_{i=1}^{N} h_i u_h(x_i) v_h(x_i), u_h, v_h \in W_h,$$

$$\|u_h\|_+^2 = \sum_{i=1}^{N} h_i (u_h(x_i))^2.$$

We introduce now the following finite difference operators:

$$D_{-x} u_h(x_i) = \frac{u_h(x_i) - u_h(x_{i-1})}{h_i}, \ i = 1,\cdots,N \tag{II.5}$$

$$D_x^* u_h(x_i) = \frac{u_h(x_{i+1}) - u_h(x_i)}{h_{i+1/2}}, \ i = 1,\cdots,N-1, \tag{II.6}$$

$$D_2 u_h(x_i) = \frac{D_{-x} u_h(x_{i+1}) - D_{-x} u_h(x_i)}{h_{i+1/2}}, i = 1,\cdots,N-1,$$

where $u_h \in W_h$.

In the next result we establish a discrete version of the *integration by parts rule* and a discrete version of the Poincaré-Friedrich's inequality.

Proposition 2.0.1 *[30] For all $u_h \in W_h$ and $v_h \in W_{h,0}$, we deduce that*

$$(-D_x^* u_h, v_h)_h = (u_h, D_{-x} v_h)_+,$$

$$(-D_2 u_h, v_h)_h = (D_{-x} u_h, D_{-x} v_h)_+,$$

and

$$\|v_h\|_h^2 \leq |\Omega| \|D_{-x} v_h\|_+^2.$$

where $|\Omega|$ *denotes the measure of* Ω.

Therefore, we are able to present the semi-discrete finite difference method. By $T_h(t)$ and $c_h(t)$ we represent the semi-discrete approximations for $T(t)$ and $c(t)$, respectively, defined by the following ordinary differential systems

$$\begin{cases} T_h'(t) = D_T D_2 T_h(t) \text{ in } \Omega_h \times (0, t_f] \\ T_h(t) = 0 \text{ on } \partial\Omega_h \times (0, t_f] \\ T_h(0) = R_h T_0 \text{ in } \Omega_h \end{cases} \tag{II.7}$$

$$\begin{cases} c_h'(t) = D_x^*(D_d(M_h T_h(t)) D_{-x} c_h(t)) \text{ in } \Omega_h \times (0, t_f] \\ c_h(t) = 0 \text{ on } \partial\Omega_h \times (0, t_f] \\ c_h(0) = R_h c_0 \text{ in } \Omega_h, \end{cases} \tag{II.8}$$

where M_h denotes the average operator

$$M_h u_h(x_i) = \frac{1}{2}(u_h(x_{i-1}) + u_h(x_i)), i = 1, \ldots, N, u_h \in W_h. \tag{II.9}$$

We notice that $T_h(t)$ and $c_h(t)$ defined by (II.7), (II.8), respectively, can be seen as fully discrete piecewise linear finite element solutions. In fact, the weak formulation of the initial boundary value problems (IBVP) (II.1), (II.2), (II.3), (II.4) is given by

$$\begin{aligned} &(T'(t), u) = D_T \left(\frac{\partial T}{\partial x}(t), \frac{\partial u}{\partial x} \right) \text{ a.e. in } (0, t_f], \forall u \in H_0^1(\Omega), \\ &(T(0), u) = (T_0, u), \forall u \in L^2(\Omega), \end{aligned} \tag{II.10}$$

and

$$\begin{aligned} &(c'(t), w) = \left(D_d(T(t)) \frac{\partial c}{\partial x}(t), \frac{\partial w}{\partial x} \right) \text{ a.e. in } (0, t_f], \forall w \in H_0^1(\Omega), \\ &(c(0), w) = (c_0, w), \forall w \in L^2(\Omega). \end{aligned} \tag{II.11}$$

In (II.10), (II.11), *a.e.* means almost everywhere and by $L^2(\Omega)$ and $H_0^1(\Omega)$ we denote the usual Sobolev spaces.

Considering that for $w_h \in W_h$, $P_h w_h$ denotes the continuous piecewise linear interpolation of w_h with respect to the partition $\overline{\Omega}_h$. The piecewise linear finite element approximations for $T(t)$ and $c(t)$, defined by (II.10), (II.11), are computed considering the piecewise linear interpolation functions $P_h T_h(t), P_h c_h(t) \in H_0^1(\Omega)$, $T_h(t), c_h(t) \in W_{h,0}$, that are solutions of the following weak problems:

$$\begin{aligned} &(P_h T_h'(t), P_h u_h) = D_T \left(\frac{\partial P_h T_h}{\partial x}(t), \frac{\partial P_h u_h}{\partial x} \right) \text{ in } (0, t_f], \forall u_h \in W_{h,0}, \\ &(P_h T_h(0), P_h u_h) = (P_h R_h T_0, P_h u_h), \forall u_h \in W_{h,0}, \end{aligned} \tag{II.12}$$

and

$$
\begin{aligned}
&(P_h c_h'(t), P_h w_h) = \left(D_d(P_h T_h(t)) \frac{\partial P_h c_h}{\partial x}(t), \frac{\partial P_h w_h}{\partial x} \right) \text{ in } (0,t_f], \forall w_h \in W_{h,0}, \\
&(P_h c_h(0), P_h w_h) = (P_h R_h c_0, P_h w_h), \forall w_h \in W_{h,0},
\end{aligned}
\tag{II.13}
$$

where $R_h : C(\overline{\Omega}) \to W_h$ denotes the restriction operator $R_h u(x_i) = u(x_i), i = 0, \dots, N$.

The two finite problems (II.12), (II.13) are then replaced by the fully discrete piecewise linear finite element approximations

$$
\begin{aligned}
&(T_h'(t), u_h)_h = -D_T(D_{-x}T_h(t), D_{-x}u_h)_+ \text{ in } (0,t_f], \forall u_h \in W_{h,0}, \\
&(T_h(0), u_h)_h = (R_h T_0, u_h)_h, \forall u_h \in W_{h,0},
\end{aligned}
\tag{II.14}
$$

and

$$
\begin{aligned}
&(c_h'(t), w_h)_h = -(D_d(M_h T_h(t)) D_{-x}c_h(t), D_{-x}w_h)_+ \text{ in } (0,t_f], \forall w_h \in W_{h,0}, \\
&(c_h(0), w_h)_h = (R_h c_0, w_h)_h, \forall w_h \in W_{h,0}.
\end{aligned}
\tag{II.15}
$$

Finally, choosing in each equation of (II.14), (II.15) a sequence of grid functions where each element is equal to one in a grid point and zero in the others we deduce the IBVP (II.7) and (II.8).

3 Convergence analysis for solutions in $C^3(\overline{\Omega})$

In this section we establish estimates for the errors $E_T(t) = R_h T(t) - T_h(t), E_c(t) = R_h c(t) - c_h(t)$, where $T_h(t), c_h(t)$ are given by (II.7), (II.8) or (II.14), (II.15).

We assume that the coefficient function D_d has bounded first derivative, which means that it belongs to $C_b^1(\mathbb{R})$ and satisfies the following assumption: $D_d \geq \beta \geq 0$ in $\mathbb{R}$.

3.1 Temperature

We start by studying the error $E_T(t) = R_h T(t) - T_h(t)$.

Theorem 3.1.1 *Let the solution of T (II.1) be in $L^2(0,t_f,C^3(\Omega))$ and let T_h be defined by (II.7), such that*

$$R_h T, T_h \in C^1([0,t_f], W_{h,0}).$$

Then there exists a positive constant Const, h and t independent, such that $E_T(t) = R_h T(t) - T_h(t)$ satisfies

$$\|E_T(t)\|_h^2 + \int_0^t \|D_{-x}E_T(s)\|_+^2 ds \leq Const h_{max}^2 \int_0^t \|T(s)\|_{C^3(\Omega)}^2 ds, \tag{II.16}$$

for $t \in [0,t_f]$ and $h \in \Lambda$.

Proof: Let $T_{r,T}(t)$ be the truncation error induced by the spatial discretization defined in (II.7). Taking into account that $D_2 R_h T(x_i,t)$ will be obtained through Taylor expansion series, for the truncation error, we get the following representation

$$\begin{aligned} T_{r,T}(x_i,t) &= D_T\left(\frac{\partial^2 T}{\partial x^2}(x_i,t) - D_2 R_h T(x_i,t)\right) \\ &= \frac{D_T}{6h_{i+\frac{1}{2}}}\Big(h_{i+1}^2\frac{\partial^3 T}{\partial x^3}(\eta_i,t) - h_i^2\frac{\partial^3 T}{\partial x^3}(\xi_i,t)\Big), \end{aligned}$$

where $\eta_i,\xi_i \in [x_{i-1},x_{i+1}], i=1,\ldots,N-1$.

For the error $E_T(t)$ we obtain successively

$$\begin{aligned} (E_T'(t),E_T(t))_h &= (R_h T'(t) - T_h'(t),E_T(t))_h \\ &= \left(D_T R_h \frac{\partial^2 T}{\partial x^2}(t) - D_T D_2 T_h(t),E_T(t)\right)_h \\ &= D_T(D_2E_T(t),E_T(t))_h + (T_{r,T}(t),E_T(t))_h \\ &= -D_T(D_{-x}E_T(t),D_{-x}E_T(t))_+ + (T_{r,T}(t),E_T(t))_h,\ t\in(0,t_f]. \end{aligned}$$

where $t\in(0,t_f]$.

Young's inequality leads to

$$\frac{1}{2}\frac{d}{dt}\|E_T(t)\|_h^2 + D_T\|D_{-x}E_T(t)\|_+^2 \le \frac{1}{4\varepsilon^2}\|T_{r,T}(t)\|_h^2 + \varepsilon^2\|E_T(t)\|_h^2,$$

where $\varepsilon \neq 0$. Considering now the discrete Poincaré inequality we get

$$\frac{d}{dt}\|E_T(t)\|_h^2 + 2(D_T - |\Omega|\varepsilon^2)\|D_{-x}E_T(t)\|_+^2 \le \frac{1}{2\varepsilon^2}\|T_{r,T}(t)\|_h^2. \tag{II.17}$$

To establish an estimation for $E_T(t)$, we compute an upper bound for $\|T_{r,T}(t)\|_h^2$. We have, successively,

$$\begin{aligned} \|T_{r,T}(t)\|_h^2 &\le \sum_{i=1}^{N-1}\frac{D_T^2}{h_{i+\frac{1}{2}}}\left(\frac{h_i^4}{18}\left(\frac{\partial^3 T}{\partial x^3}(\xi_i)\right)^2 + \frac{h_{i+1}^4}{18}\left(\frac{\partial^3 T}{\partial x^3}(\eta_i)\right)^2\right) \\ &\le \sum_{i=1}^{N-1} D_T^2\Big(\frac{h_i^3}{9}\|T\|_{C^3(\Omega)}^2 + \frac{h_{i+1}^3}{9}\|T\|_{C^3(\Omega)}^2\Big) \\ &\le \frac{2}{9}D_T^2|\Omega|\|T\|_{C^3(\Omega)}^2 h_{max}^2 \end{aligned}$$

Then, if we fix ε such that $D_T - \varepsilon^2|\Omega| > 0$, and if we take

$$Const = \frac{1}{18\varepsilon^2}\frac{D_T^2|\Omega|}{D_T - \varepsilon^2|\Omega|},$$

we establish

$$\frac{d}{dt}\Big(\|E_T(t)\|_h^2 + \int_0^t \|D_{-x}E_T(s)\|_+^2 ds - Const\, h_{max}^2\int_0^t \|T(s)\|_{C^3(\overline{\Omega})ds}^2\Big) \le 0, t\in[0,t_f]. \tag{II.18}$$

that leads to (II.16)

■

3.2 Concentration

In this section we establish an upper bound for the error $E_c(t) = R_h c(t) - c_h(t)$, where c is defined by (II.2). However, c depends on the solution T of (II.1), the upper bound for $E_c(t)$ depends on the error $E_T(t)$ as well as on the truncation error associated with the spatial discretization defined in (II.8).

Theorem 3.2.1 *Let the solutions T and c of (II.1) and (II.2), respectively, belong to $L^2(0,t_f,C^3(\overline{\Omega}))$ and let T_h, c_h be defined by (II.7) and (II.8), respectively. Let $E_T(t)$ and $E_c(t)$ be the spatial discretization errors $E_T(t) = R_h T(t) - T_h(t)$ and $E_c(t) = R_h c(t) - c_h(t)$. If*

$$R_h c, c_h \in C^1([0,t_f], W_{h,0}), R_h T, T_h \in C([0,t_f], W_{h,0}),$$

there exists a positive constant Const, space and time-independent, such that

$$\begin{aligned} \|E_c(t)\|_h^2 + \int_0^t \|D_{-x}E_c(s)\|_+^2 ds \leq Const \Bigg(\int_0^t \|E_T(s)\|_h^2 \|c(s)\|^2_{C^1(\overline{\Omega})} ds \\ + h_{max}^2 \int_0^t \|c(s)\|^2_{C^3(\overline{\Omega})} (\|T(s)\|^2_{C^2(\overline{\Omega})} + 1) ds + h_{max}^2 \Bigg), \end{aligned} \tag{II.19}$$

for $t \in [0,t_f]$ and $h \in \Lambda$.

Proof: Let $T_{r,c}(t)$ be the truncation error induced by the spatial discretization defined in (II.8)

$$T_{r,c}(x_i,t) = \frac{\partial}{\partial x}\left(D_d(T(x_i,t)) \frac{\partial c}{\partial x}(x_i,t) \right) - D_x^*(D_d(M_h R_h T(x_i,t)) D_{-x} R_h c(x_i,t)),$$

for $i = 1, \ldots, N-1$. It can be shown that $T_{r,c}(t)$ admits the representation

$$\begin{aligned} T_{r,c}(t) &= D_d'(T(x_i,t)) \left(\frac{h_i - h_{i+1}}{2} \right) \left[\frac{\partial^2 T}{\partial x^2}(x_i,t) \frac{\partial c}{\partial x}(x_i,t) + \frac{\partial T}{\partial x}(x_i,t) \frac{\partial^2 c}{\partial x^2}(x_i,t) \right] \\ &+ \frac{D_d(T(x_i,t))}{6h_{i+1/2}} \left(h_i^2 \frac{\partial^3 c}{\partial x^3}(\xi_i,t) - h_{i+1}^2 \frac{\partial^3 c}{\partial x^3}(\eta_i,t) \right) + \mathscr{O}(h_{max}^2), \end{aligned} \tag{II.20}$$

where $\mathscr{O}(h_{max}^2)$ represents a term, depending on $\|c(t)\|_{C^3(\overline{\Omega})}$ and $\|T(t)\|_{C^3(\overline{\Omega})}$, such that

$$|\mathscr{O}(h_{max}^2)| \leq Const\, h_{max}^2,$$

where $Const$ represents a positive space and time independent constant.

From (II.20), for $\|T_{r,c}(t)\|_h^2$ we have, successively,

$$\begin{aligned} \|T_{r,c}(t)\|_h^2 &\leq 4 \sum_{i=1}^{N-1} h_{i+1/2} (D_d'(T_i))^2 (h_i - h_{i+1})^2 \left[\left(\frac{\partial^2 T}{\partial x^2} \right)^2 \left(\frac{\partial c}{\partial x} \right)^2 + \left(\frac{\partial T}{\partial x} \right)^2 \left(\frac{\partial^2 c}{\partial x^2} \right)^2 \right] \\ &+ \sum_{i=1}^{N-1} \frac{D_d^2}{9h_{i+1/2}} \left[h_i^4 \left(\frac{\partial^3 c}{\partial x^3} \right)^2 + h_{i+1}^4 \left(\frac{\partial^3 c}{\partial x^3} \right)^2 \right] + \mathscr{O}(h_{max}^4) \sum_{i=1}^{N-1} h_{i+1/2} \\ &\leq \|D_d\|^2_{C_b^1(\mathbf{R})} |\Omega| \left(16 h_{max}^2 \|T\|^2_{C^2(\Omega)} \|c\|^2_{C^2(\Omega)} + \frac{4}{9} h_{max}^2 \|c\|^2_{C^3(\Omega)} + \mathscr{O}(h_{max}^4) \right) \end{aligned}$$

where, to simplify we omit the arguments of T and c. It is easy then to conclude that

$$\|T_{r,c}(t)\|_h^2 \leq Const h_{max}^2 \Big(\|T(t)\|_{C^2(\Omega)}^2 \|c(t)\|_{C^2(\Omega)}^2 + \|c(t)\|_{C^3(\Omega)}^2 + h_{max}^2\Big). \tag{II.21}$$

For the error $E_c(t)$ we have, successively,

$$\begin{aligned}(E_c'(t), E_c(t))_h &= (R_h c'(t) - c_h'(t), E_c(t))_h \\ &= -(D_d(M_h R_h T(t)) D_{-x} R_h c(t), D_{-x} E_c(t))_+ + (D_d(M_h T_h(t)) D_{-x} c_h(t), D_{-x} E_c(t))_+ \\ &\quad + (T_{r,c}(t), E_c(t))_h \\ &= -([D_d(M_h R_h T(t)) - D_d(M_h T_h(t))] D_{-x} R_h c(t), D_{-x} E_c(t))_+ \\ &\quad - ((D_d(M_h T_h(t))) D_{-x} E_c(t), D_{-x} E_c(t))_+ + (T_{r,c}(t), E_c(t))_h.\end{aligned} \tag{II.22}$$

As we have

$$\begin{aligned}&|([D_d(M_h R_h T(t)) - D_d(M_h T_h(t))] D_{-x} R_h c(t), D_{-x} E_c(t))_+| \\ &\quad \leq \|D_d\|_{C_b^1(\mathbb{R})} \sqrt{2} \|E_T(t)\|_h \|c(t)\|_{C^1(\overline{\Omega})} \|D_{-x} E_c(t)\|_+,\end{aligned}$$

for $\varepsilon_i \neq 0, i = 1,2$, considering the assumption H_1 we obtain

$$\begin{aligned}\frac{d}{dt}\|E_c(t)\|_h^2 \quad &+2(\beta - \varepsilon_1^2 - |\Omega|\varepsilon_2^2)\|D_{-x}E_c(t)\|_+^2 \\ &\leq \frac{1}{\varepsilon_1^2}\|D_d\|_{C_b^1(\mathbb{R})}\|E_T(t)\|_h^2\|c(t)\|_{C^1(\overline{\Omega})}^2 + \frac{1}{2\varepsilon_2^2}\|T_{r,c}(t)\|_h^2.\end{aligned} \tag{II.23}$$

Then, fixing constants $\varepsilon_i, i = 1,2$, such that

$$2(\beta - \varepsilon_1^2 - |\Omega|\varepsilon_2^2) > 0,$$

we guarantee the existence of a positive constant, h and t independent, such that

$$\|E_c(t)\|_h^2 + \int_0^t \|D_{-x}E_c(s)\|_+^2 ds \leq Const \int_0^t \|E_T(s)\|_h^2 \|c(s)\|_{C^1(\overline{\Omega})}^2 + \|T_{r,c}(s)\|_h^2 ds \tag{II.24}$$

finally taking into account the upper bound (II.21) we deduce (II.19). ■

Corollary 3.2.2 *Under the assumptions of Theorems 3.1.1 and 3.2.1 for $E_c(t) = R_h c(t) - c_h(t)$ we have*

$$\|E_c(t)\|_h^2 + \int_0^t \|D_{-x}E_c(s)\|_+^2 ds \leq Const\, h_{max}^2 \int_0^t \|c(s)\|_{C^3(\Omega)}^2 \big(\|T(s)\|_{C^3(\Omega)}^2 + 1\big) ds + h_{max}^2. \tag{II.25}$$

4 Convergence analysis for solutions in $C^4(\overline{\Omega})$

In Theorems 3.1.1 and 3.2.1 we establish that if the solutions $T(t)$ and $c(t)$ are in $C^3(\overline{\Omega})$, then

$$\|E_T(t)\|_h \leq Const h_{max}, \|E_c(t)\|_h \leq Const h_{max},$$

and

$$\int_0^t \|D_{-x}E_T(s)\|_+^2 ds \leq Const h_{max}^2, \int_0^t \|D_{-x}E_c(s)\|_+^2 ds \leq Const h_{max}^2,\ t \in [0,t_f],\ h \in \Lambda.$$

In this section we increase the convergence orders by increasing the hypothesis on the regularity of $T(t)$ and $c(t)$, namely we assume that $T(t), c(t) \in C^4(\overline{\Omega})$.

4.1 Temperature

Theorem 4.1.1 *Let the solution T of (II.1) be in $L^2(0,t_f,C^4(\overline{\Omega}))$ and let T_h be defined by (II.7), such that*

$$R_hT, T_h \in C^1([0,t_f], W_{h,0}).$$

Then there exists a positive constant Const, space and time independent, such that $E_T(t) = R_hT(t) - T_h(t)$ satisfies

$$\|E_T(t)\|_h^2 + \int_0^t \|D_{-x}E_T(s)\|_+^2 ds \leq Const h_{max}^4 \int_0^t \|T(s)\|_{C^4(\overline{\Omega})}^2 ds, \tag{II.26}$$

for $t \in [0,t_f]$ and $h \in \Lambda$.

Proof: As in the proof of the Theorem 3.1.1, we have

$$\frac{1}{2}\frac{d}{dt}\|E_T(t)\|_h^2 + D_T\|D_{-x}E_T(t)\|_+^2 = (T_{r,T}(t), E_T(t))_h. \tag{II.27}$$

We establish in what follows an upper bound for $(T_{r,T}(t), E_T(t))_h$. Taking into account that $T(t) \in C^4(\overline{\Omega})$, we have for $T_{r,T}(t)$ the following representation

$$T_{r,T}(x_i,t) = \frac{D_T}{3}(h_{i+1}-h_i)\frac{\partial^3 T}{\partial x^3}(x_i,t) + \mathscr{O}(h_{max}^2), \tag{II.28}$$

where $|\mathscr{O}(h_{max}^2)| \leq Const h_{max}^2 \|T(t)\|_{C^4(\overline{\Omega})}$. Then for $(T_{r,T}(t), E_T(t))_h$ we obtain

$$(T_{r,T}(t), E_T(t))_h = \frac{D_T}{6}\sum_{i=1}^{N-1} h_i^2 \left(\frac{\partial^3 T}{\partial x^3}(x_i,t)E_T(x_i,t) - \frac{\partial^3 T}{\partial x^3}(x_{i-1},t)E_T(x_{i-1},t)\right) + (\mathscr{O}(h_{max}^2), E_T(t))_h,$$

that leads to

$$\begin{aligned}
&(T_{r,T}(t), E_T(t))_h = \frac{D_T}{6}\sum_{i=1}^{N-1} h_i^2 \frac{\partial^3 T}{\partial x^3}(x_i,t)(E_T(x_i,t) - E_T(x_{i-1},t)) \\
&+\frac{D_T}{6}\sum_{i=1}^{N-1} h_i^2 \left(\frac{\partial^3 T}{\partial x^3}(x_i,t) - \frac{\partial^3 T}{\partial x^3}(x_{i-1},t)\right) E_T(x_{i-1},t)) + (\mathscr{O}(h_{max}^2), E_T(t))_h \\
&\leq \frac{D_T}{6}\underbrace{\sum_{i=1}^{N-1} h_i^3 \frac{\partial^3 T}{\partial x^3}(x_i,t)D_{-x}E_T(x_i,t)}_{A} + \frac{D_T}{6}\underbrace{\sum_{i=1}^{N-1} h_i^2 \int_{x_{i-1}}^{x_i} \frac{\partial^4 T}{\partial x^4}(x,t)dx E_T(x_{i-1},t)}_{B} + (\mathscr{O}(h_{max}^2), E_T(t))_h \\
&= \frac{D_T}{6}(A+B) + (\mathscr{O}(h_{max}^2), E_T(t))_h.
\end{aligned}$$

For A and B we have the following upper bounds

$$|A| \leq h_{max}^2 \sqrt{|\Omega|} \|T(t)\|_{C^3(\overline{\Omega})} \|D_{-x} E_T(t)\|_+, \tag{II.29}$$

$$|B| \leq h_{max}^2 \sqrt{2} \left\| \frac{\partial^4 T}{\partial x^4}(t) \right\|_{L^2(\Omega)} \|E_T(t)\|_h, \tag{II.30}$$

respectively.

Thus, applying Young's and Poincaré inequalities to the previous estimates and considering $\varepsilon_i \neq 0, i = 1,2,3$, we conclude

$$\begin{aligned}|(T_{r,T}(t), E_T(t))_h| \quad \leq h_{max}^4 \Big(\frac{D_T^2}{36} \left(\frac{1}{4\varepsilon_1^2} |\Omega| \|T(t)\|_{C^3(\overline{\Omega})}^2 + \frac{1}{\varepsilon_2^2} \left\| \frac{\partial^4 T}{\partial x^4}(t) \right\|_{L^2(\Omega)}^2 \right) \\ + Const \tfrac{1}{\varepsilon_3^2} \|T(t)\|_{C^4(\overline{\Omega})}^2 \Big) + (\varepsilon_1^2 + |\Omega|(\varepsilon_2^2 + \varepsilon_3^2)) \|D_{-x} E_T(t)\|_+^2.\end{aligned} \tag{II.31}$$

Using this last upper bound in (II.27) we deduce

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt} \|E_T(t)\|_h^2 + (D_T - (\varepsilon_1^2 + |\Omega|(\varepsilon_2^2 + \varepsilon_3^2))) \|D_{-x} E_T(t)\|_+^2 \\ &\quad \leq h_{max}^4 \Big(\frac{D_T^2}{36} \Big(\frac{1}{4\varepsilon_1^2} |\Omega| \|T(t)\|_{C^3(\overline{\Omega})}^2 + \frac{1}{\varepsilon_2^2} \left\| \frac{\partial^4 T}{\partial x^4}(t) \right\|_{L^2(\Omega)}^2 \Big) + Const \frac{1}{\varepsilon_3^2} \|T(t)\|_{C^4(\overline{\Omega})}^2 \Big),\end{aligned} \tag{II.32}$$

for $t \in (0, t_f]$. Fixing in (II.32) $\varepsilon_i \neq 0, i = 1,2,3$, such that

$$D_T - (\varepsilon_1^2 + |\Omega|(\varepsilon_2^2 + \varepsilon_3^2)) > 0,$$

we guarantee the existence of a positive constant $Const$, h and t independent, such that (II.26) holds. ■

4.2 Concentration

Theorem 4.2.1 *Let the solutions T and c belong to (II.1) and (II.2), respectively, in $L^2(0, t_f, C^4(\overline{\Omega}))$ and let T_h, c_h be defined by (II.7) and (II.8), respectively. Let $E_T(t)$ and $E_c(t)$ be the spatial-discretization errors $E_T(t) = R_h T(t) - T_h(t)$ and $E_c(t) = R_h c(t) - c_h(t)$. If*

$$R_h c, c_h \in C^1([0, t_f], W_{h,0}), R_h T, T_h \in C([0, t_f], W_{h,0}),$$

there exists a positive constant h and t independent such that

$$\begin{aligned}\|E_c(t)\|_h^2 + \int_0^t \|D_{-x} E_c(s)\|_+^2 ds \leq Const \Big(\int_0^t \|E_T(s)\|_h^2 \|c(s)\|_{C^1(\overline{\Omega})}^2 ds \\ + h_{max}^4 \int_0^t \|c(s)\|_{C^4(\overline{\Omega})}^2 (\|T(s)\|_{C^3(\overline{\Omega})}^2 + 1) ds \Big),\end{aligned} \tag{II.33}$$

for $t \in [0, t_f]$ and $h \in \Lambda$.

Proof: The truncation error induced by the spatial discretization defined in (II.8) has the representation

$$T_{r,c}(x_i,t) = (h_i - h_{i+1})\Big(\frac{1}{2}D'_d(T(x_i,t))\left[\frac{\partial^2 T}{\partial x^2}(x_i,t)\frac{\partial c}{\partial x}(x_i,t) + \frac{\partial T}{\partial x}(x_i,t)\frac{\partial^2 c}{\partial x^2}(x_i,t)\right] + \frac{1}{3}D_d(T(x_i,t))\frac{\partial^3 c}{\partial x^3}(x_i,t)\Big) + \mathscr{O}(h_{max}^2),$$

where $|\mathscr{O}(h_{max}^2)| \leq Const\|c(t)\|_{C^4(\overline{\Omega})}\big(\|T(t)\|_{C^3(\overline{\Omega})}+1\big)h_{max}^2$.

Let $g(x_i,t)$, $i=1,\ldots,N-1$, be defined by

$$g(x_i,t) = \Big(\frac{1}{3}D'_d(T(x_i,t))\left[\frac{\partial^2 T}{\partial x^2}(x_i,t)\frac{\partial c}{\partial x}(x_i,t) + \frac{\partial T}{\partial x}(x_i,t)\frac{\partial^2 c}{\partial x^2}(x_i,t)\right] + \frac{1}{3}D_d(T(x_i,t))\frac{\partial^3 c}{\partial x^3}(x_i,t)\Big).$$

Then, as in the proof of Theorem 4.1.1 we have

$$|(T_{r,c}(t),E_c(t))_h| \leq \frac{1}{2}h_{max}^2\Big(\|g(t)\|_{C(\overline{\Omega})}\sqrt{|\Omega|}\|D_{-x}E_c(t)\|_+ + \sqrt{2}\left\|\frac{\partial g}{\partial x}(t)\right\|_{L^2(\Omega)}\|E_c(t)\|_h\Big) + |\mathscr{O}(h_{max}^2)|\|E_c(t)\|_h.$$

Consequently, the discrete Poincaré-Friedrichs inequality leads to

$$\begin{aligned}|(T_{r,c}(t),E_c(t))_h| &\leq \Big(\frac{1}{2}h_{max}^2\sqrt{|\Omega|}\Big(\|g(t)\|_{C(\overline{\Omega})}+\sqrt{2}\left\|\frac{\partial g}{\partial x}(t)\right\|_{L^2(\Omega)}\Big)+|\mathscr{O}(h_{max}^2)|\sqrt{|\Omega|}\Big)\|D_{-x}E_c(t)\|_+\\
&\leq \frac{1}{4\varepsilon^2}\Big(\frac{1}{2}h_{max}^2\sqrt{|\Omega|}\Big(\|g(t)\|_{C(\overline{\Omega})}+\sqrt{2}\left\|\frac{\partial g}{\partial x}(t)\right\|_{L^2(\Omega)}\Big)+|\mathscr{O}(h_{max}^2)|\sqrt{|\Omega|}\Big)^2+\varepsilon^2\|D_{-x}E_c(t)\|_+^2\\
&\leq \frac{1}{2\varepsilon_1^2}|\Omega|\Big(h_{max}^4\Big(\|g(t)\|_{C(\overline{\Omega})}^2+2\left\|\frac{\partial g}{\partial x}(t)\right\|_{L^2(\Omega)}^2\Big)+|\mathscr{O}(h_{max}^2)|^2\Big)+\varepsilon_1^2\|D_{-x}E_c(t)\|_+^2.\end{aligned}$$

where $\varepsilon_1 \neq 0$.

Following the proof of Theorem 3.2.1, instead of the inequality (II.23), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|E_c(t)\|_h^2+(\beta-\varepsilon_1^2-\varepsilon_2^2)\|D_{-x}E_c(t)\|_+^2 &\leq \frac{1}{2\varepsilon_2^2}\|D_d\|_{C_b^1(\mathbf{R})}\|E_T(t)\|_h^2\|c(t)\|_{C^1(\overline{\Omega})}^2\\
&+\frac{1}{2\varepsilon_1^2}|\Omega|\Big(h_{max}^4\Big(\|g(t)\|_{C(\overline{\Omega})}^2+2\left\|\frac{\partial g}{\partial x}(t)\right\|_{L^2(\Omega)}^2\Big)+|\mathscr{O}(h_{max}^2)|^2\Big).\end{aligned} \tag{II.34}$$

We notice that fixing $\varepsilon \neq 0, i=1,2$, such that $\beta-\varepsilon_1^2-\varepsilon_2^2>0$, there exists a positive constant $Const$, h and t independent, such that, for $t\in(0,t_f]$,

$$\frac{d}{dt}\|E_c(t)\|_h^2+\|D_{-x}E_c(t)\|_+^2 \leq Const\Big(\|E_T(t)\|_h^2\|c(t)\|_{C^1(\overline{\Omega})}^2+h_{max}^4\|c(t)\|_{C^4(\overline{\Omega})}^2(\|T(t)\|_{C^3(\overline{\Omega})}^2+1)\Big). \tag{II.35}$$

Finally, we observe that the inequality (II.35) leads easily to (II.33).

■

From Theorems 4.1.1 and 4.2.1 we conclude the last result of this chapter where the second convergence order is established for the solution of (II.8).

Corollary 4.2.2 *Under the assumptions of the Theorems 4.1.1 and 4.2.1 for the error* $E_c(t) = R_h c(t) - c_h(t)$ *holds the following*

$$\|E_c(t)\|_h^2 + \int_0^t \|D_{-x}E_c(s)\|_+^2 ds \leq Const h_{max}^4 \int_0^t \Big(\|T(s)\|^2_{C^4(\overline{\Omega})} + \|c(s)\|^2_{C^4(\overline{\Omega})}(\|T(s)\|^2_{C^3(\overline{\Omega})} + 1)\Big) ds,$$

for $t \in [0, t_f]$ *and* $h \in \Lambda$.

5 Conclusion

In this chapter we establish error estimates for the numerical approximations for the solution of (II.1) and (II.2) which is a simpler version of the system that we would like to study in the next chapter. The numerical approximations were defined considering the finite difference methods presented in (II.7) and (II.8) on non uniform grids. As we noticed, these methods can be seen as fully discrete piecewise linear finite element methods.

Theorems 4.1.1 and 4.2.1 are the main results of this chapter. In Theorem 4.1.1 is proved that the finite difference method (II.7) leads to a second order approximations for the solution of (II.1). This result shows that the method is supraconvergent that is, though the spatial truncation error is only of first order, the method is second convergence order. In Theorem 4.2.1 we establish an error estimate for the numerical approximations defined by (II.7) and (II.8). As a corollary of this result, we conclude that the method (II.7) and (II.8) is also supraconvergent.

The regularity of the solutions of the IBVP (II.1) and (II.2), $T(t), c(t) \in C^4(\overline{\Omega})$ is the main requirement imposed in the proof of the mentioned results. In what follows we intend to obtain the estimates established in the Theorems 4.1.1 and 4.2.1 but considering $T(t), c(t) \in H^3(\Omega)$.

Chapter III

An accurate discrete model for solutions in $H^3(\Omega)$

1 Introduction

In this chapter III our goal is to study the convection-diffusion-reaction equation for a drug concentration (I.2), where T denotes the temperature, that is defined by equation (I.3). This system of PDE's is coupled with homogeneous Dirichlet boundary conditions (I.4) and initial conditions (I.5).

In Section 2 we present a study of the stability of the continuous coupled model (I.2)-(I.3). The method proposed to solve numerically the coupled problem is introduced in Section 3. In this section, the stability of the method is established under certain conditions. In Section 4, an error analysis is developed which is not based on the use of the truncation error neither on the stability of the method. Numerical experiments illustrating the convergence results and the behaviour of the concentration and temperature are included in Section 5. Finally, in Section 6 we present some conclusions.

Note that during this chapter we use $\nabla u(x,t)$ to denote $\dfrac{\partial u}{\partial x}(x,t)$.

2 The continuous model: stability analysis

In this section we study the stability of the coupled problems (I.2)-(I.3). Let $c(t), T(t) \in L^2(0,t_f,H_0^1(\Omega))$ be such that

$$(T'(t),u) = -(D_T(T(t))\nabla T(t),\nabla u) + (G(T(t)),u) \text{ a.e. in } (0,t_f], \forall u \in H_0^1(\Omega), \tag{III.1}$$

and

$$(c'(t),w) - (v(T(t))c(t),\nabla w) = -(D_d(T(t))\nabla c(t),\nabla w) + (Q(c(t)),w) \text{ a.e. in } (0,t_f], \forall w \in H_0^1(\Omega). \tag{III.2}$$

In (III.1) and (III.2), $(.,.)$ denotes the usual inner product in $L^2(\Omega)$, $\|.\|$ represents the corresponding norm. The use of a.e. in $(0,t_f]$ stands for almost everywhere in $(0,t_f]$. We assume the following conditions:

H_1 : $D_T \in C_b^1(\mathbb{R})$ and $D_T \geq \beta_0 > 0$ in $\mathbb{R}$,

H_2 : $|G(y)| \leq \beta_1 |y|$, $y \in \mathbb{R}$,

H_3 : $|v(y)| \leq \beta_2 |y|$, $y \in \mathbb{R}$,

H_4 : $D_d \in C_b^1(\mathbb{R})$ and $D_d \geq \beta_3 > 0$ in $\mathbb{R}$,

H_5 : $|Q(y)| \leq \beta_4 |y|$, $y \in \mathbb{R}$,

where $C_b^m(\mathbb{R})$ denotes the space of bounded functions with bounded m order derivatives in $\mathbb{R}$. To obtain upper bounds for the temperature and concentration, the previous assumptions will be used. To establish the stability of the weak problem (III.1), (III.2), H_2, H_3 and H_5 will be replaced by

H_2^* : $G \in C_b^2(\mathbb{R})$,

H_3^* : $v \in C_b^1(\mathbb{R})$,

H_5^* : $Q \in C_b^2(\mathbb{R})$,

respectively.

2.1 Energy estimates

We present energy estimates for the solution of the system and for the corresponding fully discretized problem.

Temperature: Taking in (III.1) $u = T(t)$, we get

$$(T'(t), T(t)) = -(D_T(T(t))\nabla T(t), \nabla T(t)) + (G(T(t)), T(t))$$

assuming H_1 and H_2, we obtain

$$\frac{1}{2}\frac{d}{dt}\|T(t)\|^2 + \beta_0 \|\nabla T(t)\|^2 \leq \beta_1 \|T(t)\|^2.$$

This inequality leads to

$$\|T(t)\|^2 + 2\beta_0 \int_0^t \|\nabla T(s)\|^2 ds \leq \|T(0)\|^2 + 2\beta_1 \int_0^t \|T(s)\|^2 ds, \tag{III.3}$$

If $T \in C^1([0,t_f], L^2(\Omega)) \cap L^2(0,t_f,H_0^1(\Omega))$, by the Gronwall Lemma we conclude

$$\|T(t)\|^2 + \int_0^t \|\nabla T(s)\|^2 ds \leq \frac{1}{\min\{1, 2\beta_0\}} e^{2\beta_1 t} \|T(0)\|^2, \ t \in [0,t_f]. \tag{III.4}$$

Concentration: Let $w = c(t)$ in (III.2).

$$(c'(t), c(t)) - (v(T(t))c(t), \nabla c(t)) = -(D_d(T(t))\nabla c(t), \nabla c(t)) + (Q(c(t)), c(t)).$$

Moreover, as $H_0^1(\Omega)$ is continuously embedded in $C^0(\overline{\Omega})$ and using H_3, H_4 and H_5, we have successively

$$-(D_d(T(t))\nabla c(t), \nabla c(t)) \leq -\beta_3 \|\nabla c(t)\|, \quad (Q(c(t)), c(t)) \leq \beta_4 \|c(t)\| \quad \text{and}$$

$$\begin{aligned} |(v(T(t))c(t),\nabla c(t))| &\leq \beta_2 \|T(t)\|_{L^\infty(\Omega)}\|c(t)\|\|\nabla c(t)\| \\ &\leq \frac{1}{4\varepsilon_1^2}\beta_2^2\|T(t)\|^2_{L^\infty(\Omega)}\|c(t)\|^2+\varepsilon_1^2\|\nabla c(t)\|^2, \end{aligned} \quad \text{(III.5)}$$

where $\varepsilon_1 \neq 0$ is an arbitrary constant. Then, applying the previous inequalities in (III.2), we easily get

$$\|c(t)\|^2+2(\beta_3-\varepsilon_1^2)\int_0^t \|\nabla c(s)\|^2 ds \leq \|c(0)\|^2+\int_0^t \Big(\frac{\beta_2^2}{2\varepsilon_1^2}\|T(s)\|^2_{L^\infty(\Omega)}+2\beta_4\Big)\|c(s)\|^2 ds. \quad \text{(III.6)}$$

If $T\in C([0,t_f],H_0^1(\Omega))$ then, for ε_1 such that $\beta_3-\varepsilon_1^2>0$, we guarantee the existence of two positive constants $\gamma_{c,i}, i=1,2$, such that

$$\|c(t)\|^2+\int_0^t \|\nabla c(s)\|^2 ds \leq \gamma_{c,1}\|c(0)\|^2 e^{\gamma_{c,2}\int_0^t \left(\|T(s)\|^2_{L^\infty(\Omega)}+1\right)ds}, t\in[0,t_f]. \quad \text{(III.7)}$$

provided that $c\in C^1([0,t_f],L^2(\Omega))\cap L^2(0,t_f,H_0^1(\Omega))$. Furthermore, as $\|T(t)\|_{L^\infty(\Omega)}\leq\|\nabla T(t)\|$, instead of (III.7), we have

$$\|c(t)\|^2+\int_0^t \|\nabla c(s)\|^2 ds \leq \gamma_{c,1}\|c(0)\|^2 e^{\gamma_{c,2}\int_0^t \left(\|\nabla T(s)\|^2+1\right)ds}, t\in[0,t_f]. \quad \text{(III.8)}$$

where the term $\int_0^t \|\nabla T(s)\|^2 ds$ in (III.4) is bounded, by (III.4) for $t\in[0,t_f]$.

However, if instead of considering the estimation (III.5) for the convection term estimate, we consider $|(v(T(t))c(t),\nabla c(t))|\leq\beta_2\|T(t)\|\|c(t)\|_{L^\infty(\Omega)}\|\nabla c(t)\|$, as $\|c(t)\|_{L^\infty(\Omega)}\leq\|\nabla c(t)\|$ we get

$$|(v(T(t))c(t),\nabla c(t))|\leq\beta_2\|T(t)\|\|\nabla c(t)\|^2, \quad \text{(III.9)}$$

then

$$\frac{1}{2}\frac{d}{dt}\|c(t)\|^2+(\beta_3-\beta_2\|T(t)\|)\|\nabla c(t)\|^2\leq\beta_4\|c(t)\|^2.$$

Consequently, if the drug convection-diffusion equation (I.2) is diffusion dominated in the sense that

$$\beta_3-\beta_2\|T(t)\|>\gamma_{c,c}>0 \text{ a.e. in } (0,t_f], \quad \text{(III.10)}$$

then c satisfies

$$\|c(t)\|^2+2\gamma_{c,c}\int_0^t \|\nabla c(s)\|^2 ds\leq\|c(0)\|^2 e^{2\beta_4 t}, t\in[0,t_f]. \quad \text{(III.11)}$$

Moreover, if the reaction term Q satisfies

H_5' : $Q\in C^1(\mathbb{R})$, and $Q(0)=0$, $Q'(c)\leq\beta_4\leq 0$ in $\mathbb{R}$,

instead of H_5, then (III.11) is replaced by

$$\|c(t)\|^2+2\gamma_{c,c}\int_0^t e^{2\beta_4(t-s)}\|\nabla c(s)\|^2 ds\leq\|c(0)\|^2 e^{2\beta_4 t}, t\in[0,t_f]. \quad \text{(III.12)}$$

2.2 Stability estimates

Let $T, \tilde{T}$ and $c, \tilde{c}$ be solutions with initial conditions $c_0, \tilde{c}_0$ and $T_0, \tilde{T}_0$, respectively. Under the assumptions specified before, for $T, \tilde{T}$ and $c, \tilde{c}$ hold the energy estimates previously established. In what follows we will obtain estimates for $T-\tilde{T}$ and $c-\tilde{c}$.

Temperature: Considering that T and $\tilde{T}$ satisfy (III.1), for $\omega_T(t) = T(t) - \tilde{T}(t)$ we obtain

$$(\omega_T'(t), \omega_T(t)) + (D_T(T(t))\nabla T(t), \nabla\omega_T(t)) - (D_T(\tilde{T}(t))\nabla\tilde{T}(t), \nabla\omega_T(t)) = (G(T(t)) - G(\tilde{T}(t)), \omega_T(t))$$

Adding and subtracting $(D_T(\tilde{T}(t))\nabla T(t), \nabla\omega_T(t))$ we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\omega_T(t)\|^2 \quad & +((D_T(T(t)) - D_T(\tilde{T}(t)))\nabla T(t), \nabla\omega_T(t)) + (D_T(\tilde{T}(t))\nabla\omega_T(t), \nabla\omega_T(t)) \\ & = (G(T(t)) - G(\tilde{T}(t)), \omega_T(t)).\end{aligned}$$

Using the assumption H_1 and H_2^*, we have successively,

$$-(D_T(\tilde{T}(t))\nabla\omega_T(t), \nabla\omega_T(t)) \leq -\beta_0\|\nabla\omega_T(t)\| \quad \text{and} \quad (G(T(t)) - G(\tilde{T}(t)), \omega_T(t)) \leq G'_{max}\|\omega_T(t)\|^2$$

and

$$\begin{aligned}&|((D_T(T(t)) - D_T(\tilde{T}(t)))\nabla T(t), \nabla\omega_T(t))| \\ &\quad \leq \|D_T'\|_{L^\infty(\mathbf{R})}\|\omega_T(t)\|\|\nabla T(t)\|_{L^\infty(\Omega)}\|\nabla\omega_T(t)\| \\ &\quad \leq \frac{1}{4\varepsilon_1^2}\|D_T'\|^2_{L^\infty(\mathbf{R})}\|\nabla T(t)\|^2_{L^\infty(\Omega)}\|\omega_T(t)\|^2 + \varepsilon_1^2\|\nabla\omega_T(t)\|^2,\end{aligned} \tag{III.13}$$

where $\varepsilon_1 \neq 0$. Consequently, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\omega_T(t)\|^2 + (\beta_0 - \varepsilon_1^2)\|\nabla\omega_T(t)\|^2 \leq \Big(G'_{max} + \frac{1}{4\varepsilon_1^2}\|D_T'\|^2_{L^\infty(\mathbf{R})}\|\nabla T(t)\|^2_{L^\infty(\Omega)}\Big)\|\omega_T(t)\|^2, \tag{III.14}$$

If $\beta_0 - \varepsilon_1^2 > 0$ and $T, \tilde{T} \in C^1([0,t_f], L^2(\Omega)) \cap L^2(0,t_f, H_0^1(\Omega) \cap W^{1,\infty}(\Omega))$, from (III.14) we obtain

$$\begin{aligned}&\frac{d}{dt}\Big(\|\omega_T(t)\|^2 e^{-\int_0^t\Big(2G'_{max} + \frac{1}{2\varepsilon_1^2}\|D_T'\|^2_{L^\infty(\mathbf{R})}\|\nabla T(\mu)\|^2_{L^\infty(\Omega)}\Big)d\mu} \\ &\qquad +2(\beta_0 - \varepsilon_1^2)\int_0^t e^{-\int_0^s\Big(2G'_{max} + \frac{1}{2\varepsilon_1^2}\|D_T'\|^2_{L^\infty(\mathbf{R})}\|\nabla T(\mu)\|^2_{L^\infty(\Omega)}\Big)d\mu}\|\nabla\omega_T(s)\|^2 ds\Big) \leq 0,\end{aligned}$$

that, applying Gronwall's Lemma, leads to

$$\begin{aligned}\|\omega_T(t)\|^2 \quad & +2(\beta_0 - \varepsilon_1^2)\int_0^t e^{\int_s^t\Big(2G'_{max} + \frac{1}{2\varepsilon_1^2}\|D_T'\|^2_{L^\infty(\mathbf{R})}\|\nabla T(\mu)\|^2_{L^\infty(\Omega)}\Big)d\mu}\|\nabla\omega_T(s)\|^2 ds \\ & \leq \|\omega_T(0)\|^2 e^{\int_0^t\Big(2G'_{max} + \frac{1}{2\varepsilon_1^2}\|D_T'\|^2_{L^\infty(\mathbf{R})}\|\nabla T(s)\|^2_{L^\infty(\Omega)}\Big)ds}, \ t \in [0,t_f].\end{aligned} \tag{III.15}$$

From (III.15), stability is established for $T \in C^1([0,t_f], L^2(\Omega)) \cap L^2(0,t_f, H_0^1(\Omega) \cap W^{1,\infty}(\Omega))$ and $\tilde{T} \in C^1([0,t_f], L^2(\Omega)) \cap L^2(0,t_f, H_0^1(\Omega))$.

Although we have already a stability result we still can obtain an estimation where the smoothness of T can be weakened, at the cost of imposing a stronger condition on $\|\nabla T(t)\|$. In fact, if rather than

(III.13), we use the following relation

$$|((D_T(T(t))-D_T(\tilde T(t)))\nabla T(t),\nabla\omega_T(t))|\le \|D_T'\|_{L^\infty(\mathbf{R})}\|\nabla T(t)\|\|\nabla\omega_T(t)\|^2.$$

we deduce, if

$$\beta_0-\|D_T'\|_{L^\infty(\mathbf{R})}\|\nabla T(t)\|\ge\gamma_T>0 \text{ a.e. in } (0,t_f], \tag{III.16}$$

for some positive constant γ_T, instead of (III.15), we conclude

$$\|\omega_T(t)\|^2+2\gamma_T\int_0^t e^{2G'_{max}(t-s)}\|\nabla\omega_T(s)\|^2ds\le\|\omega_T(0)\|^2e^{2G'_{max}t},\ t\in[0,t_f]. \tag{III.17}$$

which guarantees stability, providing that T satisfies (III.16) and $T,\tilde T\in C^1([0,t_f],L^2(\Omega))\cap L^2(0,t_f,H_0^1(\Omega))$.

Concentration: Let consider c and $\tilde c$ that satisfy (III.2), then for $\omega_c(t)=c(t)-\tilde c(t)$, we get

$$\begin{aligned}&(\omega_c'(t),\omega_c(t))-(v(T(t))c(t),\nabla\omega_c(t))+(v(\tilde T(t))\tilde c(t),\nabla\omega_c(t))\\&\quad=-(D_d(T(t))\nabla c(t),\nabla\omega_c(t))+(D_d(\tilde T(t))\nabla\tilde c(t),\nabla\omega_c(t))+(Q(c(t)),\omega_c(t))-(Q(\tilde c(t)),\omega_c(t))\end{aligned} \tag{III.18}$$

We need an estimative for each parcel. For the convection term we have

$$\begin{aligned}&|(v(T(t))c(t)-v(\tilde T(t))\tilde c(t),\nabla\omega_c(t))|\\&\quad=|((v(T(t))-v(\tilde T(t)))c(t)+v(\tilde T(t))\omega_c(t),\nabla\omega_c(t))|\\&\quad\le\|v'\|_{L^\infty(\mathbf{R})}\|\omega_T(t)\|\|c(t)\|_{L^\infty(\Omega)}\|\nabla\omega_c(t)\|+\beta_2\|\tilde T(t)\|_{L^\infty(\Omega)}\|\omega_c(t)\|\|\nabla\omega_c(t)\|\\&\quad\le\frac{1}{4\varepsilon_1^2}\|v'\|^2_{L^\infty(\mathbf{R})}\|c(t)\|^2_{L^\infty(\Omega)}\|\omega_T(t)\|^2+\frac{1}{4\varepsilon_2^2}\beta_2^2\|\tilde T(t)\|^2_{L^\infty(\Omega)}\|\omega_c(t)\|^2+(\varepsilon_1^2+\varepsilon_2^2)\|\nabla\omega_c(t)\|^2,\end{aligned} \tag{III.19}$$

with $\varepsilon_i\neq0, i=1,2$, arbitrary constants. For the diffusion term, adding and subtracting the term $(D_d(\tilde T(t))\nabla c(t),\nabla\omega_c(t))$, we get

$$\begin{aligned}&(D_d(T(t))\nabla c(t)-D_d(\tilde T(t))\nabla\tilde c(t),\nabla\omega_c(t))\\&\quad=((D_d(T(t))-D_d(\tilde T(t)))\nabla c(t)+D_d(\tilde T(t))\nabla\omega_c(t),\nabla\omega_c(t)),\end{aligned}$$

where

$$\begin{aligned}&|((D_d(T(t))-D_d(\tilde T(t)))\nabla c(t),\nabla\omega_c(t))|\\&\quad\le\|D_d'\|_{L^\infty(\mathbf{R})}\|\omega_T(t)\|\|\nabla c(t)\|_{L^\infty(\Omega)}\|\nabla\omega_c(t)\|\\&\quad\le\tfrac{1}{4\varepsilon_3^2}\|D_d'\|^2_{L^\infty(\mathbf{R})}\|\nabla c(t)\|^2_{L^\infty(\Omega)}\|\omega_T(t)\|^2+\varepsilon_3^2\|\nabla\omega_c(t)\|^2\end{aligned} \tag{III.20}$$

and

$$(D_d(\tilde T(t))\nabla\omega_c(t),\nabla\omega_c(t))\ge\beta_3\|\nabla\omega_c(t)\|^2.$$

For the reaction term, using H_5^* we deduce

$$(Q(c(t))-Q(\tilde c(t)),\omega_c)\le Q'_{max}\|\omega_c\|^2.$$

Then we obtain the differential inequality

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\omega_c(t)\|^2+(\beta_3-\varepsilon_1^2-\varepsilon_2^2-\varepsilon_3^2)\|\nabla\omega_c(t)\|^2\\
&\le\Big(\frac{\beta_2^2}{4\varepsilon_2^2}\|\nabla\tilde{T}(t)\|^2+Q'_{max}\Big)\|\omega_c(t)\|^2+\Big(\frac{1}{4\varepsilon_1^2}\|v'\|^2_{L^\infty(\mathbf{R})}\|c(t)\|^2_{L^\infty(\Omega)}+\frac{1}{4\varepsilon_3^2}\|D'_d\|^2_{L^\infty(\mathbf{R})}\|\nabla c(t)\|^2_{L^\infty(\Omega)}\Big)\\
&\|\omega_T(t)\|^2
\end{aligned}
\tag{III.21}
$$

whose solution satisfies

$$
\begin{aligned}
&\|\omega_c(t)\|^2+2(\beta_3-\varepsilon_1^2-\varepsilon_2^2-\varepsilon_3^2)\int_0^t e^{\int_s^t\left(\frac{\beta_2^2}{2\varepsilon_2^2}\|\nabla\tilde{T}(\mu)\|^2+2Q'_{max}\right)d\mu}\|\nabla\omega_c(s)\|^2ds\\
&\le\|\omega_c(0)\|^2e^{\int_0^t\left(\frac{\beta_2^2}{2\varepsilon_2^2}\|\nabla\tilde{T}(\mu)\|^2+2Q'_{max}\right)d\mu}+\int_0^t e^{\int_s^t\left(\frac{\beta_2^2}{2\varepsilon_2^2}\|\nabla\tilde{T}(\mu)\|^2+2Q'_{max}\right)d\mu}\\
&\Big(\frac{1}{2\varepsilon_1^2}\|v'\|^2_{L^\infty(\mathbf{R})}\|c(s)\|^2_{L^\infty(\Omega)}+\frac{1}{2\varepsilon_3^2}\|D'_d\|^2_{L^\infty(\mathbf{R})}\|\nabla c(s)\|^2_{L^\infty(\Omega)}\Big)\|\omega_T(s)\|^2ds,
\end{aligned}
\tag{III.22}
$$

for $t\in[0,t_f]$ and provided that $c\in C^1([0,t_f],L^2(\Omega))\cap L^2(0,t_f,H_0^1(\Omega)\cap W^{1,\infty}(\Omega))$, $\tilde{c}\in C^1([0,t_f],L^2(\Omega))\cap L^2(0,t_f,H_0^1(\Omega))$, $T\in L^2(0,t_f,L^2(\Omega))$, $\tilde{T}\in L^2(0,t_f,H_0^1(\Omega))$.

Finally for $\varepsilon_i, i=1,2,3$, such that $\beta_3-\sum_{i=1}^{3}\varepsilon_i^2>0$, we get the desired upper bound.

To conclude we recall that an upper bound for $\int_0^t\|\nabla\tilde{T}(\mu)\|^2d\mu$ is established in (III.4) and upper bounds for $\|\omega_T(t)\|^2$ are defined in (III.15) or (III.17) when $\tilde{T}\in C^1([0,t_f],L^2(\Omega))\cap L^2(0,T,H_0^1(\Omega))$ and $T\in C^1([0,t_f],L^2(\Omega))\cap L^2(0,T,H_0^1(\Omega)\cap W^{1,\infty}(\Omega))$.

From (III.22), the stability of (III.1) and (III.2) is concluded when $c\in L^\infty(0,t_f,H_0^1(\Omega)\cap W^{1,\infty}(\Omega))\cap C^1([0,t_f],L^2(\Omega))$, $T\in C^1([0,t_f],L^2(\Omega))\cap L^2(0,t_f,H_0^1(\Omega))$ and assuming that $\tilde{c},\tilde{T}\in C^1([0,t_f],L^2(\Omega))\cap L^2(0,t_f,H_0^1(\Omega))$.

Similarly to the temperature case, we can also get stability estimates under weaker assumption for the concentration. In order to do that we use $\|\omega_T(t)\|_\infty\le\|\nabla\omega_T(t)\|$, replacing respectively, (III.19) and (III.20), by

$$
\begin{aligned}
&|(v(T)c(t)-v(\tilde{T})\tilde{c}(t),\nabla\omega_c(t))|\\
&\le\Big(\frac{1}{4\varepsilon_1^2}\|v'\|^2_{L^\infty(\mathbf{R})}\|c(t)\|^2_{L^\infty(\Omega)}\|\nabla\omega_T\|^2+\frac{1}{4\varepsilon_2^2}\beta_2^2\|\nabla\tilde{T}(t)\|^2\|\omega_c(t)\|^2+(\varepsilon_1^2+\varepsilon_2^2)\|\nabla\omega_c(t)\|^2
\end{aligned}
$$

and

$$
|((D_d(T(t))-D_d(\tilde{T}(t)))\nabla c(t),\nabla\omega_c(t))|\le\frac{1}{4\varepsilon_3^2}\|D'_d\|^2_{L^\infty(\mathbf{R})}\|\nabla c(t)\|^2_{L^\infty(\Omega)}\|\nabla\omega_T(t)\|^2+\varepsilon_3^2\|\nabla\omega_c(t)\|^2,
$$

respectively.

Consequently, (III.22) is replaced by

$$\|\omega_c(t)\|^2 + 2(\beta_3 - \sum_{i=1}^{3}\varepsilon_i^2)\int_0^t e^{\int_s^t \left(\frac{\beta_2^2}{2\varepsilon_2^2}\|\nabla\tilde{T}(\mu)\|^2 + 2Q'_{max}\right)d\mu}\|\nabla\omega_c(s)\|^2 ds$$
$$\leq \|\omega_c(0)\|^2 e^{\int_0^t \left(\frac{\beta_2^2}{2\varepsilon_2^2}\|\nabla\tilde{T}(\mu)\|^2 + 2Q'_{max}\right)d\mu} + \int_0^t e^{\int_s^t \left(\frac{\beta_2^2}{2\varepsilon_2^2}\|\nabla\tilde{T}(\mu)\|^2 + 2Q'_{max}\right)d\mu}$$
$$\Big(\frac{1}{2\varepsilon_1^2}\|v'\|^2_{L^\infty(\mathbf{R})}\|c(s)\|^2_{L^\infty(\Omega)} + \frac{1}{2\varepsilon_3^2}\|D'_d\|^2_{L^\infty(\mathbf{R})}\|\nabla c(s)\|^2_{L^\infty(\Omega)}\Big)\|\nabla\omega_T(s)\|^2 ds,\ t\in[0,t_f]. \tag{III.23}$$

An upper bound for $\int_0^t \|\nabla\tilde{T}(\mu)\|^2 d\mu$ is given by (III.4). Upper bounds for $\int_0^t \|\nabla\omega_T(s)\|^2 ds$ are defined by (III.15) provided that $T\in C^1([0,t_f],L^2(\Omega))\cap L^2(0,t_f,H_0^1(\Omega)\cap W^{1,\infty}(\Omega))$ and for $\tilde{T}\in C^1([0,t_f],L^2(\Omega))\cap L^2(0,t_f,H_0^1(\Omega))$; or by (III.17) provided that (III.16) holds and $T,\tilde{T}\in C^1([0,t_f],L^2(\Omega))\cap L^2(0,t_f,H_0^1(\Omega))$.

From (III.23) we conclude the stability of the initial value problem (III.1) and (III.2) for $c\in L^\infty(0,t_f,H_0^1(\Omega)\cap W^{1,\infty}(\Omega))\cap C^1([0,t_f],L^2(\Omega))$, $\tilde{c},T,\tilde{T}\in C^1([0,t_f],L^2(\Omega))\cap L^2(0,t_f,H_0^1(\Omega))$.

3 A Finite Element Method that mimics the continuous model: stability analysis

3.1 Fully discrete method

In this section we present a fully discrete method that mimics system (III.1) and (III.2).

Regarding the discretization process, we follow the notations introduced in chapter II where the nonuniform partition of $\Omega=(a,b)$ is defined through Λ, the sequence of vectors h. As before W_h represents the space of grid functions defined in $\overline{\Omega}_h$ and $W_{h,0}=\{w_h\in W_h : w_h = 0\,\text{on}\,\partial\Omega_h\}$.

Moreover, the inner products, norms, operators and results defined in Section 1 of Chapter II hold . We observe that $x_{i+1/2}=x_i+\dfrac{h_{i+1}}{2}$, $x_{i-1/2}=x_i-\dfrac{h_i}{2}$. And notice that by $\|.\|_{1,h}$ we represent the norm $\|u_h\|_{1,h}=\left(\|u_h\|_h^2+\|D_{-x}u_h\|_+^2\right)^{1/2}$.

For $w_h\in W_h$, $P_h w_h$ denotes the continuous piecewise linear interpolation of w_h with respect to the partition $\overline{\Omega}_h$.

The piecewise linear finite element approximations for the solutions of (III.1) and (III.2) are defined as follow: $P_hT_h(t), P_hc_h(t)\in H_0^1(\Omega)$ such that $\forall u_h, w_h\in W_{h,0}$,

$$(P_hT'_h(t),P_hu_h) = -(D_T(P_hT_h(t))\nabla P_hT_h(t),\nabla P_hu_h) + (G(P_hT_h),P_hu_h), \tag{III.24}$$

and

$$(P_hc'_h(t),P_hw_h) - (v(P_hT_h)P_hc_h(t),\nabla P_hw_h) = -(D_d(P_hT_h(t))\nabla P_hc_h(t),\nabla P_hw_h) + (Q(P_hc_h(t)),P_hw_h). \tag{III.25}$$

To define the fully discrete piecewise linear approximations for the temperature and concentration we need to define approximations for the integral terms in (III.24) and (III.25).

Considering the approximation rules defined in [19], we introduce the following approximations:

$$(f,g) \simeq (R_h f, R_h g),\ f,g \in C(\overline{\Omega}), \tag{III.26}$$

where once again R_h denotes the restriction operator,

$$(a(P_h q_h)\nabla P_h u_h, \nabla P_h w_h) \simeq (a(M_h q_h)D_{-x}u_h, D_{-x}w_h)_+, q_h, u_h, w_h \in W_{h,0}, \tag{III.27}$$

and remembering that M_h is the average operator. The variational problem for the finite element approximation $P_h T_h(t)$ is defined by the following fully discrete FEM: compute $T_h(t) \in W_{h,0}$ such that

$$(T_h'(t), u_h)_h = -(D_T(M_h T_h(t))D_{-x}T_h(t), D_{-x}u_h)_+ + (G(T_h(t)), u_h)_h, \forall u_h \in W_{h,0}. \tag{III.28}$$

To define the fully discrete problem for the concentration, we need to introduce the approximation of the integral term associated with the convective term $(v(P_h T_h(t))P_h c_h(t), \nabla P_h w_h)$. We consider

$$(v(P_h T_h(t))P_h c_h(t), \nabla P_h w_h) \simeq (M_h(v(T_h(t))c_h(t)), D_{-x}w_h)_+.$$

Using quadrature rules in (III.25), we get the fully discrete FEM: compute $c_h(t) \in W_{h,0}$ such that

$$(c_h'(t), w_h)_h - (M_h(v(T_h(t))c_h(t)), D_{-x}w_h)_+ = -(D_d(M_h T_h(t))D_{-x}c_h(t), D_{-x}w_h)_+ + (Q(c_h(t)), w_h)_h, \tag{III.29}$$

$\forall w_h \in W_{h,0}$.

We remark that the coupled system (III.28), (III.29) can be rewritten as an ordinary differential system. To do that, we recall the finite difference operator $D_x^*(a(M_h q_h)D_{-x}u_h)$ defined by

$$D_x^*(a(M_h q_h)D_{-x}u_h)(x_i) = \frac{1}{h_{i+1/2}}\Big(a(M_h q_h(x_{i+1}))D_{-x}u_h(x_{i+1})a(M_h q_h(x_i))D_{-x}u_h(x_i)\Big),$$

for $i = 1, \dots, N-1$. For $q_h, u_h \in W_{h,0}$. By D_c we denote the centered finite difference operator

$$D_c(u_h)(x_i) = \frac{u_h(x_{i+1}) - u_h(x_{i-1})}{h_i + h_{i+1}}, i = 1, \dots, N-1.$$

We then have the following result.

Proposition 3.1.1 *If $u_h \in W_h$ and $w_h \in W_{h,0}$ then*

$$(D_c u_h, w_h)_h = -(M_h(u_h), D_{-x}w_h)_+, \tag{III.30}$$

and

$$\|M_h(u_h)\|_h \leq \sqrt{2}\|u_h\|_h. \tag{III.31}$$

■

We introduce now the ordinary differential systems

$$\begin{cases} T_h'(t) = F_T(T_h(t)) \text{ in } \Omega_h \times (0,t_f] \\ T_h(t) = 0 \text{ in } \partial\Omega_h \times (0,t_f] \\ T_h(0) = R_h T_0 \text{ in } \Omega_h, \end{cases} \tag{III.32}$$

and

$$\begin{cases} c_h'(t) = F_c(T_h(t), c_h(t)) \text{ in } \Omega_h \times (0,t_f] \\ c_h(t) = 0 \text{ in } \partial\Omega_h \times (0,t_f] \\ c_h(0) = R_h c_0 \text{ in } \Omega_h, \end{cases} \tag{III.33}$$

where

$$F_T(T_h(t)) = D_x^*(D_T(M_h T_h(t)) D_{-x} T_h(t)) + G(T_h(t))$$

and

$$F_c(T_h(t), c_h(t)) = D_x^*(D_d(M_h T_h(t)) D_{-x} c_h(t)) - D_c(v(T_h(t)) c_h(t)) + Q(c_h(t)).$$

Considering the inner product of the first equation of (III.32) and (III.33), with respect to $(.,.)_h$, by $u_h \in W_{h,0}$ and $w_h \in W_{h,0}$, respectively, we get (III.28) and (III.29). These results show the equivalence between the fully discrete FEM (III.28), (III.29) and the FDMs (III.32) and (III.33), respectively.

3.2 Stability

Firstly, we establish the existence of the semi-discrete approximations, at least locally, this means that there exists an interval $[0,t_f]$ and functions $T_h(t), c_h(t)$ solutions of the ordinary differential problems (III.32) and (III.33).

We observe that the previous coupled problem can be rewritten in the following equivalent form

$$\begin{cases} Z_h'(t) = F_h(Z_h(t)) \text{ in } \Omega_h \times (0,t_f], \\ Z_h(0) = Z_{0,h} \text{ in } \Omega_h, \\ Z_h(t) = 0 \text{ in } \partial\Omega_h \times (0,t_f], \end{cases} \tag{III.34}$$

where $Z_h(t) = (T_h(t), c_h(t)), Z_{0,h} = (R_h T_0, R_h c_0)$ and

$$F_h(Z_h(t)) = (F_T(T_h(t)), F_c(T_h(t), c_h(t))).$$

Proposition 3.2.1 *Under the assumptions H_1, H_2^*, H_3^*, H_4 and H_5^*,*

$$F_h : B_{\delta_T}(R_h T_0) \times B_{\delta_c}(R_h c_0) \to [W_{h,0}]^2$$

is one-side Lipschitz, where

$$B_\delta(u_h) = \{z_h \in W_{h,0} : \|z_h - u_h\|_h \leq \delta\},$$

for $u_h = R_h T_0, R_h c_0$, and $\delta = \delta_T, \delta_c$, and where $[W_{h,0}]^2 = W_{h,0} \times W_{h,0}$.

Proof: Let $Z_h=(q_h,w_h),\tilde{Z}_h=(\tilde{q}_h,\tilde{w}_h)\in B_{\delta_T}(R_hT_0)\times B_{\delta_c}(R_hc_0)$, and $\omega_q=q_h-\tilde{q}_h$, $\omega_w=w_h-\tilde{w}_h$ and $\omega=(\omega_q,\omega_w)$. We have, successively, the following

$$(F_h(Z_h)-F_h(\tilde{Z}_h),\omega)_{[W_{h,0}]^2}=(F_T(q_h)-F_T(\tilde{q}_h),\omega_q)_h+(F_c(q_h,w_h)-F_c(\tilde{q}_h,\tilde{w}_h),\omega_w)_h,$$

$$\begin{aligned}(F_T(q_h) \quad & -F_T(\tilde{q}_h),\omega_q)_h=-((D_T(M_hq_h)-D_T(M_h\tilde{q}_h))D_{-x}q_h,D_{-x}\omega_q)_+\\ & -(D_T(M_h\tilde{q}_h)D_{-x}\omega_q,D_{-x}\omega_q)_++(G(q_h)-G(\tilde{q}_h),\omega_q)_h\\ & \leq\sqrt{2}\|D_T\|_{C^1(\mathbf{R})}\|\omega_q\|_h\|D_{-x}q_h\|_{h,\infty}\|D_{-x}\omega_q\|_+-\beta_0\|D_{-x}\omega_q\|_+^2+G'_{max}\|\omega_q\|_h^2\\ & \leq(\varepsilon^2-\beta_0)\|D_{-x}\omega_q\|_+^2+\left(\frac{\|D_T\|^2_{C^1(\mathbf{R})}}{2\varepsilon^2}\|D_{-x}q_h\|^2_{h,\infty}+G'_{max}\right)\|\omega_q\|_h^2\end{aligned}$$

where $\|u_h\|_{h,\infty}=\max_{i=1,\ldots,N^*}|u_h(x_i)|$, with $N^*=N$ or $N^*=N-1$ depending on the definition of u_h.

Moreover, we establish that $\|D_{-x}q_h\|_{h,\infty}\leq\frac{4}{h^2_{min}}\Big(\delta_T+\|R_hT_0\|_h\Big)^2$, then taking $\varepsilon^2=\beta_0$, we deduce

$$\begin{aligned}(F_T(q_h)-F_T(\tilde{q}_h),\omega_q)_h \quad & \leq\Big(\frac{4}{\beta_0}\frac{1}{h^2_{min}}\Big(\delta_T+\|R_hT_0\|_h\Big)^2\|D_T\|^2_{C^1_b(\mathbf{R})}+G'_{max}\Big)\|\omega_q\|_h^2\\ & :=L_T(h)\|\omega_q\|_h^2.\end{aligned}\tag{III.35}$$

For $(F_c(q_h,w_h)-F_c(\tilde{q}_h,\tilde{w}_h),\omega_w)_h$ we establish

$$\begin{aligned}&(F_c(q_h,w_h)-F_c(\tilde{q}_h,\tilde{w}_h),\omega_w)_h=\\ &\quad=-((D_d(M_hq_h)-D_d(M_h\tilde{q}_h))D_{-x}w_h,D_{-x}\omega_w)_+-(D_d(M_h\tilde{q}_h)D_{-x}\omega_w,D_{-x}\omega_w)_+\\ &\quad+(M_h((v(q_h)-v(\tilde{q}_h))w_h),D_{-x}\omega_w)_++(M_h(v(\tilde{q}_h))\omega_w,D_{-x}\omega_w)_++Q'_{max}\|\omega_w\|_h^2\\ &\quad\leq\sqrt{2}\|D_d\|_{C^1_b(\mathbf{R})}\|\omega_q\|_h\|D_{-x}w_h\|_{h,\infty}\|D_{-x}\omega_w\|_+-\beta_3\|D_{-x}\omega_w\|_+^2\\ &\quad+\sqrt{2}\|v\|_{C^1_b(\mathbf{R})}\|\omega_q\|_h\|w_h\|_{h,\infty}\|D_{-x}\omega_w\|_++\sqrt{2}\beta_2\|\tilde{q}_h\|_{h,\infty}\|\omega_w\|_h\|D_{-x}\omega_w\|_++Q'_{max}\|\omega_w\|_h^2\end{aligned}$$

Since $\|D_{-x}w_h\|^2_{h,\infty}\leq\frac{4}{h^2_{min}}\Big(\delta_c+\|R_hc_0\|_h\Big)^2$, $\|q_h\|^2_{h,\infty}\leq\frac{2}{h_{min}}\Big(\delta_T+\|R_hT_0\|_h\Big)^2$ and $\|w_h\|^2_{h,\infty}\leq\frac{2}{h_{min}}\Big(\delta_c+\|R_hc_0\|_h\Big)^2$, thus, choosing $\varepsilon^2=\frac{1}{3}\beta_3$, we conclude that

$$\begin{aligned}(F_c(q_h,w_h)-F_c(\tilde{q}_h,\tilde{w}_h),\omega_w)_h\leq & \frac{3}{\beta_3}\Big(\frac{4}{h^2_{min}}\|D_d\|^2_{C^1_b(\mathbf{R})}+\frac{2}{h_{min}}\|v\|^2_{C^1_b(\mathbf{R})}\Big)\Big(\delta_c+\|R_hc_0\|_h\Big)^2\|\omega_q\|_h^2\\ & +\Big(\frac{6\beta_2^2}{\beta_3}\frac{1}{h_{min}}\Big(\delta_T+\|R_hT_0\|_h\Big)^2+Q'_{max}\Big)\|\omega_w\|_h^2\\ & :=L_{c,1}(h)\|\omega_q\|_h^2+L_{c,2}(h)\|\omega_w\|_h^2.\end{aligned}\tag{III.36}$$

From (III.35) and (III.36) we finally obtain

$$\begin{aligned}(F_h(Z_h) \quad & -F_h(\tilde{Z}_h),Z_h-\tilde{Z}_h)_{[W_{h,0}]^2}\leq(L_T(h)+L_{c,1}(h))\|\omega_q\|_h^2+L_{c,2}(h)\|\omega_w\|_h^2\\ & \leq\max\{L_T(h)+L_{c,1}(h),L_{c,2}(h)\}\|Z_h-\tilde{Z}_h\|^2_{[W_{h,0}]^2}.\end{aligned}\tag{III.37}$$

■

We remark that the one-side Lipschitz condition (III.37), established in Proposition 3.2.1, guarantees the existence of the semi-discrete approximations $T_h(t), c_h(t)$, at least locally.

We observe that if we use the previous result to get upper bounds for $\|\omega_h(t)\|_{[W_{h,0}]^2}$, where $\omega_h(t) = Z_h(t) - \tilde{Z}_h(t)$, $Z_h(t) = (T_h(t), c_h(t))$ is the solution of (III.34) with initial condition $Z_h(0)$, and $\tilde{Z}_h(t)$ is the solution of the same problem but with a perturbed initial condition $\tilde{Z}_h(0)$, then we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\omega_h(t)\|^2_{[W_{h,0}]^2} &= (F_h(Z_h(t)) - F_h(\tilde{Z}_h(t)), \omega_h(t))_{[W_{h,0}]^2} \\ &\leq \max\{L_T(h) + L_{c,1}(h), L_{c,2}(h)\}\|\omega_h(t)\|^2_{[W_{h,0}]^2}, \end{aligned} \tag{III.38}$$

where $L_T(h), L_{c,1}(h)$ and $L_{c,2}(h)$ are defined in Proposition 3.2.1. Consequently, we obtain

$$\|\omega_h(t)\|^2_{[W_{h,0}]^2} \leq e^{2\max\{L_T(h)+L_{c,1}(h), L_{c,2}(h)\}t}\|\omega_h(0)\|^2_{[W_{h,0}]^2}, t \geq 0. \tag{III.39}$$

The upper bound (III.39) guarantees the stability of the semi-discretization defined by F_h in bounded time intervals for each h. However, when h decreases, from this upper bound we are not able to conclude such stability behaviour. This fact is our motivation to study the stability using the energy method for each semi-discretization defined by F_T and F_c. To obtain the stability upper bounds, we start by establishing convenient upper bounds for $T_h(t)$ and $c_h(t)$ which are solutions of (III.28) and (III.29), respectively.

Energy estimates

<u>*Temperature:*</u> For the fully discretized problem, for the temperature case we find energy estimates, replacing u_h by $T_h(t)$ in (III.28). As before, we have

$$(T_h'(t), T_h(t))_h = -(D_T(M_hT_h(t))D_{-x}T_h(t), D_{-x}T_h(t))_+ + (G(T_h(t)), T_h(t))_h, \tag{III.40}$$

and following the proof of (III.4), we easily get

$$\|T_h(t)\|_h^2 + \int_0^t \|D_{-x}T_h(s)\|_+^2 ds \leq \frac{1}{\min\{1, 2\beta_0\}} e^{2\beta_1 t}\|T_h(0)\|_h^2, \tag{III.41}$$

for $t \in [0, t_f]$, provided that $T_h \in C^1([0, t_f], W_{h,0})$.

<u>*Concentration:*</u> In this case, let consider $w_h = c_h(t)$ in (III.29). As in the continuous case, we have

$$\begin{aligned} &(c_h'(t), c_h(t))_h - (M_h(v(T_h(t))c_h(t)), D_{-x}c_h(t))_+ \\ &\quad = -(D_d(M_hT_h(t))D_{-x}c_h(t), D_{-x}c_h(t))_+ + (Q(c_h(t)), c_h(t))_h. \end{aligned} \tag{III.42}$$

So establishing upper bounds for the terms we get

$$\begin{aligned} |(M_h(v(T_h(t))c_h(t)), D_{-x}c_h(t))_+| &\leq \sqrt{2}\beta_2\|T_h(t)\|_{h,\infty}\|c_h(t)\|_h\|D_{-x}c_h(t)\|_+ \\ &\leq \frac{1}{2\varepsilon_1^2}\beta_2^2\|T_h(t)\|_{h,\infty}^2\|c_h(t)\|_h^2 + \varepsilon_1^2\|D_{-x}c_h(t)\|_+^2, \end{aligned} \tag{III.43}$$

and

$$-(D_d(M_hT_h(t))D_{-x}c_h(t),D_{-x}c_h(t))_+ \leq -\beta_3\|D_{-x}c_h(t)\|_+^2 \text{ and } (Q(c_h(t)),c_h(t))_h \leq \beta_4\|c_h(t)\|_h^2$$

where $\varepsilon_1 \neq 0$ is an arbitrary constant. Hence, for ε_1 such that $\beta_3 - \varepsilon_1^2 > 0$, we easily get

$$\|c_h(t)\|_h^2 + \int_0^t \|D_{-x}c_h(s)\|_+^2 ds \leq \frac{1}{\min\{1,2(\beta_3-\varepsilon_1^2)\}}\|c_h(0)\|_h^2 e^{\int_0^t\left(\frac{\beta_2^2}{\varepsilon_1^2}\|T_h(s)\|_{h,\infty}^2+2\beta_4\right)ds}, \tag{III.44}$$

for $t \in [0,t_f]$, provided that $c_h \in C^1([0,t_f],W_{h,0})$.

From (III.41), the term $\int_0^t \|D_{-x}T_h(s)\|_+^2 ds$ is uniformly bounded in $[0,t_f]$, provided that $\|T_h(0)\|_h$ is uniformly bounded in $h \in \Lambda$. As $\int_0^t \|T_h(s)\|_{h,\infty}^2 ds \leq |\Omega| \int_0^t \|D_{-x}T_h(s)\|_+^2 ds$, we have that $\int_0^t \|T_h(s)\|_{h,\infty}^2 ds \leq \frac{1}{\min\{1,2\beta_0\}}e^{2\beta_1 t}\|T_h(0)\|_h^2$ and then

$$\|c_h(t)\|_h^2 + \int_0^t \|D_{-x}c_h(s)\|_+^2 ds \leq \gamma_{c,1}\|c_h(0)\|_h^2 e^{\frac{\beta_2^2}{\varepsilon_1^2}\frac{1}{\min\{1,2\beta_0\}}e^{2\beta_1 t}\|T_h(0)\|_h^2+2\beta_4 t}, t \in [0,t_f]. \tag{III.45}$$

On other hand, if we replace (III.43) by

$$\begin{aligned} |(M_h(v(T_h(t))c_h(t)),D_{-x}c_h(t))_+| &\leq \sqrt{2}\beta_2\|T_h(t)\|_h\|c_h(t)\|_{h,\infty}\|D_{-x}c_h(t)\|_+ \\ &\leq \|\Omega\|\sqrt{2}\beta_2\|T_h(t)\|_h\|D_{-x}c_h(t)\|_+^2. \end{aligned} \tag{III.46}$$

we may impose the discrete version of (III.10)

$$\beta_3 - \sqrt{2}\|\Omega\|\beta_2\|T_h(t)\|_h > \gamma_{c,c} > 0 \text{ a.e. in } (0,t_f),$$

for some positive constant $\gamma_{c,c}$, in order to get $c_h(t)$ satisfying the following discrete version of (III.11)

$$\|c_h(t)\|_h^2 + 2\gamma_{c,c}\int_0^t \|D_{-x}c_h(s)\|_+^2 ds \leq \|c_h(0)\|_h^2 e^{2\beta_4 t}, t \in [0,t_f]. \tag{III.47}$$

Remarking that $\|T_h(t)\|_h$ is bounded in $[0,t_f]$, according with the result (III.41).

Stability estimates

At this point we are able to establish the stability relations for the discretized problem. That consists in analysing the differences $\omega_T(t) = T_h(t) - \tilde{T}_h(t), \omega_c(t) = c(t) - \tilde{c}(t)$, when $T_h(t), \tilde{T}_h(t)$ and $c_h(t), \tilde{c}_h(t)$ are solutions of (III.29) and (III.28), respectively, with initial conditions $T_h(0), \tilde{T}_h(0)$ and $c_h(0), \tilde{c}_h(0)$, respectively.

Temperature: For $\omega_T(t) = T_h(t) - \tilde{T}_h(t)$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\omega_T(t)\|_h^2 & +((D_T(M_hT_h(t)) - D_T(M_h\tilde{T}_h(t)))D_{-x}T_h(t), D_{-x}\omega_T(t))_+ \\ & +(D_T(M_h\tilde{T}_h(t))D_{-x}\omega_T(t), D_{-x}\omega_T(t))_+ \\ & = (G(T_h(t)) - G(\tilde{T}_h(t)), \omega_T(t))_h. \end{aligned}$$

We establish the upper bounds

$$-(D_T(M_h\tilde{T}_h(t))D_{-x}\omega_T(t),D_{-x}\omega_T(t))_+ \leq -\beta_0\|D_{-x}\omega_T(t)\|_+^2 \text{ and}$$
$$(G(T_h(t))-G(\tilde{T}_h(t)),\omega_T(t))_h \leq G'_{max}\|\omega_T(t)\|_h^2,$$

but, to find an upper bound for $((D_T(M_hT_h(t))-D_T(M_h\tilde{T}_h(t)))D_{-x}T_h(t),D_{-x}\omega_T(t))_+$, we need to impose an additional condition to $D_{-x}T_h(t)$.

H_6 : $\int_0^t \|D_{-x}T_h(s)\|_{h,\infty}^2 ds$ is uniformly bounded in $h\in\Lambda$, $t\in(0,t_f]$.

This way, under the assumption H_6, we get

$$\begin{aligned}&((D_T(M_hT_h(t))-D_T(M_h\tilde{T}_h(t)))D_{-x}T_h(t),D_{-x}\omega_T(t))_+\\&\leq \sqrt{2}\|D_T\|_{C_b^1(\mathbf{R})}\|D_{-x}T_h(t)\|_{h,\infty}\|\omega_T(t)\|_h\|D_{-x}\omega_T(t)\|_+\\&\leq \frac{1}{2\varepsilon_1^2}\|D_T\|_{C_b^1(\mathbf{R})}^2\|D_{-x}T_h(t)\|_{h,\infty}^2\|\omega_T(t)\|_h^2+\varepsilon_1^2\|D_{-x}\omega_T(t)\|_+^2.\end{aligned} \tag{III.48}$$

where $\varepsilon_1 \neq 0$.

For ε_1 such that $\beta_0-\varepsilon_1^2>0$, it can be shown that

$$\begin{aligned}\|\omega_T(t)\|_h^2 \quad &+2(\beta_0-\varepsilon_1^2)\int_0^t e^{\int_s^t\left(\frac{1}{\varepsilon_1^2}\|D_T\|_{C_b^1(\mathbf{R})}^2\|D_{-x}T_h(\mu)\|_{h,\infty}^2+2G'_{max}\right)d\mu}\|D_{-x}\omega_T(s)\|_+^2 ds\\&\leq \|\omega_T(0)\|_h^2 e^{\int_0^t\left(\frac{1}{\varepsilon_1^2}\|D_T\|_{C_b^1(\mathbf{R})}^2\|D_{-x}T_h(s)\|_{h,\infty}^2+2G'_{max}\right)ds},\ t\in[0,t_f].\end{aligned} \tag{III.49}$$

The assumption H_6 guarantees that the upper bound in (III.49) is bounded by $Const.\|\omega_T(0)\|_h^2$ in $[0,t_f]$. Consequently we conclude the stability of the FEM (III.28), or equivalently, the stability of the FDM (III.32), in $T_h(t)$. We observe that it remains to analyse when H_6 effectively holds.

We can obtain another upper bound avoiding the assumption H_6. Firstly, we observe that (III.48) can be replaced by

$$\begin{aligned}&((D_T(M_hT_h(t))-D_T(M_h\tilde{T}_h(t)))D_{-x}T_h(t),D_{-x}\omega_T(t))_+\\&\leq \|D_T\|_{C_b^1(\mathbf{R})}\|D_{-x}T_h(t)\|_+\|\omega_T(t)\|_{h,\infty}\|D_{-x}\omega_T(t)\|_+\\&\leq |\Omega|\|D_T\|_{C_b^1(\mathbf{R})}\|D_{-x}T_h(t)\|_+\|D_{-x}\omega_T(t)\|_+^2,\end{aligned} \tag{III.50}$$

since $\|\omega_T(t)\|_{h,\infty}\leq|\Omega|\|D_{-x}\omega_T(t)\|_+$. Thus, we establish, by Gronwall Lemma, that

$$\|\omega_T(t)\|_h^2+2\gamma_T\int_0^t e^{2G'_{max}(t-s)}\|D_{-x}\omega_T(s)\|_+^2 ds\leq\|\omega_T(0)\|_h^2 e^{2G'_{max}t},\ t\in[0,t_f], \tag{III.51}$$

provided that

$$\beta_0-|\Omega|\|D_T\|_{C_b^1(\mathbf{R})}\|D_{-x}T_h(t)\|_+\geq\gamma_T>0 \text{ a.e. in } (0,t_f], \tag{III.52}$$

for some positive constant γ_T.

From (III.50) we conclude the stability of (III.28) or equivalently (III.32) in $T_h(t)$ provided that (III.52) holds. Condition (III.52) means that $\|T_h(t)\|_{1,h}$ is a.e bounded in $(0,t_f]$ uniformly in $h\in\Lambda$.

<u>*Concentration:*</u> Defining $\omega_c(t) = c_h(t) - \tilde{c}_h(t)$, we get

$$\begin{aligned}&(\omega_c'(t), \omega_c(t))_h - (M_h(v(T_h(t)))c_h(t), D_{-x}\omega_c(t))_+ + (M_h(v(\tilde{T}_h(t)))\tilde{c}_h(t), D_{-x}\omega_c(t))_+\\&= -(D_d(M_hT_h(t))D_{-x}c_h(t), D_{-x}\omega_c(t))_+ + (D_d(M_h\tilde{T}_h(t))D_{-x}\tilde{c}_h(t), D_{-x}\omega_c(t))_+\\&+(Q(c_h(t)) - Q(\tilde{c}_h(t)), \omega_c(t))_h\end{aligned} \tag{III.53}$$

Then, let treat each parcel separately. For the convective term, if we add and subtract the term $(M_h(v(\tilde{T}_h(t)))c_h(t), D_{-x}\omega_c(t))_+$ we deduce

$$\begin{aligned}&|(M_h\big(v(T_h(t))c_h(t) - v(\tilde{T}_h(t))\tilde{c}_h(t)\big), D_{-x}\omega_c(t))_+|\\&\quad= |(M_h\big(v(T_h(t)) - v(\tilde{T}_h(t))\big)c_h(t), D_{-x}\omega_c(t))_+ + (M_h\big(v(\tilde{T}_h(t))\big)\big(c_h(t) - \tilde{c}_h(t)\big), D_{-x}\omega_c(t))_+|\\&\quad\leq \sqrt{2}\|v\|_{C_b^1(\mathbf{R})}\|\omega_T(t)\|_h\|c_h(t)\|_{h,\infty}\|D_{-x}\omega_c(t)\|_+ + \sqrt{2}\beta_2\|\tilde{T}_h(t)\|_{h,\infty}\|\omega_c(t)\|_h\|D_{-x}\omega_c(t)\|_+\\&\quad\leq \frac{1}{2\varepsilon_1^2}\|v\|^2_{C_b^1(\mathbf{R})}\|\omega_T(t)\|_h^2\|c_h(t)\|^2_{h,\infty} + \frac{1}{2\varepsilon_2^2}\beta_2^2\|\tilde{T}_h(t)\|^2_{h,\infty}\|\omega_c(t)\|_h^2 + (\varepsilon_1^2+\varepsilon_2^2)\|D_{-x}\omega_c(t)\|_+^2,\end{aligned} \tag{III.54}$$

with $\varepsilon_i \neq 0, i = 1,2$, are arbitrary constants.

For the diffusion terms we get

$$\begin{aligned}&|((D_d(M_hT_h(t)) - D_d(M_h\tilde{T}_h(t)))D_{-x}c_h(t), D_{-x}\omega_c(t))_+|\\&\quad\leq \sqrt{2}\|D_d\|_{C_b^1(\mathbf{R})}\|\omega_T(t)\|_h\|D_{-x}c_h(t)\|_{h,\infty}\|D_{-x}\omega_c(t)\|_+\\&\quad\leq \tfrac{1}{2\varepsilon_3^2}\|D_d\|^2_{C_b^1(\mathbf{R})}\|\omega_T(t)\|_h^2\|D_{-x}c_h(t)\|^2_{h,\infty} + \varepsilon_3^2\|D_{-x}\omega_c(t)\|_+^2\end{aligned} \tag{III.55}$$

and

$$(D_d(M_h\tilde{T}_h(t))D_{-x}\omega_c(t), D_{-x}\omega_c(t))_+ \geq \beta_3\|D_{-x}\omega_c(t)\|_+^2.$$

For the reaction term, we get

$$(Q(c_h(t)) - Q(\tilde{c}_h(t)), \omega_c(t)) \leq Q'_{max}\|\omega_c(t)\|_h^2.$$

Following the steps used to establish (III.22) with the convenient adaptations, it can be show that, for $\varepsilon_i \neq 0, i = 1,2,3$, such that $\beta_3 - \sum_{i=1}^{3}\varepsilon_i^2 > 0$, we have

$$\begin{aligned}&\|\omega_c(t)\|_h^2 + \int_0^t e^{\int_s^t\left(\frac{\beta_2^2}{\varepsilon_2^2}\|\tilde{T}_h(\mu)\|^2_{h,\infty} + 2Q'_{max}\right)d\mu}\|D_{-x}\omega_c(s)\|_+^2 ds\\&\quad\leq \frac{1}{\min\{1, 2(\beta_3 - \sum_{i=1}^3\varepsilon_i^2)\}}\Big(\|\omega_c(0)\|_h^2 e^{\int_0^t\left(\frac{\beta_2^2}{\varepsilon_2^2}\|\tilde{T}_h(s)\|^2_{h,\infty} + 2Q'_{max}\right)ds} + \int_0^t e^{\int_s^t\left(\frac{\beta_2^2}{\varepsilon_2^2}\|\tilde{T}_h(\mu)\|^2_{h,\infty} + 2Q'_{max}\right)d\mu}\\&\quad\Big(\frac{1}{\varepsilon_1^2}\|v\|^2_{C_b^1(\mathbf{R})}\|c_h(s)\|^2_{h,\infty} + \frac{1}{\varepsilon_3^2}\|D_d\|^2_{C_b^1(\mathbf{R})}\|D_{-x}c_h(s)\|^2_{h,\infty}\Big)\|\omega_T(s)\|_h^2 ds\Big),\ t \in [0, t_f].\end{aligned} \tag{III.56}$$

To conclude the stability of (III.28) and (III.29), we recall that an upper bound for $\|\omega_T(t)\|_h^2$ in $[0, t_f]$ is established in (III.49) (provided that H_6 holds) or (III.51) (provided that (III.52) holds). To guarantee that the previous estimate holds, we need to assume that $T_h, \tilde{T}_h, c_h, \tilde{c}_h \in C^1([0, t_f], W_{h,0})$. However, the obtained upper bound will be h-dependent. To get a stability estimate h-independent, we need to assume that

$$\int_0^t \|\tilde{T}_h(s)\|_{h,\infty}^2 ds \text{ and } \int_0^t \Big(\|c_h(s)\|_{h,\infty}^2 + \|D_{-x}c_h(s)\|_{h,\infty}^2\Big)\|\omega_T(s)\|_h^2 ds$$

are uniformly bounded in $h \in \Lambda$. As $\int_0^t \|\tilde{T}_h(s)\|_{h,\infty}^2 ds \le \|\Omega\| \int_0^t \|D_{-x}\tilde{T}_h(s)\|_+^2 ds$, then, by (III.41) or (III.44), $\int_0^t \|\tilde{T}_h(s)\|_{h,\infty}^2 ds$ is uniformly bounded in $h \in \Lambda$. To find an upper bound to $\int_0^t \Big(\|c_h(s)\|_{h,\infty}^2 + \|D_{-x}c_h(s)\|_{h,\infty}^2\Big)\|\omega_T(s)\|_h^2 ds$, we need to guarantee that $\int_0^t \Big(\|c_h(s)\|_{h,\infty}^2 + \|D_{-x}c_h(s)\|_{h,\infty}^2\Big) ds$ is uniformly bounded in $h \in \Lambda$. As we will see later, to do that we assume an additional condition on the spatial grids $\Omega_h, h \in \Lambda$, and we show that $c_h(t)$ is a second order approximations for $c(t)$.

Analogously to the continuous case, as well as the temperature discrete case, we are able to obtain another stability estimate if instead of using the bounds (III.54) and (III.55) we use, respectively, the following

$$\begin{aligned}&|(M_h\big(v(T_h)c_h(t) - v(\tilde{T}_h)\tilde{c}_h(t)\big), D_{-x}\omega_c(t))_+|\\ &\quad\le \frac{1}{2\varepsilon_1^2}\|v\|_{C_b^1(\mathbf{R})}^2 \|D_{-x}\omega_T(t)\|_+^2 \|c_h(t)\|_h^2 + \frac{1}{2\varepsilon_2^2}\beta_2^2 \|D_{-x}\tilde{T}_h(t)\|_+^2 \|\omega_c(t)\|_h^2 + (\varepsilon_1^2+\varepsilon_2^2)\|D_{-x}\omega_c(t)\|_+^2\end{aligned} \tag{III.57}$$

considering $\|\omega_T(t)\|_\infty \le \|D_{-x}\omega_T(t)\|_+$ (for above and below) and $\|\tilde{T}(t)\|_{h,\infty} \le \|D_{-x}\tilde{T}(t)\|_+$ (for above), and

$$\begin{aligned}&|((D_d(M_hT_h) - D_d(M_h\tilde{T}_h))D_{-x}c_h(t), D_{-x}\omega_c(t))_+|\\ &\quad\le \|D_d\|_{C_b^1(\mathbf{R})}\|\omega_T(t)\|_\infty \|D_{-x}c_h(t)\|_+ \|D_{-x}\omega_c(t)\|_+\\ &\quad\le \tfrac{\|\Omega\|}{4\varepsilon_3^2}\|D_d\|_{C_b^1(\mathbf{R})}^2 \|D_{-x}\omega_T(t)\|_+^2 \|D_{-x}c_h(t)\|_+^2 + \varepsilon_3^2\|D_{-x}\omega_c(t)\|_+^2.\end{aligned} \tag{III.58}$$

Thus, in conclusion, (III.56) is replaced by

$$\begin{aligned}&\|\omega_c(t)\|_h^2 + \int_0^t e^{\int_s^t \left(\frac{\beta_2^2}{\varepsilon_2^2}\|D_{-x}\tilde{T}_h(\mu)\|_+^2 + 2Q'_{max}\right)d\mu}\|D_{-x}\omega_c(s)\|_+^2 ds\\ &\quad\le \frac{1}{\min\{1, 2(\beta_3 - \sum_{i=1}^3 \varepsilon_i^2)\}}\Big(\|\omega_c(0)\|_h^2 e^{\int_0^t \left(\frac{\beta_2^2}{\varepsilon_2^2}\|D_{-x}\tilde{T}_h(s)\|_+^2 + 2Q'_{max}\right)ds} + \int_0^t e^{\int_s^t \left(\frac{\beta_2^2}{\varepsilon_2^2}\|D_{-x}\tilde{T}_h(\mu)\|_+^2 + 2Q'_{max}\right)d\mu}\\ &\quad\Big(\frac{1}{\varepsilon_1^2}\|v\|_{C_b^1(\mathbf{R})}^2\|c_h(s)\|_h^2 + \frac{\|\Omega\|}{2\varepsilon_3^2}\|D_d\|_{C_b^1(\mathbf{R})}^2\|D_{-x}c_h(s)\|_+^2\Big)\|D_{-x}\omega_T(s)\|_+^2 ds\Big),\ t \in [0, t_f].\end{aligned} \tag{III.59}$$

In this case, upper bounds for $\int_0^t \|D_{-x}\tilde{T}_h(s)\|_+^2 ds$ can be easily obtained from (III.41). An estimate for $\int_0^t \|D_{-x}\omega_T(s)\|_+^2 ds$ is established in (III.49) or (III.51). To conclude from (III.59) the stability of (III.28) and (III.29), we need to guarantee that $\|c_h(s)\|_h^2 + \|D_{-x}c_h(s)\|_+^2$ is bounded a.e. in $(0, t_f)$, uniformly in $h \in \Lambda$. We observe that, from (III.45) or (III.47), $\|c_h(t)\|_h$, and $\int_0^t \|D_{-x}c_h(s)\|_+^2 ds$ are bounded for all $t \in [0, t_f]$, uniformly in $h \in \Lambda$, and consequently, $\|c_h(t)\|_h^2 + \|D_{-x}c_h(t)\|_+^2$ is bounded a.e. in $(0,t_f)$, uniformly in $h\in\Lambda$. In fact, if $\int_0^t \|D_{-x}c_h(s)\|_+^2 ds \le K, \forall t\in[0,t_f], \forall h\in\Lambda$, then $\operatorname{ess\,sup}_{(0,t_f)} \|D_{-x}c_h\|_+ \le K, \forall h\in\Lambda$ (Theorem 2.14, [1]).

Thus, we conclude the desired stability. It remains, now, to justify assumption H_6, in order to guarantee that (III.56) also gives us stability.

We remark that the assumption H_6 is verified if T_h and c_h verify

$$\|E_T(t)\|_h^2 + \int_0^t \|D_{-x}E_T(s)\|_+^2 ds \le Ch_{max}^4, t\in[0,t_f], \tag{III.60}$$

and

$$\|E_c(t)\|_h^2 + \int_0^t \|D_{-x}E_c(s)\|_+^2 ds \le Ch_{max}^4, t\in[0,t_f], \tag{III.61}$$

where $E_T(t) = R_hT(t) - T_h(t)$, $E_c(t) = R_hc(t) - c_h(t)$, and under the assumption on the spatial grids of the sequence Λ

$$\frac{h_{max}^4}{h_{min}} \le Const, h\in\Lambda. \tag{III.62}$$

In fact,

$$\begin{aligned}\int_0^t \|D_{-x}T_h(s)\|_{h,\infty}^2 ds &\le 2\int_0^t \|D_{-x}E_h(s)\|_{h,\infty}^2 ds + 2\int_0^t \|\nabla T(s)\|_\infty^2 ds\\ &\le \frac{2}{h_{min}}\int_0^t \|D_{-x}E_h(s)\|_+^2 ds + 2\int_0^t \|\nabla T(s)\|_\infty^2 ds\\ &\le C\frac{h_{max}^4}{h_{min}} + 2\|T\|_{L^2(0,t_f,C^1(\Omega))}^2.\end{aligned}$$

In the following proposition we summarize our stability result for (III.28) and (III.29).

Proposition 3.2.2 *Under the assumptions $H_1 - H_5$, H_2^*, H_3^* and H_5^*, if $\Omega_h, h\in\Lambda$, satisfy (III.62), $T_h, c_h \in C^1([0,t_f], W_{h,0}), h\in\Lambda$, satisfy (III.60), (III.61), respectively, then there exists a set of positive constants C_i, $i=1,\ldots,6$, h-independent, such that, for $\tilde{T}_h, \tilde{c}_h \in C^1([0,t_f], W_{h,0})$, and $\omega_T(t) = T_h(t) - \tilde{T}_h(t)$, $\omega_c(t) = c_h(t) - \tilde{c}_h(t)$, $h\in\Lambda$, we have*

$$\|\omega_T(t)\|_h^2 + \int_0^t e^{2G'_{max}(t-s)}\|D_{-x}\omega_T(s)\|_+^2 ds \le C_1\|\omega_T(0)\|_h^2 \tag{III.63}$$

$$\begin{aligned}&\|\omega_c(t)\|_h^2 + \int_0^t e^{2Q'_{max}(t-s)}\|D_{-x}\omega_c(s)\|_+^2 ds\\ &\quad\le e^{C_2\|\tilde{T}_h(0)\|_h^2 + C_3}\Big(C_4\|\omega_c(0)\|_h^2 + \|\omega_T(0)\|_h^2\big(C_5 + C_6\|c_h(0)\|_h^2\big)\Big), t\in[0,t_f].\end{aligned} \tag{III.64}$$

■

We establish in the next section the error estimates (III.60) and (III.61).

4 Convergence analysis

In this section, our goal is to introduce a new approach for the analysis of convergence, distinct from the one studied in chapter II and highlight the advantages of doing the convergence study following this line.

This comparative study will focus on the order of convergence as well as in the function's regularity demands. A majority of the results are consequence of Bramble-Hilbert Theorem ([11]), which are presented particularly as in [5] and also [4],[19], [22].

In what follows we use the following notation

$$(g)_h(x_i) = \frac{1}{|\square_i|}\int_{\square_i} g(x)dx, x_i \in \Omega_h,$$

with $\square_i = [x_{i-1/2}, x_{i+1/2}]$.

4.1 Error estimate for the temperature $T_h(t)$

Our aim is to establish the upper bound (III.60) for the error $E_T(t) = R_hT(t) - T_h(t)$, where $T_h(t)$ is defined by (III.28). We have successively

$$(E_T'(t), E_T(t))_h = ((R_hT'(t))_h, E_T(t))_h - (T_h'(t), E_T(t))_h \tag{III.65}$$

Adding and subtracting $((T'(t))_h, E_T(t))_h$ we get

$$\begin{aligned}(E_T'(t), E_T(t))_h &= ((T'(t))_h, E_T(t))_h - (T_h'(t), E_T(t))_h + \tau_d(E_T(t)) \\ &= ((\nabla(D_T(T(t))\nabla T(t)))_h, E_T(t))_h + D_T(M_h(T_h(t)))D_{-x}T_h(t), D_{-x}E_T(t))_+ \\ &+((G(T(t)))_h, E_T(t))_h - G(T_h(t)), E_T(t))_h + \tau_d(E_T(t)),\end{aligned} \tag{III.66}$$

Thus, we add and subtract the terms $(D_T(M_h(R_hT(t)))D_{-x}R_hT(t), D_{-x}E_T(t))_+$ and $(R_hG(T(t)), E_T(t))$ and organize expressions as follows

$$\begin{aligned}(E_T'(t), E_T(t))_h = &-(D_T(M_h(R_hT(t)))D_{-x}R_hT(t) - D_T(M_h(T_h(t)))D_{-x}T_h(t), D_{-x}E_T(t))_+ \\ &+(R_hG(T(t)) - G(T_h(t)), E_T(t))_h + \tau_d(E_T(t)) + \tau_{D_T}(E_T(t)) + \tau_G(E_T(t)),\end{aligned} \tag{III.67}$$

where

$$\tau_d(E_T(t)) = (R_hT'(t) - (T'(t))_h, E_T(t))_h, \tag{III.68}$$

$$\tau_{D_T}(E_T(t)) = ((\nabla(D_T(T(t))\nabla T(t)))_h, E_T(t))_h + (D_T(M_h(R_hT(t)))D_{-x}R_hT(t), D_{-x}E_T(t))_+$$

and

$$\tau_G(E_T(t)) = ((G(T(t)))_h, E_T(t))_h - (R_hG(T(t)), E_T(t))_h.$$

In order to establish an estimation to (IV.40) we need to estimate the introduced error terms. In this regard, we will start by proving the next propositions that will give us upper bound for the τ's terms.

Proposition 4.1.1 *If $T'(t) \in H^2(\Omega)$ then*

$$|\tau_d(E_T(t))| \leq Const\Big(\sum_{i=1}^{N} h_i^4 \|T'(t)\|^2_{H^2(I_i)}\Big)^{1/2} \|D_{-x}E_T(t)\|_+. \tag{III.69}$$

Proof: First of all, we consider $\tau_d(E_T(t)) = (R_hT'(t) - (T'(t))_h, E_T(t))_h$. Let rename $R_hT'(x_i,t)$ by $g(x_i,t)$, noting that our goal is to reuse later the estimations that we will find for g, just replacing it by other convenient functions.

Let, then, find an upper bound for $(g(t),E_T(t))_h - ((g(t))_h, E_T(t))_h$. We have that

$$
\begin{aligned}
&(g(t),E_T(t))_h - ((g(t))_h, E_T(t))_h \\
&= \sum_{i=1}^{N-1} h_{i+\frac{1}{2}} g(x_i,t)E_T(x_i,t) - \sum_{i=1}^{N-1} \Big(\int_{x_{i-1/2}}^{x_{i+1/2}} g(x,t)dx\Big)E_T(x_i,t) \\
&= \sum_{i=1}^{N-1} \Big(\frac{h_i}{2} g(x_i,t) - \int_{x_{i-1/2}}^{x_i} g(x,t)dx\Big)E_T(x_i,t) + \sum_{i=1}^{N-1} \Big(\frac{h_{i+1}}{2} g(x_i,t) - \int_{x_i}^{x_{i+1/2}} g(x,t)dx\Big)E_T(x_i) \\
&= \sum_{i=1}^{N} \Big(\frac{h_i}{2} g(x_i,t) - \int_{x_{i-1/2}}^{x_i} g(x,t)dx\Big)E_T(x_i,t) + \sum_{i=1}^{N} \Big(\frac{h_i}{2} g(x_{i-1},t) - \int_{x_{i-1}}^{x_{i-1/2}} g(x,t)dx\Big)E_T(x_{i-1},t),
\end{aligned}
$$

adding and subtracting the following terms,

$$
\sum_{i=1}^{N} \frac{h_i}{4} g(x_{i-1},t)E_T(x_i,t), \ \ \sum_{i=1}^{N} \frac{h_i}{4} g(x_i,t)E_T(x_{i-1},t),
$$

$$
\sum_{i=1}^{N} \frac{1}{2}\Big(\int_{x_{i-1/2}}^{x_i} g(x,t)dx\Big)E_T(x_{i-1},t), \text{ and } \sum_{i=1}^{N} \frac{1}{2}\Big(\int_{x_{i-1}}^{x_{i-1/2}} g(x,t)dx\Big)E_T(x_i,t)
$$

and organizing sums we gather information as follows,

$$
\begin{aligned}
A &= \sum_{i=1}^{N} \frac{1}{2}\left[\frac{h_i}{2}(g(x_i,t)+g(x_{i-1},t)) - \int_{x_{i-1}}^{x_i} g(x,t)dx\right](E_T(x_i,t)+E_T(x_{i-1},t)) \\
B &= \sum_{i=1}^{N} \frac{1}{2}\left[\frac{h_i}{2}(g(x_i,t)-g(x_{i-1},t)) - \Big(\int_{x_{i-1/2}}^{x_i} g(x,t)dx - \int_{x_{i-1}}^{x_{i-1/2}} g(x,t)dx\Big)\right](E_T(x_i)-E_T(x_{i-1})).
\end{aligned}
$$

Now, we reduce the problem to the estimation of A and B. For A we apply Lemma 4.0.3 (Appendix), so we have

$$
\begin{aligned}
|A| &\leq \sum_{i=1}^{N} \frac{1}{2}\left|\frac{h_i}{2}(g(x_i,t)+g(x_{i-1},t)) - \int_{x_{i-1}}^{x_i} g(x,t)dx\right| |E_T(x_i,t)+E_T(x_{i-1},t)| \\
&\leq \sum_{i=1}^{N} \frac{1}{2} C h_i^2 \|g''(t)\|_{L^1(x_{i-1},x_i)} |E_T(x_i,t)+E_T(x_{i-1},t)| \\
&\leq \frac{C}{2}\left[\sum_{i=1}^{N} h_i^2 \|g''(t)\|_{L^2(I_i)} \sqrt{h_i}|E_T(x_i,t)| + \sum_{i=1}^{N} h_i^2 \|g''(t)\|_{L^2(I_i)} \sqrt{h_i}|E_T(x_{i-1},t)|\right] \\
&\leq \frac{\sqrt{2}C}{2}\Big(\sum_{i=1}^{N} h_i^4 \|g''(t)\|_{H^2(I_i)}\Big)^{1/2} \|E_T(t)\|_h.
\end{aligned}
$$

Thus, applying Poincaré inequality we deduce

$$
|A| \leq Const\Big(\sum_{i=1}^{N} h_i^4 \|g(t)\|_{H^2(I_i)}\Big)^{1/2} \|D_{-x}E_T(t)\|_+ \tag{III.70}
$$

For B, we use Lemma 4.0.4,

$$
\begin{aligned}
|B| &\leq \sum_{i=1}^{N} \frac{1}{2} \left| \frac{h_i}{2}(g(x_i,t) - g(x_{i-1},t)) - \Big(\int_{x_{i-1/2}}^{x_i} g(x,t)dx - \int_{x_{i-1}}^{x_{i-1/2}} g(x,t)dx \Big) \right| |E_T(x_i,t) - E_T(x_{i-1},t)| \\
&\leq \sum_{i=1}^{N} \frac{1}{2} C h_i \|g'(t)\|_{L^1(x_{i-1},x_i)} |E_T(x_i,t) - E_T(x_{i-1},t)| \\
&\leq \frac{C}{2} \sum_{i=1}^{N} h_i^2 \|g'(t)\|_{L^2(I_i)} \sqrt{h_i} D_{-x} E_T(x_i,t) \\
&\leq \frac{C}{2} \Big(\sum_{i=1}^{N} h_i^4 \|g(t)\|^2_{H^1(I_i)} \Big)^{1/2} \Big(\sum_{i=1}^{N} h_i (D_{-x} E_T(x_i,t))^2 \Big)^{1/2}
\end{aligned}
$$

from here we get,

$$
|B| \leq Const \Big(\sum_{i=1}^{N} h_i^4 \|g(t)\|_{H^1(I_i)} \Big)^{1/2} \|D_{-x} E_T(t)\|_+ \tag{III.71}
$$

At this point we are able to replace back g by $R_h T'$ and obtain

$$
|\tau_d(E_T(t))| \leq Const \left[\Big(\sum_{i=1}^{N} h_i^4 \|T'(t)\|_{H^2(I_i)} \Big)^{1/2} + \Big(\sum_{i=1}^{N} h_i^4 \|T'(t)\|_{H^1(I_i)} \Big)^{1/2} \right] \|D_{-x} E_T(t)\|_+ \tag{III.72}
$$

To reach the result we remark that $H^1(\Omega) \subseteq H^2(\Omega)$.

■

Proposition 4.1.2 *If $T(t) \in H^3(\Omega) \cap H_0^1(\Omega)$, then*

$$
|\tau_{D_T}(E_T(t))| \leq Const \|D_T\|_{C_b^1(\mathbf{R})} \Big(1 + \|T(t)\|_{C^1(\overline{\Omega})} \Big) \Big(\sum_{i=1}^{N} h_i^4 \|T(t)\|^2_{H^3(I_i)} \Big)^{1/2} \|D_{-x} E_T(t)\|_+. \tag{III.73}
$$

Proof: For $\tau_{D_T}(t)$ we have

$$
\begin{aligned}
\tau_{D_T}(t) &= \Big(\Big(\nabla \Big(D_T(T(t)) \nabla T(t) \Big) \Big)_h, E_T(t) \Big)_h + (D_T(\hat{T}_h(t)) D_{-x} R_h T(t), D_{-x} E_T(t))_+ \\
&\quad - (D_T(\hat{T}_h(t)) D_{-x} R_h T(t), D_{-x} E_T(t))_+ + (D_T(M_h(R_h T(t))) D_{-x} R_h T(t), D_{-x} E_T(t))_+ \\
&:= \tau_1(t) + \tau_2(t),
\end{aligned}
$$

where

$$
\tau_1(t) = \Big(\Big(\nabla \Big(D_T(T(t)) \nabla T(t) \Big) \Big)_h, E_T(t) \Big)_h + (D_T(\hat{T}_h(t)) D_{-x} R_h T(t), D_{-x} E_T(t))_+,
$$

$$
\tau_2(t) = -(D_T(\hat{T}_h(t)) D_{-x} R_h T(t), D_{-x} E_T(t))_+ + (D_T(M_h(R_h T(t))) D_{-x} R_h T(t), D_{-x} E_T(t))_+
$$

and $\hat{T}_h(t)(x_i) = T(x_{i-1/2}, t)$.

- For $\tau_1(t)$ we have sucessively

$$
\begin{aligned}
\tau_1(t) &= (\nabla\Big(D_T(T(t))\nabla T(t))_h, E_T(t))_h + (D_T(\hat{T}_h(t))D_{-x}R_hT(t), D_{-x}E_T(t))_+ \\
&= \sum_{i=1}^{N-1}\Big(\int_{x_{i-1/2}}^{x_{i+1/2}} \nabla\left(D_T(T(t))\nabla T(x,t)\right)dx\Big)E_T(x_i,t) + \sum_{i=1}^{N} h_i D_T(\hat{T}_h(t))D_{-x}R_hT(x_i,t)D_{-x}E_T(x_i,t) \\
&= \sum_{i=1}^{N-1} D_T(T(x_{i+1/2}))\nabla T(x_{i+1/2},t)E_T(x_i,t) - D_T(T(x_{i-1/2},t))\nabla T(x_{i-1/2},t)E_T(x_i,t) \\
&+ \sum_{i=1}^{N} h_i D_T(T(x_{i-1/2},t))D_{-x}R_hT(x_i,t)D_{-x}E_T(x_i,t) \\
&= \sum_{i=1}^{N} h_i D_T(T(x_{i-1/2}))\left[D_{-x}R_hT(x_i,t) - \nabla T(x_{i-1/2},t)\right]D_{-x}E_T(x_i,t) \\
&= (D_T(\hat{T}_h(t))(D_{-x}R_hT(t) - \hat{R}_h\nabla T(t)), D_{-x}E_T(t))_+
\end{aligned}
$$

where $\hat{R}_h\nabla T(x_i,t) = \nabla T(x_{i-1/2},t)$.

Thereby, applying Lemma 4.0.2, we guarantee the existence of a constant *Const* such that

$$
\begin{aligned}
|\tau_1(t)| &\leq \sum_{i=1}^{N} h_i\left|D_T(T(x_{i-1/2},t))\right|\left|D_{-x}R_hT(x_i,t) - \nabla T(x_{i-1/2},t)\right||D_{-x}E_T(x_i,t)| \\
&\leq \sum_{i=1}^{N} Const\ h_i^2\left|D_T(T(x_{i-1/2},t))\right|\|T^{(3)}(t)\|_{L^1(x_{i-1},x_i)}|D_{-x}E_T(x_i,t)| \\
&\leq Const\Big(\sum_{i=1}^{N} h_i^4\left|D_T(T(x_{i-1/2},t))\right|^2\|T(t)\|^2_{H^3(I_i)}\Big)^{1/2}\|D_{-x}E_T(t)\|_+ \\
&\leq Const\|D_T\|_{C_b^1(\mathbf{R})}\Big(\sum_{i=1}^{N} h_i^4\|T(t)\|^2_{H^3(I_i)}\Big)^{1/2}\|D_{-x}E_T(t)\|_+
\end{aligned}
\tag{III.74}
$$

- To get an estimate for $\tau_2(t)$ first observe that

$$
\begin{aligned}
\tau_2(t) &= (D_T(M_hR_hT(t))D_{-x}R_hT(t), D_{-x}E_T(t))_+ - (D_T(\hat{T}_h(t))D_{-x}R_hT(t), D_{-x}E_T(t))_+ \\
&= \sum_{i=1}^{N} h_i(D_T(M_hR_hT(x_i,t)) - D_T(\hat{T}_h)(x_i,t))D_{-x}R_hT(x_i,t)D_{-x}E_T(x_i,t) \\
&= \sum_{i=1}^{N} h_i\Big(D_T\left(\frac{T(x_i,t)+T(x_{i-1},t)}{2}\right) - D_T(T(x_{i-1/2},t))\Big)D_{-x}R_hT(x_i,t)D_{-x}E_T(x_i,t) \\
&\leq \|D_T\|_{C_b^1(\mathbf{R})}\sum_{i=1}^{N} h_i\left(\frac{T(x_i,t)+T(x_{i-1},t)}{2} - T(x_{i-1/2},t)\right)D_{-x}R_hT(x_i,t)D_{-x}E_T(x_i,t)
\end{aligned}
\tag{III.75}
$$

Then, by Lemma 4.0.2, we have

$$
\begin{aligned}
|\tau_2(t)| &\leq \|D_T\|_{C_b^1(\mathbf{R})}\sum_{i=1}^{N} Const h_i^2 \|T^{(2)}(t)\|_{L^1(x_{i-1},x_i)}|D_{-x}R_hT(x_i)||D_{-x}E_T(x_i)| \\
&\leq Const\|D_T\|_{C_b^1(\mathbf{R})}\sum_{i=1}^{N} h_i^2 \|T^{(2)}(t)\|_{L^1(x_{i-1},x_i)}|D_{-x}R_hT(x_i)|\sqrt{h_i}|D_{-x}E_T(x_i)| \\
&\leq Const\|D_T\|_{C_b^1(\mathbf{R})}\Big(\sum_{i=1}^{N} h_i^4 \|T(t)\|^2_{H^2(I_i)}\|T(t)\|^2_{C^1(\overline{I_i})}\Big)^{1/2}\|D_{-x}E_T(t)\|_+
\end{aligned}
$$

We conclude for $|\tau_{D_T}(E_T(t))|$ the next estimate

$$
\begin{aligned}
|\tau_{D_T}(E_T(t))| \; &\leq Const\|D_T\|_{C_b^1(\mathbf{R})}\Big(\Big(\sum_{i=1}^{N} h_i^4 \|T(t)\|^2_{H^3(I_i)}\Big)^{1/2} \\
&+\Big(\sum_{i=1}^{N} h_i^4 \|T(t)\|^2_{C^1(\overline{I_i})}\|T(t)\|^2_{H^2(I_i)}\Big)^{1/2}\Big)\|D_{-x}E_T(t)\|_+,
\end{aligned}
$$

that lead to (IV.45).

■

Proposition 4.1.3 *If $G \in C_b^2(\mathbb{R})$ and $T(t) \in H^2(\Omega)\cap H_0^1(\Omega)$, then*

$$
|\tau_G(E_T(t))| \leq Const\|G\|_{C_b^2(\mathbf{R})}(1+\|T(t)\|_{C^1(\overline{\Omega})})\Big(\sum_{i=1}^{N} h_i^4 \|T(t)\|^2_{H^2(I_i)}\Big)^{1/2}\|D_{-x}E_T(t)\|_+. \qquad \text{(III.76)}
$$

Proof: We consider $\tau_G(E_T(t)) = ((G(T(t)))_h, E_T(t))_h - (R_hG(T(t)), E_T(t))_h$, we may observe that the computation of τ_G is analogous to the τ_d case, so if we consider that in (III.70) and (III.71), g is replaced by $G(T)$, we deduce the following upper bound for $\tau_G(E_T(t))$,

$$
|\tau_G(E_T(t))| \leq Const\Big(\sum_{i=1}^{N} h_i^4 \|G(T(t))\|_{H^2(I_i)}\Big)^{1/2}\|D_{-x}E_T(t)\|_+.
$$

Then applying H^2-norm definition and taking into account the assumption H_2^* we conclude (III.76).

■

Now we are able to establish the major result that guarantees the upper bound for $E_T(t)$:

Theorem 4.1.4 *We assume that*

$$
T \in L^2(0,t_f,H^3(\Omega)\cap H_0^1(\Omega))\cap H^1(0,t_f,H^2(\Omega)),
$$

$$
R_hT, T_h \in C^1([0,t_f],W_{h,0}),
$$

the coefficient functions D_T and G satisfy the assumptions H_1, H_2, respectively, as well as the assumption of Propositions 4.1.2 and 4.1.3. Then, for ε such that $\beta_0 - 4\varepsilon^2 > 0$, the error $E_T(t) =$

$R_hT(t)-T_h(t)$ satisfies the following

$$\begin{array}{l}
\|E_T(t)\|_h^2+\int_0^t e^{\int_s^t\left(\frac{1}{\varepsilon^2}\|D_T\|^2_{C_b^1(\mathbf{R})}\|D_{-x}R_hT(\mu)\|^2_\infty+2G'_{max}\right)d\mu}\|D_{-x}E_T(s)\|_+^2ds\\
\quad\leq \frac{1}{\min\{1,2(\beta_0-4\varepsilon^2)\}}e^{\int_0^t\left(\frac{1}{\varepsilon^2}\|D_T\|^2_{C_b^1(\mathbf{R})}\|D_{-x}R_hT(\mu)\|^2_{h,\infty}+2G'_{max}\right)ds}\Big(\|E_T(0)\|_h^2\\
+\frac{1}{2\varepsilon^2}\int_0^t e^{-\int_0^s\left(\frac{1}{\varepsilon^2}\|D_T\|^2_{C_b^1(\mathbf{R})}\|D_{-x}R_hT(\mu)\|^2_\infty+2G'_{max}\right)d\mu}\tau_T(s)ds\Big), t\in[0,t_f],
\end{array} \quad \text{(III.77)}$$

where

$$\tau_T(t)=Const\Big(1+\|T(t)\|_{C^1(\overline{\Omega})}\Big)^2\sum_{i=1}^N h_i^4\Big(\|T(t)\|^2_{H^3(I_i)}+\|T'(t)\|^2_{H^2(I_i)}\Big).$$

Proof: From (IV.40), applying H_1 and Mean Value Theorem, we get

- $|((D_T(M_hR_hT)-D_T(M_hT_h))D_{-x}R_hT,D_{-x}E_T)_+|\leq\|D_T\|_{C_b^1(\mathbf{R})}\sqrt{2}\|E_T\|_h\|D_{-x}R_hT\|_\infty\|D_{-x}E_T\|_+$
- $-(D_T(M_hT_h)D_{-x}E_T,D_{-x}E_T)\leq-\beta_0\|D_{-x}E_T\|_+$
- $(R_hG(T),E_T)_h-(G(T_h),E_T)_h=G'_{max}\|E_T\|_h^2$

hence, using Propositions 4.1.1, 4.1.2 and 4.1.3, we easily arrive to

$$\begin{array}{l}
\frac{1}{2}\frac{d}{dt}\|E_T(t)\|_h^2+\beta_0\|D_{-x}E_T(t)\|_+^2\\
\quad\leq\sqrt{2}\|D_T\|_{C_b^1(\mathbf{R})}\|E_T(t)\|_h\|D_{-x}R_hT(t)\|_{h,\infty}\|D_{-x}E_T(t)\|_+\\
\quad+G'_{max}\|E_T(t)\|_h^2+\frac{1}{4\varepsilon^2}\tau_T(t)+3\varepsilon^2\|D_{-x}E_T(t)\|_+^2,
\end{array} \quad \text{(III.78)}$$

where $\varepsilon\neq0$ is an arbitrary constant.
The inequality (III.78) leads to

$$\begin{array}{l}
e^{-\int_0^t\left(\frac{1}{\varepsilon^2}\|D_T\|^2_{C_b^1(\mathbf{R})}\|D_{-x}R_hT(\mu)\|^2_{h,\infty}+2G'_{max}\right)d\mu}\frac{d}{dt}\|E_T(t)\|_h^2\\
+2(\beta_0-4\varepsilon^2)e^{-\int_0^t\left(\frac{1}{\varepsilon^2}\|D_T\|^2_{C_b^1(\mathbf{R})}\|D_{-x}R_hT(s)\|^2_{h,\infty}+2G'_{max}\right)ds}\|D_{-x}E_T(t)\|_+^2\\
\quad\leq e^{-\int_0^t\left(\frac{1}{\varepsilon^2}\|D_T\|^2_{C_b^1(\mathbf{R})}\|D_{-x}R_hT(s)\|^2_{h,\infty}+2G'_{max}\right)ds}\Big(\frac{1}{\varepsilon^2}\|D_T\|^2_{C_b^1(\mathbf{R})}\|D_{-x}R_hT(t)\|^2_{h,\infty}\\
+2G'_{max}\Big)\|E_T(t)\|_h^2+\frac{1}{2\varepsilon^2}e^{-\int_0^t\left(\frac{1}{\varepsilon^2}\|D_T\|^2_{C_b^1(\mathbf{R})}\|D_{-x}R_hT(s)\|^2_{h,\infty}+2G'_{max}\right)ds}\tau_T(t),
\end{array} \quad \text{(III.79)}$$

for $t\in[0,t_f]$. Finally, choosing ε such that $\beta_0-4\varepsilon^2>0$ and considering the continuous imbedding of $H^3(\Omega)$ in $C^1(\overline{\Omega})$, we conclude (III.77).

■

Corollary 4.1.5 *Under the assumptions of Theorem 4.1.4, if $T_h(0)=R_hT(0)$ then there exists a positive constant C such that the error $E_T(t)$ satisfies*

$$\|E_T(t)\|_h^2+\int_0^t e^{2G'_{max}(t-s)}\|D_{-x}E_T(s)\|^2_{h,+}ds\leq Ch^4_{max}, t\in[0,t_f].$$

If the sequence Λ of the spatial vectors h satisfies the assumption (III.62), then the sequence of approximations for the temperature $T_h, h \in \Lambda$, is uniformly bounded in the sense

$$\|T_h(t)\|_{h,\infty} \leq C, \int_0^t e^{2G'_{max}(t-s)} \|D_{-x}T_h(s)\|_+^2 ds \leq C, t \in [0,t_f], h \in \Lambda.$$

4.2 Error estimate for the concentration $c_h(t)$

Considering the error $E_c(t) = R_h c(t) - c_h(t)$, where $T_h(t)$ and $c_h(t)$ are defined by (III.28) and (III.29) and proceeding similarly to the temperature case, we get successively

$$\begin{aligned}
&(E'_c(t), E_c(t))_h = ((c'(t))_h, E_c(t))_h - (c'_h(t), E_c(t))_h + \tau_d(E_c(t)) \\
&= -(D_d(M_h(R_hT(t)))D_{-x}R_hc(t) - D_d(M_h(T_h(t)))D_{-x}c_h(t), D_{-x}E_c(t))_+ \\
&+(M_h(R_h(v(T(t))c(t))), D_{-x}E_c(t))_+ - (M_h(v(T_h(t))c_h(t)), D_{-x}E_c(t))_+ \\
&+(R_hQ(c(t)) - Q(c_h(t)), E_c(t))_h + \tau_d(E_c(t)) + \tau_{D_d}(E_c(t)) + \tau_v(E_c(t)) + \tau_Q(E_c(t)),
\end{aligned} \tag{III.80}$$

where $\tau_d(E_c(t))$ is defined by (III.68) with $T(t)$ and $E_T(t)$ replaced by $c(t)$ and $E_c(t)$, respectively, and the other τs are given as

$$\tau_{D_d}(E_c(t)) = ((\nabla(D_d(T(t))\nabla c(t)))_h, E_c(t))_h + (D_d(M_h(R_hT(t)))D_{-x}R_hc(t), D_{-x}E_c(t))_+,$$

$$\tau_v(E_c(t)) = -(\big(\nabla.(v(T(t))c(t))\big)_h, E_c(t))_h - (M_h(R_h(v(T(t))c(t))), D_{-x}E_c(t))_+, \tag{III.81}$$

and

$$\tau_Q(E_c(t)) = ((Q(c(t)))_h, E_c(t))_h - (R_hQ(c(t)), E_c(t))_h.$$

Let establish through the next Propositions upper bounds for the previous τ's.

Proposition 4.2.1 *If $c'(t) \in H^2(\Omega)$ then*

$$|\tau_d(E_c(t))| \leq Const \Big(\sum_{i=1}^N h_i^4 \|c'(t)\|^2_{H^2(I_i)}\Big)^{1/2} \|D_{-x}E_c(t)\|_+. \tag{III.82}$$

Proof: This result follows exactly the proof of Proposition 4.1.1, just replacing T by c.

■

Proposition 4.2.2 *If $Q \in C_b^2(\mathbb{R})$ and $c(t) \in H^2(\Omega) \cap H_0^1(\Omega)$, then*

$$|\tau_Q(E_c(t))| \quad \leq Const \|Q\|_{C_b^2(\mathbf{R})} \Big(1 + \|c(t)\|_{C^1(\overline{\Omega})}\Big) \left(\sum_{i=1}^N h_i^4 \|c(t)\|^2_{H^2(I_i)}\right)^{1/2} \|D_{-x}E_c(t)\|_+. \tag{III.83}$$

Proof: This result is completely analogous to the previous result for $\tau_G(E_T(t))$, just replacing G by Q and T by c.

■

Proposition 4.2.3 *If $T(t)\in H^2(\Omega)\cap H_0^1(\Omega)$ and $c(t)\in H^3(\Omega)\cap H_0^1(\Omega)$, then*

$$
\begin{aligned}
|\tau_{D_d}(E_T(t))| \quad &\leq Const\|D_d\|_{C_b^1(\mathbf{R})}\left(\Big(\sum_{i=1}^N h_i^4\|c(t)\|^2_{H^3(I_i)}\Big)^{1/2}+\|c(t)\|_{C^1(\overline{\Omega})}\Big(\sum_{i=1}^N h_i^4\|T(t)\|^2_{H^2(I_i)}\Big)^{1/2}\right)\\
&\|D_{-x}E_c(t)\|_+.
\end{aligned}
\tag{III.84}
$$

Proof: In this proof we follow almost all the steps of the proof of Proposition 4.1.2, with relevant changes. We consider the following estimates for τ_1 and τ_2

$$
|\tau_1(t)|\leq Const\|D_d\|_{C_b^1(\mathbf{R})}\Big(\sum_{i=1}^N h_i^4\|c(t)\|^2_{H^3(I_i)}\Big)^{1/2}\|D_{-x}E_c(t)\|_+
$$

$$
|\tau_2(t)|\leq Const\|D_d\|_{C_b^1(\mathbf{R})}\Big(\sum_{i=1}^N h_i^4\|T(t)\|^2_{H^2(I_i)}\|c(t)\|^2_{C^1(\overline{I_i})}\Big)^{1/2}\|D_{-x}E_c(t)\|_+
$$

Whereby result that

$$
|\tau_{D_d}(t)|\leq Const\|D_d\|_{C_b^1(\mathbf{R})}\left[\Big(\sum_{i=1}^N h_i^4\|c(t)\|^2_{H^3(I_i)}\Big)^{1/2}+\Big(\sum_{i=1}^N h_i^4\|T(t)\|^2_{H^2(I_i)}\|c(t)\|^2_{C^1(\overline{I_i})}\Big)^{1/2}\right]\|D_{-x}E_c(t)\|_+
$$

which leads us to (III.84).

■

Proposition 4.2.4 *If $v(T(t))c(t)\in H^2(\Omega)\cap H_0^1(\Omega)$ then*

$$
|\tau_v(E_T(t))| \quad \leq Const\Big(\sum_{i=1}^N h_i^4\|v(T(t))c(t)\|^2_{H^2(I_i)}\Big)^{1/2}\|D_{-x}E_c(t)\|_+. \tag{III.85}
$$

Proof: From (III.81) we have that

$$
\begin{aligned}
\tau_v(E_c(t)) &= -\Big(\Big(\nabla(v(T(t))c(t))\Big)_h, E_c(t)\Big)_h-(M_h(R_h(v(T(t))c(t))),D_{-x}E_c(t))_+\\
&= -\sum_{i=1}^{N-1}\int_{x_{i-\frac12}}^{x_{i+\frac12}}\nabla(v(T)c)(x,t)dxE_c(x_i,t)-\sum_{i=1}^N h_i\frac{(v(T)c)(x_{i+1},t)+(v(T)c)(x_i,t)}{2}D_{-x}E_c(x_i,t)\\
&= \sum_{i=1}^N h_i(v(T)c)(x_{i-\frac12},t)D_{-x}E_c(x_i,t)-\sum_{i=1}^N h_i\frac{(v(T)c)(x_{i+1},t)+(v(T)c)(x_i,t)}{2}D_{-x}E_c(x_i,t)
\end{aligned}
$$

Then, applying modulus, and using Lemma (4.0.5) we obtain the estimate

$$
\begin{aligned}
|\tau_v(E_c(t))| &\leq \sum_{i=1}^{N} h_i \left|(v(T)c)(x_{i-\frac{1}{2}},t) - \frac{(v(T)c)(x_{i+1},t)+(v(T)c)(x_i,t)}{2}\right| |D_{-x}E_c(x_i,t)| \\
&\leq \sum_{i=1}^{N} Consth_i^2 \|(v(T)c)''(t)\|_{L^1(x_{i-1},x_i)} |D_{-x}E_c(x_i,t)| \\
&\leq \Big(\sum_{i=1}^{N} Consth_i^4 \|(v(T)c)''(t)\|^2_{L^2(x_{i-1},x_i)}\Big)^{1/2} \Big(\sum_{i=1}^{N} h_i (D_{-x}E_c(x_i,t))^2\Big)^{1/2} \\
&\leq \Big(\sum_{i=1}^{N} Consth_i^4 \|(v(T)c)''(t)\|^2_{L^2(x_{i-1},x_i)}\Big)^{1/2} \|D_{-x}E_c(t)\|_+ \\
&\leq Const\Big(\sum_{i=1}^{N} h_i^4 \|v(T)c(t)\|^2_{H^2(I_i)}\Big)^{1/2} \|D_{-x}E_c(t)\|_+
\end{aligned}
$$

Thus we conclude (III.85).

■

Theorem 4.2.5 *Let us suppose that*

$$T, c \in L^2(0,t_f,H^3(\Omega)\cap H_0^1(\Omega)) \cap H^1(0,t_f,H^2(\Omega)),$$

$$R_hT, R_hc, T_h, c_h \in C^1([0,t_f],W_{h,0}),$$

the coefficient function D_T and G satisfy the assumptions H_1, H_2, respectively, $Q, G \in C_b^2(\mathbb{R})$, v and D_d satisfy the assumption H_3, H_4, respectively, and $v(T(t))c(t) \in H^2(\Omega)\cap H_0^1(\Omega)$. If the sequence Λ of the spatial vectors h satisfies the assumption (III.62), then for the error $E_c(t) = R_hc(t) - c_h(t)$ we have

$$
\begin{aligned}
&\|E_c(t)\|_h^2 + \int_0^t e^{\int_s^t \left(\frac{1}{\varepsilon^2}\beta_2^2\|T_h(\mu)\|^2_{h,\infty}+2Q'_{max}\right)d\mu} \|D_{-x}E_c(s)\|_+^2 ds \\
&\quad \leq \frac{1}{\min\{1, 2(\beta_3 - 7\varepsilon^2)\}} e^{\int_0^t \left(\frac{1}{\varepsilon^2}\beta_2^2\|T_h(s)\|^2_{h,\infty}+2Q'_{max}\right)ds} \Big(\|E_c(0)\|_h^2 \\
&+ \textstyle\int_0^t e^{-\int_0^s \left(\frac{1}{\varepsilon^2}\beta_2^2\|T_h(\mu)\|^2_{h,\infty}+2Q'_{max}\right)d\mu} \big(\frac{1}{\varepsilon^2}\|D_d\|^2_{C_b^1(\mathbf{R})}\|D_{-x}c(s)\|^2_{h,\infty} \\
&+ \frac{1}{\varepsilon^2}\|v\|^2_{C_b^1(\mathbf{R})}\|c(s)\|^2_{h,\infty}\big)\|E_T(s)\|_h^2 ds + \frac{1}{\varepsilon^2}\int_0^t e^{-\int_0^s \left(\frac{1}{\varepsilon^2}\beta_2^2\|T_h(\mu)\|^2_{h,\infty}+2Q'_{max}\right)d\mu} \tau_c(s) ds\Big), t \in [0,t_f],
\end{aligned}
\tag{III.86}
$$

where $\varepsilon \neq 0$ is such that $\beta_3 - 7\varepsilon^2 > 0$, $\|E_T(t)\|_h^2$ is bounded in (III.77) and

$$
\begin{aligned}
\tau_c(t) &= Const\Big(\sum_{i=1}^{N} h_i^4 \|c'(t)\|^2_{H^2(I_i)} + \sum_{i=1}^{N} h_i^4 \Big(\|c(t)\|^2_{H^3(I_i)} + \|c(t)\|^2_{C^1(\overline{\Omega})}\|T(t)\|^2_{H^2(I_i)}\Big) \\
&\quad + \sum_{i=1}^{N} h_i^4 \|v(T(t))c(t)\|^2_{H^2(I_i)}\Big).
\end{aligned}
\tag{III.87}
$$

Proof: From Theorem 4.1.4, $\|T_h(t)\|_{h,\infty}, h \in \Lambda$, is uniformly bounded in $[0,t_f]$.
From (III.80), applying H_3, H_4 and the Mean Value Theorem, we get

- $([M_h(R_h(v(T))) - M_h(v(T_h))]c, D_{-x}E_c)_+ \leq \sqrt{2}\|v\|_{C_b^1(\mathbf{R})}\|E_T\|_h\|c\|_{h,\infty}\|D_{-x}E_c\|_+$

- $(M_h(v(T_h))E_c, D_{-x}E_c)_+ \leq \sqrt{2}\beta_2\|T_h\|_{h,\infty}\|E_c\|_h\|D_{-x}E_c\|_+$
- $|((D_d(M_hR_hT) - D_d(M_hc_h))D_{-x}R_hc, D_{-x}E_c)_+| \leq \|D_d\|_{C_b^1(\mathbf{R})}\sqrt{2}\|E_c\|_h\|D_{-x}R_hc\|_{h,\infty}\|D_{-x}E_c\|_+$
- $-(D_d(M_hT_h)D_{-x}E_c, D_{-x}E_c) \leq -\beta_3\|D_{-x}E_c\|_+^2$
- $(R_hQ(c), E_c)_h - (Q(c_h), E_c)_h = Q'_{max}\|E_c\|_h^2$

Thus arise from here that

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\|E_c(t)\|_h^2 + \beta_3\|D_{-x}E_c(t)\|_+^2 \\
&\leq \big(\sqrt{2}\|D_d\|_{C_b^1(\mathbf{R})}\|D_{-x}R_hc(t)\|_{h,\infty} + \|v\|_{C_b^1(\mathbf{R})}\sqrt{2}\|c(t)\|_{h,\infty}\big)\|E_T(t)\|_h\|D_{-x}E_c(t)\|_+ + Q'_{max}\|E_c(t)\|_h^2 \\
&+\sqrt{2}\beta_2\|T_h(t)\|_{h,\infty}\|E_c(t)\|_h\|D_{-x}E_c(t)\|_+ + \tau_d(E_c(t)) + \tau_{D_d}(E_c(t)) + \tau_v(E_c(t)) + \tau_Q(E_c(t)).
\end{aligned}$$

Consequently, $E_c(t)$ satisfies

$$\begin{aligned}
&\frac{d}{dt}\|E_c(t)\|_h^2 + 2(\beta_3 - 7\varepsilon^2)\|D_{-x}E_c(t)\|_+^2 \\
&\quad\leq \big(\frac{1}{\varepsilon^2}\|D_d\|_{C_b^1(\mathbf{R})}^2\|D_{-x}R_hc(t)\|_{h,\infty}^2 + \frac{1}{\varepsilon^2}\|v\|_{C_b^1(\mathbf{R})}^2\|R_hc(t)\|_{h,\infty}^2\big)\|E_T(t)\|_h^2 \\
&\quad+\big(\frac{1}{\varepsilon^2}\beta_2^2\|T_h(t)\|_{h,\infty}^2 + 2Q'_{max}\big)\|E_c(t)\|_h^2 + \frac{1}{\varepsilon^2}\tau_c(t),
\end{aligned} \tag{III.88}$$

where $\varepsilon \neq 0$ is an arbitrary constant, and $\tau_c(t)$ is given by (III.87).

The inequality (III.88) is equivalent to the following one

$$\begin{aligned}
&\|E_c(t)\|_h^2 + 2(\beta_3 - 7\varepsilon^2)\int_0^t \|D_{-x}E_c(s)\|_+^2 ds \\
&\quad\leq \|E_c(0)\|_h^2 + \int_0^t \big(\frac{1}{\varepsilon^2}\|D_d\|_{C_b^1(\mathbf{R})}^2\|D_{-x}R_hc(s)\|_{h,\infty}^2 + \frac{1}{\varepsilon^2}\|v\|_{C_b^1(\mathbf{R})}^2\|c(s)\|_{h,\infty}^2\big)\|E_T(s)\|_h^2 ds \\
&\quad+\int_0^t \big(\frac{1}{\varepsilon^2}\beta_2^2\|T_h(s)\|_{h,\infty}^2 + 2Q'_{max}\big)\|E_c(s)\|_h^2 ds + \frac{1}{\varepsilon^2}\int_0^t \tau_c(s)ds,
\end{aligned}$$

that leads to (III.86).

■

By Corollary 4.1.5, for the error $E_T(t)$ we have the following

$$\|E_T(t)\|_h^2 \leq Ch_{max}^4, t \in [0, t_f].$$

Then, from Theorem 4.2.5, we finally conclude the next estimate.

Corollary 4.2.6 *Under the assumptions of Theorems 4.1.4 and 4.2.5, if* $T_h(0) = R_hT(0)$, $c_h(0) = R_hc(0)$, *then there exists a positive constant* C *such that the error* $E_c(t)$ *satisfies*

$$\|E_c(t)\|_h^2 + \int_0^t e^{2Q'_{max}(t-s)}\|D_{-x}E_c(s)\|_+^2 ds \leq Ch_{max}^4, t \in [0, t_f].$$

5 Numerical simulations

5.1 Convergence rates

In this section our goal is to illustrate the main results in chapter III: Theorems 4.1.4 and 4.2.5. We consider $t_f = 0.1$, and we introduce in $[0,t_f]$ the uniform grid $\{t_m, m = 0,\dots,M\}$ with stepsize Δt such that $\Delta t \leq Const h_{max}^2$.

As we intend to illustrate the convergence rates established in Theorems 4.1.4 and 4.2.5, we consider the differential equations (I.2) and (I.3) with reaction terms $f_2(t)$ and $f_1(t)$, respectively. These new reaction terms generate in (III.28), (III.29), the new terms $((f_1(t))_h, u_h)_h$ and $((f_2(t))_h, w_h)_h$, respectively, or equivalently in the semi-discrete FD problems (III.32) and (III.33) the new reaction terms $(f_1(t))_h$ and $(f_2(t))_h$, respectively. These terms will be fixed in order to have problems with known solutions.

To avoid the computation of the solution of non-linear systems, the new version of the semi-discrete FE problem (III.28), (III.29), or equivalently the new version of the semi-discrete FD problem (III.32) and (III.33), are integrated in time using Euler's method following an implicit-explicit approach (IMEX approach)

$$\begin{cases} (T_h^{m+1},\phi_h)_h & +\Delta t(D_T(M_hT_h^m)D_{-x}T_h^{m+1},D_{-x}\phi_h)_+ = (T_h^m,\phi_h)_h \\ & +\Delta t(G(T_h^m),\phi_h)_h + \Delta t((f_1(t_m))_h,\phi_h)_h, \\ & m = 0,\dots,M-1, \forall \phi_h \in W_{h,0}, \\ T_h^0 = R_hT_0, & \end{cases} \tag{III.89}$$

and

$$\begin{cases} (c_h^{m+1},\psi_h)_h - \Delta t(M_h(v(T_h^m)c_h^{m+1}),D_{-x}\psi_h)_+ + \Delta t(D_d(M_hT_h^m)D_{-x}c_h^{m+1},D_{-x}\psi_h)_+ \\ \quad = (c_h^m,\psi_h) + \Delta t(Q(c_h^m),\psi_h)_h + \Delta t((f_2(t_m))_h,\psi_h)_h, \\ \qquad m = 0,\dots,M-1, \forall \psi_h \in W_{h,0}, \\ c_h^0 = R_hc_0. \end{cases} \tag{III.90}$$

The fully discrete FE discretizations (III.89), (III.90) are equivalent to the following FD discretizations

$$\begin{cases} T_h^{m+1} - \Delta t D_x^*\Big(D_T(M_hT_h^m)D_{-x}T_h^{m+1}\Big) = T_h^m + \Delta t G(T_h^m) + \Delta t(f_1)_h^m, \\ \qquad\qquad\qquad\qquad \text{in } \Omega_h,\, m = 0,\dots,M-1, \\ T_h^0 = R_hT_0 \,\text{in}\, \Omega_h, \\ T_h^m = 0 \,\text{on}\, \partial\Omega_h, m = 1,\dots,M, \end{cases} \tag{III.91}$$

$$\begin{cases} c_h^{m+1} + \Delta t D_c\big((v(T_h^{m+1})c_h^{m+1}) - \Delta t D_x^*(D_d(M_hT_h^{m+1})D_{-x}c_h^{m+1}) = c_h^m \\ \qquad\qquad +\Delta t Q(c_h^m) + \Delta t(f_2)_h^m \text{ in } \Omega_h,\, m = 0,\dots,M-1, \\ c_h^0 = R_hc_0 \,\text{in}\, \Omega_h, \\ c_h^m = 0 \,\text{on}\, \partial\Omega_h, m = 1,\dots,M. \end{cases} \tag{III.92}$$

The numerical temperature at time level $m+1$, T_h^{m+1} is evaluated solving the linear system equivalent to (III.91). As in (III.92) the temperature T_h^{m+1} is computed in the previous first step, the numerical concentration at time level $m+1$ is obtained solving the linear system equivalent to (III.92). This approach is particularly recommended in this case.

In the error tables that we present in what follows, we illustrate the behaviour of the errors

$$Error_{\ell,h} = \max_{j=1,\cdots,M}\left(\|E_\ell^j\|_h^2 + \Delta t\sum_{i=1}^{j}\|D_{-x}E_\ell^i\|_+^2\right), \ell = T, c,$$

where

$$E_\ell^j = R_h\ell(t_j) - \ell_h^j,\ j = 1,\ldots,M,$$

and ℓ_h^j is the approximation for $\ell_h(t_j)$ defined by the IMEX method (III.89), (III.90) (or (III.91), (III.92)). We also include the convergence rates $Rate_\ell$ defined considering the following formula

$$Rate_\ell = \frac{\log\left(\frac{Error_{\ell,h_{max,i}}}{Error_{\ell,h_{max,i+1}}}\right)}{\log\left(\frac{h_{max,i}}{h_{max,i+1}}\right)}, \ell = T, c,$$

where $h_{max,i}$ and $h_{max,i+1}$ are defined by two consecutive grids $\Omega_{h,i}$ and $\Omega_{h,i+1}$.

We consider $D_T(T) = D_T$, with $D_T = 10^{-3}(cm^2/s)$, and the diffusion coefficient for the concentration given by the Arrhenius equation (I.1), with $R = 8.3144621$, $E_a = 1.60217662\times 10^{-19}$, $D_0 = 10^{-1}(cm^2/s)$. Moreover, to simplify, we assume that the convective velocity is defined by $v(T) = bT(cm/s)$, where $b = 10^{-1}(cm/s^oK)$. We also take $G = Q = 0$ and $\Delta t = 10^{-6}(s)$.

In the first example we consider f_1 and f_2 such that the differential system (I.2), (I.3) has the following solution

$$\begin{aligned} &T(x,t) = e^{-D_T t}sin(\pi x),\\ &c(x,t) = e^{-t}sin(2\pi x),\ \text{ for } x\in[0,1], t\in[0,t_f]. \end{aligned} \tag{III.93}$$

In Table III.1 we include the errors for the numerical approximations for E_T and E_c obtained with (III.89), (III.90) (or (III.91), (III.92)). We also exhibit the correspondent convergence rates $Rate_\ell, \ell = T, c$. The results included in this table illustrate Theorems 4.1.4 and 4.2.5 when T and c are smooth functions.

N_{points}	h_{max}	E_T	R_T	E_c	$Rate_c$
40	3.75×10^{-2}	2.66001×10^{-5}	——	4.33994×10^{-3}	——
80	1.875×10^{-2}	1.16484×10^{-5}	1.19130	1.12487×10^{-3}	1.94791
160	9.375×10^{-3}	3.75280×10^{-6}	1.63409	2.82498×10^{-4}	1.99345
320	4.6875×10^{-3}	1.0090×10^{-6}	1.89504	7.09491×10^{-5}	1.99338
640	2.34375×10^{-3}	2.56835×10^{-7}	1.97401	1.77873×10^{-5}	1.99594
1280	1.171875×10^{-3}	6.44969×10^{-8}	1.99354	4.48864×10^{-6}	1.98649

Table III.1 Convergence rates of the numerical approximations for the smooth solutions (III.93).

To illutrate the sharpness of our convergence results, in what concerns the smoothness assumptions for the solutions, we consider now the differential system (I.2), (I.3) with solution

$$\begin{aligned} T(x,t) &= e^{-D_T t} sin(\pi x)|2x-1|^{\alpha}, \\ c(x,t) &= e^{-t} x(1-x)|2x-1|^{\alpha}, \text{ for } x \in [0,1], t \in [0,t_f]. \end{aligned} \tag{III.94}$$

We observe that $T(t), c(t) \in H^3(\Omega)$ for $\alpha > 3$, and $T(t), c(t) \in H^2(\Omega)$ for $\alpha \in (2,3]$. In Tables III.2 and III.3 we include the errors and the correspondent convergence rates obtained for $\alpha = 3.1$ and $\alpha = 2.1$, respectively. The results show that when the solutions T and c do not satisfy the smoothness assumption $T(t), c(t) \in H^3(\Omega)$, then we lose the second order convergence rates.

N_{points}	h_{max}	E_T	R_T	E_c	$Rate_c$
40	3.75×10^{-2}	8.02419×10^{-5}	——	5.20569×10^{-4}	——
80	1.875×10^{-2}	3.37646×10^{-5}	1.24885	1.33263×10^{-4}	1.96581
160	9.375×10^{-3}	1.06791×10^{-5}	1.66072	3.37786×10^{-5}	1.98009
320	4.6875×10^{-3}	2.88317×10^{-6}	1.88907	8.49696×10^{-6}	1.99109
640	2.34375×10^{-3}	7.39408×10^{-7}	1.96321	2.13042×10^{-6}	1.99581
1280	1.171875×10^{-3}	1.86910×10^{-7}	1.98403	5.32633×10^{-7}	1.99992

Table III.2 Convergence rates for the α-solution numerical approximation with $\alpha = 3.1$

N_{points}	h_{max}	E_T	R_T	E_c	R_c
40	3.75×10^{-2}	1.11053×10^{-4}	——	4.07632×10^{-4}	——
80	1.875×10^{-2}	5.09896×10^{-5}	1.12297	1.46949×10^{-4}	1.47195
160	9.375×10^{-3}	2.06318×10^{-5}	1.30533	6.00531×10^{-5}	1.29101
320	4.6875×10^{-3}	9.77867×10^{-6}	1.07715	2.65040×10^{-5}	1.18003
640	2.34375×10^{-3}	5.36280×10^{-6}	0.86665	1.21129×10^{-5}	1.12967
1280	1.171875×10^{-3}	2.90522×10^{-6}	0.88434	5.61021×10^{-6}	1.11041

Table III.3 Convergence rates for the α-solution numerical approximation with $\alpha = 2.1$

5.2 Qualitative behaviour

In this section we illustrate the effectiveness of the temperature as a drug release enhancer. We consider an isotropic and homogeneous tissue where a drug is initially dispersed. The assumptions on the properties of the tissue allow us to replace the 3D drug release problem by a 1D problem.

In order to observe the behaviour of the drug transport enhanced by the temperature variations, we will point three distinct situations:

(I) The heat source is in contact with the tissue at the boundaries,

(II) The heat source is applied to the full tissue during a defined time interval,

(III) The heat source is in contact with one of the boundaries where the flux of drug is proportional to the amount of drug that is already in the blood.

From the biological literature we know that the increase of the temperature increases the tissue permeability, the body fluid circulation, the blood vessel wall permeability, the rate-limiting membrane permeability and the drug solubility. These phenomena explain the reason why heat sources are drug delivery enhancers.

(I) *Heat source term at the boundaries*

In this first case, those phenomena are macroscopically taken into account in the convective velocity $v(T)$ and in the drug diffusion coefficient $D_d(T)$ that we assume to be given by the Arrhenius equation (I.1).

We consider that the heat sources are applied at the boundaries of the domain $\Omega = (0,1)$. Initially, the drug concentration is defined by $c(x,0) = x(1-x)$ (g/cm^3) , $x \in (0,1)$. We consider $G = 0$ and $t_f = 10^4$ (s), $D_T = 10^{-7}$ (cm^2/s), $D_0 = 10^{-4}$ (cm^2/s). We take $\Delta t = 10^{-1}$ (s) and $h = 1.25 \times 10^{-2}$ (cm).

- In what follows we consider that the heat is generated by $T(0,t) = T(1,t) = 310 + 0.1t$ (^oK) for an activation energy E_a such that $E_a/R = 4.43 \times 10^2$ (s), and $v(t) = 0$.

  In Figure III.1 we plot the temperature curves for different times. As the heat sources are localized at the boundaries, we observe an increasing of the temperature from the boundaries to the interior. The evolution of the concentration when the diffusion coefficient depends on the temperature is illustrated in Figure III.2. In this case we consider the temperature given in Figure III.1. As the time increases, an increasing on the transport near the left and right boundaries is observed. This fact is consequence of the increasing on the temperature observed in these zones. The correspondent released mass $M_r(t) = M(0) - \int_\Omega c(x,t)dx$ is plotted in Figure III.3. The released mass increases when the drug transport is enhanced by the temperature.

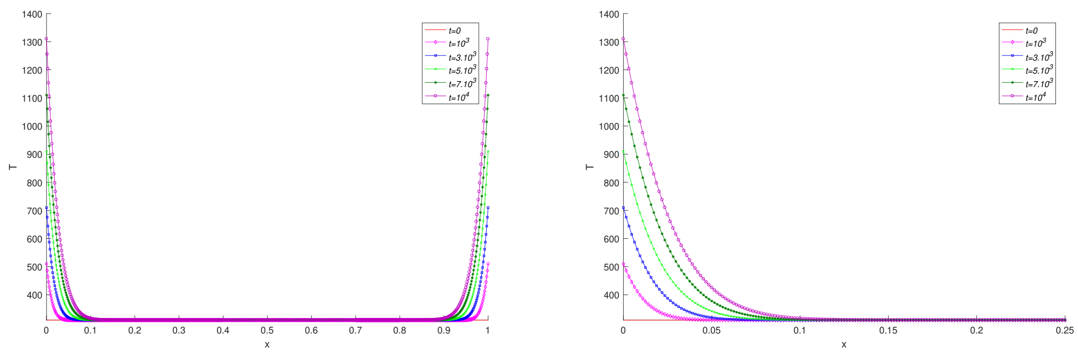

Fig. III.1 Plots of the temperature for $T(0,t) = T(1,t) = 310 + 0.1t$ at $t = 0$, 10^3, 3×10^3, 5×10^3, 7×10^3, 10^4. The right figure is a zoom of the left figure.

- We assume now that the heat is generated by $T(0,t) = T(1,t) = 310 + 5 \times 10^{-4}t$ (^oK), for $t > 0$, and the heat induces a convective transport given by $v(T) = bT$, with $b = 5 \times 10^{-4}$, (cm/s^oK). We assume that the activation energy E_a is such that $E_a/R = 10^3$ (s).

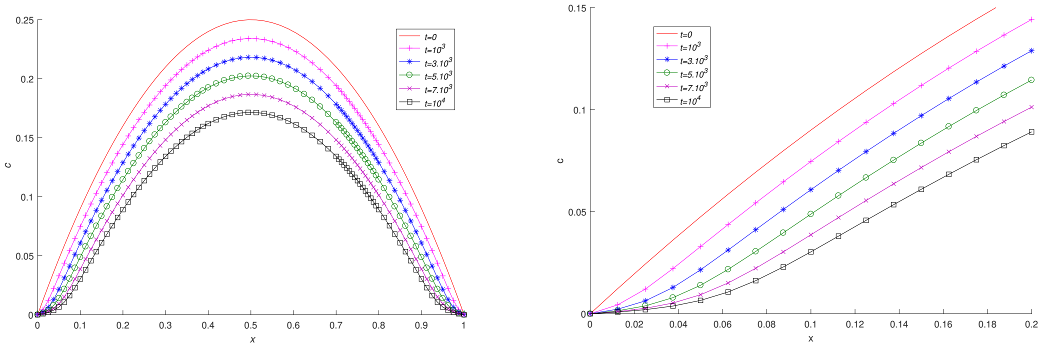

Fig. III.2 Plots of the concentration when the heat sources are defined by $T(0,t) = T(1,t) = 310 + 0.1t$ and $v = 0$ at $t = 0$, 10^3, 3×10^3, 5×10^3, 7×10^3, 10^4. The right figure is a zoom of the left figure.

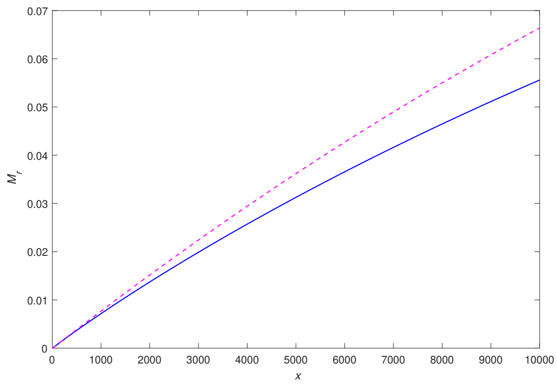

Fig. III.3 Evolution of the released drug mass M_r: under the effect of the temperature (dashed line); without the temperature effect (continuous line) for the diffusion coefficient $D_d = 10^{-4}$, $T(0,t) = T(1,t) = 310 + 0.1t$ and $v = 0$.

Figure III.4 illustrates the behaviour of the temperature. As we can see, it increases when t increases from the boundaries, where the heat sources are located, to the interior. In Figure III.5 we plot the evolution of the corresponding drug concentration. As $v(T) = bT$, with $b = 5 \times 10^{-4}$, the heat generates a convective term that induces a displacement of the drug concentration from left to the right. Moreover, the highest concentration value decreases in time due to the drug release and this value moves from the left to the right and such displacement increases with time. The behaviour of the released drug mass $M_r(t)$ is illustrated in Figure III.6. The heat, generated by the sources applied at the boundaries of the domain, increases the transport from the left to the right and, consequently, it increases the released drug. In this case, we observe that the drug mass was completely released at $t \simeq 5 \times 10^3$ while in the absence of the heat stimulus at $t = 10^4$ the drug release continues. The increase of the temperature in live tissues, like the skin, leads to an increase of the drug permeation through the tissue, an increase of the skin perfusion and an increase to the clarence to systemic circulation ([31]). These complex alterations in the tissue were mathematically translated by $D_d(T)$, a temperature depending diffusion coefficient which is an increasing function of the temperature, and by the convective term $v(T)$. The obtained results are physically sound.

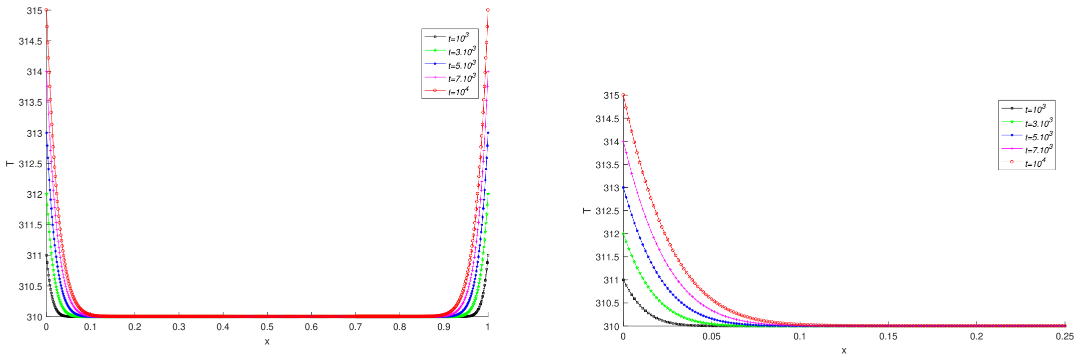

Fig. III.4 Plots of the temperature for $T(0,t) = T(1,t) = 310 + 5\times10^{-4}t$ at $t = 0$, 10^3, 3×10^3, 5×10^3,7×10^3, 10^4. The right figure is a zoom of the left figure.

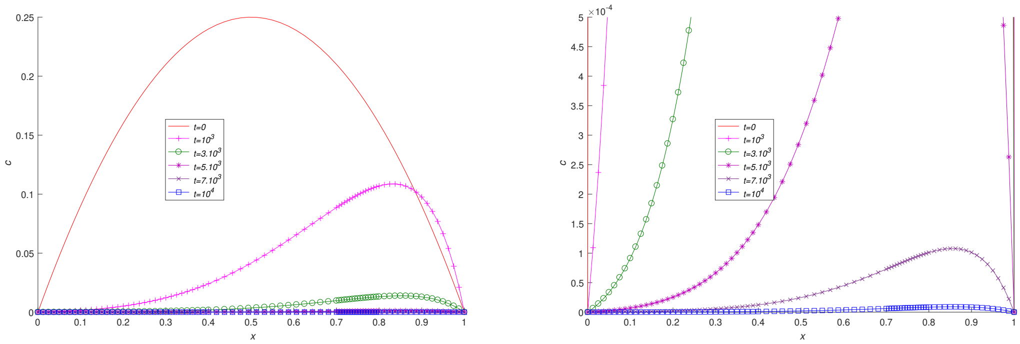

Fig. III.5 Plots of the concentration when the heat source is given by $T(0,t) = T(1,t) = 310 + 5\times10^{-4}t$ and $v(T) = 5\times10^{-4}T$ at $t = 0$, 10^3, 3×10^3, 5×10^3, 7×10^3, 10^4. The right figure is a zoom of the left figure.

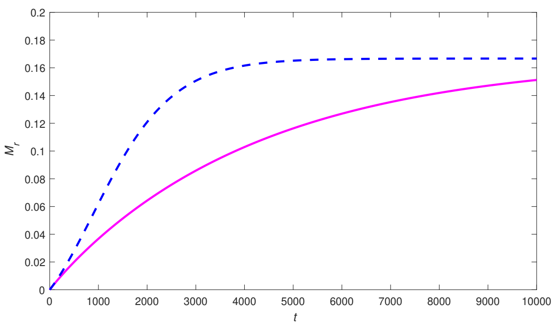

Fig. III.6 Evolution of the released drug mass M_r: under the effect of the temperature (dashed line); without the temperature effect (continuous line) for the diffusion coefficient $D_d = 10^{-4}$ and $v(T) = 5\times10^{-4}T$.

(II) *Heat source applied to the full tissue during a defined time interval*

We consider for this approach the initial temperature and concentrations given by $T(x,0) = 310x(1-x)(^oK)$ and $c(x,0) = x(1-x)(g/cm^3)$. The reaction term is inexistent. The convective term is a linear function of the temperature considering $b = 0.01$, and for the diffusion functions we assume, in this case that, both functions are given by Arrhenius equation with $E_a = 10$, $R = 8.314$, $D_{0,T} = 10^{-2}$ and $D_{0,c} = 10^{-2}$. With respect time step we consider $\Delta t = 10^{-2}$ and we take $t_f = 10$; regarding the space stepsize we consider $h = 10^{-2}$ (to simplify uniform grids are used). The source term is the following

$$G(T) = \begin{cases} 312 - 2cos(3t),\ t < 0.4 \\ 0,\ t \geq 0.4. \end{cases} \tag{III.95}$$

The effect of heat is taken into account through diffusion for temperature and through diffusion and convection for the drug concentration. We will focus on two different aspects when we consider a source term (through function $G(T)$) applied to the full spatial domain. The first aspect regards the effect on the total mass released when a heat source is applied to the entire domain during a certain time. We exhibit in (III.7) the evolution of temperature (left) and the evolution of drug concentration inside the domain. We observe that for $t = 1$ the total mass of drug, under the action of the heat source, has been totally released.

As we can observe the difference between the two cases is obvious and gives a natural justification for instance for the use of patches at its effectiveness.

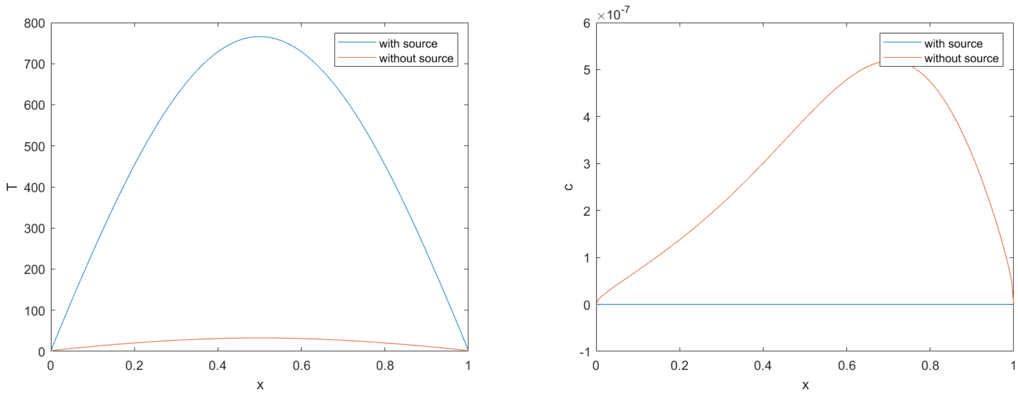

Fig. III.7 Evolution of the temperature (on the left) and concentration inside the domain (on the right) for $G(T)$ given by (III.95)

The second aspect we want to address is to illustrate the dependence of the drug release pattern on the duration of heat application. The heat source is defined in (III.95), but instead of assuming $t < 0.4$, we consider that source acts during the intervals $[0,1]$, $[0,3]$, $[4,7]$, $[1,10]$ and $[8,10]$.

Observing the Figure (III.9) we conclude that the time interval $[0,3]$ guarantees a very efficient release, the total mass of drug being totally released during the action of the heat source.

(III) *The heat source is in contact with one of the boundaries and the flux of drug is proportional to the concentration of drug in the blood*

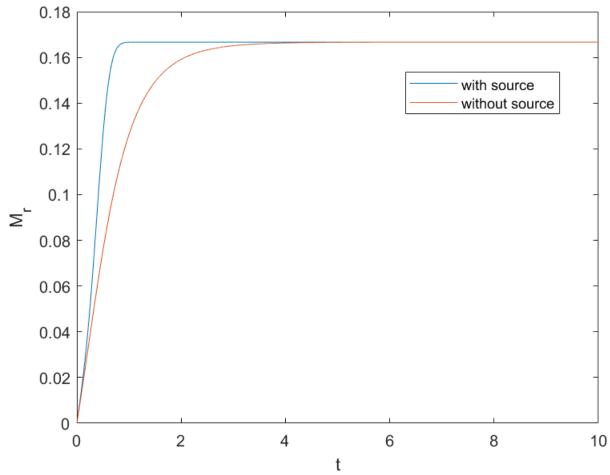

Fig. III.8 Evolution of the released drug mass $G(T)$ given by (III.95)

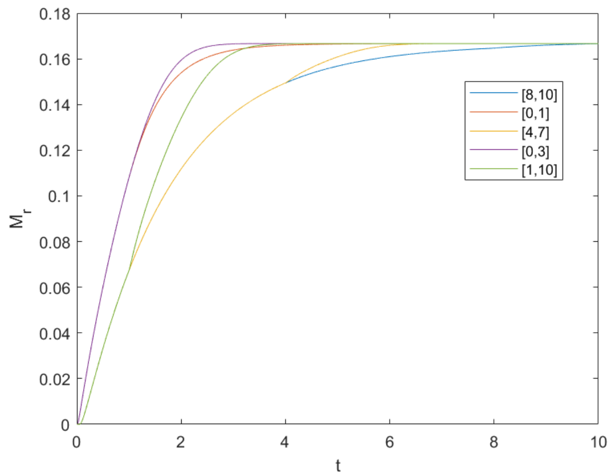

Fig. III.9 Evolution of the released drug mass for the source applied during distinct time intervals

In this case, we consider that the drug concentration that enters the blood at $x = 1$ is represented by a Robin boundary condition of type

$$-D_d(T)\frac{\partial c}{\partial x}(1,t) = \alpha(c(1,t) - c_B) \tag{III.96}$$

where α and c_B are positive constants. In (III.96) α represents the permeability of the interface drug domain/ release medium and c_B is the concentration of drug in the blood. We recall that for the boundaries the drug is immediately washed out that is we have $c(0,t) = 0$. The discrete version of the Boundary conditions is given by

$$\begin{cases} -D_d(T_N^m)\dfrac{c_N^{m+1} - c_{N-1}^{m+1}}{h^N} = \alpha(c_N^{m+1} - c_B) \\ c_0^m = 0, \ \text{for } m = 1, \cdots, M \end{cases}$$

In the numerical simulations of problem (III) we consider the initial functions $T(x,0) = 310 + 10sin(\pi x)(^oK)$ and $c(x,0) = sin(\pi x)(g/cm^3)$, for the temperature and concentration, respectively. Assuming that $G(T) = 0$, $Q(c) = 0$, $v(T) = bT$, $D_T(T) = D_d(T) = DT$, where $b = D = 0.01$. Moreover, we fix $t_f = 0.01$, $\Delta t = 0.001$ and $h = 5 \times 10^{-3}$.

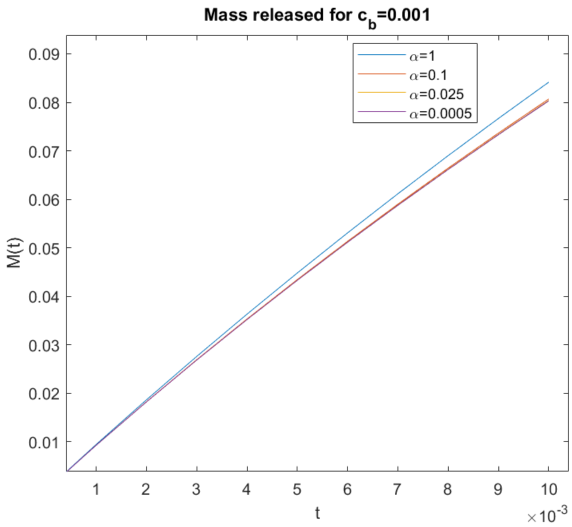

Fig. III.10 Comparison of mass released for different α's assuming $c_B = 0.001(g/cm^3)$

We observe that when we decrease α we are decreasing the membrane permeability, consequently the mass released is smaller when α goes to zero. On the other hand, as we can see in Figure (III.11), if c_B increases that will cause a decrease of the mass released, at certain time.

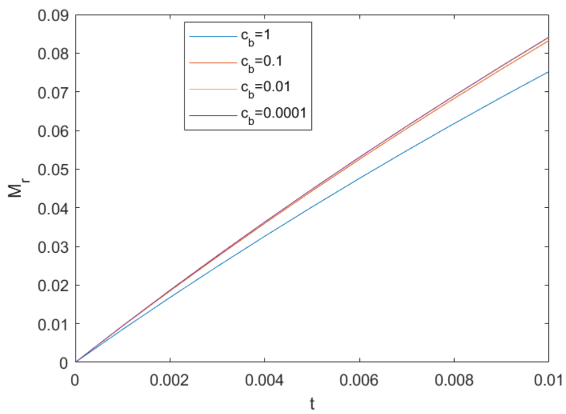

Fig. III.11 Comparison of mass released fixing $\alpha = 1$ and varying the values of c_B

6 Conclusions

The use of heat as an external stimulus to enhance drug release is nowadays a common approach in several medical applications (see [14], [15], [34], [46] and [53]). Mathematically, the drug release enhanced by the temperature is described by a diffusion-reaction equation for the temperature and a convection-diffusion-reaction equation for the drug, where the convective and the drug diffusion coefficients depend on the temperature.

In this Chapter we propose a numerical method to compute the temperature and the drug concentration. The method is based on the piecewise linear finite element method, combined with special quadrature rules. It leads to second order numerical approximations for the temperature and for the concentration provided that both solutions are in $H^3(\Omega) \cap H_0^1(\Omega)$ (Theorems 4.1.4 and 4.2.5). The proposed method mimics the continuous coupling in what concerns the stability behaviour as it was shown, in Section 2, for the continuous coupling, and, in Subsection 3.2, for the numerical coupling problem. The main stability result - Proposition 3.2.2 - establishes that the fully discrete finite element method (III.28), (III.29) or, equivalently, the finite difference method (III.32), (III.33) is stable. This result was proved under assumption H_6, that is a consequence of the second convergence order established in Theorems 4.1.4 and 4.2.5.

We reinforce the fact that the convergence analysis presented in this chapter is not based on the classical approach: consistency and stability imply convergence. The error analysis is based on the analysis of the error equation and on the use of the approach introduced by one of the authors in [5], and used later for the coupling between an elliptic equation and an integro-differential equation in [4], and for the coupling between a hyperbolic equation and a convection-diffusion equation that arises in models for drug delivery enhanced by ultrasounds in [22].

Numerical results were included to illustrate the main convergence results. The rates of convergence presented in Tables III.1, III.2 and Table III.3 illustrate the sharpness of our results in what concerns the smoothness assumptions for the temperature and concentration.

The numerical results presented in Figures III.3, III.6 and III.8 illustrate the use of heat to enhance drug release is an effective procedure. It is observed that if the drug transport is enhanced by temperature then the drug delivery attains its steady state very quickly. These results are physically sound because the increase of temperature in live tissues, like the skin, leads to an increase of the drug permeation through the tissue, an increase of the skin perfusion and an increase in the clearance to systemic circulation ([31]).

Chapter IV

Coupling nonlinear electric fields: stability and convergence analysis

1 Introduction

As it was already mentioned this system can be used to describe the drug transport through a target tissue when an electric field is used as enhancer (see for instance [6], [16], [42], [47]).

The expressions for the functions σ, D_T, v, D_d, F, G and Q were defined in chapter I. Their smoothness will be specified throughout the body of the chapter.

In what concerns the organization of the chapter we start by introducing the preliminary notations and definitions in Section 2. In Section 3 we analyse the stability and convergence properties of the discretization of the elliptic equation (I.6). The convergence properties of the semi-discrete approximations for T and c are studied in Section 4 and 5, respectively. Finally, in Section 6 we present numerical experiments illustrating the main convergence results as well as the qualitative behaviour of the solution of the IBVP (I.6)-(I.8), (I.9), (I.10). Some conclusions are presented in Section 7.

2 Preliminary notations and definitions

As mentioned before, by $L^2(\Omega)$ and $H_0^1(\Omega)$ we denote the usual Sobolev spaces and, for $m \in \mathbb{N}_0$, by $H^m(0,t_f,V)$ we represent the space of functions $w : \overline{\Omega} \times [0,t_f] \to \mathbb{R}$ such that $w^{(j)}(t) \in V, j = 0,\dots,m$, where $w^{(j)}(t)$ is the weak time derivative of order j.

The weak solution of the IBVP (I.6)-(I.8), (I.9), (I.10) is defined by

$$(\phi,T,c) \in H_0^1(\Omega) \times [L^2(0,t_f,H_0^1(\Omega)) \cap H^1(0,t_f,L^2(\Omega))]^2$$

such that

$$(\sigma(|\nabla\phi|),\nabla\psi) = (f,\psi), \forall\psi \in H_0^1(\Omega), \tag{IV.1}$$

$$(T'(t),w)+a_T(T(t),w)=(G(T(t)),w)+(F(\nabla\phi),w) \text{ a.e. in } (0,t_f), \forall w\in H_0^1(\Omega), \tag{IV.2}$$

$$(c'(t),u)+a_c(c(t),u)=(Q(c(t)),u) \text{ a.e. in } (0,t_f), \forall u\in H_0^1(\Omega), \tag{IV.3}$$

and

$$(T(0),w)=(T_0,w),\ \forall w\in L^2(\Omega),\ (c(0),u)=(c_0,u),\ \forall u\in L^2(\Omega). \tag{IV.4}$$

In (IV.2) and (IV.3) the following notation were used

$$a_T(T(t),w)=(D_T(T(t))\nabla T(t),\nabla w),$$

$$a_c(c(t),u)=(D_d(T(t))\nabla c(t),\nabla u)-(v(T(t),\nabla\phi)c(t),\nabla u).$$

To introduce the piecewise linear FE approximations for $\phi, T(t)$ and $c(t)$, we consider the sequence Λ of vectors $h=(h_1,\cdots,h_n)$ introduced in chapter II, assuming $I_i=(x_{i-1},x_i)$. We recall that W_h and $W_{h,0}$ represent, respectively, the vector space of grid functions defined in $\overline{\Omega}_h$ and the subspace of W_h of functions null on the boundary points $\partial\Omega_h=\{x_0,x_N\}$. For $u_h\in W_h$, by $P_h u_h$ we denote the piecewise linear interpolation of u_h.

The piecewise linear FE approximations for $\phi, T(t)$ and $c(t)$ are then the solution of the following system

$$(\sigma(|\nabla P_h\phi_h|),\nabla P_h\psi_h)=(f,\psi_h),\ \forall\psi_h\in W_{h,0}, \tag{IV.5}$$

$$(P_hT_h'(t),w_h)+a_T(P_hT_h(t),P_hw_h)=(G(P_hT_h(t)),P_hw_h)+(F(\nabla P_h\phi_h),P_hw_h) \text{ a.e. in } (0,t_f), \tag{IV.6}$$

$\forall w_h\in W_{h,0}$,

$$(P_hc_h'(t),u_h)+a_c(P_hc_h(t),P_hu_h)=(Q(P_hc_h(t)),P_hu_h) \text{ a.e. in } (0,t_f), \tag{IV.7}$$

$\forall u_h\in W_{h,0}$,

$$\begin{aligned}(P_hT_h(0),P_hw_h)&=(P_hR_hT_0,P_hw_h),\ \forall w_hW_{h,0},\\ (P_hc_h(0),P_hu_h)&=(P_hR_hc_0,P_hu_h)\forall u_h\in W_{h,0}.\end{aligned} \tag{IV.8}$$

In (IV.8), R_h denotes the following restriction operator $R_h: C(\overline{\Omega})\to W_h$, $R_hg(x_i)=g(x_i), i=0,\ldots,N$, $g\in C(\overline{\Omega})$.

Once more D_{-x} denotes the usual backward finite difference operator and for the operators D_x^*, D_c and M_h we consider the expressions introduced in (II.5) and (II.9). The finite difference operator D_h is given by:

$$D_hu_h(x_i)=\frac{h_{i+1}D_{-x}u_h(x_i)+h_iD_{-x}u_h(x_{i+1})}{h_{i+1}+h_i},\ i=1,\cdots,N-1,$$

where $h_{i+1/2}=\dfrac{h_i+h_{i+1}}{2}$.

If $g \in L^2(\Omega)$, then by $(g)_h$ we represent the following discrete function

$$\begin{aligned}(g)_h(x_0) &= \frac{1}{h_1}\int_{x_0}^{x_{1/2}} g(x)dx,\\ (g)_h(x_i) &= \frac{1}{h_{i+1/2}}\int_{x_{i-1/2}}^{x_{i+1/2}} g(x)dx, i=1,\ldots,N-1,\\ (g)_h(x_N) &= \frac{1}{h_N}\int_{x_{N-1/2}}^{x_N} g(x)dx.\end{aligned} \tag{IV.9}$$

In what concerns the inner products and norms we keep the notations stated previously.

Then the fully discrete FE approximations for $\phi, T(t), c(t) \in H_0^1(\Omega)$ are defined by

$$(\sigma(|D_{-x}\phi_h|)D_{-x}\phi_h, D_{-x}\psi_h)_{h,+} = ((f)_h, \psi_h)_h, \ \forall \psi_h \in W_{h,0}, \tag{IV.10}$$

$$(T_h'(t), w_h)_h + a_{T_h}(T_h(t), w_h) = (G(T_h(t)), w_h)_h + (F(D_h\phi_h), w_h)_h \text{ in } (0,t_f], \ \forall w_h \in W_{h,0}, \tag{IV.11}$$

$$(c_h'(t), u_h)_h + a_{c_h}(c_h(t), u_h) = (Q(c_h(t)), u_h)_h \text{ in } (0,t_f], \ \forall u_h \in W_{h,0}, \tag{IV.12}$$

$$T_h(0) = R_hT_0,\ c_h(0) = R_hc_0 \text{ in } \Omega_h, \tag{IV.13}$$

where

$$a_{T_h}(T_h(t), w_h) = (D_T(M_hT_h(t))D_{-x}T_h(t), D_{-x}w_h)_+,$$

and

$$a_{c_h}(c_h(t), u_h) = -(M_h(v(T_h(t), D_h\phi_h)c_h(t)), D_{-x}u_h)_+ + (D_d(M_hT_h(t))D_{-x}c_h(t), D_{-x}u_h)_+.$$

We remark that the fully discrete FE approximations defined by (IV.10)-(IV.12), (IV.13) can be obtained solving the following nonlinear finite difference system

$$-D_x^*(\sigma(|D_{-x}\phi_h|)D_{-x}\phi_h) = (f)_h \text{ in } \Omega_h, \tag{IV.14}$$

$$T_h'(t) = D_x^*(D_T(M_h(T_h(t)))D_{-x}T_h(t)) + G(T_h(t)) + F(D_h\phi_h) \text{ in } \Omega_h \times (0,t_f], \tag{IV.15}$$

$$c_h'(t) + D_c(v(T_h(t), D_h\phi_h)c_h(t)) = D_x^*(D_d(M_h(T_h(t)))D_{-x}c_h(t)) + Q(c_h(t)) \text{ in } \Omega_h \times (0,t_f], \tag{IV.16}$$

where take $D_h\phi_h(x_0) = D_{-x}\phi_h(x_1)$ and $D_h\phi_h(x_N) = D_{-x}\phi_h(x_N)$. System (IV.14)-(IV.16) is completed with the initial conditions (IV.13) and the boundary conditions

$$T_h(t) = c_h(t) = 0 \text{ on } \partial\Omega_h \times (0,t_f]. \tag{IV.17}$$

3 A Finite element method for the nonlinear elliptic equation

In this Section we study the stability and convergence properties of the piecewise linear FEM (IV.10) or equivalently the FDM (IV.14).

3.1 Stability: a first approach

To study the stability of the nonlinear finite difference operator defined by (IV.14) with Dirichlet boundary conditions we consider $\tilde{\phi}_h$ as the solution of (IV.14) with $(f)_h$ replaced by $\tilde{f}_h$. Then for $\omega_p = \phi_h - \tilde{\phi}_h$ we have

$$(\sigma(|D_{-x}\phi_h|)D_{-x}\phi_h, D_{-x}\omega_p)_+ - \sigma(|D_{-x}\tilde{\phi}_h|)D_{-x}\tilde{\phi}_h, D_{-x}\omega_p)_+ = ((f)_h - \tilde{f}_h, \omega_p)_h,$$

that leads to

$$(\sigma(|D_{-x}\tilde{\phi}_h|)D_{-x}\omega_p, D_{-x}\omega_p)_+ = ((\sigma(|D_{-x}\tilde{\phi}_h|) - \sigma(|D_{-x}\phi_h|))D_{-x}\phi_h, D_{-x}\omega_p)_+ + ((f)_h - \tilde{f}_h, \omega_p)_h.$$

We need to impose the following smoothness assumption on σ:

H_7 : $\sigma \in C_b^1(\mathbb{R}_0^+)$ and $\sigma \geq \beta_5 \geq 0$ in $\mathbb{R}_0^+$.

where $C_b^1(\mathbb{R}_0^+)$ denotes the space of real functions with bounded derivative in $\mathbb{R}_0^+$ and with norm denoted by $\|.\|_{C_b^1(\mathbb{R}_0^+)}$.

Under the assumption H_7 we conclude

$$\beta_5\|D_{-x}\omega_p\|_+^2 \leq \|\sigma\|_{C_b^1(\mathbb{R}_0^+)}\|D_{-x}\phi_h\|_\infty\|D_{-x}\omega_p\|_+^2 + \|(f)_h - \tilde{f}_h\|_h\|\omega_p\|_h, \tag{IV.18}$$

where

$$\|D_{-x}\phi_h\|_\infty = \max_{i=1,\ldots,N}|D_{-x}\phi_h(x_i)|.$$

As for $u_h \in W_{h,0}$ holds the discrete Friedrichs-Poincaré inequality $\|u_h\|_h \leq |\Omega|\|D_{-x}u_h\|_+$, where $|\Omega|$ denotes the measure of Ω, from (IV.18) we get

$$(\beta_5 - \|\sigma\|_{C_b^1(\mathbb{R}_0^+)}\|D_{-x}\phi_h\|_\infty - \varepsilon^2|\Omega|^2)\|D_{-x}\omega_p\|_+^2 \leq \frac{1}{4\varepsilon^2}\|(f)_h - \tilde{f}_h\|_h^2, \tag{IV.19}$$

where $\varepsilon \neq 0$. If we are able to fix ε such that

$$\beta_5 - \|\sigma\|_{C_b^1(\mathbb{R}_0^+)}\|D_{-x}\phi_h\|_\infty - \varepsilon^2|\Omega|^2 > 0, h \in \Lambda, \tag{IV.20}$$

then we conclude the stability of (IV.14). The condition (IV.20) requires that there exists a positive constant Const such that

$$\|D_{-x}\phi_h\|_\infty \leq Const, \, h \in \Lambda. \tag{IV.21}$$

In what follows we show that the condition (IV.21) is consequence of the accuracy of ϕ_h.

3.2 Supraconvergence-supercloseness

Theorem 3.2.1 *Let $E_\phi = R_h\phi - \phi_h$ where ϕ and ϕ_h are defined by (I.6) and (IV.14), respectively. If $\phi \in H^3(\Omega)\cap H_0^1(\Omega)$, σ satisfies the assumption H_7 and $\beta_5 - \|\sigma\|_{C_b^1(\mathbf{R}_0^+)}\|\phi\|_{C^1(\overline{\Omega})} > 0$ then there exist constants Const > 0 such that*

$$\|D_{-x}E_\phi\|_+^2 \le Const \sum_{i=1}^{N} h_i^4 \|\phi\|_{H^3(I_i)}^2. \tag{IV.22}$$

Proof: It can be shown that for the error E_ϕ holds the following equation

$$(\sigma(|D_{-x}\phi_h|)D_{-x}E_\phi, D_{-x}E_\phi)_+ = (\sigma(|D_{-x}\phi_h|)D_{-x}R_h\phi, D_{-x}E_\phi)_+ - ((f)_h, E_\phi)_h. \tag{IV.23}$$

For $((f)_h, E_\phi)_h$ we deduce

$$\begin{aligned}((f)_h, E_\phi)_h &= \sum_{i=1}^{N-1}\int_{x_{i-1/2}}^{x_{i+1/2}} -\nabla(\sigma(|\nabla\phi|)\nabla\phi)dx E_\phi(x_i)\\ &= \sum_{i=1}^{N} h_i\sigma(|\nabla\phi(x_{i-1/2})|)\nabla\phi(x_{i-1/2})D_{-x}E_\phi(x_i)\\ &= (\hat{R}_h(\sigma(|\nabla\phi|)\nabla\phi), D_{-x}E_\phi)_+\end{aligned}$$

where $\hat{R}_h g(x_i) = g(x_{i-1/2}), i = 1,\dots,N$.

Inserting the last representation in (IV.23) we obtain

$$(\sigma(|D_{-x}\phi_h|)D_{-x}E_\phi, D_{-x}E_\phi)_+ = (\sigma(|D_{-x}\phi_h|)D_{-x}R_h\phi - \hat{R}_h(\sigma(|\nabla\phi|)\nabla\phi), D_{-x}E_\phi)_+. \tag{IV.24}$$

As

$$\begin{aligned}&\sigma(|D_{-x}\phi_h|)D_{-x}R_h\phi - \hat{R}_h(\sigma(|\nabla\phi|)\nabla\phi)\\ &\quad= (\sigma(|D_{-x}\phi_h|) - \sigma(|D_{-x}R_h\phi|))D_{-x}R_h\phi + (\sigma(|D_{-x}R_h\phi| - \hat{R}_h\sigma(|\nabla\phi|))D_{-x}R_h\phi)\\ &\quad+\hat{R}_h\sigma(|\nabla\phi|)(D_{-x}R_h\phi - \hat{R}_h\nabla\phi),\end{aligned}$$

from (IV.24) we also obtain

$$(\sigma(|D_{-x}\phi_h|)D_{-x}E_\phi, D_{-x}E_\phi)_+ = \sum_{i=1}^{3}\tau_i, \tag{IV.25}$$

where

$$\tau_1 = ((\sigma(|D_{-x}\phi_h|) - \sigma(|D_{-x}R_h\phi|))D_{-x}R_h\phi, D_{-x}E_\phi)_+,$$

$$\tau_2 = ((\sigma(|D_{-x}R_h\phi|) - \hat{R}_h\sigma(|\nabla\phi|))D_{-x}R_h\phi, D_{-x}E_\phi)_+$$

and

$$\tau_3 = (\hat{R}_h\sigma(|\nabla\phi|)(D_{-x}R_h\phi - \hat{R}_h\nabla\phi), D_{-x}E_\phi)_+.$$

In what follows we deduce estimates for $\tau_i,\ i = 1,2,3$.

i) For τ_1 we easily establish

$$|\tau_1| \leq \|\phi\|_{C^1(\overline{\Omega})}\|\sigma\|_{C^1_b(\mathbf{R}^+_0)}\|D_{-x}E_\phi\|^2_+. \tag{IV.26}$$

ii) To obtain an estimate for τ_2 we start by remarking that

$$|\tau_2| \leq \|D_{-x}R_h\phi\|_\infty\|\sigma\|_{C^1_b(\mathbf{R}^+_0)}\sum_{i=1}^{N} h_i|D_{-x}\phi(x_i) - \nabla\phi(x_{i-1/2})||D_{-x}E_\phi(x_i)|$$

and

$$h_i|D_{-x}\phi(x_i) - \nabla\phi(x_{i-1/2})| = |\lambda(1) - \lambda(0) - \lambda'(\frac{1}{2})|$$

with $\lambda(\xi) = \phi(x_{i-1} + \xi h_i)$. As Bramble-Hilbert lemma leads to

$$|\lambda(1) - \lambda(0) - \lambda'(\frac{1}{2})| \leq Const\int_0^1 |\lambda^{(3)}(\xi)|d\xi \leq Const\ h_i^2\sqrt{h_i}\|\phi\|_{H^3(I_i)},$$

we conclude

$$|\tau_2| \leq Const\|D_{-x}R_h\phi\|_\infty\|\sigma\|_{C^1_b(\mathbf{R}^+_0)}\Big(\sum_{i=1}^{N} h_i^4\|\phi\|^2_{H^3(I_i)}\Big)^{1/2}\|D_{-x}E_\phi\|_+$$

that is

$$|\tau_2| \leq Const\frac{1}{\varepsilon^2}\|D_{-x}R_h\phi\|^2_\infty\|\sigma\|^2_{C^1_b(\mathbf{R}^+_0)}\sum_{i=1}^{N} h_i^4\|\phi\|^2_{H^3(I_i)} + \varepsilon^2\|D_{-x}E_\phi\|^2_+. \tag{IV.27}$$

iii) Analogously, for τ_3 we have

$$|\tau_3| \leq Const\frac{1}{\varepsilon^2}\|\sigma\|^2_{C^1_b(\mathbf{R}^+_0)}\sum_{i=1}^{N} h_i^4\|\phi\|^2_{H^3(I_i)} + \varepsilon^2\|D_{-x}E_\phi\|^2_+. \tag{IV.28}$$

Inserting (IV.26)-(IV.28) into (IV.25) we establish

$$\big(\beta_5 - 2\varepsilon^2 - \|\phi\|_{C^1(\overline{\Omega})}\|\sigma\|_{C^1_b(\mathbf{R}^+_0)}\big)\|D_{-x}E_\phi\|^2_+ \leq Const\frac{1}{\varepsilon^2}\|\sigma\|^2_{C^1_b(\mathbf{R}^+_0)}\big(1 + \|\phi\|^2_{C^1(\overline{\Omega})}\big)\sum_{i=1}^{N} h_i^4\|\phi\|^2_{H^3(I_i)}. \tag{IV.29}$$

Finally, if $\beta_5 - \|\sigma\|_{C^1_b(\mathbf{R}^+_0)}\|\phi\|_{C^1(\overline{\Omega})} > 0$ we guarantee the existence of a positive constant *Const* such that (IV.22) holds.

■

Corollary 3.2.2 *Under the assumptions of Theorem 3.2.1 we have*

$$\|R_h\phi - \phi_h\|_{1,h} \leq Const h^2_{max}. \tag{IV.30}$$

Corollary 3.2.3 *Let us suppose that the sequence of grids $\overline{\Omega}_h, h \in \Lambda$, is such that there exists a positive constant C_Λ that satisfies*

$$\frac{h_{max}}{h_{min}} \leq C_\Lambda, \tag{IV.31}$$

where $h_{min} = \min_{i=1,\ldots,N} h_i$. Then, under the assumptions of Theorem 3.2.1, there exists a positive constant Const, h independent, such that (IV.21) holds.

Proof: We remark that, for $i = 1, \ldots, N$, we have

$$\begin{aligned} |D_{-x}\phi_h(x_i)| &\leq \frac{1}{h_{min}} \sum_{j=1}^{N} h_j |D_{-x}E_\phi(x_j)| + |D_{-x}\phi(x_i)| \\ &\leq \frac{1}{h_{min}} \sqrt{|\Omega|} \|D_{-x}E_\phi\|_+ + \|\phi\|_{C^1(\overline{\Omega})} \\ &\leq \frac{h_{max}}{h_{min}} \sqrt{|\Omega|} Const\ h_{max} + \|\phi\|_{C^1(\overline{\Omega})} \end{aligned}$$

that conclude the proof. ■

3.3 Stability: second approach

As we have shown in the beginning of this section, to establish the stability of (IV.14), with Dirichlet boundary conditions, in $\phi_h, h \in \Lambda$, we need to impose the boundness of $\|D_{-x}\phi_h\|_\infty, h \in \Lambda$. Corollary 3.2.3 establishes that, under the assumptions of Theorem 3.2.1, if the sequence of grids $\overline{\Omega}_h, h \in \Lambda$, satisfies the condition (IV.31) then the boundness condition (IV.21) holds.

Theorem 3.3.1 *Under the assumptions of Theorem 3.2.1, $\overline{\Omega}_h, h \in \Lambda$, satisfies the condition (IV.31),*

$$\beta_5 - \|\sigma\|_{C_b^1(\mathbf{R}_0^+)} \|D_{-x}\phi_h\|_\infty > 0, h \in \Lambda, \tag{IV.32}$$

and $\tilde{\phi}_h \in W_{h,0}, h \in \Lambda$, is defined by (IV.14) with $(f)_h$ replaced by $\tilde{f}_h$, then there exists a positive constant Const, h independent, such that

$$\|\phi_h - \tilde{\phi}_h\|_{1,h} \leq Const \|(f)_h - \tilde{f}_h\|_h, \, h \in \Lambda. \tag{IV.33}$$

Moreover, if $\tilde{f}_h \in \bar{B}_{\rho_h}((f)_h) = \{g_h \text{ is defined in } \Omega_h : \|g_h - (f)_h\|_h \leq \rho_h\}$, $h \in \Lambda$, where $\rho_h \leq \rho h_{max}, h \in \Lambda$, then $\|D_{-x}\tilde{\phi}_h\|_\infty \leq Const, h \in \Lambda$, with Const h independent.

Proof: As we have

$$\|D_{-x}\tilde{\phi}_h\|_\infty \leq \|D_{-x}(\tilde{\phi}_h - \phi_h)\|_\infty + \|D_{-x}\phi_h\|_\infty \leq Const \frac{1}{h_{min}} \|(f)_h - \tilde{f}_h\|_h + \|D_{-x}\phi_h\|_\infty,$$

we conclude the proof using the uniform boundness of $\|D_{-x}\phi_h\|_\infty, h \in \Lambda$, and the fact $\tilde{f}_h \in \bar{B}_{\rho_h}((f)_h), h \in \Lambda$, with $\rho_h \leq \rho h_{max}, h \in \Lambda$. ■

To conclude the stability analysis of the finite difference discretization of the nonlinear elliptic operator we establish the final stability result.

Corollary 3.3.2 *Under the assumptions of Theorem 3.3.1, if* $f_h^*, \tilde{f}_h \in \bar{B}_{\rho_h}((f)_h), h \in \Lambda$, *with* $\rho_h \leq \rho h_{max}, h \in \Lambda$, *and* $\phi_h^*, \tilde{\phi}_h \in W_{h,0}$ *are solutions of the finite difference equations*

$$-D_x^*(\sigma(|D_{-x}\phi_h^*|)D_{-x}\phi_h^*) = f_h^* \text{ in } \Omega_h,$$
$$-D_x^*(\sigma(|D_{-x}\tilde{\phi}_h|)D_{-x}\tilde{\phi}_h) = \tilde{f}_h \text{ in } \Omega_h,$$

then $\|D_{-x}\phi_h^*\|_\infty \leq Const, \|D_{-x}\tilde{\phi}_h\|_\infty \leq Const, h \in \Lambda$, *with* $Const$ *h independent, and if*

$$\beta_5 - \|\sigma\|_{C_b^1(\mathbf{R}_0^+)}\|D_{-x}\phi_h^*\|_\infty > 0 \text{ or } \beta_5 - \|\sigma\|_{C_b^1(\mathbf{R}_0^+)}\|D_{-x}\tilde{\phi}_h\|_\infty > 0,$$

for $h \in \Lambda$, *then*

$$\|\phi_h^* - \tilde{\phi}_h\|_{1,h} \leq Const\|f_h^* - \tilde{f}_h\|_h, \; h \in \Lambda. \tag{IV.34}$$

4 Temperature - second order error estimates with respect to the discrete L^2-norm

4.1 Stability

We start this section by establishing energy estimates for the discrete temperature $T_h(t) \in W_{h,0}$ defined by (IV.11) or (IV.15) and with initial condition $T_h(0)$. We assume that the coefficient function D_T satisfy H_1, the assumption on G is replaced by the less restrictive condition H_2:

$H_2 : G(0) = 0, (G(u_h) - G(w_h), u_h - w_h)_h \leq \beta_1\|u_h - w_h\|_h^2, u_h, v_h \in W_{h,0}$,

and F satisfy

$H_8 : F(0) = 0$ and F is a Lipschitz function with Lipschitz constant β_6.

The previous assumptions are assumed only for theoretical purposes. For instance, if we consider Pennes' equation (I.13), we have $G(x) = -\omega_m c_b x$ that satisfies H_2. However, F defined by (I.14) satisfies H_8 only locally.

Theorem 4.1.1 *Under the assumptions* H_1,H_2 *and* H_8, *if the sequence of grids* $\overline{\Omega}_h, h \in \Lambda$, *satisfies the condition (IV.31) and* $T_h \in C^1([0,t_f], W_{h,0})$ *then*

$$\|T_h(t)\|_h^2 + 2\beta_0\int_0^t e^{(2\beta_1+1)(t-s)}\|D_{-x}T_h(s)\|_+^2 ds \leq e^{(2\beta_1+1)t}\|T_h(0)\|_h^2 + \frac{2\beta_6^2 C_\Lambda}{2\beta_1+1}\left(e^{(2\beta_1+1)t} - 1\right)\|D_{-x}\phi_h\|_+^2, \tag{IV.35}$$

for $t \in [0,t_f]$.

Proof: Fixing in (IV.11) $w_h = T_h(t)$ and considering the assumptions H_1,H_2 and H_8 and the condition (IV.31) we easily obtain

$$\frac{d}{dt}\|T_h(t)\|_h^2 + 2\beta_0\|D_{-x}T_h(t)\|_+^2 \leq 2\beta_1\|T_h(t)\|_h^2 + 2\sqrt{2}\beta_6\sqrt{C_\Lambda}\|D_{-x}\phi_h\|_+\|T_h(t)\|_h, t \in (0,t_f],$$

that leads to

$$\frac{d}{dt}\Big(e^{-(2\beta_1+1)t}\|T_h(t)\|_h^2 \quad +2\beta_0\int_0^t e^{-(2\beta_1+1)s}\|D_{-x}T_h(s)\|_+^2 ds \\ -\frac{2\beta_6^2C_\Lambda}{2\beta_1+1}\big(1-e^{-(2\beta_1+1)t}\big)\|D_{-x}\phi_h\|_+^2\Big) \le 0, t\in(0,t_f].$$

Using the smoothness assumption for $T_h(t)$ we finally conclude (IV.35). ■

The combination of Theorem 4.1.1 with Corollary 3.2.3 allow us to conclude that $\|T_h(t)\|_h^2 + 2\beta_0\int_0^t \|D_{-x}T_h(s)\|_+^2 ds$ is uniformly bounded for $t\in[0,t_f]$ and for $h\in\Lambda$, provided that $2\beta_1+1\ge 0$.

To study the stability of $T_h(t)\in W_{h,0}$ defined by (IV.11) or (IV.15) with respect to the initial condition, we consider two solutions $T_h, \tilde{T}_h \in C^1([0,t_f],W_{h,0})$ with initial condition $T_h(0)$ and $\tilde{T}_h(0)$ but both computed with $\phi_h\in W_{h,0}$, defined by the fully discrete FEM (IV.10) or equivalently the FDM (IV.14).

Let $\omega_T(t)$ be defined by $\omega_T(t) = T_h(t)-\tilde{T}_h(t)$. Following [24], for $\omega_T(t)$ we easily get

$$\frac{1}{2}\frac{d}{dt}\|\omega_T(t)\|_h^2 + (D_T(M_hT_h(t))D_{-x}\omega_T(t), D_{-x}\omega_T(t))_+ \\ \le ((D_T(M_h\tilde{T}_h(t)) - D_T(M_hT_h(t)))D_{-x}\tilde{T}_h(t), D_{-x}\omega_T(t))_+ + \beta_1\|\omega_T(t)\|_h^2,$$

and then, using H_1 and H_2, we establish

$$\frac{d}{dt}\|\omega_T(t)\|_h^2 + 2\beta_0\|D_{-x}\omega_T(t)\|_+^2 \le 2\beta_1\|\omega_T(t)\|_h^2 + 2\sqrt{2}\|D_T\|_{C_b^1(\mathbf{R})}\|D_{-x}\tilde{T}_h(t)\|_\infty\|\omega_T(t)\|_h\|D_{-x}\omega_T(t)\|_+,$$

and consequently we have

$$\frac{d}{dt}\|\omega_T(t)\|_h^2 + (2\beta_0-\varepsilon^2)\|D_{-x}\omega_T(t)\|_+^2 \le \Big(\frac{2}{\varepsilon^2}\|D_T\|_{C_b^1(\mathbf{R})}^2\|D_{-x}\tilde{T}_h(t)\|_\infty^2 + 2\beta_1\Big)\|\omega_T(t)\|_h^2, \quad \text{(IV.36)}$$

for $t\in(0,t_f]$. From (IV.36) we conclude

$$\|\omega_T(t)\|_h^2 \quad +(2\beta_0-\varepsilon^2)\int_0^t e^{\int_s^t(\frac{2}{\varepsilon^2}\|D_T\|_{C_b^1(\mathbf{R})}^2\|D_{-x}\tilde{T}_h(\mu)\|_\infty^2+2\beta_1)d\mu}\|D_{-x}\omega_T(s)\|_+^2 ds \\ \le e^{\int_0^t(\frac{2}{\varepsilon^2}\|D_T\|_{C_b^1(\mathbf{R})}^2\|D_{-x}\tilde{T}_h(\mu)\|_\infty^2+2\beta_1)d\mu}\|\omega_T(0)\|_h^2, \ t\in[0,t_f]. \quad \text{(IV.37)}$$

We notice that the stability inequality (IV.37) with $T_h(t)$ replaced by $\tilde{T}_h(t)$ can be easily established because

$$((D_T(M_h\tilde{T}_h(t)) - D_T(M_hT_h(t)))D_{-x}\tilde{T}_h(t), D_{-x}\omega_T(t))_+ \\ = -(D_T(M_h\tilde{T}_h(t))D_{-x}\omega_T(t), D_{-x}\omega_T(t))_+ + ((D_T(M_h\tilde{T}_h(t)) - D_T(M_hT_h(t)))D_{-x}T_h(t), D_{-x}\omega_T(t))_+.$$

To guarantee stability from (IV.37) we need to prove that $\int_0^t \|D_{-x}\tilde{T}_h(\mu)\|_\infty^2 d\mu$ or $\int_0^t \|D_{-x}T_h(\mu)\|_\infty^2 d\mu$ are uniformly bounded for $h\in\Lambda$ and $t\in[0,t_f]$.

4.2 Convergence analysis

Let $E_T(t) = R_hT(t) - T_h(t)$. An estimate for $\|E_T(t)\|_h$ is established in the next result whose proof follows the proof of Theorem 1 of [24].

Theorem 4.2.1 *Let T and T_h be solutions of the IBVP (I.7) and (IV.15), with homogeneous Dirichlet boundary conditions and initial values $T(0)$ and $T_h(0)$, respectively. Let ϕ and ϕ_h be solutions of the elliptic equations (I.6) and (IV.10), with homogeneous Dirichlet boundary conditions, respectively. We suppose that*

$$T \in H^1(0,t_f,H^2(\Omega)) \cap L^2(0,t_f,H^3(\Omega)\cap H_0^1(\Omega)), \phi \in H^3(\Omega)\cap H_0^1(\Omega),$$

$$R_hT, T_h \in C^1([0,t_f],W_{h,0}),\ G(T(t)) \in H^2(\Omega).$$

If the assumptions H_1,H_2 and H_8 hold and the sequence of grids $\overline{\Omega}_h, h\in\Lambda$, satisfies the condition (IV.31), then for the error $E_T(t) = R_hT(t) - T_h(t)$ holds the following

$$\begin{aligned}\|E_T(t)\|_h^2 \quad &+2(\beta_0 - 5\varepsilon^2)\int_0^t e^{\int_s^t \left(\frac{1}{\varepsilon^2}\|D_T\|^2_{C_b^1(\mathbf{R})}\|T(\mu)\|^2_{C^1(\overline{\Omega})} + 2\beta_1 + 2\delta^2\right)d\mu}\|D_{-x}E_T(s)\|_+^2\,ds\\ &\le \|E_T(0)\|_h^2 e^{\int_0^t \left(\frac{1}{\varepsilon^2}\|D_T\|^2_{C_b^1(\mathbf{R})}\|T(\mu)\|^2_{C^1(\overline{\Omega})} + 2\beta_1 + 2\delta^2\right)d\mu}\\ &+\int_0^t e^{\int_s^t \left(\frac{1}{\varepsilon^2}\|D_T\|^2_{C_b^1(\mathbf{R})}\|T(\mu)\|^2_{C^1(\overline{\Omega})} + 2\beta_1 + 2\delta^2\right)d\mu}\left(\frac{1}{\delta^2}C_\Lambda\|D_{-x}E_\phi\|_+^2 + \Gamma(s)\right)ds,\end{aligned} \tag{IV.38}$$

where $E_\phi = R_h\phi - \phi_h$, $\varepsilon \neq 0, \delta \neq 0$, and

$$\begin{aligned}|\Gamma(t)| \le Const\frac{1}{\varepsilon^2}\Big(1 + \|D_T\|^2_{C_b^1(\mathbf{R})}(1 + \|T(t)\|^2_{C^1(\overline{\Omega})})\Big)\\ \Big(\sum_{i=1}^N h_i^4\Big(\|T'(t)\|^2_{H^2(I_i)} + \|T(t)\|^2_{H^3(I_i)} + \|G(T(t))\|^2_{H^2(I_i)} + \|\phi\|^2_{H^3(I_i)}\Big) + \sum_{i=1}^{N-1}(h_i^4 + h_{i+1}^4)\|\phi\|^2_{H^3(I_i\cup I_{i+1})}\Big)\end{aligned} \tag{IV.39}$$

Proof: It can be shown that

$$\begin{aligned}(E_T'(t),E_T(t))_h =& -(D_T(M_h(R_hT(t)))D_{-x}R_hT(t) - D_T(M_h(T_h(t)))D_{-x}T_h(t), D_{-x}E_T(t))_+\\ &+(R_hG(T(t)) - G(T_h(t)), E_T(t))_h + (F(D_hR_h\phi) - F(D_h\phi_h), E_T(t))_h\\ &+\tau_d(E_T(t)) + \tau_{D_T}(E_T(t)) + \tau_G(E_T(t)) + \tau_F(E_T(t)),\end{aligned} \tag{IV.40}$$

where

$$\tau_d(E_T(t)) = (R_hT'(t) - (T'(t))_h, E_T(t))_h,$$

$$\tau_{D_T}(E_T(t)) = ((\nabla(D_T(T(t))\nabla T(t)))_h, E_T(t))_h + (D_T(M_h(R_hT(t)))D_{-x}R_hT(t), D_{-x}E_T(t))_+$$

$$\tau_G(E_T(t)) = ((G(T(t)))_h, E_T(t))_h - (R_hG(T(t)), E_T(t))_h$$

and

$$\tau_F(E_T(t)) = ((F(\nabla\phi))_h, E_T(t))_h - (F(D_hR_h\phi), E_T(t))_h.$$

We establish a convenient representation of the three first terms of the right-hand side of (IV.40). For the first term we have

$$\begin{aligned}-(D_T(M_h(R_hT(t)))D_{-x}R_hT(t) - D_T(M_h(T_h(t)))D_{-x}T_h(t), D_{-x}E_T(t))_+ \\ = -(D_T(M_hT_h(t))D_{-x}E_T(t), D_{-x}E_T(t))_+ \\ -((D_T(M_hT_h(t)) - D_T(M_hR_hT(t)))D_{-x}R_hT(t), D_{-x}E_T(t))_+\end{aligned}$$

and considering the assumption H_1 we deduce

$$\begin{aligned}-(D_T(M_h(R_hT(t)))D_{-x}R_hT(t) - D_T(M_h(T_h(t)))D_{-x}T_h(t), D_{-x}E_T(t))_+ \\ \leq -\beta_0\|D_{-x}E_T(t)\|_+^2 + \sqrt{2}\|D_T\|_{C_b^1(\mathbf{R})}\|T(t)\|_{C^1(\overline{\Omega})}\|E_T(t)\|_h\|D_{-x}E_T(t)\|_+.\end{aligned} \tag{IV.41}$$

Considering the assumption H_2, we get for the second term the estimate

$$(R_hG(T(t)) - G(T_h(t)), E_T(t))_h \leq \beta_1\|E_T(t)\|_h^2. \tag{IV.42}$$

Considering now the assumption H_8 and assuming that the sequence of spatial grids $\overline{\Omega}_h, h \in \Lambda$, satisfies the condition (IV.31), it can be shown that for the third term holds the following

$$(F(D_hR_h\phi) - F(D_h\phi_h), E_T(t))_h \leq \sqrt{2}\beta_6\sqrt{C_\Lambda}\|D_{-x}E_\phi\|_+\|E_T(t)\|_h. \tag{IV.43}$$

Estimates for $\tau_d(E_T(t))$, $\tau_{D_T}(E_T(t))$, and $\tau_G(E_T(t))$ in (IV.40) were obtained in the proof of Theorem 1 of [24].

1. For $\tau_d(E_T(t))$ we have

$$|\tau_d(E_T(t))| \leq Const\Big(\sum_{i=1}^N h_i^4\|T'(t)\|_{H^2(I_i)}^2\Big)^{1/2}\|D_{-x}E_T(t)\|_+. \tag{IV.44}$$

   provided that $T'(t) \in H^2(\Omega)$.

2. For $\tau_{D_T}(E_T(t))$ it can be shown that

$$|\tau_{D_T}(E_T(t))| \leq Const\|D_T\|_{C_b^1(\mathbf{R})}\Big(1 + \|T(t)\|_{C^1(\overline{\Omega})}\Big)\Big(\sum_{i=1}^N h_i^4\|T(t)\|_{H^3(I_i)}^2\Big)^{1/2}\|D_{-x}E_T(t)\|_+. \tag{IV.45}$$

   provided that $T(t) \in H^3(\Omega) \cap H_0^1(\Omega)$.

3. For $\tau_G(E_T(t))$ holds the following

$$|\tau_G(E_T(t))| \leq Const\Big(\sum_{i=1}^N h_i^4\|G(T(t))\|_{H^2(I_i)}^2\Big)^{1/2}\|D_{-x}E_T(t)\|_+. \tag{IV.46}$$

   provided that $G(T(t)) \in H^2(\Omega)$.

4. To estimate the error term $\tau_F(E_T(t))$ we observe that this error term admits the representation

$$\tau_F(E_T(t)) = \tau_1(t) + \tau_2(t),$$

where
$$\tau_1(t) = ((F(\nabla\phi))_h, E_T(t))_h - (R_hF(\nabla\phi), E_T(t))_h,$$
and
$$\tau_2(t) = (R_hF(\nabla\phi), E_T(t))_h - (F(D_hR_h\phi), E_T(t))_h.$$
As for $\tau_G(E_T(t))$, for $\tau_1(t)$ we have the following
$$|\tau_1(t)| \leq Const\beta_6\Big(\sum_{i=1}^{N} h_i^4\|\phi\|^2_{H^3(I_i)}\Big)^{1/2}\|D_{-x}E_T(t)\|_+,$$
provided that $\phi \in H^3(\Omega)$. To establish an estimate for $\tau_2(t)$ we remark that using the Lipschitz condition for F we have
$$\begin{aligned}|\tau_2(t)| &\leq \beta_6\sum_{i=1}^{N-1} h_{i+1/2}|\nabla\phi(x_i) - D_h\phi(x_i)||E_T(x_i,t)| \\ &\leq \beta_6\sum_{i=1}^{N-1}\frac{1}{2}|\lambda(g)||E_T(x_i,t)|\end{aligned} \tag{IV.47}$$
where
$$\lambda(g) = g'(\zeta) - \hat{\zeta}(g(1) - g(\zeta)) - \frac{1}{\hat{\zeta}}(g(\zeta) - g(0)),$$
and $g(\mu) = \phi(x_{i-1} + \mu(h_i + h_{i+1}))$, $\zeta = \dfrac{h_i}{h_i + h_{i+1}}$ and $\hat{\zeta} = \dfrac{h_i}{h_{i+1}}$. The Bramble-Hilbert lemma guarantees the existence of a positive constant $Const$ such that
$$|\lambda(g)| \leq Const\int_0^1 |g^{(3)}(\mu)|d\mu \leq Const(h_i + h_{i+1})^2\int_{x_{i-1}}^{x_{i+1}} |\phi^{(3)}(x)|dx.$$
Inserting the last upper bound in (IV.47) we easily get
$$|\tau_2(t)| \leq Const\beta_6\Big(\sum_{i=1}^{N-1}(h_i^4 + h_{i+1}^4)\|\phi\|^2_{H^3(I_i\cup I_{i+1})}\Big)^{1/2}\|D_{-x}E_T(t)\|_+$$
provided that $\phi \in H^3(\Omega)\cap H_0^1(\Omega)$.

Consequently, for $\tau_F(E_T(t))$ we conclude
$$|\tau_F(E_T(t))| \leq Const\beta_6\Big(\Big(\sum_{i=1}^{N-1}(h_i^4 + h_{i+1}^4)\|\phi\|^2_{H^3(I_i\cup I_{i+1})}\Big)^{1/2} + \Big(\sum_{i=1}^{N} h_i^4\|\phi\|^2_{H^3(I_i)}\Big)^{1/2}\Big)\|D_{-x}E_T(t)\|_+. \tag{IV.48}$$

Taking into account in (IV.40) the upper bounds (IV.41)-(IV.43) and the error estimates (IV.44), (IV.45), (IV.46), (IV.48) we arrive to
$$\begin{aligned}\frac{d}{dt}\|E_T(t)\|_h^2 &+ 2(\beta_0 - 5\varepsilon^2)\|D_{-x}E_T(t)\|_+^2 \\ &\leq \Big(\frac{2}{\varepsilon^2}\|D_T\|^2_{C_b^1(\mathbf{R})}\|T(t)\|^2_{C^1(\overline{\Omega})} + 2\beta_1 + 2\delta^2\Big)\|E_T(t)\|_h^2 + \frac{1}{\delta^2}C_\Lambda\|D_{-x}E_\phi\|_+^2 + \Gamma(t),\end{aligned} \tag{IV.49}$$

where $\varepsilon \neq 0, \delta \neq 0$, and $|\Gamma(t)|$ is bounded by (IV.39). Finally, the inequality (IV.49) leads to (IV.38). ■

Corollary 4.2.2 *Under the assumptions of Theorems 3.2.1 and 4.2.1, there exists a positive constant Const, h-independent, such that*

$$\|E_T(t)\|_h^2 + \int_0^t \|D_{-x}E_T(s)\|_+^2 ds \leq Const\left(\|E_T(0)\|_h^2 + h_{max}^4\right), \, t \in [0,t_f]. \quad \text{(IV.50)}$$

Proof: It is enough to fix in (IV.38) ε and δ such that

$$\beta_0 - 5\varepsilon^2 > 0$$

and

$$\frac{1}{\varepsilon^2}\|D_T\|^2_{C_b^1(\mathbf{R})}\|T(\mu)\|^2_{C^1(\overline{\Omega})} + 2\beta_1 + 2\delta^2 > 0.$$

■

4.3 Revisiting stability

Let us consider again (IV.37). To conclude stability, we need to guarantee that $\int_0^t \|D_{-x}\tilde{T}_h(s)\|_\infty^2 ds$ or $\int_0^t \|D_{-x}T_h(s)\|_\infty^2 ds$ are uniformly bounded in $t \in [0,t_f]$ and $h \in \Lambda$. From Corollary 4.2.2, if $\tilde{T}_h(0) \in \overline{B}_r(R_hT(0))$ or $T_h(0) \in \overline{B}_r(R_hT(0))$, with $r = \sqrt{h_{max}}$, then, from (IV.50), for $u_h(t) = T_h(t)$ or $u_h(t) = \tilde{T}_h(t)$, we obtain

$$\|R_hT(t) - u_h(t)\|_h^2 + \int_0^t \|D_{-x}(R_hT(s) - u_h(s))\|_+^2 ds \leq Const h_{max}, \, t \in [0,t_f].$$

Then

$$\begin{aligned}\int_0^t \|D_{-x}u_h(s)\|_\infty^2 ds &\leq \frac{2}{h_{min}}\int_0^t \|D_{-x}(R_hT(s) - u_h(s))\|_+^2 ds + 2\int_0^t \|T(s)\|^2_{C^1(\overline{\Omega})} ds \\ &\leq 2\frac{h_{max}}{h_{min}} + 2\int_0^t \|T(s)\|^2_{C^1(\overline{\Omega})} ds \\ &\leq 2C_\Lambda + 2\int_0^t \|T(s)\|^2_{C^1(\overline{\Omega})} ds,\end{aligned}$$

for $t \in [0,t_f]$ and $h \in \Lambda$.

From (IV.37), if $\tilde{T}_h(0) \in \overline{B}_r(R_hT(0))$ or $T_h(0) \in \overline{B}_r(R_hT(0))$, with $r = \sqrt{h_{max}}$, we conclude that for $\omega_T(t) = T_h(t) - \tilde{T}_h(t)$, where $T_h(t)$ and $\tilde{T}_h(t)$ are defined by (IV.15) with ϕ_h given by the fully discrete FEM (IV.10) or equivalently the FDM (IV.14),

$$\|\omega_T(t)\|_h^2 + \int_0^t \|D_{-x}\omega_T(s)\|_+^2 ds \leq Const\|\omega_T(0)\|_h^2, t \in [0,t_f], h \in \Lambda, \quad \text{(IV.51)}$$

provided that

$$\frac{2}{\varepsilon^2}\|D_T\|^2_{C_b^1(\mathbf{R})}\|D_{-x}\tilde{T}_h(t)\|_\infty^2 + 2\beta_1 \geq 0, t \in [0,t_f].$$

The inequality (IV.51) shows the stability of (IV.15) with Dirichlet boundary conditions.

5 Concentration - second order error estimates with respect to the discrete L^2-norm

5.1 Stability

In this section in (I.8) we impose that D_d satisfies H_4 and that the assumptions H_3 and H_5 for v and Q, respectively are replaced the following less stringent assumptions:
$H_3 : v(0,0) = 0, |v(x,y) - v(\tilde{x},\tilde{y})| \leq \beta_2(|x-\tilde{x}| + |y-\tilde{y}|), x, \tilde{x}, y, \tilde{y} \in \mathbb{R}$,
$H_5 : Q(0) = 0, (Q(u_h) - Q(w_h), u_h - w_h)_h \leq \beta_4 \|u_h - w_h\|_h^2,\ u_h, w_h \in W_{h,0}$.
The assumptions $H_3 - H_5$ were introduced only for theoretical purposes. In fact, we notice that the convective velocity v given by (I.15) does not satisfy the assumption H_3. This function is not defined at $x = 0$ and is a Lipschitz function only in a bounded set of $\mathbb{R}^2$.

We start by establishing energy estimates for $c_h(t)$ defined by (IV.14), (IV.15), (IV.16) with homogeneous Dirichlet boundary conditions (or by (IV.10), (IV.11), (IV.12)) and initial conditions $T_h(0), c_h(0)$ for the temperature and concentration, respectively.

Theorem 5.1.1 *Let us suppose that the conditions $H_3 - H_5$ hold and $c_h \in C^1([0,t_f], W_{h,0})$ and let $\phi_h, T_h(t), c_h(t) \in W_{h,0}$ defined by (IV.14), (IV.15), (IV.16) with homogeneous Dirichlet boundary conditions (or by (IV.10), (IV.11), (IV.12)) and initial conditions $T_h(0), c_h(0)$ for the temperature and concentration, respectively. We have*

$$\begin{aligned} \|c_h(t)\|_h^2 + 2(\beta_3 - \varepsilon^2)\int_0^t e^{\int_s^t \left(\frac{1}{\varepsilon^2}\beta_2^2\left(\|T_h(\mu)\|_\infty^2 + \|D_{-x}\phi_h\|_\infty^2\right) + 2\beta_4\right)d\mu} \|D_{-x}c_h(s)\|_+^2 ds \\ \leq \|c_h(0)\|_h^2 e^{\int_0^t \left(\frac{1}{\varepsilon^2}\beta_2^2\left(\|T_h(\mu)\|_\infty^2 + \|D_{-x}\phi_h\|_\infty^2\right) + 2\beta_4\right)d\mu}, t \in [0,t_f]. \end{aligned} \tag{IV.52}$$

Proof: Taking in (IV.12) $u_h = c_h(t)$ and considering the assumptions $H_3 - H_5$ we deduce

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|c_h(t)\|_h^2 + \beta_3\|D_{-x}c_h(t)\|_+^2 &\leq \sqrt{2}\beta_2\left(\|T_h(t)\|_\infty + \|D_{-x}\phi_h\|_\infty\right)\|c_h(t)\|_h\|D_{-x}c_h(t)\|_+ \\ &+\beta_4\|c_h(t)\|_h^2,\ t \in (0,t_f]. \end{aligned}$$

Then we arrive to

$$\frac{d}{dt}\|c_h(t)\|_h^2 + 2(\beta_3 - \varepsilon^2)\|D_{-x}c_h(t)\|_+^2 \leq \left(\frac{1}{\varepsilon^2}\beta_2^2\left(\|T_h(t)\|_\infty^2 + \|D_{-x}\phi_h\|_\infty^2\right) + 2\beta_4\right)\|c_h(t)\|_h^2,\ t \in (0,t_f],$$

that leads to (IV.52). ■

From Corollary 3.2.3 we conclude that $\|D_{-x}\phi_h\|_\infty, h \in \Lambda$, is bounded. Moreover, under the assumptions of Theorem 4.1.1, as $\|T_h(t)\|_\infty \leq \sqrt{|\Omega|}\|D_{-x}T_h(t)\|_+$, if $2\beta_1 + 1 \geq 0$, and $\|T_h(0)\|_h \leq Const, h \in \Lambda$, we get

$$\int_0^t \|T_h(s)\|_\infty^2 \leq Const, t \in [0,t_f], h \in \Lambda,$$

where $Const$ is h and t independent. Furthermore, if

$$\frac{1}{\varepsilon^2}\beta_2^2\left(\|T_h(t)\|_\infty^2 + \|D_{-x}\phi_h\|_\infty^2\right) + 2\beta_4 \geq 0, t \in [0,t_f], h \in \Lambda,$$

with ε such that $\beta_3 - \varepsilon^2 > 0$, then

$$\|c_h(t)\|_h^2 + \int_0^t \|D_{-x}c_h(s)\|_+^2 ds \leq Const,\ t \in [0,t_f], h \in \Lambda,$$

provided that $\|c_h(0)\|_h, h \in \Lambda$, is bounded. Consequently

$$\int_0^t \|c_h(s)\|_\infty^2 ds \leq Const, t \in [0,t_f], h \in \Lambda. \tag{IV.53}$$

Theorem 5.1.2 *Let $T_h, \tilde{T}_h, c_h, \tilde{c}_h \in C^1([0,t_f], W_{h,0})$ be defined by (IV.15), (IV.16) with homogeneous Dirichlet boundary conditions (or by (IV.11), (IV.12)) and initial conditions $T_h(0), \tilde{T}_h(0), c_h(0), \tilde{c}_h(0)$, where $\phi_h \in W_{h,0}$ is defined by (IV.14) or (IV.10). Under the assumptions $H_3 - H_5$, for $\omega_c = c_h(t) - \tilde{c}_h(t)$, $\omega_T(t) = T_h(t) - \tilde{T}_h(t)$ we have*

$$\begin{aligned}\|\omega_c(t)\|_h^2 + 2(\beta_3 - 2\varepsilon^2)\int_0^t e^{\int_s^t g_h(\mu)d\mu}\|D_{-x}\omega_c(s)\|_+^2 ds \leq \|\omega_c(0)\|_h^2 e^{\int_0^t g_h(\mu)d\mu} \\ + \frac{1}{\varepsilon^2}|\Omega|\int_0^t e^{\int_s^t g_h(\mu)d\mu}\Big(\|D_d\|_{C_b^1(\mathbf{R})}^2\|D_{-x}c_h(s)\|_+^2 + \beta_2\|\tilde{c}_h(s)\|_h^2\Big)\|D_{-x}\omega_T(s)\|_+^2 ds,\end{aligned} \tag{IV.54}$$

for $t \in [0,t_f]$ and with

$$g_h(\mu) = \big(\frac{1}{\varepsilon^2}\beta_2^2(\|T_h(\mu)\|_\infty + \|D_{-x}\phi_h\|_\infty)^2 + 2\beta_4\big), \mu \in [0,t_f].$$

Proof: It can be shown that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\omega_c(t)\|_h^2 + (D_d(M_h\tilde{T}_h(t))D_{-x}\omega_c(t), D_{-x}\omega_c(t))_+ \\ \leq ((D_d(M_hT_h(t)) - D_d(M_h\tilde{T}_h(t)))D_{-x}c_h(t), D_{-x}\omega_c(t))_+ \\ +(M_h(v_h(t)c_h(t) - \tilde{v}_h(t)\tilde{c}_h(t)), D_{-x}\omega_c(t))_+ + \beta_4\|\omega_c(t)\|_h^2,\end{aligned} \tag{IV.55}$$

where $v_h(t) = v(T_h(t), D_h\phi_h)$ and $\tilde{v}_h(t) = v(\tilde{T}_h(t), D_h\phi_h)$.

For the term $(M_h(v_h(t)c_h(t) - \tilde{v}_h(t)\tilde{c}_h(t)), D_{-x}\omega_c(t))_+$, using the assumption H_3, it is a straightforward task to show that

$$\begin{aligned}(M_h(v_h(t)c_h(t) - \tilde{v}_h(t)\tilde{c}_h(t)), D_{-x}\omega_c(t))_+ \\ \leq \sqrt{2}\beta_2\Big(\|T_h(t)\|_\infty + \|D_{-x}\phi_h\|_\infty\Big)\|\omega_c(t)\|_h + \sqrt{2}\beta_2\|\tilde{c}_h(t)\|_h\|\omega_T(t)\|_\infty\Big)\|D_{-x}\omega_c(t)\|_+.\end{aligned} \tag{IV.56}$$

As for $((D_d(M_hT_h(t)) - D_d(M_h\tilde{T}_h(t)))D_{-x}c_h(t), D_{-x}\omega_c(t))_+$ we have

$$((D_d(M_hT_h(t)) - D_d(M_h\tilde{T}_h(t)))D_{-x}c_h(t), D_{-x}\omega_c(t))_+ \leq \sqrt{2}\|D_d\|_{C_b^1(\mathbf{R})}\|D_{-x}c_h(t)\|_+\|\omega_T(t)\|_\infty\|D_{-x}\omega_c(t)\|_+ \tag{IV.57}$$

from (IV.55), taking (IV.56) and $\|\omega_T(t)\|_\infty \leq \sqrt{|\Omega|}\|D_{-x}\omega_T(t)\|_+$ we deduce

$$\begin{aligned}&\frac{d}{dt}\|\omega_c(t)\|_h^2+2(\beta_3-2\varepsilon^2)\|D_{-x}\omega_c(t)\|_+^2\\ &\qquad\leq \Big(\frac{1}{\varepsilon^2}\beta_2^2(\|T_h(t)\|_\infty+\|D_{-x}\phi_h\|_\infty)^2+2\beta_4\Big)\|\omega_c(t)\|_h^2\\ &\qquad+\frac{1}{\varepsilon^2}|\Omega|\Big(\|D_d\|^2_{C_b^1(\mathbf{R})}\|D_{-x}c_h(t)\|_+^2+\beta_2\|\tilde{c}_h(t)\|_h^2\Big)\|D_{-x}\omega_T(t)\|_+^2,\ t\in(0,t_f],\end{aligned} \tag{IV.58}$$

that leads to (IV.54).

■

From (IV.54), to conclude the stability result, we need only to notice that

$$\|D_{-x}c_h(t)\|_+,\ \|D_{-x}\omega_T(t)\|_+,\ \|\tilde{c}_h(t)\|_h$$

are uniformly bounded in Λ, a.e. in $[0,t_f]$. In fact, this conclusion holds for $\|D_{-x}c_h(t)\|_+$ and $\|\tilde{c}_h(t)\|_h$ due to the Theorem 5.1.1 provided $\|c_h(0)\|_h \leq Const, h\in\Lambda$. The inequality (IV.51) leads to the same conclusion for $\|D_{-x}\omega_T(t)\|_+$ provided that $\|\omega_T(0)\|_h \leq Const, h\in\Lambda$.

5.2 Convergence analysis

We establish in what follows an estimate for the error $E_c(t) = R_h c(t) - c_h(t)$. We follow the proof of the Theorem 2 of [24]. The novelty of the new result lies on the fact that the behaviour of $E_c(t)$ is not only determined by the error $E_T(t) = R_h T(t) - T_h(t)$, as in the Theorem 2 of [24], but also by the error $E_\phi = R_h\phi - \phi_h$.

Theorem 5.2.1 *Let*

$$T, c \in L^2(0,t_f,H^3(\Omega)\cap H_0^1(\Omega)),\ c\in H^1(0,t_f,H^2(\Omega)),$$

$$\phi \in H^3(\Omega)\cap H_0^1(\Omega),$$

be solutions of the IBVP (I.6)-(I.8), (I.9), (I.10). Let ϕ_h, T_h and c_h be the corresponding approximations defined by (IV.14)-(IV.16) with homogeneous Dirichlet boundary conditions and initial conditions $T_h(0)$ and $c_h(0)$. If

$$R_h c, c_h \in C^1([0,t_f],W_{h,0}),$$

the assumption $H_3 - H_5$ hold and $Q(c(t)) \in H^2(\Omega)$, then for $E_c(t) = R_hc(t) - c_h(t), E_T(t) = R_hT(t) - T_h(t)$ and $E_\phi = R_h\phi - \phi_h$, there exists a positive constant Const, h and t independent, such that

$$\begin{aligned}
&\|E_c(t)\|_h^2 + 2(\beta_3 - 6\varepsilon^2)\int_0^t e^{\int_s^t g_h(\mu)d\mu}\|D_{-x}E_c(s)\|_+^2 ds \\
&\quad \le \|E_c(0)\|_h^2 e^{\int_0^t g_h(\mu)d\mu} + 4\frac{\beta_2^2}{\varepsilon^2}\int_0^t e^{\int_s^t g_h(\mu)d\mu}\Big(\|E_T(s)\|_h^2 + \|D_{-x}E_\phi\|_+^2\Big)\|c(s)\|_{C^1(\overline{\Omega})}^2 ds \\
&\quad + \int_0^t e^{\int_s^t g_h(\mu)d\mu}\Gamma(s)ds, \, t \in (0, t_f],
\end{aligned} \tag{IV.59}$$

where $\varepsilon \neq 0$,

$$g_h(\mu) = 4\frac{\beta_2^2}{\varepsilon^2}\big(\|T_h(\mu)\|_\infty^2 + \|D_{-x}\phi_h\|_\infty^2\big)\|c(\mu)\|_{C^1(\overline{\Omega})}^2 + 2\beta_4,$$

$$\begin{aligned}
|\Gamma(t)| \quad &\le \frac{1}{\varepsilon^2} Const\Big(\sum_{i=1}^N h_i^4\|c'(t)\|_{H^2(I_i)}^2 \\
&+ (\|D_d\|_{C_b^1(\mathbf{R})}^2 + 1)(\|c(t)\|_{C^1(\overline{\Omega})}^2 + 1)\sum_{i=1}^N h_i^4\big(\|v(T(t), \nabla\phi)c(t)\|_{H^2(I_i)}^2 + \|\phi(t)\|_{H^3(I_i)}^2\big) \\
&+ \sum_{i=1}^{N-1}(h_i^4 + h_{i+1}^4)\|\phi(t)\|_{H^3(I_i \cup I_{i+1})}^2 + \sum_{i=1}^N h_i^4\|Q(c(t))\|_{H^2(I_i)}^2\Big).
\end{aligned} \tag{IV.60}$$

Proof: Following the proof of the Theorem 2 of [24], it can be shown that $E_c(t)$ is solution of the following differential problem

$$\begin{aligned}
(E_c'(t), E_c(t))_h = &-(D_d(M_h(R_hT(t)))D_{-x}R_hc(t) - D_d(M_h(T_h(t)))D_{-x}c_h(t), D_{-x}E_c(t))_+ \\
&+ (M_h(R_h(v(T(t), D_hR_h\phi))c(t))), D_{-x}E_c(t))_+ - (M_h(v(T_h(t), D_h\phi_h)c_h(t)), D_{-x}E_c(t))_+ \\
&+ (R_hQ(c(t)) - Q(c_h(t)), E_c(t))_h \quad + \tau_d(E_c(t)) + \tau_{D_d}(E_c(t)) + \tau_v(E_c(t)) + \tau_Q(E_c(t)),
\end{aligned} \tag{IV.61}$$

where

$$|\tau_d(E_c(t))| = |(R_hc'(t) - (c'(t))_h, E_c(t))_h| \le Const\Big(\sum_{i=1}^N h_i^4\|c'(t)\|_{H^2(I_i)}^2\Big)^{1/2}\|D_{-x}E_c(t)\|_+,$$

provided that $c'(t) \in H^2(\Omega)$,

$$\begin{aligned}
|\tau_{D_d}(E_c(t))| &= | - (D_d(\hat{R}_hT(t))\hat{R}_h\nabla c(t), D_{-x}E_c(t))_+ + (D_d(M_hR_hT(t))D_{-x}R_hc(t), D_{-x}E_c(t))_+| \\
&\le Const\|D_d\|_{C_b^1(\mathbf{R})}\big(\|c(t)\|_{C^1(\overline{\Omega})} + 1\big)\Big(\textstyle\sum_{i=1}^N h_i^4\big(\|T(t)\|_{H^2(I_i)}^2 + \|c(t)\|_{H^3(I_i)}^2\big)\Big)^{1/2}\|D_{-x}E_c(t)\|_+,
\end{aligned}$$

provided that $T(t)\in H^2(\Omega), c(t)\in H^3(\Omega)$,

$$\begin{aligned}|\tau_v(E_c(t))| &= |(v(\hat{R}_hT(t),\hat{R}_h\nabla\phi)R_hc(t),D_{-x}E_c(t))_+ - (M_h(v(\hat{R}_hT(t),D_hR_h\phi)R_hc(t)),D_{-x}E_c(t))_+|\\ &\le |(v(\hat{R}_hT(t),\hat{R}_h\nabla\phi)R_hc(t),D_{-x}E_c(t))_+ - (M_hv(\hat{R}_hT(t),R_h\nabla\phi)R_hc(t),D_{-x}E_c(t))_+|\\ &+|(M_hv(\hat{R}_hT(t),R_h\nabla\phi)R_hc(t),D_{-x}E_c(t))_+ - (M_h(v(\hat{R}_hT(t),D_hR_h\phi)R_hc(t)),D_{-x}E_c(t))_+|\\ &\le Const\left(\Big(\sum_{i=1}^{N}h_i^4\|v(T(t),\nabla\phi)c(t)\|^2_{H^2(I_i)}\Big)^{1/2}+\|c(t)\|_{C^1(\Omega)}\Big(\sum_{i=1}^{N-1}(h_i^4+h_{i+1}^4)\|\phi\|^2_{H^3(I_i\cup I_{i+1})}\Big)^{1/2}\right)\\ &\|D_{-x}E_c(t)\|_+,\end{aligned}$$

provided that and $v(T(t),\nabla\phi)c(t)\in H^2(\Omega)$ and $\phi\in H^3(\Omega)$,

$$|\tau_Q(E_c(t))| = |((Q(c(t)))_h - R_hQ(c(t)),E_c(t))_h| \le Const\Big(\sum_{i=1}^{N}h_i^4\|Q(c(t))\|^2_{H^2(I_i)}\Big)^{1/2}\|D_{-x}E_c(t)\|_+,$$

provided that $Q(c(t))\in H^2(\Omega)$.

Analogously to (IV.57) and (IV.56), we have

$$\begin{aligned}&((D_d(M_hR_hT(t)) - D_d(M_hT_h(t)))D_{-x}R_hc(t),D_{-x}E_c(t))_+\\ &\quad\le \sqrt{2}\sqrt{|\Omega|}\|D_d\|_{C_b^1(\mathbf{R})}\|c(t)\|_{C^1(\overline{\Omega})}\|D_{-x}E_T(t)\|_+\|D_{-x}E_c(t)\|_+\end{aligned}\tag{IV.62}$$

and

$$\begin{aligned}&(M_h(v(R_hT(t),D_hR_h\phi)R_hc(t)) - M_h(v(T_h(t),D_h\phi_h)c_h(t)),D_{-x}E_c(t))_+\\ &\quad\le \sqrt{2}\beta_2\Big(\Big(\|E_T(t)\|_h+\|D_{-x}E_\phi\|_+\Big)\|c(t)\|_{C^1(\overline{\Omega})}+\Big(\|T_h(t)\|_\infty+\|D_{-x}\phi_h\|_\infty\Big)\|E_c(t)\|_h\Big)\|D_{-x}E_c(t)\|_+,\end{aligned}\tag{IV.63}$$

respectively.

As
$$|(R_hQ(c(t)) - Q(c_h(t)),E_c(t))_h| \le \beta_4\|E_c(t)\|_h^2,\tag{IV.64}$$

taking (IV.62), (IV.63) and (IV.64) in (IV.61) we arrive to

$$\begin{aligned}&\frac{d}{dt}\|E_h(t)\|_h^2+2(\beta_3-6\varepsilon^2)\|D_{-x}E_c(t)\|_+^2\\ &\quad\le +\Big(4\frac{\beta_2^2}{\varepsilon^2}\big(\|T_h(t)\|_\infty^2+\|D_{-x}\phi_h\|_\infty^2\big)\|c(t)\|^2_{C^1(\overline{\Omega})}+2\beta_4\Big)\|E_c(t)\|_h^2\\ &\quad+4\frac{\beta_2^2}{\varepsilon^2}\Big(\|E_T(t)\|_h^2+\|D_{-x}E_\phi\|_+^2\Big)\|c(t)\|^2_{C^1(\overline{\Omega})}+\Gamma(t),\ t\in(0,t_f],\end{aligned}\tag{IV.65}$$

where Γ is bounded in (IV.60). Finally, the inequality (IV.65) leads to (IV.59). ■

Corollary 5.2.2 *Under the assumptions of the Theorems 3.2.1, 4.2.1 and 5.2.1, with* $T_h(0)=R_hT(0), c_h(0)=R_hc(0)$, *the error* $E_c(t)=R_hc(t)-c_h(t)$ *satisfies*

$$\|E_c(t)\|_h^2+\int_0^t\|D_{-x}E_c(s)\|_+^2ds \le Const h_{max}^4, t\in[0,t_f], h\in\Lambda.\tag{IV.66}$$

The estimate (IV.66) shows that the errors E_ϕ and $E_T(t)$ do not deteriorates de quality of the semi-discrete approximation $c_h(t)$. We notice that (IV.66) is a supraconvergence result in the finite difference community but it can be seen also as a supercloseness result in the finite element community because our finite difference discretization (IV.14)-(IV.16) is equivalent to the fully discrete finite element discretization (IV.10)-(IV.12).

6 Numerical experiments

In what follows we illustrate the qualitative behaviour of the IBVP (I.6)-(I.8), (I.9), (I.10) using the finite difference method (IV.14)-(IV.16) with the boundary and initial conditions (I.9), (I.10). The accuracy of the method was established in Theorems 3.2.1, 4.2.1 and 5.2.1. From these results we believe that the numerical plots that we present in what follows describe accurately the behaviour of the correspondent continuous model.

In $[0,t_f]$ we introduce the uniform grid $\{t_m, m=0,\ldots,M, t_0=0, t_M=t_f, \Delta t=t_m-t_{m-1}, m=1,\ldots,M\}$. We integrate in time (IV.15) and (IV.12) using the next IMEX (implicit-explicit) approach

$$\begin{cases} T_h^{m+1} = T_h^m + \Delta t D_x^*(D_T(M_h(T_h^m))D_{-x}T_h^{m+1}) + \Delta t G(T_h^m) \\ \qquad\qquad + F(D_h\phi_h) + f_{1,h}^m \text{ in } \Omega_h, m=0,\ldots,M-1, \\ T_h^0 = R_h T_0 \,\text{in}\, \Omega_h, \\ T_h^m = 0 \,\text{on}\, \partial\Omega_h, m=1,\ldots,M, \end{cases} \tag{IV.67}$$

where $G(T_h^m)(x_i) = G(T_h^m(x_i))$, $F(D_h\phi_h)(x_i) = F(D_h\phi_h(x_i)), i=1,\ldots,N-1$, and ϕ_h is defined by (IV.14),

$$\begin{cases} c_h^{m+1} + \Delta t D_c(v(T_h^{m+1}, D_h\phi_h)c_h^{m+1}) = \Delta t D_x^*(D_d(M_h(T_h^{m+1}))D_{-x}c_h^{m+1}) \\ \qquad\qquad + \Delta t Q(c_h^m) + f_{2,h}^m \text{ in } \Omega_h, m=0,\ldots,M-1, \\ c_h^0 = R_h c_0 \,\text{in}\, \Omega_h, \\ c_h^m = 0 \,\text{on}\, \partial\Omega_h, m=1,\ldots,M, \end{cases} \tag{IV.68}$$

where $Q(c_h^m)(x_i) = Q(c_h^m(x_i)), i=1,\ldots,N-1$. The grid function $f_{\ell,h}^m$, $\ell=1,2$, in (IV.67), (IV.68) are introduced only to illustrate the convergence results. In this case, these functions are such that the correspondent continuous problems have known solutions.

6.1 Convergence results

In this section we illustrate the error results obtained in this work - Theorems 3.2.1, 4.2.1 and 5.2.1. We use the following notations:

$$Error_\phi = \|D_{-x}E_\phi\|_+^2,$$

$$Error_T = \max_{j=1,\cdots,M}\left(\|E_T^j\|_h^2 + \Delta t\sum_{i=1}^M \|D_{-x}E_T^i\|_+^2\right),$$

where $E_T^i(x_\ell) = R_h T(x_j,t_i) - T_h^i(x_j), j=0,\ldots,N$,

$$Error_c = \max_{j=1,\cdots,M}\left(\|E_c^j\|_h^2 + \Delta t\sum_{i=1}^M \|D_{-x}E_c^i\|_+^2\right).$$

with $E_c^i(x_j) = R_h c(x_\ell, t_i) - c_h^i(x_j), j = 0, \dots, N$. The convergence rates $Rate_\ell$ are computed by

$$Rate_\ell = \frac{ln\left(\dfrac{Error_\ell, h_{max,i}}{Error_\ell, h_{max,i+1}}\right)}{ln\left(\dfrac{h_{max,i}}{h_{max,i+1}}\right)}, \ell = \phi, T, c,$$

where $h_{max,i}, h_{max,i+1}$ are the maximum stepsizes of the grids $\overline{\Omega}_h^{(i)}, \overline{\Omega}_h^{(i+1)}$, respectively, being the last two grids defined by the vectors $h^{(i)}, h^{(i+1)}$, where $h^{(i+1)}$ is obtained from $h^{(i)}$ introducing the middle point of each interval $[x_j, x_{j+1}]$.

- Smooth solutions: We start by considering the differential problems (I.6), (I.7), (I.8) with $t_f = 1$, σ defined by (I.11), $\sigma_0 = 2 \times 10^{-3}$, $\sigma_1 = 1.6 \times 10^{-1}$, $E_0 = 40000$, $E_1 = 90000$, $B = 30$ (see [6]) and $D_T(T) = 1$, $G(T) = 0$, $F(y) = \sigma(|y|)y$, $v(x,y) = 10^{-5} y e^{-x}$, $D_d(T) = 1$ and $Q(c) = 0$, adding to the last two equations the reaction terms f_1 and f_2 that are such that these problems have the following solutions

$$\begin{aligned} &\phi(x) = sin(\pi x)|2x-1|^\alpha, \\ &T(x,t) = e^{2t} e^x |2x-1|^\beta + 1, \\ &c(x,t) = e^t e^x |2x-1|^\gamma, x \in [0,1], t \in [0, t_f]. \end{aligned} \quad \text{(IV.69)}$$

  We remark that the coefficient functions introduced before do not satisfy all the assumptions $H_1 - H_5$ and $H_7 - H_8$. However, we will show that, even in this case, the convergence orders stated in Theorems 3.2.1, 4.2.1 and 5.2.1 are observed. To have $\phi, T(t), c(t) \in H^3(0,1) \cap H_0^1(0,1)$ we take $\alpha = 3.1$, $\beta = 3.1$, $\gamma = 3.1$. In Table IV.1 and Figure IV.1 (left side), we present the numerical results with $\Delta t = 10^{-4}$. We notice that the convergence rates $Rate_\ell, \ell = \phi, T, c$ are approximately 2 which is in agreement with the error estimates stated in Theorems 3.2.1, 4.2.1 and 5.2.1.

N	h_{max}	E_ϕ	$Rate_\phi$	E_T	$Rate_T$	E_c	$Rate_c$
50	7.3569×10^{-2}	1.5656×10^{-2}	——	5.1084×10^{-2}	——	5.1206×10^{-2}	——
100	3.6785×10^{-2}	3.9062×10^{-3}	2.0029	1.27121×10^{-2}	2.0067	1.2842×10^{-2}	1.9954
200	1.8392×10^{-2}	9.7615×10^{-4}	2.0006	3.0864×10^{-3}	2.0422	3.2168×10^{-3}	1.9972
400	9.1962×10^{-3}	2.4404×10^{-4}	1.9999	6.8725×10^{-4}	2.1669	8.0849×10^{-4}	1.9923
800	4.5981×10^{-3}	6.1006×10^{-5}	2.0001	1.49730×10^{-4}	2.1985	2.0642×10^{-4}	1.9696

Table IV.1 Convergence rates for smooth solutions ($\alpha = \beta = \gamma = 3.1$).

- Non smooth solutions: To illustrate the sharpness of the smoothness assumptions in Theorems 3.2.1, 4.2.1 and 5.2.1 we consider now the solutions (IV.69) with $\alpha = 1.6$, $\beta = 1.6$, $\gamma = 1.6$. In this case $\phi, T(t), c(t) \in H^2(0,1) \cap H_0^1(0,1)$. In Table IV.2 and Figure IV.1 (right side) we present the numerical results obtained in this case that illustrate that $Rate_\ell, \ell = \phi, T, c$ are approximately 1.

N	h_{max}	E_ϕ	$Rate_\phi$	E_T	$Rate_T$	E_c	$Rate_c$
25	1.3768×10^{-1}	3.0229×10^{-2}	——	1.3906×10^{-2}	——	5.3434×10^{-2}	——
50	6.8842×10^{-2}	7.7112×10^{-3}	1.9709	4.9451×10^{-3}	1.4916	2.1863×10^{-2}	1.2892
100	3.4421×10^{-2}	4.7356×10^{-3}	0.7034	2.4472×10^{-3}	1.0149	1.3877×10^{-2}	0.6557
200	1.7211×10^{-2}	2.2403×10^{-3}	1.0798	1.2192×10^{-3}	1.0052	6.5674×10^{-3}	1.0794
400	8.6053×10^{-3}	7.3566×10^{-4}	1.6066	6.5727×10^{-4}	0.8914	2.1729×10^{-3}	1.5956

Table IV.2 Convergence rates for non smooth solutions ($\alpha=\beta=\gamma=1.6$).

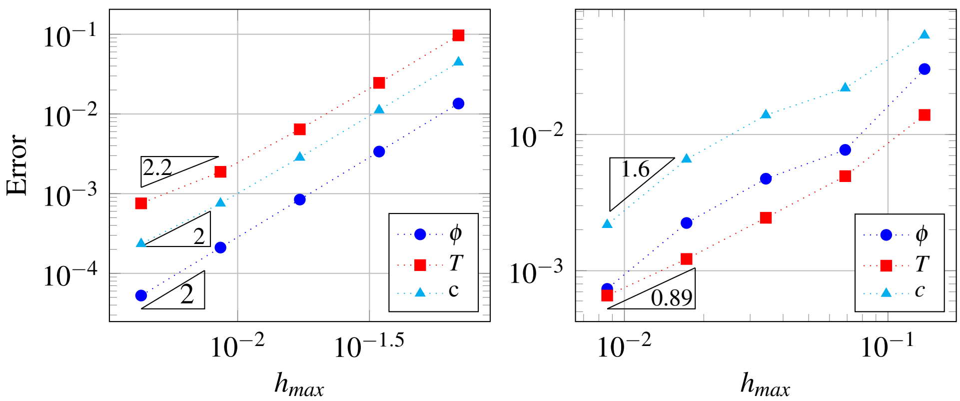

Fig. IV.1 Plots of the errors E_ϕ, E_T and E_c for $\alpha=\beta=\gamma=3.1$ (at left) and $\alpha=\beta=\gamma=1.6$ (at right)

6.2 Qualitative behaviour

In this section our aim is to illustrate the behaviour of the systems of partial differential equations studied in this chapter considering a transdermal iontophoresis application (as in Figure IV.2).

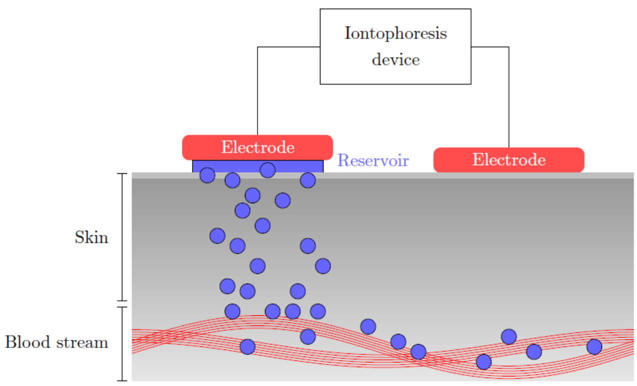

Fig. IV.2 Illustration of the process of iontophoresis [23]

To simplify, we consider the skin as a single layer defined by $[0,L]$, $L=1.1515\times10^{-3}m$ ([6]). The applied potential ϕ is defined by equation (I.6) with $f=0$ and the boundary conditions $\phi(0)=0$ and $\phi(L)=\phi_L$, where ϕ_L depends on the application protocol that we intend to illustrate. The boundary conditions for the concentration are defined by $c(0)=c_{ext}$ and $c(L)=0$ which means that at the skin surface we have a known concentration of drug and that all the drug that arrives to $x=L$ is immediately removed by the blood vessels. In this case the coefficient functions are defined as follows: the electrical conductivity σ is defined by (I.11) with $\sigma_0 = 2\times 10^{-3} S/m$, $\sigma_1 = 1.6\times 10^{-1} S/m$, $y_0 = 40000V/m$, $y_1 = 90000V/m$ and $B = 30$, $D_T(T(t)) = \dfrac{k}{\rho k_s}$, $G(T(t)) = -\dfrac{1}{\rho k_s}\omega_m c_b (T(t) - T_a)$, $F(\nabla\phi) = \dfrac{1}{\rho k_s}\sigma(|\nabla\phi|)|\nabla\phi|^2$, $D_d(T) = D$, v is defined by (I.15) with $v_b = 0$, and $Q(c) = 0$. The parameter values are included in Table IV.3 ([6], [7]).

Symbol	Definition	Value	Units
ρ	Density	1116	kg/m^3
k_s	Heat capacity (specific)	3800	J/kgK
k	Thermal conductivity	0.293	W/mK
ω_m	Perfusion	2.33	$kg/m^3 s$
c_b	Perfusion of blood	3800	J/kgK
T_a	Arterial Blood Temperature	310.15	K
D	Drug diffusivity	10^{-12}	m^2/s
F_r	Faraday constant	9.6485×10^4	C/mol
R	Gas constant	8.3144	$J/Kmol$
z	Valence	± 1	-

Table IV.3 Parameters and values

The behaviour of the drug transport, enhanced by the electric field, is illustrated in what follows considering protocols based on two scenarios:

- Low Voltage (LV)- $\phi(L) = 45V$ during $250ms$ followed by a pause of $100ms$,
- High Voltage (HV) - $\phi(L) = 500V$ during $500\mu s$ followed by a pause of $500\mu s$.

We consider the following protocols:

- $3LV$ in a total of $1.05s$,
- $3HV$ in a total of $3\times 10^{-3}s$,
- $3HV + 3LV$ in a total of $1.053s$.

In the first protocol we take $\phi(L) = 45V$ during $250ms$ followed by a pause of $100ms$. This procedure is repeated during 3 times. In the second protocol we take $\phi(L) = 500V$ during $500\mu s$ followed by a pause of $500\mu s$. This procedure is repeated 3 times. The third protocol is defined applying the first one followed by the second one. The results are compared with the drug transport through the skin without the presence of the electric field which defines the *control* scenario.

In Figure IV.3 we plot the temperature (left) and drug (right) for $t \in [0, 2s]$ when the first protocol $3LV$ is applied. The time axis is the vertical axis while space axis is the horizontal axis. The effect of the three impulses applied at $x = L$ on the temperature distribution is well illustrated by this picture as well as on the drug distribution. While the effect of the applied potential on the temperature is felt in all space domain, the corresponding effect on the drug distribution is felt only in the first part of the skin. To clarify these conclusions, in Figures IV.4 and IV.5 we plot the temperature and drug concentration evolution in several points of the spatial domain.

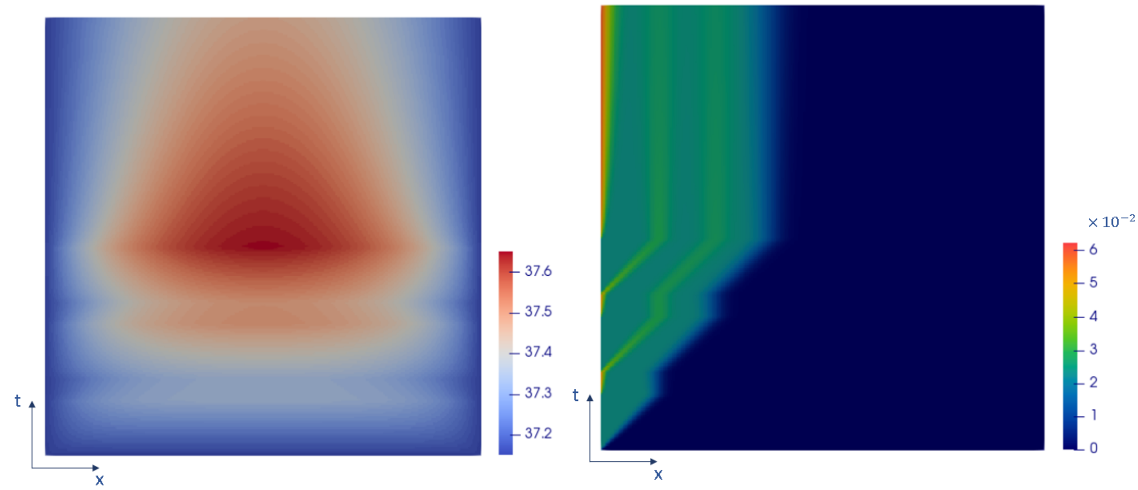

Fig. IV.3 Temperature (^oC) (left) and drug distributions (kg/m^3) (right) for $t \in [0,2s]$ enhanced by the electric field defined by the $3LV$ protocol

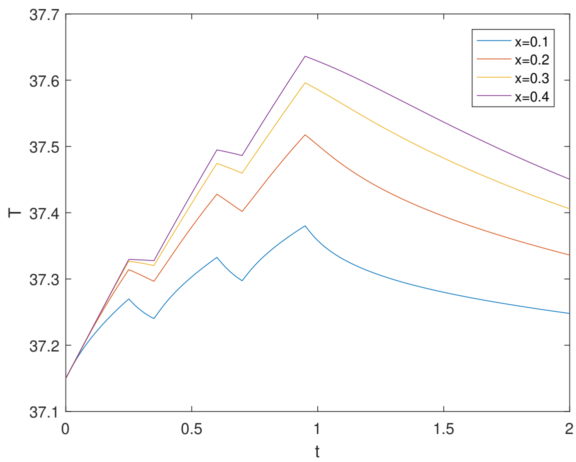

Fig. IV.4 Evolution of the temperature at $x = 0.1, 0.2, 0.3, 0.4$ for $t \in [0,2s]$ for the $3LV$ protocol.

To compare different protocols, we compute the absorbed drug mass at $x = L$,

$$M_\ell(t) = \int_0^t J_{d,\ell}(s)ds,$$

for $\ell = control, 3LV, 3HV, 3HV + 3LV$, where $J_{d,\ell}(t)$ is the drug flux at $x = L$,

$$J_{d,\ell}(t) = -(D_d(T(L,t))\nabla c(L,t) + v(T(L,t), \nabla\phi(L,t)),$$

with the potential ϕ depending on time because the potential at $x = L$ is time dependent function. In Figure IV.6 we plot the absorbed masses for the protocols $3LV$, $3HV$, $3HV + 3LV$ and $control$ for $t \in [0,6s]$. Zooming these plots, we notice that there exists a time t_1 such that for $t \geq t_1$ the

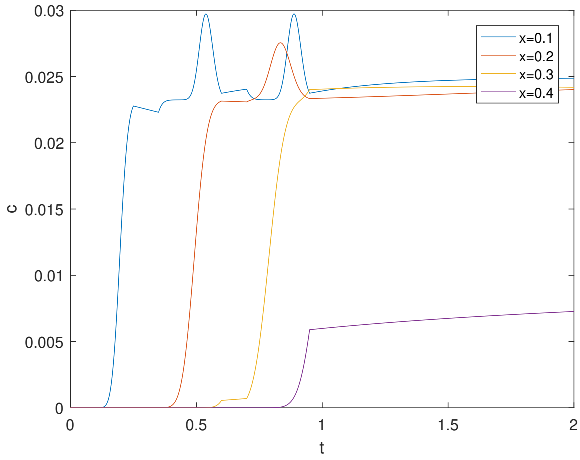

Fig. IV.5 Evolution of the concentration at $x = 0.1, 0.2, 0.3, 0.4$ for $t \in [0, 2s]$ for the $3LV$ protocol.

protocols $3LV, 3HV + 3LV$ lead to similar results and $M_\ell(t)$ for $\ell = 3HV, control$, have analogous behaviour and $M_{3LV}(t) \leq M_{3HV+3LV}(t), M_{control}(t) \leq M_{3HV}(t), t \leq t_1$ (see Figure IV.7). Moreover, for $t \in [0, t_1]$ the protocols $3HV, 3HV + 3LV$ lead to similar results and $M_{control}(t) \leq M_{3LV}(t) \leq M_\ell(t), \ell = 3HV, 3HV + 3LV$ (see Figure IV.8). From Figure IV.9 we observe that there is a time interval $[t_1, t_2]$ where $M_{3HV}(t)$ is an increasing function being its increasing smaller than the increasing of $M_{3LV}(t)$. Finally, in Figure IV.10 we plot $M_\ell(t)$, $\ell = control, 3LV, 3HV, 3HV + 3LV$, for $[t \in [0, 10min]$.

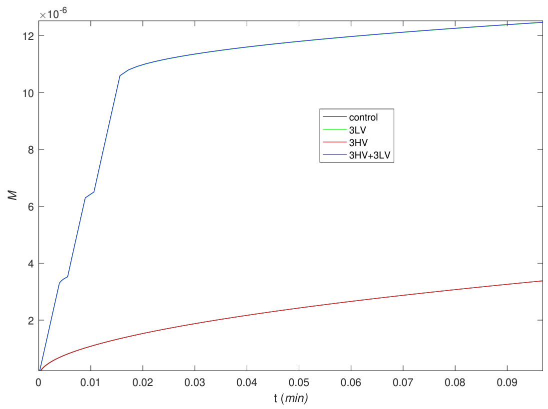

Fig. IV.6 Plots of $M_\ell(t), \ell = control, 3LV, 3HV, 3HV + 3LV$, for $t \in [0, 6s]$.

The results presented in Figure IV.10 show, as noticed before, that as time increases, the protocols $3LV$ and $3HV + 3LV$ lead to similar results while the results obtained with the protocol $3HV$ are similar to the results obtained without the electric field. Moreover, the drug mass transported through the skin is larger for the first set of protocols.

The zooms of the plots in Figure IV.10, Figures IV.11 and IV.12, show that for large times $M_{3LV}(t) \geq M_{3HV+3LV}(t)$ while $M_{control}(t) \leq M_{3HV}(t)$.

To end, we conclude the following:

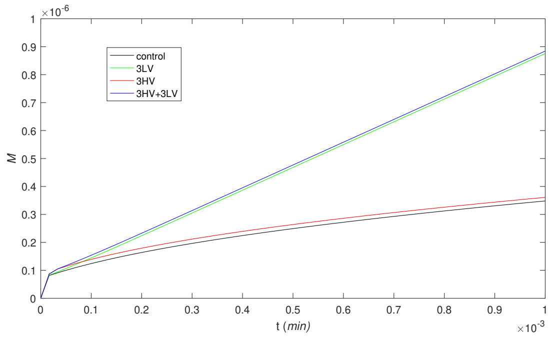

Fig. IV.7 Plots of $M_\ell(t), \ell = control, 3LV, 3HV, 3HV+3LV$, $t \in [0, 6\times 10^{-2}s]$.

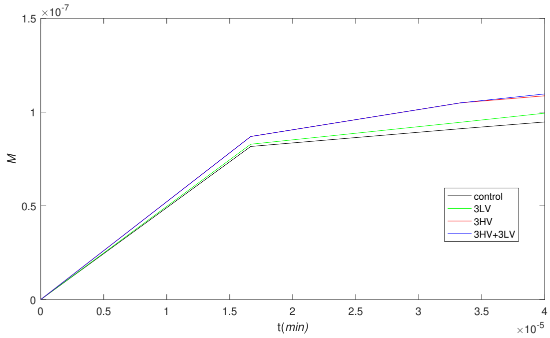

Fig. IV.8 Plot of $M_\ell(t), \ell = control, 3LV, 3HV, 3HV+3LV$, for $t \in [0, t_1]$.

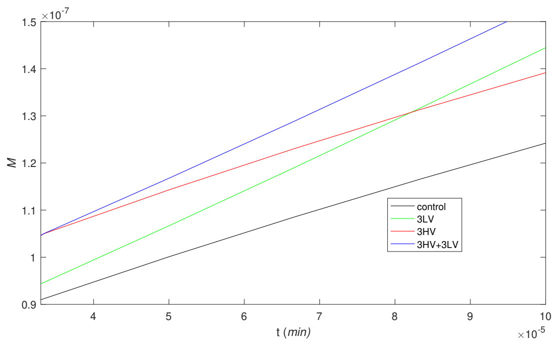

Fig. IV.9 Plots of $M_\ell(t), \ell = control, 3LV, 3HV, 3HV+3LV$, for $t \in [t_1, t_2]$.

P1 For small times, protocols defined by high intensity impulses followed by smaller intensity impulses are more effective than the protocols defined only by high intensity impulses or lower intensity impulses and protocols defined only by high intensity impulses are more effective that protocols defined by lower intensity impulses (see Figure IV.8).

P2 There exists a time interval $[t_1^*, t_2^*]$ such that, protocols defined by high intensity impulses followed by smaller intensity impulses are more effective than the protocols defined only by

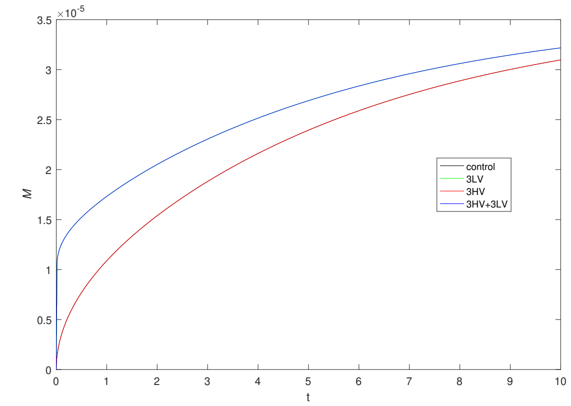

Fig. IV.10 Plots of $M_\ell(t)$, $\ell = control, 3LV, 3HV, 3HV + 3LV$, for $t \in [0, 10min]$.

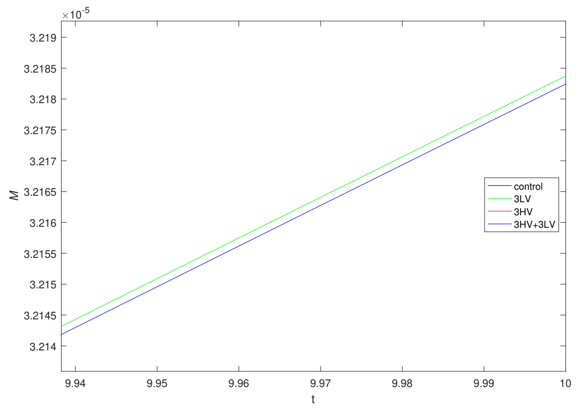

Fig. IV.11 Zoom of the plots of $M_\ell(t)$, $\ell = 3LV, 3HV + 3LV$.

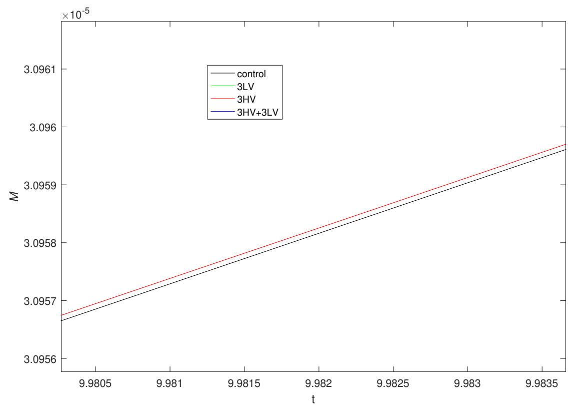

Fig. IV.12 Zoom of the plots of $M_\ell(t)$, $\ell = control, 3HV$.

high intensity impulses or lower intensity impulses and protocols defined only by lower intensity impulses are more effective that protocols defined by high intensity impulses (see Figure IV.6).

P3 For $t \geq t_2^*$, protocols based on lower intensity impulses are more effective than protocols based on high intensity impulses (see Figure IV.9).

7 Conclusions

In this chapter, the coupling between a nonlinear elliptic equation (I.6) and two parabolic equations: a diffusion equation (I.7) and a convection-diffusion equation (I.8), where the convective velocity of the second parabolic equation depends on the gradient of the solution of the elliptic equation, is studied from numerical point of view. The main problem when we solve numerically system (I.6), (I.7) and (I.8) is the computation of the numerical approximation for the elliptic problem because if its numerical gradient does not have the right convergence order then the numerical approximation for the convection-diffusion can be deteriorated.

We propose a finite difference discretization (IV.14), (IV.15), (IV.16) that can be seen as a fully finite element method (IV.10), (IV.11), (IV.12) that leads to second order approximations .More precisely for the solution of the nonlinear elliptic equation, we use a discrete version of the usual H^1-norm; for the solutions of the two parabolic equations we consider a discrete version of the usual L^2-norm. The error estimates were established in the main results of the chapter: Theorems 3.2.1, 4.2.1 and 5.2.1. These results can be interpreted as a supraconvergence results if we look to the discretizations as finite differences; otherwise they can be interpreted as supercloseness results if the discretization is seen as a fully discrete piecewise linear finite element method. In Theorem 3.2.1 we extend the results included in [5], to nonlinear problems. The convergence results were established assuming that the solution of the elliptic equation and the solutions of the parabolic equations, for each time t, are in $H^3(\Omega)$. Numerical simulations illustrating the convergence results and showing the sharpness of the smoothness assumptions were also included in this chapter.

The stability of the coupled problem (IV.14), (IV.15), (IV.16) was also studied. As we were dealing with nonlinear problems, for stationary problem (IV.14) or for the evolution problems (IV.15), (IV.16), the stability analysis presents some difficulties because the uniform boundness of the numerical approximations is required. As in [10], we get the desired results using the convergence results.

As mentioned in the introduction, the system (I.6), (I.7) (I.8) can be used to describe the drug transport through a medium, enhanced by an electric field. Numerical experiments in the scope of this application, where different protocols were used, were also included. From the numerical experiments, we observe, in short, that the application of long and low intensity protocols enhance the drug release and the use of protocols of short and high intensity does not differ significantly from the typical diffusion as can be seen in Figures IV.6 and IV.10.

Chapter V

Influence of temperature on mechanical behaviour: towards non Fickian models

1 Introduction

The mathematical modelling of the drug release from a temperature sensitive polymer was considered in [50] using a Fickian description for the drug transport. The author assumes that the temperature effects on the polymer states occurs discretely in times. Moreover, the spatial domain has two different configurations and the drug transport is described by diffusion equations. In this chapter, although considering the effect of the temperature on the behaviour of the polymer as described in [50], the viscoelastic effect of the polymeric structure on the drug transport is taken into account. In this chapter we assume that the polymer has two different states: the swollen and shrinking states that change discretely in time. Each state is characterized by different Young modulus that are associated with the density of cross-links between the polymeric chains. In Section 2 we present the mathematical model for the drug transport that is represented by an integro-differential equation. An equivalent differential equation will be established. The computation of the solution of the differential problem whose spatial domain and differential equation change discretely in time is presented in Section 3. Some conclusions are presented in Section 4.

2 A hybrid Non Fickian mathematical model

Let $\Omega(t), t \in [0,t_f]$ be the spatial domain where the drug is dispersed. As the domain is composed by a thermoresponsive polymer it changes in time: the domain swells for temperatures lower than the LCST and it shrinks for temperatures higher than the LCST. We consider $\Omega(t) = (-H(t),H(t))$ and that the concentration has a symmetric profile with respect the origin. Consequently, we take $\Omega(t) = [0,H(t)]$, where for $x = 0$ we impose symmetry boundary conditions and at $x = H(t)$ we assume that all the drug that attains the boundary is immediately removed. For $x \in \Omega(t)$, the drug concentration is described by the conservation equation

$$\frac{\partial c}{\partial t} = -\nabla(J_F(t) + J_{NF}(t)). \tag{V.1}$$

In (V.1), $J_F(t)$ is defined by (I.16) and $J_{NF}(t)$ is a non Fickian flux defined by

$$J_{NF}(t) = -D_v(T)\nabla\sigma(t)$$

where D_v stands for a viscoelastic diffusion coefficient and $\sigma(t)$ represents the polymeric stress. The liquid, in the release medium, strains the polymeric matrix that, while swelling, exerts a stress that acts as a barrier to the incoming fluid and to the release of drug (see [21] and [30]). We define the stress through a generalized Maxwell–Wiechert model with two arms as

$$\sigma(t) = -\int_0^t \left(E_0 + E_1 e^{-\dfrac{E_1(t-s)}{\mu}} \right) \frac{\partial \varepsilon}{\partial s} ds, \tag{V.2}$$

where E_0, E_1 represent the Young modulus of the spring elements, that depend on the temperature $T(t)$. In (V.2) μ denotes the viscosity of the polymer-solvent solution and ε represents the polymeric strain. If we assume that $\varepsilon = \lambda c$, then for the concentration we get

$$\begin{aligned}\frac{\partial c}{\partial t}(t) = &\ \nabla(D(T(t))\nabla c(t)) \\ &-\lambda\nabla\left[D_v(T(t))\nabla\int_0^t \left(E_0(T(t)) + E_1(T(t)) e^{\dfrac{-E_1(T(\theta))}{\mu(T(\theta))}(t-\theta)} \right) \frac{\partial c}{\partial \theta}(\theta)d\theta \right], \text{ in } (0,H(t)),\end{aligned} \tag{V.3}$$

for $t \in (0,t_f]$, completed with the boundary conditions

$$\nabla c(0,t) = 0,\ c(H(t),t) = 0,\ t \in (0,t_f]. \tag{V.4}$$

and initial condition

$$c(0) = g \text{ in } (0,H(0)). \tag{V.5}$$

In equation (V.3) the temperature T acts on the release process, through the diffusion coefficient of the drug and also through the properties of the polymeric, namely the Young modulus and the viscosity. The IBVP (V.3), (V.4), (V.5) should be coupled with (I.17) for the temperature and a mathematical law for $H(t)$.

Following [50] we consider a switch of the temperature between two different values. We further assume that this switch leads to two different values in the diffusion coefficient of the drug and the Young modulus. To keep the model analytically manageable we suppose that the polymer viscosity is constant. We also assume that a swelling and shrinking of the domain occurs. Thereby $[0,t_f]$ is split into

$$[0,t_f] = \cup_{i=0}^{n-2}[t_i,t_{i+1}) \cup [t_{n-1},t_n],\ t_0 = 0,\ t_n = t_f.$$

We suppose that the release system is in the collapsed state during the first time interval, which means that the temperature is above the critical temperature solution. Consequently, in $\cup_{i=0}[t_{2i},t_{2i+1})$ the polymeric structure is in the collapsed state and in the swollen state in $\cup_{i=1}[t_{2i-1},t_{2i})$. Let the subscripts c and s stand respectively for collapsed and swollen. For $[t_j, t_{j+1})$ we have

$$\sigma_\ell(t) = -\lambda(E_{1,\ell} + E_{0,\ell})c_\ell(t) + \lambda \left(E_{0,\ell} + E_{1,\ell} e^{-\frac{E_{1,\ell}}{\mu_\ell}(t-t_j)} \right) c_\ell(t_j) + \lambda \frac{E_{1,\ell}^2}{\mu_\ell} \int_{t_j}^{t} e^{-\frac{E_{1,\ell}}{\mu_\ell}(t-\theta)} c_\ell(\theta)d\theta \tag{V.6}$$

for $\ell = c, s$. Taking in (V.3) the expression (V.6) we deduce the following integro-differential equation

$$\begin{aligned} \frac{\partial c_\ell}{\partial t}(t) &= (D - D_{v,\ell}\lambda \hat{E}_\ell)\Delta c_\ell(t) + D_{v,\ell}\lambda \frac{E_{1,\ell}^2}{\mu_\ell} \int_{t_j}^{t} e^{-\frac{E_{1,\ell}}{\mu_\ell}(t-\theta)} \Delta c_\ell(\theta)d\theta \\ &+ D_{v,\ell}\lambda \left(E_{0,\ell} + E_{1,\ell} e^{-\frac{E_{1,\ell}}{\mu_\ell}(t-t_j)} \right) \Delta c_\ell(t_j), \end{aligned} \tag{V.7}$$

where $\hat{E}_\ell = E_{0,\ell} + E_{1,\ell}$, for $\ell = c, s$.

Finally, it is easy to show that c_ℓ satisfies

$$\frac{\partial^2 c_\ell}{\partial t^2} + \alpha_\ell \frac{\partial c_\ell}{\partial t} = D_{1,\ell}\Delta \frac{\partial c_\ell}{\partial t} + D_{2,\ell}\alpha_\ell \Delta c_\ell + \beta_\ell \alpha_\ell \Delta c_\ell(t_j) \text{ in } (0, H_\ell) \times (t_j, t_{j+1}), \tag{V.8}$$

where $\alpha_\ell = \dfrac{E_{1,\ell}}{\mu_\ell}$, $D_{1,\ell} = D - D_{v,\ell}\lambda \hat{E}_\ell$, $D_{2,\ell} = D - D_{v,\ell}\lambda E_{0,\ell}$, $\beta_\ell = D_{v,\ell}\lambda E_{0,\ell}$, $H_\ell = H(t_\ell)$ and $\ell = c, s$. Equation (V.8) is complemented with the boundary conditions

$$\nabla c_\ell(0,t) = 0, c_\ell(H_\ell, t) = 0, t \in (t_j, t_{j+1}). \tag{V.9}$$

The main problem now is the definition of the initial conditions. It is clear that when $t \in [t_0, t_1)$, the initial conditions are given by

$$\begin{cases} c_c(0) = g \\ \dfrac{\partial c_c}{\partial t}(0) = D\Delta g \text{ in } (0, H_c), \end{cases} \tag{V.10}$$

Where H_c represents the domain in the initial collapsed state. To define the initial conditions for (V.8), from (V.7) we get

$$\frac{\partial c_\ell}{\partial t}(t_j) = D\Delta c_\ell(t_j) \text{ in } (0, H_\ell).$$

A question remains without solution: What is the definition of $c_\ell(t_j)$? If in the interval (t_{j-1}, t_j) the polymeric structure is in the collapsed state, then a solution c_c defined in $[0, H_c] \times [t_{j-1}, t_j)$ is computed. However the initial conditions for (V.8) involve a function defined in $[0, H_s]$. One possibility to define the initial conditions for (V.8) is to extend c_c to $[0, H_s]$ constructing $c_{c,ext}$ such that

$$\int_0^{H_c} c_c(x, t_j)dx = \int_0^{H_s} c_{c,ext}(x)dx, \tag{V.11}$$

Equation (V.11) represents the conservation of the mass of drug when a switch in the temperature and consequently, in the volume phase occurs: the total mass at $t = t_j$ in the collapsed polymer H_c, is the initial mass for the next swollen state H_s. Then the initial conditions for (V.8) are defined by

$$\begin{cases} c_s(t_j) = c_{c,ext} \\ \dfrac{\partial c_s}{\partial t}(t_j) = D\Delta c_{c,ext} \ \text{ in } (0,H_c). \end{cases} \tag{V.12}$$

Summarizing, we introduce in the procedure proposed in [50] the viscoelastic effect of the polymer and then we propose the following algorithm to solve our problem. If the polymer is in the collapsed state at $t = 0$ then:

1. Solve the IBVP

$$\begin{cases} \dfrac{\partial^2 c_c}{\partial t^2} + \alpha_c \dfrac{\partial c_c}{\partial t} = D_{1,c}\Delta \dfrac{\partial c_c}{\partial t} + D_{2,c}\alpha_c \Delta c_c + \beta_c \alpha_c \Delta g \ \text{in}(0,H_c)\times(t_0,t_1], \\ \nabla c_c(0,t) = 0, c_c(H_c,t) = 0, \, t \in [t_0,t_1], \\ c_c(x,0) = g(x), \, \dfrac{\partial c_c}{\partial t}(x,0) = D\Delta g(x), \, x \in [0,H_c], \end{cases} \tag{V.13}$$

2. Extend $c_c(t_1)$ to $[0,H_s]$ by constructing $c_{c,ext}$ such that

$$\int_0^{H_c} c_c(x,t_1)ds = \int_0^{H_s} c_{c,ext}(x)dx. \tag{V.14}$$

3. For $i = 1,\dots,n-1$, solve the IBVP

$$\begin{cases} \dfrac{\partial^2 c_\ell}{\partial t^2} + \alpha_\ell \dfrac{\partial c_\ell}{\partial t} = D_{1,\ell}\Delta \dfrac{\partial c_\ell}{\partial t} + D_{2,\ell}\alpha_\ell \Delta c_\ell + \beta_\ell \alpha_\ell \Delta c_{ext}(t_i) \ \text{in } (0,H_\ell)\times(t_i,t_{i+1}], \\ \nabla c_\ell(0,t) = 0, \, c_\ell(H_\ell,t) = 0, \, t \in [t_i,t_{i+1}], \\ c_\ell(x,t_i) = c_{ext}(x,t_i), \dfrac{\partial c_\ell}{\partial t}(x,t_i) = D\dfrac{\partial c_{ext}}{\partial t}(x,t_i), \, x \in [0,H_\ell], \end{cases} \tag{V.15}$$

with $\ell = c$ or $\ell = s$ for i even or odd, respectively, and c_{ext} is the extension of $c_\ell(t_i)$ defined in $[0,H^*]$ with $H^* = H_s$ or $H^* = H_c$ for i even or odd, respectively, satisfying

- if i is even
$$\int_0^{H_s} c_s(x,t_i) = \int_0^{H_c} c_{s,ext}(x)dx, \tag{V.16}$$
- if i is odd
$$\int_0^{H_c} c_c(x,t_i) = \int_0^{H_s} c_{c,ext}(x)dx. \tag{V.17}$$

3 An analytic solution

In this section, using Fourier analysis, we introduce the general explicit expressions for the solutions of the IBVP's defined in the previous Section. In the first result we establish a formal representation for the solution of the IBVP (V.13).

Theorem 3.0.1 *If $g \in L^2(\Omega)$ is such that $\nabla g \in L^2(\Omega)$ and $g(0) = \nabla g(H_c) = 0$, then*

$$\begin{aligned}&\sum_{n \in I_{c,P}} cos\left(\frac{(2n+1)\pi}{2H_c}x\right)(A_n^0 e^{\omega_{+,c}t} + B_n^0 e^{\omega_{-,c}t})\\&+\sum_{n \in I_{c,H}} cos\left(\frac{(2n+1)\pi}{2H_c}x\right)e^{Re_c t}(C_n^0 cos(\omega_c t) + D_n^0 sin(\omega_c t)) - \frac{\beta_c}{D_{2,c}}g(x)\end{aligned} \tag{V.18}$$

for $x \in [0, H_c], t \in [t_0, t_1]$, defines a formal solution $c_c(x,t)$ of the IBVP (V.13).

In (V.18), $I_{c,P} = \{n \in \mathbb{N}_0 : n \geq n_+ \text{ or } n \leq n_-\}$, $I_{c,H} = \{n \in \mathbb{N}_0 : n_- < n < n_+\}$,

$$n_\pm = \frac{1}{2}\Big(\frac{2H_c\sqrt{\alpha_c}}{\pi}\frac{\sqrt{D_{2,c}} \pm \sqrt{D_{2,c} - D_{1,c}}}{D_{1,c}} - 1\Big), \tag{V.19}$$

provided that $\sqrt{1 - \frac{D_{1,c}}{D_{2,c}}} + \sqrt{D_{1,c}}\sqrt{\frac{D_{1,c}}{D_{2,c}}}\frac{\pi}{2H_c\sqrt{\alpha_c}} < 1$,

$$\gamma_c = \left(\frac{(2n+1)\pi}{2H_c}\right)^2, \tag{V.20}$$

$$\omega_{\pm,c} = \frac{-(\alpha_c + \gamma_c D_{1,c}) \pm \sqrt{(\alpha_c + \gamma_c D_{1,c})^2 - 4\gamma_c\alpha_c D_{2,c}}}{2}, \tag{V.21}$$

$$Re_c = -\frac{\alpha_c + \gamma_c D_{1,c}}{2}, \tag{V.22}$$

$$\omega_c = \sqrt{-(\alpha + \gamma_c D_{1,c})^2 + 4\gamma_c\alpha_c D_{2,c}}, \tag{V.23}$$

and the Fourier coefficients $A_n^0, B_n^0, C_n^0, D_n^0$ are given by

$$A_n^0 = \frac{D_{2,c}D\widehat{g''}(n) - \omega_{-,c}(D_{2,c} + \beta_c)\widehat{g}(n)}{D_{2,c}(\omega_{+,c} - \omega_{-,c})}, \tag{V.24}$$

$$B_n^0 = \frac{\omega_{+,c}(D_{2,c} + \beta_c)\widehat{g}(n) - D_{2,c}D_c\widehat{g''}(n)}{D_{2,c}(\omega_{+,c} - \omega_{-,c})}, \tag{V.25}$$

$$C_n^0 = \frac{(D_{2,c} + \beta_c)\widehat{g}(n)}{D_{2,c}}, \tag{V.26}$$

and

$$D_n^0 = \frac{D_{2,c}D\widehat{g''}(n) - Re_c(D_{2,c} + \beta_c)\widehat{g}(n)}{D_{2,c}\omega_c} \tag{V.27}$$

where the notation $\hat{f}(n) = \frac{2}{H_c}\int_0^{H_c} f(x)cos\left(\frac{(2n+1)\pi}{2H_c}x\right)dx$ was used.

Proof: We start by the following convenient change of variable

$$c_c(x,t) = u_c(x,t) - \frac{\beta_c c_c(x,0)}{D_{2,c}} \tag{V.28}$$

that converts the nonhomogeneous IBVP (V.13) in the homogeneous one

$$\begin{cases} \frac{\partial u_c^2}{\partial t^2} + \alpha_c \frac{\partial u_c}{\partial t} = D_{1,c}\Delta\left(\frac{\partial u_c}{\partial t}\right) + D_{2,c}\alpha_c \Delta u_c(x,t), (x,t) \in (0,H_c)\times(t_0,t_1], \\ u_c(x,0) = \left(1+\frac{\beta_c}{D_{2,c}}\right) g(x),\ \frac{\partial u_c}{\partial t}(x,0) = Dg''(x), x\in[0,H_c], \\ u_c(H_c,t) = 0,\ \nabla u_c(0,t) = 0, t\in[t_0,t_1]. \end{cases} \tag{V.29}$$

To obtain the solution of the new IBVP (V.29), we apply the method of separation of variables, defining $u_c(x,t) = X(x)T(t)$. Hence, replacing it in the partial differential equation of (V.29) we get

$$T^{''}(t)X(x) + \alpha_c X(x)T^{'}(t) = D_{1,c}X^{''}(x)T^{'}(t) + D_{2,c}\alpha_c X^{''}(x)T(t),$$

that leads to

$$\frac{T^{''}(t)+\alpha_c T^{'}(t)}{D_{1,c}T^{'}(t)+D_{2,c}\alpha_c T(t)} = \frac{X^{''}(x)}{X(x)} = -\gamma.$$

From the boundary conditions we obtain $X(H_c)T(t) = 0$ and $X^{'}(0)T(t) = 0$ and consequently, we should have $X(H_c) = 0$ and $X'(0) = 0$. Then for X we obtain the boundary value problem

$$\begin{cases} X^{''}(x) + \gamma X(x) = 0, x\in(0,H_c), \\ X'(0) = 0, X(H_c) = 0, \end{cases} \tag{V.30}$$

and for T we deduce

$$T^{''}(t) + (\alpha_c + \gamma D_{1,c})T^{'}(t) + \gamma D_{2,c}\alpha_c T(t) = 0.$$

We remark that if $\gamma \leq 0$, then $X(x) = 0$ that leads to the null solution. So, $\gamma > 0$, and

$$X(x) = A_n^0 cos(\sqrt{\gamma}x) + B_n^0 sen(\sqrt{\gamma}x).$$

As $X(H_c) = 0$ and $X^{'}(0) = 0$, we obtain

$$X(x) = cos\left(\frac{(2n+1)\pi}{2H_c}x\right), n\in\mathbb{N}_0,$$

and γ_c is given by (V.20).

On the other hand, to obtain T we notice that $z^2 + (\alpha_c + \gamma_c D_{1,c})z + \gamma_c\alpha_c D_{2,c} = 0$. Thus

$$z = \frac{-(\alpha_c+\gamma_c D_{1,c}) \pm \sqrt{(\alpha_c+\gamma_c D_{1,c})^2 - 4\gamma_c\alpha_c D_{2,c}}}{2}. \tag{V.31}$$

The definition of T depends on the nature of the roots defined by (V.31).

- If $(\alpha_c+\gamma_c D_1)^2-4\gamma_c\alpha_c D_{2,c}\geq 0$, (V.31) have the roots (V.21) and consequently, T is given by

$$T(t)=A_n^0e^{\omega_{+,c}t}+B_n^0e^{\omega_{-,c}t}. \tag{V.32}$$

- If $(\alpha_c+\gamma_c D_1)^2-4\gamma_c\alpha_c D_{2,c}<0$, then

$$T(t)=\Big(C_n^0cos(\omega_c t)+D_n^0sin(\omega_c t)\Big)e^{Re_c t} \tag{V.33}$$

  where Re_c and ω_c are given by (V.22) and (V.23).

To conclude the expression of u_c we need to specify the set of $n\in\mathbb{N}_0$ such that $(\alpha_c+\gamma_c D_1)^2-4\gamma_c\alpha_c D_{2,c}\geq 0$ or $(\alpha_c+\gamma_c D_1)^2-4\gamma_c\alpha_c D_{2,c}<0$ holds. Let n_+ and n_- be defined by (V.19) which are the real zeros of $(\alpha_c+\gamma_c D_1)^2-4\gamma_c\alpha_c D_{2,c}$. Then, for $n\in I_{c,P}=]-\infty,n_-]\cup[n_+,+\infty[$, $T(t)$ is given by (V.32) and, for $n\in I_{c,H}=]n_-,n_+[$, $T(t)$ is given by (V.33). Consequently, the candidate to u_c admits the representation

$$\begin{aligned}u_c(x,t)=&\sum_{n\in I_{c,P}}cos\left(\frac{(2n+1)\pi}{2H_c}x\right)(A_n^0e^{\omega_{+,c}t}+B_n^0e^{\omega_{-,c}t})\\&+\sum_{n\in I_{c,H}}cos\left(\frac{(2n+1)\pi}{2H_c}x\right)e^{Re_c t}(C_n^0cos(\omega_c t)+D_n^0sin(\omega_c t))\end{aligned} \tag{V.34}$$

where the constants A_n^0, B_n^0, C_n^0 and D_n^0 are computed using the initial conditions of the IBVP (V.29). Using the Fourier series of $\left(1+\dfrac{\beta_c}{D_{2,c}}\right)g$ and Dg'' we easily get the algebraic systems

$$\begin{cases}A_n^0+B_n^0=\dfrac{(D_{2,c}+\beta_c)\widehat{g}(n)}{D_{2,c}}\\ \omega_{+,c}A_n^0+\omega_{-,c}B_n^0=D\widehat{g''}(n)\end{cases}$$

and

$$\begin{cases}C_n^0=\dfrac{(D_{2,c}+\beta_c)\widehat{g}(n)}{D_{2,c}}\\ Re_cC_n^0+\omega_c D_n^0=D\widehat{g''}(n),\end{cases}$$

where the notation $\hat{f}(n)=\dfrac{2}{H_c}\displaystyle\int_0^{H_c}f(x)cos\left(\frac{(2n+1)\pi}{2H_c}x\right)dx$ was used. Solving the last linear systems we get A_n^0,B_n^0,C_n^0,D_n^0 given by (V.24), (V.25), (V.26) and (V.27), respectively, that concludes the proof.

■

We observe that the computed solution is formal because to show that it is in fact solution of the IBVP (V.18) we need to prove that the series (V.18) defines a function c_c in $[0,H_c]\times[t_0,t_1]$ that is continuous, admits the partial derivatives that arise in the partial differential equation in (V.18) and satisfies all the identities of this problem.

To obtain a solution in the time interval $[t_1,t_2]$ we need to define an extension of $c_c(t_1)$, given in Theorem 3.0.1, to $[0,H_s]$ such that (V.14) holds. We start by noting that $c_c(x,t_1)$ can be rewritten in the following equivalent form

$$c_c(x,t_1) = \sum_{n=0}^{\infty} cos\left(\frac{(2n+1)\pi}{2H_c}x\right) C_n \tag{V.35}$$

where

$$C_n = \begin{cases} A_n^0 e^{\omega_{+,c}t_1} + B_n^0 e^{\omega_{-,c}t_1} - \dfrac{\beta_c}{D_{2,c}}\hat{g}(n) & n \in I_{c,P}, \\ e^{Re_c t_1}(C_n^0 cos(\omega_c t_1) + D_n^0 sin(\omega_c t_1)) - \dfrac{\beta_c}{D_{2,c}}\hat{g}(n) & n \in I_{C,H}, \end{cases}$$

with $\hat{g}(n) = \dfrac{2}{H_c}\displaystyle\int_0^{H_c} g(x) cos\left(\dfrac{(2n+1)\pi}{2H_c}x\right) dx$. We take

$$c_{c,ext}(x) = \frac{H_c}{H_s}\sum_{n=0}^{\infty} cos\left(\frac{(2n+1)\pi}{2H_s}x\right) C_n, x \in [0,H_s]. \tag{V.36}$$

The extension $c_{c,ext}$ defined by (V.36) satisfies (V.14) and its Fourier form is convenient to obtain easily the solution of the IBVP (V.15), with $\ell = s$, $i = 1$, in $[0,H_s]\times[t_1,t_2]$. In fact, applying Theorem 3.0.1

$$\begin{aligned} c_s(x,t) = &\sum_{n\in I_{s,P}} cos\left(\frac{(2n+1)\pi}{2H_s}x\right)(A_n^1 e^{\omega_{+,s}t} + B_n^1 e^{\omega_{-,s}t}) \\ &+ \sum_{n\in I_{s,H}} cos\left(\frac{(2n+1)\pi}{2H_s}x\right) e^{Re_s t}(C_n^1 cos(\omega_s t) + D_n^1 sin(\omega_s t)) - \frac{\beta_s}{D_{2,s}} c_{c,ext}(x,t_1) \end{aligned} \tag{V.37}$$

with $I_{s,P} = \{n \in \mathbb{N}_0 : n \geq n_+ \text{ or } n \leq n_-\}$, $I_{s,H} = \{n \in \mathbb{N}_0 : n_- < n < n_+\}$,

$$n_\pm = \frac{1}{2}\Big(\frac{2H_s\sqrt{\alpha_s}}{\pi}\frac{\sqrt{D_{2,s}} \pm \sqrt{D_{2,s}-D_{1,s}}}{D_{1,s}} - 1\Big)$$

, provided that $\sqrt{1-\dfrac{D_{1,s}}{D_{2,s}}} + \sqrt{D_{1,s}}\sqrt{\dfrac{D_{1,s}}{D_{2,s}}}\dfrac{\pi}{2H_s\sqrt{\alpha_s}} < 1$,

$$\gamma_s = \left(\frac{(2n+1)\pi}{2H_s}\right)^2,$$

$$\omega_{\pm,s} = \frac{-(\alpha_s+\gamma_s D_{1,s}) \pm \sqrt{(\alpha_s+\gamma_s D_{1,s})^2 - 4\gamma_s\alpha_s D_{2,s}}}{2},$$

$$Re_s = -\frac{\alpha_s+\gamma_s D_{1,s}}{2},$$

$$\omega_s = \sqrt{-(\alpha_s+\gamma_s D_{1,s})^2 + 4\gamma_s\alpha_s D_{2,s}}, \tag{V.38}$$

and the Fourier coefficients $A_n^1, B_n^1, C_n^1, D_n^1$ are given by

$$A_n^1 = \frac{DD_{2,s}\widehat{c_{c,ext}(t_1)}''(n) - \omega_{-,s}(D_{2,s}+\beta_s)\widehat{c_{c,ext}(t_1)}(n)}{D_{2,s}(\omega_{+,s}-\omega_{-,s})},$$

$$B_n^1 = \frac{\omega_{+,s}(D_{2,s}+\beta_s)\widehat{c_{c,ext}(t_1)}(n) - DD_{2,s}\widehat{c_{c,ext}(t_1)}''(n)}{D_{2,s}(\omega_{+,s}-\omega_{-,s})},$$

$$C_n^1 = \frac{(D_{2,s}+\beta_s)\widehat{c_{c,ext}(t_1)}(s)}{D_{2,s}},$$

and

$$D_n^1 = \frac{DD_{2,s}\widehat{c_{c,ext}(t_1)}''(n) - Re_s(D_{2,s}+\beta_s)\widehat{c_{c,ext}(t_1)}(n)}{D_{2,s}\omega_s}$$

where $\widehat{c_{c,ext}(t_1)}(n) = \frac{2}{H_s}\int_0^{H_s} c_{c,ext}(x,t_1)cos\left(\frac{(2n+1)\pi}{2H_s}x\right)dx.$

To obtain the solution for $[0,H_c]\times[t_2,t_3]$ we apply again the Theorem 3.0.1 with the convenient adaptations.

4 Conclusion

The main objective of this chapter is the introduction of mathematical models for the drug release from a polymeric thermoresponsive platform. The polymer is a viscoelastic material where the Young modulus change with the temperature. The polymer has a lower critical solution temperature (LCST) and it switches from collapsed state for temperature above the LCST to swollen state for temperatures lower than the LCST.

To simulate the evolution of the polymeric platform, here it was assumed that the change in the temperature leads to two different states of the polymeric structure. A hybrid model obtained splitting the time interval into disjoint subintervals, where the polymeric domain has different lengths, is constructed where the drug transport is characterized by two sets of different values. An analytic processes based on Fourier analysis is proposed to construct the solution of this model.

The main theoretical result - Theorem 3.0.1, that allows the construction of a solution of the hybrid model, can be used to study its qualitative behaviour.

Thermoresponsive polymers are attracting an enormous scientific interest for advanced applications in drug delivery. Mathematical modelling and simulation of drug delivery, from these materials, appears as an important co-adjutant in pioneering experimental studies. Though the work included in this chapter still has an exploratory character, we think that promising numerical simulations can be obtained by using the Fourier approach presented here. In the near future we plan to develop this approach as well as the design of FEM/FDM well adapted to the moving boundary value problem.

Chapter VI

Conclusions and Future work

The study of numerical methods for systems of nonlinear differential equations of parabolic type or for systems defined by a nonlinear elliptic equation coupled with two nonlinear parabolic equations was the main objective of this work. The motivation underlying the study is the modelling of drug delivery, from polymeric matrices, enhanced by external stimuli, namely by heat. In several human diseases like cancer, the traditional treatments lead to very serious side effects. To overcome the limitations of some classical therapies, like chemotherapy in the cancer treatment, local drug delivery strategies that require drug carriers have been studied. Liposomes, dendrimers, polymeric nanoparticles, and lipoprotein drug carriers, among others, have been shown to be very promising. A large number of experimentalists have also studied new delivery systems, combined with the use of external stimuli, to enhance the drug release. Most of these new drug delivery systems are still at an experimental stage. We believe that mathematical modeling and numerical simulation, is an important co-adjutant in such pioneering experimental studies. As the differential systems studied can be used to describe the drug release in a target tissue, the numerical simulations can be used to illustrate the behaviour of the drug concentration in time and space in heat-enhanced delivery.

Regarding the mathematical aspects - the construction of numerical methods and the development of their theoretical support- we would like to highlight the main points addressed:

- two types of differential systems were considered:
  - a system of two nonlinear parabolic equations (I.3)-(I.2);
  - a systems defined by a nonlinear elliptic equations and two nonlinear parabolic equations, (I.6)-(I.8);
- The numerical methods proposed - method (III.32)-(III.33) for problem (I.3)-(I.2); (IV.14)-(IV.16) for problem (I.6)-(I.8), be viewed simultaneously as finite difference methods or completely discrete piecewise linear finite element methods.
- The main convergence outputs are defined in the following results: Theorems 4.1.4 and 4.2.5, Theorem 3.2.1 and Theorems 4.2.1 and 5.2.1. The first result shows that the numerical solutions obtained by (III.32)-(III.33) converge for the solution of the differential problem (I.3)-(I.2) with respect to the norm $\|.\|_h$ that is a discrete version of the usual L^2 norm. Theorem 3.2.1

establishes that the solution defined by (IV.14) converges to the solution of (I.6) with respect to the norm $\|.\|_{1,h}$ which is a discrete version of the usual norm in H^1. This result has an important role in the establishment of the convergence of the numerical solutions defined by (IV.15)-(IV.16) to the correspondent continuous solutions in Theorem 4.2.1 and 5.2.1.

- The main convergence results were established assuming that the solutions of the differential systems are in $H^3(\Omega)$ which is not a usual assumption in the finite difference analysis.
- The stability of the method (III.32)-(III.33) for problem (I.3)-(I.2) and(IV.14)-(IV.16) for problem (I.6)-(I.8) was studied. As we were dealing with nonlinear problems, the convergence estimates established played an important role is the stability analysis.
- The convergence results were numerically illustrated. The numerical experiments included in this work show the sharpness of the smoothness assumptions.

Regarding the applications of the models studied we present in what follows some possible medical outcomes.

In Chapter III the heat in the system was assumed to be generated by a source term, that can be located inside the spatial domain or at its boundary. The numerical experiments included can illustrate the behaviour of the drug concentration in different scenarios. In Chapter IV the heat is induced by an electric field generated by an applied potential. The model can be used to simulate iontophoresis and electroporation, techniques that are used to enhance transdermal drug delivery. A number of protocols, characterized by different potential intensities and different durations were numerically analyzed. The results obtained suggest that lower intensity protocols were more effective.

In Chapter V we consider a non-Fickian mathematical model to describe drug release from a stimuli responsive polymer. Here, the dependence of the Young modulus on the temperature was considered. Following a semi-analytic approach, we constructed the analytical solution and its qualitative behaviour was illustrated. The work in this chapter still has an exploratory character. We plan to develop, in the near future, two different strategies to build robust numerical methods to simulate drug delivery from thermoresponsive systems: approximations based on the Fourier approach and the design of FEM/FDM well adapted to the moving boundary value problem.

The systems presented in this thesis represent drug release in vitro, that is in an external medium, but they don't consider the properties of this medium. To simulate the drug release from a drug transporter and its absorption by the target tissue, enhanced by heat, it is necessary to combine all the actors of the process. Therefore the coupling of the polymeric and target tissue domains needs to be taken into account. We plan to address the problem in the near future. As human tissues are viscoelastic materials a viscoelastic version of the systems (I.3)-(I.2) and (I.6)-(I.8) considering the approach in [20], [21] and [30] will be studied. We believe that the study of in vivo release, by coupling the drug delivery systems with the living tissues, would represent a step forward a more complete comprehension of controlled drug delivery.

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Appendix A

1 Functional Spaces

- $L^2(\Omega), L^\infty(\Omega), H_0^1(\Omega), H^k(\Omega), k \in \mathbb{N}$ are the usual spaces.
- Let consider $\mathscr{B}$ a Banach space.
  - $C([0,t_f],\mathscr{B})$ denotes the space of continuous functions $b:[0,t_f] \longrightarrow \mathscr{B}$ with respect to norm $\|.\|_{\mathscr{B}}$
  - $C^k([0,t_f],\mathscr{B})$ represents the space of continuous functions $b:[0,t_f] \longrightarrow \mathscr{B}$ such that its derivatives up to order k are continuous and

$$\|b\|_{C^k([0,t_f],\mathscr{B})} = \max_{[0,t_f]} \|b^{(k)}(t)\|_{\mathscr{B}} < \infty \tag{A.1}$$

  - $L^2(0,t_f,\mathscr{B})$ denote the space of Bochner-measurable functions $b:(0,t_f) \longrightarrow \mathscr{B}$ such that

$$\|b\|_{L^2(0,t_f,\mathscr{B})} = \left(\int_0^{t_f} \|b(t)\|_{\mathscr{B}}^2 dt\right)^{1/2} < \infty \tag{A.2}$$

  - $L^\infty(0,t_f,\mathscr{B})$ represents the space of essentially bounded Bochner measurable functions

$$\|b\|_{L^\infty(0,t_f,\mathscr{B})} = ess\sup_{[0,t_f]} \|b(t)\|_{\mathscr{B}} < \infty \tag{A.3}$$

  - $H^k(0,t_f,\mathscr{B})$ denotes the space of functions b in $L^2(0,t_f,\mathscr{B})$ whose distributional time derivatives up to order k are also in $L^2(0,t_f,\mathscr{B})$ and moreover, such that

$$\|b\|_{H^k(0,t_f,\mathscr{B})} = \sum_{j=0}^{k} \int_0^{t_f} \left\|\frac{d^j b}{dt^j}\right\|_{\mathscr{B}}^2 < \infty \tag{A.4}$$

- $W^{1,\infty}(\Omega)$ denotes the Sobolev space of functions defined in Ω such that

$$\|w\|_{W^{1,\infty}(\Omega)} = \max_{|\alpha|\leq 1} ess\sup_{\Omega} |D^\alpha w| \tag{A.5}$$

with $\alpha \in \mathbb{N}_0$ and $D^\alpha w = \dfrac{\partial^{|\alpha|} w}{\partial x^\alpha}$.

2 Embeddings

Theorem 2.0.1 *[1] Let Ω be a domain in $\mathbb{R}^n$. Suppose Ω satisfies the cone condition. Let $j \geq 0$ and $m \geq 1$ be integers and let $1 \leq p < \infty$. If either $mp > n$ or $m = n$ and $p = 1$, then*

$$W^{j+m,p}(\Omega) \hookrightarrow C^j(\overline{\Omega}).$$

3 Inequalities

Gronwall's Lemma [29] Let α, β and u be real-valued functions defined on $[0,t_f]$. Assuming that β and u are continuous and that the negative part of α is integrable on every closed and bounded subinterval of $[0,t_f]$. If β is non-negative and if u satisfies the integral inequality

$$u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)ds \quad \forall t \in [0,t_f]$$

then

$$u(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s)e^{\int_s^t \beta(r)dr}ds, \; t \in [0,t_f].$$

Moreover, if the function α is non-decreasing, then

$$u(t) \leq \alpha(t)e^{\int_a^t \beta(s)ds}, \; t \in [0,t_f].$$

4 Bramble-Hilbert Results

Lemma 4.0.1 *[11] Let Ω be an open subset of $\mathbb{R}^N$ with a Lipschitz-continuous boundary. For some integer $k \geq 0$ and some number $p \in [0;1]$, let λ be a continuous linear form on the space $W^{k+1;p}(\Omega)$ with the property that*

$$\forall u \in P_k(\Omega), \; \lambda(u) = 0$$

where P_k represents the space of polynomials of degree k. Then there exists a constant $c(\Omega)$ such that

$$\forall u \in W^{k+1;p}(\Omega), \; |\lambda(u)| \leq c(\Omega)\|\lambda\|^*_{W^{k+1;p}(\Omega)}|u|_{W^{k+1;p}(\Omega)}, \tag{A.6}$$

where $\|\cdot\|^*_{W^{k+1;p}(\Omega)}$ denotes the norm in the dual space of $W^{k+1;p}(\Omega)$.

Let assume that $\Omega = (a,b)$.

Lemma 4.0.2 *Let define $u \in H^3(\Omega) \cap H^1_0(\Omega)$ and consider the nonuniform grid $\overline{\Omega}_h = \{x_i, i = 0, \cdots, N, x_i - x_{i-1} = h_i, i = 1, \ldots, N, x_0 = a, x_N = b\}$. Then for the functional $\lambda(u) = D_{-x}u(x_i) - \frac{\partial u}{\partial x}\left(x_{i-\frac{1}{2}}\right)$ there exists a constant C such that*

$$|\lambda(u)| \leq Ch_i||u'''||_{L^1(x_{i-1},x_i)}.$$

Proof: Let v be defined by $v(\varepsilon) := u(x_{i-1} + \varepsilon h_i)$, for $\varepsilon \in [0,1]$.

As $v(0) = u(x_{i-1})$, $v(1) = u(x_i)$, $v_\varepsilon(1/2) = h_i u_x(x_{i-\frac{1}{2}})$, we get

$$\begin{aligned}\lambda(u) &= \frac{u(x_i) - u_(x_{i-1}))}{h_i} - u_x(x_{i-\frac{1}{2}}) \\ &= \frac{1}{h_i}\left(u(x_i) - u(x_{i-1}) - h_i u_x(x_{i-\frac{1}{2}})\right) = \frac{1}{h_i}(v(1) - v(0) - v'(1/2)) = \frac{1}{h_i}\hat{\lambda}(v)\end{aligned}$$

where $\hat{\lambda} : W^{1,3}(\Omega) \to \mathbb{R}$ is a bounded functional that satisfies $\hat{\lambda}(\varepsilon^j) = 0, j = 0,1,2$. Thus, the Bramble-Hilbert Lemma 4.0.1 guarantees that:

$$\exists C > 0 : |\hat{\lambda}(v)| \leq C||v'''||_{L^1(\Omega)}$$

But as $||v'''||_{L^1(\Omega)} = h_i^2||u'''||_{L^1(x_{i-1},x_i)}$, we get that,

$$|\lambda(u)| \leq \frac{1}{h_i} C h_i^2 ||u'''||_{L^1(x_{i-1},x_i)} = C h_i ||u'''||_{L^1(x_{i-1},x_i)}.$$

■

Lemma 4.0.3 *Let $\overline{\Omega}_h$ be defined by $\{x_i, i = 0,\cdots,N, x_i - x_{i-1} = h_i, i = 1,\ldots,N, x_0 = a, x_N = b\}$ and let $u \in H^2(\Omega)\cap H_0^1(\Omega)$ and $\lambda(u) = \frac{h_i}{2}(u(x_i) + u(x_{i-1})) - \int_{x_{i-1}}^{x_i} u(x)dx$, there exists a constant C such that*

$$|\lambda(u)| \leq C h_i^2 ||u''||_{L^1(x_{i-1},x_i)}.$$

Proof: Let v be defined by $v(\varepsilon) := u(x_{i-1} + \varepsilon h_i)$, for $\varepsilon \in [0,1]$.
As $v(0) = u(x_{i-1})$, $v(1) = u(x_i)$, $h_i \int_0^1 v(\varepsilon)d\varepsilon = \int_{x_{i-1}}^{x_i} u(x)dx$, for $\lambda(u)$ we obtain

$$\begin{aligned}\lambda(u) &= h_i\left(\frac{1}{2}(u(x_i) + u_(x_{i-1})) - \int_{x_{i-1}}^{x_i} u(x)dx\right) \\ &= h_i \underbrace{\left(\frac{1}{2}(v(1) + v(0)) - \int_0^1 v(\varepsilon)d\varepsilon\right)}_{\hat{\lambda}(v)}\end{aligned}$$

Then the functional $\hat{\lambda} : W^{1,2}(\Omega) \to \mathbb{R}$ defined by $\hat{\lambda}(v) = \frac{1}{2}(v(1) + v(0)) - \int_0^1 v(\varepsilon)d\varepsilon, v \in W^{1,2}(\Omega)$ is bounded and $\hat{\lambda}(\varepsilon^j) = 0, j = 0,1$. Thus the Bramble-Hilbert Lemma 4.0.1 guarantees that:

$$\exists C > 0 : |\hat{\lambda}(v)| \leq C||v''||_{L^1(\Omega)}$$

But as $||v''||_{L^1(\Omega)} = h_i||u''||_{L^1(x_{i-1},x_i)}$, we get that,

$$|\lambda(u)| \leq h_i C h_i ||u''||_{L^1(x_{i-1},x_i)} = C h_i^2 ||u''||_{L^1(x_{i-1},x_i)}.$$

■

Lemma 4.0.4 *Let $\overline{\Omega}_h$ be defined by $\{x_i, i=0,\cdots,N, x_i - x_{i-1} = h_i, i=1,\ldots,N, x_0=a, x_N=b\}$ and let $u \in H^1(\Omega)\cap H_0^1(\Omega)$ and $\lambda(u) = \frac{h_i}{2}(u(x_i)-u(x_{i-1})) + \int_{x_{i-1}}^{x_{i-\frac{1}{2}}} u(x)dx - \int_{x_{i-\frac{1}{2}}}^{x_i} u(x)dx$, there exists a constant C such that*

$$|\lambda(u)| \le Ch_i \|u'\|_{L^1(x_{i-1},x_i)}.$$

Proof: Let v be defined by $v(\varepsilon) := u(x_{i-1}+\varepsilon h_i)$, for $\varepsilon \in [0,1]$.

Then having that $v(0) = u(x_{i-1})$, $v(1) = u(x_i)$, $h_i \int_0^{\frac{1}{2}} v(\varepsilon)d\varepsilon = \int_{x_{i-1}}^{x_{i-\frac{1}{2}}} u(x)dx$ and $h_i \int_{\frac{1}{2}}^1 v(\varepsilon)d\varepsilon = \int_{x_{i-\frac{1}{2}}}^{x_i} u(x)dx$, we get

$$\begin{aligned}\lambda(u) &= \frac{h_i}{2}(u(x_i)-u(x_{i-1})) + \int_{x_{i-1}}^{x_{i-\frac{1}{2}}} u(x)dx - \int_{x_{i-\frac{1}{2}}}^{x_i} u(x)dx \\ &= h_i\left(\frac{1}{2}(v(1)-v(0)) + \int_0^{\frac{1}{2}} v(\varepsilon)d\varepsilon - \int_{\frac{1}{2}}^1 v(\varepsilon)d\varepsilon\right) = h_i\hat{\lambda}(v)\end{aligned}$$

Then the functional $\hat{\lambda} : W^{1,1}(\Omega) \to \mathbb{R}$ defined by $\hat{\lambda}(v) = \frac{1}{2}(v(1)-v(0)) + \int_0^{\frac{1}{2}} v(\varepsilon)d\varepsilon - \int_{\frac{1}{2}}^1 v(\varepsilon)d\varepsilon, v \in W^{1,1}(\Omega)$ is bounded and $\hat{\lambda}(\varepsilon^j) = 0, j = 0$. Thus the Bramble-Hilbert Lemma 4.0.1 guarantees that:

$$\exists C > 0 : |\hat{\lambda}(v)| \le C||v'||_{L^1(\Omega)}$$

But as $||v'||_{L^1(\Omega)} = ||u'||_{L^1(x_{i-1},x_i)}$, we get that,

$$|\lambda(u)| \le Ch_i||u'||_{L^1(x_{i-1},x_i)}$$

■

Lemma 4.0.5 *Let $\overline{\Omega}_h$ be defined by $\{x_i, i=0,\cdots,N, x_i - x_{i-1} = h_i, i=1,\ldots,N, x_0=a, x_N=b\}$ and let $u \in H^2(\Omega)\cap H_0^1(\Omega)$ and $\lambda(u) = \frac{1}{2}(u(x_i)+u(x_{i-1})) - u(x_{i-\frac{1}{2}})$, there exists a constant C such that*

$$|\lambda(u)| \le Ch_i \|u''\|_{L^1(x_{i-1},x_i)}.$$

Proof: Let v be defined by $v(\varepsilon) := u(x_{i-1}+\varepsilon h_i)$, for $\varepsilon \in [0,1]$.
Then as $v(0) = u(x_{i-1})$, $v(1) = u(x_i)$ and $v(\frac{1}{2}) = u(x_{i-\frac{1}{2}})$, we obtain

$$\lambda(u) = \frac{1}{2}(u(x_i)+u(x_{i-1})) - u(x_{i-\frac{1}{2}}) = h_i\left(\frac{1}{2}(v(1)+v(0)) - v\left(\frac{1}{2}\right)\right) = \hat{\lambda}(v)$$

Then the functional $\hat{\lambda} : W^{2,1}(\Omega) \to \mathbb{R}$ defined by $\hat{\lambda}(v) = \frac{1}{2}(v(1)+v(0)) - v\left(\frac{1}{2}\right), v \in W^{1,2}(\Omega)$ is bounded and $\hat{\lambda}(\varepsilon^j) = 0, j = 0,1$. Thus the Bramble-Hilbert Lemma 4.0.1 guarantees that:

$$\exists C > 0 : |\hat{\lambda}(v)| \le C||v''||_{L^1(\Omega)}$$

But as $\|v''\|_{L^1(\Omega)} = h_i \|u''\|_{L^1(x_{i-1},x_i)}$, we get that,

$$|\lambda(u)| \le C h_i \|u''\|_{L^1(x_{i-1},x_i)}$$

■

Lemma 4.0.6 *Let $\overline{\Omega}_h$ be defined by $\{x_i, i = 0, \cdots, N, x_i - x_{i-1} = h_i, i = 1, \ldots, N, x_0 = a, x_N = b\}$. Let $u \in H^3(\Omega) \cap H_0^1(\Omega)$ and $\lambda(u) = \frac{\partial u}{\partial x}(x_i) - D_h u(x_i)$ there exists a constant C such that*

$$|\lambda(u)| \le C(h_i + h_{i+1}) \|u'''\|_{L^1(x_{i-1},x_i)}.$$

Proof: Let v be defined by $v(\varepsilon) := u(x_{i-1} + \varepsilon(h_i + h_{i+1}))$, for $\varepsilon \in [0,1]$.

Considering $\delta_1 = \frac{h_i}{h_i + h_{i+1}}$ and $\delta_2 = \frac{h_{i+1}}{h_i}$, we have that $v(0) = u(x_{i-1})$, $v(1) = u(x_{i+1})$, $v(\delta_1) = u(x_i)$ and $v_\varepsilon(\delta_1) = \frac{u_x(x_i)}{h_i + h_{i+1}}$. Consequently,

$$\begin{aligned}
\lambda(u) &= \frac{1}{h_i + h_{i+1}} \left((h_i + h_{i+1}) u_x(x_i) - \frac{h_{i+1}}{h_i}(u(x_i) - u(x_{i-1})) - \frac{h_i}{h_{i+1}}(u(x_{i+1}) - u(x_i)) \right) \\
&= \frac{1}{h_i + h_{i+1}} \left(v_\varepsilon(\delta_1) - \delta_2(v(\delta_1) - v(0)) - \frac{1}{\delta_2}(v(1) - v(\delta_1)) \right) = \frac{1}{h_i + h_{i+1}} \hat{\lambda}(v)
\end{aligned}$$

where $\hat{\lambda} : W^{1,3}(\Omega) \to \mathbb{R}$ is a bounded functional that satisfies $\hat{\lambda}(\varepsilon^j) = 0, j = 0,1,2$. Thus, the Bramble-Hilbert Lemma 4.0.1 guarantees that:

$$\exists C > 0 : |\hat{\lambda}(v)| \le \|v'''\|_{L^1(\Omega)} \tag{A.7}$$

However, $\|v'''\|_{L^1(\Omega)} = (h_i + h_{i+1})^2 \|u'''\|_{L^1(x_{i-1},x_{i+1})}$, so

$$|\lambda(u)| \le \frac{C}{h_i + h_{i+1}} \times (h_i + h_{i+1})^2 \|u'''\|_{L^1(x_{i-1},x_{i+1})} = (h_i + h_{i+1}) \|u'''\|_{L^1(x_{i-1},x_{i+1})} \tag{A.8}$$

■