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STRING THEORETICAL BLACK HOLES:
QUASINORMAL MODES AND GREYBODY FACTORS

Tese no âmbito do mestrado em Física Nuclear e de Partículas orientada pelo Doutor Filipe Alexandre Pedra Aguiar de Moura e apresentada ao Departamento de Física da Universidade de Coimbra.

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Abstract

In this work, we compute analytical expressions for quasinormal frequencies, associated with tensor type gravitational perturbations and complex massless scalar field perturbations, in the Callan Myers Perry black hole space time. We do this for two distinct limits. First, we compute these frequencies in the eikonal limit, resorting to WKB approximations. We then compute these frequencies in the asymptotic limit, studying two different monodromies of the perturbation, when analytically continued to the complex r-plane. Additionally, we compute analytical expressions for the greybody factors, associated with such perturbations.

Finally, we provide an analytical expression for the radius associated with the shadow cast by the Callan Myers Perry black hole space time.

Keywords— Black-Hole, Quasinormal modes, Monodromy, WKB approximation, String Theory, Greybody factor, Black hole shadow

Neste trabalho, calculamos expressões analíticas para frequências quasenormais, associadas a perturbações gravitacionais do tipo tensorial e perturbações de um campo escalar complexo não massivo, num buraco negro de Callan Myers Perry. Fazemos isto para dois limites distintos. Primeiro, calculamos estas frequências no limite eikonal, recorrendo a aproximações WKB. De seguida, calculamos estas frequências no limite assimptótico, estudando duas monodromias distintas da perturbação, quando continuada analiticamente para o plano complexo de r. Além disso, calculamos expressões analíticas para os fatores de corpo cinza, associados a estas perturbações.

Por último, obtemos uma expressão analítica para o raio associado à sombra do buraco negro de Callan Myers Perry.

Palavras-chave— Buracos negros, Modos quasenormais, Monodromia, Aproximação WKB, Teoria de Cordas, Fatores de corpo cinza, Sombra de um buraco negro

Dedication

To my parents.

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Contents

Nomenclature

$[X, Y]$ Lie bracket of the smooth vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$.

Γ Gamma function.

$\Im(z)$ Imaginary component of $z \in \mathbb{C}$.

$\Lambda^1(\mathcal{M})$ Real vector space of 1-forms in the smooth manifold $\mathcal{M}$.

$\mathcal{F}(\mathcal{M})$ Real vector space of smooth functions on the smooth manifold $\mathcal{M}$.

$\mathcal{L}_X$ Lie derivative with respect to the smooth vector field $X \in \mathfrak{X}(\mathcal{M})$.

$\mathcal{T}_l^k(\mathcal{M})$ Real vector space of smooth tensor fields of type (l, k) on the smooth manifold $\mathcal{M}$.

$\mathfrak{X}(\mathcal{M})$ Real vector space of smooth vector fields on the smooth manifold $\mathcal{M}$.

$\Re(z)$ Real component of $z \in \mathbb{C}$.

c Speed of light in the vacuum.

$C^k(\mathbb{R})$ Real vector space of real functions of real domain with continuous derivatives up to order $k+1$.

$g^{\mu\nu}$ Components of the tensor field $R \circ R\,(g) \in \mathcal{T}_0^2(\mathcal{M})$, where $g \in \mathcal{T}_2^0(\mathcal{M})$.

k Einstein gravitational constant.

Y_{lm} Laplace's spherical harmonic function, where $l \in \mathbb{N}_0$ and $-l \leq m \leq l$.

Chapter 1

Introduction

In this chapter, we do a fairly complete contextualisation of the physics and mathematics needed to understand the approaches used in this work. We will cover a broad range of areas, from a very brief introduction to general relativity and quasinormal modes to WKB theory.

1.1 A brief introduction to general relativity

Black holes are some of the most exotic solutions of the Einstein field equations

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2}\mathcal{S}g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{1.1}$$

These equations are in the core of general relativity and their job is to build pseudo-Riemannian manifolds whose geometrical properties are strongly related to the physical systems we want to study.

Before we proceed to the physics, it is fitting to provide a small introduction to pseudo-Riemannian geometry, necessary to understand the equations above.

1.1.1 Pseudo-Riemannian and Lorentzian manifolds

Pseudo-Riemannian manifolds are smooth manifolds $\mathcal{M}$, equipped with a non-degenerate smooth symmetric tensor field $g \in \mathcal{T}_2^0(\mathcal{M})$, called the metric tensor field.

Let us consider each tensor of the tensor field g. We can define the signature of these tensors as the tuple (n_0, n_+, n_-), where n_0, n_+, n_- are the number of null, positive and negative eigenvalues, respectively, of the tensor matrix representation [1]. Because g is non degenerate, the tensors are such that $n_0 = 0$. Thus, the signature can be abbreviated to (n_+, n_-).

We know the signature, as defined above, is invariant in $\mathcal{M}$ [2]. To see this, let us consider the matrix representations of the tensors $g \circ \gamma$, for some continuous curve γ in $\mathcal{M}$. The transition of the eigenvalues from positive to negative, or vice-versa, would imply the existence of a point where the quadratic form is degenerate, unless g is discontinuous somewhere and consequently not smooth.

We can define a d-dimensional Lorentzian manifold as a d-dimensional pseudo-Riemannian manifold whose signature is $(1, d-1)$. Generally, solutions of the Einstein equations (1.1) consist of metrics tensor fields associated with Lorentzian manifolds, also called Lorentzian metrics tensor fields.

1.1.2 Connections and the covariant derivative along a curve

Connections are the mathematical objects that make precise the notion of vector transportation along the tangent bundle $T\mathcal{M}$ of some smooth manifold $\mathcal{M}$. Formally, we can define a connection on $\mathcal{M}$ as a map

$$\nabla : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M}) \tag{1.2}$$

[1]Every symmetric matrix has real eigenvalues.

[2]As long as $\mathcal{M}$ is connected.

such that

$$\nabla_{X_1+X_2}Y_1 = \nabla_{X_1}Y_1 + \nabla_{X_2}Y_1 \tag{1.3}$$

$$\nabla_{X_1}(Y_1+Y_2) = \nabla_{X_1}Y_1 + \nabla_{X_1}Y_2 \tag{1.4}$$

$$\nabla_{fX_1}Y_1 = f(\nabla_{X_1}Y_1) \tag{1.5}$$

$$\nabla_{X_1}(fY_1) = f(\nabla_{X_1}Y_1) + X_1(f)Y_1 \tag{1.6}$$

for all $X_1, X_2, Y_1, Y_2 \in \mathfrak{X}(\mathcal{M})$ and $f \in \mathcal{F}(\mathcal{M})$. Let (U, ϕ) be a chart on $\mathcal{M}$. We can fully describe the connection ∇ on U as

$$\nabla_{\frac{\partial}{\partial\phi^\mu}}\frac{\partial}{\partial\phi^\nu} = \Gamma^\alpha_{\mu\nu}\frac{\partial}{\partial\phi^\alpha} \tag{1.7}$$

where the functions $\Gamma^\alpha_{\mu\nu}$ are usually called Christoffel symbols.

We are now ready to introduce the covariant derivative along some smooth curve. Let

$$V : I \subseteq \mathbb{R} \to T\mathcal{M} \tag{1.8}$$

be a smooth vector field along some smooth curve

$$\gamma : I \subseteq \mathbb{R} \to \mathcal{M}. \tag{1.9}$$

We write the covariant derivative of the vector field V, along the curve γ, as

$$\frac{DV}{Dt} = \nabla_{d\gamma(\frac{d}{dt})}V. \tag{1.10}$$

We say that a vector field, along some smooth curve, is parallel if the respective covariant derivative is zero, along the entire curve's domain.

This derivative defines a natural way to transport vectors, along any smooth curve on $\mathcal{M}$. Indeed, it can be proven that, given a smooth manifold $\mathcal{M}$ equipped with a connection ∇ and some smooth curve γ, any vector in the tangent spaces, associated with points in the image of γ, can be parallel transported by a unique vector field along γ [14].

In Riemannian manifolds and pseudo-Riemannian manifolds in particular, there is a canonical connection that can be fully extracted from the metric tensor field g, usually called the Levi-Civita connection. In order to further explain what defines this connection, let us introduce the torsion tensor field $\Theta \in \mathcal{T}^1_2(\mathcal{M})$, defined by the equation

$$\Theta(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \tag{1.11}$$

Loosely speaking, this tensor field measures how the basis of our tangent space twists as we travel in a smooth curve on $\mathcal{M}$. We say a connection is metric compatible if parallel transport preserves the pseudo scalar product, induced by the metric tensor field on the tangents spaces. The Levi-Civita connection is a metric compatible connection with a null torsion tensor field. It is shown that this two conditions alone, completely define the connection [14].

In Lorentzian manifolds, associated with the solutions of the Einstein field equations, we generally choose the Levi-Civita connection. It can be reasonably explained why we want the connection to be metric compatible. In general relativity, the metric tensor field is responsible for giving $\mathcal{M}$ the physical notion of distance. As it turns out, if the connection was not compatible with the metric tensor field, free falling objects would suffer distortions in size, as they made their way through the respective trajectories [3]. Naturally, we do not measure this effect, consequently we always consider the connection to be metric compatible. The same cannot be said about the requirement for the torsion tensor field to be null. This is, in part, because we don't have a direct physical role played by the torsion tensor field [4].

[3] In general relativity, we say that a point particle in free fall will follow a geodesic path. The geodesic path is the unique smooth curve, where the velocity vector field is parallel transported [41].

[4] Although, some gravity theories, like Einstein-Cartan theories, try to relate the torsion tensor field with the spin of particles [53].

1.1.3 Covariant derivatives of tensor fields

Here, we generalise the notion of covariant derivative to general tensor fields, with respect to vector fields, in a d-dimensional Riemannian manifold $\mathcal{M}$, equipped with a connection ∇.

Let $X \in \mathfrak{X}(\mathcal{M})$ and $f \in \mathcal{F}(\mathcal{M})$, the covariant derivative of f, with respect to the vector field X, is defined as

$$\nabla_X f = \mathcal{L}_X(f). \tag{1.12}$$

Let $Y \in \mathfrak{X}(\mathcal{M})$, the covariant derivative of Y, with respect to the vector field X, is defined as

$$\nabla_X Y. \tag{1.13}$$

Let $\sigma \in \Lambda^1(\mathcal{M})$, the covariant derivative of σ, with respect to the vector field X, is defined as $\nabla_X \sigma \in \Lambda^1(\mathcal{M})$, such that

$$\left(\nabla_X \sigma\right)(Z) = \nabla_X \sigma\,(Z) - \sigma\left(\nabla_X(Z)\right) \tag{1.14}$$

for all $Z \in \mathfrak{X}(\mathcal{M})$. Finally, the covariant derivative can be uniquely generalised to all tensors demanding that

$$\nabla_X\left(\rho \otimes \psi\right) = \nabla_X \rho \otimes \psi + \rho \otimes \nabla_X \psi \tag{1.15}$$

$$\nabla_X\left(\eta + \Omega\right) = \nabla_X \eta + \nabla_X \Omega \tag{1.16}$$

for arbitrary tensor fields ρ, ψ and for arbitrary tensor fields of the same type η, Ω. We note that both properties above agree with the definition of covariant derivatives of vector fields, 1-form fields and smooth functions.

It should be noted that we used the same symbol for the connection map and for the covariant derivative. This is fine because no ambiguity is introduced. Indeed, if we see $\nabla_X Y$ for some vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$, independently of our interpretation of the symbol ∇, the result is the same thanks to expression (1.13). If instead, we see $\nabla_X T$ for some tensor T which is not a vector field, we know that ∇ should be interpreted as the covariant derivative, as $(X, T) \notin \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M})$.

Let us consider a chart (U, ϕ), together with $X, Y \in \mathfrak{X}(\mathcal{M})$, $f \in \mathcal{F}(\mathcal{M})$, $\sigma \in \Lambda^1(\mathcal{M})$ and $T \in \mathcal{T}^k_m(\mathcal{M})$, where $k, m \in \mathbb{N}$, such that

$$X = X^\mu \frac{\partial}{\partial \phi^\mu} \tag{1.17}$$

$$Y = Y^\mu \frac{\partial}{\partial \phi^\mu} \tag{1.18}$$

$$\sigma = \sigma_\mu d\phi^\mu \tag{1.19}$$

and

$$T = T^{\nu_1 \dots \nu_k}_{\mu_1 \dots \mu_m} \frac{\partial}{\partial \phi^{\nu_1}} \otimes \dots \otimes \frac{\partial}{\partial \phi^{\nu_k}} \otimes d\phi^{\mu_1} \otimes \dots \otimes d\phi^{\mu_m}. \tag{1.20}$$

We can write the components of the covariant derivatives of the previous tensors, with respect to the chart (U, ϕ), as

$$\nabla_X f = X^\mu \frac{\partial f}{\partial \phi^\mu} \tag{1.21}$$

$$\left(\nabla_X Y\right)^\mu = X^\alpha \frac{\partial Y^\mu}{\partial \phi^\alpha} + X^\alpha Y^\beta \Gamma^\mu_{\beta\alpha} \tag{1.22}$$

$$\left(\nabla_X \sigma\right)_\mu = X^\alpha \frac{\partial \sigma_\mu}{\partial \phi^\alpha} - \sigma_\alpha X^\beta \Gamma^\alpha_{\mu\beta} \tag{1.23}$$

$$\begin{aligned}\left(\nabla_X T\right)^{\nu_1 \dots \nu_k}_{\mu_1 \dots \mu_m} = X^\alpha &\left(\frac{\partial}{\partial \phi^\alpha} T^{\nu_1 \dots \nu_k}_{\mu_1 \dots \mu_m} + \Gamma^{\nu_1}_{\beta\alpha} T^{\beta \nu_2 \dots \nu_k}_{\mu_1 \dots \mu_m} + \dots + \Gamma^{\nu_k}_{\beta\alpha} T^{\nu_1 \dots \nu_{k-1} \beta}_{\mu_1 \dots \mu_m} \right) \\ &- X^\alpha \left(\Gamma^\beta_{\mu_1 \alpha} T^{\nu_1 \dots \nu_k}_{\beta \mu_2 \dots \mu_m} + \dots + \Gamma^\beta_{\mu_m \alpha} T^{\nu_1 \dots \nu_k}_{\mu_1 \dots \mu_{m-1} \beta} \right).\end{aligned} \tag{1.24}$$

1.1.4 Tensor field space isomorphisms

Here, we introduce some canonical maps, between tensor field spaces of Riemannian manifolds, that prove to be very useful in algebraic manipulation of equations.

Let $\mathcal{M}$ be a d-dimensional Riemannian manifold and g the respective metric tensor field. We define the map

$$\begin{aligned} l\colon \mathfrak{X}(\mathcal{M}) &\to \Lambda(\mathcal{M}) \\ X &\mapsto \sigma \end{aligned}$$

such that

$$\sigma(Y) = g(X, Y) \tag{1.25}$$

for all $Y \in \mathfrak{X}(\mathcal{M})$. It can be proven that l is a $\mathcal{F}(\mathcal{M})$-linear isomorphism [41]. We can use l to extend this relation to general tensors. Indeed, let $k, m \in \mathbb{N}$, $a \leq k$ and $b \leq m$, we can define

$$\begin{aligned} L_a\colon \mathcal{T}^k_m(\mathcal{M}) &\to \mathcal{T}^{k-1}_{m+1}(\mathcal{M}) \\ T &\mapsto L_a(T) \end{aligned}$$

$$\begin{aligned} R_b\colon \mathcal{T}^k_m(\mathcal{M}) &\to \mathcal{T}^{k+1}_{m-1}(\mathcal{M}) \\ T &\mapsto R_b(T) \end{aligned}$$

where

$$L_a(T)(X_1, \ldots, X_{m+1}, \sigma_1, \ldots, \sigma_{k-1}) = T(X_1, \ldots, X_m, \sigma_1, \ldots, l(X_{m+1}), \ldots, \sigma_{k-1}) \tag{1.26}$$

$$R_b(T)(X_1, \ldots, X_{m-1}, \sigma_1, \ldots, \sigma_{k+1}) = T(X_1, \ldots, l^{-1}(\sigma_{k+1}), \ldots, X_{m-1}, \sigma_1, \ldots, \sigma_k) \tag{1.27}$$

for all $X_1, \ldots, X_{m+1} \in \mathfrak{X}(\mathcal{M})$, $\sigma_1, \ldots, \sigma_{k+1} \in \Lambda^1(\mathcal{M})$ and where $l(X_{m+1})$ and $l^{-1}(\sigma_{k+1})$ are in the positions a and b respectively. It can be proven that both maps L_a and R_b are $\mathcal{F}(\mathcal{M})$-linear isomorphisms [41]. The action of these maps in tensor fields is usually called the lowering and raising of indexes.

Finally, we introduce the metric contraction as an immediate consequence of the previous isomorphisms. Let $T \in \mathcal{T}^k_m(\mathcal{M})$ with $k, m \in \mathbb{N}$ and $1 \leq b < a \leq m$, we can define the map

$$\begin{aligned} C_{ab}\colon \mathcal{T}^k_m(\mathcal{M}) &\to \mathcal{T}^k_{m-2}(\mathcal{M}) \\ T &\mapsto C_{ab}(T) \end{aligned}$$

where $C_{ab}(T)$ is the contraction [5] over $k+1$ and b of the tensor $R_a(T)$.

1.1.5 Relevant tensor fields

Here, we introduce some of the tensor fields present in the Einstein field equations. From now on, we always assume that we are working with a d-dimensional Lorentzian manifold $\mathcal{M}$, equipped with the Levi-Civita connection ∇.

We consider the tensor field $\mathcal{R} \in \mathcal{T}^1_3(\mathcal{M})$, defined by

$$\mathcal{R}_{X,Y} Z = \nabla_X \circ \nabla_Y Z - \nabla_Y \circ \nabla_X Z - \nabla_{[X,Y]} Z \tag{1.28}$$

for all $X, Y, Z \in \mathfrak{X}(\mathcal{M})$. This tensor field is called the Riemannian curvature tensor field and, expressed as above, measures the displacement of some vector, parallel transported along infinitesimal closed loops in $\mathcal{M}$ [12].

We define the Ricci curvature tensor field $\mathrm{Ric} \in \mathcal{T}^0_2(\mathcal{M})$, as the contraction of the Riemannian tensor field over 1 and 3.

Finally, we introduce the scalar curvature $\mathcal{S} \in \mathcal{F}(\mathcal{M})$, as the metric contraction of the Ricci curvature tensor field.

[5] For a definition of contraction, we direct the reader to [41].

Let (U, ϕ) be a chart of $\mathcal{M}$. We can write the components of the tensor fields, introduced above, as

$$\mathcal{R}^{\rho}_{\sigma\mu\nu} = \frac{\partial \Gamma^{\rho}_{\nu\sigma}}{\partial \phi^{\mu}} - \frac{\partial \Gamma^{\rho}_{\mu\sigma}}{\partial \phi^{\nu}} + \left(\Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}\right) \tag{1.29}$$

$$\mathrm{Ric}_{\mu\nu} = \mathcal{R}^{\lambda}_{\mu\nu\lambda} = \left(\frac{\partial \Gamma^{\lambda}_{\lambda\mu}}{\partial \phi^{\nu}} - \frac{\partial \Gamma^{\lambda}_{\nu\mu}}{\partial \phi^{\lambda}} + \left(\Gamma^{\lambda}_{\nu\delta}\Gamma^{\delta}_{\lambda\mu} - \Gamma^{\lambda}_{\lambda\delta}\Gamma^{\delta}_{\nu\mu}\right)\right) \tag{1.30}$$

$$\mathcal{S} = g^{\alpha\beta}\mathrm{Ric}_{\beta\alpha}. \tag{1.31}$$

With this, we end our brief introduction to the elements of pseudo-Riemannian geometry, necessary to grasp the Einstein field equations (1.1).

1.1.6 Stress-Energy tensor field and the cosmological constant Λ

Here, we introduce the physical elements in the Einstein field equations.

We start with the stress energy tensor field $T \in \mathcal{T}^0_2(\mathcal{M})$. This tensor field can be seen as the physical input in the Einstein field equations and seeks to describe the density and flux of energy and momentum in every point of our space time, formally identified with the Lorentzian manifold $\mathcal{M}$. We note that this tensor is symmetric. Moreover, the usual conservation laws, applied in the context of general relativity, force the divergence of this tensor field to vanish everywhere on $\mathcal{M}$ [6].

Finally, we introduce the cosmological constant Λ. The cosmological constant $\Lambda \in \mathbb{R}$ was first introduced by Einstein, in an attempt to obtain solutions corresponding to static cosmological models of space time. It was only later that Edwin Hubble gave us evidence that the universe is expanding, making the idea of non static cosmological models of space time more plausible. Never the less, the cosmological constant kept being an interesting topic of theoretical and empirical study. Multiplied by the metric tensor field, the cosmological constant can be physically interpreted as the density and flux of energy and momentum of empty space time ($T = 0$). There is a lot that can be said about this constant. However, since we will not be doing any work related to it, we end the discussion here.

Finally, the constant G, in the Einstein field equations, is called the gravitational constant and arises as a matching requirement between general relativity and Newtonian gravity [12]. This constant depends on $\dim(\mathcal{M})$ [57].

1.1.7 Einstein-Hilbert action

Now, it would be relevant to pose the following question: Given a d-dimensional Lorentzian manifold $\mathcal{M}$, can the Einstein field equations (1.1) arise naturally from a Lagrangian formulation. For $T = 0$, the answer is yes and the corresponding d-dimensional action, also called Einstein-Hilbert action, can be locally described, in some chart (U, ϕ), as

$$\frac{1}{2k}\int_U (\mathcal{S} - 2\Lambda)\sqrt{-\det(g)}\, d\phi \tag{1.32}$$

where k denotes the Einstein gravitational constant

$$k = \frac{8\pi G}{c^4} \tag{1.33}$$

and

$$d\phi := \bigwedge_{\mu=1}^{d} d\phi^{\mu}. \tag{1.34}$$

[6]We note that we did not define the divergence, in the context of pseudo-Riemannian manifolds. However, the definition, as one would expect, employs the connection ∇ and can be easily understood after reading the previous discussions [41].

Naturally, the action should be global and hence integrate over the entire space time $\mathcal{M}$, regardless of convergence properties. Indeed, divergence of the action does not pose a problem if we only consider compact variations of the metric tensor field [47].

To derive the Einstein field equations, in the presence of a non null stress energy tensor field, one needs to add an extra term to the action (1.32). When considering fields coupled to gravity, this term often takes the form of some Lagrangian density, integrated over the same volume form, used in (1.32).

1.2 Black hole space times

Here, we discuss two black hole solutions of the Einstein field equations (1.1). The first one, called Schwarzschild black hole, is the simplest black hole solution of these equations and for that reason it is a great point to start talking about black holes in general. The second one, called the d-dimensional Tangherlini black hole, can be seen as a generalisation of the latter space time for arbitrary spacial dimensions.

1.2.1 The Schwarzschild black hole space time

The Schwarzschild black hole is a 4-dimensional Lorentzian manifold, whose metric tensor field is a solution of the Einstein field equations in the vacuum.

Physically, this manifold describes empty space in the surroundings of an uncharged spherical mass, under the assumption that the cosmological constant Λ is null. Mathematically, this means the associated Lorentzian manifold is spherically symmetric, static and asymptotically flat [7]. The topological structure of the manifold is

$$\mathcal{M} = \mathbb{R} \times \left\{ \mathbb{E}^3 - \mathcal{O} \right\} \simeq \mathbb{R} \times]0, +\infty[\times S^2 \tag{1.35}$$

where $\mathcal{O}$ denotes the origin of the 3-dimensional Euclidean space $\mathbb{E}^3$. The corresponding Lorentzian metric tensor field, expressed with respect to the topological structure above, is

$$g = -\left(1 - \frac{2Gm}{c^2 r}\right) dt \otimes dt + \left(1 - \frac{2Gm}{c^2 r}\right)^{-1} dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2(\theta) d\phi \otimes d\phi. \tag{1.36}$$

where m is the mass of the spherical object. At first glance, it appears the above tensor field has a singularity at

$$R_h \coloneqq \frac{2Gm}{c^2}. \tag{1.37}$$

However, it can be shown that this is not really a singularity [12]. Instead, it is simply a limitation of the charts used in the above description of the metric tensor field. In fact, a true singularity on the metric tensor field would violate the very definition of a Lorentzian manifold.

Although not associated with a singularity of the metric tensor field, the radius (1.37), also called event horizon radius, is still very interesting for different reasons. Indeed, inside the spherical surface defined by the event horizon radius, the geodesic curves defined by trajectories of free falling particles always approach the origin $\mathcal{O}$, as their respective parameter increases. In general relativity, the parameter associated with geodesics, describing the motion of free falling particles through space time, is interpreted as the time measured by an observer, following the trajectory. Thus, we can physically interpret the subset

$$]0, R_h[\times S^2 \tag{1.38}$$

as the spacial region of no return.

[7] A d-dimensional Lorentzian manifold is spherically symmetric if there are d-1 Killing vector fields [12], whose flows generate a group isomorphic to $\mathrm{SO}(d-1)$. If there exists some non vanishing, time-like and irrotational Killing vector field, we say the manifold is static [12]. Finally, a Lorentzian manifold is asymptotically flat if the respective metric tensor field behaves as the Minkowski space time metric tensor field, far away from the origin of the space time (where the physical object is).

Now, we can ask if this space time can be generalised to arbitrary spacial dimensions. Naturally, this is not just a question of academic interest. In fact, there are several theories, like string theories, predicting more than the usual three spacial dimensions, we are familiar with. The answer to the previous question is yes and the corresponding space time is called the d-dimensional Tangherlini black hole space time.

1.2.2 The d-dimensional Tangherlini black hole space time

The d-dimensional Tangherlini black hole space time is a d-dimensional Lorentzian manifold whose metric tensor field is a solution of the Einstein field equations in the vacuum.

Physically, this manifold describes empty $(d$-1)-dimensional space, in the surroundings of an uncharged hyperspherical mass, under the assumption that the cosmological constant Λ is null. The topological structure of this space time is

$$\mathcal{M} = \mathbb{R} \times \left\{ \mathbb{E}^{d-1} - \mathcal{O} \right\} \simeq \mathbb{R} \times]0, +\infty[\times S^{d-2}. \tag{1.39}$$

Using charts with respect to the topological structure above, the d-dimensional Tangherlini black hole metric tensor field takes the form

$$ds^2 = -\left(1 - \frac{\mu}{r^{d-3}}\right) dt \otimes dt + \left(1 - \frac{\mu}{r^{d-3}}\right)^{-1} dr \otimes dr + r^2 d^2\Omega_{d-2} \tag{1.40}$$

where μ is a mass parameter, defined as

$$\mu = \frac{16\pi G m}{(d-2)\Omega_{d-2} c^4} \tag{1.41}$$

where m represents the mass of our hyperspherical object and Ω_{d-2} stands for the surface area of the unit (d-2)-sphere

$$\Omega_{d-2} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}. \tag{1.42}$$

Finally, $d^2\Omega_{d-2}$ stands for the canonical metric tensor field of the unit $(d$-2)-sphere. This tensor field is defined as the pullback, by the inclusion function, of the canonical metric tensor field in $\mathbb{E}^{d-1}$. More explicitly, we have [18]

$$d^2\Omega_{d-2} = d\phi^1 \otimes d\phi^1 + \sum_{\mu=2}^{d-2} \left(\prod_{\nu=1}^{\mu-1} \sin^2(\phi^\nu) \right) d\phi^\mu \otimes d\phi^\mu. \tag{1.43}$$

Just like the Schwarzschild black hole space time, the d-dimensional Tangherlini black hole space time is a static, asymptotically flat and spherically symmetric Lorentzian manifold. Furthermore, it is particularly easy to see that taking $d = 4$ yields the original Schwarzschild black hole space time.

1.3 Quasinormal modes

Quasinormal modes are some of the most fascinating topics in general relativity. They are objects of extreme physical relevance due to numerous applications in many areas of physics [40, 8], some of which do not relate, in a obvious way, to general relativity [1]. Here, we do a reasonably good introduction to quasinormal modes, both physically and mathematically. Furthermore, we talk about some of the applications mentioned above.

1.3.1 Linear perturbations of fields coupled to gravity

Quasinormal modes appear naturally, in the context of general relativity, when studying linear perturbations of fields coupled to gravity in black hole space times. Forcing these perturbations to obey the least action principle yields a set of partial differential equations. These equations encode information related to the symmetries of the background space time [8]. As a consequence, it is often possible to simplify them by separation of variables. Here, one of the key features is the choice of appropriate physical boundary conditions. Indeed, black hole space times are intrinsically dissipative, hence dissipative boundary conditions are needed. Applying these boundary conditions to our system of partial differential equations often yields something very close to an eigenvalue problem. The respective eigenvectors are the quasinormal modes.

To make more clear the role of quasinormal modes, as explained above, we workout an example. For simplicity purposes, we consider a complex massless scalar field [9]

$$\Psi\colon \mathcal{M} \to \mathbb{C}$$

coupled to gravity, in a Schwarzschild black hole space time. The action can be locally expressed, for a given chart (U, ϕ), as

$$\frac{1}{2k}\int_U \sqrt{-\det(g)}\mathcal{S}\, d\phi + \int_U \sqrt{-\det(g)}\mathcal{L}\, d\phi \tag{1.44}$$

where

$$\mathcal{L} = -g^{\mu\nu}\left(\frac{\partial \Psi}{\partial \phi^\mu}\right)^*\left(\frac{\partial \Psi}{\partial \phi^\nu}\right) \in \mathcal{F}(\mathcal{M}) \tag{1.45}$$

is the Lagrangian density function of Ψ. Applying the least action principle to (1.44) yields a set of Euler-Lagrange equations for the scalar field and metric tensor field components [8]. These equations read

$$g^{\mu\nu}\left(\frac{\partial^2 \Psi}{\partial \phi^\mu \partial \phi^\nu} - \Gamma^\alpha_{\mu\nu}\frac{\partial \Psi}{\partial \phi^\alpha}\right) = 0 \tag{1.46}$$

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2}\mathcal{S}g_{\mu\nu} = \frac{8\pi}{c^4}GT_{\mu\nu} \tag{1.47}$$

where [12]

$$T_{\mu\nu} = \left(\frac{\partial \Psi}{\partial \phi^\mu}\right)^*\left(\frac{\partial \Psi}{\partial \phi^\nu}\right) - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\left(\frac{\partial \Psi}{\partial \phi^\alpha}\right)^*\left(\frac{\partial \Psi}{\partial \phi^\beta}\right). \tag{1.48}$$

Now, we consider the linear perturbations

$$g_{\mu\nu} = (g_S)_{\mu\nu} + \alpha h_{\mu\nu} \tag{1.49}$$

$$\Psi = \psi_S + \beta\psi \tag{1.50}$$

for some $\alpha, \beta \ll 1$. In the expression above, we denoted the Schwarzschild metric tensor field as g_S. Analogously, we denoted the unperturbed scalar field as ψ_S. We know that

$$\mathrm{Ric}_{\mu\nu}(g_S) - \frac{1}{2}\mathcal{S}\,(g_S)_{\mu\nu} = 0. \tag{1.51}$$

We also set

$$\psi_S = 0. \tag{1.52}$$

Using the Voss-Weyl formula [24] and discarding terms of orders greater than one in the perturbation parameters, we can rewrite equation (1.46) as

$$\frac{1}{\sqrt{-\det(g_S)}}\frac{\partial}{\partial \phi^\mu}\left(\sqrt{-\det(g_S)}(g_S)^{\mu\nu}\frac{\partial \psi}{\partial \phi^\nu}\right) = 0. \tag{1.53}$$

Using charts, associated with the Schwarzschild black hole topological structure (1.35), we notice the scalar perturbation ψ can be decomposed as

[8]Here, the precise meaning of symmetries is directly related to the Killing vector fields of the background space time in question.

[9]Although we do not state any conditions on the differentiability of Ψ, it is often assumed Ψ to be continuously differentiable, up to some order.

$$\psi(t,r,\theta,\phi) = \sum_{\omega\in\mathcal{Q}} \sum_{l=0}^{+\infty} \sum_{m=-l}^{l} \frac{e^{i\omega t}}{r} \psi_r(r) Y_{lm}(\theta,\phi) \tag{1.54}$$

where $\mathcal{Q}$ is a countable subset of $\mathbb{C}$. Indeed, rewriting equation (1.53), with respect to these charts, yields

$$\frac{\partial\psi}{\partial r} f' - \frac{1}{f}\frac{\partial^2\psi}{\partial t^2} + \frac{2f}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial r^2} f + \frac{\partial\psi}{\partial\theta}\frac{\cot(\theta)}{r^2} + \frac{\partial^2\psi}{\partial\phi^2}\frac{\csc^2(\theta)}{r^2} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0 \tag{1.55}$$

where we defined

$$f(r) \coloneqq 1 - \frac{2Gm}{c^2 r}. \tag{1.56}$$

Using the defining property of the Spherical harmonic functions

$$\frac{\partial Y_{lm}}{\partial\theta}\cot(\theta) + \frac{\partial^2 Y_{lm}}{\partial\phi^2}\csc^2(\theta) + \frac{\partial^2 Y_{lm}}{\partial\theta^2} + l(l+1)Y_{lm} = 0 \tag{1.57}$$

we can replace ansatz (1.54) in equation (1.55) yielding

$$f'\frac{d\psi_r}{dr}\frac{1}{r} - f'\frac{\psi_r}{r^2} + f\left(\frac{d^2\psi_r}{dr^2}\frac{1}{r} - 2\frac{d\psi_r}{dr}\frac{1}{r^2} + 2\frac{\psi_r}{r^3}\right) + \frac{2f}{r}\left(\frac{d\psi_r}{dr}\frac{1}{r} - \frac{\psi_r}{r^2}\right) - l(l+1)\frac{\psi_r}{r^3} + \frac{\omega^2}{f}\frac{\psi_r}{r} = 0. \tag{1.58}$$

Rearranging the terms above, we get

$$\frac{d^2\psi_r}{dr^2} f^2 + \frac{d\psi_r}{dr} f' f + \left(\omega^2 - V\right)\psi_r = 0 \tag{1.59}$$

where the apostrophe stands for derivation with respect to r and

$$V \coloneqq f\left(\frac{l(l+1)}{r^2} + \frac{f'}{r}\right). \tag{1.60}$$

Finally, we can rewrite equation (1.59) with respect to the tortoise coordinate [12]. Up to an integration constant, this coordinate can be defined as

$$dx \coloneqq \frac{dr}{f}. \tag{1.61}$$

With respect to x, the differential equation (1.59) takes the Schrödinger like form

$$\frac{d^2\psi_r}{dr^2} + (\omega^2 - V)\psi_r = 0. \tag{1.62}$$

We see that our ansatz is a solution of the equations of motion if the above equation holds true. This Schrödinger like equation is usually called the master equation of the linear perturbation ψ.

Now, we concern ourselves with boundary conditions associated to (1.62). We are interested in a solution of the scalar field perturbation ψ, propagating in the spatial region outside of the black hole. That is to say, we only seek a solution for values of r in the interval $]R_h, +\infty[$. The tortoise coordinate x, defined as we did, takes values in $\mathbb{R}$ and has the following asymptotic behaviours

$$r \to +\infty \implies x \to +\infty \tag{1.63}$$

$$r \to R_h^+ \implies x \to -\infty. \tag{1.64}$$

It is also easy to see that

$$\lim_{r\to+\infty} V(r) = 0 \tag{1.65}$$

$$\lim_{r\to R_h^+} V(r) = 0. \tag{1.66}$$

From the limits above, we gather that, in the boundaries of the region we are interested in, the master equation (1.62) becomes

$$\frac{d^2\psi_r}{dx^2} + \omega^2\psi_r = 0 \tag{1.67}$$

whose general solution is

$$\psi_r(x) = A_+ e^{i\omega x} + A_- e^{-i\omega x} \tag{1.68}$$

for some $A_\pm \in \mathbb{C}$. Because nothing should leave the spherical surface of the event horizon, we impose the boundary condition

$$\psi_r(x) \propto e^{i\omega x} \tag{1.69}$$

for $x \to -\infty$. On the other hand, as we are describing an isolated system, we want to disregard unphysical waves coming from spacial infinity. Thus, we impose the boundary condition

$$\psi_r(x) \propto e^{-i\omega x} \tag{1.70}$$

for $x \to +\infty$.

Quasinormal modes are defined as the solutions of equation (1.62), obeying the boundary conditions above for some associated quasinormal frequency $\omega \in \mathcal{Q}$.

The example above filled the role of a physical motivation for the definition of quasinormal modes and frequencies. In what follows, we redefine these in the context of general dissipative systems, using a more rigorous approach. Doing so, we formally identify the contribution of these modes to the perturbation function ψ.

1.3.2 Dissipative systems and quasinormal modes

Before properly discussing quasinormal modes, it is somewhat useful to remember what normal modes are. In order to do this, let us study the simple example of an infinite compact 1-dimensional well in quantum theory. The Hamiltonian operator, describing this physical system, takes the form

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \tag{1.71}$$

where $m \in \mathbb{R}^+$ is the mass of the particle trapped in the well and

$$V(x) := \begin{cases} 0\, ; x \in]0, L[\\ +\infty\, ;\ x \notin]0, L[\end{cases} \tag{1.72}$$

with $L \in \mathbb{R}^+$. The Hilbert space where the Hamiltonian operator $\hat{H}$ is densely defined [10] is $L^2([0, L])$. The appropriate physical boundary conditions are

$$\phi(-L) = \phi(L) = 0. \tag{1.73}$$

Using the boundary conditions above, the general solution of the Schrödinger equation

$$\hat{H}\phi = i\hbar\frac{d\phi}{dt} \tag{1.74}$$

is

$$\phi(x,t) = \sqrt{\frac{2}{L}}\sum_{n=1}^{+\infty} A_n \sin(k_n x)\, e^{-i\omega_n t} \tag{1.75}$$

where $A_n \in \mathbb{C}$ and

$$\sum_{n=1}^{+\infty} |A_n|^2 < +\infty \tag{1.76}$$

$$\omega_n := \frac{n^2\pi^2\hbar^2}{2mL^2} \qquad k_n := \frac{n\pi}{L}. \tag{1.77}$$

[10] Generally, linear operators in quantum theory are unbounded. Unbounded operators are not self-adjoint if defined in the entire Hilbert space. Therefore, linear operators in quantum theory are often defined in dense subsets of the respective Hilbert space.

We define normal modes as the elements of the set

$$\mathcal{N} := \left\{ \sqrt{\frac{2}{L}} \sin(k_n x) \;\middle|\; n \in \mathbb{N} \right\} \subset L^2\left([0, L]\right). \tag{1.78}$$

It can be shown that $\mathcal{N}$ is a maximal orthonormal subset of $L^2\left([0, L]\right)$ [48]. We call ω_n the normal frequencies of the system and they can be formally identified with the eigenvalue spectrum of the Hamiltonian operator. Indeed, we know that

$$\hat{H}\phi_n = \omega_n \phi_n \tag{1.79}$$

where we defined the eigenvectors

$$\phi_n(x) := \sqrt{\frac{2}{L}} \sin(k_n x). \tag{1.80}$$

As the eigenvalue spectrum of $\hat{H}$ is real, while intricate, the amplitude of ϕ does not have an overall decreasing behaviour with time. Hence, it is not sensible to say this problem models a physical dissipative system.

Now, let us turn our attention to physical systems, described by differential equations whose spatial variables are defined on non compact subsets of $\mathbb{R}^n$. As an example, let us picture a general 1-dimensional physical system, defined by the wave equation

$$\frac{\partial^2 \psi}{\partial t^2} + \left(V(x) - \frac{\partial^2}{\partial x^2} \right) \psi = 0 \tag{1.81}$$

where V is a continuous positive function of compact support in $\mathbb{R}$, meaning that

$$V(x) = 0 \tag{1.82}$$

for all $x \in \mathbb{R}$ such that $|x| > L$ with $L \in \mathbb{R}^+$. Let us choose general initial data of compact support

$$\psi(x, 0) = \Gamma(x) \tag{1.83}$$

$$\frac{\partial \psi}{\partial t}(x, 0) = \Omega(x) \tag{1.84}$$

for some $\Gamma \in C^2(\mathbb{R})$ and $\Omega \in C^1(\mathbb{R})$.

Now, we consider the unique solution ψ of the differential equation (1.81), with initial data as above. The Laplace transform of this solution takes the form

$$\mathcal{L}\left(\psi\right)(x, s) = \int_0^{+\infty} e^{-st} \psi(x, t) dt. \tag{1.85}$$

The differential equation satisfied by $\mathcal{L}(\psi)$ is

$$s^2 \mathcal{L}(\psi) - \frac{\partial^2 \mathcal{L}}{\partial x^2}(\psi) + V\mathcal{L}(\psi) = s\Gamma + \Omega. \tag{1.86}$$

Because we are working with initial data of compact support, we know ψ is bounded [31]. Hence, the Laplace transform $\mathcal{L}(\psi)$ is analytic for $s \in]0, +\infty[$ and admits an analytic continuation onto the complex half plane $\Re(s) > 0$ [31].

Now, let us write the Green function of the differential equation (1.86). Such function, takes the well known form [52]

$$G(s, x, x') = \begin{cases} \dfrac{f_-(s, x') f_+(s, x)}{W(s)} & ;\ x' \leq x \\ \dfrac{f_-(s, x) f_+(s, x')}{W(s)} & ;\ x' > x \end{cases} \tag{1.87}$$

where f_- and f_+ are two linearly independent solutions of the homogeneous differential equation, associated with (1.86), and W denotes the respective Wronskian. In order to simplify the notation, we define the function

$$\Lambda := s\Gamma + \Omega. \tag{1.88}$$

Using the previous definition, we can write the particular solution of the differential equation (1.86) as

$$\mathcal{L}(\psi)(s,x) = \int_{-\infty}^{+\infty} G(s,x,x')\Lambda(s,x')dx'. \tag{1.89}$$

Considering the Laplace transform definition, we know $\mathcal{L}(\psi)$ is bounded as a function of x. Indeed, since ψ is bounded, we know that $|\psi|$ has a majorant $M \in \mathbb{R}^+$. Thus, for some s in the complex half plane $\Re(s) > 0$, we have

$$|\mathcal{L}(\psi)| = \left|\int_0^{+\infty} e^{-st}\psi(x,t)dt\right| \leq \int_0^{+\infty} \left|e^{-st}\right| |\psi(x,t)| \, dt \leq M \int_0^{+\infty} e^{-\Re(s)t}dt = \frac{M}{\Re(s)}. \tag{1.90}$$

Moreover, for a fixed s in the complex half plane $\Re(s) > 0$ and for $|x| > L$, the general solution of the homogeneous differential equation, associated with (1.86), is a linear combination of the functions $e^{\pm xs}$. Since $\mathcal{L}(\psi)$ is bounded as a function of x, we are forced to choose a Green function G such that

$$f_+(s,x) \propto e^{-sx} \ ; \ x \geq L \tag{1.91}$$

$$f_-(s,x) \propto e^{sx} \ ; \ x \leq -L. \tag{1.92}$$

The above conditions uniquely define the Green function of (1.86) and the respective particular solution through (1.89).

It is within this context that quasinormal frequencies emerge. Quasinormal frequencies are defined as the complex numbers $s_n \in \mathbb{C}$ such that

$$f_+(s_n,x) = c(s_n)f_-(s_n,x) \tag{1.93}$$

for all $x \in \mathbb{R}$, where $c(s_n) \in \mathbb{C}$. For these values of s, the solutions f_+ and f_- become linearly dependent, with respect to x. When this happens, the Wronskian W vanishes and the chosen Green function becomes singular. The corresponding functions $f_-(s_n,x)$ and $f_+(s_n,x)$ are called quasinormal modes. We did not prove that such frequencies exist, but in fact they do. In [6], for the conditions we are considering, a countable family of such frequencies, in the complex half plane $\Re(s) < 0$, was proven to exist. We know $\mathcal{L}(\psi)$ admits an analytic continuation onto the complex half plane $\Re(s) > 0$. Thus, there should not exist any quasinormal frequency with positive real component. However, as solutions of the homogeneous differential equation associated with (1.86), the functions f_+ and f_- have an unique analytic continuation to the complex s-plane [31].

The quasinormal frequencies and quasinormal modes come into play when we reconsider the solution to the original differential equation (1.81). Indeed, we can use Mellin's inverse formula [17] to write ψ as

$$\psi(x,t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}\mathcal{L}(\psi)(s,x)ds = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}\int_{-\infty}^{+\infty} G(s,x,x')\Lambda(s,x')dx'ds \tag{1.94}$$

for some $a > 0$. Now, we can compute the integral over s, using the contour depicted in figure (1.1). To do this, we define

$$\mathcal{F}(s,x) := \frac{e^{st}}{2\pi i}\int_{-\infty}^{+\infty} G(s,x,x')\Lambda(s,x')dx' \tag{1.95}$$

and rewrite (1.94) as

$$\psi(x,t) = \int_{a-i\infty}^{a+i\infty} \mathcal{F}(s,x)ds. \tag{1.96}$$

Because we chose initial data of compact support, we know Λ is compactly supported on x' as well. Therefore, there can be no singularities of $\mathcal{F}$, arising from the integration. On the other hand, we

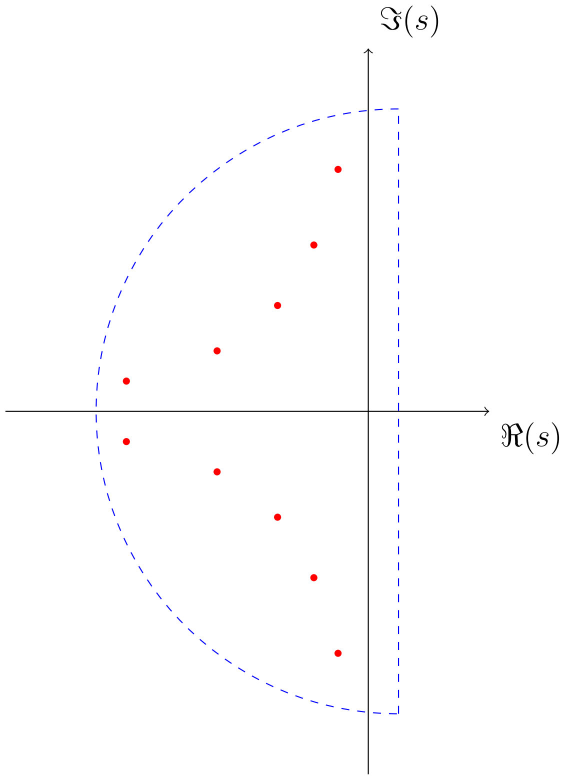

Figure 1.1: Pictorial representation of the closed contour needed to compute the integral over s in (1.94) as the blue dashed line. Furthermore, possible quasinormal frequencies are depicted by conjugated red dots. These conjugation pairs arise from conjugation of (1.93).

know G is singular for quasinormal frequencies s_n. As our contour encloses these frequencies, the residue theorem [3] yields

$$\oint_{\mathcal{D}} \mathcal{F}(s,x)ds = \int_{a-i\infty}^{a+i\infty} \mathcal{F}(s,x)ds + \int_{C} \mathcal{F}(s,x)ds = 2\pi i \sum_{n\in I} \mathrm{Res}\left(\mathcal{F}, s_n\right)(x) \tag{1.97}$$

where we denoted the blue contour and the associated arc shaped portion by $\mathcal{D}$ and $\mathcal{C}$ respectively. Furthermore, we denoted the countable index set of quasinormal frequencies by I. Rewriting the equation above, using (1.96), yields

$$\psi(x,t) = 2\pi i \sum_{n\in I} Res\left(\mathcal{F}, s_n\right)(x) - \int_{C} \mathcal{F}(s,x)ds. \tag{1.98}$$

We arrived at an expression relating ψ to the quasinormal frequencies.

The second contribution to ψ, in the equation above, is due to the contour integration along $\mathcal{C}$. We notice that $\Re(s) \ll -1$ in most of $\mathcal{C}$. Thus, the exponential factor in the definition of $\mathcal{F}$ is bound to decay rapidly as t increases and so is the contribution as a whole [15]. This renders the contribution negligible for the late time behaviour of the system. Hence, we end up with the asymptotic behaviour

$$\psi(x,t) \sim 2\pi i \sum_{n\in I} \mathrm{Res}\left(\mathcal{F}, s_n\right) \tag{1.99}$$

in this limit. Finally, the expression above can be recast in the form

$$\psi\left(x,t\right) \sim \sum_{n\in I} C_n e^{s_n t} f_+\left(s_n, x\right) \tag{1.100}$$

for finite values of x and for some $C_n \in \mathbb{C}$ [31]. The asymptotic behaviour above should be compared with equation (1.75), evidencing the similarities between normal modes and quasinormal modes.

As we can see from (1.100), the unique solution of our physical problem will decay exponentially fast in time, for spatially bounded regions. In this sense, equations such as (1.81), defined on non compact spatial domains, model physical dissipative systems. Solutions to these systems, as we just saw, cannot be fully described by normal modes. Instead, in the late time behaviour of the system, they can be well approximated by quasinormal modes. In this way, they play the analogous role of normal modes in systems of compact spatial domain. A major distinction between the two sides is that normal frequencies are real while quasinormal frequencies may be complex!

1.3.3 Dissipative systems with non compactly supported potentials

Potentials associated with master equations, describing linear perturbations of fields coupled to gravity in a black hole space time, often converge to zero for $x \to \pm\infty$. However, such potentials are usually not compactly supported, as we can see just by looking at (1.60). This changes slightly the analysis we did in the previous example. It is still true that initial data of compact support yields a bounded solution ψ [31]. Hence, we can proceed as before. It is also true that solutions f_+, f_- of the homogeneous equation, associated with (1.86), such that

$$f_+(s,x) \propto e^{-sx} \ ; \ x \to +\infty \tag{1.101}$$

$$f_-(s,x) \propto e^{sx} \ ; \ x \to -\infty \tag{1.102}$$

still do exist for $s > 0$. Furthermore, they define uniquely the correct Green function G and the respective particular solution to (1.86). However, it is no longer true that the only singularities in s, arising from the Green function G, correspond to the quasinormal frequencies. Indeed, f_+ and f_- may also have singularities in the complex s-plane. To see this, let us consider an example.

Let V be a positive function with the asymptotic expansion

$$V(x) \propto e^{-x\lambda} \tag{1.103}$$

for $x \gg 1$, where $\lambda \in \mathbb{R}^+$. Knowing that f_+ obeys the homogeneous equation associated with (1.86), allow us to express it as a solution of an integral equation. Indeed, for some fixed $s > 0$, we can write

$$f_+(s,x) = e^{-sx} - \int_x^{+\infty} \frac{\sinh\left(s(x-y)\right)}{s} V(y) f_+(s,y) dy. \tag{1.104}$$

Taking the Born approximation [23] yields the asymptotic expansion

$$f_+(s,x) \sim e^{-sx} - \int_x^{+\infty} \frac{\sinh\left(s(x-y)\right)}{s} V(y) e^{-sy} dy \tag{1.105}$$

for $x \gg 1$. Using the asymptotic expansion (1.103), we can write

$$\int_x^{+\infty} \frac{\sinh\left(s(x-y)\right)}{s} V(y) e^{-sy} dy \propto \int_x^{+\infty} \frac{\sinh(s(x-y))}{s} e^{-y(s+\lambda)} dy. \tag{1.106}$$

for $x \gg 1$. Computing the integral above yields

$$\int_x^{+\infty} \frac{\sinh(s(x-y))}{s} e^{-y(s+\lambda)} dy = \frac{e^{-x(\lambda+s)}}{(2s+\lambda)\lambda}. \tag{1.107}$$

As we can see, the analytic continuation of f_+ to the complex s-plane has a simple pole in

$$s = -\frac{\lambda}{2} \tag{1.108}$$

for $x \gg 1$.

The singularities depend heavily in the asymptotic structure of the chosen potential V. Indeed, for slower decaying non compactly supported potentials, the analytic continuation of f_+ to the complex s-plane, for $x \gg 1$, often comes with a branch point at $s = 0$ [15]. In these cases, the branch point force us to consider a branch cut, which we can chose to place in the negative real line of s. Now, no longer we can compute Mellin's inverse formula (1.94), using the contour depicted in figure (1.1). Instead, we have to consider the contour depicted in figure (1.2). Looking at this figure, we notice the arc shaped portion of our contour now wraps around the negative real line of s. This is a necessity, for we chose the branch cut to take place in that line.

Once again, using the residue theorem yields

$$\oint_{\mathcal{D}'} \mathcal{F}(s,x)ds = \int_{a-i\infty}^{a+i\infty} \mathcal{F}(s,x)ds + \int_{\mathcal{C}} \mathcal{F}(s,x)ds + \int_{\mathcal{B}} \mathcal{F}(s,x)ds = 2\pi i \sum_{n\in I} Res(\mathcal{F}, s_n) \tag{1.109}$$

where we denoted the blue contour depicted in figure (1.2) and the associated portion wrapped around the branch cut by $\mathcal{D}'$ and $\mathcal{B}$ respectively. Finally, we can write

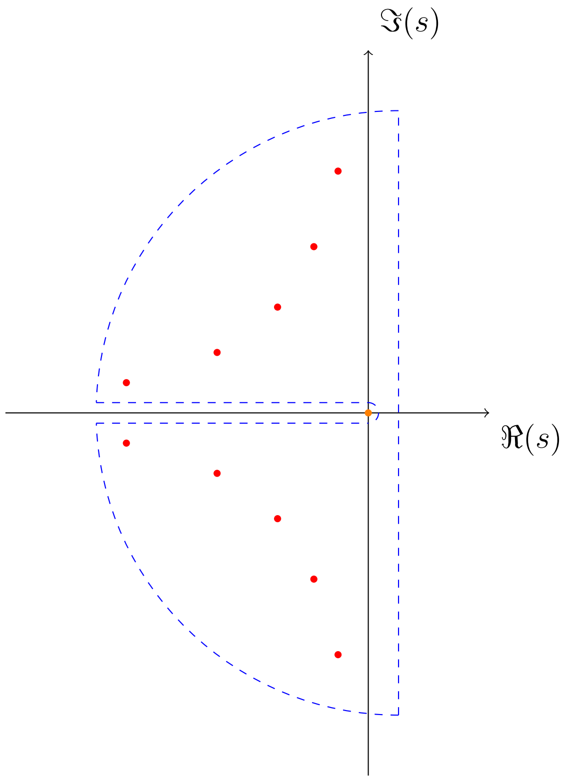

Figure 1.2: Pictorial representation of the closed contour needed to compute the integral over s in (1.94) as the blue dashed line. Possible quasinormal frequencies are depicted by conjugated red dots. Furthermore, the branch point of f_+ is depicted as an orange dot.

$$\psi(x,t) = 2\pi i \sum_{n\in I} Res\,(\mathcal{F}, s_n) - \int_{C} \mathcal{F}(s,x)ds - \int_{\mathcal{B}} \mathcal{F}(s,x)ds. \tag{1.110}$$

Overall, the contribution from the integral over $\mathcal{B}$ to ψ exists because V is non compactly supported in the spatial domain. Moreover, the exact form of this contribution depends heavily on the asymptotic structure of the potential.

In [15], the contribution of the integral over $\mathcal{B}$ is computed for a broad range of non compactly supported potentials and it is observed that, under fairly general assumptions in the late time behaviour of the system, this contribution is proportional to an inverse power of t, taking a tail like shape.

The last example revealed that systems described by an equation like (1.81), defined on a non compact spatial domain with a non compactly supported potential V, often have an extra contribution to the solution, when compared to (1.98). This contribution, often taking a tail like shape, dominates the behaviour of ψ in the very late time evolution of the system. Usually, such contribution takes place after quasinormal modes dominate the behaviour of ψ [56].

1.3.4 Quasinormal modes of black holes

We start by taking a note regarding notations. In the last two subsections, we defined quasinormal frequencies as values of s. However, it is common to write the variable associated with the Laplace transform as

$$s = i\omega. \tag{1.111}$$

Thus, quasinormal frequencies are usually defined as values of ω instead. Using this notation, it is easy to see that our latest definitions of quasinormal modes and frequencies match those first introduced, associated with the master equation (1.62).

By now, it should be much more clear why we chose the conditions (1.69) and (1.70), independently of initial data [11]. Indeed, these conditions define the asymptotic behaviour of the dominant radial contribution to ψ in the late time behaviour of the system: the quasinormal modes!

As an observer, far away from the black hole, we are interested in the limit of spatial infinity, for the late time behaviour of ψ. Looking at (1.100), we see that in order to know ψ, we first need to know the quasinormal frequencies and the associated complex constants C_n, also called quasinormal excitation coefficients. How do we compute these? Regarding the excitation coefficients, the only way to compute them is by providing the problem with initial data [8]. On the other hand, quasinormal frequencies only depend on the metric tensor field of the background black hole space time [12].

We are left with the task of computing quasinormal frequencies. In general, it is very complicated (if not impossible) to obtain an analytical expressions of them. However, there are two limiting cases where we might have an easier time looking for analytical expressions. These limiting cases target existing quasinormal frequencies with, operationally wise, desirable properties. The first limiting case targets quasinormal frequencies obeying the master equation for arbitrarily large values of the azimuthal number l [13]. This limit is usually called the eikonal limit. The second limiting case targets quasinormal frequencies ω such that $|\Im(\omega)| \gg |\Re(\omega)|$. This limit is usually called the asymptotic limit. We will be working with both these limits.

Finally, while we postpone to later chapters the introduction and discussion of actual analytical methods to compute quasinormal frequencies in both limiting cases introduced, we take one last note. In our previous example, we saw that quasinormal frequencies were restricted to the complex half plane $\Re(s) < 0$. Looking at (1.111), it is clear this restriction translates to $\Im(\omega) > 0$ in the new notation.

1.3.5 The interest of quasinormal modes

Here, we discuss, very vaguely, five of many interesting topics, associated with quasinormal modes of black holes.

The first topic concerns the AdS/CFT duality. AdS/CFT is a conjecture, fist proposed by Juan Maldacena in 1997 [34], later developed into a powerful technique. This technique provides an effective description of non perturbative and strongly coupled regime of certain gauge theories in terms of higher-dimensional gravity. Equilibrium and non-equilibrium properties of strongly coupled d-dimensional thermal gauge theories are related to the physics of $(d+1)$-dimensional black hole space times [8]. In particular, the spectrum of quasinormal frequencies, associated with the black hole space time, provides the location, in momentum space, of the poles of the retarded Green functions associated with the dual gauge theory [8].

The second topic concerns gravitational wave astronomy. As it so happens, in the context of black hole space times, quasinormal modes do not only appear in the study of interactions weak enough to be fully described by linear perturbations. Indeed, interactions too strong to be fully described by linear perturbations may also exhibit a quasinormal mode structure in the late time behaviour. A

[11] Assuming the data is of compact support.

[12] Indeed, this will be much more clear once we introduce some analytical methods to compute quasinormal frequencies.

[13] In our example, this number arises from the spherical harmonic decomposition (1.54). Naturally, this limiting case is restricted to background black hole space times whose metric tensor field allows for similar decompositions.

good example would be a collision of compact object binaries, like the collision of two black holes. These phenomena roughly develop as follows [9]:

- In an initial phase, while the black holes are sufficiently separated, they orbit relatively slowly, emitting weak gravitational waves. These waves carry away energy from the binary system, slowly pushing the black holes closer to each other. As the binary system progressively shrinks, the angular frequency, associated with the orbits of the black holes, increases and consequently stronger gravitational waves are emitted. Thus, the system as a whole is bound to lose energy progressively faster. This phase is usually called the in spiral.
- When the black holes get sufficiently close, they merge, forming a single black in the process. The strongest gravitational waves are emitted during this phase. Here, any attempts of a perturbative approach fail, as the gravitational interaction is too strong to be described by linear perturbations. Instead, methods from numerical relativity need to be employed. This phase is called the merger.
- Finally, after the merger, a single distorted black hole remains. Conveniently for us, these distortions are nice enough to be described by linear perturbations of a black hole space time. Hence, the gravitational radiation emitted, while the black hole recovers the expected symmetry, exhibits a quasinormal mode structure. This phase is called the ring down.

The interest in measuring quasinormal modes structures, in the ring down phase, comes from the fact that they encode information of the background black hole space time geometry [8]. Thus, possible developments in experimental observation of gravitational waves might lead to concrete evidence supporting the existence of rotating black hole solutions, among other amazing results.

Another topic has to do with the conjectured quantization of the black hole surface area. It was proposed that the minimal surface area of a black hole is proportional to the square of the Plank's length. Statistical physics arguments impose a constrain on the proportionality constant γ of the form

$$\gamma = 4\log(n) \tag{1.112}$$

for some $n \in \mathbb{N}$. In [28], a relation between quasinormal frequencies, in the asymptotic limit, and the constant γ is established, using Bohr's correspondence principle. This relation required the real part of the quasinormal frequencies to be proportional to $\log(n)$. This claim was supported by numerical computations showing that the real part of the Schwarzschild black hole asymptotic quasinormal frequencies is proportional to $\log(3)$ [43]. This result was later analytically confirmed by Motl and Neitzke [37]. However, the conjectured relation between γ and the asymptotic quasinormal frequencies, initially proposed by Hod, stumbled when faced with other black hole space times.

Another topic concerns non linear effects of general relativity in the structure of quasinormal modes. As we said before, quasinormal mode structures do not restrict them selves to interactions weak enough to be fully described by linear perturbations. While it is true that quasinormal modes are more often studied in the linear context, they also appear in strong gravity interactions. Thus, the natural question arises: How does the non linear character of general relativity translates itself on the quasinormal modes structures? Answers to this question should be provided by the study of higher order perturbations of fields, coupled to gravity in black hole space times. The mathematical formalism for this study, up to second order, was already developed in [20, 22, 21] and recently used to provide second order corrections to the quasinormal frequencies. The interest behind these corrections range from increasing the detectability of quasinormal modes to the eventual discovery of new properties, hidden in the strictly non linear structure of general relativity.

The last topic concerns stability analysis of black hole solutions. Quasinormal frequencies, associated with linear perturbations of a black hole space time metric tensor field, may be studied for stability test purposes [14]. Indeed, the existence of quasinormal frequencies ω, such that $\Im(\omega) < 0$, are a strong indicator that the associated black hole space time is unstable. When we introduced

[14]A metric tensor field, associated with some space time, is stable if linear perturbations with initial data of compact support are bounded in space and time.

quasinormal frequencies, we saw they were restricted to the complex half plane $\Im(\omega) > 0$. However, this restriction is only assured to hold true under the conditions we imposed on V, such as positivity. Under more general conditions, this restriction may not hold and stability may not exist.

1.4 Greybody factors

Here, we introduce, very briefly, greybody factors in the context of black hole physics and comment on the relation they share with Hawking radiation.

1.4.1 Introduction

Greybody factors are directly related to the propagation of fields in a black hole space time. The propagating fields can be, for example, complex massless scalar fields coupled to gravity, or even linear perturbations of the metric tensor field itself. Depending on the symmetries of the metric tensor field, it is often possible to reduce the equations of motion, associated with these fields, to Schrödinger type equations of the form

$$\frac{d^2\psi}{dx^2} + \left(\omega^2 - V\right)\psi = 0 \tag{1.113}$$

where x is the adequate tortoise coordinate and the potential V is such that

$$\lim_{r\to+\infty} V(r) = 0 \tag{1.114}$$

$$\lim_{r\to+R_h^+} V(r) = 0. \tag{1.115}$$

This should not come as a surprise by now, for we provided an explicit example of this reduction, concerning linear perturbations of a complex massless scalar field in a Schwarzschild black hole space time. Indeed, in that example, we also arrived at the Schrödinger type equation (1.62) with a potential agreeing with the limits above. Following the same argument given in that example, we can write the asymptotic behaviour

$$\psi(x) \sim A_+ e^{i\omega x} + A_- e^{-i\omega x} \tag{1.116}$$

in the boundaries $r \to R_h^+$ and $r \to +\infty$, for some $A_\pm \in \mathbb{C}$.

This time, we seek solutions that obey the boundary conditions

$$\psi(x) \sim T(\omega)e^{i\omega x} \tag{1.117}$$

for $r \to R_h^+$ and

$$\psi(x) \sim e^{i\omega x} + R(\omega)e^{-i\omega x} \tag{1.118}$$

for $r \to +\infty$, where $T(\omega), R(\omega) \in \mathbb{C}$ are called transmission and reflection coefficients respectively. Physically, this problem models the scattering of a propagating field in the non trivial structure of the potential V, resultant from the curvature of space time. Moreover, the problem itself differs from the quasinormal modes problem in two main ways. First of all, the functional form of the boundary condition at spatial infinity is different. Physically, this means we allow propagating waves arriving from spatial infinity, as opposed to the quasinormal modes problem. Secondly, we care about the constants multiplying the exponential functions on the boundary conditions. This is so, because they play a role in the physical interpretation and relevance of the problem. In the quasinormal modes problem, this is not the case, for the functional form of the boundary conditions alone is enough to yield plenty of information about the quasinormal frequencies [15].

In the context of this problem, the Greybody factor is defined as [26]

$$\gamma(\omega) = T(\omega)\widetilde{T}(\omega). \tag{1.119}$$

[15] We will see this later, when explicitly computing these frequencies.

where we defined the constant

$$\widetilde{T}(\omega) \coloneqq T(-\omega). \tag{1.120}$$

Naturally, the greybody factor will be a function of the frequency ω.

In this work, we compute this factor for two different fields, associated with the same black hole space time in which we compute the quasinormal frequencies.

1.4.2 Greybody factors and Hawking radiation

The greybody factor is directly connected to the asymptotic observation of Hawking radiation [44]. Indeed, at the vicinity of the black hole horizon, Hawking radiation is black body radiation [42]. However, as the radiation propagates through the outer region of the black hole space time [16], it scatters on the non trivial curvature of the latter. Thus, an asymptotic observer is bound to measure this radiation, modified in some sense.

In the famous calculation of the Hawking radiation [27], it was shown that black holes have a thermal spectrum. Moreover, the expectation value for the number of particles emitted with a certain frequency ω is

$$\Big\langle n(\omega) \Big\rangle = \frac{\gamma(\omega)}{e^{\frac{\omega}{T_{\mathcal{H}}}} \pm 1} \tag{1.121}$$

where $T_{\mathcal{H}}$ is the Hawking temperature [44] of the black hole space time and the sign $+/-$ addresses radiation composed by fermions and bosons respectively.

Integrating the expression above over the entire frequency spectrum yields the black hole emission rate.

We notice that, should the greybody factor be constant, the black hole radiation becomes black body radiation [17].

1.5 WKB theory

Here, we introduce, very briefly, the WKB theory and explain some basic results which will be extensively used throughout this work. The name WKB comes from the founders Gregor Wentzel, Hendrik Anthony Kramers, and Léon Brillouin.

1.5.1 Defining purpose and Schrödinger like equations

The broad purpose of the WKB theory is to provide approximations to general solutions of differential equations whose highest order derivative is multiplied by some small real parameter. In this work, for the most part, we will only deal with ordinary homogeneous linear differential equations. Hence, we restrict ourselves to this class of differential equations. Let us write a general differential equation of this class as

$$\epsilon \frac{d^n y}{dx^n} + \sum_{m=0}^{n-1} a_i \frac{d^m y}{dx^m} = 0 \tag{1.122}$$

where $a_i \in C^k(\mathbb{R})$, for some $k \in \mathbb{N}$. The WKB method proposes an asymptotic series expansion of the form

$$y \sim \prod_{n=0}^{+\infty} e^{S_n \delta^{n-1}}. \tag{1.123}$$

where $S_n \in C^{k_n}(\mathbb{R})$ for some $k_n \in \mathbb{N}$ and where δ is a small real parameter. The scaling of δ with ϵ is determined by substitution along with the functions S_n.

As an example, we consider the Schrödinger like equation

[16] By this, we mean outside of the region bounded by the event horizon.

[17] More precisely, the radiation would obey Fermi-Dirac statistics for fermions and Bose-Einstein statistics for bosons [45].

$$\epsilon^2 \frac{d^2 y}{dx^2} = Qy \tag{1.124}$$

where ϵ^2 is a small real parameter and $Q \in C^\infty(\mathbb{R})$. Replacing the asymptotic series (1.123) in the differential equation above yields

$$\epsilon^2 \left[\frac{1}{\delta^2} \left(\sum_{n=0}^{+\infty} \delta^n \frac{dS_n}{dx} \right)^2 + \frac{1}{\delta} \sum_{n=0}^{+\infty} \delta^n \frac{d^2 S_n}{dx^2} \right] = Q. \tag{1.125}$$

We fix $\delta = \epsilon$ and proceed to solve the differential equation perturbatively in powers of ϵ. The equation of leading order is

$$\left(\frac{dS_0}{dx} \right)^2 = Q. \tag{1.126}$$

whose solutions are

$$S_0(x) = \pm \int \sqrt{Q(x)} dx \tag{1.127}$$

up to an additive constant, which we will ignore since (1.124) is linear. The differential equation of first order in ϵ is

$$2 \frac{dS_0}{dx} \frac{dS_1}{dx} + \frac{d^2 S_0}{dx^2} = 0 \tag{1.128}$$

whose solution is

$$S_1(x) = -\frac{1}{4} \log \left(Q(x) \right) \tag{1.129}$$

up to an additive constant, which we will ignore for the same reason.

We can follow this procedure iteratively, solving differential equations of higher order in ϵ. These equations take the generic form

$$2 \frac{dS_0}{dx} \frac{dS_n}{dx} + \frac{d^2 S_{n-1}}{dx^2} + \sum_{j=1}^{n-1} \frac{dS_j}{dx} \frac{dS_{n-j}}{dx} = 0 \tag{1.130}$$

for $n \in \mathbb{Z}_{\geq 2}$.

Because (1.123) is an asymptotic series, there is usually an accuracy limit associated as the representation of the solution to the differential equation considered. Hence, only a finite amount of functions S_n are of interest to us.

Up to zeroth order in ϵ, the general solution of the differential equation (1.124) can be expressed as

$$y(x) \sim \frac{\mathcal{C}_+}{\sqrt[4]{Q(x)}} \exp \left(\frac{1}{\epsilon} \int \sqrt{Q(x)} dx \right) + \frac{\mathcal{C}_-}{\sqrt[4]{Q(x)}} \exp \left(-\frac{1}{\epsilon} \int \sqrt{Q(x)} dx \right) \tag{1.131}$$

for some $\mathcal{C}_\pm \in \mathbb{C}$.

1.5.2 Analytic continuation and Stokes lines

Previously, we introduced the WKB theory and derived the solution it provides to a Schrödinger like equation, up to second order in the asymptotic series.

Now, we study the validity domain of the solution (1.131), when analytically continued to complex x-plane. In order to do that, we rederive the solution (1.131), using a different approach. First, let us assume Q can be analytically continued onto the complex x-plane. Furthermore, let us rewrite equation (1.124) as

$$\frac{d^2 y}{dx^2} - \frac{Q}{\epsilon^2} y = 0. \tag{1.132}$$

We introduce the new dependent variable

$$\rho(x) \coloneqq y(x) \sqrt{q(x)} \tag{1.133}$$

and the new independent variable

$$w(x) := \int q(x)dx + \mathcal{C}_w \tag{1.134}$$

where q is an analytical function and $\mathcal{C}_w \in \mathbb{C}$. Rewriting our differential equation with respect to these variables yields [19]

$$\frac{d^2\rho}{dw^2} + (1+\delta)\rho = 0 \tag{1.135}$$

where we defined the function

$$\delta(x) := -\frac{Q(x)}{q^2(x)\epsilon^2} - 1 + q^{-\frac{3}{2}}(x)\frac{d^2}{dx^2}\left(\frac{1}{\sqrt{q(x)}}\right). \tag{1.136}$$

The idea is to choose a function q such that $|\delta| \sim 0$ in the region of the complex x-plane where we want to approximate the solution. Assuming we made such a choice, we can approximate our differential equation by

$$\frac{d^2\rho}{dw^2} + \rho = 0 \tag{1.137}$$

whose general solution is

$$\rho(w) = \mathcal{C}_+ e^{iw} + \mathcal{C}_- e^{-iw}. \tag{1.138}$$

for some $\mathcal{C}_\pm \in \mathbb{C}$. Now, we seek a function q such that $|\delta| \sim 0$. If we choose

$$q(x) = \frac{i\sqrt{Q(x)}}{\epsilon} \tag{1.139}$$

we can rewrite δ as

$$\delta = \frac{\epsilon^2}{4}\left(\frac{1}{Q^2}\frac{d^2Q}{dx^2} - \frac{5}{4}\frac{1}{Q^3}\frac{dQ}{dx}\right). \tag{1.140}$$

Because ϵ^2 is a small real parameter, the choice (1.139) yields a small δ everywhere in the complex x-plane, except near roots of Q.

Plugging (1.139) in (1.138) yields the approximated solution

$$y(x) = \frac{\mathcal{C}'_-}{\sqrt[4]{Q(x)}}\exp\left(-\frac{1}{\epsilon}\int\sqrt{Q(x)}dx\right) + \frac{\mathcal{C}'_+}{\sqrt[4]{Q(x)}}\exp\left(\frac{1}{\epsilon}\int\sqrt{Q(x)}dx\right) \tag{1.141}$$

for some $C'_\pm \in \mathbb{C}$. We notice the solution above is identical to (1.131)!

Now, we want to know how good the expression above is, as an approximation to the true solution of (1.124).

We already know the expression above does not represent the solution well, near roots of Q. In the remaining zones of the complex x-plane, δ is small. We will now argue that this condition alone is not enough to ensure the expression above to be a good representation of the true solution. We start by expressing a true solution of the differential equation (1.135) as

$$\rho(w) = a_+(w)e^{iw} + a_-(w)e^{-iw} \tag{1.142}$$

for some analytical functions a_+, a_-. The expression above comes with an extra degree of freedom, for we are expressing the function ρ using two arbitrary analytical functions a_+, a_-. To eliminate this freedom, we may impose the condition

$$\frac{da_+}{dw}e^{iw} + \frac{da_-}{dw}e^{-iw} = 0. \tag{1.143}$$

Using the condition above, we can write

$$\frac{d\rho}{dw} = ia_+e^{iw} - ia_-e^{-iw}. \tag{1.144}$$

The equations above allow us to rewrite the second order ordinary differential equation (1.135) as the system of two first order ordinary differential equations [19]

$$\frac{da_+}{dw} = \frac{i\delta}{2}\left(a_+ + a_- e^{-2iw}\right) \tag{1.145}$$

$$\frac{da_-}{dw} = -\frac{i\delta}{2}\left(a_- + a_+ e^{2iw}\right). \tag{1.146}$$

Furthermore, we can rewrite the system above as the matrix differential equation

$$\frac{da}{dw} = Ma \tag{1.147}$$

where we defined the matrices

$$M(w) \coloneqq \frac{i\delta}{2}\begin{bmatrix} 1 & e^{-2iw} \\ -e^{2iw} & -1 \end{bmatrix} \qquad a(w) \coloneqq \begin{bmatrix} a_+(w) \\ a_-(w) \end{bmatrix}. \tag{1.148}$$

Looking at equations (1.140) and (1.147), we notice the relative change of a with respect to w is proportional to ϵ^2. Thus, as long as the off diagonal elements of M are of the order of unity, this change is negligible. In particular, we can ask for the condition

$$\left|e^{-2iw}\right| \sim \left|e^{2iw}\right| \sim 1. \tag{1.149}$$

We found the validity domain we were looking for. Indeed, the solutions (1.138) and consequently (1.131) are good approximations of the respective true solutions, for values of x where the condition above is fulfilled.

The condition above naturally leads to the definition of Stokes lines. Stokes lines are the curves in the complex x-plane where the following equation holds

$$\Im(w) = 0. \tag{1.150}$$

Naturally, the topology of these lines depends on the constant of integration $\mathcal{C}_w$. Moreover, along these lines (1.149) holds true!

With this loose argument, we reason the solution (1.131) is a good approximation of the true solution of (1.124), along Stokes lines, sufficiently far away from roots of Q. A much more formal proof of this statement can be found in [19].

A final note should be taken regarding cases where an analytic continuation of Q onto the complex x-plane cannot be made. As a general rule, we do not expect the solution (1.131) to be a good approximation of the true solution, near singularities of Q. This is so, because δ might share the same singularities [19].

Chapter 2

Callan Myers Perry black hole

In this chapter, the Callan Myers Perry black hole space time is introduced along with some results concerning gravitational and scalar field perturbations.

2.1 Physical origin

In string theory, in order to ensure consistent string propagation, one needs local scale invariance on the string world-sheet [10]. Such condition, amounts to consistently set the beta functions [46] to zero [10]. The resulting equations set the dynamics of the dilaton scalar field ϕ and metric tensor field g, associated with the space time in which the word-sheet is embedded. Furthermore, these equations admit an expansion in powers of α', the inverse of the string tension. To first order in α', for a given chart (U, ξ), the action from which these equations emerge takes the local form [10]

$$\frac{1}{2k}\int_U \sqrt{-\det(g)}\left(\mathcal{S} - \frac{4}{d-2}g^{\nu\mu}\left(\frac{\partial\phi}{\partial\xi^\mu}\right)\left(\frac{\partial\phi}{\partial\xi^\nu}\right) + \lambda e^{-\frac{4}{d-2}\phi}C_{\mu\nu\alpha\beta}C_{\gamma\eta\rho\zeta}g^{\rho\mu}g^{\eta\nu}g^{\rho\alpha}g^{\zeta\beta}\right)d\xi \tag{2.1}$$

where $C \in \mathcal{T}^0_4(\mathcal{M})$ denotes the Weyl tensor field [55] and $\lambda = \frac{\alpha'}{2}, \frac{\alpha'}{4}, 0$ for bosonic, heterotic and type II strings respectively.

In this context, the Callan Myers Perry black hole emerges as a Lorentzian manifold whose metric tensor field solves the equations of motion of the action above. The topological structure will be that of a d-dimensional Tangherlini black hole.

The stringy correction, present in the action above, is bound to manifest itself on the metric tensor field solution, as we will see in the next section. In this fact lies the main practical interest of this solution. Indeed, the stringy correction, reflected on the metric tensor field, is bound to affect measurable quantities associated with the black hole solution. Thus, studying how these quantities change according to the stringy correction might be a viable way to provide experimental verifications of string theory. In this work, we study these corrections in properties like quasinormal modes, greybody factors and the black hole shadow.

2.2 The metric tensor field

Using charts with respect to the topological structure (1.39), allows one to write the metric tensor field of the Callan Myers Perry black hole as [10]

$$ds^2 = -f dt \otimes dt + \frac{1}{f}dr \otimes dr + r^2 d^2\Omega_{d-2} \tag{2.2}$$

where $d^2\Omega_{d-2}$ is the canonical metric tensor field of the unit (d-2)-sphere (1.43) and

$$f(r) := f_0(r)\left(1 + \lambda'\delta f\right) \tag{2.3}$$

where we defined

$$f_0(r) := 1 - \left(\frac{R_h}{r}\right)^{d-3} \tag{2.4}$$

$$\delta f(r) := -\frac{(d-3)(d-4)}{2}\left(\frac{R_h}{r}\right)^{d-3}\frac{1-\left(\frac{R_h}{r}\right)^{d-1}}{1-\left(\frac{R_h}{r}\right)^{d-3}}. \tag{2.5}$$

In the equations above, λ' stands for an adimensional version of λ, defined as

$$\lambda' := \frac{\lambda}{R_h^2}. \tag{2.6}$$

Moreover, R_h stands for the radius of the black hole event horizon, expressed as

$$R_h = (2\mu)^{\frac{1}{d-3}} \tag{2.7}$$

where the parameter μ can be related to the black hole mass m by

$$m = \left(1 + \frac{(d-3)(d-4)}{2}\lambda'\right)\frac{(d-2)\Omega_{d-2}}{k}\mu. \tag{2.8}$$

We notice R_h remained invariant under the stringy correction. This is a consequence of the coordinates used to express the metric tensor field (2.2).

Looking at the metric tensor field (2.2), we notice the associated Lorentzian manifold is spherically symmetric, static and asymptotically flat.

For future use, we consider the Hawking temperature, associated with this black hole space time. For spherically symmetric black hole space times such as ours, the Hawking temperature can be generally expressed as [39]

$$T_{\mathcal{H}} = \frac{1}{4\pi}\frac{df}{dr}(R_h). \tag{2.9}$$

Using definitions (2.4) and (2.5), we can write

$$T_{\mathcal{H}} = \frac{d-3}{4\pi R_h}\left(1 - \lambda'\frac{(d-1)(d-4)}{2}\right). \tag{2.10}$$

2.3 Gravitational perturbations

Here, we introduce, very briefly, gravitational perturbations and present some associated results in the context of the Callan Myers Perry black hole space time.

In the introduction chapter, we provided the example of quasinormal modes arising in the context of linear perturbations of a complex massless scalar field, coupled to gravity in the Schwarzschild black hole space time. However, it should be noted that quasinormal modes do not only appear associated with linear perturbations of scalar fields. For example, one can consider linear perturbations of vector fields or even linear perturbations of the metric tensor field itself. Both cases can yield a quasinormal mode structure in the late time behaviour [13, 8].

Linear perturbations of the metric tensor field are called gravitational perturbations. In the case of spherically symmetric and static black hole space times, there is more to be said about these perturbations. For example, we consider a Schwarzschild black hole space time. In the process of simplifying the equations of motion, associated with the perturbation tensor field components, one ends up with a simplified system of two master equations, instead of one [8, 36]. The independent variables associated with these equations are functions that describe two distinct types of gravitational perturbations. These types, usually called scalar and vector types, are both needed in order to fully describe general gravitational perturbations.

In spherically symmetric and static black hole space times of dimensions strictly larger than 4, such as most Tangherlini black hole space times, yet another master equation is needed to fully describe general gravitational perturbations [30]. The independent variable, associated with this master equation, is a function that describes another type of gravitational perturbations, called the tensor type.

In this work, we address quasinormal frequencies of tensor type gravitational perturbations, associated with the Callan Myers Perry black hole space time. In [39] and more generally in [38], it was shown that the master equation, associated with these perturbations, takes the Schrödinger like form

$$\frac{d^2\psi}{dx^2} + \left(\omega^2 - V\right)\psi = 0 \tag{2.11}$$

where ψ is the function describing tensor type gravitational perturbations and x is the adequate tortoise coordinate. This coordinate is defined as

$$dx := \frac{dr}{f} \tag{2.12}$$

up to an integration constant. The potential V can be expressed, with respect to r, as

$$\begin{aligned} V(r) = f(r)\left(\frac{l(l+d-3)}{r^2} + \frac{(d-2)(d-4)f(r)}{4r^2} + \frac{(d-2)f'(r)}{2r}\right) + \lambda' \\ \left(\frac{R_h}{r}\right)^2 f(r)\Bigg[\left(\frac{2l(l+d-3)}{r} + \frac{(d-4)(d-5)f(r)}{r} + (d-4)f'(r)\right)\left(2\left(\frac{1-f(r)}{r}\right) + f'(r)\right) \\ + \left(4(d-3) - (5d-16)f(r)\right)\frac{f'(r)}{r} - 4\left(f'(r)\right)^2 + (d-4)f(r)f''(r)\Bigg] \end{aligned} \tag{2.13}$$

where the apostrophe stands for derivation with respect to r.

2.4 Complex massless scalar field perturbations

We can also consider linear perturbations of a complex massless scalar field, coupled to gravity in the Callan Myers Perry black hole space time. In [8], these perturbations, in a d-dimensional Tangherlini black hole space time, are shown to be fully described by the Schrödinger like master equation

$$\frac{d^2\psi}{dz^2} + (\omega^2 - V)\psi = 0 \tag{2.14}$$

where z is the tortoise coordinate of the d-dimensional Tangherlini black hole space time. This coordinate is defined as

$$dz := \frac{dr}{f_0(r)} \tag{2.15}$$

up to an integration constant [40]. Moreover, the potential V can be expressed, with respect to r, as

$$V(r) = f_0(r)\left(\frac{l(l+d-3)}{r^2} + \frac{(d-2)(d-4)f_0(r)}{4r^2} + \frac{(d-2)f_0'(r)}{2r}\right). \tag{2.16}$$

The only effective difference between the d-dimensional Tangherlini black hole space time and the respective corrected version lies on the metric tensor field. Moreover, looking at (1.40) and (2.2), we notice these metric tensor fields are related by interchanging f with f_0, in their definitions with respect to the topological structure (1.39). Thus, linear perturbations of a complex massless scalar field, coupled to gravity in the Callan Myers Perry black hole space time, are also fully described by a Schrödinger like master equation. Furthermore, this equation takes the form

$$\frac{d^2\psi}{dx^2} + \left(\omega^2 - V\right)\psi = 0 \tag{2.17}$$

where the potential V can be expressed, with respect to r, as

$$V(r) = f(r)\left(\frac{l(l+d-3)}{r^2} + \frac{(d-2)(d-4)f(r)}{4r^2} + \frac{(d-2)f'(r)}{2r}\right). \tag{2.18}$$

We notice the potential above is identical to (2.13), without the explicit correction of first order in λ'. This is not too surprising. Indeed, linear perturbations of complex massless scalar fields and tensor type gravitational perturbations share the same quasinormal frequency spectrum, in the non string corrected limit [40].

Chapter 3

The eikonal limit

In this chapter, we analytically compute the quasinormal frequencies, in the eikonal limit, of tensor type gravitational perturbations, in a Callan Myers Perry black hole space time. The approach used here closely follows [49].

We recall the eikonal limit targets quasinormal frequencies ω obeying the master equation (2.11) for large values of the azimuthal number l.

3.1 Definition and analysis of Q

We begin by defining the function

$$Q(x) := \omega^2 - V(x). \tag{3.1}$$

For large values of l, the asymptotic expansion of Q can be written, with respect to r, as

$$Q(r) \sim \omega^2 - h(r)\left(\frac{l}{r}\right)^2 \tag{3.2}$$

where we defined

$$h(r) := f(r)\left(1 + 2\lambda'\left(\frac{R_h}{r}\right)^2\left[2(1-f(r)) + rf'(r)\right]\right). \tag{3.3}$$

Because we are working in the eikonal limit, we redefine Q as the respective large l asymptotic expansion (3.2), for convenience purposes.

Up to zeroth order in λ', we have

$$Q(r) = \omega^2 - \left[1 - \left(\frac{R_h}{r}\right)^{d-3}\right]\left(\frac{l}{r}\right)^2. \tag{3.4}$$

We notice the expression above only has one critical point, besides $r = 0$, corresponding to one minimizer. Indeed, taking the derivative yields

$$\frac{dQ}{dr}(r) = l^2\left(\frac{2}{r^3} - (d-1)R_h^{d-3}r^{-d}\right) \tag{3.5}$$

Computing the root of the expression above, we get

$$r_g := \left(\frac{d-1}{2}\right)^{\frac{1}{d-3}} R_h. \tag{3.6}$$

Furthermore, taking the second order derivative yields

$$\frac{d^2Q}{dr^2}(r_g) = \frac{2^{\frac{d+1}{d-3}}(d-3)(d-1)^{-\frac{4}{d-3}}l^2}{R_h^4}. \tag{3.7}$$

Because the expression above is strictly positive for $d \geq 4$, we gather r_g is a minimizer. When we consider (3.2), up to first order in λ', a correction of this minimizer emerges. To compute this correction, we redefine r_g as the corrected minimizer

$$r_g \mapsto r_g + \lambda' \delta r_g \tag{3.8}$$

for some unknown $\delta r_g \in \mathbb{R}$ and replace it in the derivative of Q. The resulting expression, up to first order in λ', is

$$\lambda' \frac{32^{\frac{1}{d-3}} (d-1)^{-\frac{2(d+2)}{d-3}} l^2}{R_h^4} \Bigg(\delta r_g 2^{\frac{4-d}{3-d}} (d-3)(d-1)^{\frac{2d}{d-3}} + 4((d-11)d+16)(d-1)^{\frac{d+2}{d-3}} R_h + \left(4(d+1) - 4^{\frac{1}{3-d}} (d-4)(d-3)(d-1)^{\frac{2}{d-3}}\right) (d-1)^{\frac{2d-1}{d-3}} R_h \Bigg). \tag{3.9}$$

In order to find δr_g, we equate the expression above to zero and solve for δr_g. After some algebraic manipulation, we get

$$\delta r_g = \left(\frac{d-1}{2}\right)^{\frac{1}{d-3}} \left(\frac{d-4}{2} - (2d-5) \left(\frac{2}{d-1}\right)^{\frac{d-1}{d-3}} \right) R_h. \tag{3.10}$$

Finally, we can write the corrected minimizer as

$$r_g = \left(\frac{d-1}{2}\right)^{\frac{1}{d-3}} R_h \left[1 + \lambda' \left(\frac{d-4}{2} - (2d-5) \left(\frac{2}{d-1}\right)^{\frac{d-1}{d-3}} \right) \right]. \tag{3.11}$$

3.2 WKB approximation

Here, we use the WKB theory to find an analytical relation, between quasinormal frequencies, in the eikonal limit, and the potential (2.13). To this end, we resort to the previously found minimum of Q.

We start by making the assumption that quasinormal modes ω, in the eikonal limit, are such that

$$\omega^2 \sim V(r_g). \tag{3.12}$$

If this is the case, Q can be graphically represented near $x_g = x(r_g)$, as depicted in figure (3.1).

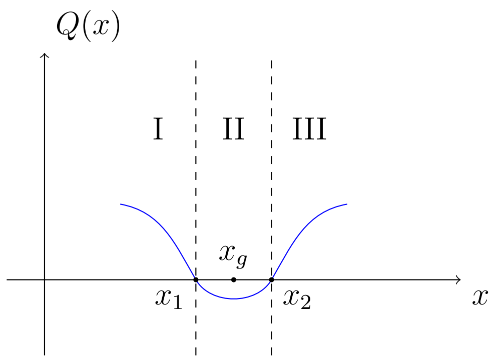

Figure 3.1: Graphical depiction of Q as the blue line. Possible roots of Q are denoted by x_1 and x_2. Moreover, the x-axis is subdivided in three regions, denoted by I, II and III.

In this figure, we notice the x-axis is subdivided in three regions, denoted by I, II and III.

Looking at (3.2), we notice $|Q|$ is very large far enough from x_g and decreases abruptly, approaching the latter. Thus, we can use WKB theory to approximate the solution of the master equation (2.11) in regions I and III. Using (1.131) yields

$$\psi_{\mathrm{I}}(x) \sim \frac{\mathcal{C}_1^+}{\sqrt[4]{Q(x)}} \exp\left(i \int_x^{x_1} \sqrt{Q(y)} dy \right) + \frac{\mathcal{C}_1^-}{\sqrt[4]{Q(x)}} \exp\left(-i \int_x^{x_1} \sqrt{Q(y)} dy \right) \tag{3.13}$$

$$\psi_{\mathrm{III}}(x) \sim \frac{\mathcal{C}_2^+}{\sqrt[4]{Q(x)}} \exp\left(i\int_{x_2}^{x}\sqrt{Q(y)}dy\right) + \frac{\mathcal{C}_2^-}{\sqrt[4]{Q(x)}} \exp\left(-i\int_{x_2}^{x}\sqrt{Q(y)}dy\right) \tag{3.14}$$

for some $\mathcal{C}_1^{\pm}, \mathcal{C}_2^{\pm} \in \mathbb{C}$. To compute an approximation for ψ_{II}, we start by considering the Taylor expansion

$$Q(x) = Q(x_g) + \frac{dQ}{dx}(x_g)(x-x_g) + \frac{d^2Q}{dx^2}(x_g)\frac{(x-x_g)^2}{2} + \mathcal{O}\left((x-x_g)^3\right). \tag{3.15}$$

Noting that

$$\frac{dQ}{dx}(x_g) = 0 \tag{3.16}$$

we can rewrite (3.15) as

$$Q(x) = Q(x_g) + \frac{d^2Q}{dx^2}(x_g)\frac{(x-x_g)^2}{2} + \mathcal{O}\left((x-x_g)^3\right). \tag{3.17}$$

Furthermore, we can neglect terms of higher orders in $(x - x_g)$ and write

$$Q(x) \sim Q(x_g) + \frac{d^2Q}{dx^2}(x_g)\frac{(x-x_g)^2}{2}. \tag{3.18}$$

in region II. This approximation is valid because region II is very slim, under the assumption (3.12). Now, defining the constants

$$k := \frac{1}{2}\frac{d^2Q}{dx^2}(x_g) \qquad \nu := -\frac{1}{2} - iQ(x_g)\left(2\frac{d^2Q}{dx^2}(x_g)\right)^{-\frac{1}{2}} \tag{3.19}$$

and the variable

$$t := \sqrt[4]{4k}e^{\frac{i\pi}{4}}(x-x_g) \tag{3.20}$$

allow us to rewrite the master equation (2.11) as

$$\frac{d^2\psi_{\mathrm{II}}}{dt^2} + \left(\nu + \frac{1}{2} - \frac{t^2}{4}\right)\psi_{\mathrm{II}} = 0 \tag{3.21}$$

in region II. The general solution of the differential equation above is [7]

$$\psi_{\mathrm{II}}(t) = \mathcal{C}_1 D_\nu(t) + \mathcal{C}_2 D_{-\nu-1}(it) \tag{3.22}$$

for some $\mathcal{C}_1, \mathcal{C}_2 \in \mathbb{C}$, where

$$D_\nu(w) = \frac{2^{\frac{\nu}{2}}e^{-\frac{w^2}{4}}\sqrt[4]{-iw}\sqrt[4]{iw}}{\sqrt{w}}\,{}_1F_1\left(-\frac{\nu}{2},\frac{1}{2},\frac{w^2}{2}\right) \tag{3.23}$$

denotes the parabolic cylinder function and

$${}_1F_1(a,b,w) = \sum_{k=0}^{+\infty}\frac{\Gamma(a+k)}{\Gamma(a)}\frac{\Gamma(b)}{\Gamma(b+k)}\frac{w^k}{k!} \tag{3.24}$$

stands for the confluent hypergeometric function of the first kind [2].

The asymptotic expansions of the parabolic cylinder function yield [7, 49]

$$\begin{aligned}\psi_{\mathrm{II}}(x) \sim\ & \mathcal{C}_2 e^{-3i\pi\frac{\nu+1}{4}}(4k)^{-\frac{\nu+1}{4}}(x-x_g)^{-(\nu+1)}e^{i\sqrt{k}\frac{(x-x_g)^2}{2}} \\ &+ \left(\mathcal{C}_1 + \mathcal{C}_2\frac{\sqrt{2\pi}e^{-i\nu\frac{\pi}{2}}}{\Gamma(\nu+1)}\right)e^{i\pi\frac{\nu}{4}}(4k)^{\frac{\nu}{4}}(x-x_g)^{\nu}e^{-i\sqrt{k}\frac{(x-x_g)^2}{2}}\end{aligned} \tag{3.25}$$

$$\begin{aligned}\psi_{\mathrm{II}}(x) \sim\ & \mathcal{C}_1 e^{-3i\pi\frac{\nu}{4}}(4k)^{\frac{\nu}{4}}(x_g-x)^{\nu}e^{-i\sqrt{k}\frac{(x-x_g)^2}{2}} \\ &+ \left(\mathcal{C}_2 - i\mathcal{C}_1\frac{\sqrt{2\pi}e^{-i\nu\frac{\pi}{2}}}{\Gamma(-\nu)}\right)e^{i\pi\frac{\nu+1}{4}}(4k)^{-\frac{\nu+1}{4}}(x_g-x)^{-(\nu+1)}e^{i\sqrt{k}\frac{(x-x_g)^2}{2}}\end{aligned} \tag{3.26}$$

for $te^{-i\frac{\pi}{4}} \gg 1$ and $te^{-i\frac{\pi}{4}} \ll -1$ respectively. Looking at (3.7), we see that $k \gg 1$. Hence, the asymptotic expansions above are valid for values of x near the respective points x_1 and x_2, where it still makes sense to match them with ψ_{I} and ψ_{III}.

Imposing the boundary conditions (1.69) and (1.70) on solutions ψ_{I} and ψ_{III} respectively yields

$$\mathcal{C}_1^+ = 0 \qquad \mathcal{C}_2^+ = 0. \tag{3.27}$$

Thus, we can rewrite ψ_{I} and ψ_{III} as

$$\psi_{\mathrm{I}}(x) \sim \frac{\mathcal{C}_1^-}{\sqrt[4]{Q(x)}} \exp\left(-i\int_x^{x_1} \sqrt{Q(y)}dy\right) \tag{3.28}$$

$$\psi_{\mathrm{III}}(x) \sim \frac{\mathcal{C}_2^-}{\sqrt[4]{Q(x)}} \exp\left(-i\int_{x_2}^{x} \sqrt{Q(y)}dy\right). \tag{3.29}$$

In the matching region, we still can use the approximation (3.18) and write

$$\psi_{\mathrm{I}}(x) \sim \frac{\mathcal{C}_1^-}{\sqrt[4]{Q(x)}} \exp\left(-i\int_x^{x_1} \sqrt{Q(x_g)+k(y-x_g)^2}dy\right) \tag{3.30}$$

$$\psi_{\mathrm{III}}(x) \sim \frac{\mathcal{C}_2^-}{\sqrt[4]{Q(x)}} \exp\left(-i\int_{x_2}^{x} \sqrt{Q(x_g)+k(y-x_g)^2}dy\right). \tag{3.31}$$

Under the assumption (3.12), we know that

$$Q(x_g) \sim 0. \tag{3.32}$$

Thus, we can write

$$\begin{aligned}\int_x^{x_1} \sqrt{Q(x_g)+k(y-x_g)^2}dy &\sim \int_x^{x_1} \sqrt{k(y-x_g)^2}dy = \sqrt{k}\int_x^{x_1} |y-x_g|dy \\ &= -\sqrt{k}\int_x^{x_1} (y-x_g)dy = \sqrt{k}\frac{(x-x_g)^2}{2} - \sqrt{k}\frac{(x_1-x_g)}{2}\end{aligned} \tag{3.33}$$

$$\begin{aligned}\int_{x_2}^{x} \sqrt{Q(x_g)+k(y-x_g)^2}dy &\sim \int_{x_2}^{x} \sqrt{k(y-x_g)^2}dy = \sqrt{k}\int_{x_2}^{x} |y-x_g|dy = \\ &\sqrt{k}\int_{x_2}^{x} (y-x_g)dy = \sqrt{k}\frac{(x-x_g)^2}{2} - \sqrt{k}\frac{(x_2-x_g)}{2}\end{aligned} \tag{3.34}$$

Using the approximations above, we can rewrite (3.30) and (3.31) as

$$\psi_{\mathrm{I}}(x) \sim \frac{\mathcal{D}_1}{\sqrt[4]{Q(x)}} e^{-i\sqrt{k}\frac{(x-x_g)^2}{2}} \tag{3.35}$$

$$\psi_{\mathrm{III}}(x) \sim \frac{\mathcal{D}_2}{\sqrt[4]{Q(x)}} e^{-i\sqrt{k}\frac{(x-x_g)^2}{2}}. \tag{3.36}$$

for some $\mathcal{D}_1, \mathcal{D}_2 \in \mathbb{C}$. We notice the exponential terms of the asymptotic expansions (3.25) and (3.26) match those of (3.36) and (3.35) respectively if and only if $\mathcal{C}_2 = 0$ and

$$\frac{1}{\Gamma(-\nu)} \sim 0. \tag{3.37}$$

The last condition demands ν to be very close to singular points of Γ. Thus, we end up with the restriction

$$\nu \in \mathbb{N}_0. \tag{3.38}$$

Using definition (3.19), we can rewrite the condition above as

$$Q(x_g)\left(2\frac{d^2Q}{dx^2}(x_g)\right)^{-\frac{1}{2}} = i\left(n+\frac{1}{2}\right) \tag{3.39}$$

for $n \in \mathbb{N}_0$. This is the condition we use to analytically compute the quasinormal frequencies.

3.3 Quasinormal frequencies computation

Here, we explicitly compute an analytical expression for the quasinormal frequencies, in the eikonal limit.

Looking at the asymptotic expression (3.2), we easily see that

$$2h(r_g) = r_g \frac{dh}{dr}(r_g). \tag{3.40}$$

Using (3.2), we can rewrite the equation (3.39) as

$$\left(\omega^2 - h(r_g)\left(\frac{l}{r_g}\right)^2\right) = i\left(n + \frac{1}{2}\right)\sqrt{2\frac{d^2Q}{dx^2}(x_g)} \tag{3.41}$$

for $n \in \mathbb{N}_0$. Rearranging some terms yields

$$\omega = \pm l\sqrt{\frac{h(r_g)}{r_g^2}}\sqrt{1 + i\left(n + \frac{1}{2}\right)\frac{r_g^2}{l^2 h(r_g)}\sqrt{2\frac{d^2Q}{dx^2}(x_g)}}. \tag{3.42}$$

Once more, using (3.2) yields

$$\begin{aligned}\frac{d^2Q}{dx^2}(x_g) = -l^2\frac{d^2}{dx^2}\left(\frac{h}{r^2}\right)(x_g) = -l^2\frac{d}{dx}\left[\left(\frac{dh}{dr}\frac{1}{r^2} - \frac{2}{r^3}h\right)\frac{dr}{dx}\right](x_g)\\ = -l^2\frac{dr}{dx}\left[\frac{d^2r}{drdx}\left(\frac{dh}{dr}\frac{1}{r^2} - \frac{2}{r^3}h\right) + \frac{dr}{dx}\left(\frac{d^2h}{dr^2}\frac{1}{r^2} - \frac{2}{r^3}\frac{dh}{dr} + \frac{6}{r^4}h - \frac{2}{r^3}\frac{dh}{dr}\right)\right](x_g).\end{aligned} \tag{3.43}$$

Using definition (2.12) and equation (3.40), we can rewrite the previous equation as

$$\frac{d^2Q}{dx^2}(x_g) = -l^2\frac{f^2(r_g)}{r_g^4}\left(\frac{d^2h}{dr^2}(r_g)r_g^2 - 2h(r_g)\right). \tag{3.44}$$

Hence, we can rewrite equation (3.42) as

$$\omega = \pm l\sqrt{\frac{h(r_g)}{r_g^2}}\sqrt{1 + \frac{i}{l}\left(n + \frac{1}{2}\right)\sqrt{2\left(\frac{f(r_g)}{h(r_g)}\right)^2\left(2h(r_g) - \frac{d^2h}{dr^2}(r_g)r_g^2\right)}} \tag{3.45}$$

for $n \in \mathbb{N}_0$. Since we are in the eikonal limit, we can Taylor expand the expression above yielding

$$\omega = l\sqrt{\frac{h(r_g)}{r_g^2}} + \frac{i}{\sqrt{2}}\left(n + \frac{1}{2}\right)\sqrt{\frac{f^2(r_g)}{r_g^2 h(r_g)}\left(2h(r_g) - \frac{d^2h}{dr^2}(r_g)r_g^2\right)} \tag{3.46}$$

where we chose the positive root, assuming stability of the black hole solution. In order to simplify the notation, we define the constants

$$\Gamma_r^g := \sqrt{\frac{h(r_g)}{r_g^2}} \qquad \Gamma_i^g := \sqrt{\frac{f^2(r_g)}{2r_g^2 h(r_g)}\left(2h(r_g) - \frac{d^2h}{dr^2}(r_g)r_g^2\right)}. \tag{3.47}$$

After some algebraic manipulation, we can write Γ_r^g and Γ_i^g, up to first order in λ', as

$$\Gamma_r^g = \sqrt{\frac{d-3}{d-1}}\left(\frac{2}{d-1}\right)^{\frac{1}{d-3}}\frac{1}{R_h}\left[1 + \frac{\lambda'}{2}\left(3(d-2)\left(\frac{2}{d-1}\right)^{\frac{d-1}{d-3}} - (d-4)\right)\right] \tag{3.48}$$

$$\Gamma_i^g = \frac{d-3}{\sqrt{d-1}}\left(\frac{2}{d-1}\right)^{\frac{1}{d-3}}\frac{1}{R_h}\left[1 - \lambda'\frac{d-4}{2}\left(1 + (d-2)\left(\frac{2}{d-1}\right)^{\frac{d-1}{d-3}}\right)\right]. \tag{3.49}$$

Rewriting the expressions above, with respect to the Hawking temperature $T_{\mathcal{H}}$, yields

$$\Gamma_r^g = \frac{4\pi T_{\mathcal{H}}}{\sqrt{(d-1)(d-3)}} \left(\frac{2}{d-1}\right)^{\frac{1}{d-3}} \left[1 + \lambda' \left(\frac{d-2}{2}\right) \left(d - 4 + 3\left(\frac{2}{d-1}\right)^{\frac{d-1}{d-3}}\right)\right] \tag{3.50}$$

$$\Gamma_i^g = \frac{4\pi T_{\mathcal{H}}}{\sqrt{d-1}} \left(\frac{2}{d-1}\right)^{\frac{1}{d-3}} \left[1 + \lambda' \frac{(d-2)(d-4)}{2} \left(1 - \left(\frac{2}{d-1}\right)^{\frac{d-1}{d-3}}\right)\right]. \tag{3.51}$$

Finally, we can rewrite the quasinormal frequencies (3.46) as

$$\omega = l\Gamma_r^g + i\left(n + \frac{1}{2}\right)\Gamma_i^g \tag{3.52}$$

for $n \in \mathbb{N}_0$. We notice they obey the starting assumption (3.12).

3.4 Unstable null circular geodesics

For some black hole space times, the real and imaginary components of quasinormal frequencies, in the eikonal limit, have solid physical interpretations [11]. Indeed, the real part can be identified as a multiple of the angular velocity, associated with the generally unique null circular geodesic of the black hole space time. As for the imaginary part, it can be identified as a multiple of the Lyapunov coefficient [51], associated with the null circular geodesic trajectory.

However, tensor type gravitational perturbations of the Callan Myers Perry black hole space time belong to a class where such identification may not be true. Fundamentally, this is so because

$$f \neq h. \tag{3.53}$$

A very good justification of why this identification might not hold for a certain class of perturbations, including ours, can be found in [33], along with an example.

Chapter 4

The asymptotic limit

In this chapter, we analytically compute the quasinormal frequencies, in the asymptotic limit, of tensor type gravitational perturbations in a Callan Myers Perry black hole space time. We recall the asymptotic limit targets quasinormal frequencies ω such that $|\Im(\omega)| \gg |\Re(\omega)|$.

First, we address the master equation (2.11) by making a crucial variable change followed by a standard perturbation theory approach.

Secondly, we set up some of the WKB theory formalism, developed in the introduction chapter, and study the associated Stokes lines topology.

Finally, we introduce and employ the monodromy method to analytically compute the quasinormal frequencies. The monodromy method, first used by Motl and Neitzk in [37], is a very powerful method allowing one to analytically compute quasinormal frequencies, in the asymptotic limit, associated with linear perturbations of fields coupled to gravity in a black hole space time.

In this chapter, we allow r to take complex values. Thus, we assume an analytical continuation of functions of r to the complex plane.

4.1 Variable change and perturbation theory

For future convenience, we want to rewrite the master equation (2.11) with respect to a simpler independent variable. The variable we seek is the tortoise coordinate of the d-dimensional Tangherlini black hole space time. Recalling (2.15), such variable is defined as

$$dz = \frac{dr}{f_0} \tag{4.1}$$

up to a complex integration constant. We allow the integration constant to be complex, because we are assuming an analytical continuation of the functions of r to the complex plane.

Looking at definitions (2.12) and (4.1), we can write

$$\frac{dz}{dx} = 1 + \lambda' \delta f. \tag{4.2}$$

We can also rewrite the second order derivative of the master equation (2.11) as

$$\frac{d^2\psi}{dx^2} = \frac{d}{dx}\left(\frac{d\psi}{dz}\frac{dz}{dx}\right) = \frac{d^2\psi}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{d\psi}{dz}\frac{d^2 z}{dx^2} = \frac{d^2\psi}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{d}{dr}\left(\frac{dz}{dx}\right)\frac{dr}{dx}\frac{d\psi}{dz}. \tag{4.3}$$

Thus, we can rewrite the master equation (2.11), with respect to z, as

$$\left(\frac{dz}{dx}\right)^2 \frac{d^2\psi}{dz^2} + \frac{d}{dr}\left(\frac{dz}{dx}\right)\frac{dr}{dx}\frac{d\psi}{dz} + \left(\omega^2 - V\right)\psi = 0. \tag{4.4}$$

Using (2.12) and (4.2), we can write, with respect to r, the expressions

$$\frac{d}{dr}\left(\frac{dz}{dx}\right)\frac{dr}{dx}(r) = \lambda' \frac{(d-4)(d-3)r^{2-2d}\left((d-1)r^4 R_h^{2d} - 2(d-2)R_h^{d+3}r^{d+1} + (d-3)R_h^4 r^{2d}\right)}{2R_h^{7-d}r^d - 2R_h^4 r^3} \tag{4.5}$$

$$\left(\frac{dz}{dx}\right)^2(r) = 1 + 2\lambda'\delta f = 1 - \lambda'(d-3)(d-4)\left(\frac{R_h}{r}\right)^{d-3}\frac{1-\left(\frac{R_h}{r}\right)^{d-1}}{1-\left(\frac{R_h}{r}\right)^{d-3}} \tag{4.6}$$

up to first order in λ'.

Now, we use perturbation theory to address the differential equation (4.4). We start by considering the expansions

$$\psi = \psi_0 + \lambda'\psi_1 \tag{4.7}$$

$$V = V_0 + \lambda' V_1. \tag{4.8}$$

where we defined the functions

$$V_0(r) := f_0(r)\left(\frac{l(l+d-3)}{r^2} + \frac{(d-2)(d-4)f_0(r)}{4r^2} + \frac{(d-2)f_0'(r)}{2r}\right). \tag{4.9}$$

$$\begin{aligned} V_1(r) := R_h^{d-7}\Bigg(&\Big[R_h^2\left(2((d-11)d+16)l^2 + 2d(d-7)^2 l - d((d-7)(d-6)d-20) - 96l + 16\right) \\ &-(d-4)(d-3)(d-2)^2 r^2\Big] R_h^{d+1} r^{d+3} + (d-4)((d-5)d(2d-7)-22) r^6 R_h^{2d} + R_h^4 r^{2d} \\ &\Big[8(d-1)R_h^2(d(l-1)+(l-3)l+4) - (d-4)(d-3)r^2\left(2(d-3)l - d + 2l^2 + 2\right)\Big]\Bigg)\frac{r^{-3d-1}}{4}. \end{aligned} \tag{4.10}$$

Here, V_0 stands for (2.13) with λ' set to zero and consequently represents the uncorrected potential. Moreover, V_1 contains all the corrections of first order in λ', present in (2.13). This includes those coming from the explicit correction present in (2.13) and those coming from the correction present in f.

Replacing these expansions in (4.4) and solving perturbatively in powers of λ' yields two distinct linear ordinary differential equations. The first one, of zeroth order in λ', is

$$\frac{d^2\psi_0}{dz^2} + \left(\omega^2 - V_0\right)\psi_0 = 0. \tag{4.11}$$

The second one, of first order in λ', is the non homogeneous differential equation

$$\frac{d^2\psi_1}{dz^2} + \left(\omega^2 - V_0\right)\psi_1 = \xi. \tag{4.12}$$

The non homogeneous term is

$$\xi = \xi_1\frac{d^2\psi_0}{dz^2} + \xi_2\frac{d\psi_0}{dz} + \xi_3\psi_0 \tag{4.13}$$

where we defined the functions

$$\xi_1(r) := -2\delta f(r) \tag{4.14}$$

$$\xi_2(r) := -f(r)\,[\delta f]'(r) \tag{4.15}$$

$$\xi_3(r) := V_1(r). \tag{4.16}$$

From now on, we restrict ourselves to the asymptotic limit. Hence, we assume that $|\Im(\omega)| \gg |\Re(\omega)|$. In the asymptotic limit, some of the WKB theory formalism, developed in the introduction chapter, can be used to give a viable approximation of the general solutions of the differential equations (4.11) and (4.12). In the next section, we set up this formalism and proceed to compute this approximation for the general solution of (4.11).

4.2 WKB theory set up

Now, we set up the WKB theory formalism, associated with the Schrödinger like differential equation (4.11).

We note it is preferable to rewrite (4.11) with respect to the coordinate r, instead of z. This is so, mainly because z is a multi valued function of r [1]. Following [4], we define a new dependent variable

$$\Psi := \sqrt{f_0}\psi_0. \tag{4.17}$$

Now, we can write

$$\frac{d^2\psi_0}{dz^2} = \frac{d}{dz}\left(\frac{d\psi_0}{dr}\frac{dr}{dz}\right) = \frac{d^2\psi_0}{dr^2}\left(\frac{dr}{dz}\right)^2 + \frac{d\psi_0}{dr}\frac{d^2r}{drdz}\frac{dr}{dz}. \tag{4.18}$$

Using definition (4.1), allow us to rewrite the previous equation as

$$\frac{d^2\psi_0}{dx^2} = \frac{d^2\psi_0}{dr^2}f_0^2 + f_0'f_0\frac{d\psi_0}{dr} \tag{4.19}$$

where the apostrophe stands for derivation with respect to r. Now, using definition (4.17), we can write

$$\frac{d\psi_0}{dr} = -\frac{1}{2}f_0^{-\frac{3}{2}}f_0'\Psi + \frac{1}{\sqrt{f_0}}\frac{d\Psi}{dr} \tag{4.20}$$

$$\frac{d^2\psi_0}{dr^2} = \frac{3}{4}f_0^{-\frac{5}{2}}\left(f_0'\right)^2\Psi - \frac{1}{2}f_0^{-\frac{3}{2}}f_0''\Psi - f_0^{-\frac{3}{2}}f_0'\frac{d\Psi}{dr} + \frac{1}{\sqrt{f_0}}\frac{d^2\Psi}{dr^2}. \tag{4.21}$$

Replacing these expressions in equation (4.19), allow us to rewrite (4.11) as

$$\frac{d^2\Psi}{dr^2} + R\Psi = 0. \tag{4.22}$$

where we defined the function

$$R := \left(\frac{\omega}{f_0}\right)^2 - \frac{V_0}{f_0^2} - \frac{1}{2}\frac{f_0''}{f_0} + \frac{1}{4}\left(\frac{f_0'}{f_0}\right)^2. \tag{4.23}$$

In the asymptotic limit, quasinormal frequencies are such that $|\omega R_h| \gg 1$. Hence, we can use (1.131) to write the WKB approximation of the general solution of (4.22) as

$$\Psi(r) \sim \mathcal{C}_+\sqrt[4]{\frac{\omega^2}{R(r)}}\exp\left(i\omega\int\sqrt{\frac{R(r)}{\omega^2}}dr\right) + \mathcal{C}_-\sqrt[4]{\frac{\omega^2}{R(r)}}\exp\left(-i\omega\int\sqrt{\frac{R(r)}{\omega^2}}dr\right) \tag{4.24}$$

for some $\mathcal{C}_\pm \in \mathbb{C}$.

Looking at (4.23), we notice R has singularities in roots of f_0 and in $r = 0$. Considering the definition (2.4), we figure these roots are solutions of the polynomial equation

$$1 - \left(\frac{R_h}{r}\right)^{d-3} = 0 \tag{4.25}$$

such as the event horizon $r = R_h$. The remaining d-4 solutions, usually called fictitious horizons, can be found equally distributed among the circle $|r| = R_h$. Furthermore, because we are considering the asymptotic limit and consequently assuming that $|\omega R_h| \gg 1$, we know the roots of R are very close to $r = 0$. Thus, we expect (4.24) to provide a good approximation of the general solution of the differential equation (4.22), in Stokes lines, far enough from $r = 0$ and from the real and fictitious horizons. In these regions, the approximation

$$R \sim \left(\frac{\omega}{f_0}\right)^2 \tag{4.26}$$

holds true. Hence, we can rewrite the WKB approximation (4.24) as

$$\Psi(r) \sim \mathcal{C}_+\sqrt{f_0(r)}\exp\left(i\omega\int\frac{dr}{f_0(r)}\right) + \mathcal{C}_-\sqrt{f_0(r)}\exp\left(-i\omega\int\frac{dr}{f_0(r)}\right) \tag{4.27}$$

[1] We will see this later.

in these regions. Using definition (4.1), we can rewrite the approximation above as

$$\Psi(r) \sim \mathcal{C}_+ \sqrt{f_0(r)} e^{i\omega z(r)} + \mathcal{C}_- \sqrt{f_0(r)} e^{-i\omega z(r)}. \tag{4.28}$$

Finally, using definition (4.17) and omitting the r dependence for simplicity purposes, we can rewrite the approximation above as

$$\psi_0(z) \sim \mathcal{C}_+ e^{i\omega z} + \mathcal{C}_- e^{-i\omega z}. \tag{4.29}$$

We will use this approximation several times during this work.

4.3 Stokes lines topology

Now, we want to know the regions of the complex r-plane, where the WKB approximation (4.29) is effective. As we saw in the introduction chapter, these regions will generally coincide with Stokes lines, associated with the approximation. Therefore, in order to know these regions well, we need to start by studying the Stokes lines topology, associated with the WKB approximation (4.29).

Considering the approximation (4.26) and definition (1.150), Stokes lines, far enough from $r = 0$ and from the real and fictitious horizons, are such that

$$\Im\left(\omega z\right) = \Re\left(z\right) = 0. \tag{4.30}$$

In the first equality above, we used the fact that we are in the asymptotic limit and consequently assuming ω to be approximately imaginary.

It is rather easy to see the topology of these lines in the surroundings of $r = 0$. Indeed, integrating (4.1) yields [40]

$$z(r) = r + \frac{1}{2}\sum_{n=0}^{d-4} \frac{1}{k_n} \log\left(1 - \frac{r}{R_n}\right) \tag{4.31}$$

where we defined

$$k_n := \frac{1}{2} f_0'(R_n) \tag{4.32}$$

and where we chose the complex integration constant, defining z, to be such that $z(r = 0) = 0$. In the expressions above, R_n denotes fictitious horizons for $1 \leq n \leq d-4$ and the real horizon for $n = 0$.

For values of r sufficiently close to the origin of the complex r-plane, we may approximate the logarithm in (4.31) by the truncated Taylor expansion

$$\log\left(1 - \frac{r}{R_n}\right) \sim -\sum_{k=1}^{I} \frac{1}{k} \left(\frac{r}{R_n}\right)^k. \tag{4.33}$$

for some $I \in \mathbb{N}$. The lowest order contribution from the Taylor expansion above to the sum in (4.31) is

$$-\frac{1}{2}\sum_{n=0}^{d-4} \frac{r}{k_n R_n}. \tag{4.34}$$

Using definitions (2.4) and (4.32), we can rewrite the sum above as

$$-\frac{1}{2}\sum_{n=0}^{d-4} \frac{r}{k_n R_n} = -\sum_{n=0}^{d-4} \frac{r}{(d-3)} = -r. \tag{4.35}$$

Thus, we notice the leading order contribution from the truncated Taylor expansion of the logarithm is cancelled by the first term in (4.31). As a consequence, the dominant term of z, near the origin of the complex r-plane, is provided by the next non zero contribution from the Taylor expansion to (4.31). It is easy to see that

$$\sum_{n=0}^{d-4} \frac{1}{k_n} \left(\frac{r}{R_n}\right)^k = \sum_{n=0}^{d-4} \frac{R_n^{d-2}}{(d-3) R_h^{d-3}} \left(\frac{r}{R_n}\right)^k = \frac{r^k}{d-3} \sum_{n=0}^{d-4} R_n^{1-k} = 0 \tag{4.36}$$

for $1 < k < d-2$. Indeed, noting the horizons are equally distributed along the circle $|r| = R_h$, the last equality above follows easily. For $k = d-2$, we have the contribution

$$-\frac{1}{d-2}\sum_{n=0}^{d-4}\frac{r^{d-2}}{d-3}R_h^{3-d} = -\frac{R_h^{3-d}}{d-2}\sum_{n=0}^{d-4}\frac{r^{d-2}}{d-3} = -\frac{R_h^{3-d}r^{d-2}}{d-2}. \tag{4.37}$$

Thus, we can write

$$z(r) \sim -\frac{R_h^{3-d}r^{d-2}}{d-2}. \tag{4.38}$$

near the surroundings of $r = 0$.

Now, we parametrize r as

$$r = \rho e^{i\theta} \tag{4.39}$$

for $\rho \in \mathbb{R}_0^+$ and $\theta \in [0, 2\pi[$. We know that

$$\Re\left(r^{d-2}\right) = \rho^{d-2}\cos\left((d-2)\theta\right). \tag{4.40}$$

Equating the expression above to zero yields

$$\theta = \frac{\pi}{d-2}\left(n + \frac{1}{2}\right) \tag{4.41}$$

for $1 \leq n \leq 2d-4$. Thus, we know there are $2d$-4 Stokes lines emerging from the origin of the complex r-plane, all equally distributed and separated by an angle of $\frac{\pi}{d-2}$ radians. In order to show the topology of Stokes lines, far from the origin of the complex r-plane, we display, in figure (4.1), several numerical plots of the these lines, for different dimensions.

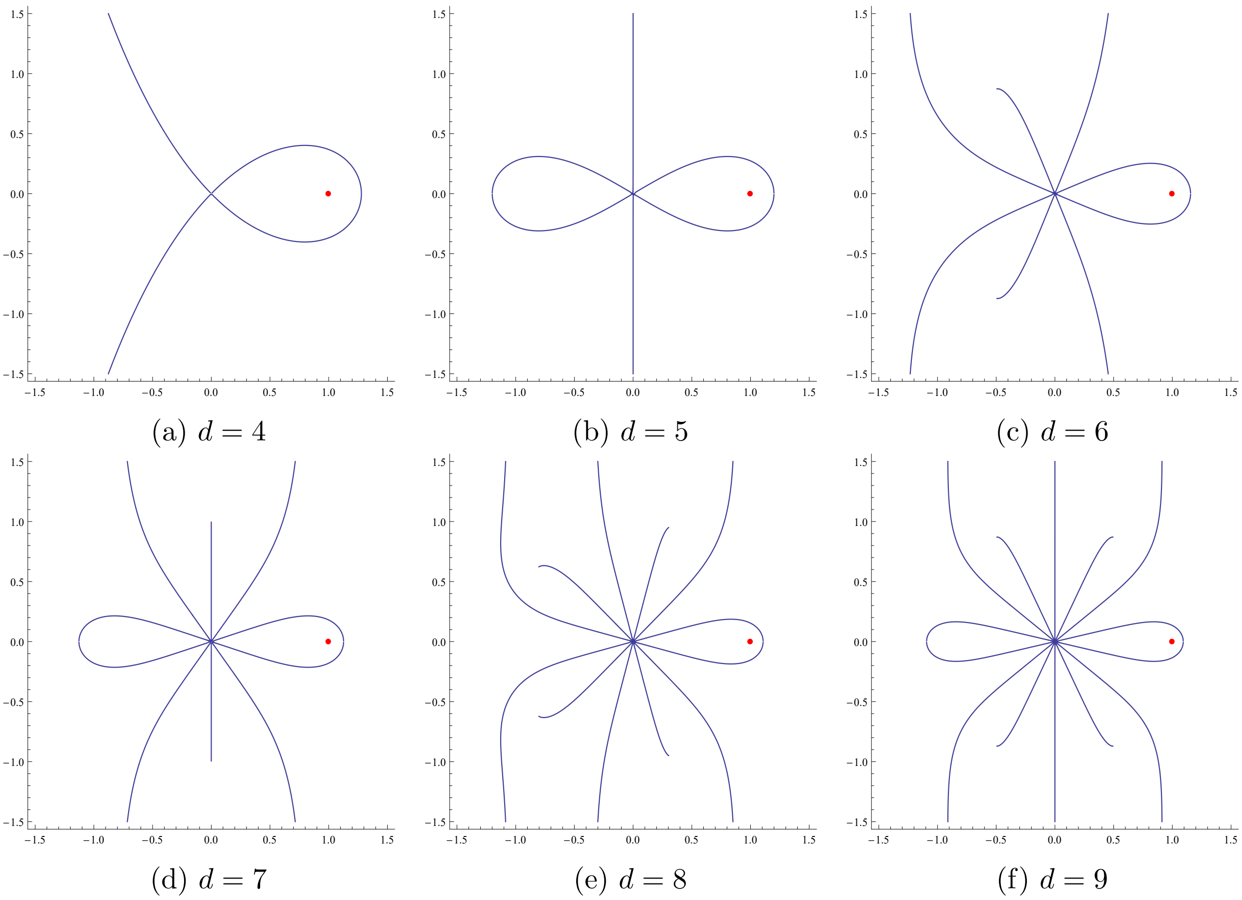

Figure 4.1: Numerical plot of the Stokes lines topology for different dimensions. The horizontal axis stands for $\Re(r)$ and the vertical axis stands for $\Im(r)$. We denoted the real horizon R_h by a red dot. These numerical plots were made using Wolfram Mathematica.

By looking at this figure, we first notice the topology of these lines, near the origin of the complex r-plane, is as we predicted. Moreover, for every dimension displayed in the figure, it appears to always exist two Stokes lines emerging from the origin of the complex r-plane, encircling the event horizon R_h. Furthermore, these lines are always followed, clock wise and counter clock wise, by two unbounded Stokes lines, asymptotically parallel to the imaginary axis. This feature does not restrict itself to the dimensions displayed in figure (4.1), being a general feature for every dimension $d \geq 4$. In [40], a good justification of this fact is presented.

Finally, for purposes that will become clear later, we also want to know the topology of the Stokes lines associated with the WKB approximation of the master equation (2.11). Although we do not show this here, it should be clear that if we were to set up the WKB formalism around (2.11), just as we did in the previous section with equation (4.11), we would end up with a WKB approximation analogous to (4.29), with z replaced by x. Hence, the Stokes lines associated with this approximation are defined by the condition

$$\Re\left(x(r)\right) = 0. \tag{4.42}$$

Integrating equation (4.2), yields the asymptotic expansion

$$z(r) \sim x(r) + \mathcal{C} \tag{4.43}$$

as $|r| \to +\infty$, for some $\mathcal{C} \in \mathbb{C}$. We set the complex integration constant, defining x, to be such that $\mathcal{C} = 0$. Hence, the Stokes lines topologies, associated with the WKB approximations of (2.11) and (4.11), match, far enough from the origin of the complex r-plane. In particular, the two previously mentioned unbounded Stokes lines are also present in the Stokes lines topology of the WKB approximation of (2.11), only differing near the origin of the complex r-plane.

4.4 Boundary conditions

Here, we discuss the boundary conditions, in spatial infinity and in the event horizon R_h.

First, we introduce a common issue in the quasinormal mode problem. It should be clear that any method devised to compute quasinormal frequencies will have to make use of the defining boundary conditions (1.69) and (1.70), such as the method we used in the last chapter. Because quasinormal frequencies are complex, gathering information of these boundary conditions amounts to distinguish between an exponentially small and an exponentially large term. Clearly, a numerical approach will face problems with such task. Moreover, any kind of analytical approximate approach will also fail to some extent. Indeed, many lower order terms of the approximation are needed in order to make sense of the exponentially decreasing term, otherwise this term might be much smaller than the approximation error and consequently needs to be disregarded.

In the last chapter, this problem was not too worrisome because quasinormal frequencies, in the eikonal limit, are approximately real. However, as we know, this is not true in general.

An elegant solution to half of this issue emerges from the moment we allow r to take complex values and consequently assume an analytic continuation of functions of r to the complex plane. Indeed, we can distinguish the two exponential terms near the event horizon, by computing the respective monodromies around it. These monodromies are non trivial because x has a branch point in the event horizon [2].

So, we are left with the task of imposing the boundary condition (1.70). As it turns out, the solution to this problem also comes with the analytic continuation of functions of r to the complex plane. Indeed, imposing the boundary condition (1.70) is equivalent to imposing the boundary condition

$$\psi(x) \propto e^{-i\omega x} \tag{4.44}$$

for $|r| \to +\infty$, in any of the two unbounded Stokes lines, mentioned in the end of the previous section [5]. Looking at (4.42), we know that

$$|e^{i\omega x}| \sim |e^{-i\omega x}| \sim 1 \tag{4.45}$$

[2] We will prove this later by explicitly computing the monodromies.

in these lines. Thus, imposing the boundary condition (4.44), no longer poses a challenge to an approximate analytical method!

In theory, we solved the problem associated with imposing the boundary conditions (1.69) and (1.70). We finish this section by relating the boundary condition (4.44) with the differential equations (4.11) and (4.12).

Looking at equation (4.43), we notice we can rewrite the boundary condition (4.44) as

$$\psi(z) \propto e^{-i\omega z} \tag{4.46}$$

for $|r| \to +\infty$, in the same Stokes lines. This boundary condition applies to the differential equation (4.4) as the latter is simply (2.11), rewritten with respect to the independent variable z. Finally, the boundary condition (4.46) also applies to (4.11) and (4.12), for both arise from a perturbative approach of (4.4). In fact, the differential equation (4.11) is the master equation associated with tensor type gravitational perturbations, in the d-dimensional Tangherlini black hole space time [40].

4.5 The monodromy method

The monodromy method uses both solutions, introduced in the previous section, to impose the appropriate boundary conditions. The general idea goes as follows:

- We pick two closed homotopic contours on the complex r-plane. Both these contours enclose only one horizon: the real one R_h. Moreover, neither one of them encloses the origin of the complex r-plane.
- One of these contours, which we name the big contour, seeks to encode information of the boundary condition (4.46) on the monodromy of ψ, associated with a full loop around it.
- The other contour, which we name the small contour, seeks to encode information of the boundary condition (1.69) on the monodromy of ψ, associated with a full loop around it.
- As both contours are homotopic, the monodromy theorem [29] asserts that the respective monodromies must be the same. Thus, equating them hopefully yields a restriction on the values of the quasinormal frequencies ω, from the complex plane to an infinite but countable subset.

This procedure should become much more clear as we approach the end of this section.

4.5.1 The big contour

In order to build the big contour, we make use of the feature we found, when studying the topology of the Stokes lines, associated with the WKB approximation (4.29). More precisely, for every dimension d, there will be two Stokes lines, emerging from the origin of the complex r-plane, encircling the event horizon R_h. Furthermore, these lines are followed, counter clock wise and clock wise, by two unbounded Stokes lines, asymptotically parallel to the imaginary axis. The big contour will follow these unbounded Stokes lines, reaching the condition $|r| \to +\infty$ twice. There, the contour abandons the Stokes lines and follows an arc shaped path enclosing it.

Overall, the big contour is well represented as depicted in figure (4.2). Looking at this figure, we notice the proportions may not be right. However, the topology of the big contour is well represented by the blue dashed line for every dimension $d \geq 5$.

The boundary condition (4.46) is to be imposed in the regions marked by D or U. Here, we choose the region D to impose it. Furthermore, we choose to follow the contour in the clockwise direction.

In the surroundings of $r = 0$, the big contour is represented as depicted in figure (4.3). Looking at equation (4.41), we notice the arc shaped portion of the big contour, depicted in this figure, sweeps an angle of $\frac{3\pi}{d-2}$ radians.

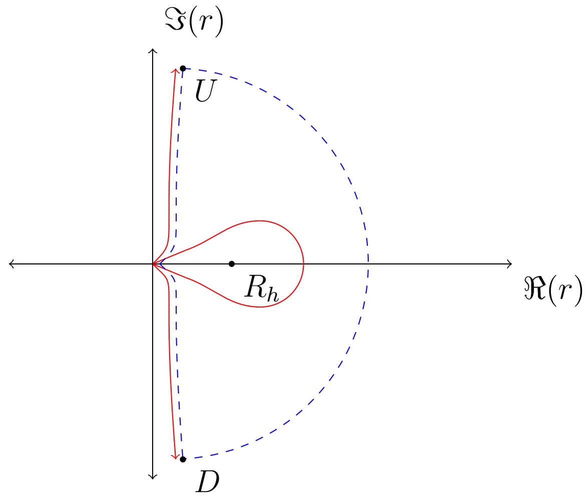

Figure 4.2: Schematic depiction of the big contour, as the blue dashed line. The Stokes lines are depicted as red curves. Naturally, not all Stokes lines are depicted. Furthermore, we marked by D and U the regions where the boundary condition (4.46) may be imposed.

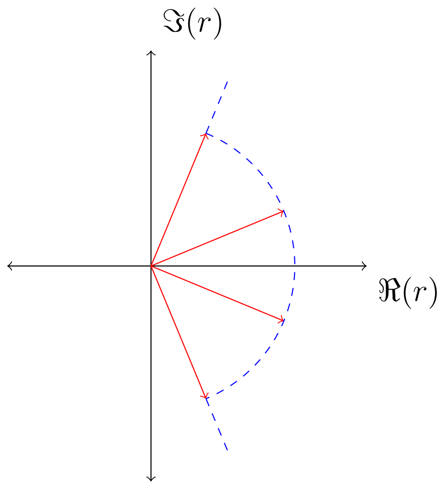

Figure 4.3: Schematic representation of the big contour, in the surroundings of $r = 0$, as the blue dashed line. The Stokes lines are represented by red curves. Naturally, not all Stokes lines are depicted.

After defining the big contour, we want to compute the monodromy of ψ, associated with a full clockwise loop around it. In order to attain reliable information on how ψ changes around the big contour, we resort to WKB theory. Indeed, we can use (4.29) in portions of the big contour, matched with Stokes lines, far enough from the the origin of the complex r-plane.

Near the origin, the WKB approximation fails as $r = 0$ is a singular point of (4.23). Thus, we need to analytically solve the differential equations (4.11) and (4.12) in this region.

Looking at definitions (4.14), (4.15), (4.16) and at the expressions (4.9) and (4.10), we can write the following asymptotic expansions

$$V_0(r) \sim -\left(\frac{d-2}{2}\right)^2 R_h^{2d-6} r^{4-2d} \tag{4.47}$$

$$\xi_1(r) \sim (d-3)(d-4)\left(\frac{R_h}{r}\right)^{d-1} \tag{4.48}$$

$$\xi_2(r) \sim \frac{(d-4)(d-3)(d-1)}{2} R_h^{2d-4} r^{3-2d} \tag{4.49}$$

$$\xi_3(r) \sim \frac{1}{4}(d-4)((d-5)d(2d-7)-22) R_h^{3d-7} r^{5-3d} \tag{4.50}$$

near the origin of the complex r-plane. Finally, we can use the asymptotic expansion (4.38) to rewrite the asymptotic expansions above as

$$V_0(z) \sim -\frac{1}{4z^2} \tag{4.51}$$

$$\xi_1(z) \sim \Gamma_1 \left(\frac{R_h}{z}\right)^{\frac{d-1}{d-2}} \tag{4.52}$$

$$\xi_2(z) \sim \Gamma_2 z^{\frac{3-2d}{d-2}} R_h^{\frac{d-1}{d-2}} \tag{4.53}$$

$$\xi_3(z) \sim \Gamma_3 z^{\frac{5-3d}{d-2}} R_h^{\frac{d-1}{d-2}} \tag{4.54}$$

near the origin of the complex r-plane, where we defined the constants

$$\Gamma_1 := (d-4)(d-3)(2-d)^{\frac{1-d}{d-2}} \tag{4.55}$$

$$\Gamma_2 := \frac{(d-4)(d-3)(d-1)}{2}(2-d)^{\frac{3-2d}{d-2}} \tag{4.56}$$

$$\Gamma_3 := \frac{1}{4}(d-4)((d-5)d(2d-7)-22)(2-d)^{\frac{5-3d}{d-2}}. \tag{4.57}$$

Using the asymptotic expansion (4.51), we can rewrite the differential equation (4.11) as

$$\frac{d^2\psi_0}{dz^2} + \left(\omega^2 + \frac{1}{4z^2}\right)\psi_0 = 0 \tag{4.58}$$

near the origin of the complex r-plane. For purposes that will become apparent soon, we rewrite the differential equation above as

$$\frac{d^2\psi_0}{dz^2} + \left(\omega^2 - \frac{j^2-1}{4z^2}\right)\psi_0 = 0 \tag{4.59}$$

and later take the limit $j \to 0$. The general solution of the differential equation above is

$$\psi_0(z) = A_+\sqrt{2\pi}\sqrt{\omega z} J_{\frac{j}{2}}(\omega z) + A_-\sqrt{2\pi}\sqrt{\omega z} J_{-\frac{j}{2}}(\omega z) \tag{4.60}$$

for some $A_\pm \in \mathbb{C}$, where

$$J_\mu(w) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\Gamma(n+\mu+1)}\left(\frac{w}{2}\right)^{2n+\mu} \tag{4.61}$$

stands for the Bessel function of the first kind [2].

The asymptotic expansion

$$J_\mu(w) \sim \sqrt{\frac{2}{\pi w}}\cos\left(w - \frac{\mu\pi}{2} - \frac{\pi}{4}\right) \tag{4.62}$$

holds for $|w| \gg 1$ [2]. Using the asymptotic expansion above, we can write the asymptotic expansion

$$\psi_0(z) \sim \left(A_+e^{i\alpha_+} + A_-e^{i\alpha_-}\right)e^{-i\omega z} + \left(A_+e^{-i\alpha_+} + A_-e^{-i\alpha_-}\right)e^{i\omega z} \tag{4.63}$$

for $|\omega z| \gg 1$, where we defined the constants

$$\alpha_\pm :== \frac{\pi}{4}(1 \pm j). \tag{4.64}$$

We notice the asymptotic expansion (4.63) makes sense. Indeed, as we are in the asymptotic limit and consequently $|\omega R_h| \gg 1$, we expect (4.63) to be valid sufficiently near the origin of the complex r-plane, where the asymptotic expansion (4.51) still makes some sense.

Following the big contour, all the way to D, we know ψ_0 will remain unchanged, relative to (4.63). This is so, because the remaining portion of the big contour, connecting the origin of the complex r-plane to D, matches with a Stokes line of the WKB approximation (4.29).

Near D, we can impose the boundary condition (4.46). Doing this yields the linear system of algebraic equations

$$\begin{cases} A_+e^{i\alpha_+} + A_-e^{i\alpha_-} = \beta \\ A_+e^{-i\alpha_+} + A_-e^{-i\alpha_-} = 0 \end{cases} \implies \begin{cases} A_- = \dfrac{\beta e^{-i\alpha_+}}{2i\sin(\alpha_- - \alpha_+)} \\ A_+ = -\dfrac{\beta e^{-i\alpha_-}}{2i\sin(\alpha_- - \alpha_+)} \end{cases} \tag{4.65}$$

for some $\beta \in \mathbb{C}$.

Now, we want to repeat this process for the non homogeneous differential equation (4.12). Using the variation of parameters method [52], we know that a particular solution of (4.12) admits the asymptotic expansion

$$\begin{aligned} \psi_1(z) \sim \sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\int \sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\frac{\xi(z)}{W(z)}dz \\ -\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\int \sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\frac{\xi(z)}{W(z)}dz \end{aligned} \tag{4.66}$$

near the origin of the complex r-plane, where W denotes the Wronskian

$$\begin{aligned} W(z) = \frac{d}{dz}\left(\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z) - \\ \frac{d}{dz}\left(\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z) = 2\pi\omega z \\ \left[\frac{d}{dz}\left(J_{-\frac{j}{2}}(\omega z)\right)J_{\frac{j}{2}}(\omega z) - \frac{d}{dz}\left(J_{\frac{j}{2}}(\omega z)\right)J_{-\frac{j}{2}}(\omega z)\right]. \end{aligned} \tag{4.67}$$

Differentiating the Bessel function yields

$$\frac{d}{dz}\left(J_{\pm\frac{j}{2}}(\omega z)\right) = \frac{\omega}{2}\left(J_{\pm\frac{j}{2}-1}(\omega z) - J_{\pm\frac{j}{2}+1}(\omega z)\right). \tag{4.68}$$

Considering the expression above, after some algebraic manipulation, we can rewrite the Wronskian (4.67) as

$$W(z) = -4\omega\sin\left(\frac{\pi j}{2}\right). \tag{4.69}$$

Motivated by definition (4.13) and by linearity of the indefinite integral, we can make the decomposition

$$\psi_1 = \sum_{k=1}^{3}\phi_k \tag{4.70}$$

near the origin of the complex r-plane, where we defined the functions

$$\begin{aligned} \phi_1(z) := \sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\int \sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\frac{\xi_1(z)}{W(z)}\frac{d^2\psi_0}{dz^2}(z)dz \\ -\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\int \sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\frac{\xi_1(z)}{W(z)}\frac{d^2\psi_0}{dz^2}(z)dz \end{aligned} \tag{4.71}$$

$$\begin{aligned} \phi_2(z) := \sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\int \sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\frac{\xi_2(z)}{W(z)}\frac{d\psi_0}{dz}(z)dz \\ -\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\int \sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\frac{\xi_2(z)}{W(z)}\frac{d\psi_0}{dz}(z)dz \end{aligned} \tag{4.72}$$

$$\begin{aligned}\phi_3(z) := \sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\int \sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\frac{\xi_3(z)}{W(z)}\psi_0(z)dz \\ -\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\int \sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\frac{\xi_3(z)}{W(z)}\psi_0(z)dz.\end{aligned} \tag{4.73}$$

To better understand the indefinite integrals above, we compute the derivatives of ψ_0. Looking at (4.60) and using (4.68), after some algebraic manipulation, we can write the asymptotic expansions

$$\begin{aligned}\frac{d\psi_0}{dz}(z) \sim \sqrt{\frac{\pi}{2}}\sqrt{\frac{\omega}{z}}A_+\left(J_{\frac{j}{2}}(\omega z) + \omega z J_{\frac{j}{2}-1}(\omega z) - \omega z J_{\frac{j}{2}+1}(\omega z)\right) \\ +\sqrt{\frac{\pi}{2}}\sqrt{\frac{\omega}{z}}A_-\left(J_{-\frac{j}{2}}(\omega z) + \omega z J_{-\frac{j}{2}-1}(\omega z) - \omega z J_{1-\frac{j}{2}}(\omega z)\right)\end{aligned} \tag{4.74}$$

$$\frac{d^2\psi_0}{dz^2}(z) \sim \sqrt{\frac{\pi}{8}}\sqrt{\frac{\omega}{z^3}}\left(j^2 - 4\omega^2 z^2 - 1\right)\left(A_+J_{\frac{j}{2}}(\omega z) + A_-J_{-\frac{j}{2}}(\omega z)\right). \tag{4.75}$$

near the origin of the complex r-plane. Using the asymptotic expansions above, we can rewrite (4.72) as

$$\begin{aligned}\phi_2(z) = \left(\Omega_2^- A_-\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\int\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)z^\delta dz \\ +\left(\Omega_2^+ A_-\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\int\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)z^\delta dz \\ +\left(\Omega_2^- A_+\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\int\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)z^\delta dz \\ +\left(\Omega_2^+ A_+\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\int\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)z^\delta dz \\ +\left(\Omega_2^- A_-\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\int\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{-\frac{j}{2}-1}(\omega z)\right)\omega z^{\delta+1}dz \\ +\left(\Omega_2^+ A_-\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\int\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{-\frac{j}{2}-1}(\omega z)\right)\omega z^{\delta+1}dz \\ +\left(\Omega_2^- A_+\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\int\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{\frac{j}{2}-1}(\omega z)\right)\omega z^{\delta+1}dz \\ +\left(\Omega_2^+ A_+\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\int\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{\frac{j}{2}-1}(\omega z)\right)\omega z^{\delta+1}dz \\ -\left(\Omega_2^- A_-\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\int\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{-\frac{j}{2}+1}(\omega z)\right)\omega z^{\delta+1}dz \\ -\left(\Omega_2^+ A_-\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\int\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{-\frac{j}{2}+1}(\omega z)\right)\omega z^{\delta+1}dz \\ -\left(\Omega_2^- A_+\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\int\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{\frac{j}{2}+1}(\omega z)\right)\omega z^{\delta+1}dz \\ -\left(\Omega_2^+ A_+\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\int\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{\frac{j}{2}+1}(\omega z)\right)\omega z^{\delta+1}dz\end{aligned} \tag{4.76}$$

where we defined the constants

$$\Omega_2^\pm := \pm\frac{\pi\Gamma_2 R_h^{\frac{d-1}{d-2}}}{4\omega}\csc\left(\frac{\pi j}{2}\right) \qquad \delta := \frac{5-3d}{d-2} \tag{4.77}$$

in order to simplify the notation. In the same fashion, we can rewrite (4.71) and (4.73) as

$$\begin{aligned}\phi_1(z) = \left(\Omega_1^- A_-\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\int\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\left(j^2-4\omega^2z^2-1\right)z^\delta dz \\ +\left(\Omega_1^+ A_-\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\int\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\left(j^2-4\omega^2z^2-1\right)z^\delta dz \\ +\left(\Omega_1^- A_+\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\int\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\left(j^2-4\omega^2z^2-1\right)z^\delta dz\end{aligned}$$

$$+\left(\Omega_1^+ A_+\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\int\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\left(j^2-4\omega^2 z^2-1\right)z^\delta dz \tag{4.78}$$

$$\begin{aligned}\phi_3(z) = &\left(\Omega_3^- A_-\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\int\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)z^\delta dz\\ &+\left(\Omega_3^+ A_-\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\int\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)z^\delta dz\\ &+\left(\Omega_3^- A_+\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\int\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)z^\delta dz\\ &+\left(\Omega_3^+ A_+\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\int\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)z^\delta dz\end{aligned} \tag{4.79}$$

where we analogously defined the constants

$$\Omega_1^\pm := \pm\frac{\pi\Gamma_1 R_h^{\frac{d-1}{d-2}}}{8\omega}\csc\left(\frac{\pi j}{2}\right) \qquad \Omega_3^\pm := \pm\frac{\pi\Gamma_3 R_h^{\frac{d-1}{d-2}}}{2\omega}\csc\left(\frac{\pi j}{2}\right). \tag{4.80}$$

Just as we did when considering ψ_0, we want to study the behaviour of ψ_1, in the portion of the big contour connecting the origin of the complex r-plane to D.

We start by looking for an asymptotic expansion of ψ_1 in the limit $\omega z \to +\infty$ [3]. To this end, we consider the generic indefinite integral

$$P_{\mu\nu\epsilon}(w) := \int w^\epsilon J_\mu(w)J_\nu(w)dw \tag{4.81}$$

for $\mu,\nu\in\mathbb{R}$ and $\epsilon<0$. This indefinite integral, although complicated at first sight, has a rather simple asymptotic expansion. Indeed, one can write

$$P_{\mu\nu\epsilon}(w) \sim \frac{\Gamma\left(\frac{1}{2}-\frac{\epsilon}{2}\right)\Gamma\left(-\frac{\epsilon}{2}\right)\Gamma\left(\frac{\epsilon}{2}+\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2}\right)}{2\sqrt{\pi}\Gamma\left(-\frac{\epsilon}{2}+\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2}\right)\Gamma\left(-\frac{\epsilon}{2}+\frac{\nu}{2}-\frac{\mu}{2}+\frac{1}{2}\right)\Gamma\left(-\frac{\epsilon}{2}+\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2}\right)} \tag{4.82}$$

for $w\to+\infty$. In order to simplify the notation, we define the constants

$$\mathcal{H}(\mu,\nu,\epsilon) := \frac{\Gamma\left(\frac{1}{2}-\frac{\epsilon}{2}\right)\Gamma\left(-\frac{\epsilon}{2}\right)\Gamma\left(\frac{\epsilon}{2}+\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2}\right)}{2\sqrt{\pi}\Gamma\left(-\frac{\epsilon}{2}+\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2}\right)\Gamma\left(-\frac{\epsilon}{2}+\frac{\nu}{2}-\frac{\mu}{2}+\frac{1}{2}\right)\Gamma\left(-\frac{\epsilon}{2}+\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2}\right)}. \tag{4.83}$$

Using (4.82) and (4.62), allow us to write the asymptotic expansions

$$\phi_k(z) \sim \left(\Theta_k^+ e^{i\alpha_+}+\Theta_k^- e^{i\alpha_-}\right)e^{-i\omega z}+\left(\Theta_k^+ e^{-i\alpha_+}+\Theta_k^- e^{-i\alpha_-}\right)e^{i\omega z} \tag{4.84}$$

for $\omega z\gg 1$, with $k=1,2,3$. In the asymptotic expansions above, we defined the constants

$$\begin{aligned}\Theta_1^+ := &\ \Omega_1^+\omega^{-\delta-1}\left(j^2-1\right)\left[A_-\mathcal{H}\left(-\frac{j}{2},-\frac{j}{2},\delta+1\right)+A_+\mathcal{H}\left(-\frac{j}{2},\frac{j}{2},\delta+1\right)\right] \qquad (4.85)\\ &-4\Omega_1^+\omega^{-\delta-1}\left[A_-\mathcal{H}\left(-\frac{j}{2},-\frac{j}{2},\delta+3\right)+A_+\mathcal{H}\left(-\frac{j}{2},\frac{j}{2},\delta+3\right)\right]\\ \Theta_1^- := &\ \Omega_1^-\omega^{-\delta-1}\left(j^2-1\right)\left[A_-\mathcal{H}\left(\frac{j}{2},-\frac{j}{2},\delta+1\right)+A_+\mathcal{H}\left(\frac{j}{2},\frac{j}{2},\delta+1\right)\right] \qquad (4.86)\\ &-4\Omega_1^-\omega^{-\delta-1}\left[A_-\mathcal{H}\left(\frac{j}{2},-\frac{j}{2},\delta+3\right)+A_+\mathcal{H}\left(\frac{j}{2},\frac{j}{2},\delta+3\right)\right]\\ \Theta_2^+ := &\ \Omega_2^+\omega^{-\delta-1}\left[A_-\mathcal{H}\left(-\frac{j}{2},-\frac{j}{2},\delta+1\right)+A_+\mathcal{H}\left(-\frac{j}{2},\frac{j}{2},\delta+1\right)\right] \qquad (4.87)\end{aligned}$$

[3]Indeed, this is the limit directing to the lower portion of the big contour. This is so, because we are considering that $\Im(\omega)>0$.

$$
\begin{aligned}
&+\Omega_2^+\omega^{-\delta-1}\left[A_-\mathcal{H}\left(-\frac{j}{2},-\frac{j}{2}-1,\delta+2\right)+A_+\mathcal{H}\left(-\frac{j}{2},\frac{j}{2}-1,\delta+2\right)\right]\\
&-\Omega_2^+\omega^{-\delta-1}\left[A_-\mathcal{H}\left(-\frac{j}{2},1-\frac{j}{2},\delta+2\right)+A^+\mathcal{H}\left(-\frac{j}{2},\frac{j}{2}+1,\delta+2\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\Theta_2^- &:= \Omega_2^-\omega^{-\delta-1}\left[A_-\mathcal{H}\left(\frac{j}{2},-\frac{j}{2},\delta+1\right)+A_+\mathcal{H}\left(\frac{j}{2},\frac{j}{2},\delta+1\right)\right]\\
&+\Omega_2^-\omega^{-\delta-1}\left[A_-\mathcal{H}\left(\frac{j}{2},-\frac{j}{2}-1,\delta+2\right)+A_+\mathcal{H}\left(\frac{j}{2},\frac{j}{2}-1,\delta+2\right)\right]\\
&-\Omega_2^-\omega^{-\delta-1}\left[A_-\mathcal{H}\left(\frac{j}{2},1-\frac{j}{2},\delta+2\right)+A^+\mathcal{H}\left(\frac{j}{2},\frac{j}{2}+1,\delta+2\right)\right]
\end{aligned}
\tag{4.88}
$$

$$
\Theta_3^+ := \Omega_3^+\omega^{-\delta-1}\left[A_-\mathcal{H}\left(-\frac{j}{2},-\frac{j}{2},\delta+1\right)+A_+\mathcal{H}\left(-\frac{j}{2},\frac{j}{2},\delta+1\right)\right] \tag{4.89}
$$

$$
\Theta_3^- := \Omega_3^-\omega^{-\delta-1}\left[A_-\mathcal{H}\left(\frac{j}{2},-\frac{j}{2},\delta+1\right)+A_+\mathcal{H}\left(\frac{j}{2},\frac{j}{2},\delta+1\right)\right] \tag{4.90}
$$

in order to simplify the notation. Now, defining the constants

$$
\Lambda_I^\pm := \sum_{k=1}^{3}\Theta_k^\pm \tag{4.91}
$$

allow us to write the simple asymptotic expansion

$$
\psi_1(z) \sim \left(\Lambda_I^+ e^{i\alpha_+}+\Lambda_I^- e^{i\alpha_-}\right)e^{-i\omega z}+\left(\Lambda_I^+ e^{-i\alpha_+}+\Lambda_I^- e^{-i\alpha_-}\right)e^{i\omega z} \tag{4.92}
$$

for $\omega z \gg 1$. We notice the asymptotic expansion above is a linear combination of plane waves, just like (4.63).

Now, we can ask if ψ_1 remains unchanged, relative to the expression above, while following the remaining portion of the big contour, connecting the origin of the complex r-plane to D. This turns out to be true. Indeed, from arguments previously stated, we know ψ_0 will remain unchanged, relative to (4.63), along this portion of the big contour. Hence, up to an additive linear combination of plane waves, such as (4.92), we can write ψ_1 as

$$
\psi_1(z) \sim e^{-i\omega z}\int e^{i\omega z}\frac{\xi(z)}{W(z)}dz - e^{i\omega z}\int e^{-i\omega z}\frac{\xi(z)}{W(z)}dz \tag{4.93}
$$

along the remaining portion of the big contour, connecting the origin of the complex r-plane to D. Here, the Wronskian can be expressed as

$$
W(z) \sim \frac{d}{dz}\left(e^{-i\omega z}\right)e^{i\omega z} - \frac{d}{dz}\left(e^{i\omega z}\right)e^{-i\omega z} = -2i\omega. \tag{4.94}
$$

Looking at (4.13) and (4.63), we can write

$$
\begin{aligned}
\xi(z) \sim -\omega^2\xi_1(z)&\left[\left(A_+e^{i\alpha_+}+A_-e^{i\alpha_-}\right)e^{-i\omega z}+\left(A_+e^{-i\alpha_+}+A_-e^{-i\alpha_-}\right)e^{i\omega z}\right]\\
\omega i\xi_2(z)&\left[\left(A_+e^{-i\alpha_+}+A_-e^{-i\alpha_-}\right)e^{i\omega z}-\left(A_+e^{i\alpha_+}+A_-e^{i\alpha_-}\right)e^{-i\omega z}\right]\\
\xi_3(z)&\left[\left(A_+e^{i\alpha_+}+A_-e^{i\alpha_-}\right)e^{-i\omega z}+\left(A_+e^{-i\alpha_+}+A_-e^{-i\alpha_-}\right)e^{i\omega z}\right]
\end{aligned}
\tag{4.95}
$$

in this portion of the big contour. Looking at the previous two expressions, we notice (4.93) is at most proportional to ω. However, we know the asymptotic expansion (4.92) is proportional to $\omega^{-\delta-2}$. Therefore, the relative contribution from (4.93) to (4.92), along the remaining portion of the big contour, connecting the origin of the complex r-plane to D, is of order

$$
\delta+3 = \frac{5-3d}{d-2}+3 = -\frac{1}{d-2} < 0 \tag{4.96}
$$

in ω. As we are considering the asymptotic limit and consequently assuming $|\omega R_h| \gg 1$, we can ignore this contribution, assuming ψ_1 to remain constant relative to (4.92), along the remaining portion of the big contour, connecting the origin of the complex r-plane to D [4].

Near D, we notice the expression (4.92) does not agree with the boundary condition (4.46). In order to fix this, we need to redefine the particular solution ψ_1 in an appropriate fashion. We do this by adding an adequate solution of the homogeneous differential equation, associated with (4.12). Near the origin of the complex r-plane, the general solution of this equation is

$$\chi(z) \sim B_+ \sqrt{2\pi}\sqrt{\omega z} J_{\frac{j}{2}}(\omega z) + B_- \sqrt{2\pi}\sqrt{\omega z} J_{-\frac{j}{2}}(\omega z) \tag{4.97}$$

for some $B_\pm \in \mathbb{C}$. Now, we redefine ψ_1 as

$$\psi_1 \mapsto \psi_1 + \chi. \tag{4.98}$$

Considering the asymptotic expansions (4.62) and (4.92), we can write the redefined asymptotic expansion

$$\begin{aligned}\psi_1(z) \sim & \left[\left(\Lambda_I^+ + B_+\right) e^{i\alpha_+} + \left(\Lambda_I^- + B_-\right) e^{i\alpha_-}\right] e^{-i\omega z} \\ & + \left[\left(\Lambda_I^+ + B_+\right) e^{-i\alpha_+} + \left(\Lambda_I^- + B_-\right) e^{-i\alpha_-}\right] e^{i\omega z}.\end{aligned} \tag{4.99}$$

for $\omega z \gg 1$. In order to fix the boundary condition problem, we choose [5] $B_\pm$ as the unique solution of the linear system of algebraic equations

$$\begin{cases} B_+ e^{i\alpha_+} + B_- e^{i\alpha_-} = 0 \\ B_+ e^{-i\alpha_+} + B_- e^{-i\alpha_-} = -\left(\Lambda_I^+ e^{-i\alpha_+} + \Lambda_I^- e^{-i\alpha_-}\right) \end{cases} \implies \begin{cases} B_- = \dfrac{e^{i\alpha_+}\left(\Lambda_I^+ e^{-i\alpha_+} + \Lambda_I^- e^{-i\alpha_-}\right)}{2i \sin\left(\alpha_- - \alpha_+\right)} \\ B_+ = -\dfrac{e^{i\alpha_-}\left(\Lambda_I^+ e^{-i\alpha_+} + \Lambda_I^- e^{-i\alpha_-}\right)}{2i \sin\left(\alpha_- - \alpha_+\right)}. \end{cases} \tag{4.100}$$

Using (4.100), we can rewrite the asymptotic expansion (4.99) as

$$\psi_1(z) \sim \left(\Lambda_I^+ e^{i\alpha_+} + \Lambda_I^- e^{i\alpha_-}\right) e^{-i\omega z}. \tag{4.101}$$

Considering (4.7), we can write

$$\begin{aligned}\psi(z) &\sim \left(A_+ e^{i\alpha_+} + A_- e^{i\alpha_-}\right) e^{-i\omega z} + \lambda' \left(\Lambda_I^+ e^{i\alpha_+} + \Lambda_I^- e^{i\alpha_-}\right) e^{-i\omega z} \\ &= \left(A_+ e^{i\alpha_+} + A_- e^{i\alpha_-}\right) e^{-i\omega z} \left[1 + \lambda' \left(\frac{\Lambda_I^+ e^{i\alpha_+} + \Lambda_I^- e^{i\alpha_-}}{A_+ e^{i\alpha_+} + A_- e^{i\alpha_-}}\right)\right]\end{aligned} \tag{4.102}$$

near D.

Now, we want to understand how ψ changes in the portion of the big contour close to the origin of the complex r-plane. There, the contour performs a $\frac{3\pi}{d-2}$ radians rotation around the origin. Thus, we want to understand how ψ_0 and ψ_1 change under such rotation. First, we need to translate this rotation to the complex z-plane, since both ψ_0 and ψ_1 are written with respect to this variable. Looking at (4.38), we notice that a $\frac{3\pi}{d-2}$ radians rotation, around the origin of the complex r-plane, corresponds to a 3π radians rotation around the origin of the complex z-plane. Taking a look at (4.61), we can write the Bessel functions as

$$J_\mu(w) = \left(\frac{w}{2}\right)^\mu \mathcal{J}_\mu(w) \tag{4.103}$$

where

$$\mathcal{J}_\mu(w) := \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\Gamma\left(n+\mu+1\right)} \left(\frac{w}{2}\right)^{2n} \tag{4.104}$$

[4]We notice this consideration is consistent with the way we computed the asymptotic expansion (4.92). Indeed, the constants (4.85) and (4.86) are well defined only if $\delta + 3 < 0$.

[5]By choosing a specific particular solution, using χ defined by the linear system (4.100), we are losing some generality in the argument. Indeed, there is an uncountable family of particular solutions ψ_1 for which the boundary condition (4.46) holds. In appendix A, we prove all such solutions yield the same monodromy. Thus, supporting the loss of generality.

is an even holomorphic function of w. Moreover, the indefinite integral (4.81) evaluates to

$$P_{\mu\nu\epsilon}(w) = \frac{2^{-\mu-\nu} w^{\epsilon+\mu+\nu+1}}{\Gamma(\mu+1)\Gamma(\nu+1)(\epsilon+\mu+\nu+1)} \mathcal{P}_{\mu\nu\epsilon}(w) \tag{4.105}$$

where we defined the function

$$\mathcal{P}_{\mu\nu\epsilon}(w) \coloneqq {}_3F_4\left(\frac{\gamma}{2}, \frac{\gamma+1}{2}, \frac{\epsilon+\gamma}{2}; \mu+1, \frac{\epsilon+\gamma+2}{2}, \nu+1, \gamma; -w^2\right) \tag{4.106}$$

and the constant

$$\gamma \coloneqq \mu+\nu+1. \tag{4.107}$$

Here, the function

$${}_3F_4\left(a,b,c;d,e,f,g;w\right) = \sum_{n=0}^{+\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c+n)\Gamma(d)\Gamma(e)\Gamma(f)\Gamma(g)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d+n)\Gamma(e+n)\Gamma(f+n)\Gamma(g+n)} \frac{w^n}{n!} \tag{4.108}$$

stands for a generalised hypergeometric function [2]. This function is holomorphic in the complex w-plane if

$$d, e, f, g \notin \mathbb{Z}_{\leq 0}. \tag{4.109}$$

Hence, we know (4.106) is holomorphic and even in the complex w-plane, for all values of μ, ν and ϵ we are interested in.

Performing a 3π radians rotation around the origin of the complex w-plane yields

$$\begin{aligned} J_\mu\left(e^{3\pi i}w\right) &= e^{3\pi i\mu}\left(\frac{w}{2}\right)^\mu \mathcal{J}_\mu(e^{3\pi i}w) = e^{3\pi i\mu}\left(\frac{w}{2}\right)^\mu \mathcal{J}_\mu(e^{\pi i}w) \\ &= e^{3\pi i\mu}\left(\frac{w}{2}\right)^\mu \mathcal{J}_\mu(w) = e^{3\pi i\mu} J_\mu(w) \end{aligned} \tag{4.110}$$

where in the second equality we used the fact that $\mathcal{J}_\mu$ is holomorphic in $w=0$ and in the third equality we used the fact that $\mathcal{J}_\mu$ is even. Analogously, we see that

$$P_{\mu\nu\epsilon}(e^{3\pi i}w) = e^{3\pi i(\epsilon+\mu+\nu+1)} P_{\mu\nu\epsilon}(w). \tag{4.111}$$

Looking at (4.60) and using (4.110), we can write

$$\begin{aligned} \psi_0(e^{3\pi i}z) &= A_+\sqrt{2\pi}\sqrt{\omega e^{3\pi i}z}e^{3\pi i\frac{j}{2}}J_{\frac{j}{2}}(\omega z) + A_-\sqrt{2\pi}\sqrt{\omega e^{3\pi i}z}e^{-3\pi i\frac{j}{2}}J_{-\frac{j}{2}}(\omega z) \\ &= A_+\sqrt{2\pi}\sqrt{\omega z}e^{\frac{3\pi i}{2}}e^{3\pi i\frac{j}{2}}J_{\frac{j}{2}}(\omega z) + A_-\sqrt{2\pi}\sqrt{\omega z}e^{\frac{3\pi i}{2}}e^{-3\pi i\frac{j}{2}}J_{-\frac{j}{2}}(\omega z) \\ &= A_+\sqrt{2\pi}\sqrt{\omega z}e^{6i\alpha_+}J_{\frac{j}{2}}(\omega z) + A_-\sqrt{2\pi}\sqrt{\omega z}e^{6i\alpha_-}J_{-\frac{j}{2}}(\omega z) \end{aligned} \tag{4.112}$$

near the origin of the complex r-plane. In the last equality above, we used the definitions (4.64). Looking at (4.76), (4.110) and (4.111), we can write

$$\begin{aligned} \phi_2(e^{3\pi i}z) = &\left(\Omega_2^- A_- e^{6i\alpha_-}\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)e^{3\pi i(\delta+2)}\int\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)z^\delta dz \\ &+ \left(\Omega_2^+ A_- e^{6i\alpha_+}\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)e^{3\pi i(\delta+2-j)}\int\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)z^\delta dz \\ &+ \left(\Omega_2^- A_+ e^{6i\alpha_-}\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)e^{3\pi i(\delta+2+j)}\int\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)z^\delta dz \\ &+ \left(\Omega_2^+ A_+ e^{6i\alpha_+}\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)e^{3\pi i(\delta+2)}\int\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)z^\delta dz \\ &+ \left(\Omega_2^- A_- e^{6i\alpha_-}\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)e^{3\pi i(\delta+2)}\int\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{-\frac{j}{2}-1}(\omega z)\right)\omega z^{\delta+1}dz \\ &+ \left(\Omega_2^+ A_- e^{6i\alpha_+}\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)e^{3\pi i(\delta+2-j)}\int\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{-\frac{j}{2}-1}(\omega z)\right)\omega z^{\delta+1}dz \end{aligned}$$

$$+\left(\Omega_2^- A_+ e^{6i\alpha_-}\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)e^{3\pi i(\delta+2+j)}\int\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{\frac{j}{2}-1}(\omega z)\right)\omega z^{\delta+1}dz$$
$$+\left(\Omega_2^+ A_+ e^{6i\alpha_+}\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)e^{3\pi i(\delta+2)}\int\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{\frac{j}{2}-1}(\omega z)\right)\omega z^{\delta+1}dz$$
$$-\left(\Omega_2^- A_- e^{6i\alpha_-}\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)e^{3\pi i(\delta+4)}\int\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{-\frac{j}{2}+1}(\omega z)\right)\omega z^{\delta+1}dz$$
$$-\left(\Omega_2^+ A_- e^{6i\alpha_+}\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)e^{3\pi i(\delta+4-j)}\int\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{-\frac{j}{2}+1}(\omega z)\right)\omega z^{\delta+1}dz$$
$$-\left(\Omega_2^- A_+ e^{6i\alpha_-}\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)e^{3\pi i(\delta+4+j)}\int\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{\frac{j}{2}+1}(\omega z)\right)\omega z^{\delta+1}dz$$
$$-\left(\Omega_2^+ A_+ e^{6i\alpha_+}\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)e^{\delta+4}\int\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{\frac{j}{2}+1}(\omega z)\right)\omega z^{\delta+1}dz \tag{4.113}$$

near the origin of the complex r-plane. In the same fashion, we can write

$$\phi_1(e^{3\pi i}z)=\left(\Omega_1^- A_- e^{6\alpha_-}\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)e^{3\pi i(\delta+2)}\int\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)z^{\delta}(j^2-1)dz$$
$$+\left(\Omega_1^+ A_- e^{6\alpha_+}\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)e^{3\pi i(\delta+2-j)}\int\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)z^{\delta}(j^2-1)dz$$
$$+\left(\Omega_1^- A_+ e^{6\alpha_-}\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)e^{3\pi i(\delta+2+j)}\int\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)z^{\delta}(j^2-1)dz$$
$$+\left(\Omega_1^+ A_+ e^{6\alpha_+}\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)e^{3\pi i(\delta+2)}\int\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)z^{\delta}(j^2-1)dz$$
$$-\left(\Omega_1^- A_- e^{6\alpha_-}\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)e^{3\pi i(\delta+4)}\int\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)z^{\delta+2}4\omega^2dz$$
$$-\left(\Omega_1^+ A_- e^{6\alpha_+}\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)e^{3\pi i(\delta+4-j)}\int\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)z^{\delta+2}4\omega^2dz$$
$$-\left(\Omega_1^- A_+ e^{6\alpha_-}\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)e^{3\pi i(\delta+4+j)}\int\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)z^{\delta+2}4\omega^2dz$$
$$-\left(\Omega_1^+ A_+ e^{6\alpha_+}\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)e^{3\pi i(\delta+4)}\int\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)z^{\delta+2}4\omega^2dz \tag{4.114}$$

$$\phi_3(e^{3\pi i}z)=\left(\Omega_3^- A_- e^{6\alpha_-}\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)e^{3\pi i(\delta+2)}\int\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)z^{\delta}dz$$
$$+\left(\Omega_3^+ A_- e^{6\alpha_+}\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)e^{3\pi i(\delta+2-j)}\int\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)z^{\delta}dz$$
$$+\left(\Omega_3^- A_+ e^{6\alpha_-}\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)e^{3\pi i(\delta+2+j)}\int\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)z^{\delta}dz$$
$$+\left(\Omega_3^+ A_+ e^{6\alpha_+}\right)\left(\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)e^{3\pi i(\delta+2)}\int\left(\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z)\right)\left(\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)\right)z^{\delta}dz \tag{4.115}$$

near the origin of the complex r-plane.

Now, we want to know how ψ_0 and ψ_1 behave along the portion of the big contour, connecting the origin of the complex r-plane to U. First, we study the limit

$$e^{3\pi i}\omega z\to-\infty \tag{4.116}$$

as this is the limit directing towards the upper portion of the big contour. However, this limit is equivalent to $\omega z\to+\infty$. Thus, we still can use the asymptotic expansion (4.82). Using (4.62) on (4.112), yields the asymptotic expansion

$$\psi_0(e^{3\pi i}z)\sim\left(A_+e^{5i\alpha_+}+A_-e^{5i\alpha_-}\right)e^{i\omega z}+\left(A_+e^{7i\alpha_+}+A_-e^{7i\alpha_-}\right)e^{-i\omega z} \tag{4.117}$$

for $\omega z \gg 1$. Analogously, using (4.82) and (4.62) on (4.113), (4.114) and (4.115) yields the asymptotic expansions

$$\phi_k(e^{3\pi i}z) \sim \left(\Xi_k^+ e^{5i\alpha_+} + \Xi_k^- e^{5i\alpha_-}\right) e^{i\omega z} + \left(\Xi_k^+ e^{7i\alpha_+} + \Xi_k^- e^{7i\alpha_-}\right) e^{-i\omega z} \tag{4.118}$$

for $\omega z \gg 1$, with $k = 1, 2, 3$. In the asymptotic expansions above, we defined the constants

$$\begin{aligned}
\Xi_1^+ &:= \Omega_1^+ \omega^{-\delta-1} e^{3\pi i(\delta+2)} \left(j^2-1\right) \left[A_- e^{-3\pi i j} \mathcal{H}\left(-\frac{j}{2}, -\frac{j}{2}, \delta+1\right) + A_+ \mathcal{H}\left(-\frac{j}{2}, \frac{j}{2}, \delta+1\right)\right] && (4.119)\\
&\quad -4\Omega_1^+ \omega^{-\delta-1} e^{3\pi i(\delta+4)} \left[A_- e^{-3\pi i j} \mathcal{H}\left(-\frac{j}{2}, -\frac{j}{2}, \delta+3\right) + A_+ \mathcal{H}\left(-\frac{j}{2}, \frac{j}{2}, \delta+3\right)\right] \\
\Xi_1^- &:= \Omega_1^- \omega^{-\delta-1} e^{3\pi i(\delta+2)} \left(j^2-1\right) \left[A_- \mathcal{H}\left(\frac{j}{2}, -\frac{j}{2}, \delta+1\right) + A_+ e^{3\pi i j} \mathcal{H}\left(\frac{j}{2}, \frac{j}{2}, \delta+1\right)\right] && (4.120)\\
&\quad -4\Omega_1^- \omega^{-\delta-1} e^{3\pi i(\delta+4)} \left[A_- \mathcal{H}\left(\frac{j}{2}, -\frac{j}{2}, \delta+3\right) + A_+ e^{3\pi i j} \mathcal{H}\left(\frac{j}{2}, \frac{j}{2}, \delta+3\right)\right] \\
\Xi_2^+ &:= \Omega_2^+ \omega^{-\delta-1} e^{3\pi i(\delta+2)} \left[A_- e^{-3\pi i j} \mathcal{H}\left(-\frac{j}{2}, -\frac{j}{2}, \delta+1\right) + A_+ \mathcal{H}\left(-\frac{j}{2}, \frac{j}{2}, \delta+1\right)\right] && (4.121)\\
&\quad +\Omega_2^+ \omega^{-\delta-1} e^{3\pi i(\delta+2)} \left[A_- e^{-3\pi i j} \mathcal{H}\left(-\frac{j}{2}, -\frac{j}{2}-1, \delta+2\right) + A_+ \mathcal{H}\left(-\frac{j}{2}, \frac{j}{2}-1, \delta+2\right)\right] \\
&\quad -\Omega_2^+ \omega^{-\delta-1} e^{3\pi i(\delta+4)} \left[A_- e^{-3\pi i j} \mathcal{H}\left(-\frac{j}{2}, 1-\frac{j}{2}, \delta+2\right) + A^+ \mathcal{H}\left(-\frac{j}{2}, \frac{j}{2}+1, \delta+2\right)\right] \\
\Xi_2^- &:= \Omega_2^- \omega^{-\delta-1} e^{3\pi i(\delta+2)} \left[A_- \mathcal{H}\left(\frac{j}{2}, -\frac{j}{2}, \delta+1\right) + A_+ e^{3\pi i j} \mathcal{H}\left(\frac{j}{2}, \frac{j}{2}, \delta+1\right)\right] && (4.122)\\
&\quad +\Omega_2^- \omega^{-\delta-1} e^{3\pi i(\delta+2)} \left[A_- \mathcal{H}\left(\frac{j}{2}, -\frac{j}{2}-1, \delta+2\right) + A_+ e^{3\pi i j} \mathcal{H}\left(\frac{j}{2}, \frac{j}{2}-1, \delta+2\right)\right] \\
&\quad -\Omega_2^- \omega^{-\delta-1} e^{3\pi i(\delta+4)} \left[A_- \mathcal{H}\left(\frac{j}{2}, 1-\frac{j}{2}, \delta+2\right) + A^+ e^{3\pi i j} \mathcal{H}\left(\frac{j}{2}, \frac{j}{2}+1, \delta+2\right)\right] \\
\Xi_3^+ &:= \Omega_3^+ \omega^{-\delta-1} e^{3\pi i(\delta+2)} \left[A_- e^{-3\pi i j} \mathcal{H}\left(-\frac{j}{2}, -\frac{j}{2}, \delta+1\right) + A_+ \mathcal{H}\left(-\frac{j}{2}, \frac{j}{2}, \delta+1\right)\right] && (4.123)\\
\Xi_3^- &:= \Omega_3^- \omega^{-\delta-1} e^{3\pi i(\delta+2)} \left[A_- \mathcal{H}\left(\frac{j}{2}, -\frac{j}{2}, \delta+1\right) + A_+ e^{3\pi i j} \mathcal{H}\left(\frac{j}{2}, \frac{j}{2}, \delta+1\right)\right] && (4.124)
\end{aligned}$$

in order to simplify the notation. Finally, using (4.62) on (4.97) yields the asymptotic expansion

$$\chi(e^{3\pi i}z) \sim \left(B_+ e^{5i\alpha_+} + B_- e^{5i\alpha_-}\right) e^{i\omega z} + \left(B_+ e^{7i\alpha_+} + B_- e^{7i\alpha_-}\right) e^{-i\omega z}. \tag{4.125}$$

We define the constants

$$\Lambda_F^\pm := \sum_{k=1}^{3} \Xi_k^\pm + B_\pm. \tag{4.126}$$

Using the definitions above together with (4.118) and (4.125) yields the simple asymptotic expansion

$$\psi_1(e^{3\pi i}z) \sim \left(\Lambda_F^+ e^{5i\alpha_+} + \Lambda_F^- e^{5i\alpha_-}\right) e^{i\omega z} + \left(\Lambda_F^+ e^{7i\alpha_+} + \Lambda_F^- e^{7i\alpha_-}\right) e^{-i\omega z} \tag{4.127}$$

for $\omega z \gg 1$. Finally, redefining z as

$$z \mapsto e^{3\pi i}z = -z \tag{4.128}$$

allow us to rewrite the asymptotic expansions (4.117) and (4.127) as

$$\psi_0(z) \sim \left(A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}\right) e^{-i\omega z} + \left(A_+ e^{7i\alpha_+} + A_- e^{7i\alpha_-}\right) e^{i\omega z} \tag{4.129}$$

$$\psi_1(z) \sim \left(\Lambda_F^+ e^{5i\alpha_+} + \Lambda_F^- e^{5i\alpha_-}\right) e^{-i\omega z} + \left(\Lambda_F^+ e^{7i\alpha_+} + \Lambda_F^- e^{7i\alpha_-}\right) e^{i\omega z} \tag{4.130}$$

for $\omega z \ll -1$.

Just as before, we want to know if both ψ_0 and ψ_1 remain unchanged, relative to the asymptotic expansions above, while following the remaining portion of the big contour, connecting the origin of the complex r-plane to U. This turns out to be true for the same reasons provided to justify that ψ_0 and ψ_1 remain approximately unchanged relative to (4.63) and (4.92), respectively, while following the remaining portion of the big contour, connecting the origin of the complex r-plane to D. Thus, the expressions above are still valid near U.

Considering (4.7), we can write

$$\begin{aligned}\psi(z) \sim \left(A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}\right) e^{-i\omega z} \left[1 + \lambda' \left(\frac{\Lambda_F^+ e^{5i\alpha_+} + \Lambda_F^- e^{i5\alpha_-}}{A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}}\right)\right] \\ + \left(A_+ e^{7i\alpha_+} + A_- e^{7i\alpha_-}\right) e^{i\omega z} \left[1 + \lambda' \left(\frac{\Lambda_F^+ e^{7i\alpha_+} + \Lambda_F^- e^{7i\alpha_-}}{A_+ e^{7i\alpha_+} + A_- e^{7i\alpha_-}}\right)\right]\end{aligned} \tag{4.131}$$

near U.

Now, we want to understand how ψ behaves, while following the arc shaped portion of the big contour. To this end, we recall equation (4.43) to point out that we may rewrite the expression above as

$$\begin{aligned}\psi(x) \sim \left(A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}\right) e^{-i\omega x} \left[1 + \lambda' \left(\frac{\Lambda_F^+ e^{5i\alpha_+} + \Lambda_F^- e^{i5\alpha_-}}{A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}}\right)\right] \\ + \left(A_+ e^{7i\alpha_+} + A_- e^{7i\alpha_-}\right) e^{i\omega x} \left[1 + \lambda' \left(\frac{\Lambda_F^+ e^{7i\alpha_+} + \Lambda_F^- e^{7i\alpha_-}}{A_+ e^{7i\alpha_+} + A_- e^{7i\alpha_-}}\right)\right]\end{aligned} \tag{4.132}$$

near U. This is to be expected. Indeed, ψ is a solution of the master equation (2.11), expanded up to first order in λ'. Thus, it makes sense that we are able express ψ, near U, as a linear combination of plane waves $e^{\pm i\omega x}$, for this is the general solution provided by the WKB approximation of (2.11), in the respective Stokes lines [6].

We notice the arc shaped portion of the big contour no longer matches with a Stokes line of the WKB approximation of (2.11). Thus, as we abandon U and start following this portion of the big contour, we can no longer be sure that ψ will remain unchanged relative to (4.132). On the other hand, we know this portion of the contour is such that $\Re(x) > 0$. Looking at equation (1.146), we notice the term proportional to $e^{-i\omega x}$ will remain approximately unchanged, as opposed to the term proportional to $e^{i\omega x}$ [7].

From the considerations above, together with equation (4.43), we figure the term proportional to $e^{-i\omega z}$ will remain unchanged, while following the arc shaped portion of the big contour, as opposed to the term proportional to $e^{i\omega z}$. Hence, after following this portion of the big contour and returning to D, we can write

$$\psi(z) \sim \left(A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}\right) e^{-i\omega z} \left[1 + \lambda' \left(\frac{\Lambda_F^+ e^{i5\alpha_+} + \Lambda_F^- e^{i5\alpha_-}}{A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}}\right)\right] + \mathcal{K} e^{i\omega z} \tag{4.133}$$

near D, for some unknown $\mathcal{K} \in \mathbb{C}$.

Before computing the monodromy of ψ, we need to address one last detail. Looking at (4.31), we notice z has a branch point in the real and fictitious horizons. Since the big contour encloses the real horizon, a full loop around it is bound to cross a branch cut somewhere. Thus, the expression above is written with respect to a variable z, defined on a branch of the Riemann surface of z different from the branch where z is defined for (4.102). In order to related these two variables, we simply need to compute the monodromy of z, associated with one full clockwise loop around the event horizon. Looking at (4.31), we can write the asymptotic expansion

$$z(r) \sim \frac{1}{f_0'(R_h)} \log\left(1 - \frac{r}{R_h}\right) \tag{4.134}$$

near $r = R_h$. Making the parameterization

[6] We remind that the Stokes lines, associated with the WKB approximation of the master equation (2.11), match those of the WKB approximation (4.29) for $|r| \to +\infty$.

[7] We remind the variable w, used in (1.147), relates to x as $\omega x = w$. Moreover, we are assuming that $\Im(\omega) > 0$.

$$1-\frac{r}{R_h}=\rho e^{i\theta} \tag{4.135}$$

for $\rho \in \mathbb{R}_0^+$ and $\theta \in [0, 2\pi[$, allow us to rewrite the asymptotic expansion (4.134) as

$$z(r) \sim \frac{1}{f_0'(R_h)}\left(\log\left(\rho\right)+i\theta\right). \tag{4.136}$$

near $r = R_h$. Following a full clockwise loop around $r = R_h$ is equivalent to let θ run from 2π to 0. Therefore, the monodromy of z, associated with a full clockwise loop around $r = R_h$, is

$$\Delta_z \coloneqq -\frac{2\pi i}{f_0'(R_h)}. \tag{4.137}$$

Using the monodromy above, we can relate the variables previously mentioned by redefining the one used in (4.133) as

$$z \mapsto z + \Delta_z. \tag{4.138}$$

Thus, we can rewrite (4.133) as

$$\psi(z) \sim \left(A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}\right) e^{-i\omega\Delta_z} e^{-i\omega z}\left[1+\lambda'\left(\frac{\Lambda_F^+ e^{i5\alpha_+}+\Lambda_F^- e^{i5\alpha_-}}{A_+ e^{5i\alpha_+}+A_- e^{5i\alpha_-}}\right)\right] + \mathcal{K} e^{i\omega\Delta_z} e^{i\omega z} \tag{4.139}$$

near D.

We are finally in position to compute the monodromy of ψ. In the next section, we show that the monodromy of ψ, around the big contour, is multiplicative, as a consequence of the boundary condition in the event horizon. From this, we gather that $\mathcal{K} = 0$. Hence, we can rewrite (4.139) as

$$\psi(z) \sim \left(A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}\right) e^{-i\omega\Delta_z} e^{-i\omega z}\left[1+\lambda'\left(\frac{\Lambda_F^+ e^{i5\alpha_+}+\Lambda_F^- e^{i5\alpha_-}}{A_+ e^{5i\alpha_+}+A_- e^{5i\alpha_-}}\right)\right] \tag{4.140}$$

near D. At last, comparing the expression above with (4.102), yields the multiplicative monodromy

$$\mathcal{M}_1 = \left(\frac{A_+ e^{5i\alpha_+}+A_- e^{5i\alpha_-}}{A_+ e^{i\alpha_+}+A_- e^{i\alpha_-}}\right) e^{-i\omega\Delta_z}\left(1+\lambda'\delta\mathcal{M}_1\right) \tag{4.141}$$

where we defined the constant

$$\delta\mathcal{M}_1 \coloneqq \frac{\Lambda_F^+ e^{5i\alpha_+}+\Lambda_F^- e^{5i\alpha_-}}{A_+ e^{5i\alpha_+}+A_- e^{5i\alpha_-}} - \frac{\Lambda_I^+ e^{i\alpha_+}+\Lambda_I^- e^{i\alpha_-}}{A_+ e^{i\alpha_+}+A_- e^{i\alpha_-}}. \tag{4.142}$$

Taking the limit $j \to 0$ and performing a large amount of algebraic manipulation yields the rather simple expression

$$\delta\mathcal{M}_1 = (\omega R_h)^{\frac{d-1}{d-2}}\Pi_g \tag{4.143}$$

where we defined the constant

$$\Pi_g \coloneqq \frac{2\sqrt{\pi}}{3}\frac{(d(d-5)+2)(d-4)}{(d-1)(d-2)^{\frac{d-1}{d-2}}} e^{-\frac{2i\pi}{d-2}} \frac{\Gamma\left(\frac{1}{2(d-2)}\right)\Gamma\left(\frac{d-3}{2(d-2)}\right)}{\Gamma\left(\frac{d-1}{2(d-2)}\right)^2}\sin\left(\frac{\pi}{d-2}\right) \tag{4.144}$$

in order to simplify the notation. Analogously, taking the limit $j \to 0$ yields

$$\frac{A_+ e^{i5\alpha_+}+A_- e^{i5\alpha_-}}{A_+ e^{i\alpha_+}+A_- e^{i\alpha_-}} = -3. \tag{4.145}$$

Finally, we can write

$$\mathcal{M}_1 = -3(1+\lambda'\delta\mathcal{M}_1)e^{-i\omega\Delta_z}. \tag{4.146}$$

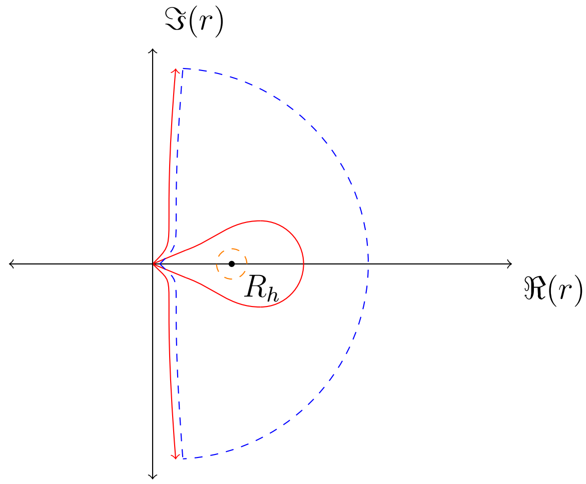

Figure 4.4: Schematic depiction of the small and big contours as the orange and blue dashed lines respectively. The orange contour is to be interpreted as arbitrarily close to R_h. The Stokes lines are depicted by red curves. Naturally, not all Stokes lines are depicted.

4.5.2 The small contour

The small contour is remarkably simple, when compared to the big contour. Indeed, we build an arbitrarily small closed contour around the event horizon R_h. Such contour can be represented as depicted in figure (4.4). In this contour, we wont need to perturbatively approach the master equation (2.11) nor change the independent variable. Indeed, as the contour is arbitrarily small, we can simply solve (2.11) Taylor expanded around the event horizon.

Looking at (2.13), we see that

$$V(r) \sim 0 \tag{4.147}$$

near $r = R_h$. Therefore, we can rewrite (2.11) as

$$\frac{d^2\psi}{dx^2} + \omega^2\psi = 0 \tag{4.148}$$

near R_h. The general solution of the differential equation above is

$$\psi(x) = \mathcal{C}_+ e^{i\omega x} + \mathcal{C}_- e^{-i\omega x} \tag{4.149}$$

for some $\mathcal{C}_\pm \in \mathbb{C}$. Imposing the boundary condition (1.69), yields the restriction

$$C_- = 0. \tag{4.150}$$

Thus, we can write

$$\psi(x) \sim C_+ e^{i\omega x} \tag{4.151}$$

in the small contour.

Now, we want to compute the monodromy of ψ, associated with a full clockwise loop around the small contour. We do this by computing the corresponding monodromy of x. Indeed, analogously to z, the variable x has a branch point in the event horizon R_h. To see this, we start by expanding (2.12), up to first order in λ'. Doing so yields

$$dx = \frac{dr}{f_0}\left(1 - \lambda'\delta f\right). \tag{4.152}$$

Considering (2.5), we can write

$$\lim_{r\to R_h}\left[\delta f(r)\right] = -\frac{(d-3)(d-4)}{2}\lim_{r\to R_h}\left[\frac{1-\left(\frac{R_h}{r}\right)^{d-1}}{1-\left(\frac{R_h}{r}\right)^{d-3}}\right] = -\frac{(d-1)(d-4)}{2} \tag{4.153}$$

where in the last equality we used the l'Hôpital rule for complex valued functions [35]. Using the limit above, we can rewrite (4.152) as

$$dx = \frac{dr}{f_0}\left(1 + \frac{\lambda'}{2}(d-4)(d-1)\right) \tag{4.154}$$

in the small contour. Integrating the equation above, using (4.1), yields

$$x(r) \sim \left(1 + \frac{\lambda'}{2}(d-4)(d-1)\right) z(r) + \mathcal{C} \tag{4.155}$$

in the small contour, for some $\mathcal{C} \in \mathbb{C}$. Thus, the monodromy of x, associated with a full clockwise loop around the small contour, is

$$\Delta_x \coloneqq \left(1 + \frac{\lambda'}{2}(d-4)(d-1)\right) \Delta_z. \tag{4.156}$$

Finally, the monodromy of ψ, associated with a full clockwise loop around the small contour, is multiplicative and given by

$$\mathcal{M}_2 \coloneqq e^{i\omega\Delta_x}. \tag{4.157}$$

4.5.3 Equating monodromies

Now, we want to relate the monodromies $\mathcal{M}_1$ and $\mathcal{M}_2$.

We notice the big contour is homotopic to the small one. This is so, because one can continuously deform the big contour into the small one. Thus, using the monodromy theorem, we know the monodromies of ψ, associated with the full clockwise loops around the big and the small contours, are equal [8]. Hence, the equation

$$\mathcal{M}_1 = \mathcal{M}_2 \tag{4.158}$$

must hold. Using (4.146) and (4.157), we can rewrite the equation above as

$$-3(1 + \lambda'\delta\mathcal{M}_1)e^{-i\omega(\Delta_x + \Delta_z)} = 1. \tag{4.159}$$

Taking the logarithm on both sides of the equation above yields

$$\log(3) + (2k+1)\pi i - i\omega(\Delta_x + \Delta_z) + \log(1 + \lambda'\delta\mathcal{M}_1) = 0 \tag{4.160}$$

for all $k \in \mathbb{Z}$. Taylor expanding the last logarithm in the equation above, up to first order in λ', yields

$$\log(3) + (2k+1)\pi i - i\omega(\Delta_x + \Delta_z) + \lambda'\delta\mathcal{M}_1 = 0 \tag{4.161}$$

for all $k \in \mathbb{Z}$.

Considering (4.137) and (4.156), we can write

$$\Delta_x + \Delta_z = -\left(2 + \lambda'\frac{(d-4)(d-1)}{2}\right)\frac{2\pi i}{f_0'(R_h)}. \tag{4.162}$$

Using (2.4), we can rewrite the equation above as

$$\Delta_x + \Delta_z = -\left(2 + \lambda'\frac{(d-4)(d-1)}{2}\right)\frac{2\pi i}{d-3}R_h. \tag{4.163}$$

Using (2.10), we can write

$$\Delta_x + \Delta_z = -\left(2 + \lambda'\frac{(d-4)(d-1)}{2}\right)\left(1 - \lambda'\frac{(d-1)(d-4)}{2}\right)\frac{i}{2T_{\mathcal{H}}}. \tag{4.164}$$

[8] This validates the assumption that the monodromy of ψ, associated with a full clockwise loop around the big contour, is multiplicative.

Expanding the expression above, up to first order in λ', yields

$$\Delta_x + \Delta_z = -\frac{i}{T_{\mathcal{H}}}\left(1 - \lambda' \frac{(d-1)(d-4)}{4}\right). \tag{4.165}$$

Using the equation above and (4.143) , we can rewrite (4.161) as

$$\log(3) + (2k+1)\pi i = \frac{\omega}{T_{\mathcal{H}}}\left(1 - \lambda' \frac{(d-1)(d-4)}{4}\right) - \lambda' \left(R_h \omega\right)^{\frac{d-1}{d-2}} \Pi_g \tag{4.166}$$

for all $k \in \mathbb{Z}$. Using (2.10), we can write

$$\left(R_h \omega\right)^{\frac{d-1}{d-2}} = \left[\frac{\omega(d-3)}{T_{\mathcal{H}} 4\pi}\right]^{\frac{d-1}{d-2}} \left(1 - \lambda' \frac{(d-1)(d-4)}{2}\right)^{\frac{d-1}{d-2}}. \tag{4.167}$$

Taylor expanding the expression above, up to first order in λ', yields

$$\left(R_h \omega\right)^{\frac{d-1}{d-2}} \sim \left[\frac{\omega(d-3)}{T_{\mathcal{H}} 4\pi}\right]^{\frac{d-1}{d-2}} \left(1 - \lambda' \frac{(d-1)^2(d-4)}{2(d-2)}\right). \tag{4.168}$$

Finally, replacing the expansion above in (4.166) yields

$$\log(3) + (2k+1)\pi i = \frac{\omega}{T_{\mathcal{H}}}\left(1 - \lambda' \frac{(d-1)(d-4)}{4}\right) - \lambda' \left[\frac{\omega(d-3)}{T_{\mathcal{H}} 4\pi}\right]^{\frac{d-1}{d-2}} \Pi_g \tag{4.169}$$

for all $k \in \mathbb{Z}$. We ended with a transcendental equation that certainly restricts the possible values of ω from the complex plane to an infinite but countable subset.

Chapter 5

Greybody factor

In this chapter, we compute an analytical expression for the greybody factor, associated with tensor type gravitational perturbations in the Callan Myers Perry black hole space time. We only do this computation for the limit $|\Im(\omega R_h)| \gg 1$. First, we discuss an issue associated with imposing the boundary conditions (1.117) and (1.118), analogous to an issue we faced on the quasinormal modes problem.

Finally, we compute an analytical expression for the greybody factor, using a method very similar to the monodromy method. As such, we reuse plenty of definitions and results from the previous chapter.

The general outline of this procedure will closely follow [42].

5.1 Boundary conditions

Like the quasinormal modes problem, computing greybody factors requires one to gather information from the boundary conditions. In this case, these are (1.117) and (1.118). In general, ω can be complex, just as in the quasinormal modes problem. Thus, the boundary conditions (1.117) and (1.118) are very difficult to impose, both for numerical and analytical approximate methods. This is so, because we are once again dealing with an exponential large and an exponential small term on the boundaries of the problem. This issue can be partially solved in the same way we solved the analogous issue, in the quasinormal modes problem. Indeed, by allowing r to take complex values and consequently assuming the analytic continuation of every function of r to the complex plane, we can gather information from (1.117), by appealing to the monodromy of ψ around the event horizon. Moreover, imposing the boundary condition

$$\psi(x) \sim e^{i\omega x} + R(\omega)e^{-i\omega x} \tag{5.1}$$

for $r \to +\infty$, in the same Stokes line we imposed the boundary condition (4.44), is equivalent to imposing the boundary condition (1.118). Using equation (4.43), we can rewrite the boundary condition above as

$$\psi(z) \sim e^{i\omega z} + R(\omega)e^{-i\omega z} \tag{5.2}$$

for $r \to +\infty$, on the same Stokes line. As before, the boundary condition above is much easier to impose than (1.118).

5.2 Greybody factor computation

The strategy we use to compute the greybody factor consists in building a linear system of three algebraic equations where one of the independent variables is the reflection coefficient $R(\omega)$. After solving this system and consequently finding $R(\omega)$, we repeat the process to find

$$\widetilde{R}(\omega) \coloneqq R(-\omega). \tag{5.3}$$

Finally, we relate these coefficients with the Greybody factor $\gamma(\omega)$, using the equation

$$R(\omega)\widetilde{R}(\omega)+\gamma(\omega)=1 \tag{5.4}$$

valid for all asymptotically flat space times [26].

5.2.1 Computation of $R(\omega)$

In order to build the system we seek, we use the big and small contours, defined in the previous chapter.

From (4.60), we know the general solution of the differential equation (4.11) has the asymptotic expansion

$$\psi_0(z)\sim A_+\sqrt{2\pi}\sqrt{\omega z}J_{\frac{j}{2}}(\omega z)+A_-\sqrt{2\pi}\sqrt{\omega z}J_{-\frac{j}{2}}(\omega z) \tag{5.5}$$

near the origin of the complex r-plane, for some $A_\pm\in\mathbb{C}$. Moreover, for some fixed $A_\pm$, the particular solution of the differential equation (4.12) has the asymptotic expansion

$$\psi_1(z)\sim\sum_{k=1}^{3}\phi_k(z) \tag{5.6}$$

near the origin of the complex r-plane, where the functions ϕ_k are defined by (4.71), (4.72) and (4.73). In this case, we saw that

$$\begin{aligned}\psi(z)\sim&\left(A_+e^{i\alpha_+}+A_-e^{i\alpha_-}\right)e^{-i\omega z}\left[1+\lambda'\left(\frac{\Lambda_I^+e^{i\alpha_+}+\Lambda_I^-e^{i\alpha_-}}{A_+e^{i\alpha_+}+A_-e^{i\alpha_-}}\right)\right]\\&+\left(A_+e^{-i\alpha_+}+A_-e^{-i\alpha_-}\right)e^{i\omega z}\left[1+\lambda'\left(\frac{\Lambda_I^+e^{-i\alpha_+}+\Lambda_I^-e^{-i\alpha_-}}{A_+e^{-i\alpha_+}+A_-e^{-i\alpha_-}}\right)\right]\end{aligned} \tag{5.7}$$

near D, where the constants $\alpha_\pm$ and $\Lambda_I^\pm$ are defined by (4.64) and (4.91) respectively. Imposing the boundary condition (5.2) on the expression above yields the algebraic equations

$$\left(A_++\lambda'\Lambda_I^+\right)e^{i\alpha_+}+\left(A_-+\lambda'\Lambda_I^-\right)e^{i\alpha_-}=R(\omega) \tag{5.8}$$

$$\left(A_++\lambda'\Lambda_I^+\right)e^{-i\alpha_+}+\left(A_-+\lambda'\Lambda_I^-\right)e^{-i\alpha_-}=1. \tag{5.9}$$

Now, we only need one equation to complete our system. This equation will encode exclusively functional information of the boundary condition (1.117), turning a blind eye to $T(\omega)$. We build such equation, using once more the monodromy theorem. Indeed, we will equate two monodromies of ψ, associated with full clockwise loops around the big and small contours.

This time, we do not need to redefine the particular solution of (4.12), in order to obey the boundary condition at spatial infinity. Thus, to extract the monodromy of ψ, associated with a full clockwise loop around the big contour, we can proceed as in last chapter, without needing the redefinition (4.98). Hence, the multiplicative monodromy [1] of ψ, associated with a full clockwise loop around the big contour, is

$$\mathcal{N}_1:=\left(\frac{A_+e^{5i\alpha_+}+A_-e^{5i\alpha_-}}{A_+e^{i\alpha_+}+A_-e^{i\alpha_-}}\right)e^{-i\omega\Delta_z}\left(1+\lambda'\delta\mathcal{N}_1\right) \tag{5.10}$$

where we defined the constants

$$\delta\mathcal{N}_1:=\frac{\Lambda_G^+e^{5i\alpha_+}+\Lambda_G^-e^{5i\alpha_-}}{A_+e^{5i\alpha_+}+A_-e^{5i\alpha_-}}-\frac{\Lambda_I^+e^{i\alpha_+}+\Lambda_I^-e^{i\alpha_-}}{A_+e^{i\alpha_+}+A_-e^{i\alpha_-}} \tag{5.11}$$

$$\Lambda_G^\pm:=\sum_{k=1}^{3}\Xi_k^\pm. \tag{5.12}$$

The constants Ξ_k and Δ_z are defined in $(4.119-4.124)$ and (4.137) respectively.

[1] We assume the monodromy to be multiplicative for the same reason we provided in the last chapter.

The monodromy of ψ, associated with a full clockwise loop around the small contour, is unchanged from (4.157). This is so, because the boundary conditions (1.69) and (1.117) share the same functional form. Hence, we can write the monodromy of ψ, associated with a full clockwise loop around the small contour, as

$$\mathcal{N}_2 := \mathcal{M}_2 \tag{5.13}$$

where $\mathcal{M}_2$ is defined in (4.157).

Because the big and small contours are homotopic, the monodromy theorem yields the equation

$$\mathcal{N}_1 = \mathcal{N}_2. \tag{5.14}$$

Using (4.157), (5.8) and (5.10), we can rewrite the equation above as

$$R(\omega) = e^{-i\omega(\Delta_z+\Delta_x)} \left[\left(A_+ + \lambda' \Lambda_G^+\right) e^{5i\alpha_+} + \left(A_- + \lambda' \Lambda_G^-\right) e^{5i\alpha_-} \right]. \tag{5.15}$$

The equations (5.8), (5.9) and (5.15) make up the linear system we seek. The independent variables are the reflection coefficient $R(\omega)$ together with the complex constants $A_\pm$.

In order to solve this system, we use standard perturbation theory. Thus, we start by considering the expansions

$$R(\omega) = R_0(\omega) + \lambda' R_1(\omega) \tag{5.16}$$

$$A_\pm = A_\pm^0 + \lambda' A_\pm^1. \tag{5.17}$$

Furthermore, using (4.165) allow us to write the Taylor expansion

$$e^{-i\omega(\Delta_z+\Delta_x)} \sim e^{-\frac{\omega}{T_\mathcal{H}}} \left(1 + \lambda' \frac{\omega}{T_\mathcal{H}} \frac{(d-4)(d-1)}{4}\right) \tag{5.18}$$

up to first order in λ'. Replacing the expansions above in the equations of our system and solving them perturbatively in powers of λ' yields two distinct linear systems of algebraic equations. The first one, of zeroth order in λ', is

$$\begin{cases} A_+^0 e^{5i\alpha_+} + A_-^0 e^{5i\alpha_-} = R_0(\omega) e^{\frac{\omega}{T_\mathcal{H}}} \\ A_+^0 e^{i\alpha_+} + A_-^0 e^{i\alpha_-} = R_0(\omega) \\ A_+^0 e^{-i\alpha_+} + A_-^0 e^{-i\alpha_-} = 1 \end{cases} \tag{5.19}$$

whose solution is

$$R_0(\omega) = \frac{2i\cos\left(\frac{\pi j}{2}\right)}{e^{\frac{\omega}{T_\mathcal{H}}} + 2\cos(\pi j) + 1} \tag{5.20}$$

$$A_+^0 = \frac{e^{i\alpha_-} - R_0(\omega) e^{-i\alpha_-}}{2i\sin(\alpha_- - \alpha_+)} \tag{5.21}$$

$$A_-^0 = \frac{R_0(\omega) e^{-i\alpha_+} - e^{i\alpha_+}}{2i\sin(\alpha_- - \alpha_+)}. \tag{5.22}$$

The second linear system of algebraic equations, of first order in λ', is

$$\begin{cases} \left(A_+^1 + \sigma_+\right) e^{5i\alpha_+} + \left(A_-^1 + \sigma_-\right) e^{5i\alpha_-} = e^{\frac{\omega}{T_\mathcal{H}}} \left(R_1(\omega) - L\omega T_\mathcal{H}^{-1} R_0(\omega)\right) \\ \left(A_+^1 + \zeta_+\right) e^{i\alpha_+} + (A_-^1 + \zeta_-) e^{i\alpha_-} = R_1(\omega) \\ \left(A_+^1 + \zeta_+\right) e^{-i\alpha_+} + (A_-^1 + \zeta_-) e^{-i\alpha_-} = 0 \end{cases} \tag{5.23}$$

where we defined the constant

$$L := \frac{(d-4)(d-1)}{4} \tag{5.24}$$

in order to simplify the notation used. Additionally, we defined $\zeta_\pm$ as the constants resulting from switching A_+ and A_- for (5.21) and (5.22) respectively, in the definitions of $\Lambda_I^\pm$. Analogously, $\sigma^\pm$ are defined as the constants resulting from switching A_+ and A_- for (5.21) and (5.22) respectively, in the definitions of $\Lambda_G^\pm$. Moreover, we can define the constants

$$\Sigma_1 := -\left(\zeta_+ e^{i\alpha_+} + \zeta_- e^{i\alpha_-}\right) \tag{5.25}$$

$$\Sigma_2 := -\left(\zeta_+ e^{-i\alpha_+} + \zeta_- e^{-i\alpha_-}\right) \tag{5.26}$$

$$\Sigma_3 := -L\omega T_{\mathcal{H}}^{-1} e^{\frac{\omega}{T_{\mathcal{H}}}} R_0(\omega) - \left(\sigma_+ e^{5i\alpha_+} + \sigma_- e^{5i\alpha_-}\right) \tag{5.27}$$

and rewrite the system (5.23) in the simple form

$$\begin{cases} A_+^1 e^{5i\alpha_+} + A_-^1 e^{5i\alpha_-} - R_1(\omega) e^{\frac{\omega}{T_{\mathcal{H}}}} = \Sigma_3 \\ A_+^1 e^{i\alpha_+} + A_-^1 e^{i\alpha_-} - R_1(\omega) = \Sigma_1 \\ A_+^1 e^{-i\alpha_+} + A_-^1 e^{-i\alpha_-} = \Sigma_2. \end{cases} \tag{5.28}$$

Solving this system for $R_1(\omega)$ yields

$$R_1(\omega) = -\frac{2\Sigma_1 \cos(\pi j) - 2i\Sigma_2 \cos\left(\frac{\pi j}{2}\right) + \Sigma_1 + \Sigma_3}{e^{\frac{\omega}{T_{\mathcal{H}}}} + 2\cos(\pi j) + 1}. \tag{5.29}$$

Considering (5.16) and the equation above, we can write

$$R(\omega) = R_0(\omega)\left(1 + \lambda' \delta R_g(\omega)\right) \tag{5.30}$$

where we defined the constant

$$\delta R_g(\omega) := -\frac{2\Sigma_1 \cos(\pi j) - 2i\Sigma_2 \cos\left(\frac{\pi j}{2}\right) + \Sigma_1 + \Sigma_3}{R_0(\omega)\left(e^{\frac{\omega}{T_{\mathcal{H}}}} + 2\cos(\pi j) + 1\right)}. \tag{5.31}$$

Taking the limit $j \to 0$ and performing a large amount of algebraic manipulation yields

$$R_0(\omega) = \frac{2i}{e^{\frac{\omega}{T_{\mathcal{H}}}} + 3} \tag{5.32}$$

$$\delta R_g(\omega) = \frac{1}{e^{\frac{\omega}{T_{\mathcal{H}}}} + 3}\left[\frac{(d-4)(d-1)}{4}\frac{\omega}{T_{\mathcal{H}}} e^{\frac{\omega}{T_{\mathcal{H}}}} + \left(\frac{\omega}{T_{\mathcal{H}}}\right)^{\frac{d-1}{d-2}}\left(\frac{d-3}{4\pi}\right)^{\frac{d-1}{d-2}} \Upsilon_g\right] \tag{5.33}$$

where we defined the constant

$$\Upsilon_g := -\frac{\pi^2 (4-2d)^{\frac{1}{2-d}} (d-4)(d-1)^3((d-5)d+2) e^{\frac{i\pi(5-3d)}{2(d-2)}} \Gamma\left(\frac{1}{d-2}\right)\left(e^{\frac{i\pi(5-3d)}{d-2}}\left(e^{\frac{\omega}{T_{\mathcal{H}}}} - 1\right) - 4\right)}{16(d-2)^5 \Gamma\left(\frac{3d-5}{2d-4}\right)^4} \tag{5.34}$$

in order to simplify the notation used.

5.2.2 Computation of $\widetilde{R}(\omega)$

Here, we want to compute $\widetilde{R}(\omega)$, using the same strategy used in the last subsection. Under the transformation $\omega \mapsto -\omega$, the boundary condition (5.2) becomes

$$\psi(z) \sim e^{-i\omega z} + \widetilde{R}(\omega) e^{i\omega z}. \tag{5.35}$$

for $r \to +\infty$, on the same Stokes line.

In order to recycle work from the previous chapter, as we did in the last subsection, we still write ψ with respect to ω instead of $-\omega$. Indeed, this can be done, as equations (2.11), (4.11) and (4.12) are invariant under the transformation $\omega \mapsto -\omega$. Thus, we still can write the asymptotic expansion

$$\psi_0(z) \sim \widetilde{A}_+ \sqrt{2\pi}\sqrt{\omega z} J_{\frac{j}{2}}(\omega z) + \widetilde{A}_- \sqrt{2\pi}\sqrt{\omega z} J_{-\frac{j}{2}}(\omega z) \tag{5.36}$$

near the origin, for some $\widetilde{A}_\pm \in \mathbb{C}$. In this case, we can write

$$\begin{aligned}\psi(z) \sim & \left(\widetilde{A}_+ e^{i\alpha_+} + \widetilde{A}_- e^{i\alpha_-}\right) e^{-i\omega z} \left[1 + \lambda' \left(\frac{\widetilde{\Lambda}_I^+ e^{i\alpha_+} + \widetilde{\Lambda}_I^- e^{i\alpha_-}}{\widetilde{A}_+ e^{i\alpha_+} + \widetilde{A}_- e^{i\alpha_-}}\right)\right] \\ & + \left(\widetilde{A}_+ e^{-i\alpha_+} + \widetilde{A}_- e^{-i\alpha_-}\right) e^{i\omega z} \left[1 + \lambda' \left(\frac{\widetilde{\Lambda}_I^+ e^{-i\alpha_+} + \widetilde{\Lambda}_I^- e^{-i\alpha_-}}{\widetilde{A}_+ e^{-i\alpha_+} + \widetilde{A}_- e^{-i\alpha_-}}\right)\right]\end{aligned} \tag{5.37}$$

near D, where $\widetilde{\Lambda}_I^\pm$ are the constants resulting from switching $A_\pm$ for $\widetilde{A}_\pm$, in the definitions (4.91). Imposing the boundary condition (5.35) on the expression above yields the algebraic equations

$$\left(\widetilde{A}_+ + \lambda' \widetilde{\Lambda}_I^+\right) e^{i\alpha_+} + \left(\widetilde{A}_- + \lambda' \widetilde{\Lambda}_I^-\right) e^{i\alpha_-} = 1 \tag{5.38}$$

$$\left(\widetilde{A}_+ + \lambda' \widetilde{\Lambda}_I^+\right) e^{-i\alpha_+} + \left(\widetilde{A}_- + \lambda' \widetilde{\Lambda}_I^-\right) e^{-i\alpha_-} = \widetilde{R}(\omega). \tag{5.39}$$

Analogously to (5.10), the monodromy of ψ, associated with a full clock wise loop around the big contour, is

$$\widetilde{\mathcal{N}}_1 := \left(\frac{\widetilde{A}_+ e^{5i\alpha_+} + \widetilde{A}_- e^{5i\alpha_-}}{\widetilde{A}_+ e^{i\alpha_+} + \widetilde{A}_- e^{i\alpha_-}}\right) e^{-i\omega\Delta_z} \left(1 + \lambda' \delta\widetilde{\mathcal{N}}_1\right) \tag{5.40}$$

where we defined the constant

$$\delta\widetilde{\mathcal{N}}_1 := \frac{\widetilde{\Lambda}_G^+ e^{i5\alpha_+} + \widetilde{\Lambda}_G^- e^{i5\alpha_-}}{\widetilde{A}_+ e^{5i\alpha_+} + \widetilde{A}_- e^{5i\alpha_-}} - \frac{\widetilde{\Lambda}_I^+ e^{i\alpha_+} + \widetilde{\Lambda}_I^- e^{i\alpha_-}}{\widetilde{A}_+ e^{i\alpha_+} + \widetilde{A}_+ e^{i\alpha_-}}. \tag{5.41}$$

In the expression above, $\widetilde{\Lambda}_G^\pm$ denotes the constants resulting from switching $A_\pm$ for $\widetilde{A}_\pm$ in the definition (5.12).

Under the transformation $\omega \mapsto -\omega$, the boundary condition (1.117) becomes

$$\psi(x) \sim \widetilde{T}(\omega) e^{-i\omega x} \tag{5.42}$$

for $r \to R_h^+$. Thus, following the exact same procedure used in the last chapter, yields

$$\widetilde{\mathcal{N}}_2 := e^{-i\omega\Delta_x} \tag{5.43}$$

as the multiplicative monodromy of ψ, associated with a full clock wise loop around the small contour.

Using the monodromy theorem yields the equation

$$\widetilde{\mathcal{N}}_1 = \widetilde{\mathcal{N}}_2. \tag{5.44}$$

Using (5.38), (5.40) and (5.43), we can rewrite the equation above as

$$1 = e^{-i\omega(\Delta_z - \Delta_x)} \left[\left(\widetilde{A}_+ + \lambda' \widetilde{\Lambda}_G^+\right) e^{5i\alpha_+} + \left(\widetilde{A}_- + \lambda' \widetilde{\Lambda}_G^-\right) e^{5i\alpha_-}\right]. \tag{5.45}$$

Furthermore, we can write the Taylor expansion

$$e^{-i\omega(\Delta_z - \Delta_x)} \sim 1 + \lambda' L \frac{\omega}{T_{\mathcal{H}}} \tag{5.46}$$

up to first order in λ'.

The equations (5.38), (5.39) and (5.45) make up the linear system of algebraic equations we seek. Once again, we address this system using standard perturbation theory. Thus, we start by considering the expansions

$$\widetilde{R}(\omega) = \widetilde{R}_0(\omega) + \lambda' \widetilde{R}_1(\omega). \tag{5.47}$$

$$\widetilde{A}_\pm = \widetilde{A}_\pm^0 + \lambda' \widetilde{A}_\pm^1 \tag{5.48}$$

Plugging the expansions above in the equations of our system and solving them perturbatively in powers of λ' yields two distinct linear systems of algebraic equations. The first one, of zeroth order in λ', is

$$\begin{cases} \widetilde{A}^0_+ e^{5i\alpha_+} + \widetilde{A}^0_- e^{5i\alpha_-} = 1 \\ \widetilde{A}^0_+ e^{i\alpha_+} + \widetilde{A}^0_- e^{i\alpha_-} = 1 \\ \widetilde{A}^0_+ e^{-i\alpha_+} + \widetilde{A}^0_- e^{-i\alpha_-} = \widetilde{R}(\omega) \end{cases} \tag{5.49}$$

whose solution is

$$\widetilde{R}_0(\omega) = -2i\cos\left(\frac{\pi j}{2}\right) \tag{5.50}$$

$$\widetilde{A}^0_+ = \frac{-e^{-i\alpha_-} + \widetilde{R}_0(\omega)e^{i\alpha_-}}{2i\sin(\alpha_- - \alpha_+)} \tag{5.51}$$

$$\widetilde{A}^0_- = \frac{-\widetilde{R}_0(\omega)e^{i\alpha_+} + e^{-i\alpha_+}}{2i\sin(\alpha_- - \alpha_+)}. \tag{5.52}$$

The second system, of first order in λ', is

$$\begin{cases} \left(\widetilde{A}^1_+ + \widetilde{\sigma}_+\right) e^{5i\alpha_+} + \left(\widetilde{A}^1_- + \widetilde{\sigma}_-\right) e^{5i\alpha_-} = -L\omega T_{\mathcal{H}}^{-1} \\ \left(\widetilde{A}^1_+ + \widetilde{\zeta}_+\right) e^{i\alpha_+} + \left(\widetilde{A}^1_- + \widetilde{\zeta}_-\right) e^{i\alpha_-} = 0 \\ \left(\widetilde{A}^1_+ + \widetilde{\zeta}_+\right) e^{-i\alpha_+} + \left(\widetilde{A}^1_- + \widetilde{\zeta}_-\right) e^{-i\alpha_-} = \widetilde{R}_1(\omega) \end{cases} \tag{5.53}$$

where we defined $\widetilde{\zeta}_\pm$ as the constants resulting from switching $\widetilde{A}_+$ and $\widetilde{A}_-$ for (5.51) and (5.52) respectively, in the definitions of $\widetilde{\Lambda}^\pm_I$. Analogously, $\widetilde{\sigma}^\pm$ are defined as the constants resulting from switching $\widetilde{A}_+$ and $\widetilde{A}_-$ for (5.51) and (5.52) respectively, in the definitions of $\widetilde{\Lambda}^\pm_G$. Moreover, defining the constants

$$\widetilde{\Sigma}_1 := -\left(\widetilde{\zeta}_+ e^{i\alpha_+} + \widetilde{\zeta}_- e^{i\alpha_-}\right) \tag{5.54}$$

$$\widetilde{\Sigma}_2 := -\left(\widetilde{\zeta}_+ e^{-i\alpha_+} + \widetilde{\zeta}_- e^{-i\alpha_-}\right) \tag{5.55}$$

$$\widetilde{\Sigma}_3 := -L\omega T_{\mathcal{H}}^{-1} - \widetilde{\sigma}_+ e^{5i\alpha_+} - \widetilde{\sigma}_- e^{5i\alpha_-} \tag{5.56}$$

allow us to rewrite (5.53) as the simple system

$$\begin{cases} A^1_+ e^{5i\alpha_+} + A^1_- e^{5i\alpha_-} = \Sigma_3 \\ A^1_+ e^{i\alpha_+} + A^1_- e^{i\alpha_-} = \Sigma_1 \\ A^1_+ e^{-i\alpha_+} + A^1_- e^{-i\alpha_-} - \widetilde{R}_1(\omega) = \Sigma_2. \end{cases} \tag{5.57}$$

Solving this system for $\widetilde{R}_1(\omega)$ yields

$$\widetilde{R}_1(\omega) = -\frac{i}{2}\sec\left(\frac{\pi j}{2}\right)\left(2\widetilde{\Sigma}_1\cos(\pi j) - 2i\widetilde{\Sigma}_2\cos\left(\frac{\pi j}{2}\right) + \widetilde{\Sigma}_1 + \widetilde{\Sigma}_3\right). \tag{5.58}$$

Using (5.47) and the equation above, we can write

$$\widetilde{R}(\omega) = \widetilde{R}_0(\omega)\left(1 + \lambda'\delta\widetilde{R}_g(\omega)\right) \tag{5.59}$$

where we defined the constant

$$\delta\widetilde{R}_g(\omega) := -\frac{i}{2\widetilde{R}_0(\omega)}\sec\left(\frac{\pi j}{2}\right)\left(2\widetilde{\Sigma}_1\cos(\pi j) - 2i\widetilde{\Sigma}_2\cos\left(\frac{\pi j}{2}\right) + \widetilde{\Sigma}_1 + \widetilde{\Sigma}_3\right). \tag{5.60}$$

Taking the limit $j \to 0$ and performing a large amount of algebraic manipulation yields

$$\widetilde{R}_0(\omega) = -2i \tag{5.61}$$

$$\delta\widetilde{R}_g(\omega) = -\frac{(d-1)(d-4)}{16}\frac{\omega}{T_{\mathcal{H}}} + \left(\frac{\omega}{T_{\mathcal{H}}}\right)^{\frac{d-1}{d-2}}\left(\frac{d-3}{4\pi}\right)^{\frac{d-1}{d-2}}\widetilde{\Upsilon}_g \tag{5.62}$$

where we defined the constant

$$\widetilde{\Upsilon}_g \coloneqq -\frac{\pi^2(4-2d)^{\frac{1}{2-d}}(d-4)((d-5)d+2)e^{\frac{i\pi(5-3d)}{2(d-2)}}\Gamma\left(\frac{d-1}{d-2}\right)}{(d-1)\Gamma\left(\frac{d-1}{2d-4}\right)^4}. \tag{5.63}$$

5.2.3 Computation of $\gamma(\omega)$

Finally, using (5.4) we can write the greybody factor as

$$\gamma(\omega) = 1 - R(\omega)\widetilde{R}(\omega) = \gamma_0(\omega)\left(1 + \lambda'\delta\gamma_g(\omega)\right) \tag{5.64}$$

where we defined

$$\gamma_0(\omega) \coloneqq \frac{e^{\frac{\omega}{T_{\mathcal{H}}}} - 1}{3 + e^{\frac{\omega}{T_{\mathcal{H}}}}} \tag{5.65}$$

$$\delta\gamma_g(\omega) \coloneqq \frac{4}{1 - e^{\frac{\omega}{T_{\mathcal{H}}}}}\left(\delta\widetilde{R}_g(\omega) + \delta R_g(\omega)\right). \tag{5.66}$$

After some algebraic manipulation, we can write

$$\delta\gamma_g(\omega) = -\frac{4}{3 + e^{\frac{\omega}{T_{\mathcal{H}}}}}\left[\frac{3}{16}(d-4)(d-1)\frac{\omega}{T_{\mathcal{H}}} + \left(\frac{\omega}{T_{\mathcal{H}}}\right)^{\frac{d-1}{d-2}}\left(\frac{d-3}{4\pi}\right)^{\frac{d-1}{d-2}}\varrho_g\right] \tag{5.67}$$

where we defined the constant

$$\varrho_g \coloneqq -\frac{\pi^{\frac{3}{2}}2^{\frac{d-1}{d-2}}(4-2d)^{\frac{1}{2-d}}(d-4)((d-5)d+2)e^{\frac{i\pi(5-3d)}{d-2}}\Gamma\left(\frac{2d-3}{2d-4}\right)}{(d-1)\Gamma\left(\frac{d-1}{2d-4}\right)^3}\cos\left(\pi\left(\frac{3d-5}{2d-4}\right)\right). \tag{5.68}$$

Chapter 6

Scalar field and black hole shadow

In this chapter, once again, we provide analytical expressions for quasinormal frequencies in the eikonal limit and in the asymptotic limit. However, this time, the quasinormal frequencies we seek are associated with linear perturbations of a complex massless scalar field, coupled to gravity in the Callan Myers Perry black hole space time. Furthermore, we provide the analytical expression of the greybody factor, associated with the same kind of perturbations.

In short, we provide the results analogous to those computed in the last three chapters for linear perturbations of a complex massless scalar field, coupled to gravity in the Callan Myers Perry black hole space time.

Comparing the master equations (2.11) and (2.17), considering the respective potentials (2.13) and (2.18), we notice they are not so different. Indeed, the only difference lies in the explicit correction, of first order in λ', present in the potential (2.13). Thus, the procedures needed to compute the results, analogous to those provided in the previous three chapters, are almost identical. In fact, the difference amounts to a simplification of the potential used in the master equation. Thus, we will not reproduce the computations here. Instead, we explain how this simplification changes the procedures used in the last three chapters.

Finally, we provide an analytical expression for the radius associated with the shadow cast by the Callan Myers Perry black hole space time. Furthermore, we comment on the strong relation of such radius with the eikonal quasinormal frequencies, associated with complex massless scalar field perturbations.

6.1 The eikonal limit

The procedure used in the third chapter for computing quasinormal frequencies in the eikonal limit remains almost unchanged in this context. Indeed, the difference between the potentials (2.13) and (2.18) only affects the asymptotic expansion of Q, for large values of l. Moreover, this difference amounts to switching h with f, when considering computations, associated with linear perturbations of the complex massless scalar field. In this case, the quasinormal frequencies take the form

$$\omega = l\sqrt{\frac{f(r_s)}{r_s^2}} + \frac{i}{\sqrt{2}}\left(n+\frac{1}{2}\right)\sqrt{\frac{f(r_s)}{r_s^2}\left(2f(r_s) - \frac{d^2 f}{dr^2}(r_s)r_s^2\right)} \tag{6.1}$$

for $n \in \mathbb{N}_0$, where

$$r_s = R_h\left(\frac{d-1}{2}\right)^{\frac{1}{d-3}}\left[1+\lambda'\left(\frac{d-4}{2} - (d-4)\left(\frac{2}{d-1}\right)^{\frac{d-1}{d-3}}\right)\right] \tag{6.2}$$

is the minimizer of the asymptotic expansion of Q, for large values of l. In order to simplify the notation, we define the constants

$$\Gamma_r^s := \sqrt{\frac{f(r_s)}{r_s^2}} \qquad \Gamma_i^s := \sqrt{\frac{f^2(r_s)}{2r_s^2}\left(2f(r_s) - \frac{d^2 f}{dr^2}(r_s)r_s^2\right)}. \tag{6.3}$$

After some algebraic manipulation, we can write Γ_r^s and Γ_i^s, up to first order in λ', as

$$\Gamma_r^s = \frac{4\pi T_{\mathcal{H}}}{\sqrt{(d-1)(d-3)}}\left(\frac{2}{d-1}\right)^{\frac{1}{d-3}}\left[1-\lambda'\left(\frac{d-4}{2}\right)\left(2-d-\left(\frac{2}{d-1}\right)^{\frac{d-1}{d-3}}\right)\right] \tag{6.4}$$

$$\Gamma_i^s = \frac{4\pi T_{\mathcal{H}}}{\sqrt{d-1}}\left(\frac{2}{d-1}\right)^{\frac{1}{d-3}}\left[1+\lambda'\frac{(d-4)(d-2)}{2}\left(1-\left(\frac{2}{d-1}\right)^{\frac{d-1}{d-3}}\right)\right] \tag{6.5}$$

Finally, we can rewrite the quasinormal frequencies (6.1) as

$$\omega = l\Gamma_r^s + i\left(n+\frac{1}{2}\right)\Gamma_i^s \tag{6.6}$$

for $n \in \mathbb{N}_0$.

On a ending remark, it is worth noting that Γ_r^s and Γ_i^s have solid physical interpretations. Indeed, Γ_r^s is the angular velocity of the unique null circular geodesic of the Callan Myers Perry black hole space time. As for Γ_i^s, it can be identified with the Lyapunov coefficient of the null circular geodesic trajectory. A proof of this identification can be found in [11].

6.2 The asymptotic limit

The procedure used in the fourth chapter for computing quasinormal frequencies in the asymptotic limit is almost unchanged in this case. Indeed, the difference between the potentials (2.13) and (2.18) only affects the non homogeneous term (4.11). More precisely, when considering computations, associated with linear perturbations of the complex massless scalar field, we have a different ξ_3. In this case, the procedure remains identical to the one used in the fourth chapter, if one redefines Γ_3 as

$$\Gamma_3 \mapsto \frac{1}{4}(d-4)(d-3)(d-2)(2d-3)(2-d)^{\frac{5-3d}{d-2}}. \tag{6.7}$$

The transcendental equation, defining the quasinormal modes in the asymptotic limit, is

$$\log(3)+(2k+1)\pi i = \frac{\omega}{T_{\mathcal{H}}}\left(1-\lambda'\frac{(d-1)(d-4)}{4}\right)-\lambda'\left[\frac{\omega(d-3)}{T_{\mathcal{H}}4\pi}\right]^{\frac{d-1}{d-2}}\Pi_s \tag{6.8}$$

for $k \in \mathbb{Z}$, where we defined the constant

$$\Pi_s := \frac{8\pi^2(4-2d)^{\frac{1}{2-d}}(d-4)(d-3)e^{\frac{i\pi(5-3d)}{d-2}}}{3(d-1)}\frac{\Gamma\left(\frac{1}{d-2}\right)}{\Gamma\left(\frac{d-1}{2d-4}\right)^4}\sin\left(\left(\frac{7-4d}{2d-4}\right)\pi\right). \tag{6.9}$$

6.3 Greybody factor

The procedure used in the fifth chapter for analytically computing the greybody factor in the limit $|\Im(\omega)| \to +\infty$ is also almost unchanged in this case. In fact, since that procedure recycled most of the results from the fourth chapter, the difference amounts to performing the redefinition (6.7). In this case, we have

$$R(\omega) = R_0(\omega)\left(1+\lambda'\delta R_s(\omega)\right) \tag{6.10}$$

where we defined the constants

$$\delta R_s(\omega) := \frac{1}{e^{\frac{\omega}{T_{\mathcal{H}}}}+3}\left[\frac{(d-4)(d-1)}{4}\frac{\omega}{T_{\mathcal{H}}}e^{\frac{\omega}{T_{\mathcal{H}}}}+\left(\frac{\omega}{T_{\mathcal{H}}}\right)^{\frac{d-1}{d-2}}\left(\frac{d-3}{4\pi}\right)^{\frac{d-1}{d-2}}\Upsilon_s\right] \tag{6.11}$$

$$\Upsilon_s := -\frac{\pi^2(4-2d)^{\frac{1}{2-d}}(d-4)(d-3)e^{\frac{i\pi(5-3d)}{2(d-2)}}\Gamma\left(\frac{1}{d-2}\right)\left(e^{\frac{i\pi(5-3d)}{d-2}}\left(e^{\frac{\omega}{T_{\mathcal{H}}}}-1\right)-4\right)}{(d-1)\Gamma\left(\frac{d-1}{2d-4}\right)^4} \tag{6.12}$$

in order to simplify the notation used. Additionally, we have

$$\widetilde{R}(\omega) = \widetilde{R}_0(\omega)\left(1 + \lambda'\delta\widetilde{R}_s(\omega)\right) \tag{6.13}$$

where we defined the constants

$$\delta\widetilde{R}_s(\omega) := -\frac{(d-1)(d-4)}{16}\frac{\omega}{T_{\mathcal{H}}} + \left(\frac{\omega}{T_{\mathcal{H}}}\right)^{\frac{d-1}{d-2}}\left(\frac{d-3}{4\pi}\right)^{\frac{d-1}{d-2}}\widetilde{\Upsilon}_s \tag{6.14}$$

$$\widetilde{\Upsilon}_s := -\frac{\pi^2(4-2d)^{\frac{1}{2-d}}(d-4)(d-3)(d-1)^2 e^{\frac{i\pi(5-3d)}{2(d-2)}}\Gamma\left(\frac{2d-3}{d-2}\right)}{16(d-2)^2\Gamma\left(\frac{3d-5}{2d-4}\right)^4}. \tag{6.15}$$

Finally, we get

$$\gamma(\omega) = R(\omega)\widetilde{R}(\omega) = \gamma_0(\omega)\left(1 + \lambda'\delta\gamma_s(\omega)\right) \tag{6.16}$$

where we defined the constants

$$\delta\gamma_s(\omega) := -\frac{4}{e^{\frac{\omega}{T_{\mathcal{H}}}}+3}\left(\frac{3}{16}(d-4)(d-1)\frac{\omega}{T_{\mathcal{H}}} + \left(\frac{\omega}{T_{\mathcal{H}}}\right)^{\frac{d-1}{d-2}}\left(\frac{d-3}{4\pi}\right)^{\frac{d-1}{d-3}}\varrho_s\right) \tag{6.17}$$

$$\varrho_s := -\frac{\pi^{\frac{3}{2}}2^{\frac{1}{d-2}}(4-2d)^{\frac{1}{2-d}}(d-4)(d-3)e^{\frac{i\pi(5-3d)}{d-2}}\Gamma\left(\frac{1}{2(d-2)}\right)}{(d-1)\Gamma\left(\frac{d-1}{2d-4}\right)^3}\cos\left(\left(\frac{5-3d}{2d-4}\right)\pi\right). \tag{6.18}$$

6.4 Black hole shadow

In this section, we provide an analytical expression for the radius associated with the black hole shadow cast by the Callan Myers Perry black hole space time.

6.4.1 Brief introduction

In order to define the black hole shadow, it is useful to introduce some 4-dimensional concepts first. Let us consider some 4-dimensional asymptotically flat black hole space time, together with an observer located at spatial infinity. As the black hole space time is asymptotically flat, an observer at spatial infinity can set up an Euclidean spatial coordinate system [54]. In this framework, the plane that simultaneously contains the black hole and is normal to the line connecting it to the observer is called the observer's sky. Now, we consider a light source behind the black hole, from the observer's perspective. As the photons, emitted by the source, travel through space time, some will get trapped inside the event horizon surface while some will eventually reach the observer. The absence of photons that ended trapped inside the event horizon surface will make the observer perceive some sort of shadow in the observer's sky. In the case of spherical symmetric black hole space times, this shadow is a disk. However, shadows associated with non spherically symmetric space times are not [54].

Generalising the reasoning above to a d-dimensional asymptotically flat space time, the observer's sky becomes a d-2 dimensional hyperplane. In this case, the shadow cast by the black hole will be a submanifold of such hyperplane.

6.4.2 Shadow computation

Let us consider a d-dimensional spherically symmetric, static and asymptotically flat black hole space time $\mathcal{M}$. The topological structure of such space time is identical to (1.39). The respective metric tensor field can be expressed, with respect to such topological structure, as

$$ds^2 = -g(r)dt \otimes dt + \frac{1}{g(r)}dr \otimes dr + r^2 d\Omega^2_{d-2} \tag{6.19}$$

for some real function g.

The shadow cast by this black hole space time, under a suitable projection, forms a disk with radius

$$\mathfrak{S} = \frac{r_p}{\sqrt{g(r_p)}} \tag{6.20}$$

where r_p is the radius of the generally unique null circular geodesic of $\mathcal{M}$ [16, 32, 50, 25]. This radius is defined by the equation [11]

$$2g(r_p) = r_p g'(r_p) \tag{6.21}$$

where the apostrophe stands for derivation with respect to r.

The Callan Myers Perry black hole space time falls into this class of black holes. Thus, we can make use of the formulas above, by replacing g with f.

As it turns out, we already computed r_p for the Callan Myers Perry black hole space time. Indeed, we know that r_s, expressed in (6.2), is the solution of (3.40) with h replaced by f, which is precisely the equation (6.21) for the Callan Myers Perry black hole space time. As a consequence, we can write

$$r_p = r_s = R_h \left(\frac{d-1}{2}\right)^{\frac{1}{d-3}} \left[1 + \lambda' \left(\frac{d-4}{2} - (d-4)\left(\frac{2}{d-1}\right)^{\frac{d-1}{d-3}}\right)\right]. \tag{6.22}$$

Furthermore, looking at (6.3) and (6.20), we can write

$$\mathfrak{S} = \frac{1}{\Gamma_r^s} = \frac{\sqrt{(d-1)(d-3)}}{4\pi T_{\mathcal{H}}} \left(\frac{d-1}{2}\right)^{\frac{1}{d-3}} \left[1 + \lambda' \left(\frac{d-4}{2}\right)\left(2 - d - \left(\frac{2}{d-1}\right)^{\frac{d-1}{d-3}}\right)\right]. \tag{6.23}$$

The equation above displays a strong relation between $\mathfrak{S}$ and the real part of the eikonal quasinormal frequencies, associated with linear perturbations of complex massless scalar fields. In the non corrected limit, this relation extends to all eikonal quasinormal frequencies, for they are identical. A recent study about this relation can be found in [16].

Chapter 7

Concluding remarks

In this chapter, we make some concluding remarks concerning our results.

7.1 Eikonal limit

Here, we address the quasinormal frequencies computation in the eikonal limit.

For tensor type gravitational perturbations, we computed the quasinormal frequencies (3.52). For linear perturbations of a complex massless scalar field, we computed the quasinormal frequencies (6.6). In the limit $\alpha' \to 0$, we recover the quasinormal frequencies of the d-dimensional Tangherlini black hole space time [8][1].

Comparing expressions (3.51) and (6.5), we notice they are the same. This means the imaginary components of the quasinormal frequencies spectra, associated with both linear perturbations of a complex massless scalar field and tensor type gravitational perturbations, are identical! This is somewhat remarkable. Indeed, in a d-dimensional Tangherlini black hole space time, these perturbations share the same quasinormal frequencies spectrum [40][2]. Thus, we found the stringy correction preserves this relation for the imaginary component of the spectra. In the appendix B, we prove this relation in a more explicit way. Doing so, we find that such relation does not hold for all explicit corrections of the potential (2.18). Indeed, the structure of the explicit correction in (2.13) is crucial in the proof we give.

7.2 Asymptotic limit

Now, we address the computation of quasinormal frequencies in the asymptotic limit. In this limit, we did not compute an explicit expression for the quasinormal frequencies. Instead, we built a transcendental equation whose independent variable was ω.

For tensor type gravitational perturbations, this equation is (4.169). For linear perturbations of a complex massless scalar field, this equation is (6.8). We notice these equations are different. Thus, we figure the stringy correction does not preserve the identity relation, found in the d-dimensional Tangherlini black hole space time, between the quasinormal frequencies spectra of these perturbations.

In the limit $\alpha' \to 0$, we recover the asymptotic quasinormal frequencies for the d-dimensional Tangherlini black hole space time [40, 37].

For fixed values of ω, we notice our results may loose precision as d increases. Indeed, while computing the monodromy, needed to build the transcendental equations, we made an approximation by discarding terms of order

$$\left(\frac{\omega}{T_{\mathcal{H}}}\right)^{-\frac{1}{d-2}}. \tag{7.1}$$

[1] We warn that [8] defines quasinormal frequencies as $-\omega$, when compared with our definition.

[2] Not only in the eikonal limit, but in general.

This is fine for the theoretical asymptotic limit, in which we assume that $|\omega| \to +\infty$. However, one must take care, when applying our results for very large, but not arbitrary so, values of $|\omega|$. Indeed, suppose one uses our transcendental equations to compute values of ω such that

$$\left|\frac{\omega}{T_{\mathcal{H}}}\right| \sim 10^8. \tag{7.2}$$

Then, those values were computed assuming that $10^{-\frac{8}{d-2}}$ is negligible. However, taking $d = 5, 7, 9$ yields

$$10^{-\frac{8}{3}} \approx 0.002 \qquad 10^{-\frac{8}{5}} \approx 0.03 \qquad 10^{-\frac{8}{7}} \approx 0.07 \tag{7.3}$$

respectively. As we can see, the approximation grows worst as we increase d. Thus, our results for progressively higher dimensions only apply to progressively higher values of $\left|\frac{\omega}{T_{\mathcal{H}}}\right|$.

7.3 Greybody factors

Here, we address the computation of the greybody factors in the limit $|\Im(\omega)| \gg 1$.

For tensor type gravitational perturbations, we computed the greybody factor (5.64). For linear perturbations of a complex massless scalar field, we computed the greybody factor (6.16). Taking the limit $\alpha' \to 0$, we recover the expression for the greybody factor in the d-dimensional Tangherlini black hole space time [26, 42]. Moreover, in the limit $\alpha' \to 0$, the reflection coefficients match those of the d-dimensional Tangherlini black hole [26, 42].

The computed greybody factors share the same issue, present in the results concerning the asymptotic quasinormal frequencies. More precisely, for fixed values of ω, our results may loose precision as d increases. This is a consequence of the same approximation, used in the computation of the asymptotic quasinormal frequencies.

7.4 Black hole shadow

Here, we consider the radius associated with the black hole shadow cast by the Callan Myers Perry black hole space time, expressed in (6.23). Taking the limit $\alpha' \to 0$, we recover the radius associated with the shadow cast by the d-dimensional Tangherlini black hole space time [50], as expected.

Appendix A

Monodromy invariance on $\mathfrak{F}$

Here, we prove there exists an uncountable family $\mathfrak{F}$ of particular solutions of (4.12), for a fixed solution of (4.11), obeying the boundary condition (4.46). Moreover, we prove that (4.141) does not depend on the choice of a representative of that family.

For a fixed solution ψ_0 of (4.11), it is easy to prove there exists an uncountable family of particular solutions ψ_1 of (4.12), obeying the boundary condition (4.46). Indeed, let us choose the constants B_- and B_+, as the unique solution of the linear system

$$\begin{cases} B_+ e^{i\alpha_+} + B_- e^{i\alpha_-} = \gamma \\ B_+ e^{-i\alpha_+} + B_- e^{-i\alpha_-} = -\left(\Lambda_I^+ e^{-i\alpha_+} + \Lambda_I^- e^{-i\alpha_-}\right) \end{cases} \tag{A.1}$$

for some $\gamma \in \mathbb{C}$. Using (4.62), we notice the resultant χ still makes the redefinition (4.98) agree with the boundary condition (4.46). Thus, the family of particular solutions ψ_1, obeying the boundary condition (4.46) for some fixed ψ_0, is parameterized by $\gamma \in \mathbb{C}$. More precisely, $\mathfrak{F}$ is in bijection with $\mathbb{C}$.

Now, we prove (5.2) does not depend on the choice of a representative of $\mathfrak{F}$. Computing the solution of $(A.1)$ wields

$$B_- = \frac{e^{i\alpha_+}\left(\Lambda_I^+ e^{-i\alpha_+} + \Lambda_I^- e^{-i\alpha_-}\right)}{2i\sin\left(\alpha_- - \alpha_+\right)} + \frac{\gamma e^{-i\alpha_+}}{2i\sin(\alpha_- - \alpha_+)} \tag{A.2}$$

$$B_+ = -\frac{e^{i\alpha_-}\left(\Lambda_I^+ e^{-i\alpha_+} + \Lambda_I^- e^{-i\alpha_-}\right)}{2i\sin\left(\alpha_- - \alpha_+\right)} - \frac{\gamma e^{-i\alpha_-}}{2i\sin(\alpha_- - \alpha_+)}. \tag{A.3}$$

Choosing χ as defined by the constants above, we can rewrite (4.102) as

$$\psi(z) \sim \left(A_+ e^{i\alpha_+} + A_- e^{i\alpha_-}\right) e^{-i\omega z}\left[1 + \lambda'\left(\frac{\Lambda_I^+ e^{i\alpha_+} + \Lambda_I^- e^{i\alpha_-}}{A_+ e^{i\alpha_+} + A_- e^{i\alpha_-}} + \frac{B_+ e^{i\alpha_+} + B_- e^{i\alpha_-}}{A_+ e^{i\alpha_+} + A_- e^{i\alpha_-}}\right)\right] \tag{A.4}$$

near D. Moreover, using the definitions (4.126) with respect to the original values of B_+ and B_- [1], we can rewrite (4.140) as

$$\psi(z) \sim \left(A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}\right) e^{-i\omega m} e^{-i\omega z}\left[1 + \lambda'\left(\frac{\Lambda_F^+ e^{5i\alpha_+} + \Lambda_F^- e^{5i\alpha_-}}{A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}} + \frac{K_+ e^{5i\alpha_+} + K_- e^{5i\alpha_-}}{A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}}\right)\right] \tag{A.5}$$

near D, where we defined the constants

$$K_\pm := \mp\frac{\gamma e^{-i\alpha_\mp}}{2i\sin(\alpha_- - \alpha_+)}. \tag{A.6}$$

This time, comparing $(A.4)$ and $(A.5)$, wields the multiplicative monodromy

$$\mathcal{M}_1 = \left(\frac{A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}}{A_+ e^{i\alpha_+} + A_- e^{i\alpha_-}}\right) e^{-i\omega\Delta_z}\left(1 + \lambda'\delta\mathcal{M}_1\right) \tag{A.7}$$

[1] The unique solution of the linear system (4.100).

where we defined

$$\delta\mathcal{M}_1 \coloneqq \frac{\Lambda_F^+ e^{i5\alpha_+} + \Lambda_F^- e^{i5\alpha_-}}{A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}} + \frac{K_+ e^{5i\alpha_+} + K_- e^{5i\alpha_-}}{A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}} - \frac{\Lambda_I^+ e^{i\alpha_+} + \Lambda_I^- e^{i\alpha_-}}{A_+ e^{i\alpha_+} + A_+ e^{i\alpha_-}} - \frac{B_+ e^{i\alpha_+} + B_- e^{i\alpha_-}}{A_+ e^{i\alpha_+} + A_- e^{i\alpha_-}}. \tag{A.8}$$

Now, we show the expression above is identical to (4.142). That is the case if and only if

$$\frac{K_+ e^{5i\alpha_+} + K_- e^{5i\alpha_-}}{A_+ e^{5i\alpha_+} + A_- e^{5i\alpha_-}} - \frac{B_+ e^{i\alpha_+} + B_- e^{i\alpha_-}}{A_+ e^{i\alpha_+} + A_- e^{i\alpha_-}} = 0. \tag{A.9}$$

Using $(A.2)$, $(A.3)$, $(A.6)$ and (4.65), we can rewrite the equation above as

$$\frac{\gamma}{\beta}\left[\frac{e^{(5\alpha_- - \alpha_+)i} - e^{(5\alpha_+ - \alpha_-)i}}{e^{(5\alpha_- - \alpha_+)i} - e^{(5\alpha_+ - \alpha_-)i}}\right] - \frac{\gamma}{\beta} = \frac{\gamma}{\beta} - \frac{\gamma}{\beta} = 0. \tag{A.10}$$

Thus, we proved the monodromy (4.141) is independent on the choice of a representative of $\mathfrak{F}$.

Appendix B

Remarkable equality

Here, we provide an analytical proof of the remarkable equality between the imaginary components of the eikonal quasinormal frequencies, associated with tensor type gravitational perturbations and scalar field perturbations.

Effectively, we want to show that (3.49) is equal to (6.5), in a more explicit way. This task amounts to prove that

$$\frac{f^2(r_s)}{f(r_s)r_s^2}\left[2f(r_s)-f''(r_s)r_s^2\right]=\frac{f^2(r_g)}{h(r_g)r_g^2}\left[2h(r_g)-h''(r_g)r_g^2\right] \tag{B.1}$$

where the apostrophe stands for derivation with respect to r. Before we start, it is convenient to restate the following equations

$$2f(r_s)=r_sf'(r_s) \tag{B.2}$$

$$2h(r_g)=r_gh'(r_g) \tag{B.3}$$

$$2f_0(r_0)=r_0f_0'(r_0) \tag{B.4}$$

where r_0 comes from the expansions

$$r_g=r_0+\lambda'\delta r_g \tag{B.5}$$

$$r_s=r_0+\lambda'\delta r_s. \tag{B.6}$$

We begin by performing the Taylor expansions

$$f(r_s)=f(r_g)+f'(r_g)(r_s-r_g) \tag{B.7}$$

$$f''(r_s)=f''(r_g)+f'''(r_g)(r_s-r_g) \tag{B.8}$$

up to first order in λ'. Using these expansions, we can write

$$\begin{aligned}f(r_s)f''(r_s)&=\left[f(r_g)+f'(r_g)(r_s-r_g)\right]\left[f''(r_g)+f'''(r_g)(r_s-r_g)\right]\\&=f(r_g)f''(r_g)+f(r_g)f'''(r_g)(r_s-r_g)+f'(r_g)f''(r_g)(r_s-r_g)\\&=f(r_g)f''(r_g)+(r_s-r_g)\left[f(r_g)f'''(r_g)+f'(r_g)f''(r_g)\right]\\&=f(r_g)f''(r_g)+(r_s-r_g)\left[f_0(r_0)f_0'''(r_0)+f_0'(r_0)f_0''(r_0)\right]\end{aligned} \tag{B.9}$$

where in the last equality, we discarded contributions of order $\mathcal{O}(\lambda'^2)$.

Now, let us rewrite h as

$$h(r)=f(r)+\Delta f(r) \tag{B.10}$$

where we defined the function

$$\Delta f(r):=2\lambda' f(r)\left(\frac{R_h}{r}\right)^2\left[2(1-f(r))+rf'(r)\right]. \tag{B.11}$$

Now, we can write

$$h(r_g)=f(r_g)\left(1+\frac{\Delta f(r_g)}{f(r_g)}\right)\implies\frac{1}{h(r_g)}=\frac{1}{f(r_g)}\left(1-\frac{\Delta f(r_g)}{f(r_g)}\right)=\frac{1}{f(r_g)}-\frac{\Delta f(r_g)}{f^2(r_g)} \tag{B.12}$$

where in the second equality, we discarded contributions of order $\mathcal{O}(\lambda'^2)$. Moreover, we can write

$$\begin{aligned}\frac{f^2(r_g)}{h(r_g)}h''(r_g) &= f^2(r_g)\left(\frac{1}{f(r_g)}-\frac{\Delta f(r_g)}{f^2(r_g)}\right)h''(r_g) = \left(f(r_g)-\Delta f(r_g)\right)\left(f''(r_g)+\Delta f''(r_g)\right)\\ &= f(r_g)f''(r_g)-\Delta f(r_g)f''(r_g)+f(r_g)\Delta f''(r_g) = f(r_g)f''(r_g)-\Delta f(r_0)f_0''(r_0)+f_0(r_0)\Delta f''(r_0)\end{aligned} \tag{B.13}$$

where in last equality, we discarded contributions of order $\mathcal{O}(\lambda'^2)$.

We can write

$$\begin{aligned}2\frac{f^2(r_s)}{r_s^2}-2\frac{f^2(r_g)}{r_g^2} &= \frac{2}{r_s^2}\left(f^2(r_g)+2f(r_g)f'(r_g)(r_s-r_g)\right)-2\frac{f^2(r_g)}{r_g^2}\\ &= 2f^2(r_g)\left(\frac{1}{r_s^2}-\frac{1}{r_g^2}\right)+\frac{4}{r_0^2}f_0(r_0)f_0'(r_0)(r_s-r_g).\end{aligned} \tag{B.14}$$

where in the last equality, we discarded contributions of order $\mathcal{O}(\lambda'^2)$. We see that

$$\frac{1}{r_s^2}-\frac{1}{r_g^2}=\left(\frac{1}{r_s}-\frac{1}{r_g}\right)\left(\frac{1}{r_s}+\frac{1}{r_g}\right)=\frac{2}{r_0}\left(\frac{1}{r_s}-\frac{1}{r_g}\right)=\frac{2}{r_0}\left(\frac{r_g-r_s}{r_sr_g}\right)=-\frac{2}{r_0^3}(r_s-r_g) \tag{B.15}$$

where in the second and fourth equalities, we discarded contributions of order $\mathcal{O}(\lambda'^2)$. Using the equation above, allow us to rewrite $(B.14)$ as

$$2\frac{f^2(r_s)}{r_s^2}-2\frac{f^2(r_g)}{r_g^2}=(r_s-r_g)\left[\frac{4}{r_0^2}f_0(r_0)f_0'(r_0)-\frac{4}{r_0^3}f_0^2(r_0)\right]=(r_s-r_g)\frac{4}{r_0^3}f_0^2(r_0) \tag{B.16}$$

where in the last equality, we used $(B.4)$.

Using $(B.9)$ and $(B.13)$, we can write

$$\begin{aligned}f^2(r_g)h^{-1}(r_g)h''(r_g)-f(r_s)f''(r_s) &= -f(r_g)f''(r_g)-(r_s-r_g)\left[f_0(r_0)f_0'''(r_0)+f_0'(r_0)f_0''(r_0)\right]\\ &+f(r_g)f''(r_g)-\Delta f(r_0)f_0''(r_0)+f_0(r_0)\Delta f''(r_0) = f_0(r_0)\Delta f''(r_0)-\Delta f(r_0)f_0''(r_0)\\ &-(r_s-r_g)\left[f_0(r_0)f_0'''(r_0)+f_0'(r_0)f_0''(r_0)\right].\end{aligned} \tag{B.17}$$

Using the equation above together with $(B.16)$, allow us to rewrite equation $(B.1)$ as

$$(r_s-r_g)\left[\frac{4}{r_0^3}f_0^2(r_0)-f_0(r_0)f_0'''(r_0)-f_0'(r_0)f_0''(r_0)\right]+f_0(r_0)\Delta f''(r_0)-\Delta f(r_0)f_0''(r_0)=0. \tag{B.18}$$

From (3.11) and (6.2), we gather that

$$r_s-r_g=2\lambda'\frac{R_h^2}{r_0}. \tag{B.19}$$

Hence, we can rewrite $(B.18)$ as

$$\lambda' R_h^2\left[\frac{8}{r_0^4}f_0^2(r_0)-\frac{2}{r_0}f_0(r_0)f_0'''(r_0)-\frac{2}{r_0}f_0'(r_0)f_0''(r_0)\right]+f_0(r_0)\Delta f''(r_0)-\Delta f(r_0)f_0''(r_0)=0. \tag{B.20}$$

Now, we need to attend to the structure of Δf. Using $(B.11)$, we can write

$$\begin{aligned}\Delta f'(r) &= 2\lambda' f_0'(r)\left(\frac{R_h}{r}\right)^2\left[2(1-f_0(r))+rf_0'(r)\right]-4\lambda' f_0(r)\frac{R_h^2}{r^3}\left[2(1-f_0(r))+rf_0'(r)\right]\\ &+2\lambda' f_0(r)\left(\frac{R_h}{r}\right)^2\left[rf_0''(r)-f_0'(r)\right]\end{aligned} \tag{B.21}$$

$$\Delta f''(r) = 2\lambda' f_0''(r)\left(\frac{R_h}{r}\right)^2\left[2(1-f_0(r))+rf_0'(r)\right] - 8\lambda' f_0'(r)\frac{R_h^2}{r^3}\left[2(1-f_0(r))+rf_0'(r)\right]$$
$$+4\lambda' f_0'(r)\left(\frac{R_h}{r}\right)^2\left[rf_0''(r)-f_0'(r)\right] + 12\lambda' f_0(r)\frac{R_h^2}{r^4}\left[2(1-f_0(r))+rf_0'(r)\right] \tag{B.22}$$
$$-8\lambda' f_0(r)\frac{R_h^2}{r^3}\left[rf_0''(r)-f_0'(r)\right] + 2\lambda' f_0(r)\left(\frac{R_h}{r}\right)^2 rf_0'''(r).$$

Using $(B.4)$ yields

$$2\lambda' f_0''(r_0)\left(\frac{R_h}{r_0}\right)^2\left[2(1-f_0(r_0))+r_0f_0'(r_0)\right] = 4\lambda' f_0''(r_0)\left(\frac{R_h}{r_0}\right)^2 \tag{B.23}$$

$$8\lambda' f_0'(r_0)\frac{R_h^2}{r_0^3}\left[2(1-f_0(r_0))+r_0f_0'(r_0)\right] = 16\lambda' f_0'(r_0)\frac{R_h^2}{r_0^3} \tag{B.24}$$

$$12\lambda' f_0(r_0)\frac{R_h^2}{r_0^4}\left[2(1-f_0(r_0))+r_0f_0'(r_0)\right] = 24\lambda' f_0(r_0)\frac{R_h^2}{r_0^4}. \tag{B.25}$$

Thus, we can rewrite $(B.22)$ as

$$\Delta f''(r_0) = 4\lambda' R_h^2\Bigg[\frac{f_0''(r_0)}{r_0^2} - 4\frac{f_0'(r_0)}{r_0^3} + \frac{f_0'(r_0)}{r_0^2}\left(r_0f_0''(r_0)-f_0'(r_0)\right) + 6\frac{f_0(r_0)}{r_0^4} -$$
$$2\frac{f_0(r_0)}{r_0^3}\left(r_0f_0''(r_0)-f_0'(r_0)\right) + \frac{1}{2}\frac{f_0(r_0)}{r_0}f_0'''(r_0)\Bigg] = 4\lambda' R_h^2\left[\frac{f_0''(r_0)}{r_0^2} - 2\frac{f_0(r_0)}{r_0^4} + \frac{1}{2}\frac{f_0(r_0)}{r_0}f_0'''(r_0)\right] \tag{B.26}$$

and consequently, we see that

$$f_0(r_0)\Delta f''(r_0) - \Delta f(r_0)f_0''(r_0) = 4\lambda' R_h^2\left[\frac{f_0''(r_0)f_0(r_0)}{r_0^2} - 2\frac{f_0^2(r_0)}{r_0^4} + \frac{1}{2}\frac{f_0^2(r_0)}{r_0}f_0'''(r_0)\right]$$
$$-4\lambda' f_0(r_0)f_0''(r_0)\left(\frac{R_h}{r_0}\right)^2 = \lambda' R_h^2\left[2\frac{f_0^2(r_0)}{r_0}f_0'''(r_0) - 8\frac{f_0^2(r_0)}{r_0^4}\right]. \tag{B.27}$$

Plugging the equation above in $(B.20)$ yields

$$\lambda' R_h^2\left[2\frac{f_0^2(r_0)}{r_0}f_0'''(r_0) - \frac{2}{r_0}f_0(r_0)f_0'''(r_0) - \frac{2}{r_0}f_0'(r_0)f_0''(r_0)\right] = 0$$
$$\iff f_0^2(r_0)f_0'''(r_0) - f_0(r_0)f'''(r_0) - f_0'(r_0)f_0''(r_0) = 0$$
$$\iff f_0^2(r_0)f_0'''(r_0) - f_0(r_0)f'''(r_0) - \frac{2}{r_0}f_0(r_0)f_0''(r_0) = 0 \tag{B.28}$$
$$\iff f_0(r_0)f_0'''(r_0) - f'''(r_0) - \frac{2}{r_0}f_0''(r_0) = 0$$
$$\iff f_0'''(r_0)\left(f_0(r_0)-1\right) - \frac{2}{r_0}f_0''(r_0) = 0.$$

We found an equation equivalent to $(B.1)$, much simpler than the latter!

Now, let us define the function

$$\mathcal{F}(r) \coloneqq f_0'''(r)\left(f_0(r)-1\right) - \frac{2}{r}f_0''(r). \tag{B.29}$$

Equation $(B.28)$ holds true if and only if r_0 is a root of $\mathcal{F}$. Using (2.4), we can write

$$\mathcal{F}(r) = -(d-3)(d-2)(d-1)\frac{R_h^{d-3}}{r^d}\left(\frac{R_h}{r}\right)^{d-3} + \frac{2}{r}(d-3)(d-2)\frac{R_h^{d-3}}{r^{d-1}}$$
$$= \left(\frac{R_h}{r}\right)^{d-3}(d-3)(d-2)\left[\frac{2}{r^3} - (d-1)\frac{R_h^{d-3}}{r^d}\right]. \tag{B.30}$$

Computing the roots of $\mathcal{F}$ yields

$$\begin{aligned} \mathcal{F}(r) = 0 \iff \frac{2}{r^3} - (d-1)\frac{R_h^{d-3}}{r^d} = 0 \\ \iff \frac{2}{d-1} = \left(\frac{R_h}{r}\right)^{d-3} \implies r = \left(\frac{d-1}{2}\right)^{\frac{1}{d-3}} R_h = r_0. \end{aligned} \tag{B.31}$$

Thus, completing the proof.

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