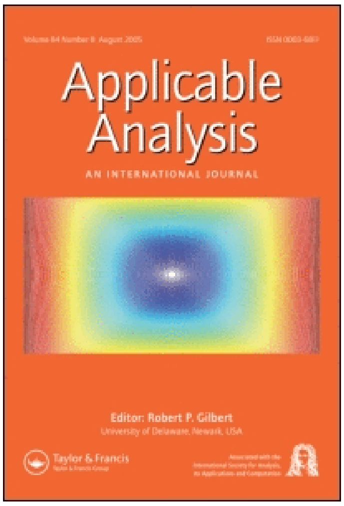

Applicable Analysis
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713454076

Memory effects and random walks in reaction-transport systems

J. A. Ferreira [a]; P. De Oliveira [a]
[a] Department of Mathematics, University of Coimbra, Portugal

First Published:January2007

To cite this Article Ferreira, J. A. and De Oliveira, P.(2007)'Memory effects and random walks in reaction-transport systems',Applicable Analysis,86:1,99 — 118

To link to this Article: DOI: 10.1080/00036810601110638

URL: http://dx.doi.org/10.1080/00036810601110638

Applicable Analysis
Vol. 86, No. 1, January 2007, 99–118

Memory effects and random walks in reaction-transport systems

J. A. FERREIRA* and P. DE OLIVEIRA

Department of Mathematics, University of Coimbra, Portugal

Communicated by P. Ciarlet

(*Received 6 March 2006; in final form 10 November 2006*)

In this article, we study continuous and discrete models to describe reaction transport systems with memory and long range interaction. In these models the transport process is described by a non-Brownian random walk model and the memory is induced by a waiting time distribution of the gamma type. Numerical results illustrating the behavior of the solution of discrete models are also included.

Keywords: Integro-differential equation; Non-Brownian motion; Stability; Numerical methods

2000 Mathematics Subject Classifications: 35B35; 35K57; 65M06; 65M12

1. Introduction

Reaction–diffusion models are currently used to describe the dynamics of problems that involve dispersal and reaction phenomena. These problems arise in a wide variety of contexts as for example population structure, propagation of epidemics or combustion waves.

From a chronological point of view the first models found in the literature are differential models. The simplest one is the well known Fisher–Kolmogorov–Petrosvskii–Piskunov (FKPP) equation

$$\frac{\partial v}{\partial t} = -\frac{\partial J}{\partial x} + f(v), \tag{1}$$

where $J = -D(\partial v/\partial x)$, D is the diffusion coefficient and f represents the reaction term [3,11,12]. This equation presents, however, a serious drawback – which is related to its

*Corresponding author. Email: ferreira@mat.uc.pt

Applicable Analysis ISSN 0003-6811 print/ISSN 1563-504X online © 2007 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/00036810601110638

parabolic character – that can be roughly defined as an "infinite speed of heat/mass transfer". As a consequence the propagation rate of traveling wave solutions, given by $\sqrt{4DU}$ for $f(v) = U(1-v)v$, exhibits the unphysical property of becoming arbitrarily large when U goes to infinity.

To overcome this difficulty several modifications of (1) have been proposed in the literature. A first modification takes into account the boundness of the transport process by introducing a relaxation parameter τ which represents the waiting time between two successive jumps of the particles whose movement we want to describe [1,2,4,6,7]. The FKPP equation is then replaced by the integro-differential equation

$$\frac{\partial v}{\partial t}(x,t) = \frac{D}{\tau}\int_0^t e^{-(t-s)/\tau}\frac{\partial^2 v}{\partial x^2}(x,s)\mathrm{d}s + f(v(x,t)). \tag{2}$$

We note that equation (2) can be obtained from (1) by defining the flux J as the solution of the first order differential equation

$$\frac{\partial J}{\partial t} + \frac{1}{\tau}J = \frac{D}{\tau}\frac{\partial v}{\partial x}.$$

Another generalization of FKPP equation results from considering the form of the particular random walk model underlying the transport process. This approach leads to the establishment of integro-difference equations of type

$$v(x, t+\tau) = \int_{\mathbb{R}} v(x+\Delta, t)\phi(\Delta)\mathrm{d}\Delta + \tau f(v) \tag{3}$$

where the kernel $\phi(\Delta)$ represents the probability distribution function of jumps length [8,13]. An equivalent continuous version of (3), up to the second order in τ, is the integro-differential equation

$$\frac{\partial v}{\partial t}(x,t) = \frac{1}{\tau}\left(\int_{\mathbb{R}} v(x+\Delta,t)\phi(\Delta)\mathrm{d}\Delta - v(x,t)\right) + f(v(x,t)), \quad x \in \mathbb{R}. \tag{4}$$

A natural generalization of both (3) and (4) consists in considering a model where the memory effects associated with random process are also present. To describe this simultaneous effect of randomness and memory integro-differential equations of type

$$\frac{\partial v}{\partial t}(x,t) = \int_0^t \alpha(t-s)\left(\int_{\mathbb{R}} v(x+\Delta,s)\phi(\Delta)\mathrm{d}\Delta - v(x,s)\right)\mathrm{d}s + f(v(x,t)), \quad x \in \mathbb{R}, \tag{5}$$

have been proposed in [9] and [10]. We observe that when the time kernel $\alpha(t-s)$ is defined by $\alpha(t-s) = \lambda\delta(t-s)$, where δ stands for the Dirac delta function, we obtain the "memoryless" equation (4) with $\lambda = (1/\tau)$. On the other hand if isotropic kernels ϕ are considered in (5) then up to the second order in Δ an equivalent "memoryfull" deterministic equation of type (2) is obtained. In the case $\alpha(t-s) = \lambda^2 e^{-2\lambda(t-s)}$, which corresponds to a waiting time density defined by a member of the family of gamma distributions, where the parameter λ is given by $\lambda = (2/\tau)$ and τ stands for the mean time between successive jumps, we have a model represented by the integro-differential equation

$$\frac{\partial v}{\partial t}(x,t) = \lambda^2 \int_0^t e^{-2\lambda(t-s)} \left(\int_{\mathbb{R}} v(x+\Delta, s)\,\phi(\Delta)\mathrm{d}\Delta - v(x,s) \right) \mathrm{d}s + f(v(x,t)), \quad x \in \mathbb{R}, t > 0. \tag{6}$$

From a practical point of view models of type (6) are very useful because both memory effects and random walks represent significant features in many areas of physics, chemistry and biology. As far as such models are concerned the speed of traveling waves has been computed for various time and space kernels in [9] and [10]. However in these articles there is no reference to the well-posedness of the model nor to the stability of steady states. One of our aims in this article is to study these last problems. In this sense, in section 2, we establish an energy estimate which leads to the stability of the model. In section 3 the stability of the steady states is studied by using an equivalent second-order equation which is a generalization of the telegrapher's equation. In section 4 we study the qualitative properties – steepness and width – of the front connecting the stable state with the unstable state. The energy estimate established for equation (6) is then used in Section 5 to design a numerical method exhibiting discrete analogous energy properties. Finally, in section 6 we present some numerical examples.

2. The stability of the model

In the main result of this section – Theorem 7 – we study the behavior of the solution of problem (6) with initial condition

$$v(x,0) = v_0(x), \quad x \in \mathbb{R}. \tag{7}$$

The stability of the model presented in Theorem 2 is then a straightforward consequence of Theorem 1.

Theorem 1 *Let v be solution of* (6), (7). *If the source term f is a differentiable function that satisfies*

$$f(0) = 0,\ f'(y) \le M_{f'},\ y \in [c,d], \tag{8}$$

where $[c,d]$ is such that $v(y,t) \in [c,d],\ y \in \mathbb{R}, t \ge 0,$ then

$$\|v(\cdot,t)\|^2_{L^2(\mathbb{R})} + \lambda^2 \left\| \int_0^t e^{-2\lambda(t-s)} v(\cdot,s)\mathrm{d}s \right\|^2_{L^2(\mathbb{R})} \le e^{\max\{1-4\lambda,\ \lambda^2+2M_{f'}\}t} \|v_0\|^2_{L^2(\mathbb{R})}. \tag{9}$$

Proof Multiplying equation (6) by v with respect to the $L^2(\mathbb{R})$ inner product, it it can be shown that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(\cdot,t)\|^2_{L^2(\mathbb{R})} = \lambda^2\int_{\mathbb{R}\times\mathbb{R}}\int_0^t e^{-2\lambda(t-s)}v(z,s)\mathrm{d}s\phi(z-y)\mathrm{d}zv(y,t)\mathrm{d}y \\ -\lambda^2\int_{\mathbb{R}}\left(\int_0^t e^{-2\lambda(t-s)}v(y,s)\mathrm{d}s\right)v(y,t)\mathrm{d}y + \int_{\mathbb{R}} f(v(y,t))v(y,t)\mathrm{d}y. \quad (10)$$

Let us represent respectively by Q_1 and Q_2 the first and the second terms of the right-hand side of this last equation.

For Q_1 we have

$$Q_1 \le \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}}\left(\int_0^t e^{-2\lambda(t-s)}v(y,s)\mathrm{d}s\right)^2\phi(z-y)\mathrm{d}z\,\mathrm{d}y + \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}} v^2(y,t)\phi(z-y)\mathrm{d}z\,\mathrm{d}y \\ \le \frac{1}{2}\int_{\mathbb{R}}\left(\int_0^t e^{-2\lambda(t-s)}v(y,s)\mathrm{d}s\right)^2\mathrm{d}y + \frac{1}{2}\int_{\mathbb{R}} v^2(y,t)\mathrm{d}y \\ = \frac{1}{2}\left\|\int_0^t e^{-2\lambda(t-s)}v(\cdot,s)\mathrm{d}s\right\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\|v(\cdot,t)\|^2_{L^2(\mathbb{R})}, \quad (11)$$

and for Q_2 it can be shown that

$$Q_2 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\int_0^t e^{-2\lambda(t-s)}v(\cdot,s)\mathrm{d}s\right\|^2_{L^2(\mathbb{R})} + 2\lambda\left\|\int_0^t e^{-2\lambda(t-s)}v(\cdot,s)\mathrm{d}s\right\|^2_{L^2(\mathbb{R})}. \quad (12)$$

As we have

$$\int_{\mathbb{R}} f(v(y,t))v(y,t)\mathrm{d}y \le M_{f'}\|v(\cdot,t)\|^2_{L^2(\mathbb{R})}, \quad (13)$$

we easily establish from (10)–(13)

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(\cdot,t)\|^2_{L^2(\mathbb{R})} + \lambda^2\frac{\mathrm{d}}{\mathrm{d}t}\left\|\int_0^t e^{-2\lambda(t-s)}v(\cdot,s)\mathrm{d}s\right\|^2_{L^2(\mathbb{R})} \le (1-4\lambda)\lambda^2\left\|\int_0^t e^{-2\lambda(t-s)}v(\cdot,s)\mathrm{d}s\right\|^2_{L^2(\mathbb{R})} \\ + \left(\lambda^2 + 2M_{f'}\right)\|v(\cdot,t)\|^2_{L^2(\mathbb{R})} \quad (14)$$

which allows us to conclude (9). ■

As a consequence of Theorem 1, we establish in what follows the stability of (6)–(7).

THEOREM 2 *Let v and $\tilde{v}$ be two solutions of* (6) *with initial conditions v_0 and $\tilde{v}_0$ respectively. If $v(y,t)$, $\tilde{v}(y,t) \in [c,d]$, $y \in \mathbb{R}$, $t \ge 0$, and the differentiable source function f*

satisfies (8), *then*

$$\|(v-\tilde{v})(\cdot,t)\|^2_{L^2(\mathbb{R})}+\lambda^2\left\|\int_0^t e^{-2\lambda(t-s)}(v-\tilde{v})(\cdot,s)\mathrm{d}s\right\|^2_{L^2(\mathbb{R})}\leq e^{\max\{1-4\lambda,\,\lambda^2+2M_{f'}\}t}\|v_0-\tilde{v}_0\|^2_{L^2(\mathbb{R})}. \tag{15}$$

Proof For $w = v-\tilde{v}$ we have

$$\frac{\partial w}{\partial t}(x,t)=\lambda^2\int_0^t e^{-2\lambda(t-s)}\Big(\int_{\mathbb{R}} w(x+\Delta,s)\phi(\Delta)\mathrm{d}\Delta - w(x,s)\Big)\mathrm{d}s$$
$$+ f'(\theta v(x,t)+(1-\theta\tilde{v}(x,t))w(x,t),\quad x\in\mathbb{R},$$

with $\theta\in(0,1)$. Proceeding as in Theorem 1 we then establish (15). ■

Remark 1

(1) Let us assume that λ and $M_{f'}$ satisfy

$$1-4\lambda\leq\lambda^2+2M_{f'}, \tag{16}$$

which is equivalent to

$$\lambda\in\Big[-2+\sqrt{5-2M_{f'}},\,+\infty\Big) \tag{17}$$

provided that $M_{f'}\leq(5/2)$.

(a) If

$$\lambda^2+2M_{f'}<0, \tag{18}$$

then (6)–(7) is stable.

(b) If $\lambda^2+2M_{f'}>0$ then we conclude that

$$\|(v-\tilde{v})(\cdot,t)\|^2_{L^2(\mathbb{R})}+\lambda^2\left\|\int_0^t e^{-2\lambda(t-s)}(v-\tilde{v})(\cdot,s)\mathrm{d}s\right\|^2_{L^2(\mathbb{R})} \tag{19}$$

is bounded in bounded time intervals.

(2) In the case λ and $M_{f'}$ do not satisfy (16) that is

$$1-4\lambda>\lambda^2+2M_{f'}, \tag{20}$$

and $M_{f'}\leq(5/2)$ then as λ is a positive constant we have

$$\lambda\in(0,-2+\sqrt{5-2M_{f'}}).$$

Two particular subcases of (20) can be considered.

(a) If $\lambda<14$ then (6)–(7) is stable.

(b) If $\lambda>14$ then (19) is bounded in bounded time intervals.

(3) Finally we consider a source function such that $M_{f'} > 5/2$. As $\lambda^2 + 4\lambda + 2M_{f'} - 1 \geq 0$ we conclude in this case that (19) is bounded in bounded time intervals.

Remark 2 As a consequence of Theorem 2 we conclude that if (6)–(7) has a solution then such a solution is unique.

3. The stability of the steady states

In this section we prove that the solution of (6)–(7) is solution of a telegrapher's initial value problem provided that such a solution is smooth enough. Using this result we can characterize the stability of the steady states of (6).

THEOREM 3 *Let v be the solution of* (6) *with* $v(x,0) = v_0(x)$, $x \in \mathbb{R}$. *Then v satisfies the telegrapher's equation*

$$\frac{\partial^2 v}{\partial t^2}(x,t) = \frac{\partial v}{\partial t}(x,t)(f'(v(x,t)) - 2\lambda) + \lambda^2\left(\int_{\mathbb{R}} v(x+\Delta,t)\phi(\Delta)\mathrm{d}\Delta - v(x,t)\right) + 2\lambda f(v(x,t)), \quad x \in \mathbb{R},\ t > 0, \tag{21}$$

and the initial conditions

$$\begin{cases} \dfrac{\partial v}{\partial t}(x,0) = f(v_0(x)), & x \in \mathbb{R}, \\ v(x,0) = v_0(x), & x \in \mathbb{R}, \end{cases} \tag{22}$$

provided that $(\partial^2 v/\partial t^2)$ *exists. Otherwise, if v is a solution of* (21)–(22) *then v is a solution of* (6) *and* $v(x,0) = v_0(x)$, $x \in \mathbb{R}$.

Proof We remark that from (6) we have

$$\frac{\partial^2 v}{\partial t^2}(x,t) = \lambda^2\left(\int_{\mathbb{R}} v(x+\Delta,t)\phi(\Delta)\mathrm{d}\Delta - v(x,t)\right) + f'(v(x,t))\frac{\partial v}{\partial t}(x,t) - 2\lambda^3\int_0^t e^{-2\lambda(t-s)}\left(\int_{\mathbb{R}} v(x+\Delta,s)\phi(\Delta)\mathrm{d}\Delta - v(x,s)\right)\mathrm{d}s$$

which combined with (6) enable us to conclude that v satisfies the integro-differential equation (21). ■

It is easy to show that with a source function given by

$$f(v) = Uv(1-v), \tag{23}$$

equation (6) has the steady states $v = 1$ and $v = 0$. In the following we study the stability properties of such steady states considering that the probability density function of jumps length is represented by the Gauss density function

$$\phi(\Delta) = \frac{1}{r\sqrt{\pi}} e^{-(\Delta^2/r^2)}, \tag{24}$$

or the Laplace density function

$$\phi(\Delta) = \frac{1}{2r} e^{-(|\Delta|/r)}. \tag{25}$$

THEOREM 4 *Let the source function f be defined by* (23). *If the kernel ϕ is defined by* (24) *or* (25), *then the steady state $v = 1$ is stable.*

Proof Following [14] we consider the linearized method. For $v = 1$ we have the linearized integro-differential equation

$$\frac{\partial^2 w}{\partial t^2}(x,t) + \frac{\partial w}{\partial t}(x,t)(2\lambda + U) = \lambda^2 \left(\int_{\mathbb{R}} w(x+\Delta, t)\phi(\Delta) \mathrm{d}\Delta - w(x,t) \right) - 2\lambda U. \tag{26}$$

Let us take $w(x,t) = \exp(ikx + \gamma t), k \in \mathbb{Z}$. For γ we obtain the algebraic equation

$$\gamma^2 + \gamma(2\lambda + U) - \lambda^2 \left(\sqrt{2\pi}\mathcal{F}(\phi)(k) - 1 \right) + 2\lambda U = 0, \tag{27}$$

where $\mathcal{F}(\phi)(k)$ denotes the Fourier transform of the density function ϕ. As

(1) for the Gauss kernel we have $\sqrt{2\pi}\mathcal{F}(\phi)(k) = e^{-k^2 r^2/4}$,
(2) for the Laplace kernel we have $\sqrt{2\pi}\mathcal{F}(\phi)(k) = 1/(1 + r^2 k^2)$,

we conclude in both cases that γ is a negative real number for all $k \in \mathbb{Z}$, and consequently $v = 1$ is a stable steady state of (6). ∎

THEOREM 5 *Let the source function f be defined by* (23). *If the kernel ϕ is defined by* (24) *or* (25), *then the steady state $v = 0$ is unstable.*

Proof Let us consider initial value problem (21), (22) with the initial condition $v_0(x) = \epsilon$. We compute a solution of this problem of form $v(x,t) = \epsilon w(t)$. For w we obtain the ordinary differential equation

$$w'' + (2\lambda - U(1 - 2\epsilon w))w' - 2\lambda U(1 - \epsilon w)w = 0, \tag{28}$$

which is equivalent to the system

$$\begin{cases} w' = z \\ z' = -(2\lambda - U(1 - 2\epsilon w))z + 2\lambda U(1 - \epsilon w)w. \end{cases} \tag{29}$$

System (29) has equilibrium points $P_1 = (0,0)$ and $P_2 = ((1/\epsilon), 0)$ which are unstable and stable points respectively. We then conclude that v converges to 1 when $t \to \infty$, which allow us to establish that $v = 0$ is an unstable steady state [5]. ∎

4. Steepness and width of the wave front

Let us consider equation (5) and a traveling wave solution v connecting $v=1$ and $v=0$. Assuming that v is C^∞, equation (5) can also be written as

$$\frac{\partial v}{\partial t}(x,t) = \int_0^t \alpha(t-s) \sum_{\ell=0}^{\infty} \frac{\langle \Delta^\ell \rangle}{\ell!} \frac{\partial^\ell v}{\partial x^\ell}(x,s)\mathrm{d}s + f(v(x,t)), \quad x \in \mathbb{R}, \tag{30}$$

where $< \Delta^\ell >= \int_{\mathbb{R}} \Delta^\ell \phi(\Delta)\mathrm{d}\Delta$. Then for isotropic kernels we get

$$\frac{\partial v}{\partial t}(x,t) = \int_0^t \alpha(t-s) \sum_{\ell=1}^{\infty} \frac{\langle \Delta^{2\ell} \rangle}{(2\ell)!} \frac{\partial^{2\ell} v}{\partial x^{2\ell}}(x,s)\mathrm{d}s + f(v(x,t)), \quad x \in \mathbb{R}. \tag{31}$$

For each t let $\overline{x}(t)$ be the point where $\partial v/\partial x$ attains it maximum and the partial derivatives $\partial^{2\ell} v/\partial x^{2\ell}$ are null, which means that the travel wave v presents an inflection point $\overline{x}$. We have

$$\frac{\partial v}{\partial t}(\overline{x}(t),t) = f(v(\overline{x}(t),t)). \tag{32}$$

Considering now Lagrangian coordinates moving with the speed $\overline{V}$ of the front that is (z,t) with $z = x - \overline{V}t$ we deduce that

$$\frac{\partial \overline{v}}{\partial t}(\overline{x}(t),t) = -\overline{V}\frac{\partial \overline{v}}{\partial z}(\overline{x}(t),t) \tag{33}$$

with $\partial\overline{v}/\partial z(\overline{x}(t),t) = (\partial v/\partial z)(\overline{x}(t) - \overline{V}t, t)$.

From (32) and (33) we conclude that

$$\frac{\partial \overline{v}}{\partial z}(\overline{x}(t),t) = -\frac{f(\overline{v}(\overline{x}(t),t))}{\overline{V}}.$$

As $(\partial\overline{v}/\partial z)(\overline{x}(t),t)$ measures the steepness of the front we can define its width $\overline{W}(t)$ as in [13], by the module of the inverse of the steepness that is

$$\overline{W}(t) = \frac{\overline{V}}{|f(\overline{v}(\overline{x}(t),t))|}.$$

If $f(\overline{v}) = U\overline{v}(1-\overline{v})$ we can explicitly compute $\overline{W}(t)$ obtaining

$$\overline{W}(t) = \frac{\overline{V}}{U\overline{v}(\overline{x}(t),t))\big(1-\overline{v}(\overline{x}(t),t)\big)}.$$

As the gradient $\partial v/\partial x$ attains its maximum for $x = \overline{x}(t)$ and $\partial v/\partial x = \partial v/\partial z$ then $\partial v/\partial z$ attains a maximum at $\overline{z} = \overline{x}(t) - \overline{V}t$. Considering that

$$\frac{\partial^2 \overline{v}}{\partial z^2}(\overline{x}(t),t) = -\frac{f'(\overline{v}(\overline{x}(t),t))}{V}\frac{\partial \overline{v}}{\partial z}(\overline{x}(t),t),$$

we have $f'((\overline{v}(\overline{x}(t),t))=0$. In the case of the logistic reaction $f'(\overline{v}(\overline{x}(t),t))=0$ for $\overline{v}(\overline{x}(t),t)=1/2$ and consequently $f(\overline{v}(\overline{x}(t),t))=U/4$. The width $\overline{W}$ of the front can finally be represented by

$$\overline{W}=\frac{4\overline{V}}{U}.$$

5. Discrete models of non-Brownian type

In this section, we study numerical methods for equation (6).

Let us consider (6) with $t\in(0,T]$ where we define the grid $\{t_j, j=0,\ldots,M\}$ with $t_0=0,\ t_{j+1}-t_j=\Delta t,$ for $j=0,\ldots,M-1$. In $\mathbb{R}$ we introduce the uniform grid $\mathbb{R}_h=\{x_i, i\in\mathbb{Z}\}$, with $x_0=0,\ x_i=ih,\ i\in\mathbb{Z}$. By $v_{i\hat{}j}$ we denote an approximation to $v(x_i,t_j)$ defined by

$$v_i^{n+1}=v_i^n+\Delta t^2\lambda^2\sum_{j=1}^{n+1}e^{-2\lambda(t_{n+1}-t_j)}\left(h\sum_{k\in\mathbb{Z}}v_{i+k}^j\phi(zh)-v_i^j\right)+f(v_i^{n+1}),\ i\in\mathbb{Z}, \tag{34}$$

with initial condition

$$v_i^0=v_0(x_i),\quad i\in\mathbb{Z}. \tag{35}$$

We study in what follows the behavior of the grid function v_h^n, defined in the grid $\mathbb{R}_h$, with respect to the norm

$$\|w_h\|^2_{L^2(\mathbb{R}_h)}=h\sum_{i\in\mathbb{Z}}v_h(x_i)^2 \tag{36}$$

induced by the inner product

$$(u_h,w_h)_h=h\sum_{i\in\mathbb{Z}}v_h(x_i)w_h(x_i), \tag{37}$$

where u_h and w_h are grid functions taking values in $\mathbb{R}_h$.

Theorem 6 *Let ϕ be a probability density function of jumps length defined by* (24) *or* (25). *If the source function f satisfies* (8) *then the solution of* (34) *satisfies*

$$\left\|v_h^{n+1}\right\|^2_{L^2(\mathbb{R}_h)}+\lambda^2\left\|\Delta t\sum_{j=1}^{n+1}e^{-2\lambda(t_{n+1}-t_j)}v_h^j\right\|^2_{L^2(\mathbb{R}_h)}\le C_I^n(1+\Delta t^2\lambda^2)C_0\left\|v_h^0\right\|^2_{L^2(\mathbb{R}_h)} \tag{38}$$

with

$$\begin{aligned}C_I&=\frac{1}{\min\{1-\Delta t(2M_{f'}+\lambda^2);1-\Delta t\}},\\ C_0&=\frac{1}{1-2\Delta tM_{f'}},\end{aligned} \tag{39}$$

and provided that

$$1-\Delta t>0\,,1-\Delta t(2M_{f'}+\lambda^2)>0\,,1-2\Delta tM_{f'}>0. \tag{40}$$

Proof

(1) Let us assume first that $n \geq 1$.
Multiplying (34) by v_h and considering the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}\|v_h^{n+1}\|^2_{L^2(\mathbb{R}_h)} \leq \frac{1}{2}\|v_h^{n+1}\|^2_{L^2(\mathbb{R}_h)}+\frac{1}{2}\|v_h^{n}\|^2_{L^2(\mathbb{R}_h)}+h\Delta t\sum_{i\in\mathbb{Z}}f(v_i^{n+1})v_i^{n+1}\\ +h^2\Delta t^2\lambda^2\sum_{i,k\in\mathbb{Z}}\sum_{j=1}^{n+1}e^{-2\lambda(t_{n+1}-t_j)}v_{i+k}^j\phi(kh)v_i^{n+1}\\ -h\Delta t^2\lambda^2\sum_{i\in\mathbb{Z}}\sum_{j=1}^{n+1}e^{-2\lambda(t_{n+1}-t_j)}v_i^jv_i^{n+1}.\end{aligned} \tag{41}$$

As the source function f satisfies (8) we easily deduce

$$h\sum_{i\in\mathbb{Z}}f(v_i^{n+1})v_i^{n+1} \leq M_{f'}\|v_h^{n+1}\|^2_{L^2(\mathbb{R}_h)}. \tag{42}$$

Let Q_3 and Q_4 be defined respectively by

$$Q_3=h^2\Delta t^2\sum_{i,k\in\mathbb{Z}}\sum_{j=1}^{n+1}e^{-2\lambda(t_{n+1}-t_j)}v_{i+k}^j\phi(kh)v_i^{n+1},$$

$$Q_4=h\Delta t^2\sum_{i\in\mathbb{Z}}\sum_{j=1}^{n+1}e^{-2\lambda(t_{n+1}-t_j)}v_i^jv_i^{n+1}.$$

As far as Q_3 is concerned we have

$$\begin{aligned}Q_3&=h^2\Delta t^2\sum_{i,m\in\mathbb{Z}}\sum_{j=1}^{n+1}e^{-2\lambda(t_{n+1}-t_j)}v_m^j\phi((m-i)h)v_i^{n+1}\\ &=h^2\Delta t^2\sum_{m\in\mathbb{Z}}\sum_{j=1}^{n+1}e^{-2\lambda(t_{n+1}-t_j)}v_m^j\phi((m-i)h)v_i^{n+1}\\ &\leq\frac{\Delta t}{2}h\sum_{m\in\mathbb{Z}}\left(\Delta t\sum_{j=1}^{n+1}e^{-2\lambda(t_{n+1}-t_j)}v_m^j\right)^2h\sum_{i\in\mathbb{Z}}\phi((m-i)h)\\ &\quad+\frac{\Delta t}{2}h\sum_{i\in\mathbb{Z}}(v_i^{n+1})^2h\sum_{m\in\mathbb{Z}}\phi((m-i)h).\end{aligned} \tag{43}$$

Attending that ϕ represents the Gauss density function (24) or Laplace density function (25), we conclude that

$$Q_3 \leq \frac{\Delta t}{2}\left\|\Delta t\sum_{j=1}^{n+1} e^{-2\lambda(t_{n+1}-t_j)}v_h^j\right\|^2_{L^2(\mathbb{R}_h)} + \frac{\Delta t}{2}\|v_h^{n+1}\|^2_{L^2(\mathbb{R}_h)}. \tag{44}$$

For Q_4 holds the following representation

$$Q_4 = \frac{1}{2}\left\|\Delta t\sum_{j=1}^{n+1} e^{-2\lambda(t_{n+1}-t_j)}v_h^j\right\|^2_{L^2(\mathbb{R})} - \frac{e^{-4\lambda\Delta t}}{2}\left\|\Delta t\sum_{j=1}^{n} e^{-2\lambda(t_n-t_j)}v_h^j\right\|^2_{L^2(\mathbb{R})} + \frac{\Delta t^2}{2}\|v_h^{n+1}\|^2_{L^2(\mathbb{R})}. \tag{45}$$

Considering (42)–(45) in (41) we obtain

$$\begin{aligned}\left(1+\Delta t^2\lambda^2 - \Delta t\left(2M_{f'}+\lambda^2\right)\right)\|v_h^{n+1}\|^2_{L^2(\mathbb{R}_h)} + \lambda^2(1-\Delta t)\left\|\Delta t\sum_{j=1}^{n+1} e^{-2\lambda(t_{n+1}-t_j)}v_h^j\right\|^2_{L^2(\mathbb{R}_h)}\\ \leq \lambda^2 e^{-4\lambda\Delta t}\left\|\Delta t\sum_{j=1}^{n} e^{-2\lambda(t_n-t_j)}v_h^j\right\|^2_{L^2(\mathbb{R}_h)} + \|v_h^n\|^2_{L^2(\mathbb{R}_h)}.\end{aligned} \tag{46}$$

which implies

$$\begin{aligned}\min\{1-\Delta t(2M_{f'}+\lambda^2)\,;1-\Delta t\}\left(\|v_h^{n+1}\|^2_{L^2(\mathbb{R}_h)} + \lambda^2\left\|\Delta t\sum_{j=1}^{n+1} e^{-2\lambda(t_{n+1}-t_j)}v_h^j\right\|^2_{L^2(\mathbb{R}_h)}\right)\\ \leq \lambda^2\left\|\Delta t\sum_{j=1}^{n} e^{-2\lambda(t_n-t_j)}v_h^j\right\|^2_{L^2(\mathbb{R}_h)} + \|v_h^n\|^2_{L^2(\mathbb{R}_h)}.\end{aligned} \tag{47}$$

Then choosing Δt such that (40) holds we obtain, for $n \geq 1$,

$$\begin{aligned}\|v_h^{n+1}\|^2_{L^2(\mathbb{R}_h)} + \lambda^2\left\|\Delta t\sum_{j=1}^{n+1} e^{-2\lambda(t_{n+1}-t_j)}v_h^j\right\|^2_{L^2(\mathbb{R}_h)}\\ \leq C_I\left(\lambda^2\left\|\Delta t\sum_{j=1}^{n} e^{-2\lambda(t_n-t_j)}v_h^j\right\|^2_{L^2(\mathbb{R}_h)} + \|v_h^n\|^2_{L^2(\mathbb{R}_h)}\right).\end{aligned} \tag{48}$$

(2) We consider now $n = 0$. It is easy to establish in this case that

$$(1-2\Delta t M_{f'})\|v_h^1\|^2_{L^2(\mathbb{R}_h)} \leq \|v_h^0\|^2_{L^2(\mathbb{R}_h)} + 2\Delta t^2\lambda^2\left(h^2\sum_{i,k\in\mathbb{Z}} v_{i+k}^1 v_i^1\phi(kh) - \|v_h^1\|^2_{L^2(\mathbb{R}_h)}\right).$$

As we have

$$h^2 \sum_{i,k\in\mathbb{Z}} v^1_{i+k} v^1_i \phi(kh) = h^2 \sum_{i,m\in\mathbb{Z}} v^1_m v^1_i \phi((m-i)h)$$
$$\leq \frac{h}{2}\sum_{m\in\mathbb{Z}} (v^1_m)^2 h \sum_{i\in\mathbb{Z}} \phi((m-i)h)$$
$$+\frac{h}{2}\sum_{i\in\mathbb{Z}} (v^1_i)^2 h \sum_{m\in\mathbb{Z}} \phi((m-i)h)$$
$$\leq \left\| v^1_h \right\|^2_{L^2(\mathbb{R}_h)}$$

we conclude that

$$\left\| v^1_h \right\|^2_{L^2(\mathbb{R}_h)} \leq C_0 \left\| v^0_h \right\|^2_{L^2(\mathbb{R}_h)} \tag{49}$$

with C_0 given by (39), provided that Δt satisfies $1 - 2\Delta t M_{f'} > 0$.
Finally from (48) and (49) we obtain (38). ■

Remark 3 In order to establish stability bounds for Δt_0 the coefficients C_I and C_0 in (38) can be analyzed with some detail.

(1) Let

$$2M_{f'} + \lambda^2 > 1, \tag{50}$$

which implies that

$$C_I = \frac{1}{1 - \Delta t(2M_{f'} + \lambda^2)}.$$

As $\lambda = 2/\tau$ where τ stands for the mean time between successive jumps, inequality (50) is verified by a large class of source function because λ^2 can be very large. Let Δt_0 be fixed such that

$$1 - \Delta t_0(2M_{f'} + \lambda^2) > 0, \tag{51}$$

and

$$1 - 2\Delta t_0 M_{f'} > 0. \tag{52}$$

If $M_{f'} < 0$ then (52) holds and Δt_0 is fixed only by (51). In this case $M_{f'} > 0$, and Δt_0 is defined by

$$\Delta t_0 = \frac{1}{\max\{2M_{f'}; 2M_{f'} + \lambda^2\}}.$$

As for $\Delta t \leq \Delta t_0$ we have

$$C_I \leq 1 + \Delta t \frac{2M_{f'} + \lambda^2}{1 - \Delta t_0(2M_{f'} + \lambda^2)},$$

we conclude from (38)

$$\begin{aligned} &\left\| v_h^{n+1} \right\| + \lambda^2 \left\| \Delta t \sum_{j=1}^{n+1} e^{-2\lambda(t_{n+1}-t_j)} v_h^j \right\|^2_{L^2(\mathbb{R}_h)} \\ &\quad \leq e^{n\Delta t(2M_{f'}+\lambda^2)/(1-\Delta t_0(2M_{f'}+\lambda^2))} \frac{1+\lambda^2 \Delta t_0^2}{1-2\Delta t_0 M_{f'}} \left\| v_h^0 \right\|^2_{L^2(\mathbb{R}_h)}. \end{aligned} \tag{53}$$

(2) If λ and $M_{f'}$ do not satisfy (50) then

$$C_I = \frac{1}{1-\Delta t}.$$

Let Δt_0 be such that

$$\Delta t_0 = \frac{1}{\max\{1, 2M_{f'}\}}. \tag{54}$$

Then for $\Delta t \leq \Delta t_0,$ we conclude from (38)

$$\left\| v_h^{n+1} \right\| + \lambda^2 \left\| \Delta t \sum_{j=1}^{n+1} e^{-2\lambda(t_{n+1}-t_j)} v_h^j \right\|^2_{L^2(\mathbb{R}_h)} \leq e^{n\Delta t(1/1-\Delta t_0)} \frac{1+\lambda^2 \Delta t_0^2}{1-2\Delta t_0 M_{f'}} \left\| v_h^0 \right\|^2_{L^2(\mathbb{R}_h)}. \tag{55}$$

Inequalities (53), (55) enable us to conclude the stability of method (34). ■

If the reaction is stiff the implicit discretization (34) should be used. For non-stiff reactions the implicit–explicit discretization

$$v_i^{n+1} = v_i^n + \Delta t^2 \lambda^2 \sum_{j=1}^{n+1} e^{-2\lambda(t_{n+1}-t_j)} \left(h \sum_{k \in \mathbb{Z}} v_{i+k}^j \phi(zh) - v_i^j \right) + f(v_i^n),\ i \in \mathbb{Z}, \tag{56}$$

can be used. We establish in what follows a stability result for method (56).

If source function f satisfies (8) then

$$h \sum_{i \in \mathbb{Z}} f(v_i^n) v_i^{n+1} \leq \frac{M_{f'}^2}{2} \left\| v_h^n \right\|^2_{L^2(\mathbb{R}_h)} + \frac{1}{2} \left\| v_h^{n+1} \right\|^2_{L^2(\mathbb{R}_h)} \tag{57}$$

but also

$$h\sum_{i\in\mathbb{Z}} f(v_i^n)v_i^{n+1} \le \frac{1}{2}\left\|v_h^n\right\|_{L^2(\mathbb{R}_h)}^2 + \frac{M_{f'}^2}{2}\left\|v_h^{n+1}\right\|_{L^2(\mathbb{R}_h)}^2. \tag{58}$$

Inequalities (57) and (58) enable us to conclude that for $n \ge 1$, (48) holds with C_I replaced now by C_E defined by

$$C_E = \frac{\max\{1+\Delta t; 1+\Delta t M_{f'}^2\}}{\min\{1-\Delta t(1+\lambda^2); 1-\Delta t(M_{f'}^2+\lambda^2)\}}, \tag{59}$$

provided that

$$1-\Delta t(1+\lambda^2) > 0, 1-\Delta t(M_{f'}^2+\lambda^2) > 0. \tag{60}$$

As

$$\begin{aligned}\left\|v_h^1\right\|_{L^2(\mathbb{R}_h)}^2 &\le \frac{1+\Delta t M_{f'}^2}{1-\Delta t}\left\|v_h^0\right\|_{L^2(\mathbb{R}_h)}^2,\\ \left\|v_h^1\right\|_{L^2(\mathbb{R}_h)}^2 &\le \frac{1+\Delta t}{1-\Delta t M_{f'}^2}\left\|v_h^0\right\|_{L^2(\mathbb{R}_h)}^2,\end{aligned} \tag{61}$$

we deduce that

$$\left\|v_h^1\right\|_{L^2(\mathbb{R}_h)}^2 \le C_0\left\|v_h^0\right\|_{L^2(\mathbb{R}_h)}^2 \tag{62}$$

where

$$C_0 = \frac{\max\{1+\Delta t M_{f'}; 1+\Delta t\}}{\min\{1-\Delta t; 1-\Delta t M_{f'}^2\}}. \tag{63}$$

Estimate (62) is analogous to estimate (49) established for implicit method (34).

Following the proof of Theorem 37 we conclude next stability result:

THEOREM 7 *Let ϕ be a probability density function of jumps length defined by* (24) *or* (25). *If the source function f satisfies* (8) *then the numerical solution defined by* (56) *satisfies*

$$\left\|v_h^{n+1}\right\| + \lambda^2\left\|\Delta t\sum_{j=1}^{n+1} e^{-2\lambda(t_{n+1}-t_j)}v_h^j\right\|_{L^2(\mathbb{R}_h)}^2 \le C_E^n(1+\lambda^2\Delta t^2)C_0\left\|v_h^0\right\|_{L^2(\mathbb{R}_h)}^2 \tag{64}$$

provided that (60) *holds and with C_E and C_0 defined by* (59), (63) *respectively*.

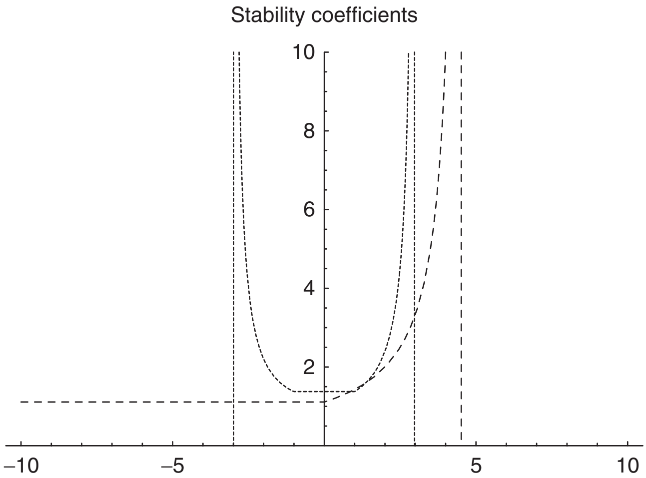

Figure 1. Stability coefficients $C_E(\ldots)$ and $C_I(---)$ for $\lambda = 1,\ \Delta t = 0.1$.

If Δt_0 is a fixed value such that

$$1 - \Delta t_0(1 + \lambda^2) > 0,\ 1 - \Delta t_0(M_{f'}^2 + \lambda^2) > 0, \tag{65}$$

then for $\Delta t < \Delta t_0$ we easily obtain

$$\begin{aligned} &\| v_h^{n+1} \|_{L^2(\mathbb{R}_h)}^2 + \lambda^2 \left\| \Delta t \sum_{j=1}^{n+1} e^{-2\lambda(t_{n+1}-t_j)} v_h^j \right\|_{L^2(\mathbb{R}_h)}^2 \\ &\quad \le e^{n\Delta t(1+\lambda^2+M_{f'}^2/1-\Delta t_0(1+\lambda^2))} C_0(1 + \lambda^2 \Delta t_0^2) \| v_h^0 \|_{L^2(\mathbb{R}_h)}^2. \end{aligned} \tag{66}$$

Estimate (66) guarantees stability for method (56).

In figure 1 we plot the stability coefficients C_I and C_E as functions of $M_{f'}$ for $\lambda = 1$, $\Delta t = 0.1$.

6. Numerical examples

The purpose of this section is two-fold: firstly to illustrate the stability behavior of implicit method (34) and implicit–explicit method (56) and secondly to analyze the dependence on f, ϕ and λ of the speed propagation and the steepness of the front.

The computational results have been obtained with a reaction term of type $f(v) = U(1 - v)v$, with probability density functions ϕ defined by (24) and (25), and an initial condition v_0 given by

$$v_0(x) = \begin{cases} 1, & x \le 50, \\ 0, & x > 50. \end{cases}$$

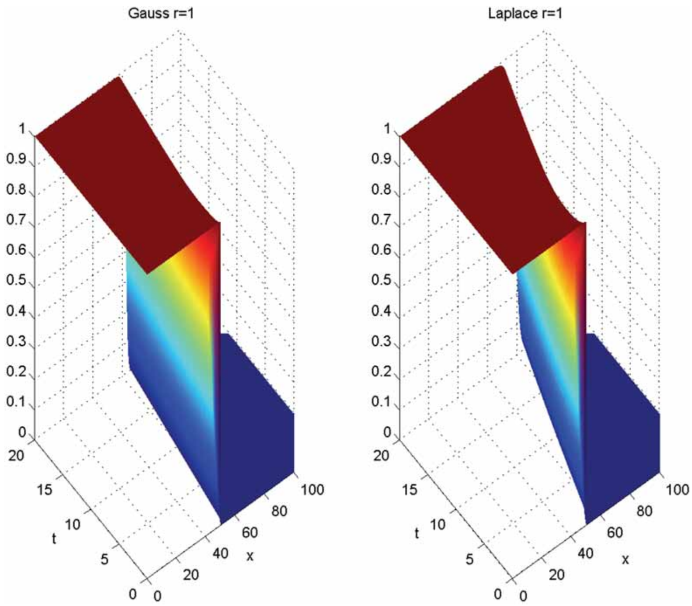

Figure 2. Numerical solutions computed with method (56) for $U = \lambda = r = 1$ and $h = \Delta t = 0.1$.

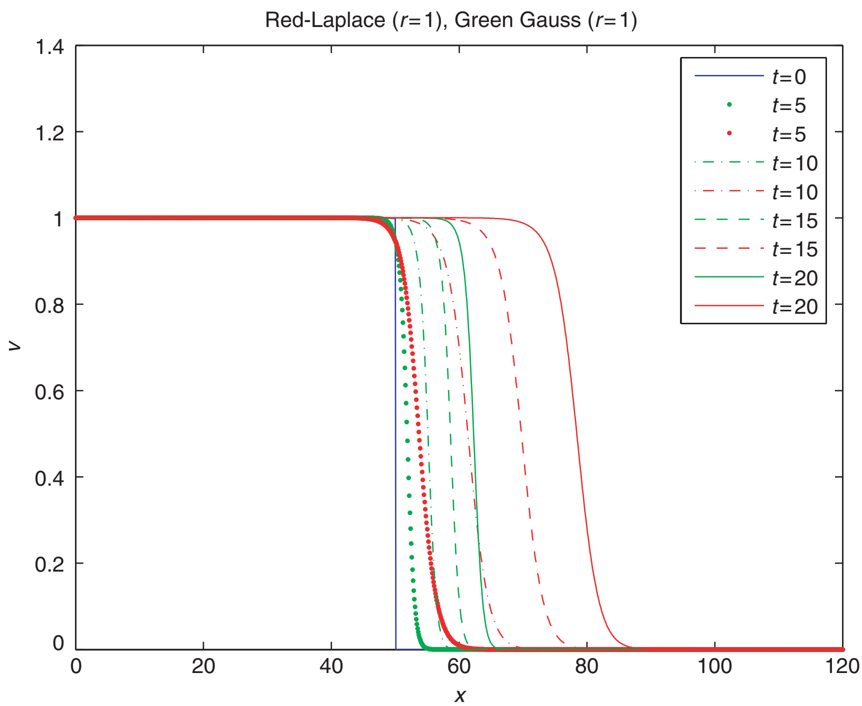

Figure 3. Numerical solutions computed with method (56) for $U = \lambda = r = 1$ and $h = \Delta t = 0.1$.

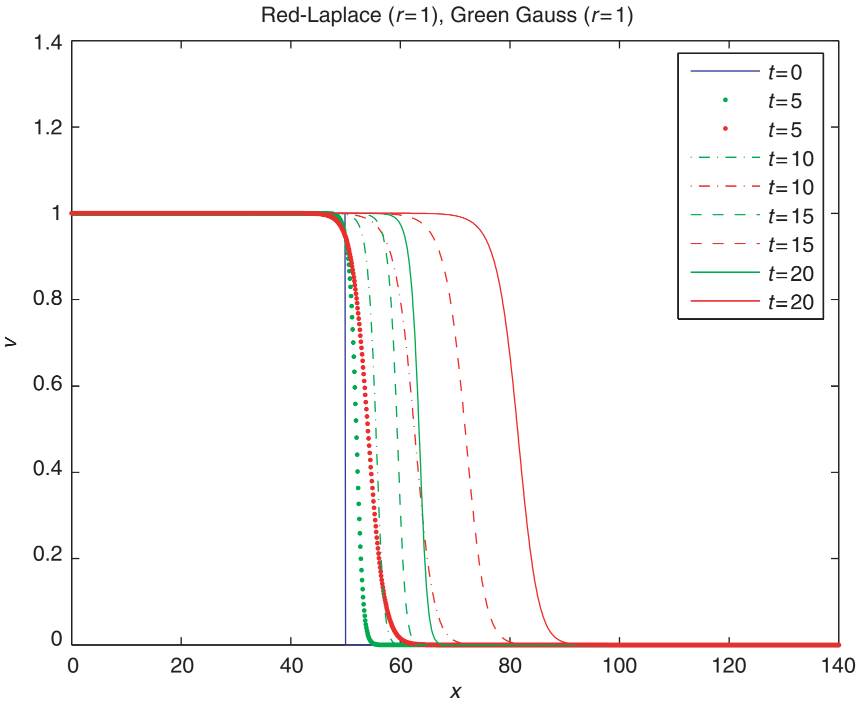

Figure 4. Numerical solutions computed with method (34) for $U = r = 1, h = \Delta t = 0.1$ and $\lambda = 1$.

Considering $h = \Delta t = 0.1$ both methods present a stable behavior for $U = 1$ (figures 2–4). This result was expected because both methods (34) and (56) are stable as a consequence of (40) and (65). For $U = 6$ implicit-explicit method (56) exhibits an unstable behavior as was expected from the fact that (65) does not holds (figures 5 and 6).

Let us consider in what follows $U = 1$ and again $h = \Delta t = 0.1$. We illustrate now the influence of space and time memory on the speed of propagation and on the steepness of the front. As for $U = 1$, $h = \Delta t = 0.1$ both methods give analogous solutions we just exhibit in what follows the results obtained with the less computationally expensive method (56). In figures 2 and 3 can be observed the influence of Gauss and Laplace probability density functions in the speed of propagation. Laplace probability density function induces a greater speed of propagation and leads to a smoother solution.

In figure 7 we show the numerical solution computed using Laplace and Gauss kernels but using now a parameter $r = 0.5$. As expected from an intuitive point of view if we decrease r the speed of the front decreases and its steepness increases.

Finally in figure 8 we illustrate the influence of the waiting time τ between two successive jumps with $\lambda = 10$, $U = 1, h = 0.1$ and $\Delta t = 0.01$. As $\lambda = 2/\tau$ it is expected that as λ increases the speed increases and the steepness decreases.

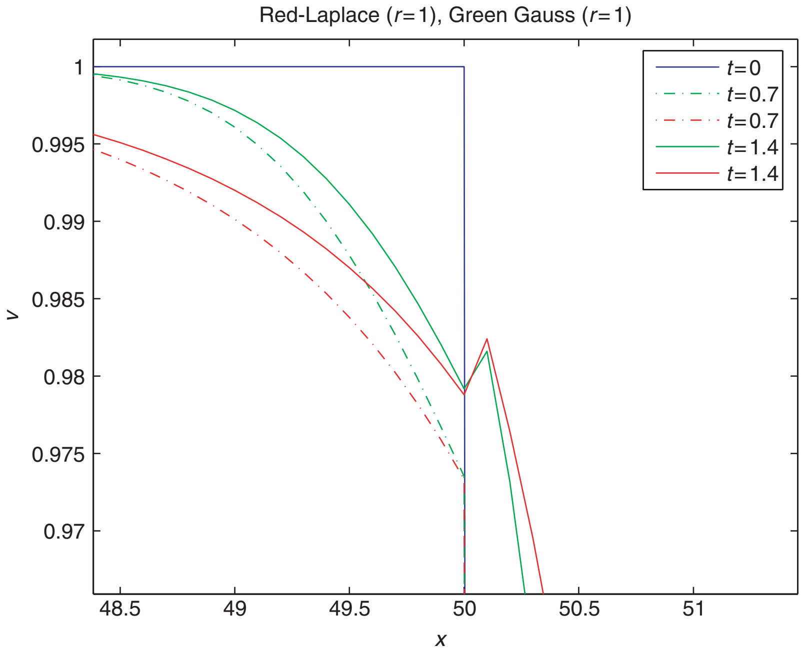

Figure 5. Numerical solutions computed with method (56) for $h = \Delta t = 0.1$, $r = \lambda = 1$ and $U = 6$.

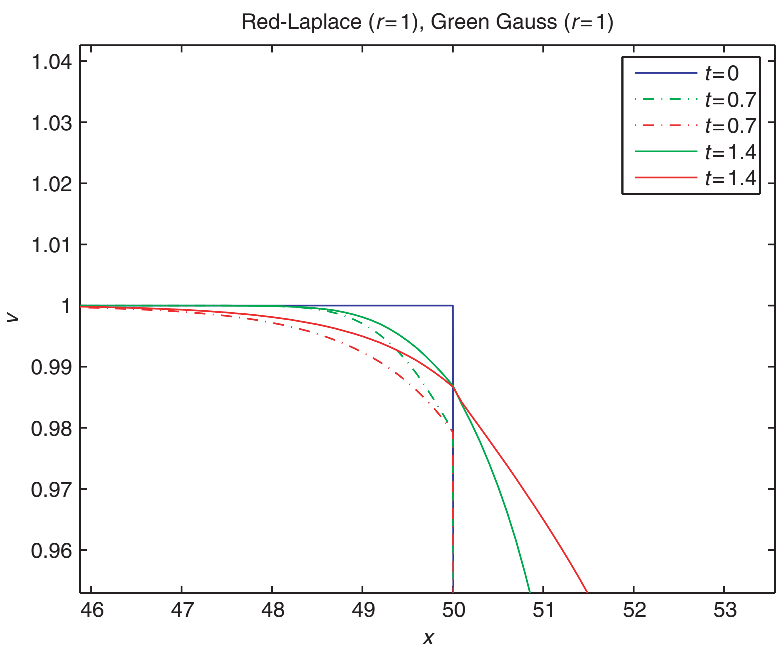

Figure 6. Numerical solutions computed with method (34) for $h = \Delta t = 0.1$, $r = \lambda = 1$ and $U = 6$.

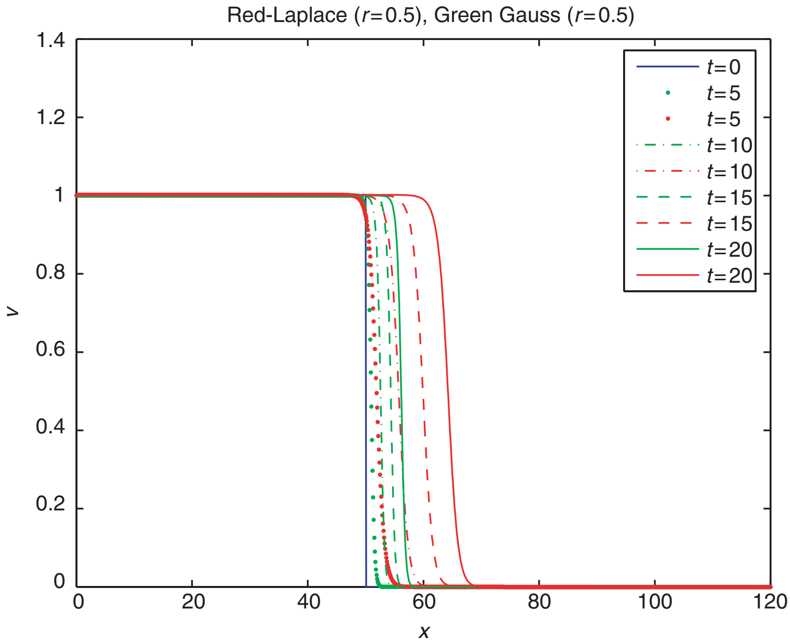

Figure 7. Numerical solutions computed with method (56) for $U = \lambda = 1, h = \Delta t = 0.1$ and $r = 0.5$.

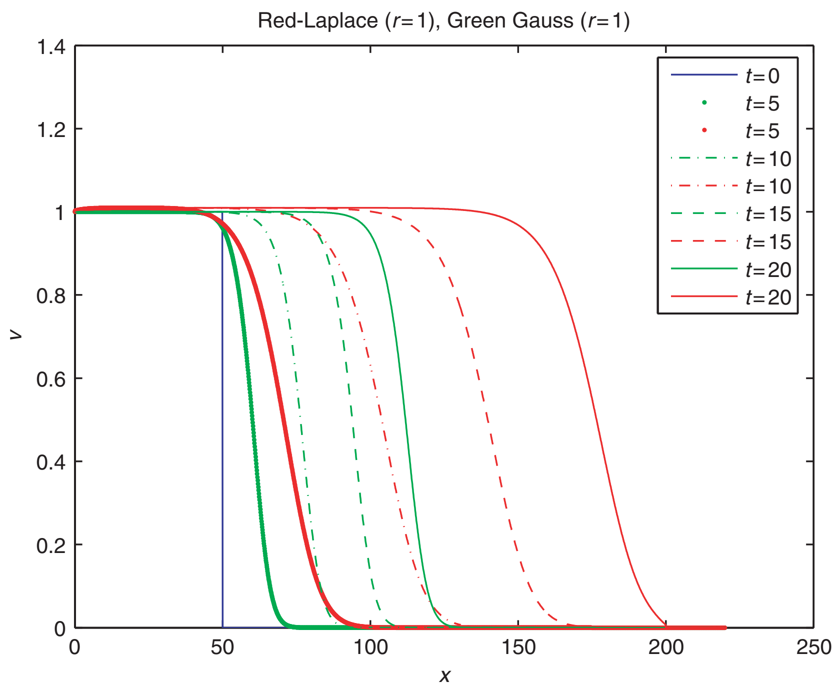

Figure 8. Numerical solutions computed with method (56) for $U = r = 1, h = 0.1,\ \Delta t = 0.01$ and $\lambda = 10$.

Acknowldgement

This work was supported by Centro de Matemática da Universidade de Coimbra.

References

[1] Araújo, A., Ferreira, J.A. and Oliveira, P., 2005, Qualitative behaviour of numerical traveling waves solutions for reaction diffusion equations with memory. *Applicable Analysis*, **84**, 1231–1246.
[2] Araújo, A., Ferreira, J.A. and Oliveira, P., 2006, The effect of memory terms in the qualitative behaviour of the solution of diffusion equations. *Journal of Computational Mathematics*, **24**, 91–102.
[3] Aronson, D.G. and Weinberger, H.F., 1978, Multidimensional nonlinear diffusion in population genetics. *Advanced Mathematics*, **30**, 33–76.
[4] Branco, J.R., Ferreira, J.A. and Oliveira, P., Numerical methods for generalized Fisher–Kolmogorov–Petrovskii–Piskunov equation. *Applied Numerical Mathematics* (To appear).
[5] Debnath, L., 1997, *Nonlinear Partial Differential Equations for Scientists and Engineers* (Boston: Birkhuser).
[6] Fedotov, S., 1998, Traveling waves in a reaction – diffusion system: diffusion with finite velocity and Kolmogorov–Petrovski–Piskunov kinectics. *Physical Review E*, **5**(4), 5143–5145
[7] Fedotov, S., 1999, Nonuniform reaction rate distribution for the generalized Fisher equation: ignition ahead of the reaction front. *Physical Review E*, **60**(4), 4958–4961
[8] Fedotov, S., 2001, Front propagation into an unstable state of reaction – transport systems. *Physical Review Letter*, **86**(5), 926–929
[9] Fedotov, S. and Okuda, Y., 2002, Non Markovian random process and traveling front in a reaction transport system with memory and long-range interactions. *Physical Review E*, **66**, 021113–021119
[10] Fedotov, S. and Okuda, Y., 2004, Waves in a reaction-transport system with memory, long-range interactions and transmutations. *Physical Review E*, **70**, 051108–051117.
[11] Fisher, R.A., 1937, The wave of advance of advantageous genes. *Annals of Eugenics*, **7**, 353–369.
[12] Kolmogorov, A., Petrovskii, I. and Piskunov, N., 1937, Étude de l'equation de la diffusion avec croissance de la matière et son application à un problème biologique. Moscow University, *Bulletien Mathematics*, **1**, 1–25.
[13] Mendez, V., Pujol, T. and Fort, J., 2002, Dispersal probability distributions and the wave-front speed problem. *Physical Review E*, **65**, 041109–041114.
[14] Zauderer, E., 1983, *Partial Differential Equations of Applied Mathematics* (New York: John Wiley Sons).