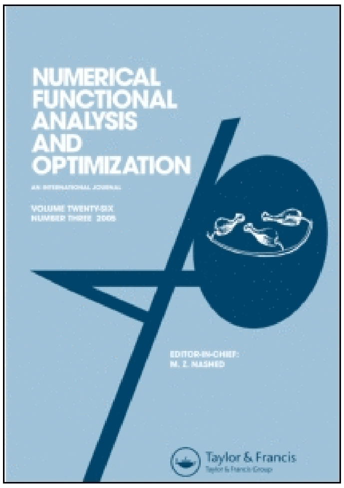

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SUPRACONVERGENCE AND SUPERCLOSENESS OF A SCHEME FOR ELLIPTIC EQUATIONS ON NONUNIFORM GRIDS

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□ *In this paper, we study the convergence of a finite difference scheme on nonuniform grids for the solution of second-order elliptic equations with mixed derivatives and variable coefficients in polygonal domains subjected to Dirichlet boundary conditions. We show that the scheme is equivalent to a fully discrete linear finite element approximation with quadrature. It exhibits the phenomenon of supraconvergence, more precisely, for $s \in [1, 2]$ order $O(h^s)$-convergence of the finite difference solution, and its gradient is shown if the exact solution is in the Sobolev space $H^{1+s}(\Omega)$. In the case of an equation with mixed derivatives in a domain containing oblique boundary sections, the convergence order is reduced to $O(h^{3/2-\epsilon})$ with $\epsilon > 0$ if $u \in H^3(\Omega)$. The second-order accuracy of the finite difference gradient is in the finite element context nothing else than the superconeness of the gradient. For $s \in \{1, 2\}$, the given error estimates are strictly local.*

Keywords Finite difference scheme; Finite element method; Nonuniform grids; Stability; Supercloseness of gradient; Supraconvergence.

AMS Subject Classification 65N06; 65N30; 65N12.

1. INTRODUCTION

We consider the discretization of the differential equation

$$Au := -(au_x)_x - (bu_x)_y - (bu_y)_x - (cu_y)_y + (du)_x + (eu)_y + fu = g \quad \text{in } \Omega \subset \mathbb{R}^2 \tag{1.1}$$

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subject to Dirichlet boundary conditions

$$u = \psi \quad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a simple polygonal domain. The discretization is obtained by a standard finite difference method (FDM) on a nonuniform rectangular grid $\overline{\Omega}_H$ subdividing Ω. The resulting discrete problem is equivalent to a fully discrete linear finite element method (FEM) defined on the triangulation of Ω generated by $\overline{\Omega}_H$.

Our aim is to study the behavior of the scheme for a sequence of grids $\overline{\Omega}_H$, $H \in \Lambda$, with maximal mesh-size $H_{\max}$ converging to zero without any restriction on the nonuniformity of $\overline{\Omega}_H$. In this case, the FDM scheme is only first order consistent but we will show that nevertheless the approximate solutions u_H, $H \in \Lambda$, and their (discrete) gradients are more accurate. This property is usually called supraconvergence and was considered, without being exhaustive, in [3–7, 10, 16, 21, 23, 24]. Finite difference methods on nonuniform meshes for the Laplacian in a square with solutions $u \in H^{1+s}(\Omega)$ are considered in [29] for $s = 2$ and in [2] and [14] for $s \in [1, 2]$. The idea in these papers is, as in [4], to add a correction to the standard finite difference scheme on uniform grids that makes the scheme second-order accurate also on nonuniform meshes. A result of the current paper is that no correction is needed to prove the same convergence order as on uniform meshes. Supraconvergence results have been obtained by the authors in [3] for general second-order elliptic equations in polygonal domains subjected to Dirichlet boundary conditions assuming that $u \in C^4(\overline{\Omega})$. In the one-dimensional case for general boundary condition, it is proved in [1] that for $s \in (1/2, 2]$, the approximations and its gradients exhibit an error of optimal order $O(h^s)$ provided u is in the Sobolev space $H^{1+s}(\Omega)$.

Our main result is Theorem 6.1 and its corollaries in Section 6. For domains having no oblique boundary sections, the H^1-norm of $P_H(R_H u - u_H)$, the linearly interpolated error $R_H u - u_H$ on the grid, is of order $O(H_{\max}^s)$ provided $u \in H^{1+s}(\Omega), s \in [1, 2]$. This convergence order holds also true for differential operators containing no mixed derivatives in general polygonal domains, while otherwise the convergence order for $s \in (1, 2]$ is reduced to $O(H_{\max}^{(s+1)/2-\epsilon})$ with $\epsilon > 0$ arbitrarily small. The error estimates we prove in the case $s \in \{1, 2\}$ are strictly local, which is desirable when working with nonuniform grids.

The fully discrete piecewise linear finite element approximation is constructed by associating with the rectangular grid $\overline{\Omega}_H$ a triangulation $\mathcal{T}_H$ of the domain and applying a special quadrature formula to the corresponding linear FEM. The larger than first order convergence of the gradient of $P_H(R_H u - u_H)$ we prove is in this context called the supercloseness of the gradient (see [30, p. 80]). Several recovery techniques for the gradient are based on the supercloseness property (see [8, 9, 17, 18, 22, 26, 31, 32] and the bibliography [19]). In the supercloseness results involved in these papers, the meshes are either completely uniform or a smooth transformation of a uniform mesh, whereas we work on nonuniform meshes. We want to point out the significant difference in the behavior of the scheme on uniform and nonuniforms grids, which can be well seen from the finite difference presentation: whereas on the former grids the truncation error is second-order and smoothly varying from grid point to grid point, it is first order and strongly oscillating on the latter. In [22], the finite element scheme considered is also fully discrete. It is obtained with the aid of a second-order accurate quadrature formula, whereas our quadrature formulas are only of first order.

An advantage of the relation between the FDM and the FEM is that it allows one to technically simplify the analysis of the former. In this way, we can work with the usual norms in Sobolev spaces in place of the not so comfortable discrete norms for grid functions.

Of course, it has always been known that the linear finite element approximation can be written as a finite difference scheme, especially for the Laplacian. (But our specially tuned FEM which is equivalent to the standard FDM in (3.1) seems to be new.) So it appears natural, as our results show, that the H^1 error estimates obtained for the FDM are closely related to supercloseness of the FEM. But the literature gives the impression that there exist the two separated communities of the FEM and FDM people (see [12, 13, 15, 28] and the overview in [11] for the latter) and the relation of those results has not been considered in that respect.

The paper is organized as follows. In Section 2, we present the variational formulation of the boundary value problem (1.1), (1.2) and the fully discrete nonstandard piecewise linear FEM. In Section 3, the corresponding finite difference scheme is introduced. A main ingredient in the convergence analysis is the stability of the scheme in Section 4. In Section 5, the essential estimate of the truncation error is given from which the main results are derived through a series of lemmas in Section 6.

2. A FULLY DISCRETE GALERKIN APPROXIMATION

It is convenient to start with the familiar Galerkin formulation of our boundary value problem and its discretization by linear finite elements with quadrature. In the next section, it will be shown that the method is equivalent to the standard FDM (3.1) for solving (1.1), (1.2).

We will work with the usual Sobolev spaces $W_p^r(\Omega)$ for $r \in \mathbb{N} \cup \{0\}$ and $p \in [2, \infty]$ with semi-norms and norms, respectively, given by

$$|v|_{W_p^r(\Omega)} = \left(\sum_{|\alpha|=r} \|D^\alpha v\|_{L_p(\Omega)}^p \right)^{1/p}, \quad \|v\|_{W_p^r(\Omega)} = \left(\sum_{j=0}^{r} |v|_{W_p^r(\Omega)}^p \right)^{1/p},$$

with the usual interpretation in the case $p = \infty$ and $\|\cdot\|_{L_p(\Omega)}$ denoting the usual norm in the Sobolev space $L_p(\Omega)$. We often write shorter $H^r(\Omega)$ in place of $W_2^r(\Omega)$ and $\|\cdot\|_r$ for its norm. By $(\cdot, \cdot)_0$ we denote the standard inner product on $L_2(\Omega)$.

We now write down the familiar variational formulation of (1.1), (1.2). Let $\Omega \subset \mathbb{R}^2$ be a bounded simple polygonal domain, i.e., the boundary $\partial\Omega$ of Ω is the union of straight line segments that form no cuts. The variational formulation of our problem is

$$\text{find } u \in H^1(\Omega) \text{ such that}$$
$$a(u, v) = (g, v)_0 \text{ for } v \in H_0^1(\Omega) \text{ and } u = \psi \text{ on } \partial\Omega, \tag{2.1}$$

where $a(\cdot, \cdot)$ is the sesquilinear form defined by

$$\begin{aligned} a(v, w) = {} & (av_x, w_x)_0 + (bv_x, w_y)_0 + (bv_y, w_x)_0 + (cv_y, w_y)_0 \\ & - (dv, w_x)_0 - (ev, w_y)_0 + (fv, w)_0 \quad \text{for } v, w \in H^1(\Omega). \end{aligned} \tag{2.2}$$

The coefficients of the given problem (1.1) are assumed to be smooth enough, i.e., that they are in the Sobolev space $W_\infty^s(\Omega)$ for the case $s \in \{1, 2\}$, respectively. Schemes for less regular coefficients (on uniform grids) are also known [11, 12, 15, 20, 28], which are based on earlier work by Samarskij [27]. We also impose the general assumption that the homogeneous problem (2.1), i.e., with $g = 0$ and $\psi = 0$, has only the solution $u = 0$.

The discretization of (2.2) is obtained in the following way. We first introduce a nonequidistant rectangular grid in $\overline{\Omega}$. Let $\mathbf{h} = (h_j)_{\mathbb{Z}}$ and $\mathbf{k} = (k_\ell)_{\mathbb{Z}}$ be two sequences of mesh-sizes, i.e., of positive numbers. We define the grid

$$\mathbb{R}_{\mathbf{h}} = \{x_j \in \mathbb{R} : x_{j+1} = x_j + h_j, j \in \mathbb{Z}\}$$

with $x_0 \in \mathbb{R}$ given and a corresponding grid $\mathbb{R}_{\mathbf{k}}$ with the mesh-size vector $\mathbf{k}$ in place of $\mathbf{h}$ and y_0 in place of x_0. Let $\mathbb{R}_H$ be the two-dimensional rectangular grid

$$\mathbb{R}_H = \mathbb{R}_{\mathbf{h}} \times \mathbb{R}_{\mathbf{k}} \subset \mathbb{R}^2$$

and define

$$\Omega_H := \Omega \cap \mathbb{R}_H, \quad \partial\Omega_H := \partial\Omega \cap \mathbb{R}_H, \quad \overline{\Omega}_H = \overline{\Omega} \cap \mathbb{R}_H.$$

The grid $\overline{\Omega}_H$ is assumed to satisfy the following geometric condition with respect to the region Ω:

(Geom) The intersection of $\partial\Omega$ with the rectangles $\square := (x_j, x_{j+1}) \times (y_\ell, y_{\ell+1})$ spanned by points (x_j, y_ℓ), $(x_{j+1}, y_{\ell+1})$ of $\mathbb{R}_H$ is either empty or it is a diagonal of $\square$.

By W_H we denote the space of grid functions on $\overline{\Omega}_H$ and by $W_{0,H}$ the subspace of grid functions vanishing on $\partial\Omega_H$. For convenience, we assume that functions in W_H are also defined outside of $\overline{\Omega}_H$ with function values equal to zero. For $(x_j, y_\ell) \in \overline{\Omega}_H$ let $\square_{j,\ell} := (x_{j-1/2}, x_{j+1/2}) \times (y_{\ell-1/2}, y_{\ell+1/2}) \cap \Omega$ and $\omega_{j,\ell} := |\square_{j,\ell}|$, the measure of $\square_{j,\ell}$. Then

$$(v_H, w_H)_H := \sum_{(x_j,y_\ell)\in\overline{\Omega}_H} \omega_{j,\ell} v_{j,\ell} \bar{w}_{j,\ell} \quad \text{for } v_H, w_H \in W_H$$

defines an inner product on W_H. Let R_H denote the operator of pointwise restriction to the grid in question. The discrete problem has the form:

find $u_H \in W_H$ such that

$$a_H(u_H, v_H) = (g_H, v_H)_H \ \text{ for } v_H \in W_{0,H} \ \text{ and } \ u_H = R_H\psi \text{ on } \partial\Omega_H. \tag{2.3}$$

We assume that at least $\psi \in C^0(\partial\Omega)$. Because ψ is the restriction of u to $\partial\Omega$, higher regularity for ψ will follow from the later regularity assumption for u. In (2.3) $a_H(\cdot,\cdot)$ is a sesquilinear form, which we are now going to define.

Let $\mathcal{T}_H$ be a triangulation of Ω using the set $\overline{\Omega}_H$ as vertices. By $P_H v_H$ we denote the continuous piecewise linear interpolation of v_H with respect to $\mathcal{T}_H$. Then $a_H(\cdot,\cdot)$ is given as a sum

$$a_H = a + b + c + d + e + f \tag{2.4}$$

of sesquilinear forms corresponding to the different terms in the continuous variational problem (2.2). They are all constructed in a similar way on the basis of linear triangular finite elements combined with an individual quadrature, where the discretization of the mixed derivative terms requires special attention (see below).

Let $\Delta \in \mathcal{T}_H$. We define $a_{\Delta,x}$ to be the value of the coefficient a in the midpoint of the side of Δ parallel to the x-axis. Then let

$$a(v_H, w_H) := \sum_{\Delta\in\mathcal{T}_H} a_{\Delta,x} \int_\Delta (P_H v_H)_x (P_H \bar{w}_H)_x dx\, dy. \tag{2.5}$$

Similarly, with $c_{\Delta,y}$ denoting the value of c in the midpoint of the side of Δ parallel to the y-axis,

$$c(v_H, w_H) := \sum_{\Delta\in\mathcal{T}_H} c_{\Delta,y} \int_\Delta (P_H v_H)_y (P_H \bar{w}_H)_y dx\, dy. \tag{2.6}$$

The approximation of the first-order terms is achieved by

$$d(v_H, w_H) := -\sum_{\Delta\in\mathcal{T}_H} [P_H(dv_H)]_{\Delta,x} \int_\Delta (P_H \bar{w}_H)_x dx\, dy, \tag{2.7}$$

$$e(v_H, w_H) := -\sum_{\Delta\in\mathcal{T}_H} [P_H(ev_H)]_{\Delta,y} \int_\Delta (P_H \bar{w}_H)_y dx\, dy. \tag{2.8}$$

Finally,

$$f(v_H, w_H) := \big((R_H f) v_H, w_H\big)_H. \tag{2.9}$$

The function g on the right-hand side of (1.1) is discretized by the grid function

$$g_H(x_j, y_\ell) := \frac{1}{\omega_{j,\ell}} \int_{\square_{j,\ell}} g(x,y)\, dx\, dy, \quad (x_j, y_\ell) \in \Omega_H. \tag{2.10}$$

In Section 6, we will also consider the possibility of taking $g_H = R_H g$.

For the discretization of the mixed derivatives, we need some preparations. We consider two special triangulations of Ω, which we call $\mathcal{T}_H^{(1)}$ and $\mathcal{T}_H^{(2)}$. They are obtained from the disjoint decomposition

$$\mathbb{R}_H = \mathbb{R}_H^{(1)} \,\dot{\cup}\, \mathbb{R}_H^{(2)},$$

where the sum $j+\ell$ of the indices of the points (x_j, y_ℓ) in $\mathbb{R}_H^{(1)}$ and in $\mathbb{R}_H^{(2)}$ is even and odd, respectively. To simplify the following definition we introduce $\mathbb{R}_H^{(3)} := \mathbb{R}_H^{(1)}$. With each point $(x_j, y_\ell) \in \mathbb{R}_H$ we associate the (open) triangles $\Delta_{j,\ell}^{(i)}$, $i = 1, 2, 3, 4$, which have an angle $\pi/2$ at (x_j, y_ℓ) and two of the four horizontal/vertical neighbor grid points of (x_j, y_ℓ) as further vertices. We then define for $v \in \{1, 2\}$ the triangulations

$$\begin{aligned} \mathcal{T}_{H,1}^{(v)} &:= \left\{ \Delta_{j,\ell}^{(i)} \subset \Omega : (x_j, y_\ell) \in \mathbb{R}_H^{(v)}, i \in \{1,2,3,4\} \right\}, \\ \mathcal{T}_{H,2}^{(v)} &:= \left\{ \Delta_{j,\ell}^{(i)} \subset \left(\Omega \setminus \cup \{\Delta \mid \Delta \in \mathcal{T}_{H,1}^{(v)}\}\right) : (x_j, y_\ell) \in \mathbb{R}_H^{(v+1)}, i \in \{1,2,3,4\} \right\}, \\ \mathcal{T}_H^{(v)} &:= \mathcal{T}_{H,1}^{(v)} \cup \mathcal{T}_{H,2}^{(v)}. \end{aligned} \tag{2.11}$$

By $\mathcal{T}_H^{obl}$ we denote the set of triangles that have one side on the oblique part of $\partial\Omega$. $\mathcal{T}_H^{obl}$ is empty for a domain Ω, which is the union of rectangles. Figure 1 shows an example of a triangulation. For $v = 1, 2$ the continuous piecewise linear interpolation $P_H^{(v)} v_H$ of a grid function $v_H \in W_H$ with respect to the triangulations $\mathcal{T}_H^{(v)}$ is well-defined.

The approximation of the mixed derivatives requires special attention near oblique sections of the boundary. This problem is related to the existence of the irregularly orientated triangles in $\mathcal{T}_{H,2}^{(v)}$, which are also responsible for a loss in accuracy. The problem is handled by suitably discretizing the coefficient b. For a triangle Δ in a triangulation denote by (x_Δ, y_Δ) the vertex of Δ associated with the angle $\pi/2$ of Δ and by $(\tilde{x}_\Delta, y_\Delta)$ and $(x_\Delta, \tilde{y}_\Delta)$ the other vertex of Δ with the same y- and x-coordinate, respectively. Then, for $v \in \{1, 2\}$,

$$b_{\Delta,x} := \begin{cases} b(x_\Delta, y_\Delta) & \text{if } \Delta \in \mathcal{T}_{H,1}^{(v)} \\ b(\tilde{x}_\Delta, y_\Delta) & \text{if } \Delta \in \mathcal{T}_{H,2}^{(v)}, \end{cases} \qquad b_{\Delta,y} := \begin{cases} b(x_\Delta, y_\Delta) & \text{if } \Delta \in \mathcal{T}_{H,1}^{(v)} \\ b(x_\Delta, \tilde{y}_\Delta) & \text{if } \Delta \in \mathcal{T}_{H,2}^{(v)}, \end{cases}$$

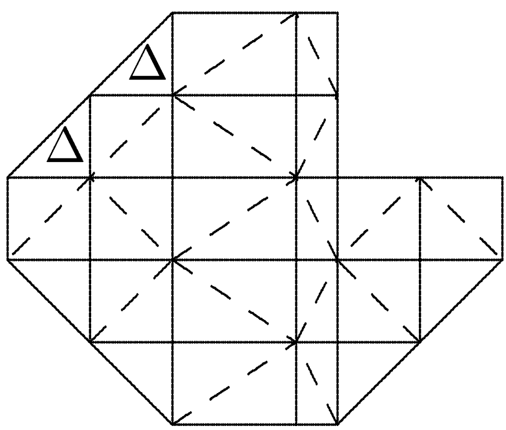

FIGURE 1 Triangulation $\mathcal{T}_H^{(v)}$. Δ indicates triangles of $\mathcal{T}_{H,2}^{(v)}$.

and

$$b(v_H, w_H) := \frac{1}{2}\big(b^{(1)}(v_H, w_H) + b^{(2)}(v_H, w_H)\big) \quad \text{for } v_H \in W_H, w_H \in W_{0,H}, \tag{2.12}$$

where

$$\begin{aligned} b^{(\nu)}(v_H, w_H) &:= \sum_{\Delta \in \mathcal{T}_H^{(\nu)}} \int_\Delta \Big[b_{\Delta,x}(P_H^{(\nu)} v_H)_x (P_H^{(\nu)} \bar{w}_H)_y + b_{\Delta,y}(P_H^{(\nu)} v_H)_y (P_H^{(\nu)} \bar{w}_H)_x \Big] dx\, dy \\ &=: b_{xy}^{(\nu)}(v_H, w_H) + b_{yx}^{(\nu)}(v_H, w_H). \end{aligned} \tag{2.13}$$

3. THE FINITE DIFFERENCE SCHEME

The discretized variational problem (2.3) is equivalent to a standard FDM for the differential operator A on a nonuniform grid, which we will derive in this section. It is this relation that shows that our later supraconvergence theorem is a supercloseness result for the finite element scheme (2.3).

The FDM belonging to (2.3) is obtained by choosing grid functions v_H adequately. For its formulation we use the centered finite difference quotients

$$\delta_x^{(1/2)} v_{j,\ell} = \frac{v_{j+1/2,\ell} - v_{j-1/2,\ell}}{x_{j+1/2} - x_{j-1/2}}, \quad \delta_x^{(1/2)} v_{j+1/2,\ell} = \frac{v_{j+1,\ell} - v_{j,\ell}}{x_{j+1} - x_j},$$
$$\delta_x v_{j,\ell} = \frac{v_{j+1,\ell} - v_{j-1,\ell}}{x_{j+1} - x_{j-1}}$$

in x-direction and also correspondingly defined quantities in y-direction, which make sense for $u_H \in W_H$ in the way they are applied in (3.1). Now choosing v_H to vanish in all but one grid point in Ω_H and collecting the terms arising from (2.3), it is straightforward to obtain the equations

$$\begin{aligned} A_H u_H := &-\delta_x^{(1/2)}(a\delta_x^{(1/2)} u_H) - \delta_y(b\delta_x u_H) - \delta_x(b\delta_y u_H) - \delta_y^{(1/2)}(c\delta_y^{(1/2)} u_H) \\ &+ \delta_x(du_H) + \delta_y(eu_H) + fu_H = g_H \quad \text{in } \Omega_H. \end{aligned} \tag{3.1}$$

If the operator A contains mixed derivatives then A_H acts, next to oblique parts of the boundary, on grid points outside $\overline{\Omega}_H$. In this case, the missing quantities in forming $A_H u_H$ are determined by auxiliary variables that are obtained by a kind of antisymmetric extension. For example, let $(x_j, y_\ell) \in \Omega_H$ be a grid point such that $(x_{j-1}, y_{\ell+1}) \notin \overline{\Omega}_H$. In the approximation of $(bu_x)_y$ the auxiliary value $u_{j-1,\ell+1}$ is then determined by

$$u_{j-1,\ell+1} - \psi_{j-1,\ell} = -(u_{j,\ell} - \psi_{j,\ell+1}). \tag{3.2}$$

The approximation of the differential operator obtained from (2.4) has the expected finite difference form (3.1), which is expressed in the following proposition.

Proposition 3.1. *Let the sesquilinear form* $a_H(\cdot, \cdot)$ *and the operator* A_H *be defined by* (2.4) *and* (3.1)*, respectively. Then the following relation holds:*

$$a_H(v_H, w_H) = (A_H v_H, w_H)_H \quad \textit{for } v_H \in W_H, \;\; w_H \in W_{0,H}.$$

4. INVERSE STABILITY

We now consider a sequence of grids $\mathbb{R}_H$ such that $H_{\max} := \max\{h_j, k_\ell, j, \ell \in \mathbb{Z}\}$, the maximal mesh-size, tends to zero. We use the symbol "Λ" for the sequence of mesh-size vectors and write "$(H \in \Lambda)$" for the convergence with respect to H running through this sequence.

One main ingredient in the convergence analysis is the following inverse stability result. Here and in the sequel, C denotes a generic constant independent of significant quantities.

Proposition 4.1. *Assume that the homogeneous variational problem* (2.1) *has only the solution* $u = 0$. *For each* $H \in \Lambda$, *let* $\mathcal{T}_H$ *be a triangulation of* Ω *generated by* Ω_H *and denote by* P_H *the corresponding piecewise linear interpolation operator. Then there exists a constant* C *such that for* $H \in \Lambda$ *with* $H_{\max}$ *small enough*

$$\|P_H v_H\|_1 \le C \sup_{0 \ne w_H \in W_{0,H}} \frac{|a_H(v_H, w_H)|}{\|P_H w_H\|_1} \quad \textit{for } v_H \in W_{0,H}. \tag{4.1}$$

The proof of this theorem differs only in minor details from the one of theorem 2 in [3] and can be taken from there.

5. ESTIMATING THE TRUNCATION ERROR

Our error estimates are based on the inverse stability inequality in Proposition 4.1 applied to the global discretization error $R_H u - u_H$ in place of v_H. Note that pointwise evaluation of u makes sense because $H^2(\Omega)$ is continuously embedded in $C(\overline{\Omega})$. Also, $R_H u - u_H \in W_{0,H}$. Hence, because u_H solves (2.3), we have to estimate the truncation error

$$\tau_H(v_H) := a_H(R_H u, v_H) - (g_H, v_H)_H \tag{5.1}$$

in terms of $\|P_H v_H\|_1$, which is the aim of this section.

Before going into the details, we recall that v_H is defined by zero outside of Ω_H. It is convenient to take this fact into account and extend the range of sums over the whole space if v_H happens to be a factor. This will be done without further notice in the sequel. As a consequence, boundary terms are avoided when summing by parts. To keep things well-defined, we also extend u, the coefficients of A and g outside of Ω. The specific way of the extension does not matter because there is the multiplication by the factor zero. In later calculations, it will be convenient to choose a specific extension that we define where it is needed. In the proofs, we will use the simple forward differences

$$\Delta_x v_{j,\ell} := v_{j+1,\ell} - v_{j,\ell} \quad \text{and} \quad \Delta_y v_{j,\ell} := v_{j,\ell+1} - v_{j,\ell}. \tag{5.2}$$

Our starting point is the quantity $(g_H, v_H)_H$ in (5.1). According to the definition of g_H in (2.10) we have

$$(g_H, v_H)_H = \sum_{(x_j,y_\ell)\in\Omega_H} \int_{\square_{j,\ell}} (Au)(x,y)\,dx\,dy\,\bar{v}_{j,\ell}. \tag{5.3}$$

We consider each single contribution of Au [see (1.1)] in (5.2) separately. We start with the term $-(au_x)_x$. We want to remark that the estimate in Lemma 5.1 for the case $s=1$ seems to be obvious from the known finite element analysis, but normally there are regular triangulations considered while the triangles here may have interior angles converging to zero.

Lemma 5.1. *Let $s \in \{1,2\}$, $u \in H^{1+s}(\Omega)$ and the coefficient $a \in W^s_\infty(\Omega)$. Then the part*

$$\tau_H^{(a)}(v_H) := a(R_H u, v_H) - \sum_{(x_j,y_\ell)\in\Omega_H} \int_{\square_{j,\ell}} (-au_x)_x\,dx\,dy\,\bar{v}_{j,\ell}$$

of the truncation error τ_H satisfies the estimate

$$\left|\tau_H^{(a)}(v_H)\right| \le C\left(\sum_{\Delta\in\mathcal{T}_H} (\operatorname{diam}\Delta)^{2s}\|u_x\|^2_{H^s(\Delta)}\right)^{1/2} \|P_H v_H\|_1 \quad \text{for } v_H \in W_{0,H}.$$

Proof. We introduce the intervals $I_\ell := (y_{\ell-1/2}, y_{\ell+1/2})$. Using the definition (2.5) of $a(\cdot,\cdot)$ and integrating in (5.2) one time by parts, we obtain with the aid of a partial summation with respect to j (recall that the summation is over all $j,\ell \in \mathbb{Z}$) the representation

$$\begin{aligned}
\tau_H^{(a)}(v_H) &= \sum_{j,\ell} |I_\ell|\big(a\delta_x^{(1/2)}u\big)(x_{j+1/2}, y_\ell)\Delta_x\bar{v}_{j,\ell} \\
&\quad + \sum_{j,\ell}\int_{I_\ell}\big((au_x)(x_{j+1/2},y) - (au_x)(x_{j-1/2},y)\big)\,dy\,\bar{v}_{j,\ell} \\
&= \sum_{j,\ell}\left[|I_\ell|\,\big(a\delta_x^{(1/2)}u\big)(x_{j+1/2},y_\ell) - \int_{I_\ell}(au_x)(x_{j+1/2},y)\,dy\right]\Delta_x\bar{v}_{j,\ell}.
\end{aligned} \tag{5.4}$$

Case $s = 1$. In a first step, we want to replace u_x in (5.4) by $\delta_x^{(1/2)} u$ and estimate the resulting error. Fix $(x_{j+1/2}, y_\ell) \in \Omega$ and define $\widetilde{\square}_{j,\ell} := (x_j, x_{j+1}) \times I_\ell \cap \Omega$. Next to oblique parts of $\partial\Omega$, we extend u from $\widetilde{\square}_{j,\ell}$ to $H^2((x_j, x_{j+1}) \times I_\ell)$ boundedly [map $(x_j, x_{j+1}) \times I_\ell$ affinely to the unit square Q, use the Calderón extension operator [25] to extend the transformed u boundedly into an element of $H^2(Q)$ and then map back]. For almost all $y \in I_\ell$, the function $u(\cdot, y)$ is an element of $H^2(x_j, x_{j+1})$. For each such y

$$F_1(u) := \big(au_x - a\delta_x^{(1/2)} u\big)(x_{j+1/2}, y) \tag{5.5}$$

is a bounded linear functional with respect to $u(\cdot, y) \in H^2(x_j, x_{j+1})$ that vanishes for the functions 1 and x. The Bramble–Hilbert Lemma furnishes in the usual way combined with a suitable scaling argument

$$|F_1(u)| \leq C \sup_{\widetilde{\square}_{j,\ell}} |a(x, y)| h_j^{1/2} \left(\int_{x_j}^{x_{j+1}} |u_{xx}(x, y)|^2 dx \right)^{1/2}. \tag{5.6}$$

Integrating with respect to y over the intervals $(y_{\ell-1/2}, y_\ell)$ and $(y_\ell, y_{\ell+1/2})$ separately and applying Schwarz's inequality for integrals yields

$$\begin{aligned} &\left| \int_{I_\ell} \big(au_x - a\delta_x^{(1/2)} u\big)(x_{j+1/2}, y)\, dy \right| \\ &\quad \leq C h_j^{1/2} \big(k_{\ell-1}^{1/2} \| u_{xx} \|_{H^0(\hat{\square}_{j,\ell-1/2})} + k_\ell^{1/2} \| u_{xx} \|_{H^0(\hat{\square}_{j,\ell})} \big), \end{aligned}$$

where $\hat{\square}_{j,\ell} := (x_j, x_{j+1}) \times (y_\ell, y_{\ell+1/2})$. Thus, with an application of Schwarz's inequality for sums

$$\begin{aligned} &\left| \sum_{j,\ell} \int_{I_\ell} \big(au_x - a\delta_x^{(1/2)} u\big)(x_{j+1/2}, y)\, dy \Delta_x \bar{v}_{j,\ell} \right|^2 \\ &\quad \leq C \sum_{(x_{j+1/2}, y_\ell) \in \Omega} h_j^2 \big(\| u_{xx} \|^2_{H^0(\hat{\square}_{j,\ell-1/2})} + \| u_{xx} \|^2_{H^0(\hat{\square}_{j,\ell})} \big) \\ &\qquad \times \sum_{(x_{j+1/2}, y_\ell) \in \Omega} h_j (k_{\ell-1} + k_\ell) \left| \frac{\Delta_x v_{j,\ell}}{h_j} \right|^2 \\ &\quad \leq C \sum_{(x_{j+1/2}, y_\ell) \in \Omega} h_j^2 \big(\| u_{xx} \|^2_{H^0(\hat{\square}_{j,\ell-1/2})} + \| u_{xx} \|^2_{H^0(\hat{\square}_{j,\ell})} \big) \| (P_H v_H)_x \|_0^2 \\ &\quad \leq C \sum_{\Delta \in \mathcal{T}_H} (\operatorname{diam} \Delta)^2 \| u_x \|^2_{H^1(\Delta)} \| P_H v_H \|_1^2. \end{aligned} \tag{5.7}$$

We continue with estimating $\tau_H^{(a)}(v_H)$ from (5.4). In view of the quantity already bounded in (5.7), we consider

$$F_2(u) := |I_\ell|\big(a\delta_x^{(1/2)}u\big)(x_{j+1/2},y_\ell) - \int_{I_\ell}\big(a\delta_x^{(1/2)}u\big)(x_{j+1/2},y)\,dy$$
$$= \frac{1}{h_j}\int_{x_j}^{x_{j+1}}\left(|I_\ell|a(x_{j+1/2},y_\ell)u_x(x,y_\ell) - \int_{I_\ell}a(x_{j+1/2},y)u_x(x,y)\,dy\right)dx. \tag{5.8}$$

For almost all x, the inner bracket is the error of a rectangle rule for integrating $a(x_{j+1/2},\cdot)u_x(x,\cdot)$ over the intervals $(y_{\ell-1/2},y_\ell)$ and $(y_\ell,y_{\ell+1/2})$, which can be bounded with the aid of the Bramble–Hilbert Lemma by

$$C\left(k_{\ell-1}^3\int_{y_{\ell-1/2}}^{y_\ell}\left|\big(a(x_{j+1/2},y)u_x(x,y)\big)_y\right|^2dy + k_\ell^3\int_{y_\ell}^{y_{\ell+1/2}}\left|\big(a(x_{j+1/2},y)u_x(x,y)\big)_y\right|^2dy\right)^{1/2}. \tag{5.9}$$

We use (5.9) in (5.8), apply the product differentiation rule to $(au_x)_y$ and obtain

$$|F_2(u)| \le Ch_j^{-1/2}\left(k_{\ell-1}^3\|u_x\|^2_{H^1(\hat{\square}_{j,\ell-1/2})} + k_\ell^3\|u_x\|^2_{H^1(\hat{\square}_{j,\ell})}\right)^{1/2}. \tag{5.10}$$

To prove the asserted bound for $\tau_H^{(a)}(v_H)$ in (5.4), we estimate the contribution coming along with $F_2(u)$ in the same way as in (5.7) and then combine with the estimate (5.7).

Case $s=2$. In the representation of the truncation error in the case $s=1$, we found a rectangle rule that does not allow the second-order estimate we want to prove now. We will derive a different representation that is more suitable. We start with a similar preliminary step as in the case $s=1$ replacing this time $\delta_x^{(1/2)}u(x_{j+1/2},y_\ell)$ in (5.4) by $u_x(x_{j+1/2},y_\ell)$, which now makes sense because $H^3(\Omega)\hookrightarrow C^1(\overline{\Omega})$ continuously. We consider

$$F_3(u) := \big(a\delta_x^{(1/2)}u\big)(x_{j+1/2},y_\ell) - (au_x)(x_{j+1/2},y_\ell)$$
$$= a(x_{j+1/2},y_\ell)\left(\frac{1}{h_j}\int_{x_j}^{x_{j+1}}u_x(x,y_\ell)\,dx - u_x(x_{j+1/2},y_\ell)\right) \tag{5.11}$$

as a linear bounded functional in the function $u_x\in H^2(\widetilde{\square}_{j,\ell})$ that vanishes for the functions $1,x$ and y. The Bramble–Hilbert Lemma furnishes the bound

$$|F_3(u)| \le C\sup_{\widetilde{\square}_{j,\ell}}|a(x,y)|\,(h_j|I_\ell|)^{-1/2}\big(h_j^2+|I_\ell|^2\big)|u_x|_{H^2(\widetilde{\square}_{j,\ell})}$$

and we obtain in a similar way as in (5.7)

$$\left|\sum_{j,\ell}|I_\ell|\big(a\delta_x^{(1/2)}u-au_x\big)(x_{j+1/2},y_\ell)\Delta_x\bar v_{j,\ell}\right|\le C\bigg(\sum_{\Delta\in\mathcal{T}_H}(\operatorname{diam}\Delta)^4|u_x|^2_{H^2(\Delta)}\bigg)^{\frac12}\|P_Hv_H\|_1. \tag{5.12}$$

Next we define the quantities

$$T^{(1)}_{j,\ell}:=\frac{k_\ell}{2}(au_x)(x_{j+1/2},y_\ell)-\int_{y_\ell}^{y_{\ell+1/2}}(au_x)(x_{j+1/2},y)\,dy \tag{5.13}$$

and

$$T^{(2)}_{j,\ell-1}:=\frac{k_{\ell-1}}{2}(au_x)(x_{j+1/2},y_\ell)-\int_{y_{\ell-1/2}}^{y_\ell}(au_x)(x_{j+1/2},y)\,dy. \tag{5.14}$$

Note that $|I_\ell|=(k_{\ell-1}+k_\ell)/2$. A summation by parts leads to the identity

$$\begin{aligned}
&\sum_{j,\ell}\left[|I_\ell|\,(au_x)(x_{j+1/2},y_\ell)-\int_{I_\ell}(au_x)(x_{j+1/2},y)\,dy\right]\Delta_x\bar v_{j,\ell}\\
&\quad=\sum_{j,\ell}\big(T^{(1)}_{j,\ell}\Delta_x\bar v_{j,\ell}+T^{(2)}_{j,\ell}\Delta_x\bar v_{j,\ell+1}\big)\\
&\quad=\sum_{j,\ell}\big(T^{(1)}_{j,\ell}+T^{(2)}_{j,\ell}\big)\frac{\Delta_x\bar v_{j,\ell}+\Delta_x\bar v_{j,\ell+1}}{2}+\sum_{j,\ell}\big(T^{(1)}_{j,\ell}-T^{(2)}_{j,\ell}\big)\frac{\Delta_x\bar v_{j,\ell}-\Delta_x\bar v_{j,\ell+1}}{2}\\
&\quad=:Q_1+Q_2.
\end{aligned}$$

We start estimating Q_1 and note that

$$T^{(1)}_{j,\ell}+T^{(2)}_{j,\ell}=\frac{k_\ell}{2}((au_x)(x_{j+1/2},y_\ell)+(au_x)(x_{j+1/2},y_{\ell+1}))-\int_{y_\ell}^{y_{\ell+1}}(au_x)(x_{j+1/2},y)\,dy.$$

This is nothing else than the error in the trapezoidal rule applied to the function $(au_x)(x_{j+1/2},\cdot)$. The Bramble–Hilbert Lemma furnishes

$$\begin{aligned}
\big|T^{(1)}_{j,\ell}+T^{(2)}_{j,\ell}\big|&\le C\bigg(\frac{k_\ell}{h_j}\bigg)^{1/2}\big(h_j^2+k_\ell^2\big)\,|(au_x)|_{H^2(\hat\Box_{j,\ell})}\\
&\le C\bigg(\frac{k_\ell}{h_j}\bigg)^{1/2}\big(h_j^2+k_\ell^2\big)\|u_x\|_{H^2(\hat\Box_{j,\ell})},
\end{aligned}$$

where $\hat{\square}_{j,\ell} := (x_j, x_{j+1}) \times (y_\ell, y_{\ell+1})$ and we took $a \in W^2_\infty(\Omega)$ into account. It follows in a similar way as in (5.7) that

$$|Q_1| \le C\Big(\sum_{\Delta\in\mathcal{T}_H} (\operatorname{diam}\Delta)^4 \|u_x\|^2_{H^2(\Delta)}\Big)^{1/2} \|P_H v_H\|_1. \tag{5.15}$$

We are now going to estimate Q_2. A summation by parts with respect to j leads to the representation

$$Q_2 = \frac{1}{2}\sum_{j,\ell}\Big(T^{(1)}_{j-1,\ell} + T^{(2)}_{j,\ell} - T^{(2)}_{j-1,\ell} - T^{(1)}_{j,\ell}\Big)\Delta_y \bar{v}_{j,\ell}. \tag{5.16}$$

With (5.13) and (5.14) it is seen that

$$T^{(1)}_{j,\ell} - T^{(1)}_{j-1,\ell} + T^{(2)}_{j-1,\ell} - T^{(2)}_{j,\ell} = \int_{x_{j-1/2}}^{x_{j+1/2}}\bigg(\int_{y_{\ell+1/2}}^{y_{\ell+1}} (au_x)_x dy - \int_{y_\ell}^{y_{\ell+1/2}} (au_x)_x dy \\ + \frac{k_\ell}{2}(au_x)_x(x,y_\ell) - \frac{k_\ell}{2}(au_x)_x(x,y_{\ell+1})\bigg)dx.$$

Using the same ideas as in deriving (5.9) from (5.8), we obtain for almost all $x \in (x_{j-1/2}, x_{j+1/2})$

$$\bigg|\int_{y_{\ell+1/2}}^{y_{\ell+1}} (au_x)_x(x,y)\,dy - \int_{y_\ell}^{y_{\ell+1/2}} (au_x)_x(x,y)\,dy \\ + \frac{k_\ell}{2}(au_x)_x(x,y_{\ell+1}) - \frac{k_\ell}{2}(au_x)_x(x,y_\ell)\bigg| \\ \le Ck_\ell^{3/2}\bigg(\int_{y_\ell}^{y_{\ell+1}} \big|\big(a(x,y)u_x(x,y)\big)_{xy}\big|^2 dy\bigg)^{1/2}.$$

After integration with respect to x and an application of Schwarz's inequality for integrals

$$\big|T^{(1)}_{j,\ell} - T^{(1)}_{j-1,\ell} + T^{(2)}_{j-1,\ell} - T^{(2)}_{j,\ell}\big| \le Ck_\ell^{3/2}\Big(h_{j-1}^{1/2}\|u_x\|_{H^2(\tilde{\square}_{j-1/2,\ell})} + h_j^{1/2}\|u_x\|_{H^2(\tilde{\square}_{j,\ell})}\Big)$$

follows. Hence, it is easily seen that Q_2 satisfies the same bound as Q_1 in (5.15). The derived estimates altogether show that the assertion holds also true in the case $s = 2$. □

The contribution $\tau_H^{(c)}(v_H)$ of the second order y-derivative part $-(cu_y)_y$ of A to the truncation error $\tau_H(v_H)$ in (5.1) allows the same bound as $\tau_H^{(a)}(v_H)$ has with u_x replaced by u_y. Let us now consider the mixed derivatives part. The sesquilinear form $b(\cdot,\cdot)$ is defined in (2.12).

Lemma 5.2. *Let $u \in H^2(\Omega)$ and the coefficient $b \in W^1_\infty(\Omega)$. Then the part*

$$\tau_H^{(b)}(v_H) := b(R_H u, v_H) - \sum_{(x_j,y_\ell)\in\Omega_H} \int_{\Box_{j,\ell}} \big((-bu_y)_x + (-bu_x)_y\big)\,dx\,dy\,\bar{v}_{j,\ell} \tag{5.17}$$

of the truncation error $\tau_H(v_H)$ satisfies the estimate

$$\left|\tau_H^{(b)}(v_H)\right| \le C\bigg(\sum_{\Delta\in\mathcal{T}_H} (\operatorname{diam}\Delta)^2\big(\|u_x\|^2_{H^1(\Delta)} + \|u_y\|^2_{H^1(\Delta)}\big)\bigg)^{1/2} \|P_H v_H\|_1$$

for $v_H \in W_{0,H}$.

Proof. We concentrate on estimating the error in the discretization of $(bu_y)_x$; the estimates for $(bu_x)_y$ are similar. By a partial integration and a summation by parts we obtain, using the notation in the proof of Lemma 5.1,

$$\tilde{b}_{yx}(u, v_H) := \sum_{j,\ell} \int_{\Box_{j,\ell}} (-bu_y)_x\,dx\,dy\,\bar{v}_{j,\ell} = \sum_{j,\ell} \int_{I_\ell} (bu_y)(x_{j+1/2}, y)\,dy\,\Delta_x \bar{v}_{j,\ell}. \tag{5.18}$$

Next we want to evaluate

$$b_{yx}(R_H u, v_H) := \frac{1}{2}\big(b_{yx}^{(1)}(R_H u, v_H) + b_{yx}^{(2)}(R_H u, v_H)\big) \quad \text{for } v_H \in W_{0,H},$$

where $b_{yx}^{(\nu)}(R_H u, v_H)$ was defined in (2.13). The inconvenient contributions coming from the triangles $\Delta \in \mathcal{T}_H^{obl}$ are taken into account in dealing with them in the form of a perturbation. For example, let $\Delta \in \mathcal{T}_{H,2}^{(\nu)}$ have the vertices $(x_j, y_\ell), (x_j, y_{\ell+1})$ and (x_{j-1}, y_ℓ). Then we write

$$\begin{aligned} b_{\Delta,y} \int_\Delta (P_H^{(\nu)} u)_y (P_H^{(\nu)} \bar{v}_H)_x\,dx\,dy &= \frac{k_\ell}{2} b_{j-1,\ell} (\delta_y^{(1/2)} u)_{j,\ell+1/2} \bar{v}_{j,\ell} \\ &= \left[\frac{k_\ell}{2} b_{j-1,\ell} (\delta_y^{(1/2)} u)_{j-1,\ell+1/2} + \frac{1}{2} b_{\Delta,y} \operatorname{sign}(\Delta) \sum_{i=1}^{4} (-1)^i u_\Delta^{(i)}\right] \bar{v}_{j,\ell}, \end{aligned} \tag{5.19}$$

where we used the notation $u_\Delta^{(i)} := u(P^{(i)})$ with $P^{(i)}$ for the clockwise numbered vertices of $\hat{\Box}_{j-1,\ell} = (x_{j-1}, x_j) \times (y_\ell, y_{\ell+1})$ and $\operatorname{sign}(\Delta)$ for the (for our purpose not important) plus or minus sign depending on the location of Δ. Thus we obtain, evaluating the integrals in the definition of $b_{yx}(\cdot,\cdot)$,

$$\begin{aligned} b_{yx}(R_H u, v_H) &= \frac{1}{4}\sum_{j,\ell}\Big[-b_{j+1,\ell}\big(k_\ell\big(\delta_y^{(1/2)}u\big)_{j+1,\ell+1/2} + k_{\ell-1}\big(\delta_y^{(1/2)}u\big)_{j+1,\ell-1/2}\big) \\ &\qquad + b_{j-1,\ell}\big(k_\ell\big(\delta_y^{(1/2)}u\big)_{j-1,\ell+1/2} + k_{\ell-1}\big(\delta_y^{(1/2)}u\big)_{j-1,\ell-1/2}\big)\Big]\bar{v}_{j,\ell} \\ &\quad + \frac{1}{4}\sum_{\Delta\in\mathcal{T}_H^{obl}} b_{\Delta,y}\operatorname{sign}(\Delta)\sum_{i=1}^{4}(-1)^i u_\Delta^{(i)}\,\bar{v}_H(x_\Delta, y_\Delta) \\ &=: B_1(u, v_H) + B_2(u, v_H). \end{aligned} \tag{5.20}$$

Changing indices in the summation, it is easy to see that

$$\begin{aligned} B_1(u, v_H) = \frac{1}{4}\sum_{j,\ell}\Big[&k_\ell\big(b_{j+1,\ell}\big(\delta_y^{(1/2)}u\big)_{j+1,\ell+1/2} + b_{j,\ell}\big(\delta_y^{(1/2)}u\big)_{j,\ell+1/2}\big) \\ &+ k_{\ell-1}\big(b_{j+1,\ell}\big(\delta_y^{(1/2)}u\big)_{j+1,\ell-1/2} + b_{j,\ell}\big(\delta_y^{(1/2)}u\big)_{j,\ell-1/2}\big)\Big]\Delta_x\bar{v}_{j,\ell}. \end{aligned} \tag{5.21}$$

The desired bound for $\tilde{b}_{yx}(u, v_H) - B_1(u, v_H)$ is now obtained with the same reasoning as in the proof of Lemma 5.1. We are left with the estimate of $B_2(u, v_H)$, which is provided in the next lemma.

In the statement of the following lemma the known fact is used that $H^3(\Omega) \hookrightarrow W_p^2(\Omega)$ is continuously, embedded for all $p \in [2,\infty)$.

Lemma 5.3. *Let $s \in \{1, 2\}, u \in H^{1+s}(\Omega)$ and the coefficient $b \in W_\infty^s(\Omega)$. Then, for all $p \in [2,\infty)$ and $v_H \in W_{0,H}$, the third quantity of* (5.20) *satisfies*

$$\Big|\sum_{\Delta\in\mathcal{T}_H^{obl}} b_{\Delta,y}\operatorname{sign}(\Delta)\sum_{i=1}^{4}(-1)^i u_\Delta^{(i)}\,\bar{v}_H(x_\Delta, y_\Delta)\Big|$$

$$\leq \begin{cases} C\Big(\sum_{\Delta\in\mathcal{T}_H^{obl}}(\operatorname{diam}\Delta)^2|u|^2_{H^2(\Delta)}\Big)^{1/2}\|P_H v_H\|_1 & \text{if } s = 1, \\ C\Big(\sum_{\Delta\in\mathcal{T}_H^{obl}}(\operatorname{diam}\Delta)^{4(1-1/p)}|u|^2_{W_p^2(\Delta)}\Big)^{1/2}\|P_H v_H\|_1 & \text{if } s = 2. \end{cases}$$

Proof. Originally, u is defined on Δ only. We extend u outside Δ as described before (5.5) [the extension of u is not globally in $H^2(\mathbb{R}^2)$ but this is not needed in the following]. Coming now to the proof of the asserted estimate, we see that (recall $\hat{\Box}_{j-1,\ell} = (x_{j-1}, x_j) \times (y_\ell, y_{\ell+1})$)

$$\left|\sum_{i=1}^{4}(-1)^i u_\Delta^{(i)}\right| = \left|\int_{x_{j-1}}^{x_j}\int_{y_\ell}^{y_{\ell+1}} u_{xy}\,dx\,dy\right| \le C|\hat{\Box}_{j-1,\ell}|^{1/2}\left(\int_{\hat{\Box}_{j-1,\ell}}|u_{xy}|^2\,dx\,dy\right)^{1/2}$$
$$\le C|\Delta|^{1/2}|u|_{H^2(\Delta)}.$$

Together with the corresponding estimates for the remaining triangles in $\mathcal{T}_H^{obl}$ we obtain

$$\left|\sum_{\Delta\in\mathcal{T}_H^{obl}} b_{\Delta,y}\operatorname{sign}(\Delta)\sum_{i=1}^{4}(-1)^i u_\Delta^{(i)}\bar{v}_H(x_\Delta, y_\Delta)\right|$$
$$\le C\left(\sum_{\Delta\in\mathcal{T}_H^{obl}}|\Delta||u|^2_{H^2(\Delta)}\right)^{1/2}\left(\sum_{\Delta\in\mathcal{T}_H^{obl}}|v_H(x_\Delta, y_\Delta)|^2\right)^{1/2}.$$

Because v_H has zero boundary conditions, the last factor admits the estimate

$$\sum_{\Delta\in\mathcal{T}_H^{obl}}|v_H(x_\Delta, y_\Delta)|^2 \le \frac{1}{2}\sum_{\Delta\in\mathcal{T}_H^{obl}}\left(\frac{h_{j-1}}{k_\ell} + \frac{k_\ell}{h_{j-1}}\right)|v_H(x_\Delta, y_\Delta)|^2$$
$$\le \sum_{\Delta\in\mathcal{T}_H^{obl}}\left(|\Delta|\left|\left((P_H v_H)_x\right)_{|\Delta}\right|^2 + |\Delta|\left|\left((P_H v_H)_y\right)_{|\Delta}\right|^2\right)$$
$$\le C\|P_H v_H\|_1^2$$

and the proof for the case $s = 1$ is done. If $s = 2$, Schwarz's inequality for integrals furnishes

$$|u|^2_{H^2(\Delta)} \le |\Delta|^{(p-2)/p}|u|^2_{W_p^2(\Delta)} \le (\operatorname{diam}\Delta)^{2(p-2)/p}|u|^2_{W_p^2(\Delta)}$$

and the result follows from the already proved one for $s = 1$. □

Lemma 5.4. *Let $u \in H^3(\Omega)$ and the coefficient $b \in W_\infty^2(\Omega)$. Then the part $\tau_H^{(b)}$ from (5.17) of the truncation error $\tau_H(v_H)$ satisfies for each $p \in [2, \infty)$ the estimate*

$$\left|\tau_H^{(b)}(v_H)\right| \le C\left(\sum_{\Delta\in\mathcal{T}_H}(\operatorname{diam}\Delta)^4\|u\|^2_{H^3(\Delta)} + \sum_{\Delta\in\mathcal{T}_H^{obl}}(\operatorname{diam}\Delta)^{4(1-1/p)}|u|^2_{W_p^2(\Delta)}\right)^{1/2}$$
$$\times\|P_H v_H\|_1 \quad \text{for } v_H \in W_{0,H}.$$

Proof. The first part of the proof coincides with that of Lemma 5.2 until formula (5.21) and we continue there. In a first step, we rewrite with the aid of a summation by parts and replacing $k_\ell \delta_y^{(1/2)} u$ by an integration of u_y with respect to y

$$
\begin{aligned}
B_1(u, v_H) &= \frac{1}{4} \sum_{j,\ell} k_\ell \Big[\big(b_{j+1,\ell} \big(\delta_y^{(1/2)} u\big)_{j+1,\ell+1/2} + b_{j,\ell} \big(\delta_y^{(1/2)} u\big)_{j,\ell+1/2} \big) \Delta_x \bar{v}_{j,\ell} \\
&\quad + \big(b_{j+1,\ell+1} \big(\delta_y^{(1/2)} u\big)_{j+1,\ell+1/2} + b_{j,\ell+1} \big(\delta_y^{(1/2)} u\big)_{j,\ell+1/2} \big) \Delta_x \bar{v}_{j,\ell+1} \Big] \\
&= \sum_{j,\ell} \frac{1}{8} \int_{y_\ell}^{y_{\ell+1}} \Big[\big(b_{j+1,\ell} + b_{j+1,\ell+1} \big) u_y(x_{j+1}, y) \\
&\qquad + \big(b_{j,\ell} + b_{j,\ell+1} \big) u_y(x_j, y) \Big] dy \big(\Delta_x \bar{v}_{j,\ell} + \Delta_x \bar{v}_{j,\ell+1} \big) \\
&\quad + \sum_{j,\ell} \frac{1}{8} \int_{y_\ell}^{y_{\ell+1}} \Big[\big(b_{j+1,\ell} - b_{j+1,\ell+1} \big) u_y(x_{j+1}, y) \\
&\qquad + \big(b_{j,\ell} - b_{j,\ell+1} \big) u_y(x_j, y) \Big] dy \big(\Delta_x \bar{v}_{j,\ell} - \Delta_x \bar{v}_{j,\ell+1} \big) \\
&= \sum_{j,\ell} \frac{1}{8} \int_{y_\ell}^{y_{\ell+1}} \Big[\big(b_{j+1,\ell} + b_{j+1,\ell+1} \big) u_y(x_{j+1}, y) \\
&\qquad + \big(b_{j,\ell} + b_{j,\ell+1} \big) u_y(x_j, y) \Big] dy \big(\Delta_x \bar{v}_{j,\ell} + \Delta_x \bar{v}_{j,\ell+1} \big) \\
&\quad + \sum_{j,\ell} \frac{1}{8} \int_{y_\ell}^{y_{\ell+1}} \Big[\big(b_{j+1,\ell+1} - b_{j+1,\ell} \big) u_y(x_{j+1}, y) \\
&\qquad - \big(b_{j-1,\ell+1} - b_{j-1,\ell} \big) u_y(x_{j-1}, y) \Big] dy \, \Delta_y \bar{v}_{j,\ell} \\
&=: \sum_{j,\ell} B_{j,\ell}^{(1)} \big(\Delta_x \bar{v}_{j,\ell} + \Delta_x \bar{v}_{j,\ell+1} \big) + \sum_{j,\ell} B_{j,\ell}^{(2)} \Delta_y \bar{v}_{j,\ell}. \qquad (5.22)
\end{aligned}
$$

Again with the aid of summations by parts we rewrite, starting from (5.18),

$$
\begin{aligned}
\tilde{b}_{yx}(u, v_H) &= \sum_{j,\ell} \left(\int_{y_{\ell-1/2}}^{y_\ell} + \int_{y_\ell}^{y_{\ell+1/2}} \right) (b u_y)(x_{j+1/2}, y) \, dy \, \Delta_x \bar{v}_{j,\ell} \\
&= \sum_{j,\ell} \frac{1}{2} \int_{y_\ell}^{y_{\ell+1}} (b u_y)(x_{j+1/2}, y) \, dy \big(\Delta_x \bar{v}_{j,\ell+1} + \Delta_x \bar{v}_{j,\ell} \big) \\
&\quad + \sum_{j,\ell} \frac{1}{2} \left(\int_{y_{\ell+1/2}}^{y_{\ell+1}} - \int_{y_\ell}^{y_{\ell+1/2}} \right) (b u_y)(x_{j+1/2}, y) \, dy \big(\Delta_x \bar{v}_{j,\ell+1} - \Delta_x \bar{v}_{j,\ell} \big) \\
&= \sum_{j,\ell} \frac{1}{2} \int_{y_\ell}^{y_{\ell+1}} (b u_y)(x_{j+1/2}, y) \, dy \big(\Delta_x \bar{v}_{j,\ell+1} + \Delta_x \bar{v}_{j,\ell} \big)
\end{aligned}
$$

$$+\sum_{j,\ell}\frac{1}{2}\left(\int_{y_{\ell+1/2}}^{y_{\ell+1}}-\int_{y_\ell}^{y_{\ell+1/2}}\right)$$
$$\times\left((bu_y)(x_{j-1/2},y)-(bu_y)(x_{j+1/2},y)\right)dy\,\Delta_y\bar{v}_{j,\ell}$$
$$=:\sum_{j,\ell}S_{j,\ell}^{(1)}(\Delta_x\bar{v}_{j,\ell+1}+\Delta_x\bar{v}_{j,\ell})+\sum_{j,\ell}S_{j,\ell}^{(2)}\Delta_y\bar{v}_{j,\ell}. \tag{5.23}$$

Now we begin with estimating the corresponding quantities in $\tilde{b}_{yx}(u,v_H)-B_1(u,v_H)$ starting with $S_{j,\ell}^{(1)}-B_{j,\ell}^{(1)}$. First we concentrate on $B_{j,\ell}^{(1)}$ alone and pick

$$b_{j+1,\ell}u_y(x_{j+1},y)+b_{j,\ell}u_y(x_j,y)=F_\ell(x_{j+1},y)+F_\ell(x_j,y), \tag{5.24}$$

where $F_\ell(x,y):=b(x,y_\ell)u_y(x,y)$. An application of the Bramble–Hilbert Lemma and taking $b\in W_\infty^2(\Omega)$ into account yields that uniformly for $y\in(y_\ell,y_{\ell+1})$

$$|F_\ell(x_{j+1},y)+F_\ell(x_j,y)-2F_\ell(x_{j+1/2},y)|\le Ch_j^{3/2}|F(\cdot,y)|_{H^2(x_j,x_{j+1})}.$$

Integration of the last inequality over $(y_\ell,y_{\ell+1})$ provides an additional factor $k_\ell^{1/2}$ and we end up with

$$\left|\int_{y_\ell}^{y_{\ell+1}}\left[b_{j+1,\ell}u_y(x_{j+1},y)+b_{j,\ell}u_y(x_j,y)-2b_{j+1/2,\ell}u_y(x_{j+1/2},y)\right]dy\right|$$
$$\le Ch_j^{3/2}k_\ell^{1/2}\|F\|_{H^2(\hat{\square}_{j,\ell})}\le C\big(h_j^2+k_\ell^2\big)\|u_y\|_{H^2(\hat{\square}_{j,\ell})}. \tag{5.25}$$

The same bound holds if we consider the left-hand side of (5.24) evaluated at $y_{\ell+1}$ in place of y_ℓ.

Next we consider $S_{j,\ell}^{(1)}$ and derive the following estimates, where the Bramble–Hilbert Lemma is applied to the appearing midpoint rule:

$$\left|\int_{y_\ell}^{y_{\ell+1}}(bu_y)(x_{j+1/2},y)dy-k_\ell(bu_y)_{j+1/2,\ell+1/2}\right|\le C\big(h_j^2+k_\ell^2\big)\left(\frac{k_\ell}{h_j}\right)^{1/2}|bu_y|_{H^2(\hat{\square}_{j,\ell})}$$
$$\le C\big(h_j^2+k_\ell^2\big)\left(\frac{k_\ell}{h_j}\right)^{1/2}\|u_y\|_{H^2(\hat{\square}_{j,\ell})}, \tag{5.26}$$

$$\left|b_{j+1/2,\ell+1/2}\int_{y_\ell}^{y_{\ell+1}}u_y(x_{j+1/2},y)dy-k_\ell(bu_y)_{j+1/2,\ell+1/2}\right|$$
$$\le C\big(h_j^2+k_\ell^2\big)\left(\frac{k_\ell}{h_j}\right)^{1/2}|u_y|_{H^2(\hat{\square}_{j,\ell})} \tag{5.27}$$

and

$$\left|\big(b_{j+1/2,\ell+1} + b_{j+1/2,\ell} - 2b_{j+1/2,\ell+1/2}\big)\int_{y_\ell}^{y_{\ell+1}} u_y(x_{j+1/2}, y)\,dy\right|$$
$$\leq Ck_\ell^2\left(\frac{k_\ell}{h_j}\right)^{1/2}\|u_y\|_{H^1(\hat{\square}_{j,\ell})}. \tag{5.28}$$

Combining the bounds (5.25)–(5.28), it follows in the same way as in (5.7) that

$$\left|\sum_{j,\ell}\big(S_{j,\ell}^{(1)} - B_{j,\ell}^{(1)}\big)(\Delta_x\bar{v}_{j,\ell+1} + \Delta_x\bar{v}_{j,\ell})\right| \leq C\bigg(\sum_{\Delta\in\mathcal{T}_H}(\operatorname{diam}\Delta)^4\|u_y\|^2_{H^2(\Delta)}\bigg)^{1/2}\|P_H v_H\|_1. \tag{5.29}$$

We are now going to estimate $S_{j,\ell}^{(2)}$ and $B_{j,\ell}^{(2)}$. Starting from the definition (5.23) of $S_{j,\ell}^{(2)}$, we obtain with the aid of the Bramble–Hilbert Lemma [recall $\hat{\square}_{j,\ell} = (x_j, x_{j+1})\times(y_\ell, y_{\ell+1})$]

$$|S_{j,\ell}^{(2)}| = \frac{1}{2}\left|\int_{x_{j-1/2}}^{x_{j+1/2}}\left(\int_{y_{\ell+1/2}}^{y_{\ell+1}} - \int_{y_\ell}^{y_{\ell+1/2}}\right)(bu_y)_x(x,y)\,dy\,dx\right|$$
$$\leq C(h_j + k_\ell)(h_jk_\ell)^{1/2}|(bu_y)_x|_{H^1(\hat{\square}_{j,\ell})} + (h_j + k_\ell)(h_{j-1}k_\ell)^{1/2}|(bu_y)_x|_{H^1(\hat{\square}_{j-1,\ell})}.$$

We use $|(bu_y)_x|_1 \leq C\|u_y\|_2$ and derive as in (5.7) the bound

$$\left|\sum_{j,\ell} S_{j,\ell}^{(2)}\,\Delta_y\bar{v}_{j,\ell}\right| \leq C\bigg(\sum_{\Delta\in\mathcal{T}_H}(\operatorname{diam}\Delta)^4\|u_y\|^2_{H^2(\Delta)}\bigg)^{1/2}\|P_H v_H\|_1. \tag{5.30}$$

Recalling the definition (5.22) of $B_{j,\ell}^{(2)}$, it is seen that

$$\big|B_{j,\ell}^{(2)}\big| = \frac{1}{8}\left|\int_{x_{j-1}}^{x_{j+1}}\int_{y_\ell}^{y_{\ell+1}}\big[(b(x,y_\ell) - b(x,y_{\ell+1}))u_y(x,y)\big]_x\,dy\,dx\right|$$
$$\leq C\int_{x_{j-1}}^{x_{j+1}}\int_{y_\ell}^{y_{\ell+1}} k_\ell\|b\|_{2,\infty}\big(|u_y(x,y)| + |u_{yx}(x,y)|\big)\,dy\,dx$$
$$\leq Ck_\ell\big((h_{j-1}k_\ell)^{1/2}\|u_y\|_{H^1(\hat{\square}_{j-1,\ell})} + (h_jk_\ell)^{1/2}\|u_y\|_{H^1(\hat{\square}_{j,\ell})}\big).$$

It follows the same way as before

$$\left|\sum_{j,\ell} B_{j,\ell}^{(2)}\,\Delta_y\bar{v}_{j,\ell}\right| \leq C\bigg(\sum_{\Delta\in\mathcal{T}_H}(\operatorname{diam}\Delta)^4\|u_y\|^2_{H^1(\Delta)}\bigg)^{1/2}\|P_H v_H\|_1. \tag{5.31}$$

With (5.29)–(5.31), the desired bound for $\tau_H^{(b)}(v_H) = \tilde{b}_{yx}(u, v_H) - B_1(u, v_H)$ is proved. □

Let us now consider the contribution of the approximation of $(du)_x$ to the truncation error.

Lemma 5.5. *Let* $s \in \{1, 2\}, u \in H^{1+s}(\Omega)$ *and the coefficient* $d \in W_\infty^s(\Omega)$. *Then the part*

$$\tau_H^{(d)}(v_H) := d(R_H u, v_H) - \sum_{(x_j, y_\ell) \in \Omega_H} \int_{\square_{j,\ell}} (du)_x dx\, dy\, \bar{v}_{j,\ell}$$

of the truncation error $\tau_H(v_H)$ *satisfies the estimate*

$$\left|\tau_H^{(d)}(v_H)\right| \leq C \left(\sum_{\Delta \in \mathcal{T}_H} (\operatorname{diam} \Delta)^{2s} \|u\|^2_{H^{1+s}(\Delta)} \right)^{1/2} \|P_H v_H\|_1 \quad \text{for } v_H \in W_{0,H}.$$

Proof. In the same way as in the proof of Lemma 5.1, using the notation from there, we obtain the representation

$$\tau_H^{(d)}(v_H) = \sum_{j,\ell} \left[\int_{I_\ell} (du)(x_{j+1/2}, y) dy - |I_\ell| \frac{(du)_{j,\ell} + (du)_{j+1,\ell}}{2} \right] \Delta_x \bar{v}_{j,\ell}.$$

The proof now follows the lines of the proofs before. □

The contribution $\tau_H^{(e)}(v_H)$ coming from the approximation of $(eu)_y$ satisfies the same bound.

We are left with estimating the approximation of fu. As a preparation, we provide the following lemma.

Lemma 5.6. *The following identity holds for* $a_j, b_j \in \mathbb{C}, j = 1, \ldots, 4$:

$$\begin{aligned} 4 \sum_{i=1}^4 a_i b_i = \sum_{i=1}^4 a_i \sum_{i=1}^4 b_i &+ (a_1 + a_2 - a_3 - a_4)(b_1 + b_2 - b_3 - b_4) \\ &+ (a_1 - a_2 + a_3 - a_4)(b_1 - b_2 + b_3 - b_4) \\ &+ (a_1 - a_2 - a_3 + a_4)(b_1 - b_2 - b_3 + b_4). \end{aligned}$$

Proof. The assertion follows applying the identity $2(ab + cd) = (a + c)(b + d) + (a - c)(b - d)$ to $2(a_1 b_1 + a_2 b_2)$ and $2(a_3 b_3 + a_4 b_4)$ and then another time to the resulting terms. □

Lemma 5.7. *Let $s \in \{1,2\}$, $u \in H^2(\Omega)$ and $f \in W^s_\infty(\Omega)$. Then the part*

$$\tau_H^{(f)}(v_H) := f(R_H u, v_H) - \sum_{(x_j,y_\ell)\in\Omega} \int_{\square_{j,\ell}} fu\,dx\,dy\,\bar{v}_{j,\ell} \tag{5.32}$$

of the truncation error $\tau_H(v_H)$ satisfies the estimate

$$\left|\tau_H^{(f)}(v_H)\right| \le C\biggl(\sum_{\Delta\in\mathcal{T}_H} (\operatorname{diam}\Delta)^{2s}\|u\|^2_{H^2(\Delta)}\biggr)^{1/2} \|P_H v_H\|_1 \quad \textit{for } v_H \in W_{0,H}.$$

Proof. We give the proof for the case $s = 2$ only. Recall that the sum in (5.32) can be extended over $\mathbb{R}_H$ in place of Ω_H without changing its value and that we can also consider u and f to be extended outside of Ω as described in the proof of Lemma 5.3. Fix j, ℓ and consider the rectangle $\hat{\square} := (x_j, x_{j+1}) \times (y_\ell, y_{\ell+1})$. We subdivide $\hat{\square}$ in four congruent rectangles $\hat{\square}^{(i)}$, $i = 1, \ldots, 4$, of equal size and denote by w_i the value of a function w in the common vertex of $\hat{\square}^{(i)}$ and $\hat{\square}$. The part of $\tau_H^{(f)}(v_H)$ related to $\hat{\square}$ is

$$E(\hat{\square}) := \sum_{i=1}^{4} \rho_i \bar{v}_i, \quad \rho_i := \int_{\hat{\square}^{(i)}} fu\,dx\,dy - |\hat{\square}^{(i)}|(fu)_i.$$

We apply Lemma 5.6 to $E(\hat{\square})$ and estimate the resulting four terms. Because $|\hat{\square}| = 4|\hat{\square}^{(i)}|$, the first one is

$$E_1(\hat{\square}) := \sum_{i=1}^{4} \rho_i \sum_{i=1}^{4} \bar{v}_i = \biggl(\int_{\hat{\square}} fu\,dx\,dy - \frac{|\hat{\square}|}{4} \sum_{i=1}^{4} (fu)_i\biggr) \sum_{i=1}^{4} \bar{v}_i.$$

The Bramble–Hilbert Lemma furnishes

$$\left|\int_{\hat{\square}} fu\,dx\,dy - \frac{|\hat{\square}|}{4}\sum_{i=1}^{4}(fu)_i\right| \le C\|f\|_{2,\infty}(\operatorname{diam}\hat{\square})^2|\hat{\square}|^{1/2}\|u\|_{H^2(\hat{\square})}$$

and we obtain

$$|E_1(\hat{\square})| \le C(\operatorname{diam}\hat{\square})^2\|u\|_{H^2(\hat{\square})}\|P_H v_H\|_{H^0(\hat{\square})}.$$

Hence,

$$\sum_{\hat{\square}} |E_1(\hat{\square})| \le C\biggl(\sum_{\Delta\in\mathcal{T}_H} (\operatorname{diam}\Delta)^4\|u\|^2_{H^2(\Delta)}\biggr)^{1/2} \|P_H v_H\|_0. \tag{5.33}$$

The next term coming from the application of Lemma 5.6 to $E(\hat{\Box})$ has the form (we assume that the numbering of the $\hat{\Box}^{(i)}$ was done accordingly)

$$\begin{aligned} E_2(\hat{\Box}) &:= (\rho_1 + \rho_2 - \rho_3 - \rho_4)(\bar{v}_1 + \bar{v}_2 - \bar{v}_3 - \bar{v}_4) \\ &= (\rho_1 + \rho_2 - \rho_3 - \rho_4)(\bar{v}_{j,\ell} - \bar{v}_{j,\ell+1} + \bar{v}_{j+1,\ell} - \bar{v}_{j+1,\ell+1}) \\ &= -(\rho_1 + \rho_2 - \rho_3 - \rho_4)k_\ell\big((P_H\bar{v}_H)_y(x_j, y_{\ell+1/2}) + (P_H\bar{v}_H)_y(x_{j+1}, y_{\ell+1/2})\big). \end{aligned}$$

With the aid of the Bramble–Hilbert Lemma follows

$$\left|\rho_1 + \rho_2 - \rho_3 - \rho_4\right| \le C\|f\|_{1,\infty}(\operatorname{diam}\hat{\Box})|\hat{\Box}|^{1/2}\|u\|_{H^1(\hat{\Box})}$$

which leads in a similar way as before to

$$\sum_{\hat{\Box}}|E_2(\hat{\Box})| \le C\bigg(\sum_{\Delta\in\mathcal{T}_H}(\operatorname{diam}\Delta)^4\|u\|^2_{H^1(\Delta)}\bigg)^{1/2}\|P_H v_H\|_1.$$

The remaining two terms of E have the same bound, and together with (5.33) the proof is complete. $\Box$

6. DISCRETIZATION ERROR ESTIMATES

Theorem 6.1. *Let the grids $\overline{\Omega}_H$, $H \in \Lambda$, satisfy condition (Geom). Assume that the homogeneous variational problem* (2.1) *is uniquely solvable. Then the discretized problem* (3.1), *or equivalently* (2.3), *has a unique solution $u_H \in W_H$ for $H \in \Lambda$ with H_{max} sufficiently small. Let $s \in \{1, 2\}$, $u \in H^{1+s}(\Omega)$ and assume that the coefficients of the differential operator are in $W^s_\infty(\Omega)$. Then the error estimate*

$$\|P_H(R_H u - u_H)\|_1 \le C\bigg(\sum_{\Delta\in\mathcal{T}_H}(\operatorname{diam}\Delta)^2\|u\|^2_{H^2(\Delta)}\bigg)^{1/2} \le CH_{\max}\|u\|_{H^2(\Omega)}$$

holds for $s = 1$ while for $s = 2$ and each $p \in [2, \infty)$

$$\begin{aligned} &\|P_H(R_H u - u_H)\|_1 \\ &\quad \le C\bigg(\sum_{\Delta\in\mathcal{T}_H}(\operatorname{diam}\Delta)^4\|u\|^2_{H^3(\Delta)} + \sigma(b)\sum_{\Delta\in\mathcal{T}_H^{obl}}(\operatorname{diam}\Delta)^{4(1-1/p)}|u|^2_{W^2_p(\Delta)}\bigg)^{1/2} \\ &\quad \le C\Big(H^2_{\max}\|u\|_{H^3(\Omega)} + \sigma(b)H_{\max}^{3/2-1/p}\|u\|_{W^2_p(\Omega_H^{obl})}\Big) \\ &\quad \le C\Big(H^2_{\max} + \sigma(b)H_{\max}^{3/2-1/p}\Big)\|u\|_{H^3(\Omega)} \end{aligned}$$

holds, where $\Omega_H^{obl} := \cup\{\Delta \mid \Delta \in \mathcal{T}_H^{obl}\}$ and $\sigma(b) = 1$ or 0 for $b \ne 0$ or $b = 0$, respectively.

Proof. We have already noted at the beginning of Section 5 that the discretization error bound follows from the corresponding bound of the truncation error $\tau_H(v_H)$ from (5.1), which we have split in the form

$$\tau_H(v_H) = \tau_H^{(a)}(v_H) + \tau_H^{(b)}(v_H) + \tau_H^{(c)}(v_H) + \tau_H^{(d)}(v_H) + \tau_H^{(e)}(v_H) + \tau_H^{(f)}(v_H).$$

The particular estimates for these quantities are proved in the form we need them in Lemmas 5.1, 5.2, 5.4, 5.5, 5.7, taking into account that $\tau_H^{(c)}(v_H)$ and $\tau_H^{(e)}(v_H)$ have corresponding bounds as $\tau_H^{(a)}(v_H)$ and $\tau_H^{(d)}(v_H)$, respectively. The second last inequality follows from the one before with the aid of Hölder's inequality for sums taking $\sum_{\Delta\in\mathcal{T}_H^{obl}} \operatorname{diam}\Delta \le C$ into account. □

Note that also for $s = 2$ there is no error term of order $(3/2 - 1/p)$ in Theorem 6.1 if the boundary $\partial\Omega$ has no oblique sections or if $b = 0$. From Theorem 6.1 we have the following corollaries.

Corollary 6.2. *Let $u \in H^3(\Omega)$ and assume that there exists a neighborhood Ω_0 of the oblique part of $\partial\Omega$ such that $u \in C^2(\overline{\Omega} \cup \Omega_0)$. Assume that the coefficients of the differential operator are in $W_\infty^2(\Omega)$. Then the unique solution u_H in Theorem* 6.1 *satisfies the error estimate*

$$\begin{aligned}\|P_H(R_H u - u_H)\|_1 &\le C\Bigg(\sum_{\Delta\in\mathcal{T}_H} (\operatorname{diam}\Delta)^4 \|u\|_{H^3(\Delta)}^2 + \sum_{\Delta\in\mathcal{T}_H^{obl}} (\operatorname{diam}\Delta)^4 |u|_{C^2(\overline{\Delta})}^2\Bigg)^{1/2}\\ &\le C\big(H_{\max}^2 \|u\|_{H^3(\Omega)} + H_{\max}^{3/2} |u|_{C^2(\Omega_H^{obl})}\big)\\ &\le CH_{\max}^{3/2}\big(\|u\|_{H^3(\Omega)} + |u|_{C^2(\Omega_H^{obl})}\big).\end{aligned}$$

Proof. The first bound follows from the case $s = 2$ in Theorem 6.1 by estimating $|u|_{W_p^2(\Delta)}$ with the maximum norm. The second one then follows from $\sum_{\Delta\in\mathcal{T}_H^{obl}} \operatorname{diam}\Delta \le C$. □

Corollary 6.3. *Let $s \in [1, 2]$, $u \in H^{1+s}(\Omega)$ and assume that the coefficients of the differential operator are in $W_\infty^2(\Omega)$. Then for each $p \in [2, \infty)$, the unique solution u_H in Theorem* 6.1 *satisfies the error estimate*

$$\|P_H(R_H u - u_H)\|_1 \le \begin{cases} CH_{\max}^{1+(s-1)(1/2-1/p)} \|u\|_{H^{1+s}(\Omega)} & \text{in general},\\ CH_{\max}^{s} \|u\|_{H^{1+s}(\Omega)} & \text{if } \mathcal{T}_H^{obl} = \emptyset \text{ or } b = 0.\end{cases}$$

Proof. The result is derived by interpolation between the case $s = 1$ and $s = 2$ in Theorem 6.1. □

Remark 6.4. The discretization of the right-hand side g as an integral average (2.9) can be replaced by the pointwise restriction to the grid without changing the convergence rates if $g \in H^2(\Omega)$. This can be seen from Lemma 5.7 as the difference of the right-hand side in both kinds of discretization is of second-order.

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