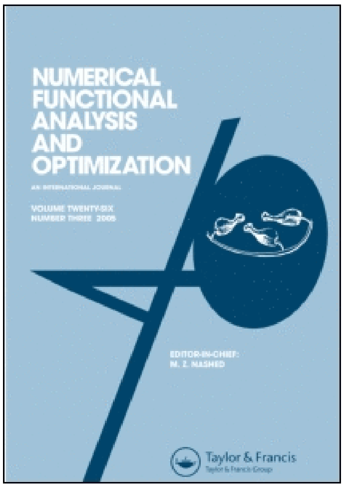

Numerical Functional Analysis and Optimization
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597287

Discretely Compact Embeddings of Spaces of Cell-Centered Grid Functions

Sílvia Barbeiro [a]
[a] Center for Mathematics, University of Coimbra, Coimbra, Portugal

Online Publication Date: 01 May 2008

To cite this Article Barbeiro, Sílvia(2008)'Discretely Compact Embeddings of Spaces of Cell-Centered Grid Functions',Numerical Functional Analysis and Optimization,29:5,491 — 506

To link to this Article: DOI: 10.1080/01630560802099092

URL: http://dx.doi.org/10.1080/01630560802099092

Numerical Functional Analysis and Optimization, 29(5–6):491–506, 2008

ISSN: 0163-0563 print/1532-2467 online
DOI: 10.1080/01630560802099092

DISCRETELY COMPACT EMBEDDINGS OF SPACES OF CELL-CENTERED GRID FUNCTIONS

Sílvia Barbeiro

Center for Mathematics, University of Coimbra, Coimbra, Portugal

□ *Compactness of embeddings in discrete counterparts of Sobolev spaces is considered. We study the embeddings in spaces of cell-centered grid functions, in one- and two-dimensional domains. No restrictions are made on the mesh-ratios of the underlying meshes.*

Keywords Cell-centered grid functions; Compactness; Discrete spaces; Embeddings.

AMS Subject Classification 46E39; 74S20.

1. INTRODUCTION

Results in compactness of embeddings of spaces of grid functions can play a primary role in the study of stability and convergence of finite difference schemes. In particular, they are important technical tools in order to establish supraconvergence results for schemes on nonuniform meshes (see, e.g., [3–6, 9]). The discrete convergence theory was introduced by Stummel, in [11], and later considered also by Grigorieff and Reinhardt (see, e.g., [7, 8, 10, 12]) among others.

In this paper, we consider spaces of cell-centered grid functions. We prove discrete compactness of embeddings in discrete versions of the Sobolev spaces L^p, $W_0^{1,p}$ and $W^{2,p} \cap W_0^{1,p}$, $1 \leq p \leq \infty$, in one-dimensional domains. In two-dimensional domains we prove similar results for the cases L^p and $W_0^{1,p}$, $0 \leq p < \infty$. Grigorieff gives, in [8], correspondent results for spaces of vertex-centered grid functions in one-dimensional domains. In the case of nonuniform grids, the normed spaces that we consider in

Address correspondence to Sílvia Barbeiro, Center for Mathematics, University of Coimbra, Apartado 3008, Coimbra 3001-454, Portugal; E-mail: silvia@mat.uc.pt

this paper do not coincide with those defined in [8], and different kinds of proofs are needed.

We obtain the compactness result for the discrete version of the embeddings $W_0^{1,p} \to L^q$, in the one-dimensional case, using a correspondent result in the continuous case. Functions defined in all the domains that coincide with grid functions in the grid points are considered. In the two-dimensional case we could not find appropriate continuous prolongations of the grid functions, and the proof of the discrete compactness result uses the Kolmogorov compactness theorem. This last strategy could also be applied to the one-dimensional case, but we chose to present the proof that we consider more natural. The proof of discrete compactness of discrete versions of the embeddings $W_0^{1,p} \cap W^{2,p} \to W_0^{1,q}$ we present is also specific for the one-dimensional case, and there is not an immediate extension for the two-dimensional case.

The results we present in this paper play an important role in the stability analysis of a cell-centered finite difference scheme for second-order elliptic equations in [3].

2. DISCRETE APPROXIMATIONS

In this section, we start by introducing the discrete counterparts of the Sobolev spaces $L^p(0,R)$, $W_0^{1,p}(0,R)$ and $W^{2,p}(0,R) \cap W_0^{1,p}(0,R)$. We define the partition G_h of the domain $[0,R]$,

$$G_h := \{0 = x_0 < x_1 < \cdots < x_N = R\}.$$

The set of the cell-centers is given by

$$S_h := \{x_{1/2}, x_{3/2}, \ldots, x_{N-1/2}\},$$

where

$$x_{j-1/2} := \frac{x_{j-1} + x_j}{2}, \quad j = 1, \ldots, N.$$

For the grid functions v_h and w_h defined on $\overline{S}_h := S_h \cup \{x_0, x_N\}$ and G_h, respectively, the centered difference quotients are given by

$$(\delta v_h)_j := \frac{v_{j+1/2} - v_{j-1/2}}{h_{j-1/2}}, \quad j = 0, \ldots, N,$$

and

$$(\delta w_h)_{j-1/2} := \frac{w_j - w_{j-1}}{h_{j-1}}, \quad j = 1, \ldots, N,$$

where $x_{-1/2} := x_0$, $x_{N+1/2} := x_N$ and

$$h_{j-1/2} := x_{j+1/2} - x_{j-1/2}, \quad j = 0, \ldots, N,$$
$$h_{j-1} := x_j - x_{j-1}, \quad j = 1, \ldots, N.$$

We also consider $x_{-1} := x_0$, $x_{N+1} := x_N$, $h_{-1} := h_N := 0$. Let Λ be a sequence of mesh sizes $h = (h_0, \ldots, h_{N-1})$ such that

$$h_{\max} := \max\{h_{j-1}, j = 1, \ldots, N\}$$

converges to zero.

Let $\overset{\circ}{L}{}_h^p$, $\overset{\circ}{W}{}_h^{1,p}$ and $W_h^{2,p} \cap \overset{\circ}{W}{}_h^{1,2}$, respectively, be the spaces of grid functions on $\overline{S}_h$, which are zero in 0 and R, equipped with the norms $\|\cdot\|_{0,p,h}$, $\|\cdot\|_{1,p,h}$ and $\|\cdot\|_{2,p,h}$, respectively, where, if $p \in [1, \infty[$ then

$$\|v_h\|_{m,p,h} := \left(\sum_{\ell=0}^{m} |v_h|_{\ell,p,h}^p \right)^{1/p},$$

with

$$|v_h|_{0,p,h}^p := \sum_{j=1}^{N} h_{j-1} |v_{j-1/2}|^p,$$

$$|v_h|_{1,p,h}^p := \sum_{j=0}^{N} h_{j-1/2} |(\delta v_h)_j|^p,$$

$$|v_h|_{2,p,h}^p := \sum_{j=1}^{N} h_{j-1} |(\delta^2 v_h)_{j-1/2}|^p.$$

If $p = \infty$ then

$$\|v_h\|_{m,\infty,h} := \max_{0 \le \ell \le m} |v_h|_{\ell,\infty,h},$$

where

$$|v_h|_{0,\infty,h} := \max_{1 \le j \le N} |(v_h)_{j-1/2}|,$$

$$|v_h|_{1,\infty,h} := \max_{0 \le j \le N} |(\delta v_h)_j|,$$

$$|v_h|_{2,\infty,h} := \max_{1 \le j \le N} |(\delta^2 v_h)_{j-1/2}|.$$

The pointwise restriction of the function v to the grid S_h will be denoted by $R_h v$.

The symbol $\|\cdot\|_{0,p,h}$ does not always represent a norm in spaces of grid functions. In order to overcome this fact, we restrict the definition to spaces of grid functions that are zero in $\{0, R\}$. Sometimes, for the easiness of the writing, we use the notation $\overset{\circ}{W}{}_h^{0,p}$ for $\overset{\circ}{L}{}_h^p$. The space $\overset{\circ}{L}{}_h^2$ is endowed by the inner product

$$(v_h, w_h)_h := \sum_{j=1}^{N} h_{j-1} v_{j-1/2} w_{j-1/2},$$

which is a discrete version of the usual $L^2(0, R)$-inner product, $(\cdot, \cdot)_0$.

The discrete spaces introduced $\overset{\circ}{W}{}_h^{m,p}$, $1 \leq p < \infty$, and $\overset{\circ}{W}{}_h^{m,\infty}$, $m = 0, 1$, form discrete approximations to $W_0^{m,p}(0, R)$ and $C^m(0, R)$, respectively, in the sense explained in what follows ([11, 12]). A sequence $(v_h)_{h\in\Lambda}$ is said to converge discretely in $(L^p(0, R), \Pi \overset{\circ}{L}{}_h^p)$ to an element $v \in L^p(0, R)$,

$$v_h \to v \text{ in } \left(L^p(0, R), \Pi \overset{\circ}{L}{}_h^p\right) \quad (h \in \Lambda),$$

if for each $\epsilon > 0$ there exists $\varphi \in C^\infty(0, R)$ such that

$$\|v - \varphi\|_{L^p(0,R)} \leq \epsilon, \quad \limsup\{\|v_h - R_h\varphi\|_{0,p,h}, h \in \Lambda\} \leq \epsilon;$$

it is said to converge discretely in $(W_0^{1,p}(0, R), \Pi \overset{\circ}{W}{}_h^{1,p})$ to an element $v \in W_0^{1,p}(0, R)$,

$$v_h \to v \text{ in } \left(W_0^{1,p}(0, R), \Pi \overset{\circ}{W}{}_h^{1,p}\right) \quad (h \in \Lambda),$$

if for each $\epsilon > 0$ there exists $\varphi \in C^\infty(0, R)$ such that

$$\|v - \varphi\|_{W^{1,p}(0,R)} \leq \epsilon, \quad \limsup\{\|v_h - R_h\varphi\|_{1,p,h}, h \in \Lambda\} \leq \epsilon;$$

it is said to converge discretely in $(C(0, R), \Pi \overset{\circ}{L}{}_h^\infty)$ to an element $v \in C(0, R)$,

$$v_h \to v \text{ in } \left(C(0, R), \Pi \overset{\circ}{L}{}_h^\infty\right) \quad (h \in \Lambda),$$

if

$$\|v_h - R_h v\|_{0,\infty,h} \to 0 \quad (h \in \Lambda);$$

it is said to converge discretely in $(C^1(0, R), \Pi \overset{\circ}{W}{}_h^{1,\infty})$ to an element $v \in C^1(0, R)$,

$$v_h \to v \text{ in } \left(C^1(0, R), \Pi \overset{\circ}{W}{}_h^{1,\infty}\right) \quad (h \in \Lambda),$$

if

$$\|v_h - R_h v\|_{1,\infty,h} \to 0 \quad (h \in \Lambda).$$

A sequence $(v_h)_\Lambda$ converges weakly to v in $(L^2(0,R), \Pi \mathring{L}_h^2)$, $v_h \rightharpoonup v$ in $(L^2(0,R), \Pi \mathring{L}_h^2)(H \in \Lambda)$, if

$$(w_h, v_h)_h \to (w, v)_0 \quad (h \in \Lambda)$$

for all $w \in L^2(0,R)$ and for all $(w_h)_\Lambda \in \Pi \mathring{L}_h^2$ such that $w_H \to w$ in $(L^2(0,R), \Pi \mathring{L}_h^2)$.

In main result of this section, Theorem 2.1, asserts that every bounded sequence in $\mathring{W}_h^{1,p}$ has a convergent subsequence in $(L^q(0,R), \Pi \mathring{L}_h^q)$ and in $(C(0,R), \Pi \mathring{L}_h^\infty)$ and that every bounded sequence in $W_h^{2,p} \cap \mathring{W}_h^{1,p}$ has a convergent subsequence in $(W_0^{1,q}(0,R), \Pi \mathring{W}_h^{1,q})$ and in $(C^1(0,R), \Pi \mathring{W}_h^{1,\infty})$, with p, q satisfying certain conditions.

Corresponding natural embeddings in Sobolev spaces are given by the Rellich–Kondrachov theorem (see, e.g., [1]).

Theorem 2.1. *The sequence of natural embeddings*

$$J_h : \mathring{W}_h^{1,p} \to L_h^q \quad \textit{and} \quad J_h : W_h^{2,p} \cap \mathring{W}_h^{1,p} \to \mathring{W}_h^{1,q}, \quad h \in \Lambda,$$

for $1 \le p, q \le \infty$, *with* $q < \infty$ *if* $p = 1$, *are discretely compact.*

Proof. We first consider the embedding

$$J_h : \mathring{W}_h^{1,p} \to \mathring{L}_h^q, \quad h \in \Lambda.$$

Let $(v_h)_\Lambda \in \Pi \mathring{W}_h^{1,p}$ be a bounded sequence. For each $h \in \Lambda$, we consider the function w_h, which is linear in each interval $[x_{j-1/2}, x_{j+1/2}], j = 0, \ldots, N$, satisfying

$$w_h(x_{j-1/2}) := v_h(x_{j-1/2}), \quad j = 0, \ldots, N+1.$$

Then the sequence $(w_h)_\Lambda \in \Pi W_0^{1,p}(0,R)$ is bounded in $\Pi W_0^{1,p}(0,R)$. In fact, if $p = \infty$, then

$$\|w_h\|_{W^{1,p}(0,R)} = \|v_h\|_{1,p,h}.$$

If $1 \leq p < \infty$, then

$$|w_h|_{W^{1,p}(0,R)} = \left(\int_0^R |w'(x)|^p \, dx \right)^{1/p} = \left(\sum_{j=0}^{N} h_{j-1/2} |(\delta v_h)_j|^p \right)^{1/p} = |v_h|_{1,p,h}$$

and, by Friedrich's inequality, there exists a constant C, depending only on R and p, such that

$$\|w_h\|_{W^{1,p}(0,R)} \leq C |w_h|_{W^{1,p}(0,R)}.$$

We note that the embedding $W^{1,p}(0,R) \to C[0,R]$, $1 < p < \infty$, is compact and consequently we can find a subsequence $\Lambda' \subseteq \Lambda$ and a function $w \in C(0,R)$ such that

$$\max_{x \in [0,R]} |w_h(x) - w(x)| \to 0 \quad (h \in \Lambda').$$

Hence,

$$v_h \to w \text{ in } (C(0,R), \Pi \mathring{L}_h^\infty) \quad (h \in \Lambda').$$

Because for $1 \leq q < \infty$

$$\|v_h - R_h w\|_{0,q,h} \leq R^{1/q} \|v_h - R_h w\|_{0,\infty,h},$$

we conclude the convergence

$$v_h \to w \text{ in } (L^q(0,R), \Pi \mathring{L}_h^q) \quad (h \in \Lambda').$$

Let us now consider $p = 1$. The embedding $W^{1,1}(0,R) \to L^q(0,R)$ is compact and subsequently

$$w_h \to w \text{ in } L^q(0,R) \quad (h \in \Lambda')$$

for some subsequence $\Lambda' \subseteq \Lambda$ and $w \in L^q(0,R)$. We are now going to prove that $v_h \to w$ in $(L^q(0,R), \Pi \mathring{L}_h^q)(h \in \Lambda')$. For each $\epsilon > 0$ and $c > 0$, it is possible to find $\varphi \in C_0^\infty(0,R)$ such that

$$\|w - \varphi\|_{L^q(0,R)} \leq \frac{\epsilon}{c}.$$

Let us consider the function ψ_h, which is linear in each interval $[x_{j-1/2}, x_{j+1/2}]$, $j = 0, \ldots, N$, and satisfies

$$\psi_h(x) := \varphi(x_{j-1/2}), \quad j = 0, \ldots, N+1.$$

There exists a constant $C > 0$, depending on R and q, such that

$$C^q \|v_h - R_h\varphi\|^q_{0,q,h} \leq \int_0^R |w_h - \psi_h|^q \, dx.$$

Because $\psi_h \to \varphi$ in $L^q(0, R)$, then

$$\int_0^R |w_h - \psi_h|^q \, dx \to \int_0^R |w - \varphi|^q \, dx \quad (h \in \Lambda').$$

Consequently, taking $c = C$, holds

$$\limsup\{\|v_h - R_h\varphi\|_{0,q,h}, h \in \Lambda'\} \leq \epsilon.$$

This concludes the first part of the proof, i.e., the sequence $J_h : \mathring{W}_h^{1,p} \to \mathring{L}_h^q$, $h \in \Lambda$, is discretely compact.

We consider now the sequence of embeddings $(J_h)_\Lambda$, $J_h : W_h^{2,p} \cap \mathring{W}_h^{1,p} \to \mathring{W}_h^{1,q}$. Let $(v_h)_\Lambda \in \Pi(W_h^{2,p} \cap \mathring{W}_h^{2,p})$ be bounded. The sequence $(w_h)_\Lambda$, where w_h is defined by

$$w_h(x_j) := (\delta v_h)_j, \quad j = 0, \ldots, N,$$

linear in each interval $[x_j, x_{j+1}]$, $j = 0, \ldots, N-1$, is bounded in $\Pi W^{1,p}(0, R)$. For $p > 1$, then the Rellich–Kondrachov theorem gives the existence of $\Lambda' \subset \Lambda$ and $w_1 \in C[0, R]$ such that

$$\max_{0 \leq j \leq N} |(\delta v_h)_j - w_1(x_j)| \to 0 \quad (h \in \Lambda').$$

Let $w_0(x) := \int_0^x w_1(t)\,dt$. Taking $v_0 = w_0(0) = 0$ into account, we have

$$\begin{aligned} |v_{j-1/2} - w_0(x_{j-1/2})| &\leq \sum_{i=0}^{N} |(\delta v)_i - w_1(x_i)| + \sum_{i=0}^{N} |w_1(x_i) - (\delta w_0)_i| \\ &\leq R \max_{0 \leq i \leq N} |(\delta v)_i - w_1(x_i)| + R \max_{0 \leq i \leq N} |w_0'(x_i) - (\delta w_0)_i|, \end{aligned}$$

$j = 0, \ldots, N+1$. Hence the convergence

$$\max_{0 \leq j \leq N+1} |v_{j-1/2} - w_0(x_{j-1/2})| \to 0 \quad (h \in \Lambda''),$$

follows and we conclude that

$$v_h \to w_0 \quad (h \in \Lambda') \text{ in } (C^1(0, R), \Pi\, \mathring{W}_h^{1,\infty}).$$

For the case $p = 1$, the proof is analogous. □

The next lemma is helpful in the proof of the compactness embedding theorem for the two-dimensional case.

Lemma 2.2. *Let $(v_h)_\Lambda \in \Pi\, \overset{\circ}{W}{}_h^{1,p}$ be a bounded sequence, with $1 \le p < \infty$. For any $\tau \in \mathbb{R}$, the step function defined by*

$$w_h(x) := v_h(x_{j-1/2}), \quad x \in]x_{j-1}, x_j], \quad j = 1, \ldots, N,$$

and zero outside of these intervals, satisfies

$$\int_I |w_h(x+\tau) - w_h(x)|^p \, dx \le 3(|\tau| + h_{\max})^p |v_h|_{1,p,h}^p, \tag{2.1}$$

where I is any interval containing (x_0, x_N).

Proof. Let $\tau > 0$. Then

$$\int_I |w_h(x+\tau) - w_h(x)|^p \, dx \le \int_0^{R-\tau} |w_h(x+\tau) - w_h(x)|^p \, dx + \int_{-\tau}^0 |w_h(x+\tau)|^p \, dx + \int_{R-\tau}^R |w_h(x)|^p \, dx.$$

For f defined by

$$f(x) := j, \quad x \in]x_{j-1}, x_j],$$

we have, using Hölder's inequality,

$$\int_0^{R-\tau} |w_h(x+\tau) - w_h(x)|^p \, dx \le \int_0^{R-\tau} \left(\sum_{k=f(x)}^{f(x+\tau)-1} |w_h(x_{k+1/2}) - w_h(x_{k-1/2})| \right)^p dx$$
$$\le \int_0^{R-\tau} \left(\sum_{k=f(x)}^{f(x+\tau)-1} h_{k-1/2} \right)^{p-1} \sum_{k=f(x)}^{f(x+\tau)-1} h_{k-1/2} |(\delta v_h)_k|^p \, dx.$$

Because $\sum_{k=f(x)}^{f(x+\tau)-1} h_{k-1/2} \le \tau + h_{\max}$, we obtain

$$\int_0^{R-\tau} |w_h(x+\tau) - w_h(x)|^p \, dx \le (\tau + h_{\max})^{p-1} \sum_{j=0}^{N_{R-\tau}} \left(h_{j-1} \sum_{k=j}^{f(x_j+\tau)-1} h_{k-1/2} |(\delta v_h)_k|^p \right)$$
$$\le (\tau + h_{\max})^{p-1} \sum_{k=0}^{N} \left(h_{k-1/2} |(\delta v_h)_k|^p \sum_{j=s(k)}^{k} h_{j-1} \right),$$

where $N_{R-\tau}$ and $s(k)$, are the biggest integer and the smallest integers, respectively, such that $\sum_{i=1}^{N_{R-\tau}} h_{i-1} \leq R - \tau$ and $f(x_{s(k)} + \tau) - 1 \geq k$. From

$$x_k - x_{s(k)} < \tau \quad \sum_{j=s(k)}^{k} h_{j-1} < \tau + h_{\max},$$

we conclude that

$$\int_0^{R-\tau} |w_h(x+\tau) - w_h(x)|^p \, dx \leq (\tau + h_{\max})^p |v_h|_{1,p,h}^p. \tag{2.2}$$

On the other hand,

$$\int_{-\tau}^0 |w_h(x+\tau)|^p \, dx = \int_0^\tau |w_h(x)|^p \, dx \leq \sum_{j=1}^{N_\tau} h_{j-1} |w_h(x_{j-1/2})|^p,$$

with N_τ the smallest integer such that $\sum_{i=1}^{N_\tau} h_{i-1} \geq \tau$, and then

$$\begin{aligned}\int_{-\tau}^0 |w_h(x+\tau)|^p \, dx &\leq \sum_{j=1}^{N_\tau} h_{j-1} \left(\sum_{k=0}^{j-1} |v_h(x_{k+1/2}) - v_h(x_{k-1/2})| \right)^p \\ &= \sum_{j=1}^{N_\tau} h_{j-1} \left(\sum_{k=0}^{j-1} h_{k-1/2} |(\delta v_h)_k| \right)^p.\end{aligned}$$

Because $\sum_{j=1}^{N_\tau} h_{j-1} \leq \tau + h_{\max}$ and for $j \leq N_\tau$, $\sum_{k=0}^{j-1} h_{k-1/2} \leq \tau + h_{\max}$, it follows by an application of Hölder's inequality

$$\int_{-\tau}^0 |w_h(x+\tau)|^p \, dx \leq (\tau + h_{\max})^p |v_h|_{1,p,h}^p. \tag{2.3}$$

In the same way as before, we have

$$\int_{R-\tau}^R |w_h(x)|^p \, dx \leq \sum_{j=N_{R-\tau}}^{N} h_{j-1} \left(\sum_{k=j}^{N} h_{k-1/2} |(\delta v_h)_k| \right)^p$$

and consequently

$$\int_{R-\tau}^R |w_h(x)|^p \leq (\tau + h_{\max})^p |v_h|_{1,p,h}^p. \tag{2.4}$$

From (2.2)–(2.4) we obtain (2.1).

The case $\tau < 0$ can be proved analogously. $\square$

The next lemma gives some more information about the embeddings considered in the Theorem 2.1. Corresponding results for spaces of continuous functions are well-known (see, e.g., [2], Theorem 3.12).

Lemma 2.3. *If* $(v_h)_\Lambda \in \Pi \overset{\circ}{W}{}_h^{1,2}$ *is bounded and weakly convergent to* v *in* $(L^2(0,R), \Pi \overset{\circ}{L}{}_h^2)$, *then* $v \in W_0^{1,2}(0,R)$.

We address the demonstration to the proof of Lemma 3.4, where a correspondent result for two-dimensional domains is considered.

3. DISCRETE APPROXIMATION OF $L^p(\Omega)$ AND $W_0^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}^2$

We now need norms for functions on two-dimensional grids. To this end we introduce discrete versions of the Sobolev spaces $W_0^{m,p}(\Omega)$, $m = 0, 1$, $p \in [1, \infty[$, where Ω is a union of rectangles.

Let us first introduce the nonuniform grid G_H. In a rectangle $R = (x_{-1}, x_{N+1}) \times (y_{-1}, y_{M+1})$ that contains Ω, we define the subset $G_H := R_1 \times R_2$, where

$$R_1 := \{x_{-1} < x_0 < \cdots < x_N < x_{N+1}\}$$

and

$$R_2 := \{y_{-1} < y_0 < \cdots < y_M < y_{M+1}\}.$$

The grid G_H is assumed to satisfy the following condition: the vertices of Ω are in the centers of the rectangles formed by G_H.

In the case of a rectangular domain $\Omega = (x_0, x_N) \times (y_0, y_M)$, we allow both $R = \Omega$ and $R \supset \Omega$, i.e., we consider $x_{-1} \leq x_0$, $x_N \leq x_{N+1}$, $y_{-1} \leq y_0$ and $y_M \leq y_{M+1}$.

Let

$$S_H := \{(x_{j-1/2}, y_{\ell-1/2}) : j = 0, \ldots, N+1, \ell = 0, \ldots, M+1\},$$

where $x_{j-1/2} := (x_{j-1} + x_j)/2$, $y_{\ell-1/2} := (y_{\ell-1} + y_\ell)/2$, and $\Omega_H := S_H \cap \Omega$, $\partial\Omega_H := S_H \cap \partial\Omega$, $\overline{\Omega}_H := \Omega_H \cup \partial\Omega_H$.

In the definition of the discrete norms, we use the following centered divided differences in x-direction

$$(\delta_x v_H)_{j,\ell+1/2} := \frac{v_{j+1/2,\ell+1/2} - v_{j-1/2,\ell+1/2}}{h_{j-1/2}},$$

$$(\delta_x w_H)_{j-1/2,\ell+1/2} := \frac{w_{j,\ell+1/2} - w_{j-1,\ell+1/2}}{h_{j-1}},$$

where $h_{j-1/2} := x_{j+1/2} - x_{j-1/2}$, $h_{j-1} := x_j - x_{j-1}$. Correspondingly, the finite central difference with respect to the variable y are defined, with the mesh size vector k in place of h.

We denote by $\overset{\circ}{W}{}_H^{m,p}(R)$, $m = 0, 1$, $p \in [1, \infty[$, the space of grid functions defined in C_H that are zero on the set

$$\{(x_{j-1/2}, y_{\ell-1/2}) : j = 0, N+1, \ell = 0, \ldots, M+1 \vee j = 1, \ldots, N, \ell = 0, M+1\},$$

and equipped with the norm

$$\|v_H\|_{W_H^{m,p}(R)} := \left(\sum_{r=0}^{m} |v_H|_{r,H}^p \right)^{1/p},$$

where

$$|v_H|_{0,p,H}^p := \sum_{j=1}^{N} \sum_{\ell=1}^{M} h_{j-1} k_{\ell-1} |v_{j-1/2,\ell-1/2}|^p,$$

$$|v_H|_{1,p,H}^p := \sum_{j=0}^{N} \sum_{\ell=1}^{M} h_{j-1/2} k_{\ell-1} |(\delta_x v_H)_{j,\ell-1/2}|^p$$
$$+ \sum_{j=1}^{N} \sum_{\ell=0}^{M} h_{j-1} k_{\ell-1/2} |(\delta_y v_H)_{j-1/2,\ell}|^p.$$

Let P_{S_H} be the following operator that extends a grid function v_H in $\overline{\Omega}_H$ to S_H,

$$P_{S_H} v_H := v_H \quad \text{in } \overline{\Omega}_H, \quad P_{S_H} v_H := 0 \quad \text{in } S_H \backslash \overline{\Omega}_H.$$

We denote by $\overset{\circ}{W}{}_H^{m,p}$, $m = 0, 1$, the space of functions defined in $\overline{\Omega}_H$ that are zero on $\partial\Omega_H$, equipped with the norm

$$\|v_H\|_{m,p,H} := \left(\sum_{r=0}^{m} |P_{S_H} v_H|_{r,H}^p \right)^{1/p}, \quad m = 0, 1.$$

The space $\overset{\circ}{W}{}_H^{0,p}$ is also denoted by $\overset{\circ}{L}{}_H^p$. $\overset{\circ}{L}{}_H^2$ is endowed by the inner product

$$(v_H, w_H)_H := \sum_{j=1}^{N} \sum_{\ell=1}^{M} h_{j-1} k_{\ell-1} (P_{S_H} v_H)_{j-1/2,\ell-1/2} (P_{S_H} \bar{w}_H)_{j-1/2,\ell-1/2}.$$

When it is clear from the context that we use the extended function, we omit the notation P_{S_H}.

Let R_H be the operator that defines the restriction to $\overline{\Omega}_H$.

The discrete spaces introduced above form discrete approximations of their continuous counterparts in the sense that we explain in what follows.

Let Λ be a sequence of positive vectors $H = (h, k)$ of step-sizes such that the maximum step-size $H_{\max}$ converges to zero. A sequence $(v_H)_\Lambda \in \Pi \overset{\circ}{L}{}_H^p$ converges discretely to $v \in L^p(\Omega)$ in $(L^p(\Omega), \Pi \overset{\circ}{L}{}_H^p)$, $v_H \to v$ in $(L^p(\Omega), \Pi \overset{\circ}{L}{}_H^p)$ $(H \in \Lambda)$, if for each $\epsilon > 0$ there exists $\varphi \in C^\infty(\Omega)$ such that

$$\|v - \varphi\|_{L^p(\Omega)} \leq \epsilon, \quad \lim_{H_{\max}\to 0} \sup\{\|v_H - R_H\varphi\|_{0,p,H}\} \leq \epsilon.$$

A sequence $(v_H)_\Lambda \in \Pi \overset{\circ}{W}{}_H^{1,p}$ converges discretely to $v \in W_0^{1,p}(\Omega)$ in $(W_0^{1,p}(\Omega), \Pi \overset{\circ}{W}{}_H^{1,p})$, $v_H \to v$ in $(W_0^{1,p}(\Omega), \Pi \overset{\circ}{W}{}_H^{1,p})$ $(H \in \Lambda)$, if for each $\epsilon > 0$ there exists $\varphi \in C^\infty(\Omega)$ such that

$$\|v - \varphi\|_{W^{1,p}(\Omega)} \leq \epsilon, \quad \lim_{H_{\max}\to 0} \sup\{\|v_H - R_H\varphi\|_{1,p,H}\} \leq \epsilon.$$

A sequence $(v_H)_\Lambda$ converges weakly to v in $(L^2(\Omega), \Pi \overset{\circ}{L}{}_H^2)$, $v_H \rightharpoonup v$ in $(L^2(\Omega), \Pi \overset{\circ}{L}{}_H^2)$ $(H \in \Lambda)$, if

$$(w_H, v_H)_H \to (w, v)_0 \quad (H \in \Lambda)$$

for all $w \in L^2(\Omega)$ and $(w_H)_\Lambda \in \Pi \overset{\circ}{L}{}_H^2$ such that $w_H \to w$ in $(L^2(\Omega), \Pi \overset{\circ}{L}{}_H^2)$.

The following theorem was proved by Stummel in [11].

Theorem 3.1. *Let $(v_H)_\Lambda$ be a bounded sequence in $\Pi \overset{\circ}{L}{}_H^2$. Then, there exists a subsequence Λ' of Λ and $v \in L^2(\Omega)$, such that*

$$v_H \rightharpoonup v \quad \text{in } (L^2(\Omega), \Pi \overset{\circ}{L}{}_H^2) \quad (H \in \Lambda').$$

The proof of the discrete compactness result that we present in the following is based in the Kolmogorov compactness theorem ([1, 13]) and uses the next lemma.

Lemma 3.2. *Let $(v_H)_\Lambda \in \Pi \overset{\circ}{W}{}_H^{1,p}$ be a bounded sequence, with $1 \leq p < \infty$. Let us consider the step function w_H defined by*

$$w_H(x, y) := v_{j+1/2,\ell+1/2}, \quad (x, y) \in (x_j, x_{j+1}) \times (y_\ell, y_{\ell+1}) \subset \Omega, \tag{3.1}$$

and zero on $\mathbb{R}^2\backslash\Omega$. *Let* Q *be a set containing* Ω. *Then, for all* $\tau = (\tau_1, \tau_2) \in \mathbb{R}^2$ *the following estimate holds*

$$\int_Q |w_H(x+\tau_1, y+\tau_2) - w_H(x,y)|^p \, dx \, dy \\ \leq \frac{3}{2} 2^p (|\tau_1| + |\tau_2| + h_{\max} + k_{\max})^p |v_H|^p_{1,p,H}. \tag{3.2}$$

Proof. For $\tau = (\tau_1, \tau_2) \in \mathbb{R}^2$ holds

$$\int_Q |w_H(x+\tau_1, y+\tau_2) - w_H(x,y)|^p \, dx \, dy \\ \leq 2^{p-1} \int_Q |w_H(x+\tau_1, y+\tau_2) - w_H(x, y+\tau_2)|^p \, dx \, dy \\ + 2^{p-1} \int_Q |w_H(x, y+\tau_2) - w_H(x,y)|^p \, dx \, dy.$$

Because

$$\int_Q |w_H(x+\tau_1, y+\tau_2) - w_H(x, y+\tau_2)|^p \, dx \, dy \\ \leq \sum_{\ell=1}^{M} k_{\ell-1} \int_{x_0-\tau_1}^{x_{N+\tau_1}} |w_H(x+\tau_1, y_{\ell-1/2}) - w_H(x, y_{\ell-1/2})|^p \, dx,$$

then from Lemma 2.2, we obtain

$$\int_Q |w_H(x+\tau_1, y+\tau_2) - w_H(x, y+\tau_2)|^p \, dx \, dy \\ \leq 3(|\tau_1| + h_{\max})^p \sum_{\ell=1}^{M} k_{\ell-1} \sum_{j=0}^{N} h_{j-1/2} |(\delta_x(P_{C_H} v_H))_{j,\ell-1/2}|^p \\ \leq 3(|\tau_1| + h_{\max})^p |v_H|^p_{1,p,H}.$$

Analogously,

$$\int_Q |w_H(x, y+\tau_2) - w_H(x,y)|^p \, dx \, dy \leq 3(|\tau_2| + k_{\max})^p |v_H|^p_{1,p,H}.$$

We conclude that

$$\int_Q |w_H(x+\tau_1, y+\tau_2) - w_H(x,y)|^p \, dx \, dy \\ \leq \frac{3}{2} \times 2^p [(|\tau_1| + h_{\max})^p + (|\tau_2| + k_{\max})^p] |v_H|^p_{1,p,H}. \qquad \square$$

Theorem 3.3. *The sequence of embeddings $(J_H)_\Lambda$,*

$$J_H : \overset{\circ}{W}{}_H^{1,p} \to \overset{\circ}{L}{}_H^{p} \quad (H \in \Lambda), \tag{3.3}$$

$1 \le p < \infty$, *is discretely compact.*

Proof. Let $(v_H)_\Lambda \in \Pi\, \overset{\circ}{W}{}_H^{1,p}$ be a bounded sequence. There exists M independent of H such that

$$\|v_H\|_{1,p,H} \le M.$$

For $(w_H)_\Lambda$ defined by (3.1)

$$\int_\Omega |w_H(x+\eta_1, y+\eta_2) - w_H(x,y)|^p \, dx\, dy \le C(|\tau_1| + |\tau_2| + h_{\max} + k_{\max})^p M^p.$$

Because

$$\|w_H\|_{L^p(\Omega)} = |v_H|_{0,p,H} \le M,$$

then $(w_H)_\Lambda$ is uniformly bounded in $\Pi L^p(\Omega)$. Using the Kolmogorov compactness theorem, we conclude that the sequence $(w_H)_\Lambda$ is relatively compact in $L^p(\Omega)$. There exists a sequence $\Lambda' \subseteq \Lambda$ and $w \in L^p(\Omega)$ such that

$$w_H \to w \text{ in } L^p(\Omega) \quad (H \in \Lambda').$$

In order to conclude the proof, we need to prove that

$$v_H \to w \text{ in } (L^p(\Omega), \Pi\, \overset{\circ}{L}{}_H^{p}) \quad (H \in \Lambda').$$

Let $\epsilon > 0$. There exists $\varphi \in C_0^\infty(\Omega)$ such that

$$\|w - \varphi\|_{L^p(\Omega)} \le \epsilon.$$

For the step function ψ_H defined by

$$\psi_H(x,y) := \varphi(x_{j+1/2}, y_{\ell+1/2}), \quad (x,y) \in (x_j, x_{j+1}) \times (y_\ell, y_{\ell+1}) \subset \Omega,$$

null otherwise, we have

$$\|\psi_H - \varphi\|_{L^p(\Omega)} \to 0 \quad (H \in \Lambda)$$

and then

$$\|v_H - R_H\varphi\|_{0,p,H} = \|w_H - \psi_H\|_{L^p(\Omega)} \to \|w - \varphi\|_{L^p(\Omega)}. \qquad \square$$

Lemma 3.4. *If* $(v_H)_\Lambda \in \Pi \overset{\circ}{W}{}_H^{1,2}$ *is bounded and weakly convergent to* v *in* $(L^2(\Omega), \Pi \overset{\circ}{L}{}_H^2)$*, then* $v \in W_0^{1,2}(\Omega)$.

Proof. Let $(v_H)_\Lambda$ be a bounded sequence in $\Pi \overset{\circ}{W}{}_H^{1,2}$ such that

$$v_H \rightharpoonup v \text{ in } (L^2(\Omega), \Pi \overset{\circ}{L}{}_H^2) \quad (H \in \Lambda). \tag{3.4}$$

We consider $(w_H)_\Lambda$ from Lemma 3.2. From the proof of the last theorem, we know that $(w_H)_\Lambda$ converges to $w \in L^2(\Omega)$, for some $\Lambda' \subseteq \Lambda$. Let us consider the sequence $(\tilde{w}_H)_{\Lambda'}$ defined by

$$\tilde{w}_H := w_H \quad \text{in } \Omega, \quad \tilde{w}_H := 0 \quad \text{in } \mathbb{R}^2 \backslash \Omega,$$

and the prolongation to $\mathbb{R}^2$ of w

$$\tilde{w} := w \quad \text{in } \Omega, \quad \tilde{w} := 0 \quad \text{in } \mathbb{R}^2 \backslash \Omega.$$

We note that $\tilde{w}_H \to \tilde{w}$ in $L^2(\mathbb{R}^2)$ $(H \in \Lambda')$. For $\varphi \in C_0^\infty(\mathbb{R}^2)$ and all $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$, $\eta \neq 0$, from Lemma 3.2 we have

$$\int_{\mathbb{R}^2} |(\tilde{w}_H(x+\eta_1, y+\eta_2) - \tilde{w}_H(x,y))\varphi(x,y)|\,dx\,dy \leq C(|\eta| + H_{\max})\|\varphi\|_{L^2(\mathbb{R}^2)}.$$

Taking the limit when $H_{\max} \to 0$, results in

$$\int_{\mathbb{R}^2} |(\tilde{w}_H(x+\eta_1, y+\eta_2) - \tilde{w}_H(x,y))\varphi(x,y)|\,dx\,dy \leq C|\eta|\|\varphi\|_{L^2(\mathbb{R}^2)},$$

and consequently

$$\int_{\mathbb{R}^2} \frac{|\varphi(x-\eta_1, y-\eta_2) - \varphi(x,y)|}{|\eta|}|\tilde{w}(x,y)|\,dx\,dy \leq C\|\varphi\|_{L^2(\mathbb{R}^2)}.$$

Considering $\eta = \varepsilon(1,0)$ and the limit $\varepsilon \to 0$, we conclude that

$$\int_{\mathbb{R}^2} |\varphi_x(x,y)\tilde{w}(x,y)|\,dx\,dy \leq C\|\varphi\|_{L^2(\mathbb{R}^2)},$$

for $\varphi \in C_0^\infty(\mathbb{R}^2)$. Analogously, taking $\eta = \varepsilon(0,1)$, we obtain

$$\int_{\mathbb{R}^2} |\varphi_y(x,y)\tilde{w}(x,y)|\,dx\,dy \leq C\|\varphi\|_{L^2(\mathbb{R}^2)},$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2)$. Consequently, $\tilde{w} \in W^{1,2}(\mathbb{R}^2)$. Because w is a restriction of $\tilde{w}$ to Ω and $\tilde{w} = 0$ in $\mathbb{R}^2 \backslash \Omega$, then $w \in W_0^{1,2}(\Omega)$.

Let us finally prove that $v = w$. Let $r \in L^2(\Omega)$ and $(r_H)_\Lambda \in \Pi \overset{\circ}{L}{}_H^2$, such that $r_H \to r$ in $(L^2(\Omega), \Pi \overset{\circ}{L}{}_H^2)$ $(H \in \Lambda)$. For the step function defined by

$$s_H(x,y) := r_H(x_{j+1/2}, y_{\ell+1/2}), \quad (x,y) \in (x_j, x_{j+1}) \times (y_\ell, y_{\ell+1}) \subset \Omega,$$

zero in $\mathbb{R}^2 \backslash \Omega$, we have

$$(v_H, r_H)_H = (w_H, s_H)_0 \to (w, r)_0 \quad (H \in \Lambda).$$

Finally,

$$v_H \rightharpoonup w \text{ in } \left(L^2(\Omega), \Pi \overset{\circ}{L}{}^2_H\right) \quad (H \in \Lambda).$$

Considering (3.4), we conclude that $v = w$. $\square$

ACKNOWLEDGMENTS

The research work compiled in this paper benefited from the suggestions of Rolf Dieter Grigorieff to whom the author wishes to express sincere gratitude. The author also thanks the helpful comments of José Augusto Ferreira and the paper reviewers.

The author gratefully acknowledges the support of this work by the Centro de Matemática da Universidade de Coimbra and Fundação para a Ciência e Tecnologia.

REFERENCES

1. R.A. Adams (1975). *Sobolev Spaces*. Academic Press, New York.
2. S. Agmon (1965). *Lectures on Elliptic Boundary Value Problems*. Van Nostrand Mathematical Studies. Van Nostrand, New York.
3. S. Barbeiro (2007). Supraconvergent cell-centered scheme for two dimensional elliptic problems. *Appl. Num. Math.* (in press).
4. J.A. Ferreira (1997). On the convergence on nonrectangular grids. *J. Comp. Appl. Math.* 85: 333–344.
5. J.A. Ferreira and R.D. Grigorieff (1998). On the supraconvergence of elliptic finite difference schemes. *Appl. Num. Math.* 28:275–292.
6. J.A. Ferreira and R.D. Grigorieff (2006). Supraconvergence and supercloseness of a scheme for elliptic equations on non-uniform grids. *Numer. Funct. Anal. Optim.* 27(5–6):539–564.
7. R.D. Grigorieff (1972). Diskret kompakte Einbettungen in Sobolewschen Räumen. *Math. Ann.* 197:71–85.
8. R.D. Grigorieff (1981/1982). Zur diskreten Kompaktheit von Funktionen auf nichtäquidistanten Gittern in $\mathbb{R}$. *Numer. Funct. Anal. Optim.* 4:383–395.
9. R.D. Grigorieff (1988). Some stability inequalities for compact finite difference schemes. *Math. Nachr.* 135:93–101.
10. H.J. Reinhardt (1985). *Analysis of Approximation Methods for Differential and Integral Equations*. Applied Mathematical Sciences 57. Springer, New York.
11. F. Stummel (1970). Diskrete konvergenz linearer Operatoren. *I. Math. Ann.* 190:45–92.
12. F. Stummel (1972). *Discrete Convergence of Mappings*. Topics in Numerical Analysis. Academic Press, London, pp. 285–310.
13. J. Wloka (1987). *Partial Differential Equations*. Cambridge University Press, Cambridge.