CARTESIAN CLOSED EXACT COMPLETIONS IN TOPOLOGY

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Abstract. Using generalized enriched categories, in this paper we show that Rosický's proof of cartesian closedness of the exact completion of the category of topological spaces can be extended to a wide range of topological categories over **Set**, like metric spaces, approach spaces, ultrametric spaces, probabilistic metric spaces, and bitopological spaces. In order to do so we prove a sufficient criterion for exponentiability of $(\mathbb{T},V)$-categories and show that, under suitable conditions, every injective $(\mathbb{T},V)$-category is exponentiable in $(\mathbb{T},V)$-**Cat**.

1. Introduction

As Lawvere has shown in his celebrated paper [Law73], when V is a closed category the category V-**Cat** of V-enriched categories and V-functors is also monoidal closed. This result extends neither to the cartesian structure nor to the more general setting of $(\mathbb{T},V)$-categories. Indeed, cartesian closedness of V does not guarantee cartesian closedness of V-**Cat**: take for instance the category of (Lawvere's) metric spaces P_+-**Cat**, where P_+ is the complete real half-line, ordered with the $\geq$ relation, and equipped with the monoidal structure given by addition $+$; P_+ is cartesian closed but P_+-**Cat** is not (see [CH06] for details); and, even when the monoidal structure of V is the cartesian one, the category $(\mathbb{T},V)$-**Cat** of $(\mathbb{T},V)$-categories and $(\mathbb{T},V)$-functors (see [CT03]) does not need to be cartesian closed, as it is the case of the category **Top** of topological spaces and continuous maps, that is $(\mathbb{U},2)$-**Cat** for $\mathbb{U}$ the ultrafilter monad.

Rosický showed in [Ros99] that **Top** is weakly cartesian closed, and, consequently, that its exact completion is cartesian closed. Weak cartesian closedness of **Top** follows from the existence of enough injectives in its full subcategory $\mathbf{Top}_0$ of $T0$-spaces and the fact that they are exponentiable, and this feature, together with several good properties of **Top**, gives cartesian closedness of its exact completion. More precisely, Rosický has shown in [Ros99] the following theorem.

Theorem 1.1. *Let* $\mathbf{C}$ *be a complete, infinitely extensive and well-powered category with (reg epi, mono)-factorizations such that* $f\times 1$ *is an epimorphism whenever* f *is a regular epimorphism. Then the exact completion of* $\mathbf{C}$ *is cartesian closed provided that* $\mathbf{C}$ *is weakly cartesian closed.*

In this paper we use the setting of $(\mathbb{T},V)$-categories, for a quantale V and a **Set**-monad $\mathbb{T}$ laxly extended to V-**Rel** to conclude, in a unified way, that several topological categories over **Set** share with **Top** the cartesian closedness of the exact completion. This was recently used by Adámek and Rosický in the study of free completions of categories [AR18]. In fact, the category $(\mathbb{T},V)$-**Cat** is topological over **Set** [CH03, CT03], hence complete and with (reg epi, mono)-factorizations such that $f\times 1$ is an epimorphism whenever f is, and it is infinitely extensive [MST06]. To assure weak

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cartesian closedness of $(\mathbb{T},V)$-**Cat** we consider two distinct scenarios, either restricting to the case that V is a frame – so that its monoidal structure is the cartesian one – or considering the case that the lax extension is determined by a $\mathbb{T}$-algebraic structure on V, as introduced in [Hof07] under the name of topological theory. In the latter case the proof generalizes Rosický's proof for $\mathbf{Top}_0$, after observing that, using the Yoneda embedding of [CH09, Hof11], every separated $(\mathbb{T},V)$-category can be embedded in an injective one, and, moreover, these are exponentiable in $(\mathbb{T},V)$-**Cat**. For general $(\mathbb{T},V)$-categories one proceeds again as in [Ros99], using the fact that the reflection of $(\mathbb{T},V)$-**Cat** into its full subcategory of separated $(\mathbb{T},V)$-categories preserves finite products. As observed by Rosický, the exact completion of **Top** relates to the cartesian closed category of equilogical spaces [BBS04]. Analogously, our approach leads to the study of generalized equilogical spaces, as developed in [Rib18].

The paper is organized as follows. In Section 2 we introduce $(\mathbb{T},V)$-categories and list their properties used throughout the paper. In Section 3 we revisit the exponentiability problem in $(\mathbb{T},V)$-**Cat**, establishing a sufficient criterion for exponentiability which generalizes the results obtained in [Hof07, HS15]. In Section 4 we study the properties of injective $(\mathbb{T},V)$-categories which will be used in the forthcoming section to conclude that, under suitable assumptions, injective $(\mathbb{T},V)$-categories are exponentiable (Theorem 5.8). This result will allow us to conclude, in Theorem 6.3, that $(\mathbb{T},V)$-**Cat** is weakly cartesian closed, and, finally, thanks to Theorem 1.1, that the exact completion of $(\mathbb{T},V)$-**Cat** is cartesian closed. We conclude our paper with a section on examples, which include, among others, metric spaces, approach spaces, probabilistic metric spaces, and bitopological spaces.

2. THE CATEGORY OF $(\mathbb{T},V)$-CATEGORIES

Throughout V *is a commutative and unital quantale*, i.e V is a complete lattice with a symmetric and associative tensor product $\otimes$, with unit k and right adjoint hom, so that $u\otimes v\le w$ if, and only if, $v\le \hom(u,w)$, for all $u,v,w\in V$. Further assume that V *is a Heyting algebra*, so that $u\wedge -$ also has a right adjoint, for every $u\in V$. We denote by V-**Rel** the 2-category of V*-relations* (or V-matrices), having as objects sets, as 1-cells V-relations $r:X\to\!\!\!\!\!\!\;\mapsto Y$, i.e. maps $r:X\times Y\to V$, and 2-cells $\varphi:r\to r'$ given by componentwise order $r(x,y)\le r'(x,y)$. Composition of 1-cells is given by relational composition. V-**Rel** has an involution, given by transposition: the transpose of $r:X\to\!\!\!\!\!\!\;\mapsto Y$ is $r^\circ:Y\to\!\!\!\!\!\!\;\mapsto X$ with $r^\circ(y,x)=r(x,y)$.

We fix a *non-trivial monad* $\mathbb{T}=(T,m,e)$ *on* **Set** *satisfying (BC)*, i.e. T preserves weak pullbacks and the naturality squares of the natural transformation m are weak pullbacks (see [CHJ14]). In general we do not assume that T preserves products. Later we will make use of the comparison map $\mathrm{can}_{X,Y}:T(X\times Y)\to TX\times TY$ defined by $\mathrm{can}_{X,Y}(\mathfrak{w})=(T\pi_X(\mathfrak{w}),T\pi_Y(\mathfrak{w}))$ for all $\mathfrak{w}\in T(X\times Y)$, where π_X and π_Y are the product projections. Moreover, we assume that $\mathbb{T}$ *has an extension to* V-**Rel**, which we also denote by $\mathbb{T}$, in the following sense:

- there is a lax functor $T:V\text{-}\mathbf{Rel}\to V\text{-}\mathbf{Rel}$ which extends $T:\mathbf{Set}\to\mathbf{Set}$;
- $T(r^\circ)=(Tr)^\circ$ for all V-relations r;

- the natural transformations $e : 1_{V\text{-}\mathbf{Rel}} \to T$ and $m : T^2 \to T$ become op-lax; that is, for every $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$,

$$e_Y \cdot r \leq Tr \cdot e_X, \qquad\qquad m_Y \cdot TTr \leq Tr \cdot m_X.$$

$$\begin{tikzcd}
X \arrow[r, "e_X"] \arrow[d, "r"', "|" marking] \arrow[dr, phantom, "\leq"] & TX \arrow[d, "Tr", "|" marking] \\
Y \arrow[r, "e_Y"'] & TY
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
TTX \arrow[r, "m_X"] \arrow[d, "TTr"', "|" marking] \arrow[dr, phantom, "\leq"] & TX \arrow[d, "Tr", "|" marking] \\
TTY \arrow[r, "m_Y"'] & TY
\end{tikzcd}$$

We note that our conditions are stronger than those used in [HST14].

A $(\mathbb{T}, V)$-*category* is a pair (X, a) where X is a set and $a : TX \relbar\joinrel\mapstochar\joinrel\rightarrow X$ is a V-relation such that

$$\begin{tikzcd}
X \arrow[r, "e_X"] \arrow[dr, "1_X"'] \arrow[r, phantom, shift right=4, "\leq"'] & TX \arrow[d, "a", "|" marking] \\
& X
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
T^2X \arrow[r, "m_X"] \arrow[d, "Ta"', "|" marking] \arrow[dr, phantom, "\leq"] & TX \arrow[d, "a", "|" marking] \\
TX \arrow[r, "a"', "|" marking] & X
\end{tikzcd}$$

that is, the map $a : TX \times X \to V$ satisfies the conditions:

(R) for each $x \in X$, $k \leq a(e_X(x), x)$;

(T) for each $\mathfrak{X} \in T^2X, \mathfrak{x} \in TX, x \in X$, $Ta(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) \leq a(m_X(\mathfrak{X}), x)$.

Given $(\mathbb{T}, V)$-categories (X, a), (Y, b), a $(\mathbb{T}, V)$-*functor* $f : (X, a) \to (Y, b)$ is a map $f : X \to Y$ such that

$$\begin{tikzcd}
TX \arrow[r, "Tf"] \arrow[d, "a"', "|" marking] \arrow[dr, phantom, "\leq"] & TY \arrow[d, "b", "|" marking] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

that is, for each $\mathfrak{x} \in TX$ and $x \in X$, $a(\mathfrak{x}, x) \leq b(Tf(\mathfrak{x}), f(x))$; f is said to be *fully faithful* when this inequality is an equality.

$(\mathbb{T}, V)$-categories and $(\mathbb{T}, V)$-functors form the category $(\mathbb{T}, V)$-**Cat**. If $(X, a : TX \relbar\joinrel\mapstochar\joinrel\rightarrow X)$ satisfies (R) (and not necessarily (T)), we call it a $(\mathbb{T}, V)$-*graph*. The category $(\mathbb{T}, V)$-**Gph**, of $(\mathbb{T}, V)$-graphs and $(\mathbb{T}, V)$-functors, contains $(\mathbb{T}, V)$-**Cat** as a full reflective subcategory.

We present the examples in detail in the last section. We mention here, however, that the leading examples are obtained when one considers the quantale $2 = (\{0, 1\}, \leq, \&, 1)$ and Lawvere's real half-line $P_+ = ([0, \infty], \geq, +, 0)$, the identity monad $\mathbb{I}$ and the ultrafilter monad $\mathbb{U}$ on **Set**. Thus we obtain the following examples:

- $(\mathbb{I}, V)$-**Cat** is the category of V-categories and V-functors; in particular, $(\mathbb{I}, 2)$-**Cat** is the category **Ord** of (pre)ordered sets and monotone maps, while $(\mathbb{I}, P_+)$-**Cat** is the category **Met** of Lawvere's metric spaces and non-expansive maps (see [Law73]).
- $(\mathbb{U}, 2)$-**Cat** is the category **Top** of topological spaces and continuous maps.
- $(\mathbb{U}, P_+)$-**Cat** is the category **App** of Lowen's approach spaces and non-expansive maps (see [Low97]).

We recall (see [AHS90, Definition 21.1]) that a functor $G : \mathbf{A} \to \mathbf{B}$ is said to be *topological* if every source $(f_i : B \to GA_i)_{i \in I}$ in **B** has a unique G-initial lift $(\overline{f_i} : A \to A_i)_{i \in I}$. The following was proved in [CH03] (see also [CT03]).

Theorem 2.1. *The forgetful functors* $(\mathbb{T}, V)$-**Cat** $\to$ **Set** *and* $(\mathbb{T}, V)$-**Gph** $\to$ **Set** *are topological.*

This shows, in particular, that (see [AHS90, Chapter 21] for details):

- $(\mathbb{T}, V)$-**Cat** is complete and cocomplete.

- Monomorphisms in $(\mathbb{T},V)$-**Cat** are the morphisms whose underlying map is injective; therefore, since the $(\mathbb{T},V)$-structures on any set form a set, $(\mathbb{T},V)$-**Cat** is well-powered.
- Every topological category over **Set** has two factorization systems, (reg epi, mono) and (epi, reg mono); in $(\mathbb{T},V)$-**Cat** the former one is in general not stable (that is, regular epimorphisms need not be stable under pullback – **Top** is such an example), but the latter one is. Indeed, epimorphisms in $(\mathbb{T},V)$-**Cat** are the $(\mathbb{T},V)$-functors which are surjective as maps, the forgetful functor $(\mathbb{T},V)$-**Cat** $\to$ **Set** preserves pullbacks, and surjective maps are stable under pullback in **Set**. Therefore, as $f\times 1_Z$ is the pullback of $f:X\to Y$ along $\pi_Y:Y\times Z\to Y$, we conclude that $f\times 1_Z$ is an epimorphism provided f is.

$(\mathbb{T},V)$-**Cat** has a natural structure of 2-category: for $(\mathbb{T},V)$-functors $f,g:(X,a)\to(Y,b)$, $f\le g$ if $g\cdot a\le b\cdot Tf$. This condition can be equivalently written as $k\le b(e_Y(f(x)),g(x))$ for every $x\in X$ (see [CT03] for details). We write $f\simeq g$ if $f\le g$ and $g\le f$.

Extensivity of $(\mathbb{T},V)$-**Cat** was studied in [MST06]:

Theorem 2.2. $(\mathbb{T},V)$-**Cat** *is infinitely extensive.*

In general $(\mathbb{T},V)$-**Cat** is not cartesian closed, while $(\mathbb{T},V)$-**Gph** is. In fact, the following was proved in [CHT03]:

Theorem 2.3. $(\mathbb{T},V)$-**Gph** *is a quasi-topos.*

We also note that the tensor product of V induces a canonical structure c on $X\times Y$ defined by

$$c(\mathfrak{w},(x,y))=a(T\pi_X(\mathfrak{w}),x)\otimes b(T\pi_Y(\mathfrak{w}),y),$$

where $\mathfrak{w}\in T(X\times Y)$, $x\in X$, $y\in Y$. We put

$$(X,a)\otimes(Y,b)=(X\times Y,c),$$

and this construction is in an obvious way part of a functor $\otimes:(\mathbb{T},V)\text{-}\mathbf{Gph}\times(\mathbb{T},V)\text{-}\mathbf{Gph}\to(\mathbb{T},V)\text{-}\mathbf{Gph}$. However, the tensor product of two $(\mathbb{T},V)$-categories is in general not a $(\mathbb{T},V)$-category (see [Hof07, Lemma 6.1]).

Weak cartesian closedness of $(\mathbb{T},V)$-**Cat** needs a thorough study of injective $(\mathbb{T},V)$-categories and some extra conditions. This is the subject of the following sections.

3. Exponentiable $(\mathbb{T},V)$-categories

Recall that an object C of a category $\mathbf{C}$ with finite products is *exponentiable* whenever the functor $C\times -:\mathbf{C}\to\mathbf{C}$ has a right adjoint. The category $\mathbf{C}$ is *cartesian closed* if every object C of $\mathbf{C}$ is exponentiable. Equivalently, if for each pair of objects A,B of $\mathbf{C}$ there exists an object $\langle A,B\rangle$ and a morphism $\mathrm{ev}:\langle A,B\rangle\times A\to B$ such that, for each morphism $f:C\times A\to B$ there exists a unique morphism $\overline{f}:C\to\langle A,B\rangle$ with $\mathrm{ev}\cdot(\overline{f}\times 1_A)=f$. Dropping uniqueness of $\overline{f}$ gives the notion of *weakly cartesian closed category.*

In this section we present a sufficient condition for a $(\mathbb{T},V)$-category X to be exponentiable in $(\mathbb{T},V)$-**Cat**, which generalises [Hof06, Theorem 4.3] and [Hof07, Theorem 6.5]. To start, we recall that $(\mathbb{T},V)$-**Cat** can be fully embedded into the cartesian closed category $(\mathbb{T},V)$-**Gph**. Here, for $(\mathbb{T},V)$-graphs (X,a) and (Y,b), the exponential $\langle(X,a),(Y,b)\rangle$ has as underlying set

$$Z:=\{h:(X,a)\times(1,e_1^\circ)\to(Y,b)\mid h\text{ is a }(\mathbb{T},V)\text{-functor}\},$$

which becomes a $(\mathbb{T},V)$-graph when equipped with the largest structure b^a making the evaluation map

$$\mathrm{ev}:Z\times X\to Y,\ (h,x)\mapsto h(x)$$

a $(\mathbb{T}, V)$-functor: for $\mathfrak{p} \in TZ$ and $h \in Z$, put

$$b^a(\mathfrak{p}, h) = \bigvee \{v \in V \mid \forall \mathfrak{q} \in (T\pi_Z)^{-1}(\mathfrak{p}), x \in X \,.\, a(T\pi_X(\mathfrak{q}), x) \wedge v \leq b(T\mathrm{ev}(\mathfrak{q}), h(x))\},$$

where π_X and π_Z are the product projections. Note that the supremum above is even a maximum since $- \wedge -$ distributes over suprema.

Given V-relations $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow X'$ and $s : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Y'$, we define in V-**Rel** $r \owedge s : X \times Y \relbar\joinrel\mapstochar\joinrel\rightarrow X' \times Y'$ by $(r \owedge s)((x, y), (x', y')) = r(x, x') \wedge s(y, y')$. That is, $r \owedge s = (\pi_{X'}^\circ \cdot r \cdot \pi_X) \wedge (\pi_{Y'}^\circ \cdot s \cdot \pi_Y)$ in the ordered set V-$\mathbf{Rel}(X \times Y, X' \times Y')$.

Theorem 3.1. *Assume that the diagram*

$$\begin{tikzcd}
T(X \times Y) \arrow[r, "\mathrm{can}_{X,Y}"] \arrow[d, "|" marking, "T(r \owedge s)"'] & TX \times TY \arrow[d, "|" marking, "(Tr) \owedge (Ts)"] \\
T(X' \times Y') \arrow[r, "\mathrm{can}_{X',Y'}"'] & TX' \times TY'
\end{tikzcd} \tag{3.i}$$

commutes, for all V-relations $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow X'$ and $s : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Y'$. Then a $(\mathbb{T}, V)$-category (X, a) is exponentiable provided that

$$\bigvee_{\mathfrak{x} \in TX} (Ta(\mathfrak{X}, \mathfrak{x}) \wedge u) \otimes (a(\mathfrak{x}, x) \wedge v) \geq a(m_X(\mathfrak{X}), x) \wedge (u \otimes v), \tag{3.ii}$$

for all $\mathfrak{X} \in TTX$, $x \in X$ and $u, v \in V$.

Proof. We show that the $(\mathbb{T}, V)$-graph structure b^a on Z is transitive, for each $(\mathbb{T}, V)$-category (Y, b). To this end, let $\mathfrak{P} \in TTZ$, $\mathfrak{p} \in TZ$, $h \in Z$, $x \in X$ and $\mathfrak{w} \in T(Z \times X)$ with $T\pi_Z(\mathfrak{w}) = m_Z(\mathfrak{P})$. We have to show that

$$(T(b^a)(\mathfrak{P}, \mathfrak{p}) \otimes b^a(\mathfrak{p}, h)) \wedge a(T\pi_X(\mathfrak{w}), x) \leq b(T\,\mathrm{ev}(\mathfrak{w}), h(x)).$$

Since m has (BC), there is some $\mathfrak{Q} \in TT(Z \times X)$ with $TT\pi_Z(\mathfrak{Q}) = \mathfrak{P}$ and $m_{Z \times X}(\mathfrak{Q}) = \mathfrak{w}$. Hence, $m_X(TT\pi_X(\mathfrak{Q})) = T\pi_X(\mathfrak{w})$, and we calculate:

$$\begin{aligned}
&(T(b^a)(\mathfrak{P}, \mathfrak{p}) \otimes b^a(\mathfrak{p}, h)) \wedge a(T\pi_X(\mathfrak{w}), x) \\
&\qquad \leq \bigvee_{\mathfrak{x} \in TX} ((T(b^a)(TT\pi_Z(\mathfrak{Q}), \mathfrak{p}) \wedge Ta(TT\pi_X(\mathfrak{Q}), \mathfrak{x})) \otimes (b^a(\mathfrak{p}, h) \wedge a(\mathfrak{x}, x)) && \text{(by (3.ii))} \\
&\qquad \leq \bigvee_{\mathfrak{x} \in TX} \bigvee_{\mathfrak{q} \in \mathrm{can}^{-1}(\mathfrak{p}, \mathfrak{x})} T(b^a \owedge a)(T\,\mathrm{can}_{Z,X}(\mathfrak{Q}), \mathfrak{q}) \otimes (b^a \owedge a)(\mathrm{can}_{Z,X}(\mathfrak{q}), (h, x)) && \text{(using (3.i))} \\
&\qquad = \bigvee_{\mathfrak{q} \in (T\pi_Z)^{-1}(\mathfrak{p})} T(b^a \owedge a)(T\,\mathrm{can}_{Z,X}(\mathfrak{Q}), \mathfrak{q}) \otimes (b^a \owedge a)(\mathrm{can}_{Z,X}(\mathfrak{q}), (h, x)) \\
&\qquad = \bigvee_{\mathfrak{q} \in (T\pi_Z)^{-1}(\mathfrak{p})} T(b^a \times a)(\mathfrak{Q}, \mathfrak{q}) \otimes (b^a \times a)(\mathfrak{q}, (h, x)) \\
&\qquad \leq \bigvee_{\mathfrak{q} \in (T\pi_Z)^{-1}(\mathfrak{p})} Tb(TT\,\mathrm{ev}(\mathfrak{Q}), T\,\mathrm{ev}(\mathfrak{q})) \otimes b(T\,\mathrm{ev}(\mathfrak{q}), h(x)) \\
&\qquad \leq b(m_Y \cdot TT\,\mathrm{ev}(\mathfrak{Q}), h(x)) = b(T\,\mathrm{ev}(\mathfrak{w}), h(x)).
\end{aligned}$$

□

Remark 3.2. We note that the inequality $\mathrm{can}_{X',Y'}\cdot T(r \oslash s) \leq ((Tr) \oslash (Ts))\cdot \mathrm{can}_{X,Y}$ is automatically true. Firstly, this inequality is equivalent to $T(r \oslash s) \leq \mathrm{can}^\circ_{X',Y'}\cdot((Tr) \oslash (Ts))\cdot \mathrm{can}_{X,Y}$; secondly,

$$\begin{aligned} T(r \oslash s) &= T((\pi^\circ_{X'}\cdot r\cdot \pi_X) \wedge (\pi^\circ_{Y'}\cdot s\cdot \pi_Y)) \\ &\leq T(\pi^\circ_{X'}\cdot r\cdot \pi_X) \wedge T(\pi^\circ_{Y'}\cdot s\cdot \pi_Y) \\ &\leq \mathrm{can}^\circ_{X',Y'}\cdot((Tr) \oslash (Ts))\cdot \mathrm{can}_{X,Y}\,. \end{aligned}$$

It is worthwhile to notice that, when V is a frame, that is $\otimes = \wedge$, the condition above is equivalent to

$$\bigvee_{\mathfrak{x}\in TX} Ta(\mathfrak{X},\mathfrak{x}) \wedge a(\mathfrak{x},x) \geq a(m_X(\mathfrak{X}),x),$$

for all $\mathfrak{X} \in TTX$ and $x \in X$. Therefore:

Corollary 3.3. *When V is a frame and* (3.i) *commutes for all V-relations $r : X \to\!\!\!\!\!\!+\; X'$ and $s : Y \to\!\!\!\!\!\!+\; Y'$, a $(\mathbb{T}, V)$-category (X, a) is exponentiable provided that*

$$a \cdot m_X = a \cdot Ta.$$

4. Injective and representable $(\mathbb{T}, V)$-categories

In this section we recall an important class of $(\mathbb{T}, V)$-categories, the so-called *representable* ones. More information on this type of $(\mathbb{T}, V)$-categories can be found in [CCH15, HST14]. We also recall from [CH09, Hof07, Hof11] that every injective $(\mathbb{T}, V)$-category is representable.

Based on the lax extension of the **Set**-monad $\mathbb{T} = (T, m, e)$ to V-**Rel**, $\mathbb{T}$ admits a natural extension to a monad on V-**Cat**, in the sequel also denoted by $\mathbb{T} = (T, m, e)$ (see [Tho09]). Here the functor $T : V\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Cat}$ sends a V-category (X, a_0) to (TX, Ta_0), and $e_X : X \to TX$ and $m_X : TTX \to TX$ become V-functors for each V-category X. The Eilenberg–Moore algebras for this monad can be described as triples (X, a_0, α) where (X, a_0) is a V-category and (X, α) is an algebra for the **Set**-monad $\mathbb{T}$ such that $\alpha : T(X, a_0) \to (X, a_0)$ is a V-functor. For $\mathbb{T}$-algebras (X, a_0, α) and (Y, b_0, β), a map $f : X \to Y$ is a homomorphism $f : (X, a_0, \alpha) \to (Y, b_0, \beta)$ precisely if f preserves both structures, that is, whenever $f : (X, a_0) \to (Y, b_0)$ is a V-functor and $f : (X, \alpha) \to (Y, \beta)$ is a $\mathbb{T}$-homomorphism.

There are canonical adjoint functors

$$\begin{tikzcd} (V\text{-}\mathbf{Cat})^{\mathbb{T}} \arrow[r, bend left, "K"] \arrow[r, phantom, "\top"] & (\mathbb{T}, V)\text{-}\mathbf{Cat}. \arrow[l, bend left, "M"] \end{tikzcd}$$

The functor K associates to each $X = (X, a_0, \alpha)$ in $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$ the $(\mathbb{T}, V)$-category $KX = (X, a)$, where $a = a_0 \cdot \alpha$, and keeps morphisms unchanged. Its left adjoint $M : (\mathbb{T}, V)\text{-}\mathbf{Cat} \to (V\text{-}\mathbf{Cat})^{\mathbb{T}}$ sends a $(\mathbb{T}, V)$-category (X, a) to $(TX, Ta \cdot m^\circ_X, m_X)$ and a $(\mathbb{T}, V)$-functor f to Tf. Via the adjunction $M \dashv K$ one obtains a lifting of the **Set**-monad $\mathbb{T} = (T, m, e)$ to a monad on $(\mathbb{T}, V)$-**Cat**, also denoted by $\mathbb{T} = (T, m, e)$.

In this setting we can define 'duals' in $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$ and carry them into $(\mathbb{T}, V)$-**Cat**. Indeed, since $T : V\text{-}\mathbf{Rel} \to V\text{-}\mathbf{Rel}$ commutes with the involution $(-)^\circ$: for every $\mathbb{T}$-algebra $X = (X, a_0, \alpha)$ also (X, a°_0, α) is a $\mathbb{T}$-algebra. Moreover, if (X, a) is a $(\mathbb{T}, V)$-category, we define X^{op} by mapping (X, a) into $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$ via M, dualizing the image in $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$, and then carrying it back to $(\mathbb{T}, V)$-**Cat**; that is,

$$X^{\mathrm{op}} = K((M(X, a))^{\mathrm{op}}) = (TX, m_X \cdot (Ta)^\circ \cdot m_X).$$

Since the monad $\mathbb{T} = (T, m, e)$ on $(\mathbb{T}, V)$-**Cat** is lax idempotent (i.e, of Kock-Zöberlein type), an algebra structure $\alpha : TX \to X$ on a $(\mathbb{T}, V)$-category X is left adjoint to the unit $e_X : X \to TX$.

We call a $(\mathbb{T},V)$-category X *representable* whenever $e_X : X \to TX$ has a left adjoint in $(\mathbb{T},V)$-**Cat**; equivalently, whenever there is some $(\mathbb{T},V)$-functor $\alpha : TX \to X$ with $\alpha \cdot e_X \simeq 1_X$, since then

$$e_X \cdot \alpha = T\alpha \cdot e_{TX} \geq T\alpha \cdot Te_X \simeq 1_{TX}.$$

However, a left adjoint $\alpha : TX \to X$ to e_X is in general only a pseudo-algebra structure on X, that is,

$$\alpha \cdot e_X \simeq 1_X \qquad \text{and} \qquad \alpha \cdot T\alpha \simeq \alpha \cdot m_X.$$

For every representable $(\mathbb{T},V)$-category (X,a), the structure $a : TX \nrightarrow X$ can be decomposed as $a = a_0 \cdot \alpha$, where $a_0 = a \cdot e_X$ denotes the underlying V-category structure.

A $(\mathbb{T},V)$-category X is *injective* whenever, for each fully faithful $h : A \to B$ in $(\mathbb{T},V)$-**Cat** and each $(\mathbb{T},V)$-functor $f : A \to X$, there is a $(\mathbb{T},V)$-functor $g : B \to X$ with $g \cdot h \simeq f$.

Proposition 4.1. *Every injective $(\mathbb{T},V)$-category is representable.*

Proof. Let X be an injective $(\mathbb{T},V)$-category. The $(\mathbb{T},V)$-functor $e_X : (X,a) \to (TX, Ta\cdot m_X^\circ \cdot m_X)$ is an embedding. Indeed, e_X is injective because the monad T is non-trivial, and it is fully faithful:

$$e_X^\circ \cdot Ta \cdot m_X^\circ \cdot m_X \cdot Te_X \leq a \cdot Ta \cdot m_X^\circ \leq a \cdot m_X \cdot m_X^\circ \leq a.$$

Hence, there is a $(\mathbb{T},V)$-functor $\alpha : TX \to X$ with $\alpha \cdot e_X \simeq 1_X$, and so X is representable. □

5. INJECTIVE $(\mathbb{T},V)$-CATEGORIES ARE EXPONENTIABLE

In Section 6 we will show that, under some conditions, $(\mathbb{T},V)$-**Cat** is weakly cartesian closed. Notably, we will use that every $(\mathbb{T},V)$-category can be embedded into an injective one; which, by the main result of this section, implies that every $(\mathbb{T},V)$-category can be embedded into an exponentiable one. We hasten to remark that this is easily seen to be fulfilled for $\mathbb{T}$ being the identity monad, witnessed by the *Yoneda embedding* (see [Law73])

$$y_X : X \to PX := V^{X^{\mathrm{op}}}.$$

Here PX is the free cocompletion of X; being cocomplete, PX is injective.

To treat the general case, we *will consider from now on only* extensions of the monad $\mathbb{T}$ to V-**Rel** given by a $\mathbb{T}$-algebra structure $\xi : TV \to V$ on V, so that we are dealing with a *strict topological theory* in the sense of [Hof07]. In this case, the extension of $T : \mathbf{Set} \to \mathbf{Set}$ to V-**Rel** is defined by

$$\begin{aligned} &Tr : TX \times TY \to V \\ &\qquad (\mathfrak{x},\mathfrak{y}) \mapsto \bigvee \left\{ \xi \cdot Tr(\mathfrak{w}) \;\middle|\; \mathfrak{w} \in T(X \times Y), T\pi_X(\mathfrak{w}) = \mathfrak{x}, T\pi_Y(\mathfrak{w}) = \mathfrak{y} \right\} \end{aligned}$$

for each V-relation $r : X \times Y \to V$.

In order to obtain a Yoneda embedding, we consider the $\mathbb{T}$-algebra $(V, \hom, \xi)$ which is mapped by K into the important $(\mathbb{T},V)$-category $(V, \hom_\xi)$, where $\hom_\xi = \hom \cdot \xi$ (see Section 4). The proof of the following result can be found in [CH09] and [Hof11].

Theorem 5.1. *If the extension of $\mathbb{T}$ to V-**Rel** is induced by a strict topological theory, then, for every $(\mathbb{T},V)$-category (X,a), the V-relation $a : TX \nrightarrow X$ defines a $(\mathbb{T},V)$-functor*

$$a : X^{\mathrm{op}} \otimes X \to (V, \hom_\xi).$$

Moreover, the $\otimes$-exponential mate $y_X = \ulcorner a \urcorner : X \to V^{X^{\mathrm{op}}}$ of a is fully faithful, and the $(\mathbb{T},V)$-category $PX = V^{X^{\mathrm{op}}}$ is injective.

*In fact, this construction defines a functor $P : (\mathbb{T},V)$-**Cat** $\to (\mathbb{T},V)$-**Cat** and $y = (y_X)_X$ is a natural transformation $y : 1_{(\mathbb{T},V)\text{-}\mathbf{Cat}} \to P$.*

Since y_X is fully faithful, when X is injective there exists a $(\mathbb{T},V)$-functor $\mathrm{Sup}_X : PX \to X$ such that $\mathrm{Sup}_X \cdot y_X \simeq 1_X$. As shown in [Hof11, Theorem 2.7], $\mathrm{Sup}_X \dashv y_X$. Moreover, for each $(\mathbb{T},V)$-category (X,a), y_X is one-to-one if, and only if, (X,a) is *separated*, i.e. for every $f,g : (Y,b) \to (X,a)$, $f \simeq g$ only if $f = g$ (see [HT10], for example). It follows immediately that, for an injective $(\mathbb{T},V)$-functor $f : X \to Y$ where Y is separated, also X is.

Lemma 5.2. *The $\otimes$-exponential Y^X is separated, for every separated $(\mathbb{T},V)$-category Y and every representable $(\mathbb{T},V)$-category X; in particular, PX is separated, for every $(\mathbb{T},V)$-category X.*

Proof. See [HT10, Corollary 4.12 (2)]. □

Corollary 5.3. *Every separated $(\mathbb{T},V)$-category embeds into an injective $(\mathbb{T},V)$-category.*

In Section 2 we introduced the tensor product $X \otimes Y$ of $(\mathbb{T},V)$-graphs X and Y. We remark that, in the setting of a strict topological theory, $X \otimes Y$ is a $(\mathbb{T},V)$-category provided that X and Y are so (see [Hof07]).

The result promised in the title of this section was shown in [Hof14, Proposition 2.7] for the special case of $\otimes = \wedge$:

Proposition 5.4. *If the quantale V is a frame and* (3.i) *commutes for all V-relations $r : X \nrightarrow X'$ and $s : Y \nrightarrow Y'$, then every representable $(\mathbb{T},V)$-category is exponentiable. In particular, in this case every injective $(\mathbb{T},V)$-category is exponentiable.*

To treat the general case, we will make use of the following conditions:

Assumptions 5.5.
1. The diagram (3.i) commutes, for all V-relations $r : X \nrightarrow X'$ and $s : Y \nrightarrow Y'$.
2. For all $u,v,w \in V$,
$$w \wedge (u \otimes v) = \bigvee\{u' \otimes v' \mid u' \leq u, v' \leq v, u' \otimes v' \leq w\};$$
or, equivalently, every injective V-category is exponentiable: see [HR13, Theorem 5.3].
3. For every V-relation $a : X \nrightarrow Y$ and $u \in V$,
$$T(a \otimes u) = Ta \otimes u,$$
where $a \otimes u$ is the V-relation defined by $(a \otimes u)(x,y) = a(x,y) \otimes u$.
4. The maps $V \otimes V \xrightarrow{\otimes} V$ and $X \xrightarrow{(-,u)} X \otimes V$ are $(\mathbb{T},V)$-functors, for all $u \in V$.

These morphisms induce an interesting action of V on every injective $(\mathbb{T},V)$-category (X,a) as follows. The $(\mathbb{T},V)$-functor
$$X^{\mathrm{op}} \otimes X \otimes V \xrightarrow{a \otimes 1} V \otimes V \xrightarrow{\otimes} V$$
induces a $(\mathbb{T},V)$-functor $\tilde{a} : X \otimes V \to PX$. We denote the composite
$$X \otimes V \xrightarrow{\tilde{a}} PX \xrightarrow{\mathrm{Sup}_X} X$$
by $\oplus$, and
$$X \xrightarrow{(-,u)} X \otimes V \xrightarrow{\tilde{a}} PX \xrightarrow{\mathrm{Sup}_X} X,$$
assigning to each $x \in X$ an element $x \oplus u$ in X, by $- \oplus u$.

Analogously we will write $\mathfrak{x} \oplus u$ for $T(- \oplus u)(\mathfrak{x})$, for every $\mathfrak{x} \in TX$ and $u \in V$. Note that $(\mathbb{T},V)$-functoriality of $- \oplus u$ can be written as
$$a(\mathfrak{x},y) \leq a(\mathfrak{x} \oplus u, y \oplus u),$$
for every $\mathfrak{x} \in TX$ and $y \in X$.

Lemma 5.6. *Assuming 5.5 (4), for an injective* $(\mathbb{T},V)$*-category* (X,a)*, with* $a=a_0\cdot\alpha$ *as usual, the following holds, for every* $x,y\in X$*,* $\mathfrak{x}\in TX$ *and* $u\in V$*:*

(1) $a_0(x\oplus u,y)=\hom(u,a_0(x,y))$*;*

(2) $a_0(x,y\oplus u)\geq a_0(x,y)\otimes u$*;*

(3) $a(\mathfrak{x}\oplus u,y)\geq\hom(u,a(\mathfrak{x},y))$*;*

(4) $a(\mathfrak{x},y\oplus u)\geq a(\mathfrak{x},y)\otimes u$*.*

Moreover, if, in addition, 5.5 (3) *holds, then, for every* $\mathfrak{X}\in T^2X$*,* $\mathfrak{y}\in TX$*,* $u\in V$*,*

(5) $Ta(\mathfrak{X},\mathfrak{y}\oplus u)\geq Ta(\mathfrak{X},\mathfrak{y})\otimes u$*.*

Proof. (1) For every $x,y\in X$ and $u\in V$,

$$\begin{aligned}
a_0(x\oplus u,y)&=a_0(\mathrm{Sup}_X(\tilde{a}(x,u)),y) && \text{(by definition of }\oplus)\\
&=[\tilde{a}(x,u),y_X(y)] && \text{(because }\mathrm{Sup}_X\dashv y_X)\\
&=\bigwedge_{\mathfrak{x}\in TX}\hom(\tilde{a}(x,u)(\mathfrak{x}),y_X(y)(\mathfrak{x})) && \text{(by definition of [,])}\\
&=\bigwedge_{\mathfrak{x}\in TX}\hom(a(\mathfrak{x},x)\otimes u,a(\mathfrak{x},y)) && \text{(by definition of }\tilde{a}\text{ and }y_X(y))\\
&=\hom(u,a_0(x,y)),
\end{aligned}$$

because, using the fact that $a=a_0\cdot\alpha$ and

$$a_0(\alpha(\mathfrak{x}),x)\otimes u\otimes\hom(u,a_0(x,y))\leq a_0(\alpha(\mathfrak{x}),x)\otimes a_0(x,y)\leq a_0(\alpha(\mathfrak{x}),y),$$

for $\mathfrak{x}\in TX$, we can conclude that

$$\hom(u,a_0(x,y))\leq\bigwedge_{\mathfrak{x}\in TX}\hom(a_0(\alpha(\mathfrak{x}),x)\otimes u,a_0(\alpha(\mathfrak{x}),y)).$$

Taking $\mathfrak{x}=e_X(x)$, we see that this inequality is in fact an equality as claimed.

(2) Since, by hypothesis, $-\oplus u$ is a $(\mathbb{T},V)$-functor, and so, in particular, a V-functor $(X,a_0)\to(X,a_0)$,

$$a_0(x,y)\leq a_0(x\oplus u,y\oplus u)=\hom(u,a_0(x,y\oplus u)),$$

and then

$$a_0(x,y)\otimes u\leq\hom(u,a_0(x,y\oplus u))\otimes u\leq a_0(x,y\oplus u).$$

(3) One has

$$\begin{aligned}
k&\leq a_0(\alpha(\mathfrak{x}),\alpha(\mathfrak{x}))=a(\mathfrak{x},\alpha(\mathfrak{x}))\\
&\leq a(\mathfrak{x}\oplus u,\alpha(\mathfrak{x})\oplus u)\\
&=a_0(\alpha(\mathfrak{x}\oplus u),\alpha(\mathfrak{x})\oplus u).
\end{aligned}$$

Using (1) we conclude that

$$\begin{aligned}
\hom(u,a(\mathfrak{x},y))&=a_0(\alpha(\mathfrak{x})\oplus u,y)\\
&\leq a_0(\alpha(\mathfrak{x}\oplus u),\alpha(\mathfrak{x})\oplus u)\otimes a_0(\alpha(\mathfrak{x})\oplus u,y)\\
&\leq a_0(\alpha(\mathfrak{x}\oplus u),y)=a(\mathfrak{x}\oplus u,y).
\end{aligned}$$

(4) follows directly from (2), while (5) follows from (4). ☐

Lemma 5.7. *Let* $\varphi:V\to W$ *be a surjective quantale homomorphism; that is,* φ *preserves the tensor, the neutral element, and suprema. Then, if* V *satisfies condition 5.5 (2), so does* W*.*

Theorem 5.8. *Under Assumptions 5.5, every injective* $(\mathbb{T},V)$*-category is exponentiable in* $(\mathbb{T},V)$-**Cat**.

Proof. Let $\mathfrak{X} \in T^2X$, $x \in X$ and $u, v \in V$. In order to conclude that

$$\bigvee_{\mathfrak{x} \in TX} (Ta(\mathfrak{X}, \mathfrak{x}) \wedge u) \otimes (a(\mathfrak{x}, x) \wedge v) \geq a(m_X(\mathfrak{X}), x) \wedge (u \otimes v),$$

we make use of Hypothesis 5.5 (2). Let $u', v' \in V$ with $u' \leq u$, $v' \leq v$ and $u' \otimes v' \leq a(m_X(\mathfrak{X}), x)$. First we note that

$$\begin{aligned} Ta(\mathfrak{X}, T\alpha(\mathfrak{X}) \oplus u') \wedge u &\geq (Ta(\mathfrak{X}, T\alpha(\mathfrak{X})) \otimes u') \wedge u && \text{(by 5.6 (5))} \\ &= (Ta_0(T\alpha(\mathfrak{X}), T\alpha(\mathfrak{X})) \otimes u') \wedge u \\ &\geq (k \otimes u') \wedge u = u', \end{aligned}$$

and

$$\begin{aligned} a(T\alpha(\mathfrak{X}) \oplus u', x) &\geq \hom(u', a(T\alpha(\mathfrak{X}), x)) && \text{(by 5.6 (3))} \\ &= \hom(u', a_0(\alpha(T\alpha(\mathfrak{X})), x)) \\ &= \hom(u', a_0(\alpha(m_X(\mathfrak{X})), x)) \\ &= \hom(u', a(m_X(\mathfrak{X}), x)). \end{aligned}$$

Now, from $u' \otimes v' \leq a(m_X(\mathfrak{X}), x)$ and $v' \leq v$ we get

$$v' \leq \hom(u', a(m_X(\mathfrak{X}), x)) \wedge v \leq a(T\alpha(\mathfrak{X}) \oplus u', x) \wedge v,$$

hence

$$u' \otimes v' \leq (Ta(\mathfrak{X}, T\alpha(\mathfrak{X}) \oplus u') \wedge u) \otimes (a(T\alpha(\mathfrak{X}) \oplus u', x) \wedge v).$$

Therefore $a(m_X(\mathfrak{X}), x) \wedge (u \otimes v) \leq \bigvee_{\mathfrak{x} \in TX} (Ta(\mathfrak{X}, \mathfrak{x}) \wedge u) \otimes (a(\mathfrak{x}, x) \wedge v).$ ☐

Remark 5.9. Under Assumptions 5.5, it follows from Lemma 5.2 that the exponential $\langle (X, a), (Y, b) \rangle$ is separated, for all separated injective $(\mathbb{T}, V)$-categories (X, a) and (Y, b). In fact, with $a = a_0 \cdot \alpha$, the epimorphism $(X, \alpha) \to (X, a)$ in $(\mathbb{T}, V)$-**Cat** is mapped to the monomorphism

$$\langle (X, a), (Y, b) \rangle \longrightarrow \langle (X, \alpha), (Y, b) \rangle = (Y, b)^{(X, \alpha)},$$

which proves that $\langle (X, a), (Y, b) \rangle$ is separated.

6. $(\mathbb{T}, V)$-**Cat** IS WEAKLY CARTESIAN CLOSED

Building on the results of the previous section, in this section we show that, under some conditions, $(\mathbb{T}, V)$-**Cat** is weakly cartesian closed. We start by proving this property for the full subcategory $(\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{sep}}$ of $(\mathbb{T}, V)$-**Cat** of separated $(\mathbb{T}, V)$-categories.

Theorem 6.1. *Under Assumptions 5.5,* $(\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{sep}}$ *is weakly cartesian closed.*

Proof. For X, Y separated $(\mathbb{T}, V)$-categories, consider the Yoneda embeddings $y_X : X \to PX$ and $y_Y : Y \to PY$, and the exponential $\langle PX, PY \rangle$. The elements of its underlying set can be identified with $(\mathbb{T}, V)$-functors $E \times PX \to PY$ (where $E = (1, e_1^\circ)$ is the generator of $(\mathbb{T}, V)$-**Cat**), and the universal morphism $\mathrm{ev} : \langle PX, PY \rangle \times PX \to PY$ with the evaluation map: $\mathrm{ev}(\varphi, \mathfrak{x}) = \varphi(\mathfrak{x})$ (where, for simplicity, we identify the set $E \times PX$ with PX). We can therefore consider

$$\ll X, Y \gg = \{\varphi : E \times PX \to PY \mid \varphi(y_X(X)) \subseteq y_Y(Y)\},$$

with the initial structure with respect to the inclusion $\iota : \ll X, Y \gg \to \langle PX, PY \rangle$. Moreover, the morphism

$$\ll X, Y \gg \times X \xrightarrow{\iota \times y_X} \langle PX, PY \rangle \times PX \xrightarrow{\mathrm{ev}} PY$$

factors through y_Y via a morphism

$$\ll X, Y \gg \times X \xrightarrow{\widetilde{\mathrm{ev}}} Y.$$

Next we show that this is a weak exponential in $(\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{sep}}$.

Given any separated $(\mathbb{T}, V)$-category Z, and a $(\mathbb{T}, V)$-functor $f : Z \times X \to Y$, by injectivity of PY there exists a $(\mathbb{T}, V)$-functor $f' : Z \times PX \to PY$ making the square below commute. Then, by universality of the evaluation map ev, there exists a unique $(\mathbb{T}, V)$-functor $\overline{f} : Z \to \langle PX, PY\rangle$ making the bottom triangle commute.

$$\begin{tikzcd}
Z \times X \arrow[r, "f"] \arrow[d, "1_Z\times y_X"'] & Y \arrow[d, "y_Y"] \\
Z \times PX \arrow[r, "f'"] \arrow[d, "\overline{f}\times 1_{PX}"'] & PY \\
\langle PX, PY\rangle \times PX \arrow[ur, "\mathrm{ev}"'] &
\end{tikzcd}$$

The map $\overline{f} : Z \to \langle PX, PY\rangle$, assigning to each $z \in Z$ a map $\overline{f}(z) : PX \to PY$, is such that $\overline{f}(z)(y_X(x)) = \mathrm{ev}(\overline{f}(z), y_X(x)) = y_Y(f(z, x))$; that is, $\overline{f}(z)(y_X(X)) \subseteq y_Y(Y)$, and this means that $\overline{f}(z) \in \ll X, Y \gg$. Hence we can consider the corestriction $\tilde{f}$ of $\overline{f}$ to $\ll X, Y \gg$, which is again a $(\mathbb{T}, V)$-functor since $\ll X, Y \gg$ has the initial structure with respect to $\langle PX, PY\rangle$, so that the following diagram commutes.

$$\begin{tikzcd}
\ll X, Y \gg \times X \arrow[r, "\widetilde{\mathrm{ev}}"] & Y \\
Z \times X \arrow[u, "\tilde{f}\times 1_X"] \arrow[ur, "f"'] &
\end{tikzcd}$$

□

In order to show that $(\mathbb{T}, V)$-**Cat** is weakly cartesian closed, we follow the proof of [Ros99]. Hence, first we show that:

Proposition 6.2. *The reflector* $R : (\mathbb{T}, V)$-$\mathbf{Cat} \to (\mathbb{T}, V)$-$\mathbf{Cat}_{\mathrm{sep}}$ *preserves finite products.*

Proof. We recall that, for any $(\mathbb{T}, V)$-category (X, a), $R(X, a) = (\tilde{X}, \tilde{a})$, with $\tilde{X} = X/\sim$, where $x \sim y$ if $k \le a(e_X(x), y) \wedge a(e_X(y), x)$, and $\tilde{a} = \eta_X \cdot a \cdot (T\eta_X)^\circ$, with $\eta_X : X \to \tilde{X}$ the projection. This structure makes η_X both an initial and a final morphism (see [HST14] for details).

Let $f : R(X \times Y) \to RX \times RY$ be the unique morphism such that $f \cdot \eta_{X\times Y} = \eta_X \times \eta_Y$.

$$\begin{tikzcd}
(X \times Y, c) \arrow[r, "\eta_{X\times Y}"] \arrow[dr, "\eta_X\times\eta_Y"'] & (R(X \times Y), \tilde{c}) \arrow[d, "f"] \\
& (RX \times RY, d)
\end{tikzcd}$$

From $c(e_{X\times Y}(x, y), (x', y')) = a(e_X(x), x')\wedge b(e_Y(y), y')$ it is immediate that $(x, y) \sim (x', y')$ in $X\times Y$ if, and only if, $x \sim x'$ in X and $y \sim y'$ in Y. Therefore, f is a bijection. Assuming the Axiom of Choice, so that T preserves surjections, we have, for every $\mathfrak{z} \in T(R(X \times Y))$, $(x, y) \in X \times Y$,

$$\begin{aligned}
\tilde{c}(\mathfrak{z}, [(x, y)]) &= c(\mathfrak{w}, (x, y)) && \text{(for any } \mathfrak{w} \in (T\eta_{X\times Y})^{-1}(\mathfrak{z})) \\
&= d(T(\eta_X \times \eta_Y)(\mathfrak{w}), ([x], [y])) && \text{(because } \eta_X \times \eta_Y \text{ is initial)} \\
&= d(Tf(\mathfrak{z}), ([x], [y]);
\end{aligned}$$

that is, f is initial and therefore an isomorphism. □

Theorem 6.3. *Under Assumptions 5.5,* $(\mathbb{T},V)$-**Cat** *is weakly cartesian closed.*

Proof. Given $(\mathbb{T},V)$-categories (X,a), (Y,b), to build the weak exponential $\ll X,Y\gg$ we will show the *cosolution set condition* for the functor $-\times(X,a)$.

For each $(\mathbb{T},V)$-functor $f:(Z,c)\times(X,a)\to(Y,b)$ we consider its reflection $Rf:RZ\times RX\cong R(Z\times X)\to RY$ and we factorise it through the weak evaluation in $(\mathbb{T},V)$-$\mathbf{Cat}_{\mathrm{sep}}$, $Rf=\widetilde{\mathrm{ev}}\cdot(\overline{Rf}\times 1_{RX})$, so that in the diagram below the outer rectangle commutes.

Then we define $Z_f=Z/\sim$ by

$$z\sim z' \text{ if both } f(z,x)=f(z',x),\text{ for every } x\in X,\text{ and } \overline{Rf}(\eta_Z(z))=\overline{Rf}(\eta_Z(z')),$$

and equip it with the final structure for the projection $q_f:Z\to Z_f$. Then $h_f:Z_f\to\ll RX,RY\gg$, with $h_f([z])=\overline{Rf}(\eta_Z(z))$, is a $(\mathbb{T},V)$-functor since its composition with q_f is $\overline{Rf}\cdot\eta_Z$ and q_f is final. Then we factorise f via the surjection $q_f\times 1_X:Z\times X\to Z_f\times X$ as in the diagram below. Moreover, the map $\hat{f}:Z_f\times X\to Y$, with $\hat{f}([z],x)=f(z,x)$, is a $(\mathbb{T},V)$-functor because $\eta_Y\cdot\hat{f}=\widetilde{\mathrm{ev}}\cdot(h_f\times\eta_X)$ is and η_Y is initial.

$$\begin{tikzcd}
Z\times X \arrow[rrr, "f"] \arrow[d, "\eta_Z\times 1_X"'] \arrow[dr, "q_f\times 1_X"] & & & Y \arrow[dd, "\eta_Y"] \\
RZ\times X \arrow[d, "\overline{Rf}\times 1_X"'] & Z_f\times X \arrow[urr, "\hat{f}"] \arrow[r] \arrow[dl, "h_f\times 1_X"] & (\coprod_g Z_g\times X)\cong(\coprod_g Z_g)\times X \arrow[ur, "\mathrm{ev}"] & \\
\ll RX,RY\gg\times X \arrow[r, "1\times\eta_X"'] & \ll RX,RY\gg\times RX \arrow[rr, "\widetilde{\mathrm{ev}}"'] & & RY
\end{tikzcd}$$

Since the cardinality of Z_f is bounded by the cardinality of the set $|\ll RX,RY\gg|\times|Y|^{|X|}$, as witnessed by the injective map

$$\begin{array}{rcl} Z_f & \to & |\ll RX,RY\gg|\times|Y|^{|X|}, \\ {[z]} & \mapsto & (\overline{Rf}(\eta_Z(z)),f(z,-)) \end{array}$$

there is only a set of possible $(\mathbb{T},V)$-categories Z_f. Hence we can form its coproduct, as in the diagram above, and consider the induced $(\mathbb{T},V)$-functor $\mathrm{ev}:(\coprod_g Z_g)\times X\cong\coprod_g(Z_g\times X)\to Y$ (note that the isomorphism follows from extensivity of $(\mathbb{T},V)$-**Cat**). ☐

7. EXAMPLES

In this section we use Theorem 6.3 to present examples of weakly cartesian closed categories. Hence, in conjunction with the following theorem established in [Ros99], we obtain examples of categories with cartesian closed exact completion since all other conditions of that theorem are trivially satisfied in these examples.

Theorem 7.1. *Let* $\mathbf{C}$ *be a complete, infinitely extensive and well-powered category in which every morphism factorizes as a regular epi followed by a mono, and where* $f\times 1$ *is an epimorphism for every regular epimorphism* $f:A\to B$ *in* $\mathbf{C}$. *Then, if* $\mathbf{C}$ *is weakly cartesian closed, the exact completion* $\mathbf{C}_{\mathrm{ex}}$ *of* $\mathbf{C}$ *is cartesian closed.*

We note that, in order to conclude that $(\mathbb{T},V)$-**Cat** is weakly cartesian closed, we have to check whether V and $\mathbb{T}$ satisfy Assumptions 5.5.

First we analyse examples where $\mathbb{T}$ is the identity monad. In this particular setting we only have to check that 5.5 (2) holds. The category V-**Cat** is always monoidal closed, as shown in [Law73]. Therefore, when V is a frame considered as a quantale, then V-**Cat** is cartesian closed. This is the case of 2, and so one concludes that **Ord** *is cartesian closed.* Moreover, for V the lattice $([0,\infty],\geq)$ with $\otimes=\wedge$, V-**Cat** is *the category of ultrametric spaces*, which *is* therefore also *cartesian closed.*

When $V=P_+$, V-**Cat** is the category **Met** of Lawvere's metric spaces [Law73], which is not cartesian closed (see [CH06] for details). However, the quantale P_+ satisfies 5.5 (2), and so from Theorem 6.3 it follows that **Met** *is weakly cartesian closed.*

Metric and ultrametric spaces can be also viewed as categories enriched in a quantale based on the complete lattice $[0,1]$ with the usual "less or equal" relation $\leq$, which is isomorphic to $[0,\infty]$ via the map $[0,1]\to[0,\infty]$, $u\mapsto-\ln(u)$ where $-\ln(0)=\infty$. More in detail, we consider the following quantale operations on $[0,1]$ with neutral element 1.

(1) For $\otimes=*$ being the ordinary multiplication, via the isomorphism $[0,1]\simeq[0,\infty]$, this quantale is isomorphic to the quantale P_+, hence $[0,1]$-**Cat** $\simeq$ **Met**.

(2) For the tensor $\otimes=\wedge$ being infimum, the isomorphism $[0,1]\simeq[0,\infty]$ establishes an equivalence between $[0,1]$-**Cat** and the category of ultrametric spaces and non-expansive maps.

(3) Another interesting multiplication on $[0,1]$ is the *Łukasiewicz tensor* $\otimes=\odot$ given by $u\odot v=\max(0,u+v-1)$. Via the lattice isomorphism $[0,1]\to[0,1]$, $u\mapsto 1-u$, this quantale is isomorphic to the quantale $[0,1]$ with "greater or equal" relation $\geq$ and tensor $u\otimes v=\min(1,u+v)$ truncated addition. Therefore $[0,1]$-**Cat** is equivalent to the *category of bounded-by-*1 *metric spaces and non-expansive maps.* Moreover, with respect to the "greater or equal" relation and truncated addition on $[0,1]$, the map

$$[0,\infty]\to[0,1],\; u\mapsto\min(1,u)$$

is a surjective quantale morphism; therefore, by Lemma 5.7, also $[0,1]$ with the Łukasiewicz tensor satisfies 5.5 (2).

(4) More generally, every continuous quantale structure $\otimes$ on the lattice $[0,1]$ (with Euclidean topology and the usual "less or equal" relation) with neutral element 1 satisfies 5.5 (2). This can be shown using the fact, proven in [Fau55] and [MS57], that every such tensor $\otimes:[0,1]\times[0,1]\to[0,1]$ is a combination of the three operations on $[0,1]$ described above. More precise:

   (a) For $u,v\in[0,1]$ and $e\in[0,1]$ idempotent with $u\leq e\leq v$: $u\otimes v=\min(u,v)=u$.

   (b) For every non-idempotent $u\in[0,1]$, there exist idempotents e and f with $e<u<f$ and such that the interval $[e,f]$ (with the restriction of the tensor on $[0,1]$ and with neutral element f) is isomorphic to $[0,1]$ either with multiplication or Łukasiewicz tensor.

Now let $w,u,v\in[0,1]$. We may assume $u\leq v$. If $u\otimes v\leq w$, then clearly

$$w\wedge(u\otimes v)=u\otimes v=\bigvee\{u'\otimes v'\mid u'\leq u, v'\leq v, u'\otimes v'\leq w\}.$$

We consider now $w<u\otimes v\leq u\leq v$. If w is idempotent, then

$$w=w\otimes v,\quad w\leq u,\quad v\leq v;$$

otherwise there are idempotents e and f with $e<w<f$ and $[e,f]$ is isomorphic to $[0,1]$ either with multiplication or Łukasiewicz tensor.

Case 1: $v\leq f$. Then 5.5 (2) holds since $w,u\otimes v,u,v\in[e,f]$.

Case 2: $f < v$. Then $w = w \wedge v = w \otimes v$, $w \leq u$ and $v \leq v$.

We conclude that $[0,1]$-**Cat** *is weakly cartesian closed*, for every continuous quantale structure $\otimes$ on the lattice $[0,1]$ with neutral element 1.

Now let $V = \Delta$ be the *quantale of distribution functions* (see [HR13, CH17] for details). As observed in [HR13], it verifies 5.5 (2), and so we can conclude from Theorem 6.3 that *the category* Δ-**Cat** *of probabilistic metric spaces and non-expansive maps is weakly cartesian closed.*

When $\mathbb{T}$ is not the identity monad, some further work is need to guarantee Assumptions 5.5.

Theorem 7.2. *(1) The tensor product on the quantale V defines a $(\mathbb{T}, V)$-functor $\otimes : V \otimes V \to V$.*

(2) Let $u \in V$ satisfying $u\cdot ! \geq \xi \cdot Tu$.

$$\begin{tikzcd} T1 \arrow[r, "Tu"] \arrow[d, "!"'] \arrow[dr, phantom, "\geq"] & TV \arrow[d, "\xi"] \\ 1 \arrow[r, "u"'] & V \end{tikzcd}$$

Then $(-, u) : X \to X \times V$ is a $(\mathbb{T}, V)$-functor, for every $(\mathbb{T}, V)$-category X.

(3) Let $u \in V$ satisfying $u\cdot ! = \xi\cdot Tu$. Then $T(r \otimes u) = (Tr) \otimes u$, for every V-relation $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$.

Proof. The first assertion is [Hof11, Proposition 1.4(1)]. To see (2), assume that $u \in V$ with $u\cdot ! \geq \xi \cdot Tu$. Let (X, a) be a $(\mathbb{T}, V)$-category, $\mathfrak{x} \in TX$ and $x \in X$. Considering the map $X \xrightarrow{!} 1 \xrightarrow{u} V$, we have to show that

$$a(\mathfrak{x}, x) \leq a(\mathfrak{x}, x) \otimes \hom(T(u\cdot !)(\mathfrak{x}), u),$$

which follows immediately from $u\cdot ! \geq \xi \cdot Tu$. Finally, to prove (3), let $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ be a V-relation and $u \in V$ with $u\cdot ! = \xi \cdot Tu$. Note that the V-relation $r \otimes u : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ is given by

$$X \times Y \xrightarrow{\ r\ } V \xrightarrow{\langle 1_V, u\cdot ! \rangle} V \times V \xrightarrow{\ \otimes\ } V.$$

Hence, applying the **Set**-functor T to the functions $r : X \times Y \to V$ and $r \otimes u : X \times Y \to V$, we obtain

$$\begin{aligned} \xi \cdot T(r \otimes u) &= \xi \cdot T(\otimes) \cdot T\langle 1_V, u\cdot ! \rangle \cdot Tr \\ &= \otimes \cdot (\xi \times \xi) \cdot \mathrm{can}_{X,Y} \cdot T\langle 1_V, u\cdot ! \rangle \cdot Tr \\ &= \otimes \cdot \langle \xi, u\cdot ! \cdot \xi \rangle \cdot Tr \\ &= \otimes \cdot \langle 1_V, u\cdot ! \rangle \cdot \xi \cdot Tr. \end{aligned}$$

Therefore, returning to V-relations, we conclude that $T(r \otimes u) = (Tr) \otimes u$. □

Remark 7.3. If $T1 = 1$, then $u\cdot ! = \xi \cdot Tu$ for every $u \in V$.

In order to guarantee Assumptions 5.5 (1), we need an extra condition on ξ.

Proposition 7.4. *Assume that*

$$\begin{tikzcd} T(V \times V) \arrow[r, "T(\wedge)"] \arrow[d, "\langle \xi\cdot T\pi_1, \xi\cdot T\pi_2 \rangle"'] \arrow[dr, phantom, "\leq"] & TV \arrow[d, "\xi"] \\ V \times V \arrow[r, "\wedge"'] & V. \end{tikzcd}$$

Then, for all V-relations $r : X \nrightarrow X'$ and $s : Y \nrightarrow Y'$,

$$\begin{tikzcd}
T(X\times Y) \arrow[r, "\mathrm{can}_{X,Y}"] \arrow[d, "T(r\owedge s)"', "\shortmid" marking] \arrow[dr, phantom, "\geq"] & TX\times TY \arrow[d, "Tr\owedge Ts", "\shortmid" marking] \\
T(X'\times Y') \arrow[r, "\mathrm{can}_{X',Y'}"'] & TX'\times TY'.
\end{tikzcd}$$

Proof. First we note that, from the preservation of weak pullbacks by T, it follows that the commutative diagram

$$\begin{tikzcd}
T(A\times B) \arrow[r, "T(f\times g)"] \arrow[d, "\mathrm{can}_{A,B}"'] & T(X\times Y) \arrow[d, "\mathrm{can}_{X,Y}"] \\
TA\times TB \arrow[r, "Tf\times Tg"'] & TX\times TY
\end{tikzcd}$$

is also a weak pullback.

Let $\mathfrak{w}\in T(X\times Y)$, $\mathfrak{x}'\in TX'$ and $\mathfrak{y}'\in TY'$. Put $(\mathfrak{x},\mathfrak{y})=\mathrm{can}_{X,Y}(\mathfrak{w})$. By the definition of the extension of T and since V is a Heyting algebra,

$$Tr(\mathfrak{x},\mathfrak{x}')\wedge Ts(\mathfrak{y},\mathfrak{y}')=\bigvee\left\{\xi\cdot Tr(\mathfrak{w}_1)\wedge\xi\cdot Ts(\mathfrak{w}_2)\;\middle|\;\begin{matrix}\mathfrak{w}_1\in T(X\times X'):\mathfrak{w}_1\mapsto\mathfrak{x},\mathfrak{w}_1\mapsto\mathfrak{x}'\\ \mathfrak{w}_2\in T(Y\times Y'):\mathfrak{w}_2\mapsto\mathfrak{y},\mathfrak{w}_2\mapsto\mathfrak{y}'\end{matrix}\right\}.$$

Note that in

$$\begin{tikzcd}
& T(X\times Y\times X'\times Y') \arrow[d, "\cong"'] & & \\
T(X\times Y) \arrow[d, "\mathrm{can}"'] & T(X\times X'\times Y\times Y') \arrow[l, "T(\pi_X\times\pi_Y)"'] \arrow[r, "T(r\times s)"] \arrow[d, "\mathrm{can}"'] & T(V\times V) \arrow[r, "T(\wedge)"] \arrow[d, "\mathrm{can}"'] & TV \arrow[dd, "\xi"] \\
TX\times TY & T(X\times X')\times T(Y\times Y') \arrow[l, "T\pi_X\times T\pi_Y"] \arrow[r, "Tr\times Ts"'] & TV\times TV \arrow[d, "\xi\times\xi"'] \arrow[dr, phantom, "\leq"] & \\
& & V\times V \arrow[r, "\wedge"'] & V
\end{tikzcd}$$

the left hand side is a weak pullback, the middle diagram commutes, and in the right hand side we have "lower path" $\leq$ "upper path" as indicated. Therefore, for such $\mathfrak{w}_1\in T(X\times X')$ and $\mathfrak{w}_2\in T(Y\times Y')$, there exists some $\mathfrak{v}\in T(X\times X'\times Y\times Y')$ which projects to $\mathfrak{w}\in T(X\times Y)$ and to $(\mathfrak{w}_1,\mathfrak{w}_2)\in T(X\times X')\times T(Y\times Y')$. Hence, taking also into account the definition of the V-relation $T(r\owedge s)$,

$$\begin{aligned}Tr(\mathfrak{x},\mathfrak{x}')\wedge Ts(\mathfrak{y},\mathfrak{y}')&\leq\bigvee\left\{\xi\cdot T(\wedge)\cdot T(r\times s)(\mathfrak{v})\;\middle|\;\mathfrak{v}\in T(X\times Y\times X'\times Y');\;\begin{matrix}\mathfrak{v}\mapsto\mathfrak{w}\\ \mathfrak{v}\mapsto\mathfrak{x}',\mathfrak{v}\mapsto\mathfrak{y}'\end{matrix}\right\}\\ &\leq\bigvee\{T(r\owedge s)(\mathfrak{w},\mathfrak{w}')\mid\mathfrak{w}'\in T(X'\times Y'),\ \mathrm{can}_{X',Y'}(\mathfrak{w}')=(\mathfrak{x}',\mathfrak{y}')\}.\end{aligned}$$

□

Remark 7.5. We note that the inequality

$$\begin{tikzcd}
T(V\times V) \arrow[r, "T(\wedge)"] \arrow[d, "\langle\xi\cdot T\pi_1,\xi\cdot T\pi_2\rangle"'] \arrow[dr, phantom, "\geq"] & TV \arrow[d, "\xi"] \\
V\times V \arrow[r, "\wedge"'] & V
\end{tikzcd}$$

is always true.

Corollary 7.6. *If the quantale V satisfies Assumption 5.5 (2) and the diagrams*

$$\begin{tikzcd} T(V\times V) \arrow[r, "T(\wedge)"] \arrow[d, "\langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle"'] & TV \arrow[d, "\xi"] \\ V\times V \arrow[r, "\wedge"'] & V \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T1 \arrow[r, "Tu"] \arrow[d, "!"'] & TV \arrow[d, "\xi"] \\ 1 \arrow[r, "u"'] & V \end{tikzcd}$$

commute, for all $u \in V$, then all Assumptions 5.5 are satisfied.

Let $\mathbb{T}$ be the ultrafilter monad $\mathbb{U} = (U, m, e)$. Then, when V is any of the quantales listed above but Δ, all the needed conditions are satisfied. Therefore, in particular we can conclude that:

Examples 7.7. (1) *The category* **Top** $= (\mathbb{U}, 2)$-**Cat** *of topological spaces and continuous maps is weakly cartesian closed* (as shown by Rosický in [Ros99]).

(2) *The category* **App** $= (\mathbb{U}, P_+)$-**Cat** *of approach spaces and non-expansive maps is weakly cartesian closed.*

(3) In fact, for each continuous quantale structure on the lattice $([0, 1], \leq) \simeq ([0, \infty], \geq)$, $(\mathbb{U}, [0, 1])$-**Cat** is weakly cartesian closed. In particular, *the category of non-Archimedean approach spaces and non-expansive maps studied in* [CVO17] *is weakly cartesian closed.*

(4) If V is a completely distributive complete lattice with $\otimes = \wedge$, then, with

$$\xi : UV \to V,\ \mathfrak{x} \mapsto \bigwedge_{A\in\mathfrak{x}} \bigvee A,$$

all the conditions of Theorem 6.3 are satisfied (see [Hof07, Theorem 3.3]) and therefore $(\mathbb{U}, V)$-**Cat** is weakly cartesian closed. In particular, with $V = P2$ being the powerset of a 2-element set, we obtain that *the category* **BiTop** *of bitopological spaces and bicontinuous maps is weakly cartesian closed* (see [HST14]).

Remark 7.8. For $V = \Delta$ the quantale of distribution functions, we do not know whether there is an appropriate compact Hausdorff topology $\xi : UV \to V$ satisfying the conditions of this section.

Now let $\mathbb{T}$ be the free monoid monad $\mathbb{W} = (W, m, e)$. For each quantale V, we consider

$$\xi : WV \to V,\ (v_1, \dots, v_n) \mapsto v_1 \otimes \cdots \otimes v_n,\ () \mapsto k$$

which induces the extension $W : V$-**Rel** $\to V$-**Rel** sending $r : X \to\!\!\!\!\!\shortmid\ Y$ to the V-relation $Wr : WX \to\!\!\!\!\!\shortmid\ WY$ given by

$$Wr((x_1, \dots, x_n), (y_1, \dots, y_m)) = \begin{cases} r(x_1, y_1) \otimes \cdots \otimes r(x_n, y_n) & \text{if } n = m, \\ \bot & \text{if } n \neq m. \end{cases}$$

The category $(\mathbb{W}, 2)$-**Cat** is equivalent to the category **MultiOrd** of *multi-ordered sets* and their morphisms (see [HST14]), more generally, $(\mathbb{W}, V)$-categories can be interpreted as multi-V-categories and their morphisms. The representable multi-ordered sets are precisely the ordered monoids, which is a special case of [Her00, Her01] describing monoidal categories as representable multi-categories (see also [CCH15]). We recall that the separated injective multi-ordered sets are precisely the quantales (see [LBKR12] and also [Sea10]), and we conclude:

Proposition 7.9. *Every quantale is exponentiable in* **MultiOrd**.

Theorem 7.10. *If the quantale V is a frame, then* $(\mathbb{W}, V)$-**Cat** *is weakly cartesian closed. In particular,* **MultiOrd** *is weakly cartesian closed.*

Finally, for a monoid $(H, \cdot, h)$, we consider the monad $\mathbb{H} = (- \times H, m, e)$, with $m_X : X \times H \times H \to X \times H$ given by $m_X(x, a, b) = (x, a \cdot b)$ and $e_X : X \to X \times H$ given by $e_X(x) = (x, h)$. Here we consider

$$\xi : V \times H \to V,\ (v, a) \mapsto v,$$

which leads to the extension $- \times H : V\text{-}\mathbf{Rel} \to V\text{-}\mathbf{Rel}$ sending the V-relation $r : X \nrightarrow Y$ to the V-relation $r \times H : X \times H \nrightarrow Y \times H$ with

$$r \times H((x, a), (y, b)) = \begin{cases} r(x, y) & \text{if } a = b, \\ \bot & \text{if } a \neq b. \end{cases}$$

In particular, $(\mathbb{H}, 2)$-categories can be interpreted as *H-labelled ordered sets* and *equivariant maps*.

For every quantale V and every $v : 1 \to V$, the diagrams

$$\begin{tikzcd} V \times V \times H \arrow[r, "\wedge \times 1_H"] \arrow[d, "\pi_{1,2}"'] & V \times H \arrow[d, "\xi = \pi_1"] \\ V \times V \arrow[r, "\wedge"'] & V \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} 1 \times H \arrow[r, "v \times 1_H"] \arrow[d, "!"'] & V \times H \arrow[d, "\xi"] \\ 1 \arrow[r, "v"'] & V \end{tikzcd}$$

commute, therefore we obtain:

Theorem 7.11. *For every quantale V satisfying Assumption 5.5* (2)*, the category* $(\mathbb{H}, V)$-**Cat** *is weakly cartesian closed.*

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