Outside options and confidence in Zeuthen-Hicks bargaining

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Abstract

The Zeuthen-Hicks bargaining model connects strategic and axiomatic bargaining models by providing a description of the behavior of each party, and showing that the entire process leads to the axiomatically founded Nash bargaining solution. In its original formulation, the model treats parties asymmetrically by considering different decision alternatives of the focal party (who can either accept the opponent's offer or make a counteroffer, but not quit the negotiation) and the opponent (who can accept the focal party's offer or quit the negotiation, but not make a counteroffer). We extend the model to consider the full set of possible actions from both sides, which requires explicit modeling of the expectations of the parties concerning outcomes and outside options that become available during the process. We show analytically that under the assumption of concave utilities of both parties, the bargaining process converges to the nonsymmetric Nash bargaining solution. This result provides a new interpretation of the parameters of the nonsymmetric Nash bargaining solution, linking them to behavior in the bargaining process. Furthermore, we perform a simulation study to analyze the outcomes for non-concave utilities.

Keywords: Zeuthen-Hicks bargaining, Nonsymmetric Nash bargaining solution, negotiator confidence

1 Introduction

Two-party bargaining is a classical negotiation problem with a clear practical relevance (commercial contracts, labor contracts, corporate mergers, etc.). Having been studied for a long time (Nash, 1950; Harsanyi, 1956), it still attracts

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the attention of many scholars (e.g., Bastianello and LiCalzi, 2019; Dias and Vetschera, 2019a; Schweighofer-Kodritsch, 2018). A useful abstraction of the two-party bargaining problem is to consider bargaining over one single issue. Although many real-life bargaining problems involve several issues, aggregation of multiple issues to utility values leads to a formally similar structure, if only efficient solutions are considered. The present paper therefore, following the literature, considers a single issue bilateral bargaining problem.

The literature often makes a distinction between axiomatic and strategic bargaining models (e.g., Sutton, 1986). The former are mostly concerned with an axiomatic characterization of bargaining solutions, such as the well-known Nash Bargaining Solution (NBS) (Nash, 1950) or the Raiffa-Kalai-Smorodinsky solution (Kalai and Smorodinsky, 1975), but do not discuss the process of reaching that solution. The latter emphasize the process of bargaining and the strategic actions of parties, studying how parties decide which offers to exchange from one round to the next. Among strategic bargaining models we can refer for instance to the classical models of Zeuthen-Hicks (Zeuthen, 1930; Harsanyi, 1956; Bishop, 1964) and Rubinstein (1982). Strategic and axiomatic models are often related, as the Zeuthen-Hicks (Z-H) and Rubinstein models have been shown to converge to the NBS under some assumptions.

In this work, we focus on the Z-H bargaining model, which is particularly interesting as it considers the possibility of several negotiation rounds. Thus this model has been used to study bargaining processes empirically (Fandel, 1985; Vetschera, 2019). Because of its dynamic nature, it raises the issue of whether the process will converge to the theoretically optimal solution or not (Dias and Vetschera, 2019a). According to this model, the two parties (for illustrative purposes we refer to them as the seller and the buyer) make alternating offers, in which they change their position on the issue under negotiation. For illustrative purposes, we refer to the issue as the price.

In each step of the process, the focal party needs to choose between accepting the opponent's offer or making a counter-offer. For example, the seller has to decide whether to accept the price offered by the buyer, or demand a higher price. The latter option is a gamble, as the buyer can concede and accept to pay that price, or it may quit and both parties get a worse disagreement outcome. Obviously, there is a critical probability at which the seller would be indifferent between the two options. This critical probability is seen as an indicator of each party's current strength in the bargaining process: The party that can tolerate a higher probability that the opponent terminates the negotiation has a better position. Consequently, the party having the lower critical probability will make a concession (weaken its demand), to obtain a critical probability that is higher than the opponent's. A brief overview of the formal structure of the model will be presented in section 3.

Since the core of the Z-H model is the decision of one focal negotiator, it considers the two parties initially in an asymmetric way, which leads to two somewhat paradoxical situations. Each party considers a probability that the other party quits, but both are contemplating a choice between accepting the other party's offer or making a new offer, i.e., none of the parties contemplates quitting. In this article, we will address this issue by extending the Z-H model in a way that explicitly models the possibility of quitting by both sides. As we will show, this extension requires that a party will quit the bargaining process only if an attractive outside option appears, which was previously not available. The model thus applies to negotiations in which this can occur. The second paradox of the standard Z-H model is that this model on the one hand prescribes that the weaker party should make a concession, i.e. change its offers, yet the only reactions of the other party that each party anticipates are accepting that offer or terminating the negotiation. In reality, one will expect that after making a counter-offer, the negotiation process will go on for some more rounds in which the other party will also change its position. The agreement that is eventually reached will therefore likely be somewhere in-between the two offers currently on the table. Our extended Z-H model therefore explicitly introduces the expectations a party might have about the ultimate outcome of the bargaining process. These expectations depend on factors such as the confidence each party has in its own bargaining skills versus those of the opponent, or contextual factors such as time pressure. A more confident party will expect the final outcome to be close to its own offer, a less confident party might expect the outcome to be closer to the opponent's current position.

In this paper, we show that under the assumption that the two parties have concave utility functions, our extended Z-H model provides a full characterization of the exchange of successive offers, as well as the predicted outcome. The analysis of the model yields the nonsymmetric (or asymmetric) NBS, i.e., the solution obtained in Nash's framework without the symmetry axiom (see, inter alia, Muthoo, 1999; Roth, 1979), in which the utilities are exponentially weighted. Therefore, this work maintains a linkage between the strategic and axiomatic interpretations of the Z-H model. We also show that the exponents in the nonsymmetric NBS can be interpreted as parameters reflecting the confidence of the negotiators, thus providing a new interpretation for the exponents of the nonsymmetric NBS. This adds a process-related perspective to the usual interpretation as "bargaining strength".

These results hold only for concave utility functions. As a further contribution of this article, we conduct a simulation study to examine the outcome for utility functions which are not concave. In particular, this study assesses the impact of negotiator confidence on the results, in terms of outcome for each party, joint utility, and by how much the theoretical solution is missed.

Ensuing this introduction, Section 2 describes the standard Z-H model and the proposed extensions. Section 3 provides analytical results for the extended model, allowing to characterize the mutually optimal solution and the process of reaching that solution when the utilities are concave. Section 4 presents a simulation study for the general case in which utility functions are not necessarily concave. Section 5 presents the main conclusions and suggests some of the future research paths opened by this work.

2 Model overview

We consider a negotiation between a buyer and a seller about one single transaction. Without loss of generality, we assume that the good to be traded has a value of zero to the seller, and a value of one to the buyer. Trading thus creates a value of one, that can be split among the two parties. The only issue to be negotiated is the price, which thus describes how the potential gain from trade of one is allocated to the parties. We denote the seller's offer by s and the buyer's offer by b. The seller's utility of some price x is $u_s(x)$, and the buyer's utility is $u_b(x)$. Buyer and seller have opposing preferences: the seller prefers a higher price and the buyer a lower price. Thus we assume throughout the paper that $u'_s(x) > 0$ and $u'_b(x) < 0$.

As we have already outlined in the introduction, the standard Z-H model treats the two parties asymmetrically and considers different actions for both parties. While the focal party has to make a decision between accepting the opponent's offer and continuing the bargaining process with his or her own offer, the opponent's choices are whether to accept the focal party's offer or to quit the negotiation.

Before introducing our extensions, we briefly review the standard model from the point of view of the seller as focal party. The seller has to decide whether to accept the buyer's offer b, which would provide a utility of $u_s(b)$, or to make a counter-offer s, which can either be rejected or accepted by the buyer. If the buyer rejects and terminates the negotiation, both parties receive a disagreement outcome d. In the buyer-seller example we consider, the disagreement outcome is that no trade takes place, and both parties therefore receive zero profit from trade. The probability that the buyer rejects the seller's offer s is p_b. A seller who maximizes expected utility will be willing to take the risk of rejection and make the offer s if

$$p_b u_s(d) + (1 - p_b) u_s(s) > u_s(b) \tag{1}$$

From this inequality, the critical probability p_b, which would make the seller indifferent between the two options, is given by

$$p_b = \frac{u_s(s) - u_s(b)}{u_s(s) - u_s(d)} \tag{2}$$

Similarly, the critical probability from the buyer's perspective (the probability that the seller will reject the buyer's offer) is

$$p_s = \frac{u_b(b) - u_b(s)}{u_b(b) - u_b(d)} \tag{3}$$

To gain the lead over the buyer, the seller wants to achieve

$$p_b > p_s \tag{4}$$

By substituting (2) and (3) and taking into account that disagreement is the worst outcome for both parties and therefore $u_s(d) = u_b(d) = 0$, condition (4) becomes

$$u_b(s) u_s(s) > u_b(b) u_s(b) \tag{5}$$

If the seller has managed to establish condition (5), the buyer will then try to reverse it to gain the lead again. Therefore, each side wants to maximize the product of utilities of its offer. If both parties continue to increase that product, the process will converge to the NBS. If the utility functions of both sides are concave, it can be shown (Dias and Vetschera, 2019a) that the function $u_b(x)u_s(x)$ is quasiconcave and therefore has a single maximum. Otherwise, if parties use only local information, they might get stuck in a local maximum and fail to reach the NBS.

In the framework shown in Figure 1, we extend this model to take into account that both parties have the three options of quitting, accepting their opponent's offer, or continuing the negotiation with some counter-offer.

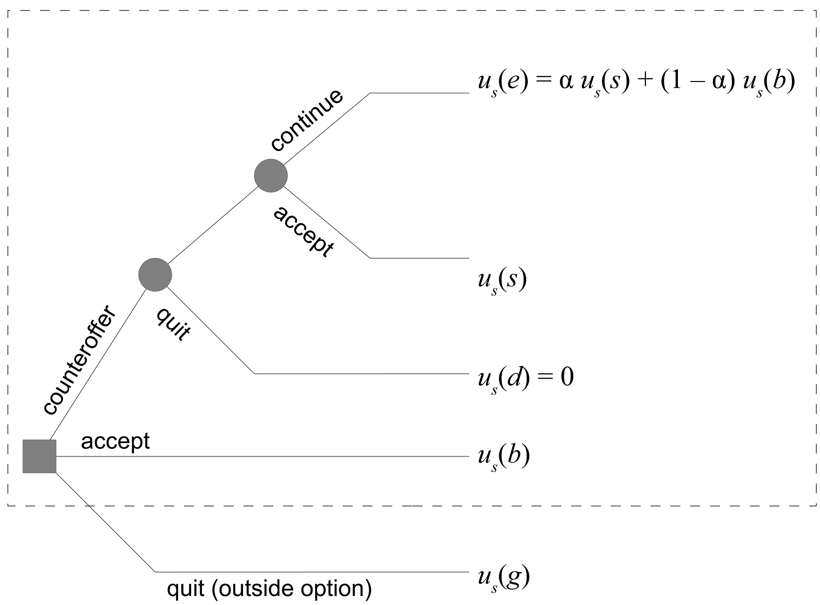

Figure 1: Framework: Decisions in one negotiation round (Seller's perspective)

In the standard Z-H model, terminating the negotiation (which only the opponent can do) leads to the disagreement outcome d, which is a bad outcome for both parties. In our extended model, we consider that quitting the negotiation and termination by the opponent are possibly different. Note that if in Figure 1, g (the outcome for the seller if the seller quits the negotiation) and d (the outcome if the buyer quits) would be the same, then quitting would always be either dominated by continuing the negotiation (if the expected outcome from continuation were better), or dominate it (otherwise). Thus, for a constant disagreement outcome, one of the strategies would be eliminated from consideration. In particular, if the disagreement outcome is indeed the worst possible outcome for both sides, then quitting would also be a dominated strategy for the opponent. Therefore the whole logic of the Z-H model, which is built on the possibility that the opponent terminates the negotiation, would break down.

However, outside options might come up during an ongoing negotiation. A new potential trading partner could be found, or become active. If the prospects of dealing with the new trading partner seem to be more attractive than the partner with whom one negotiates (and bargaining in good faith requires that one negotiates with only one potential partner at a given time), this might lead to termination of the negotiation. Considering the possibility that outside options appear thus makes it possible that the opponent terminates the negotiation, but it is uncertain whether that will actually happen. It is also noteworthy that connecting the opponent's decision to quit to an uncertain outside option also avoids the need for a subjective interpretation of this probability (on the relevance of removing this need, we refer to the debate between Kadane and Larkey (1982) and Harsanyi (1982)).

In our analysis, we assume that at this moment no such dominating outside offer exists from the perspective of the focal party (which we identify with the seller). Thus the focal party only considers the part of the decision tree that is enclosed in the dashed rectangle in Figure 1.

Our second extension of the standard Z-H model concerns the possibility that the opponent continues the negotiation by making a counter-offer. Thus, we do not consider the offer s made by the focal negotiator as a final take-it-or-leave-it offer. The opponent might accept the focal negotiator's offer, or make a counter-offer and eventually the negotiation will then end in a compromise e. The opponent might also terminate the negotiation, in which case the focal party would receive the disagreement utility $u(d)$ (assuming that no better outside option for the focal party becomes available). In order to be consistent with the notion that bargaining might continue for several rounds after making the offer s, the probability p_b^q now has to refer to the possibility that the opponent (the buyer) will quit the negotiation at any future step (not necessarily immediately after receiving offer s).

Since e is somewhere between the offers of the two parties currently on the table, we can write the utility of e to the seller as a linear combination of the utilities of the two offers:

$$u_s(e) = \alpha_s u_s(s) + (1 - \alpha_s) u_s(b) \tag{6}$$

The buyer could also accept the seller's offer rather than continue bargaining. We denote the probability that the buyer accepts the seller's offer by p_b^a. The expected outcome of the negotiation in case that the buyer decides not to quit the negotiation is

$$p_b^a u_s(s) + (1 - p_b^a)[\alpha_s u_s(s) + (1 - \alpha_s) u_s(b)] = \gamma_s u_s(s) + (1 - \gamma_s) u_s(b) \tag{7}$$

where

$$\gamma_s = p_b^a + (1 - p_b^a)\alpha_s \tag{8}$$

is a parameter that can be interpreted as the seller's confidence in its own bargaining strength and skills. A negotiator who is very confident (has a high

γ) will assume that the negotiation will end somewhere near its own offer (high α), or that he/she is able to convince the opponent to immediately accept the offer, thus increasing p_b^a. In contrast, a more pessimistic negotiator will assume the opposite, i.e. that eventually the opponent will prevail. We denote the certainty equivalent of lottery (7) by z_s and the expected utility that the seller will obtain if the buyer does not quit by $u_s(z_s)$.

By introducing the notion of confidence, we add a descriptive element to the model. This extension towards a more behavioral model is in line with empirical literature utilizing the Z-H model (such as Svejnar, 1986), which explains asymmetry in bargaining outcomes by differences in individual characteristics of the parties such as risk attitudes.

We can now set up the model in a way very similar to the standard Z-H model. The seller will prefer to continue the negotiation rather than directly accept the buyer's offer iff

$$(1 - p_b^q)u_s(z_s) + p_b^q u_s(d) > u_s(b) \tag{9}$$

where p_b^q is the probability that the buyer quits the negotiation (e.g. because he or she receives a better outside offer). Note that this probability refers to the possibility that the buyer receives an outside offer that would cause him or her to quit the negotiation. It is therefore not related to the seller's confidence in his or her own bargaining skills, but to the environment in which the negotiation takes place. After rearrangement of terms, we obtain a critical probability for the seller as

$$p_b^q = \frac{u_s(z_s) - u_s(b)}{u_s(z_s) - u_s(d)} \tag{10}$$

Similarly, we can calculate a critical probability for the buyer as

$$p_s^q = \frac{u_b(z_b) - u_b(s)}{u_b(z_b) - u_b(d)} \tag{11}$$

In accordance with the standard model, we interpret these probabilities as indicators of the strength of the current bargaining position of each party. The seller wishes to obtain a stronger position versus the buyer and wants to establish the relation $p_b^q > p_s^q$, i.e.,

$$\frac{u_s(z_s) - u_s(b)}{u_s(z_s) - u_s(d)} > \frac{u_b(z_b) - u_b(s)}{u_b(z_b) - u_b(d)} \tag{12}$$

Since we can, without loss of generality, set the utility of the disagreement outcome for both parties to zero, $u_s(d) = u_b(d) = 0$, the inequality above reduces to

$$u_s(z_s)u_b(z_b) - u_s(b)u_b(z_b) > u_s(z_s)u_b(z_b) - u_s(z_s)u_b(s) \tag{13}$$

and therefore

$$u_s(z_s)u_b(s) > u_s(b)u_b(z_b) \tag{14}$$

As long as condition (14) does not hold, the seller will try to establish it. Note that if both parties have the maximum level of confidence, i.e., $\gamma_s = \gamma_b = 1$, condition (14) is equal to the condition (5) in the standard Z-H model. In (5), only the left hand side can be influenced by the seller, so the seller will aim at increasing the product of utilities of his or her offer, which eventually leads to the NBS. In contrast, both sides of the inequality (14) depend on the offers of both parties. Still, as we will show in the next section, there are situations in which the process will lead to the NBS even if the parties are not fully optimistic.

3 Model analysis

The extended model presented in the previous section contains the standard Z-H model as a special case for $\gamma_s = \gamma_b = 1$. In the opposite extreme case, if both parties are purely pessimistic, they both assume that continuing the bargaining process will lead them exactly to the same outcome as the opponent's offer that is already on the table. Thus, they are indifferent between immediately accepting the opponent's offer and a continuation of bargaining, and their critical probabilities are zero. For the rest of our analysis, we ignore this possibility and assume that the confidence parameters for both sides are strictly positive and less than one.

The NBS can be obtained even if the two parties are not fully confident, as the following proposition shows:

Proposition 1 *The bargaining process will converge to the (symmetric) NBS if the confidence levels of both parties are equal and strictly positive.*

Proof: We denote the common level of confidence by $\gamma_s = \gamma_b = \gamma$. By substituting the definitions of $u_s(z_s)$ and $u_b(z_b)$, inequality (14) becomes

$$[\gamma u_s(s) + (1-\gamma)u_s(b)]u_b(s) > u_s(b)[\gamma u_b(b) + (1-\gamma)u_b(s)] \tag{15}$$

Since the term $(1-\gamma)u_s(b)u_b(s)$ cancels out and $\gamma > 0$, this is equivalent to (5). q.e.d.

Proposition 1 already provides a hint that the outcome of the process depends on the relative magnitude of the two confidence parameters rather than on their values. The following proposition shows that this is indeed the case:

Proposition 2 *If the confidence parameters of both parties are strictly positive, the outcome of an extended ZH-bargaining model depends only on the ratio of confidence parameters, not on their values.*

Proof: Let $r = \gamma_s/\gamma_b$ denote the ratio of the two confidence parameters. Then inequality (15) becomes

$$[r\gamma_b u_s(s) + (1-r\gamma_b)u_s(b)]u_b(s) > [\gamma_b u_b(b) + (1-\gamma_b)u_b(s)]u_s(b) \tag{16}$$

$$\Leftrightarrow r\gamma_b u_s(s)u_b(s)+u_s(b)u_b(s)-r\gamma_b u_s(b)u_b(s) > \gamma_b u_s(b)u_b(b)+u_s(b)u_b(s)-\gamma_b u_b(s)u_s(b) \tag{17}$$

$$\Leftrightarrow r\gamma_b u_s(s)u_b(s) - r\gamma_b u_s(b)u_b(s) > \gamma_b u_s(b)u_b(b) - \gamma_b u_b(s)u_s(b) \tag{18}$$

Dividing this inequality by $\gamma_b > 0$ yields the equivalent condition which depends on r rather than γ_s and γ_b:

$$r(u_s(s) - u_s(b))u_b(s) > (u_b(b) - u_b(s))u_s(b) \tag{19}$$

q.e.d.

Note that as long as the seller's offer is better for the seller than the buyer's offer, $u_s(s) - u_s(b) > 0$. Thus, the larger r, the easier it is to establish that inequality. This means a seller who is more confident will make smaller concessions (and end up at a better value). As $s - b$ decreases, both sides of the above inequality tend to 0.

For the remainder of this section, we assume that both utilities are concave. As the simulation results in the following section will show, this assumption is crucial for many properties of the model, and no clear predictions about the outcome of the process can be made for non-concave utilities.

For our further analysis, we define a function $f(b, s)$ as the difference between the left hand and right hand side of (19):

$$f(b, s) = r(u_s(s) - u_s(b))u_b(s) - (u_b(b) - u_b(s))u_s(b) \tag{20}$$

As long as $f(b, s) < 0$, the seller will change s to increase its value above zero, and if $f(b, s) > 0$, the buyer will change b to decrease f below zero. We first analyze whether these changes actually correspond to concessions of the respective parties. A concession by the seller means lowering the price s demanded by the seller, a concession by the buyer means increasing b. Thus, the seller has an incentive to make a concession (decrease s in order to increase f) if

$$\frac{\partial f}{\partial s} < 0 \tag{21}$$

and the buyer has a incentive to make a concession (increase b in order to decrease f) if

$$\frac{\partial f}{\partial b} < 0 \tag{22}$$

For our analysis, we represent the problem in b/s space and analyze in which regions of that space the above conditions are fulfilled. Note that as soon as $s \leq b$, both sides would accept the other side's offer, so we have to consider only situations in which $s > b$. Furthermore, $f(b, s) = 0$ for $s = b$. We now show that for $r < 1$, the area defined by $\{(b, s) : 0 \leq b \leq 1, 0 \leq s \leq 1, b < s\}$ can be partitioned as shown in Figure 2.

In the regions labeled A and B, the buyer needs to make a concession, as $f(b, s) > 0$. Since in both regions $\partial f/\partial b < 0$, the buyer here needs to increase the offered price in order to decrease f. Similarly, the seller will make a concession and decrease the demanded price in regions C and D.

The partitioning shown in Figure 2 is characterized by the following properties:

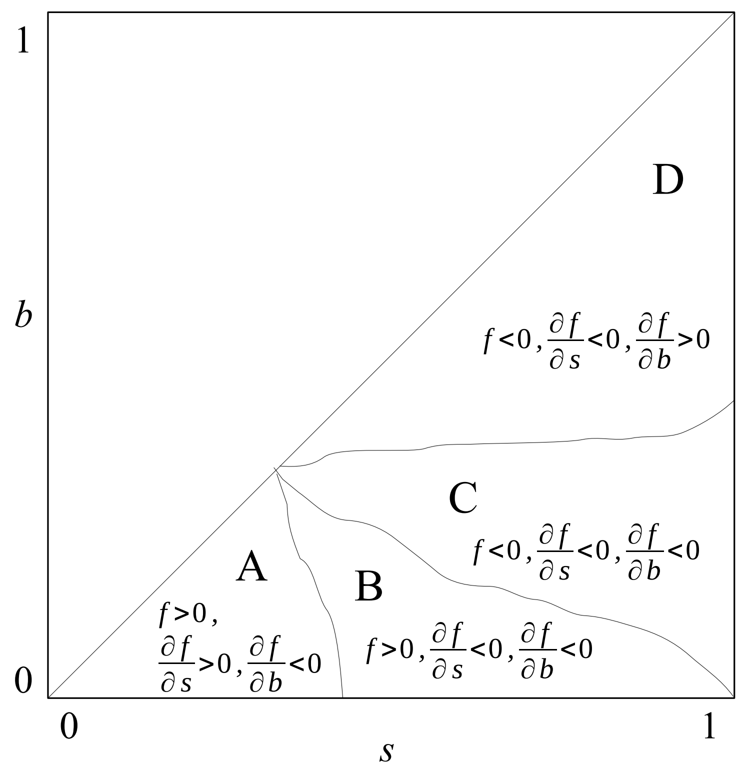

Figure 2: Partitioning of b/s space when the utilities are concave and $r < 1$

- The curve separating regions A and B, i.e., the curve along which $\partial f/\partial s = 0$, is monotonically decreasing and intersects with the line $b = 0$ at $s = N$, where N is the price in the NBS.
- The curve separating regions B and C, i.e., the curve along which $f = 0$, passes through the point $(b = 0, s = 1)$ and is monotonically decreasing.
- The curve separating regions C and D, i.e., the curve along which $\partial f/\partial b = 0$, is monotonically increasing and intersects with the line $s = 1$ at $b = N$, where N is the price in the NBS.
- All three curves intersect with the line $s = b$ at the same point.

Similarly, if $r > 1$, the partitioning is characterized by the following properties:

- The curve separating regions A and B, i.e. the curve along which $\partial f/\partial s = 0$, is monotonically increasing and intersects with the line $b = 0$ at $s = N$, where N is the price in the NBS.
- The curve separating regions B and C, i.e. the curve along which $f = 0$, passes through the point $(b = 0, s = 1)$ and is monotonically decreasing
- The curve separating regions C and D, i.e. the curve along which $\partial f/\partial b = 0$, is monotonically decreasing and intersects with the line $s = 1$ at $b = N$, where N is the price in the NBS.
- All three curves intersect with the line $s = b$ at the same point.

Proofs of these properties are provided in the appendix. Obviously, the fact that the lines at which partial derivatives are zero intersect with the boundaries at the NBS, and their monotonicity together imply that for $r > 1$, the three lines will intersect at a price which is larger than the Nash outcome (i.e., better than the NBS for the seller), while for $r < 1$, they will intersect at a price lower than the Nash outcome (i.e., better for the buyer).

This partitioning now allows us to make predictions about the bargaining process. If both parties start out with their extreme positions (i.e., the buyer offering zero, the seller demanding one), the process begins at the bottom right corner in Figure 2. At this point, $f(b, s) = 0$. The buyer wants f to be negative, the seller wants f to be positive, so both sides have an incentive to make a concession. If the buyer makes a small concession to $b = \Delta$, the process moves into region C where the seller has an incentive to make a concession to move from region C to B. Similarly, if the seller initially makes a concession to $s = 1 - \Delta$, the buyer then has an incentive to move from region B to C. The process thus consists of alternating concessions so that offers oscillate between regions B and C around the curve $f = 0$. Ultimately, the process moves toward the point at which all three curves intersect. Here, $s = b$, thus, the two parties agree on the price.

The process thus proceeds to the point in which all three curves intersect. This point has several properties. The fact that both derivatives are zero at a point in which the offers of both sides are identical implies that this outcome is stable, since no party has a incentive to deviate from this point. We define

Definition 1: A locally stable agreement (LSA) is a value x so that for $s = x$: $\partial f(x, s)/\partial s = 0$ and for $b = x$: $\partial f(b, x)/\partial b = 0$.

We use the term "locally stable agreement" rather than "equilibrium" since the underlying incentives are different. We do not consider the moves of parties that would lead to maximizing their utility given the opponent's strategy, but the moves that would lead to a reversal of the sign of f. Note that if $u_i(d) = 0$ for $i \in \{s, b\}$, accepting any agreement $s = b = x \in]0, 1[$ dominates quitting the negotiation.

We also note that our definition of an LSA involves changes of offers in both directions. In fact, once a point on the line $s = b$ is reached, it would not be rational for the parties to make any further concession, since the opponent would be willing to accept the currently stated price. Our definition of an LSA thus also considers whether parties would have an incentive to retract a concession and toughen their position (i.e., whether the buyer has an incentive to ask for a lower and the seller to ask for a higher price). Although such behavior would violate the principle of bargaining in good faith, we still consider the absence of incentives for such behavior an important characteristic of a stable bargaining outcome. Indeed, from the definition of f in (20), we derive:

$$\partial f(b, s)/\partial s = r u_s'(s) u_b(s) + u_b'(s) u_s(b) \tag{23}$$

$$\partial f(b, s)/\partial b = -r u_b(s) u_s'(b) - u_s(b) u_b'(b) \tag{24}$$

Thus, if $s = b = x$, the derivatives have opposite sign. If x is not an LSA, then either $\partial f(x, s)/\partial s|_{s=x} > 0$, which means the seller has an incentive to ask for a higher price, or $\partial f(b, x)/\partial b|_{b=x} > 0$, which means the buyer has an incentive to ask for a lower price.

As we show in Proposition 8 in the appendix, the curves where $\partial f/\partial s = 0$ and $\partial f/\partial b = 0$ intersect at a point where $s = b$, therefore that point is an LSA. We now provide a characterization of an LSA in terms of the agreement value:

Proposition 3 *An agreement $s = b = x$ is a locally stable agreement if and only if x is a local extremum of $W(x) = u_s(x)^r u_b(x)$ (or equivalently $W(x) = u_s(x)^{\gamma_s} u_b(x)^{\gamma_b}$ or $W(x) = u_s(x) u_b(x)^{1/r}$)*

Proof: Since the power is a monotonic function, the three functions stated in Proposition 3 obviously take their maximum or minimum at the same values of x. We therefore consider function $u_s(x)^r u_b(x)$.

An LSA implies that $\partial f(b, s)/\partial s = 0$. From the definition of f in (20), we obtain

$$\partial f(b, s)/\partial s = r u_s'(s) u_b(s) + u_b'(s) u_s(b) = 0 \tag{25}$$

At the intersection with $s = b$, therefore

$$r u_s'(s) u_b(s) + u_b'(s) u_s(s) = 0 \tag{26}$$

must hold. Similarly

$$\partial f(b,s)/\partial b = -ru_b(s)u'_s(b) - u_s(b)u'_b(b) = 0 \tag{27}$$

and at the intersection with $s = b$.

$$-ru'_s(s)u_b(s) - u'_b(s)u_s(s) = 0 \tag{28}$$

At an extremum of $W(x)$, the first order condition $W'(x) = 0$ must hold.

$$W'_r(x) = ru'_s(x)u_s(x)^{r-1}u_b(x) + u_s(x)^r u'_b(x) = 0 \tag{29}$$

Dividing by $u_s(x)^{r-1} > 0$ gives (26) as well as (28), so the two conditions are equivalent. q.e.d.

Note that Proposition 3 does not require the assumption that both utility functions are concave. However, this assumption is needed to demonstrate that this LSA is unique.

Proposition 4 *If the utility functions of both parties are concave, there exists only one LSA, which corresponds to the unique maximum of $W(x)$, denoted x^*. Moreover no party has incentives to unilaterally deviate from x^*, i.e., $f(x^*,s) < 0, \forall s > x^*$ and $f(b,x^*) > 0, \forall b < x^*$.*

Proof: Here we consider the function $u_s(x)^r u_b(x)$ for $r \leq 1$, the proof for $r \geq 1$ using $u_s(x)u_b(x)^{1/r}$ is analogous. First, we show there exists a unique LSA. The second derivative of W is:

$$\begin{aligned} W''_r(x) = &\, r(r-1)u_s(x)^{r-2}u'_s(x)^2 u_b(x) + \\ &+ ru_s(x)^{r-1}u''_s(x)u_b(x) + \\ &+ 2[ru_s(x)^{r-1}u'_s(x)u'_b(x)] + \\ &+ u_s(x)^r u''_b(x) \end{aligned}$$

Since $r \leq 1$ and all utilities and $u'_s(x)$ are positive, the first term is not positive. The second and fourth term are negative by concavity of the utility functions and the third term is negative because $u'_b(x) < 0$. Thus, at least three terms are negative and none is positive. This means that $W(x)x$ is a concave function, which has a single maximum at the point x where $W'(x) = 0$. In Proposition 3, we have already shown that for any utility functions, this first order condition is fulfilled at an LSA, and therefore this unique maximum corresponds to an LSA. Furthermore, we have shown in Proposition 8 in the appendix that only one point can exist at which both derivatives $\partial f/\partial s$ and $\partial f/\partial b$ are zero. So the LSA at this maximum is also the only LSA that exists for concave utilities.

Let us now show that no party has an incentive to revert this agreement. We show (in the Appendix) in Lemma 4 that for $b < s$, $\partial^2 f/\partial^2 s < 0$ and in Lemma 3 that $\partial^2 f/\partial^2 b > 0$. The value $s = x^*$ is a maximum for $f(x^*,s)$ as a function of s, since $\partial f(b,s)/\partial s|_{b=s=x^*} = 0$ and $\partial^2 f(b,s)/\partial s^2 < 0$, and therefore $f(x^*,s)$ is less than $f(x^*, x^*) = 0$ for $s > x^*$. Similarly, the value $s = x^*$ is a minimum for $f(b, x^*)$ as a function of b , since $\partial f(b, s)/\partial b|_{b=s=x^*} = 0$ and $\partial^2 f(b, s)/\partial b^2 > 0$, and therefore $f(b, x^*)$ is greater than $f(x^*, x^*) = 0$ for $b < x^*$. q.e.d.

Thus, if the parties' utilities are concave, the bargaining process converges to this unique LSA, which is the nonsymmetric NBS $\text{argmax}_x\, u_s(x)^{\gamma_s} u_b(x)^{\gamma_b}$, which coincides with $\text{argmax}_x\, u_s(x)^r u_b(x)$ as $\gamma_b > 0$.

If at least one of the two utility functions is not concave, $W(x)$ is no longer necessarily concave and thus it might have multiple local maxima and minima. Proposition 3 implies that the two parties do not have an incentive to move away from a local minimum, so a local minimum would also be an LSA. However, we note that from a point that is neither a local maximum nor a local minimum, it is more likely that the parties move towards the local maximum of $W(x)$ rather than towards the local minimum.

Consider a local maximum $x^{\max}$ and the two neighboring local minima below and above this maximum, denoted $x^{\min_1}$ and $x^{\min_2}$, so that $x^{\min_1} < x^{\max} < x^{\min_2}$ and the interval $(x^{\min_1}, x^{\min_2})$ does not contain any other extrema of $W(x)$. Let x^a be some arbitrary point $x^{\max} < x^a < x^{\min_2}$. Since x^a is located above the maximum of $W(x)$, $W'(x^a) < 0$, which implies $\partial f/\partial b > 0$. Therefore this solution would be unattractive to the buyer, who would be overpaying as $f(b, x^a)$ would be negative for $b < x^a$ (the buyer would have an incentive to retract from a possible agreement $s = b = x^a$ and ask for a lower price).[1] Now suppose that x^a is such that $x^{\min_1} < x^a < x^{\max}$. Since x^a is located between below the maximum of $W(x)$, $W'(x^a) > 0$ implies $\partial f/\partial s > 0$. Therefore this solution would be unattractive to the seller, who would be conceding too much as $f(x^a, s)$ would be positive for $s > x^a$ (the seller would have an incentive to retract from a possible agreement $s = b = x^a$ and ask for a higher price).[2]

In summary, these results provide a characterization of the negotiation process and the final outcome when the buyer's and the seller's utility functions are concave. If the buyer and the seller are equally optimistic ($r = 1$) then the process converges to the well-known NBS. Otherwise, it converges to the nonsymmetric NBS, placing more weight on the utility of the more confident party. However, these results do not allow reaching conclusions about the outcome of the negotiation if the utilities are not concave, where the process might e.g. lead to a local minimum of W. This motivates the simulation study presented in the following section.

---

[1]Suppose the buyer asks for price $x^a - \Delta$. This creates a situation in which $f(x^a - \Delta, x^a) < 0$, so now the seller needs to move away from x^a, too. Obviously, selecting $s = x^a - \Delta$ would bring the negotiation to $f(x^a - \Delta, x^a - \Delta) = 0 > f(x^a - \Delta, x^a)$. However, it is not clear whether a seller acting only on local information would make that choice. For $r > 1$, the seller will necessarily have to make a regular concession. From Lemma 5 in the appendix (which does not depend on the utility functions being concave), we know that for $r > 1$, $\partial^2 f/\partial s \partial b > 0$. Since $\partial f/\partial s$ is already negative at $s = b = x^a$, decreasing b will further decrease it. Therefore, the seller then will locally increase f by reducing s. However, for $r < 1$, it is not clear whether the seller will have a local incentive to make a concession, or to make a reverse concession.

[2]In this case, Lemma 5 in the appendix allows saying that the buyer would need to accept the price increase if $r < 1$, but this would not be guaranteed if $r > 1$.

4 Simulation

We use a simulation analysis to explore the properties of the model for nonconcave utility functions. Specifically, we want to study whether the process still converges to a local maximum of the nonsymmetric Nash function identified in Proposition 3, or how often it fails to do so. Furthermore, we use the simulation to quantify the effects of the confidence parameters on the individual outcomes of the parties, and on other properties of the agreement. The analysis of the preceding section has shown that in the agreement, the party having the higher level of confidence will be better off than in the (symmetric) NBS. In the simulation study, we can quantify this effect and verify whether it will also hold when the utility functions are not concave. Furthermore, we can study how far the solutions deviate from the NBS, and to quantify the possible loss in efficiency (joint utility) that results from this deviation.

For the simulation, arbitrary pairs of monotonic (but not necessarily concave) utility functions were generated using the bisection approach of Dias and Vetschera (2019b). Utility functions for both sides are represented as utility values for equally spaced prices in the zero-one interval. Since the method is based on a bisection approach, it works most efficiently if the number of intervals is a power of two. For the present simulation, $2^9 = 512$ intervals were used.

Each pair of utility functions defines a problem. In total, two sets of 100,000 problems each were generated using different random number streams to check the stability of results. For each problem, the two confidence parameters γ_s and γ_b were varied from 0.05 to 1 in steps of 0.05, thus generating 400 combinations of confidence parameters. Since Proposition 2 holds for arbitrary (and not just for concave) utility functions, some of these combinations should generate identical outcomes. They were nevertheless all included in the simulation to test the numerical stability of the simulation, and the results confirmed that.

The bargaining process was simulated in the following way: The initial offer of one party was set to the best outcome for that party (1 for the seller or 0 for the buyer), and the initial offer of the other party was set to that party's second best outcome ($1-1/512$ for the seller or $1/512$ for the buyer). Between problems, the party starting with the second best outcome was alternated. Given the two offers, the value of f was calculated to decide which party needs to make a concession. If $f < 0$, the seller has to make a concession, otherwise, the buyer. The conceding party makes a concession by moving to the first discrete price level that would revert the sign of f (i. e., the seller moves to the highest price smaller than its current offer that leads to $f > 0$, the buyer to the lowest price above its previous offer that would lead to $f < 0$). Denote the current offer of the buyer and seller by s_t and b_t and the set of possible prices by $X = \{x_i\} = \{0, 1/512, 2/512, ...1\}$. The next offer s_{t+1} of the seller is then given by

$$s_{t+1} = \max_{x_i : x_i < s_t \wedge f(b_t, x_i) > 0} x_i \tag{30}$$

Concessions of the buyer are determined in an analogous way. The process terminates when

$$s_{t+1} \leq b_t \vee b_{t+1} \geq s_t \tag{31}$$

i.e., when one party cannot do better than accept the offer from the other party. Since each step moves the offer of one side towards the offer of the other side, and there is only a finite number of possible offers, the process will always converge to a solution. This is consistent with our model, since in the model a negotiation is only terminated if one party obtains an outside offer that is better than the expected outcome of the negotiation.

Note that this convergence property relies on the assumption that both parties bargain in good faith, i.e., they do not retract from a previously made offer. For non-concave utilities, it could be the case that e.g. for a given offer b_t of the buyer, there exists a value $s' > s_t$ so that $f(b_t, s') > 0$. Thus the seller could achieve $f > 0$ by increasing, rather than decreasing, the price he or she demands. The assumption of bargaining in good faith excludes such moves, which could cause the process to oscillate infinitely between some offers.

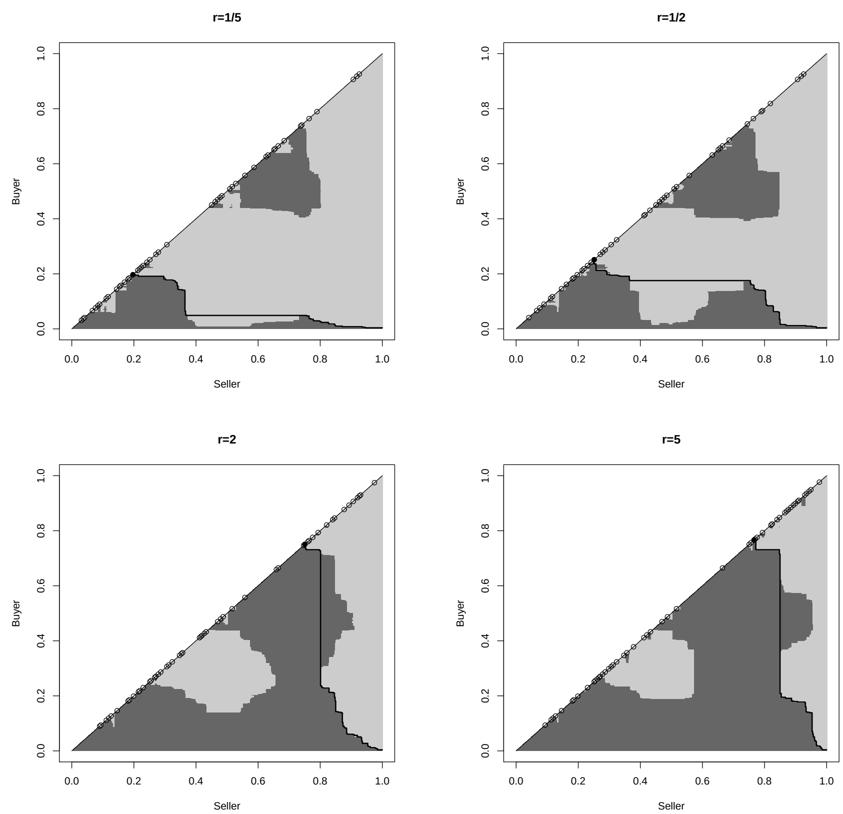

Figure 3: Partitioning or b/s space and negotiation path for non-convex utilities and different values of r

	γ_b							
	0.05		0.35		0.65		0.95	
γ_s	global	local	global	local	global	local	global	local
0.05	0.00	0.00	27.45	0.43	33.92	0.92	37.65	1.57
0.35	27.81	0.46	0.00	0.00	9.45	0.08	14.90	0.04
0.65	33.17	0.84	9.73	0.07	0.00	0.00	6.01	0.01
0.95	36.99	1.32	14.74	0.05	5.56	0.06	0.00	0.00

Table 1: Fraction (in %) of cases in which the process did not converge to the global or to a local maximum of the nonsymmetric Nash function

As Figure 3 shows, the partitioning of b/s space into compact subspaces, that existed for concave utilities, no longer exists if utilities are not concave. In Figure 3, regions in which $f > 0$ (corresponding to regions A and B in Figure 2) are marked in dark gray, regions in which $f < 0$ (corresponding to C and D) in light gray. Within these regions, the parts in which the partial derivatives of f with respect to s and b are positive and negative are also scattered (we did not depict these regions in the figure to reduce its complexity). Hollow circles along the line $s = b$ mark local maxima of the nonsymmetric Nash function $u_s(x)^r u_b(x)$, the solid circle marks the global maximum. Figure 3 shows a problem in which the process converged to the global maximum for all values of r. Clearly, the maximum depends on r, the price in the agreement increases for a higher r, i.e., the more confident the seller is. In the two examples with $r < 1$, reaching the agreement in some cases required quite large concessions from the seller to move across a region in which $f < 0$ (the long black line across the light gray region), for example, in the negotiation with $r = 1/5$, the seller had to decrease the price from about 0.76 to 0.37 in one step. Similarly, in the negotiations represented in the lower part of Figure 3, the buyer in one step had to make a large concession (across the dark gray region).

However, the process does not always converge to the global and sometimes not even to a local maximum of the nonsymmetric Nash function, as the example in Figure 4 shows. Here, the global maximum is located at approximately $s = b = 0.12$, but the process converged to $s = b = 0.44$. The reason for this deviation is also quite obvious from the figure: The large region around $s = 0.8$ in which $f > 0$ (marked in dark gray) forces the buyer to increase its offer up to the level of 0.44, while the seller has to make only small concessions. Once that level is reached, however, there is no more possibility for the seller to achieve $f > 0$ at a price higher than the buyer's offer, so the seller accepts the offer from the buyer.

Table 1 shows the fraction of cases in which the process did not converge to a local or the global maximum of the nonsymmetric Nash function. Here we present only a few selected values of the two confidence parameters, the intermediate values do not offer much additional insight and can be obtained from the authors upon request.

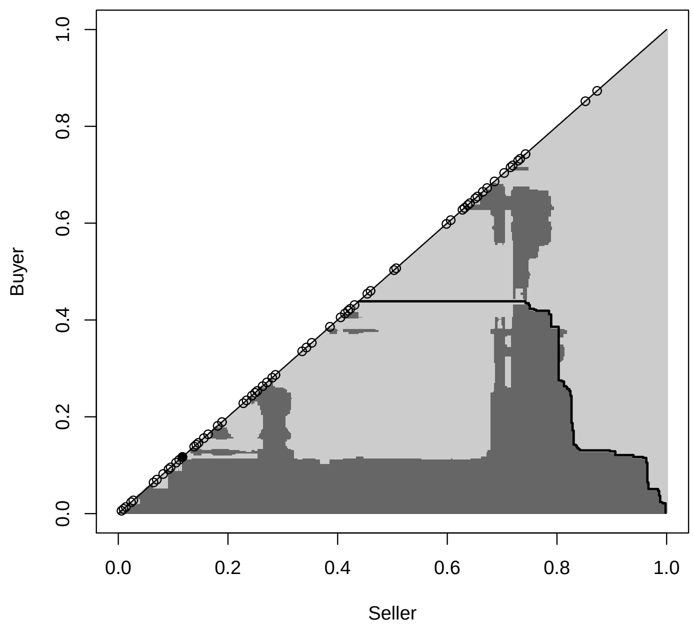

Figure 4: Example of a negotiation not converging to a maximum of the non-symmetric Nash function

	γ_b			
γ_s	0.05	0.35	0.65	0.95
0.05	100.0	88.0	82.0	78.2
0.35	87.5	100.0	98.4	96.2
0.65	82.1	98.5	100.0	99.4
0.95	78.8	96.0	99.4	100.0

Table 2: Ratio of the nonsymmetric Nash function achieved in the agreement to the value in the global optimum

Given the high number of local optima shown in Figures 3 and 4, it is not surprising that the global maximum was not reached in a considerable number of cases. As we have already shown, if the confidence parameters of the two parties are equal, the process will always converge to the NBS (which in that case is also equal to the nonsymmetric NBS), so in these cases, all entries are zero. However, it should be noted that the simulation assumes that both parties have full information about both utility functions and therefore will make large concessions if such concessions are needed to reach the global maximum. Otherwise, the existence of local maxima could prevent them from converging to an agreement. This situation is similar to the standard Z-H model, where local maxima could also prevent convergence to the NBS (Dias and Vetschera, 2019a). If the two confidence parameters are more imbalanced, the process fails more often. The highest value is reached when one party is highly confident (0.95), and the other has extremely low confidence (0.05). The situation of Figure 4, in which the process also fails to reach a local maximum, is quite rare. Even in very unbalanced settings, it occurs in only about 1.5% of all cases.

To study whether failing to find the global maximum actually implies a large loss in performance, we compare the value of the nonsymmetric Nash objective that was achieved in the agreement to its global maximum.

As Table 2 shows,[3] the agreement reached achieves well over 80% of the global optimum expect when the difference between the two parties is extreme. If both sides have a confidence parameter of more than 0.35, the process converges to a solution that provides more than 95% of the global maximum and thus in effect performs almost as well as the global optimum of the nonsymmetric NBS.

As a final analysis, we now study the effect of differences in the confidence values on individual and collective outcomes.

Table 3 shows that the party having the lower confidence parameter will never be able to perform better than in the (symmetric) NBS, while the party having the higher confidence parameter will always achieve at least the outcome it would obtain in the NBS. The performance effect of being the weaker party

[3]The value of the global maximum is not the same for $r = 1/2$ and $r = 2$. Thus, to compute the loss in performance in a balanced way, Table 2 uses the function $u_s(x)^{\gamma_s/\gamma_b} u_b(x)$ for $\gamma_s > \gamma_b$ and $u_s(x) u_b(x)^{\gamma_b/\gamma_s}$ for $\gamma_s < \gamma_b$.

γ_s		γ_b 0.05	0.35	0.65	0.95
0.05	min	100.0	25.1	14.4	10.1
	avg	100.0	60.6	53.0	48.9
	max	100.0	100.0	100.0	100.0
0.35	min	100.0	100.0	69.9	53.8
	avg	128.9	100.0	84.1	75.3
	max	175.0	100.0	100.0	100.0
0.65	min	100.0	100.0	100.0	81.0
	avg	131.5	114.5	100.0	90.1
	max	185.5	130.2	100.0	100.0
0.95	min	100.0	100.0	100.0	100.0
	avg	132.5	121.0	109.4	100.0
	max	189.8	146.2	119.1	100.0

Table 3: Performance of the seller relative to the seller's outcome in the NBS (in %) for different levels of the confidence parameters

is quite strong, both on average and in the worst case. Conversely, a party that is much stronger can achieve almost twice the outcome it would obtain in the NBS.

γ_s	γ_b 0.05	0.35	0.65	0.95
0.05	99.7	94.1	91.5	90.0
0.35	94.1	99.7	98.9	97.7
0.65	91.5	98.9	99.7	99.4
0.95	90.0	97.7	99.4	99.7

Table 4: Efficiency of the joint utility of the agreement (in % of the possible maximum), for different combinations of confidence parameters

As Table 4 shows, the overall efficiency of the outcome is not affected very strongly by differences in the confidence parameters. The table shows the joint utility (sum of utilities of the two parties) obtained in the agreement as a fraction of the maximum joint utility that could have been obtained in any agreement. Since the NBS does not maximize the sum of the two utilities, the main diagonal of that table (where the process always converges to the NBS) is below 100%.

5 Conclusions

In this paper, we have extended the original Z-H model to consider the parties in a more symmetric way. This symmetric treatment solves the apparent paradox that it is on the one hand not rational to quit a negotiation, and on the other hand a rational party considers it possible that the other (also rational) party quits. Furthermore, we not only consider the focal party's option to make a new offer, but also introduce the focal party's anticipation of future offers of the opponent. This extension leads to the introduction of parameters γ_s and γ_b that summarize the confidence of each party taking into account their bargaining skills and strengths.

The analysis of the extended Z-H model presented in this article provides a full characterization of the exchange of successive offers, as well as the predicted outcome, if the two parties have concave utility functions. In such cases, the final outcome will be a nonsymmetric NBS, in which the utility functions of both parties are exponentially weighted by their confidence parameters, or equivalently, the utility of one party is weighted by the ratio of the confidence parameters. This extension adds a descriptive element to the Z-H model and shows how negotiator characteristics such as their confidence in their own bargaining skills can systematically influence the outcome of a negotiation.

In addition, this analysis contributes another possible explanation for obtaining the nonsymmetric NBS. Kalai (1977) has shown that the nonsymmetric NBS can be obtained by n-person replications of Nash's original setting, considering that two parties represent two groups of different size. Other paths to obtain the nonsymmetric NBS have been proposed in the context of Rubinstein (1982)'s model, namely considering asymmetry in the parties' discount rates, preferences, or beliefs about determinants in the environment (Binmore et al., 1986; Muthoo, 1999). In our case, the path to reach the nonsymmetric NBS is based on the logic of the Z-H model, and asymmetry derives from unequal confidence factors.

If the utilities are not concave, we can still show that any local maximum of the nonsymmetric Nash product is a locally stable agreement, but there might be many such maxima and the bargaining process might miss the global maximum (or even a local maximum). The simulation study in Section 4 sheds light on the possible outcomes of bargaining processes in such situations. The results show that the process misses the global maximum in a considerable number of cases, especially if there is a large imbalance in confidence between the parties. A local maximum is reached in most of the cases, even when confidence is highly unbalanced.

The fact that the global optimum is missed, however, might not entail a large loss. The simulation results indicate that in most cases the bargaining outcome reached by the parties is fairly high even in unbalanced cases, both in terms of ratio of the nonsymmetric Nash function (compared to the global maximum) and in terms of joint utility (sum of utilities). The loss in both cases decreases when confidence gets more balanced.

Finally, the theoretical results obtained for the concave utilities case, as well as the empirical results obtained in the simulations for general utility functions, confirm the general conclusion that the more confident party has an advantage. The party with higher confidence will always get a result which is better than the (symmetric) NBS (and the reverse occurs for the other party).

This article will potentially renew the interest in the strategic and axiomatic aspects of the Z-H model, opening multiple paths for future research. One such path will possibly be concerned with the confidence parameters γ_s and γ_b. Future theoretical and behavioral studies can address the way these parameters are driven by bargaining experience, expectations about the appearance of attractive outside options, or other elements defining bargaining strength (time pressure, linkage with other negotiations, etc). In the present paper, we have assumed that these parameters are constant throughout the process. Future developments might consider that these parameters change during the bargaining process, possibly even as a function of how the process progresses. Finally, more analytical studies could show that some specific types of non-concave utility functions also lead to a single-peaked nonsymmetric Nash product, as it was possible to observe in the original Z-H model (Dias and Vetschera, 2019a).

The model presented here could also be useful for empirical studies. It provides a connection between negotiator characteristics and bargaining outcomes. Thus it could on the one hand be used to infer negotiator confidence from bargaining outcomes. On the other hand, if measures of confidence (or overconfidence, which has been shown to affect negotiations, see Neale and Bazerman, 1985) are available, the model makes clear predictions how these will influence the bargaining process and its outcomes, and these predictions could also be tested empirically.

Appendix

Proofs that b/s space can be partitioned as shown in Figure 2. This means that for $r < 1$, we have to show the following properties:

- The curve separating regions A and B, i.e. the curve along which $\partial f/\partial s = 0$, is monotonically decreasing.
- The curve separating regions B and C, i.e. the curve along which $f = 0$, passes through the point $(b = 0, s = 1)$ and is monotonically decreasing.
- The curve separating regions C and D, i.e. the curve along which $\partial f/\partial b = 0$, is monotonically increasing.
- All three curves intersect with the line $s = b$ at the same point.

Preliminaries

Lemma 1 *For $b = 0$, $\partial f/\partial s = 0$ when s corresponds to the NBS.*

Proof:
$$\frac{\partial f}{\partial s} = r[u_s'(s)u_b(s) + u_s(s)u_b'(s)] + (1-r)u_s(b)u_b'(s) \tag{32}$$
Since $u_s(0) = 0$, (32) reduces to r times $u_s'(s)u_b(s) + u_s(s)u_b'(s)$, which is null if and only if $[u_s(s)u_b(s)]'$ is null. Thus, this is the point which maximizes $u_s(s)u_b(s)$, i.e. the NBS.

Lemma 2 *For $s = 1$, $\partial f/\partial b = 0$ when b corresponds to the NBS.*

Proof:
$$\frac{\partial f}{\partial b} = (1-r)u_s'(b)u_b(s) - u_b'(b)u_s(b) - u_b(b)u_s'(b) \tag{33}$$
Since $u_b(1) = 0$, (33) reduces again to the first derivative of the Nash objective function with respect to b.

To show the relevant properties of the lines separating the different regions in Figure 2, we have to consider the second derivatives of f:

Lemma 3 *For $s > b$: $\partial^2 f/\partial^2 b > 0$*

Proof: From (33), we obtain
$$\partial^2 f/\partial^2 b = u_s''(b)[(1-r)u_b(s) - u_b(b)] - u_b''(b)u_s(b) - 2u_b'(b)u_s'(b) \tag{34}$$
Since we assume concave utilities, all second derivatives are negative. By definition of the utilities, $u_b'(x) < 0$ and $u_s'(x) > 0$. Therefore the second and third term are negative, since they have also a negative sign, their contribution to the sum is positive. Since $s > b$ and the buyer's utility decreases, the second factor of the first term is negative, so the entire term is positive. q.e.d.

Lemma 4 *For $s > b$: $\partial^2 f/\partial^2 s < 0$*

Proof: From (32), we obtain
$$\partial^2 f/\partial^2 s = u_b''(s)[ru_s(s) + (1-r)u_s(b)] + ru_s''(s)u_b(s) + 2ru_s(s)u_b'(s) \tag{35}$$

By a similar argument as above, the last two terms are negative. The second factor of the first term is a weighted combination of utilities and therefore positive, the first factor is negative, so the entire term is negative. q.e.d.

Lemma 5 *The sign of $\partial^2 f/\partial b\partial s = \partial^2 f/\partial s\partial b$ depends on r. For $r > 1$, it is positive, for $r < 1$, it is negative.*

Proof: From the first derivatives, we obtain
$$\partial^2 f/\partial b\partial s = \partial^2 f/\partial s\partial b = (1-r)u_s'(b)u_b'(s) \tag{36}$$
The product of the two derivatives is negative, thus the sign of $(1-r)$ determines the sign as indicated. q.e.d.

Separation between regions A and B

Proposition 5 *The curve in r/s space at which $\partial f/\partial s = 0$ is monotonically decreasing for $r < 1$ and monotonically increasing for $r > 1$*

Proof: We have shown in Lemma 4 that $\partial^2 f/\partial^2 s < 0$. For $r < 1$, according to Lemma 5, $\partial^2 f/\partial b \partial s < 0$ Thus, an increase in s must be matched by a decrease in b to keep the value of $\partial f/\partial s$ at zero. For $r > 1$, according to Lemma 5, $\partial^2 f/\partial b \partial s > 0$ Thus, an increase in s must be matched by an increase in b to keep the value at zero.

Corollary 1 *The point at which this curve intersects the line $s = b$ is above the NBS ($b = s = N$) for $r > 1$ and below the NBS for $r < 1$.*

Proof: As shown in Lemma 2 the curve intersects the boundary $b = 0$ at $s = N$. When b increases, s will increase or decrease according to the above proposition, leading to the indicated outcome.

Separation between regions C and D

Proposition 6 *The curve in r/s space at which $\partial f/\partial b = 0$ is monotonically increasing for $r < 1$ and monotonically decreasing for $r > 1$.*

Proof: We have shown in Lemma 3 that $\partial^2 f/\partial^2 b > 0$. For $r < 1$, according to Lemma 5, $\partial^2 f/\partial b \partial s < 0$ Thus, an increase in s must be matched by an increase in b to keep the value of $\partial f/\partial b$ at zero. For $r > 1$, according to Lemma 3, $\partial^2 f/\partial b \partial s > 0$ Thus, an increase in s must be matched by a decrease in b to keep the value at zero.

Corollary 2 *The point at which this curve intersects the line $s = b$ is above the NBS ($b = s = N$) for $r > 1$ and below the NBS for $r < 1$.*

Proof: The curve intersects with $s = 1$ at $b = N$. Decreasing s will lead to an increase or decrease in b according to the above theorem, leading to the indicated outcome.

Separation between regions B and C

Proposition 7 *The curve $f = 0$ in r/s space is monotonically decreasing.*

Proof: Obviously, $f(0, 1) = 0$, therefore the curve ends at $b = 0, s = 1$. There, both derivatives are negative, any decrease in s must therefore be matched by an increase in b. As we have just shown, the two curves where the two derivatives are zero can only intersect at the line $s = b$. Thus, for $s > b$, there is always a corridor between the two curves in which both derivatives are negative.

Intersection of separating lines

Proposition 8 *The curves $\partial f/\partial b = 0$ and $\partial f/\partial s = 0$ intersect at a point where $s = b$ holds.*

Proof: We can rewrite the two curves from (33) and (32) as

$$\partial f/\partial b = (1-r)u_s'(b)u_b(s) - N'(b) = 0 \quad (37)$$

and

$$\partial f/\partial s = rN'(s) + (1-r)u_s(b)u_b'(s) = 0 \quad (38)$$

where $N(x)$ is the Nash objective function $u_s(x)u_b(x)$, and $N'(x)) = u_s'(x)u_b(x) + u_s(x)u_b'(x)$. Now assume that $\partial f/\partial b = 0$ intersects the line $s = b$ at a point where $s = b = x$. Then, we can write (37) as

$$\begin{aligned}
\partial f/\partial b &= (1-r)u_s'(x)u_b(x) - N'(x) = 0 \\
&\Leftrightarrow (1-r)u_s'(x)u_b(x) - N'(x) + (1-r)N'(x) - (1-r)N'(x) = 0 \\
&\Leftrightarrow (1-r)u_s'(x)u_b(x) - rN'(x) - (1-r)N'(x) = 0 \\
&\Leftrightarrow -rN'(x) - (1-r)u_s(x)u_b'(x) = 0 \\
&\Leftrightarrow \partial f/\partial s = 0.
\end{aligned}$$

Therefore, at $s = b = x$ both $\partial f/\partial b = 0$ and $\partial f/\partial s = 0$. Since we have already shown that both curves are monotonic in the opposite direction, they can intersect only at one point, so there can be no intersection at which $s \neq b$.

Proposition 9 *The curve where $f = 0$ and the two curves at which the derivatives are zero all intersect with the line $s = b$ at the same point.*

Proof: We have already shown that the curves at which the derivatives are zero intersect at the line $s = b$. Note that at the line $s = b$, also $f = 0$ holds. Thus we have an intersection of two lines at which $f = 0$. The line $s = b$ is monotonically increasing, which implies that the derivatives of f with respect to the two variables must have opposite signs. The other curve at which $f = 0$ holds is monotonically decreasing, so along that curve the two derivatives must have the same sign. This can only happen at the same time if both derivatives are zero, so the two other lines at which the derivatives are zero must also go through that point.

References

Bastianello, L. and LiCalzi, M. (2019). The probability to reach an agreement as a foundation for axiomatic bargaining. *Econometrica*, 87(3):837–865.

Binmore, K., Rubinstein, A., and Wolinsky, A. (1986). The Nash bargaining solution in economic modelling. *The RAND Journal of Economics*, 17(2):176–188.

Bishop, R. L. (1964). A Zeuthen-Hicks theory of bargaining. *Econometrica*, 32:410–417.

Dias, L. C. and Vetschera, R. (2019a). Multiple local optima in Zeuthen-Hicks bargaining: An analysis of different preference models. *EURO Journal on Decision Processes*, 1-2:33–53.

Dias, L. C. and Vetschera, R. (2019b). On generating utility functions in Stochastic Multicriteria Acceptability Analysis. *European Journal of Operational Research*, 278:672–685.

Fandel, G. (1985). On the applicability of group decision-making concepts to wage bargaining. In Haimes, Y. Y. and Chankong, V., editors, *Decision Making with Multiple Objectives*, pages 532–548. Springer, Berlin et al.

Harsanyi, J. C. (1956). Approaches to the bargaining problem before and after the theory of games: A critical discussion of Zeuthen's, Hicks', and Nash's theories. *Econometrica*, 24(2):144–157.

Harsanyi, J. C. (1982). Subjective probability and the theory of games: Comments on Kadane and Larkey's paper. *Management Science*, 28(2):120–124.

Kadane, J. B. and Larkey, P. D. (1982). Subjective probability and the theory of games. *Management Science*, 28(2):113–120.

Kalai, E. (1977). Nonsymmetric Nash solutions and replications of 2-person bargaining. *International Journal of Game Theory*, 6(3):129–133.

Kalai, E. and Smorodinsky, M. (1975). Other solutions to Nash's bargaining problem. *Econometrica*, 43(3):513–518.

Muthoo, A. (1999). *Bargaining theory with applications*. Cambridge University Press, Cambridge.

Nash, J. F. (1950). The bargaining problem. *Econometrica*, 18(2):155–162.

Neale, M. A. and Bazerman, M. H. (1985). The effects of framing and negotiator overconfidence on bargaining behaviors and outcomes. *Academy of Management Journal*, 28(1):34–49.

Roth, A. (1979). *Axiomatic models of bargaining*. Springer Verlag, Berlin.

Rubinstein, A. (1982). Perfect equilibrium in a bargaining model. *Econometrica*, 50(1):97–109.

Schweighofer-Kodritsch, S. (2018). Time preferences and bargaining. *Econometrica*, 86(1):173–217.

Sutton, J. (1986). Non-cooperative bargaining theory: An introduction. *The Review of Economic Studies*, 53(5):709–724.

Svejnar, J. (1986). Bargaining power, fear of disagreement, and wage settlements: Theory and evidence from U.S. industry. *Econometrica*, 54(5):1055–1078.

Vetschera, R. (2019). Zeuthen-Hicks bargaining in electronic negotiations. *Group Decision and Negotiation*, 28:255–274.

Zeuthen, F. (1930). *Problems of monopoly and economic warfare*. Routledge, London.