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Campanato Spaces and applications in Partial Differential Equations

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Abstract

In this work, we start off by studying the $\mathscr{L}^{(q,\lambda)}(\Omega,\delta)$ space, which generalizes the Campanato space $\mathscr{L}^{(q,\lambda)}(\Omega)$ introduced by S. Campanato in 1963. We prove that, for $\lambda > 1$, they are equivalent to the spaces of Hölder continuous functions $C^{0,\alpha}(\overline{\Omega})$, with $\alpha = \frac{m}{q}(\lambda - 1)$, where m depends on the metric. In this chapter, we follow the article [3] of G. Da Prato.

In the following chapter, we apply these results in the context of partial differential equations, obtaining quantitative results regarding the regularity of the weak solutions of the degenerate non-homogeneous p-Laplace equation. We follow the article [4].

In the final chapter, we start by studying some essential results in the theory of partial differential equations, such as expansion of positivity and the Harnack Inequality, and use them to get the Hölder continuity of solutions to the porous medium equation. We follow the book [9] and the article [13].

Contents

Chapter 1

Introduction

The $\mathcal{L}^{(q,\lambda)}(\Omega)$ spaces, introduced by S. Campanato in 1963, are of extreme importance, in particular when applied in the context of Partial Differential Equations. Campanato proved that, for $n < \lambda \leq n+q$, equivalent to the space of Hölder continuous functions with exponent $\alpha = \frac{\lambda - n}{q}$ (cfr. [1]). However, it is sometimes necessary to equip $\mathbb{R}^n$ with a non-euclidean metric. By its definition, it is clear that these spaces depend on the metric of $\mathbb{R}^n$.

In the first part of this thesis, we prove that if we equip $\mathbb{R}^n$ with a metric that satisfies certain properties, we can get similar results. We will therefore consider the spaces $\mathcal{L}^{(q,\lambda)}(\Omega,\delta)$, $q \geq 1$, $\lambda > 1$, which generalize Campanato's spaces. We prove that they are equivalent to the spaces of Hölder continuous functions $C^{0,\alpha}(\Omega,\delta)$, for this metric. We will define Campanato's spaces in an alternative way to that of [1], which justifies the different intervals of λ.

In the following chapter, we will study the regularity of the weak solutions to the inhomogeneous partial differential equation

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f \in L^{(q,r)}, \quad p \geq 2.$$

We show that the solutions are Hölder continuous, with exponent α given by

$$\alpha = \frac{(pq-n)r - pq}{q[(p-1)r-(p-2)]},$$

proving that they belong to $\mathcal{L}^{(p,\lambda)}(\Omega,\delta)$, for some λ and δ and using the Campanato-Da Prato characterization of the Hölder continuity.

In the final chapter, we want to get similar results for the PME

$$u_t - \operatorname{div}(m|u|^{m-1}\nabla u) = f, \quad m > 1.$$

We conclude that bounded weak solutions of (4.0.1) are locally of class $C^{0,\gamma,\frac{\gamma}{\theta}}$, with

$$\gamma = \frac{\alpha}{m}, \qquad \alpha = \min\left\{\alpha_0^-, \frac{m[(2q-n)r-2q]}{q[mr-(m-1)]}\right\},$$

where $0 < \alpha_0 \leq 1$ denotes the optimal Hölder exponent of (4.0.1) with $f \equiv 0$ and

$$\theta = \alpha\left(1 + \frac{1}{m}\right) + (1-\alpha)2. \tag{1.0.1}$$

A crucial result in getting this regularity is the general local Hölder continuity of these solutions. Attempting to make this text as self-contained as possible, we decided to include a proof of this continuity, which in turn required some crucial results such as expansion of positivity and Harnack Inequality. Taking the risk of going out of context, we decided these results were worth studying because of their incredible usefulness in studying PDE's.

We would also like to point out that the final proof does not directly use Campanato's characterization of Hölder continuity, since this result is trivial in the case of the L^∞ norm.

Chapter 2

$\mathcal{L}^{(q,\lambda)}(\Omega, \delta)$ Spaces

2.1 Basic Definitions

Consider a metric $\delta(x, y)$ defined on $\mathbb{R}^n$ which preserves the structure of the vector space, and let $B(x, \rho), x \in \mathbb{R}^n, \rho > 0$, be the open ball in $\mathbb{R}^n$ with center x and radius ρ, according to this metric. Furthermore, we will assume the following properties:

1. $B(0, \rho)$ is convex, $\forall \rho > 0$;

2. there exist positive numbers M_1, M_2 and $m \geq n$ such that

$$M_1 \rho^m \leq |B(0, \rho)| \leq M_2 \rho^m. \tag{2.1.1}$$

From the first property, we can conclude that the topology induced by this metric is equivalent to the euclidean and that $\delta(x, y)$ is continuous in $\mathbb{R}^n \times \mathbb{R}^n$. Let $\Omega(x, \rho) = \Omega \cap B(x, \rho)$. In the following, we adapt the usual definition of these spaces to the new metric.

Definition 1 (Hölder Spaces). *Let $C^{0,\alpha}(\Omega, \delta)$ be the space of Hölder continuous functions with exponent α over $\overline{\Omega}$ w.r.t. δ, equipped with the norm*

$$\begin{aligned} ||u||_{C^{0,\alpha}(\overline{\Omega},\delta)} &= \sup_{x \in \overline{\Omega}} |u(x)| + \sup_{x,y \in \overline{\Omega}} \frac{|u(x) - u(y)|}{\delta^\alpha(x, y)} = \\ &= \sup_{x \in \overline{\Omega}} |u(x)| + [u]_{C^{0,\alpha}(\overline{\Omega},\delta)}. \end{aligned} \tag{2.1.2}$$

Definition 2 (Campanato's Spaces). *Let $\mathcal{L}^{(q,\lambda)}(\Omega, \delta)$, with $q \geq 1$, $\lambda > 0$, be the space of funtions $u \in L^q(\Omega)$ such that*

$$[u]_{\mathcal{L}^{(q,\lambda)}(\Omega,\delta)} = \left(\sup_{\substack{x_0 \in \overline{\Omega} \\ \rho \in (0, d(\Omega)]}} |\Omega(x_0, \rho)|^{-\lambda} \int_{\Omega(x_0,\rho)} |u(x) - u_{x_0,\rho}|^q dx \right)^{\frac{1}{q}} < +\infty,$$

where $u_{x_0,\rho}$ is the averaged integral of u over $\Omega(x_0, \rho)$.

Then $\mathcal{L}^{(q,\lambda)}(\Omega, \delta)$ is Banach when equipped with the norm

$$||u||_{\mathcal{L}^{(q,\lambda)}(\Omega,\delta)} = ||u||_{L^q(\Omega)} + [u]_{\mathcal{L}^{(q,\lambda)}(\Omega,\delta)}.$$

Finally, let us assume that Ω satisfies the property *(A)*:

$$|\Omega(x,\rho)| \geq A|B(x,\rho)|, \forall x \in \Omega, 0 < \rho \leq \rho_0, \tag{2.1.3}$$

for some $A > 0$.

2.2 Initial properties

We want to study the behavior of $u_{x_0,\rho}$ when $\rho \to 0$. For this purpose, we start with the following lemmas.

Lemma 1. *Consider $u \in \mathcal{L}^{(q,\lambda)}(\Omega,\delta), q \geq 1, \lambda \geq 0$, and let $0 < \sigma < \rho$. Then there exists a constant $K_1 > 0$ such that, for almost all $x_0 \in \overline{\Omega}$, it holds*

$$|u_{x_0,\rho} - u_{x_0,\sigma}| \leq K_1 \Big(\frac{\rho^{\lambda m} - \sigma^{\lambda m}}{\sigma^m}\Big)^{\frac{1}{q}} [u]_{\mathcal{L}^{(q,\lambda)}(\Omega,\delta)}. \tag{2.2.1}$$

Proof. Let $0 < \sigma < \rho$ and $x_0 \in \overline{\Omega}$ be arbitrary. Then, for almost all $x \in \Omega(x_0,\sigma)$, it holds

$$|u_{x_0,\rho} - u_{x_0,\sigma}|^q \leq 2^q\big(|u(x) - u_{x_0,\rho}|^q + |u(x) - u_{x_0,\sigma}|^q\big).$$

Integrating over $\Omega(x_0,\sigma)$ w.r.t. x, we get

$$|\Omega(x_0,\sigma)|\,|u_{x_0,\rho} - u_{x_0,\sigma}|^q \leq 2^q\Big(\int_{\Omega(x_0,\rho)} |u(x) - u_{x_o,\rho}|^q dx +$$
$$+ \int_{\Omega(x_0,\rho)} |u(x) - u_{x_0,\sigma}|^q dx\Big).$$

Using (2.1.1) and property *(A)*, it follows

$$|u_{x_0,\rho} - u_{x_0,\sigma}|^q \leq \frac{2^q M_2^\lambda}{AM_1}\Big(\frac{\rho^{\lambda m} + \sigma^{\lambda m}}{\sigma^m}\Big)[u]^q_{\mathcal{L}^{(q,\lambda)}(\Omega,\delta)}$$

as pretended.

□

Lemma 2. *Let $u \in \mathcal{L}^{(q,\lambda)}(\Omega,\delta), q \geq 1, \lambda \geq 0$ and h be a positive integer. Then there exists a constant $K_2 > 0$ such that, for almost all $x \in \overline{\Omega}$ and $0 < \rho < d$, it holds*

$$|u_{x_0,\rho} - u_{x_0,2^{-h}\rho}| \leq K_2 \rho^\alpha (1 - 2^{-\alpha h})[u]_{\mathcal{L}^{(q,\lambda)}(\Omega,\delta)}, \tag{2.2.2}$$

where $\alpha = \frac{m}{q}(\lambda - 1)$.

Proof. We apply the previous lemma, with $\sigma = \frac{\rho}{2}$. Then, we immediately get

$$|u_{x_0,\rho} - u_{x_0,\frac{\rho}{2}}|^q \leq \frac{2^q M_2^\lambda}{AM_1}\Big(\frac{\rho^{\lambda m} + (\rho/2)^{\lambda m}}{(\rho/2)^m}\Big)[u]^q_{\mathcal{L}^{(q,\lambda)}(\Omega,\delta)}.$$

Noting that

$$\frac{\rho^{\lambda m} + (\rho/2)^{\lambda m}}{(\rho/2)^m} = \rho^{q\alpha} 2^{-q\alpha}(1 + 2^{\alpha m}),$$

we get

$$|u_{x0,\rho} - u_{x0,\frac{\rho}{2}}| \leq 2\Big(\frac{M_2^\lambda}{AM_1}\Big)^{\frac{1}{q}} \rho^\alpha 2^{-\alpha}(1+2^{\alpha m})^{\frac{1}{q}}[u]_{\mathcal{L}^{(q,\lambda)}(\Omega,\delta)} = K(u)\rho^\alpha.$$

Therefore,

$$\begin{aligned} |u_{x0,\rho} - ux0, 2^{-h}\rho| &\leq \sum_{j=i}^{h} |u_{x0,2^{1-j}\rho} - u_{x0,2^{-j}\rho}| \leq \\ &\leq K(u)\rho^\alpha \sum_{j=1}^{h} 2^{\alpha(1-j)} \leq K(u)\frac{1-2^{-\alpha h}}{1-2^{-\alpha}}\rho^\alpha \end{aligned}$$

which concludes the proof, with $K_2 = \frac{K(u)}{1-2^{-\alpha}}$.

□

Lemma 3. *Let $u \in \mathcal{L}^{(1,\lambda)}(\Omega, \delta)$, with $\lambda > 1$. Then, for almost all $x_0 \in \overline{\Omega}$, it exists and is finite the limit*

$$\lim_{\rho\to 0} u_{x0,\rho} =: \tilde{u}(x_0)$$

and satisfies

$$|u_{x0,\rho} - \tilde{u}(x_0)| \leq K_3 \rho^{m(\lambda-1)}[u]\mathcal{L}^{(1,\lambda)}(\Omega,\delta), \tag{2.2.3}$$

where K_3 is independent of x_0.

Proof. We start by proving that $\{u_{x_0,2^{-h}\rho}\}_h$ is Cauchy.

Consider $l, m \in \mathbb{N}$ and assume, without loss of generality, that $l < m$. Then, for $\rho_0 = 2^{-l}\rho$,

$$\begin{aligned} &|u_{x_0,2^{-l}\rho} - u_{x_0,2^{-m}\rho}| = |u_{x_0,\rho_0} - u_{x_0,2^{l-m}\rho_0}| \leq \\ &\leq \frac{K(u)}{1-2^{-\alpha}}\rho_0^{-\alpha}(1-2^{\alpha(l-m)}) \leq \frac{K(u)}{1-2^{-\alpha}}\rho^\alpha 2^{-l\alpha} \to 0, \quad (l\to\infty) \end{aligned}$$

where the first inequality is justified by Lemma 2 and for the second, we note that $1 - 2^{\alpha(l-m)} \leq 1$. Then, $\{u_{x_0,2^{-h}\rho}\}_h$ is Cauchy and therefore convergent, for each choice of $x_0 \in \overline{\Omega}$ and $0 < \rho < d$.

Now we fix ρ and consider $\tilde{u}(x) = \lim_{h\to\infty} u_{x,2^{-h}\rho}$. We want to prove that this limit is independent of the choice of ρ. Let $0 < \sigma < d$ and assume, without loss of generality, that $\sigma < \rho$. Then it suffices to prove that

$$\lim_{h\to\infty} |\tilde{u}(x) - u_{x,2^{-h}\sigma}| = 0,$$

i.e., that

$$\lim_{h\to\infty} |u_{x,2^{-h}\rho} - u_{x,2^{-h}\sigma}| = 0.$$

But Lemma 1 gives

$$|u_{x_0,2^{-h}\rho} - u_{x_0,2^{-h}\sigma}| \leq K_1\Big(\frac{(2^{-h}\rho)^{\lambda m} + (2^{-h}\sigma)^{\lambda m}}{(2^{-h}\sigma)^m}\Big)[u]_{\mathcal{L}^{(1,\lambda)}(\Omega,\delta)} \leq$$
$$\leq K_1\Big(\frac{\rho^{\lambda m} + \sigma^{\lambda m}}{\sigma^m}\Big)(2^{-h})^{m(\lambda-1)}[u]_{\mathcal{L}^{(1,\lambda)}(\Omega,\delta)} \to 0, \quad (h\to\infty).$$

Therefore $\tilde{u}(x)$ is independent of the choice of ρ and

$$\tilde{u}(x) = \lim_{\rho\to 0} u_{x_0,\rho}.$$

The thesis follows from Lemma 2, by considering $h\to\infty$.

□

Lemma 4. *Let $u \in \mathcal{L}^{(1,\lambda)}(\Omega,\delta)$, $\lambda > 1, x, y \in \overline{\Omega}$ and $\rho = 2\delta(x,y)$. Then there exists a constant $K_4 > 0$ such that*

$$|u_{x,\rho} - u_{y,\rho}| \leq K_4\rho^{m(\lambda-1)}[u]_{\mathcal{L}^{(1,\lambda)}(\Omega,\delta)}. \tag{2.2.4}$$

Proof. Fix $x, y \in \overline{\Omega}$ and consider $\rho = 2\delta(x,y)$. Since $\Omega(x,\rho/2) \subset \Omega(x,\rho)\cap\Omega(y,\rho)$ and for almost all $z \in \Omega(x,\rho/2)$,

$$|u_{x,\rho} - u_{y,\rho}| \leq |u(z) - u_{x,\rho}| + |u(z) + u_{y,\rho}|.$$

Then, by the inclusion,

$$\int_{\Omega(x,\rho/2)} |u_{x,\rho} - u_{y,\rho}|dz \leq \int_{\Omega(x,\rho)} |u(z) - u_{x,\rho}|dz + \int_{\Omega(y,\rho)} |u(z) - u_{y,\rho}|dz$$

and we can proceed as in the proof of Lemma 2. □

2.3 Regularity properties

Theorem 1. *Let Ω be a set satisfying (A) and $q \geq 1$. If $\lambda > 1$, then $\mathcal{L}^{(q,\lambda)}(\Omega,\delta) \cong C^{0,\alpha}(\overline{\Omega},\delta)$, with $\alpha = \frac{m}{q}(\lambda-1)$*

Proof. Now we prove *2.* We start by proving that $C^{0,\alpha}(\overline{\Omega},\delta) \subset \mathcal{L}^{(q,\lambda)}(\Omega,\delta)$.

Let $u \in C^{0,\alpha}(\overline{\Omega},\delta)$, $x_0 \in \overline{\Omega}, \rho > 0$. Then

$$\int_{\Omega(x_0,\rho)} |u(x) - u_{x_0,\rho}|^q dx = \frac{1}{|\Omega(x_0,\rho)|^q}\int_{\Omega(x_0,\rho)}\Big|\int_{\Omega(x_0,\rho)} u(x) - u(y)dy\Big|^q dx \leq$$
$$\leq \frac{1}{|\Omega(x_0,\rho)|^q}\int_{\Omega(x_0,\rho)}\Big(\int_{\Omega(x_0,\rho)} |u(x) - u(y)|dy\Big)^q dx \leq$$
$$\leq \frac{1}{|\Omega(x_0,\rho)|^q}\int_{\Omega(x_0,\rho)}\Big(\int_{\Omega(x_0,\rho)} K\delta^\alpha(x,y)dy\Big)^q dx \leq$$
$$\leq K^q\, 2^{\alpha q}|\Omega(x_0,\rho)|\rho^{\alpha q}.$$

Defining $K' = K^q 2^{\alpha q}$, we get

$$|\Omega(x_0,\rho)|^{-\lambda} \int_{\Omega(x_0,\rho)} |u(x) - u_{x_0,\rho}|^q dx \leq K'|\Omega(x_0,\rho)|^{1-\lambda}\rho^{\alpha q}. \tag{2.3.1}$$

Noting property (2.1.1) of the metric and property *(A)* of Ω, since $1-\lambda<0$, it holds

$$\begin{aligned}|\Omega(x_0,\rho)|^{1-\lambda}\rho^{\alpha q} &\leq A^{1-\lambda}|B(0,\rho)|^{1-\lambda}\rho^{m(\lambda-1)} \leq \\ &\leq A^{1-\lambda}M_1^{1-\lambda}\rho^{m(1-\lambda)}\rho^{m(\lambda-1)} = (AM_1)^{1-\lambda}.\end{aligned}$$

Replacing in (2.3.1) and taking the supreme in $x_0 \in \overline{\Omega}, \rho > 0$, we conclude that $u \in \mathcal{L}^{(q,\lambda)}(\Omega,\delta)$.

Finally, we prove the opposite inclusion. We can restrict this study to the case $q=1$, since the following inclusion holds $\mathcal{L}^{(q,\lambda)}(\Omega,\delta) \subset \mathcal{L}^{(1,1+\frac{1}{q}(\lambda-1))}(\Omega,\delta)$. In fact, if $u \in \mathcal{L}^{(q,\lambda)}(\Omega,\delta)$, then

$$\sup_{\substack{x_0\in\overline{\Omega}\\ \rho\in[0,d(\Omega)]}} \left(|\Omega(x_0,\rho)|^{-\lambda}\int_{\Omega(x_0,\rho)}|u(x)-u_{x_0,\rho}|^q dx\right) < +\infty.$$

Hence, by Hölder inequality,

$$\begin{aligned}&|\Omega_{x_0,\rho}|^{-(1+\frac{\lambda-1}{q})}\int_{\Omega(x_0,\rho)}|u(x)-u_{x_0,\rho}|dx \leq \\ &\leq \left(\int_{\Omega(x_0,\rho)}|u(x)-u_{x_0,\rho}|^q dx\right)^{\frac{1}{q}} |\Omega(x_0,\rho)|^{1-\frac{1}{q}}\,|\Omega(x_0,\rho)|^{-(1+\frac{\lambda-1}{q})} = \\ &= \left(|\Omega(x_0,\rho)|^{-\lambda}\int_{\Omega(x_0,\rho)}|u(x)-u_{x_0,\rho}|^q dx\right)^{\frac{1}{q}}.\end{aligned}$$

Taking the supreme, we conclude the inclusion. Also, if the theorem holds for $q=1$ and $u \in \mathcal{L}^{(q,\lambda)}(\Omega,\delta)$, then $u \in \mathcal{L}^{(1,1+\frac{1}{q}(\lambda-1))}(\Omega,\delta)$, from which we conclude $u \in C^{0,\alpha}(\overline{\Omega},\delta)$.

Suppose therefore that $u \in \mathcal{L}^{(1,\lambda)}(\Omega,\delta)$. We start by proving that $\tilde{u}$, defined on Lemma 3, is Hölder continuous with exponent α. So we consider $x,y \in \overline{\Omega}$ and $\rho = 2\delta(x,y)$. It holds

$$|\tilde{u}(x)-\tilde{u}(y)| \leq |\tilde{u}(x)-u_{x,\rho}| + |u_{x,\rho}-u_{y,\rho}| + |\tilde{u}(y)-u_{y,\rho}|.$$

By Lemmas 3 and 4, we get

$$\begin{aligned}|\tilde{u}(x)-\tilde{u}(y)| &\leq (2K_3+K_4)[u]_{\mathcal{L}^{(1,\lambda)}(\Omega,\delta)}\rho^{m(\lambda-1)} = \\ &= 2^{m(\lambda-1)}(2K_3+K_4)[u]_{\mathcal{L}^{(1,\lambda)}(\Omega,\delta)}\big(\delta(x,y)\big)^{m(\lambda-1)}.\end{aligned}$$

Now we prove that $u_{x,\rho}$ converges, in $L^1(\Omega)$, to u, when $\rho \to 0$. In fact,

$$\begin{aligned}\int_{\Omega} |u(x) - u_{x,\rho}|dx &= \frac{1}{|\Omega(x,\rho)} \int_{\Omega} \Big| \int_{\Omega(x,\rho)} \big(u(x) - u(t)\big)dt \Big| dx \leq \\ &\leq \frac{1}{A|B(0,\rho)|} \int_{\Omega(x,\rho)} \int_{\Omega} |u(x) - u(t)|dxdt \leq \\ &\leq \frac{1}{A|B(0,\rho)|} \int_{B(0,\rho)} \int_{\Omega} |u(x) - u(x+t)|dxdt.\end{aligned}$$

Using an argument of density of continuous functions in $L^1(\Omega)$, we get that $\int_\Omega |u(x) - u(x+t)|dx \to 0$ when $|t| \to 0$. Since the topology induced by the metric δ is equivalent to the euclidean, this is equivalent to $\delta(0,t) \to 0$, i.e., $\rho \to 0$.

Finally, by Lemma 3, $u_{x,\rho}$ converges uniformly to $\tilde{u}(x)$ almost everywhere in $\overline{\Omega}$. Therefore $u(x) = \tilde{u}(x)$ a.e.. $\square$

Chapter 3

Application to the degenerate inhomogeneous p-Laplace equation

In this chapter, we will apply the previous results to study the regularity of weak solutions to the partial differential equation

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f \in L^{q,r}, \quad p > 2.$$

We will get an explicit expression for the Hölder exponent of the weak solutions u of (3.1.1), using Theorem 1. It suffices to show that $u \in \mathcal{L}^{(q,\lambda)}(\Omega, \delta)$ for a suitable metric δ defined on $\mathbb{R}^n \times \mathbb{R}$, and $\lambda > 1$, which shall be defined explicitly later on.

For this purpose, we use a method based on the geometric tangential analysis, which consists in considering the homogeneous problem as being tangential to the non-homogeneous. After studying the regularity of the weak solutions to the homogeneous problem, we will use geometric iterations to transport these properties to the solution of the non-homogeneous problem.

In order to be in the conditions to use Caccioppoli's estimate to control each iteration, we need to consider the so-called θ-parabolic cylinders defined by
$G_\tau := (-\tau^\theta, 0) \times B_\tau(0)$, with θ defined later.

Since we wish to use Theorem 1, we must consider a metric in which the balls correspond to the θ-parabolic cylinders.

So we define the metric $\delta : (\mathbb{R}^n \times \mathbb{R}) \times (\mathbb{R}^n \times \mathbb{R}) \to \mathbb{R}_0^+$ given by

$$\delta\Big((x,t),(y,s)\Big) = \max(|x-y|, |t-s|^{1/\theta}), \tag{3.0.1}$$

where $|x-y|$ is the euclidean metric in $\mathbb{R}^n$. To see that this is indeed a metric, it suffices to prove that $|t+s|^r \leq |t|^r + |s|^r$, for a given $0 < r < 1$. Indeed, since the function $f(x) = 1 + x^r - (1+x)^r$, $x \in [0,1]$, is increasing and $f(0) = 0$, assuming $|t| > |s|$ and defining $x = \frac{|s|}{|t|}$, we conclude the triangular inequality.

Since the balls are given by $B'_r(x,t) = \{(y,s) \in \mathbb{R}^n \times \mathbb{R} : |x-y| < r \text{ and } |t-s| < r^\theta\}$, it is clear that they correspond exactly to the θ-parabolic cylinders.

We now define universal constants which depend solely on the data. Let

$$\alpha := \frac{p\left(1 - \frac{1}{r} - \frac{n}{pq}\right)}{\left(\frac{2}{r} + \frac{n}{q} - 1\right) + p\left(1 - \frac{1}{r} - \frac{n}{pq}\right)}, \tag{3.0.2}$$

and

$$\theta := 2\alpha + (1-\alpha)p. \tag{3.0.3}$$

By (3.1.2) and (3.1.3), we get $0 < \alpha < 1$ and so, since $p > 2$, we have $2 < \theta < p$.

We will prove that these solutions live in the space $C^{0,\alpha,\alpha/\theta}$, i.e., the solutions are Hölder continuous with exponents α in space and α/θ in time.

3.0.1 Preliminaries

We start by introducing a few preliminary results, omitting their proofs. First some Embeddings.

Let $m, p > 1$ and consider the Banach spaces

$$V^{m,p}(U_T) = L^\infty(0,T; L^m(U)) \cap L^p(0,T; W^{1,p}(U))$$
$$V_0^{m,p}(U_T) = L^\infty(0,T; L^m(U)) \cap L^p(0,T; W_0^{1,p}(U))$$

both equipped with the norm

$$||v||_{V^{m,p}(U_T)} = \operatorname*{ess\,sup}_{0<t<T} ||v(.,t)||_{m,U} + ||Dv||_{p,U_T}.$$

Proposition 1. *There exists a constant γ, depending only on N, p and m such that, for every $v \in V_0^{m,p}(U_T)$,*

$$\int\int_{U_T} |v|^q \, dxdt \le \gamma^q \left(\int\int_{U_T} |Dv|^p \, dxdt\right) \left(\operatorname*{ess\,sup}_{0<t<T} \int_U |v|^m dx\right)^{\frac{p}{N}}$$

where $q = p\frac{N+m}{N}$.

Proposition 2. *There exists a constant γ depending only upon N such that, for every $v \in V_0^p(U_T)$,*

$$||v||_{q,r,U_T} \le \gamma ||u||_{V^p(\Omega_T)},$$

where the numbers q, r are linked by

$$\frac{1}{r} + \frac{N}{pq} = \frac{N}{p^2}$$

Lemma 5. *Let $v \in V^{m,p}(U_T)$. Then $(v-k)_\pm \in V^{m,p}(U_T)$ for all $k \in \mathbb{R}$.*

The next result is a lemma on fast convergence.

Lemma 6. *Let $\{Y_n\}$ be a sequence of positive numbers satisfying the recursive inequalities*

$$Y_{n+1} \leq Cb^n Y_n^{1+\alpha}$$

where $C, b > 1$ and $\alpha > 0$ are given numbers. If

$$Y_0 \leq C^{-1/\alpha} b^{-1/\alpha^2},$$

then $Y_n \to 0$ as $n \to \infty$.

3.1 Weak solutions and Caccioppoli's estimate

Let $U \subset \mathbb{R}^n$ be open and bounded and $T > 0$. Consider the space-time domain $U_T = U \times (0,T)$. We wish to study the equation

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f \qquad \text{em } U_T, \tag{3.1.1}$$

with $f \in L^{q,r}(U_T) = L^r(0,T;L^q(U))$. Note that $L^r(0,T;L^q(U))$ corresponds to the space of functions $g : (0,T) \to L^q(U)$ which are r-integrable. This space is equipped with the norm

$$||g||_{q,r} = \Big(\int_0^T ||g(t)||_q^r dt \Big)^{\frac{1}{r}}$$

which makes it a Banach space. We will consider the following restrictions

$$\frac{1}{r} + \frac{n}{pq} < 1 \tag{3.1.2}$$

and

$$\frac{2}{r} + \frac{n}{q} > 1. \tag{3.1.3}$$

We start by defining weak solution to the equation (3.1.1).

Definition 3. *We say a function*

$$u \in C_{loc}(0,T;L^2(U)) \cap L^p_{loc}(0,T;W^{1,p}(U))$$

is a weak solution to the equation (3.1.1) *if, for each compact $K \subset U$ and each subinterval $[t_1,t_2] \subset (0,T]$, it holds*

$$\int_K u\varphi dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \big(-u\varphi_t + |\nabla u|^{p-2}\nabla u \nabla\varphi \big) dxdt = \int_{t_1}^{t_2}\int_K f\varphi dxdt, \tag{3.1.4}$$

for every $\varphi \in H^1_{loc}(0,T;L^2(K)) \cap L^p_{loc}(0,T;W_0^{1,p}(K))$.

Alternatively, one can make use of a smoothing process to circumvent difficulties regarding time regularity, however as we will see in the next section, the time derivative is well defined (in a weak sense) and belongs to some Lebesgue space. For this, define the Steklov average of a function $v \in L^1(U_T)$ as

$$v_h(x,t) = \begin{cases} \frac{1}{h}\int_t^{t+h} v(x,\tau)d\tau & \text{if } t \in (0, T-h] \\ 0 & \text{if } t \in (T-h, T] \end{cases}$$

and we can define weak solutions in an alternative fashion

Definition 4. *We say a function*

$$u \in C_{loc}(0,T; L^2(U)) \cap L^p_{loc}(0,T; W^{1,p}(U))$$

is a weak solution to the equation (3.1.1) *if, for each compact $K \subset U$ and every $0 < t < T-h$, it holds*

$$\int_{K\times\{t\}} \left((u_h)_t\phi + (|\nabla u|^{p-2}\nabla u)_h.\nabla\phi\right)dx = \int_{K\times\{t\}} f_h\phi dx, \tag{3.1.5}$$

for all $\phi \in W^{1,p}_0(K)$.

An essential tool to control the behavior of the weak solutions of (3.1.1) is the Caccioppoli's energy estimate.

Lemma 7 (Caccioppoli's estimate)**.** *Let u be a weak solution of* (3.1.1)*. Given $K\times[t_1,t_2] \subset U\times(0,T]$, there exists a constant C, depending only on n, p and $K\times[t_1,t_2]$, such that*

$$\begin{aligned} &\sup_{t_1<t<t_2}\int_K u^2\xi^p dx + \int_{t_1}^{t_2}\int_K |\nabla u|^p\xi^p dxdt \leq \qquad (3.1.6)\\ &\leq C\left(\int_{t_1}^{t_2}\int_K |u|^p(\xi^p + |\nabla\xi|^p)dxdt + \int_{t_1}^{t_2}\int_K u^2\xi^{p-1}|\xi_t|dxdt + ||f||_{q,r}\right). \end{aligned}$$

Proof. We choose the test function $\phi = u_h\xi^p$ (which is an admissible one, since the derivative $(u_h)_t$ is well defined) and replace in (3.1.5). Integrating in both sides over $(t_1,t) \subset (t_1,t_2)$ we get

$$\begin{aligned} &\int_{t_1}^t\int_K \left((u_h)_t u_h\xi^p + (|\nabla u|^{p-2}\nabla u)_h(\nabla u_h\xi^p + u_h p\xi^{p-1}\nabla\xi)\right)dxd\tau = \\ &= \int_{t_1}^t\int_K f_h u_h\xi^p\, dxd\tau \iff \\ \iff &\int_K \frac{1}{2}u_h^2\xi^p\, dx\Big|_{t_1}^t + \int_{t_1}^t\int_K \left(|\nabla u|^{p-2}\nabla u\right)_h \nabla u_h\xi^p\, dxd\tau = \\ &= \int_{t_1}^t\int_K \left(p(|\nabla u|^{p-2}\nabla u)_h u_h\xi^{p-1}\nabla\xi + \frac{1}{2}p\xi^{p-1}\xi_t u_h^2 + f_h u_h\xi^p\right)dxd\tau \end{aligned}$$

now we take $h \to 0$ and consider the norm of the second member to get

$$\int_k \frac{1}{2} u^2 \xi^p dx \Big|_{t_1}^{t} + \int_{t_1}^{t} \int_K |\nabla u|^p \xi^p dx dt \leq \tag{3.1.7}$$
$$\leq \int_{t_1}^{t_2} \int_K \left(p |\nabla u|^{p-1} |u| \xi^{p-1} |\nabla \xi| + \frac{1}{2} p \xi^{p-1} |\xi_t| u^2 + |f||u| \xi^p \right) dx dt.$$

Studying term by term, by Young's inequality we get

$$\int_{t_1}^{t_2} \int_K p |\nabla u|^{p-1} |u| \xi^{p-1} |\nabla \xi| dx dt \leq$$
$$\leq \epsilon^{p'} (p-1) \left(\int_{t_1}^{t_2} \int_K |\nabla u \xi|^p dx dt \right) + \epsilon^{-p} \int_{t_1}^{t_2} \int_K |u \nabla \xi|^p dx dt,$$

with $p' = \frac{p}{p-1}$. Considering ϵ small enough, the first term gets absorbed by the first member.

The term containing f is treated in the standard way, (cfr. [10]).

Now we instead integrate over $(t, t_2) \subset (t_1, t_2)$. Adding both estimates, we get the desired inequality.

□

3.2 Time regularity

We want to prove that the weak solutions of (3.1.1) have time derivatives, in the sense of Sobolev, belonging to some Lebesgue space. Note that the following identity holds for weak solutions to (3.1.1)

$$\int_0^T \int_K u \phi_t \, dx dt = \int_0^T \int_K (\operatorname{div}(|\nabla u|^{p-2} \nabla u) - f) \phi \, dx dt \tag{3.2.1}$$

for all test functions $\phi \in C^\infty$, assuming the space derivatives exists in the weak sense. We wish to prove that the derivatives

$$v_{i,j} := \frac{\partial}{\partial x_j} \left(|\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \right)$$

exist in the sense of Sobolev, belonging to some space L^{q_1}. Since $f \in L^q$, then $v_{i,j} - f \in L^{q_1} \cap L^q = L^{\min(q_1, q)}$, from which we conclude $u_t = \frac{\partial}{\partial x_j} \left(|\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \right) - f$. In this section, we will prove the following main theorem.

Theorem 2. *Let $2 \leq p < \infty$. If $u = u(x,t)$ is a weak solution of* (3.1.1), *then the time derivative u_t exists in the sense of Sobolev and lives in a Lebesgue space.*

3.2.1 Energy estimate

We start by defining weak solutions of (3.1.1) in a different way.

Definition 5. *Let $\Omega \subset \mathbb{R}^n$, $\Omega_T := \Omega \times (0,T)$ and $p \geq 2$. We say that $u \in C_{loc}(0,T;L^2(U)) \cap L^p_{loc}(0,T;W^{1,p}(U))$ is a weak solution of (3.1.1) if*

$$\int_0^T \int_\Omega (-u\phi_t + |\nabla u|^{p-2}\nabla u \nabla \phi)\, dxdt = \int_0^T \int_\Omega f\phi\, dxdt \tag{3.2.2}$$

for all $\phi \in C_0^1(\Omega_T)$.

As we saw, we want to study the space derivatives of u. For such, we require a variation of the Caccioppoli's estimate for the difference $u(x+h,t) - u(x,t)$, with $|h|$ small enough. Clearly, for each test function ϕ with support compactly contained in Ω, the function $v(x,t) := u(x+h,t)$ satisfies (3.2.2), as long as $|h|$ is small enough.

Subtracting the equations for $v(x,t)$ and $u(x,t)$, we get

$$\begin{aligned}&\int_0^T \int_\Omega |(\nabla u(x+h,t)|^{p-2}\nabla u(x+h,t) - \\ &- |\nabla u(x,t)|^{p-2}\nabla u(x,t))\nabla\phi(x,t)\, dxdt = \\ =&\int_0^T \int_\Omega (u(x+h,t) - u(x,t))\phi_t\, dxdt\end{aligned} \tag{3.2.3}$$

The usual procedure to obtain such an energy estimate is to consider the test function

$$\phi(x,t) = \eta(t)\xi(x)^p(u(x+h,t) - u(x,t)),$$

with $\xi \in C_0^\infty(\Omega), 0 \leq \xi \leq 1$, and $\eta(t) \in [0,1]$ is a cutoff function. However, this is not an admissible one, since when computing the time derivative ϕ_t, the forbidden derivative u_t will appear. A formal calculation yields

$$\begin{aligned}\nabla\phi(x,t) &= \eta(t)\xi(x)^p\big(\nabla u(x+h,t) - \nabla u(x,t)\big) + \\ &\quad + \eta(t)p\xi(x)^{p-1}\nabla\xi(x)\big(u(x+h,t) - u(x,t)\big), \\ \phi_t(x,t) &= \eta'(t)\xi(x)^p\big(u(x+h,t) - u(x,t)\big) + \\ &\quad + \eta(t)\xi(x)^p\big(u(x+h,t) - u(x,t)\big)_t.\end{aligned}$$

Substituting in (3.2.3), the second member becomes

$$\begin{aligned}&\int_0^T \int_\Omega \big(u(x+h,t) - u(x,t)\big)^2\eta'(x,t)\xi(x)^p + \\ &+\frac{1}{2}\Big[\big(u(x+h,t) - u(x,t)\big)^2\Big]_t \eta(t)\xi(x)^p\, dxdt = \\ &=\frac{1}{2}\int_0^T \int_\Omega \big(u(x+h,t) - u(x,t)\big)^2\eta'(x,t)\xi(x)^p\, dxdt.\end{aligned}$$

So we get

$$\int_0^T \int_\Omega \eta(t)\xi(x)^p(|\nabla u(x+h,t)|^{p-2}\nabla u(x+h,t)-$$
$$-|\nabla u(x,t)|^{p-2}\nabla u(x,t))(\nabla u(x+h,t)-\nabla u(x,t))\,dxdt=$$
$$=-\,p\int_0^T \int_\Omega \eta(t)\xi(x)^{p-1}(x)\big(u(x+h,t)-u(x,t)\big) \tag{3.2.4}$$
$$\big(|\nabla u(x+h,t)|^{p-2}\nabla u(x+h,t)-|\nabla u(x,t)|^{p-2}\nabla u(x,t)\big)\,\nabla\xi(x)\,dxdt+$$
$$+\frac{1}{2}\int_0^T \int_\Omega \eta'(t)\xi(x)^p\big(u(x+h,t)-u(x,t)\big)^2\,dxdt$$

which is the desired estimate. So now we wish to get to the same identity, but starting with an admissible test function. For this purpose, consider the convolution

$$\begin{aligned} f(x,t)^* &= \int_0^T \int_\Omega f(y,\tau)\rho_\sigma(x-y,t-\tau)dyd\tau \\ &= \int_{B_\sigma} f(x-y,t-\tau)\rho_\sigma(y,\tau)dyd\tau \end{aligned}$$

with $(x,t)\in\big(\Omega_T\big)_\sigma := \{(y,\tau)\in\Omega_T : \operatorname{dist}\big((y,\tau),\partial\Omega_T\big)>\sigma\}$ and where ρ_σ is a smooth non-negative function with support in the ball $B_\sigma\subset\mathbb{R}^{n+1}$.

Now we note that, if u satisfies (3.2.2), then so does u^* in its domain. In fact,

$$\int_\sigma^{T-\sigma}\int_{\Omega_\sigma}\big((|\nabla u|^{p-2}\nabla u)^*\big)\,\nabla\phi\,dxdt=$$
$$=\int_\sigma^{T-\sigma}\int_{\Omega_\sigma}\left(\int_{B_\sigma}|\nabla u|^{p-2}\nabla u(x-y,t-\tau)\rho_\sigma(y,\tau)dydt\right)\nabla\phi\,dxdt=$$
$$=\int_\sigma^{T-\sigma}\int_{\Omega_\sigma}\Big(\int_{B_\sigma}\big(|\nabla u|^{p-2}\nabla u(x-y,t-\tau)\big)\,\nabla\phi\rho_\sigma(y,\tau)dyd\tau\Big)\,dxdt=$$
$$=\int_{B_\sigma}\Big(\int_\sigma^{T-\sigma}\int_{\Omega_\sigma}\big(|\nabla u|^{p-2}\nabla u(x-y,t-\tau)\big)\,\nabla\phi\,dxdt\Big)\rho_\sigma(y,\tau)\,dyd\tau=$$
$$=\int_{B_\sigma}\Big(\int_0^T\int_\Omega\big(|\nabla u|^{p-2}\nabla u(x-y,t-\tau)\big)\,\nabla\overline{\phi}\,dxdt\Big)\rho_\sigma(y,\tau)\,dyd\tau=$$
$$=\int_{B_\sigma}\Big(\int_0^T\int_\Omega u(x-y,t-\tau)\overline{\phi}_t\,dxdt\Big)\,dyd\tau=$$
$$=\int_\sigma^{T-\sigma}\int_{\Omega_\sigma}\Big(\int_{B_\sigma}u(x-y,t-\tau)\rho_\sigma(y,\tau)\,dyd\tau\Big)\phi_t\,dxdt=$$
$$=\int_\sigma^{T-\sigma}\int_{\Omega_\sigma}u^*\phi_t\,dxdt$$

where $\overline{\phi}$ is an extension of ϕ to Ω_T that is zero outside of $(\Omega_T)_\sigma$.

Therefore u^* and v^* also satisfy (3.2.2). So we can repeat the previous argument with u^* and v^* instead of u and v, respectively. From (3.2.3) we can write

$$\begin{aligned} \int_0^T\int_\Omega\big((|\nabla v|^{p-2}\nabla v)^*-(|\nabla u|^{p-2}\nabla u)^*\big)\,(\nabla\phi(x,t))\,dxdt \\ =\int_0^T\int_\Omega(v^*-u^*)\phi_t\,dxdt \end{aligned} \tag{3.2.5}$$

as long as the parameter σ is small enough.

We can now consider the test function

$$\phi(x,t) = \eta(t)\xi(x)^p(u^*(x+h\,t) - u^*(x,t)),$$

which is now an admissible one, since the derivative u_t^* is well defined, and a similar calculation yields

$$\int_\sigma^{T-\sigma}\int_{\Omega_\sigma} \eta(t)\xi(x)^p \left((|\nabla v|^{p-2}\nabla v)^* - (|\nabla u|^{p-2}\nabla u)^*\right)(\nabla v^* - \nabla u^*)\, dxdt =$$
$$= -\,p\int_\sigma^{T-\sigma}\int_{\Omega_\sigma} \eta(t)\xi(x)^{p-1}(x)\big(v^* - u^*\big)\big((|\nabla v|^{p-2}\nabla v)^* -$$
$$-(|\nabla u|^{p-2}\nabla u)^*\big)(\nabla\xi(x))\, dxdt + \frac{1}{2}\int_\sigma^{T-\sigma}\int_{\Omega_\sigma} \eta'(t)\xi(x)^p\big(v^* - u^*\big)^2\, dxdt.$$

Now we can let $\sigma \to 0$ and we arrive once again at the energy estimate (3.2.4), now starting with a valid test function.

3.2.2 Bounding of difference quotients and final proof

We wish to apply the following result.

Lemma 8. *Let $D^h u$ be the vector with the difference quotients of size h. Then*

1. *Suppose $1 \le p < \infty$ and $u \in W^{1,p}(U)$. Then for each $V \subset\subset U$*

$$||D^h u||_{L^p(V)} \le C||Du||_{L^p(U)}$$

for some constant C and all $0 < |h| < \frac{1}{2}dist(V, \partial U)$.

2. *Assume $1 < p < \infty$, $u \in L^p(U)$, and there exists a constant C such that*

$$||D^h u||_{L^p(V)} \le C$$

for all $0 < |h| < \frac{1}{2}dist(V, \partial U)$. Then

$$u \in W^{1,p}(V),\ \textit{with}\ ||Du||_{L^p(V)} \le C$$

To do this, we first define the vector field

$$F(x,t) := |\nabla u(x,t)|^{\frac{p-2}{2}}\nabla u(x,t)$$

and write DF for the Jacobian matrix with the elements

$$\Big(\frac{\partial}{\partial x_j}\big(|\nabla u|^{\frac{p-2}{2}}\frac{\partial u}{\partial x_i}\big)\Big)_{i,j}.$$

Lemma 9. *Let $p > 2$. The derivatives DF exist in the Sobolev sense and $DF \in L^2_{loc}(\Omega_T)$. The estimate*

$$\int_\tau^T \int_\Omega \xi(x)^p |DF|^2 dxdt \leq \frac{C}{\tau} \int_0^\tau \int_\Omega \xi(x)^p |\nabla u(x,t)|^2 dxdt + \tag{3.2.6}$$
$$+ \int_0^T \int_\Omega (\xi(x)^p + |\nabla \xi(x)|^p) |\nabla u(x,t)|^p dxdt$$

holds when $\tau > 0$. Here $\xi \in C_0^\infty(\Omega)$, $\xi(x) \geq 0$.

Proof. We will make use of the following useful inequalities

$$\frac{4}{p^2} \Big| |b|^{\frac{p-2}{2}} b - |a|^{\frac{p-2}{2}} a \Big|^2 \leq < |b|^{p-2} b - |a|^{p-2} a, b - a >, \tag{3.2.7}$$

$$\Big| |b|^{p-2} b - |a|^{p-2} a \Big| \leq (p-1) \Big(|b|^{\frac{p-2}{2}} + |a|^{\frac{p-2}{2}} \Big) \Big| |b|^{\frac{p-2}{2}} - |a|^{\frac{p-2}{2}} a \Big|, \tag{3.2.8}$$

for vectors a, b and $p \geq 2$.

We now apply inequality (3.2.7) and recall the energy estimate (3.2.4) to get

$$\frac{4}{p^2} \int_0^T \int_\Omega \eta(t) \xi(x)^p \big| F(x+h,t) - F(x,t) \big|^2 dxdt = \tag{3.2.9}$$
$$= \frac{4}{p^2} \int_0^T \int_\Omega \eta(t) \xi(x)^p \Big| |\nabla v|^{\frac{p-2}{2}} \nabla v - |\nabla u|^{\frac{p-2}{2}} \nabla u \Big|^2 dxdt \leq$$
$$\leq \int_0^T \int_\Omega \eta(t) \xi(x)^p \left(|\nabla v|^{p-2} \nabla v - |\nabla u|^{\frac{p-2}{2}} \nabla u \right) (\nabla v - \nabla u) \; dxdt =$$
$$\leq \frac{1}{2} \int_0^T \int_\Omega \eta'(t) \xi(x)^p (v-u)^2 dxdt +$$
$$+ p \int_0^T \int_\Omega \eta(t) \xi(x)^{p-1} |v-u| \big| |\nabla v|^{p-2} \nabla v - |\nabla u|^{p-2} \nabla u \big| |\nabla \xi(x)| dxdt.$$

Now we focus on the last integral. By (3.2.8),

$$p \int_0^T \int_\Omega \eta(t) \xi(x)^{p-1} |v-u| \big| |\nabla v|^{p-2} \nabla v - |\nabla u|^{p-2} \nabla u \big| |\nabla \xi(x)| dxdt \leq$$
$$\leq p(p-1) \int_0^T \int_\Omega \Big(\eta(t)^{\frac{1}{2}} \xi(x)^{\frac{p}{2}} |F(x+h,t) - F(x,t)| \Big) .$$
$$. \Big(\eta(t)^{\frac{1}{2}} |v-u| |\nabla \xi| \Big) \Big(|\nabla v|^{\frac{p-2}{2}} + |\nabla u|^{\frac{p-2}{2}} \Big) \xi(x)^{\frac{p-2}{2}} dxdt.$$

Substituting in (3.2.9) and dividing both members by $|h|^2$, we get

$$\begin{aligned}
&\frac{4}{p^2}\int_0^T\int_\Omega \eta(t)\xi(x)^p\left|\frac{F(x+h,t)-F(x,t)}{|h|}\right|^2 dxdt \leq\\
\leq&\frac{1}{2}\int_0^T\int_\Omega \eta'(t)\xi(x)^p\left|\frac{v-u}{|h|}\right|^2 dxdt+\\
&+p(p-1)\int_0^T\int_\Omega\left(\eta(t)^{\frac{1}{2}}\xi(x)^{\frac{p}{2}}\left|\frac{F(x+h,t)-F(x,t)}{|h|}\right|\right)\\
&\left(\eta(t)^{\frac{1}{2}}\frac{|v-u|}{|h|}|\nabla\xi|\right)\left(|\nabla v|^{\frac{p-2}{2}}+|\nabla u|^{\frac{p-2}{2}}\right)\xi(x)^{\frac{p-2}{2}}dxdt\leq\\
\leq&\frac{1}{2}\int_0^T\int_\Omega \eta'(t)\xi(x)^p\left|\frac{v-u}{|h|}\right|^2 dxdt+\\
&+\frac{p(p-1)\epsilon^2}{2}\int_0^T\int_\Omega \eta(t)\xi(x)^p\left|\frac{F(x+h,t)-F(x,t)}{|h|}\right|^2 dxdt+\\
&+(p-1)\epsilon^{-p}\int_0^T\int_\Omega \eta(t)^{\frac{p}{2}}\left|\frac{v-u}{|h|}\right|^p|\nabla\xi(x)|^p dxdt+\\
&+c_p\int_0^T\int_\Omega \xi(x)^p\left(|\nabla v|^p+|\nabla u|^p\right)dxdt
\end{aligned}$$

where we used the trivial inequality

$$abc\leq\frac{\epsilon^2a^2}{2}+\frac{\epsilon^{-p}b^p}{p}+\frac{p-2}{2p}c^{\frac{2p}{p-2}}.$$

Now we take ϵ such that $\frac{p(p-1)\epsilon^2}{2}=\frac{2}{p^2}$ to get

$$\begin{aligned}
&\frac{2}{p^2}\int_0^T\int_\Omega \eta(t)\xi(x)^p\left|\frac{F(x+h,t)-F(x,t)}{|h|}\right|^2 dxdt\leq\\
\leq&\frac{1}{2}\int_0^T\int_\Omega \eta'(t)\xi(x)^p\left|\frac{u(x+h,t)-u(x,t)}{|h|}\right|^2 dxdt+\\
&+a_p\int_0^T\int_\Omega\left|\frac{u(x+h,t)-u(x,t)}{|h|}\right|^p|\nabla\xi(x)|^p dxdt+\\
&+c_p\int_0^T\int_\Omega \xi(x)^p\left(|\nabla u(x+h,t)|^p+|\nabla u(x,t)|^p\right)dxdt.
\end{aligned}$$

Finally, we choose suiting cutoff functions. Let η be a 3-piecewise linear cutoff function such that $\eta(t)=1$ for $\tau\leq t\leq T-\beta$ for small enough $\beta>0$. Then we clearly see that $\eta'(t)>0$ if and only if $t\in(0,\tau)$, so we can remove the other portion of the integral. Furthermore, in this interval $\eta'(t)=\frac{1}{\tau}$. We arrive at the estimate

$$\begin{aligned}
&\frac{2}{p^2}\int_\tau^T\int_\Omega \xi(x)^p\left|\frac{F(x+h,t)-F(x,t)}{|h|}\right|^2 dxdt\leq\\
\leq&\frac{1}{2\tau}\int_0^\tau\int_\Omega \xi(x)^p\left|\frac{u(x+h,t)-u(x,t)}{|h|}\right|^2 dxdt+\\
&+a_p\int_0^T\int_\Omega\left|\frac{u(x+h,t)-u(x,t)}{|h|}\right|^p|\nabla\xi(x)|^p dxdt+\\
&+c_p\int_0^T\int_\Omega \xi(x)^p\left(|\nabla u(x+h,t)|^p+|\nabla u(x,t)|^p\right)dxdt.
\end{aligned}$$

We can furthermore choose ξ such that $\xi(x) = 1$ in some compact V_1 and such that $|\nabla\xi| < C$ for some $C > 0$. It follows immediately, recalling the initial assumptions on ξ, that both ξ and $\nabla\xi$ have supports compactly contained in Ω, call them S_1 and S_2 respectively. We get

$$\begin{aligned}
&\frac{2}{p^2}\int_\tau^T\int_{V_1}\left|\frac{F(x+h,t)-F(x,t)}{|h|}\right|^2 dxdt \leq \\
\leq&\frac{1}{2\tau}\int_0^\tau\int_{S_1}\left|\frac{u(x+h,t)-u(x,t)}{|h|}\right|^2 dxdt+ \\
&+a_pC\int_0^T\int_{S_2}\left|\frac{u(x+h,t)-u(x,t)}{|h|}\right|^p dxdt+ \\
&+c_p\int_0^T\int_{S_1}\left(|\nabla u(x+h,t)|^p+|\nabla u(x,t)|^p\right)dxdt.
\end{aligned}$$

We can finally apply Lemma 8. Since $u \in W^{1,p}(U)$, we use the first part of the Lemma to get

$$\begin{aligned}
&\frac{2}{p^2}||D^hF||^2_{L^{2,2}(V_1\times(\tau,T))} \leq \frac{1}{2\tau}||D^hu||^2_{L^{2,2}(S_1\times(0,\tau))}+a_pC||D^hu||^p_{L^{p,p}(S_2\times(0,T))}+ \\
&+c_p(||Du||^p_{L^{p,p}((S_1+h)\times(0,T))}+||Du||^p_{L^{p,p}(S_1\times(0,T))}) \leq \\
&\leq C_1||Du||^2_{L^{2,2}(S_1\times(0,\tau))}+C_2||Du||^p_{L^{p,p}(S\times(0,T))} \leq C||Du||^p_{L^{p,p}(S\times(0,T))}.
\end{aligned}$$

Now we use second part of the Lemma to conclude that $F \in L^2_{loc}(0,T;W^{1,2}(\Omega))$. Finally, taking $|h| \to 0$ yields the desired estimate. □

We are in a position to prove **Theorem 2**.

Proof. Note that by our definition of F, it holds

$$|F|^2 = |\nabla u|^p, \quad |\nabla u|^{p-2}\nabla u = |F|^{1-\frac{2}{p}}F.$$

We compute

$$\frac{\partial}{\partial x_j}\left(|\nabla u|^{p-2}\nabla u\right) = \frac{\partial}{\partial x_j}\left(|F|^{1-\frac{2}{p}}F\right) = \left(1-\frac{2}{p}\right)|F|^{-1-\frac{2}{p}} < \frac{\partial F}{\partial x_j}, F > +|F|^{1-\frac{2}{p}}\frac{\partial F}{\partial x_j}$$

therefore, taking the module and the power $\frac{p}{p-1}$, we get

$$\begin{aligned}
\left|\frac{\partial}{\partial x_j}\left(|\nabla u|^{p-2}\nabla u\right)\right|^{\frac{p}{p-1}} &\leq \left(2-\frac{2}{p}\right)^{\frac{p}{p-1}}\left|\frac{\partial F}{\partial x_j}\right|^{\frac{p}{p-1}}|F|^{\frac{p-2}{p-1}} \leq \\
&\leq \left(2-\frac{2}{p}\right)^{\frac{p}{p-1}}\left(\frac{p}{2(p-1)}\left|\frac{\partial F}{\partial x_j}\right|^2+\frac{p-2}{2(p-1)}|F|^2\right)
\end{aligned}$$

and we can use Lemma 9 to complete the proof.

□

3.3 Hölder regularity

Our next result uses a compactness property to conclude that if the source term f has a small enough norm in $L^{q,r}$, then the solution u to (3.1.1) is close in norm to the solution of the homogeneous problem in an inner subdomain. We will make use of the following classical compactness result

Lemma 10. *Let $X \hookrightarrow B \subset Y$ be Banach spaces. Let $F \subset L^p(0,T;X)$ be bounded, where $1 \leq p < \infty$, and $\partial F/\partial t := \{\partial f/\partial t : f \in F\}$ be bounded in $L^1(0,T;Y)$. Then F is relatively compact in $L^p(0,T;B)$.*

Lemma 11 (Approximation to p-caloric functions)**.** *For every $\delta > 0$, there exists $0 < \epsilon \ll 1$ such that if $||f||_{L^{q,r}(G_1)} \leq \epsilon$ and u is a local weak solution of (3.1.1), with $||u||_{p,avg,G_1} \leq 1$, then there exists a p-caloric function ϕ in $G_{1/2}$, in the sense that*

$$\phi_t - div(|\nabla\phi|^{p-2}\nabla\phi) = 0, \text{ in } G_{1/2}, \tag{3.3.1}$$

and moreover satisfies

$$||u-\phi||_{p,avg,G_{1/2}} \leq \delta. \tag{3.3.2}$$

Proof. The proof is done by contradiction. Suppose that, for some $\delta_0 > 0$, there exists a sequence of functions $(u^j)_j$ and $(f^j)_j$ that satisfy the hypothesis but not the thesis. That is, for every j,

$$u^j \in C_{loc}(-1,0;L^2_{loc}(B_1)) \cap L^p_{loc}(-1,0;W^p_{loc}(B_1)) \tag{3.3.3}$$

and $(f^j)_j \in L^{q,r}(G_1)$ with

$$u^j_t - \text{div}(|\nabla u^j|^{p-2}\nabla u^j) = f^j \text{ in } G_1, \tag{3.3.4}$$

$$||u^j||_{p,avg,G_1} \leq 1, \tag{3.3.5}$$

$$||f^j||_{L^{q,r}(G_1)} \leq \frac{1}{j}, \tag{3.3.6}$$

but still, for any j and any p-caloric function ϕ in $G_{1/2}$,

$$||u^j - \phi||_{p,avg,G_{1/2}} > \delta_0. \tag{3.3.7}$$

Define, for simplicity, the notation

$$V(I \times U) := L^\infty(I;L^2(U)) \cap L^p(I;W^{1,p}(U)).$$

equipped with the norm

$$||u^j||_{V(G_{1/2})} = ||u^j||_{L^{2,\infty}(G_{1/2})} + ||u^j||_{L^p(-1/2^\theta,0;W^{1,p}(B_{1/2}))}. \tag{3.3.8}$$

We will prove that our assumptions imply that the sequence $(u^j)_j$ is bounded in $V(G_{1/2})$.

We note the following inequality

$$a + b \leq 2max(a, b),$$

so that

$$(a + b)^p \leq 2^p \max(a, b)^p \leq 2^p(a^{p_1} + b^{p_2}).$$

Hence

$$a + b \leq 2(a^{p_1} + b^{p_2})^{\frac{1}{p*}},$$

where $p* = p_1$ if $a \geq b$ and $p* = p_2$ otherwise, where p_1, p_2 can be any positive numbers. So we can write

$$||u^j||_{V(G_{1/2})} \leq C\left(||u^j||^2_{L^{2,\infty}(G_{1/2})} + ||u^j||^p_{L^p(-1/2^\theta,0;W^{1,p}(B_{1/2}))}\right)^{\frac{1}{p*}} \leq$$

$$\leq C\left(\sup_{-1/2^\theta<t<0}\int_{B_{1/2}} |u^j|^2 dx + \int_{-\frac{1}{2^\theta}}^0\int_{B_{1/2}} |u^j|^p + |\nabla u^j|^p dxdt\right)^{\frac{1}{p*}} \leq$$

$$\leq C\left(\sup_{-1<t<0}\int_{B_1} |u^j|^2\xi^p dx + ||u^j||^p_{L^p(G_1)} + \int_{-1}^0\int_{B_{1/2}} |\nabla u^j|^p\xi^p dxdt\right)^{\frac{1}{p*}}.$$

We can absorb the term $||u^j||^p_{L^pG_1}$ in the constant, since (3.3.5) holds. Caccioppoli's estimate yields

$$||u^j||_{V(G_{1/2})} \leq C\Big(\int_{-1}^0\int_{B_1}\left(|u^j|^p(\xi^p + |\nabla\xi|^p) + (u^j)^2\xi^{p-1}|\xi_t|\right) dxdt +$$
$$+ ||f^j||_{L^{q,r}(G_1)}\Big)^{\frac{1}{p*}}$$

a control on the derivatives of the test function gives

$$||u^j||_{V(G_{1/2})} \leq C\left(||u^j||^p_{p,avg,G_1} + ||u^j||^2_{2,avg,G_1} + \frac{1}{j}\right)^{\frac{1}{p*}} \leq$$
$$\leq C.$$

In these calculations, we also proved that $(u^j)_j$ is bounded in $L^p(-1/2^\theta, 0; W^{1,p}(B_{1/2}))$ (this is trivial from (3.3.8)).

By Theorem 2,

$$||u^j_t||_{L^{s,1}(G_{1/2})} \leq c,$$

with $s = \min\{q, p/(p-1)\} < p$. Since it holds

$$W^{1,p} \hookrightarrow L^p \subset L^s$$

we invoke Lemma 10, with $X = W^{1,p}(G_{1/2})$, $B = L^p(G_{1/2})$ and $Y = L^s(G_{1/2})$, which guarantees that the sequence $(u^j)_j$ is a precompact in $L^p(G_{1/2})$ which implies that we can extract a subsequence that converges strongly to a function $\psi \in L^p(G_{1/2})$. The weak convergence guarantees that $\psi \in V(G_{1/2})$. So we can take the limit in (3.3.4) to find that

$$\psi_t - \operatorname{div}(|\nabla\psi|^{p-2}\nabla\psi) = 0 \text{ in } G_{1/2},$$

i.e., ψ is p-caloric. This, together with the convergence in the $L^p(G_{1/2})$ norm, contradicts assumption (3.3.7) which completes the proof. □

Next we wish to use the previous Lemma to transport properties of the p-caloric functions to the weak solution u, considering an even smaller domain.

Lemma 12. *Let $0 < \alpha < 1$ be fixed. There exists $\epsilon > 0$ and $0 < \lambda \ll 1/2$, depending only on p, n and α such that if $||f||_{L^{q,r}(G_1)} \leq \epsilon$ and u is a local weak solution of (3.1.1) in G_1, with $||u||_{p,avg,G_1} \leq 1$, then there exists a universally bounded constant c_0 such that*

$$||u - c_0||_{p,avg,G_\lambda} \leq \lambda^\alpha. \tag{3.3.9}$$

Proof. Take $0 \leq \delta \leq 1$ to be chosen later, and apply the last Lemma to get $0 \leq \epsilon \ll 1$ and a p-caloric function ϕ in $G_{1/2}$ such that

$$||u - \phi||_{p,avg,G_{1/2}} \leq \delta.$$

We start by noting that

$$||u||^p_{p,avg,G_{1/2}} \leq \frac{|G_1|}{|G_{1/2}|}||u||^p_{p,avg,G_1} \leq 2^{n+\theta}$$

so that

$$||\phi||_{p,avg,G_{1/2}} \leq ||u-\phi||_{p,avg,G_{1/2}} + ||u||_{p,avg,G_{1/2}} \leq \delta + 2^{\frac{n+\theta}{p}} \leq C. \tag{3.3.10}$$

Since ϕ is p-caloric, it follows from standard theory that ϕ is universally $C^{0,1/2}_{loc}$ in time and $C^{0,1}_{loc}$ in space (cfr. [8]). So we see that for all $(x,t) \in G_\lambda$ with $\lambda \ll 1/2$ to be chosen,

$$\begin{aligned}|\phi(x,t) - \phi(0,0)| \leq& |\phi(x,t) - \phi(0,t)| + |\phi(0,t) - \phi(0,0)| \leq \\ \leq& C'|x-0| + C''|t-0|^{1/2} \leq \\ \leq& C'\lambda + C''\lambda^{\theta/2} \leq C\lambda\end{aligned}$$

since $\theta > 2$ and $\lambda < 1$.

Therefore, we can write

$$\sup_{(x,t)\in G_\lambda} |\phi(x,t) - \phi(0,0)| \leq C\lambda.$$

where the constant C is universal in the sense that it does not depend on the point (x,t) or λ.

Hence there holds the estimate

$$||u(x,t) - \phi(0,0)||_{p,avg,G_\lambda} \leq ||u(x,t) - \phi(x,t)||_{p,avg,G_\lambda} + ||\phi(x,t) - \phi(0,0)||_{p,avg,G_\lambda} \leq \tag{3.3.11}$$

$$\leq \left(\frac{1}{2\lambda}\right)^{\frac{\theta+n}{p}} \delta + C\lambda. \tag{3.3.12}$$

We choose $\lambda \ll 1/2$ and so $G_\lambda \subset G_{1/2}$. We put $c_0 := \phi(0,0)$ observing that due to (3.3.10) and the fact that ϕ is p-caloric, that c_0 is universally bounded. Finally we fix the constants. Recalling that $0 < \alpha < 1$, we can choose $\lambda \ll 1/2$ so small that

$$C\lambda \leq \frac{1}{2}\lambda^\alpha$$

and then we define

$$\delta = \frac{1}{2}\lambda^\alpha (2\lambda)^{\frac{n+\theta}{p}}.$$

Finally, from (3.3.11)

$$||u(x,t) - c_0||_{p,avg,G_\lambda} \leq \frac{\lambda^\alpha}{2} + C\lambda \leq \lambda^\alpha$$

□

Next we iterate Lemma 12 in the appropriate geometric setting.

Lemma 13. *Under the conditions of the previous lemma, there exists a convergent sequence of real numbers $(c_k)_k$ with*

$$|c_k - c_{k+1}| \leq C(n,p)\,(\lambda^\alpha)^k \tag{3.3.13}$$

such that

$$||u - c_k||_{p,avg,G_{\lambda^k}} \leq \left(\lambda^k\right)^\alpha \tag{3.3.14}$$

Proof. The proof is done by induction on $k \in \mathbb{N}$. For $k = 1$, (3.3.14) holds due to Lemma 12, with $c_1 = c_0$. Suppose the conclusion holds for k, and we proceed to prove it then holds for $k+1$. We start by defining the function $v : G_1 \to \mathbb{R}$ by

$$v(x,t) = \frac{u(\lambda^k x, \lambda^{k\theta} t) - c_k}{\lambda^{\alpha k}}, \tag{3.3.15}$$

which is clearly well defined.

We readily see that

$$
\begin{aligned}
&\lambda^{\alpha k}||v||_{p,avg,G_1} = ||u(\lambda^k x, \lambda^{k\theta} t) - c_k||_{p,avg,G_1} = \\
=&|G_1|^{-1/p} \left(\int_{-1}^{0} \int_{B_1} |u(\lambda^k x, \lambda^{k\theta} t) - c_k|^p dxdt \right)^{\frac{1}{p}} = \\
=&|G_1|^{-1/p} \lambda^{-k(n+\theta)/p} ||u - c_k||_{L^p(G_{\lambda^k})} = \\
=&||u - c_k||_{p,avg,G_{\lambda^k}},
\end{aligned}
\tag{3.3.16}
$$

so that

$$
||v||_{p,avg,G_1} \leq 1. \tag{3.3.17}
$$

We compute

$$
v_t(x,t) = \lambda^{k(\theta-\alpha)} u_t(\lambda^k x, \lambda^{k\theta} t).
$$

and

$$
\text{div}\left(|\nabla v(x,t)|^{p-2}\nabla(x,t)\right) = \lambda^{pk-(p-1)\alpha k}\text{div}\left(|\nabla u(\lambda^k x, \lambda^{k\theta} t)|^{p-2}\nabla u(\lambda^k x, \lambda^{k\theta} t)\right)
$$

We conclude, recalling (3.0.3), that

$$
v_t - \text{div}\left(|\nabla v(x,t)|^{p-2}\nabla(x,t)\right) = \lambda^{pk-(p-1)\alpha k} f(\lambda^k x, \lambda^{k\theta} t) =: \tilde{f}(x,t).
$$

A computation similar to (3.3.16) gives

$$
\begin{aligned}
&||\tilde{f}||_{L^{q,r}(G_1)} = \\
=&\lambda^{((pk-(p-1)\alpha k)q-kn)\frac{r}{q}-k\theta} \int_{-\lambda^{k\theta}} \left(\int_{B_{\lambda^k}} |f(x,t)|^q dx \right)^{\frac{r}{q}} dt.
\end{aligned}
$$

Due to the choices (3.0.2) and (3.0.3),

$$
\left((pk-(p-1)\alpha k)\, q - kn\right) \frac{r}{q} - k\theta = 0
$$

so that

$$
||\tilde{f}||_{L^{q,r}(G_1)} = ||f||_{L^{q,r}(G_{\lambda^k})} \leq ||f||_{[L^{q,r}(G_1)} \leq \epsilon
$$

which entitles v to Lemma 12. It then follows that there exists a constant $\tilde{c_0}$, with $|\tilde{c_0}| \leq C(n,p)$ such that

$$
||v - \tilde{c_0}||_{p,avg,G_\lambda} \leq \lambda^\alpha.
$$

Again, a computation similar to (3.3.16) gives

$$||u - c_{k+1}||_{p,avg,G_{\lambda^{k+1}}} \leq \lambda^{\alpha(k+1)}$$

for $c_{k+1} := c_k + \tilde{c_0}\lambda^{\alpha k}$, which concludes the induction. Clearly, it holds

$$|c_{k+1} - c_k| \leq C(n,p)\left(\lambda^{\alpha}\right)^k.$$

□

Now we present the main result of this chapter.

Theorem 3. *A locally bounded weak solution of* (3.1.1)*, with $f \in L^{q,r}$, satisfying* (3.1.2) *and* (3.1.3) *is locally Hölder continuous with exponents α in space; and α/θ in time.*

Proof. We start by noting that the conditions $||u||_{p,avg,G_1} \leq 1$ and $||f||_{L^{q,r}(G_1)} \leq \epsilon$ in Lemma 13 are not restrictive, since we can fall into that framework by scaling and contraction. Indeed, given a solution u, let

$$v(x,t) = \rho u(\rho^a x, \rho^{(p-2)+ap}t)$$

with a, ρ to be fixed, with is a solution to (3.1.1) with

$$\tilde{f}(x,t) = \rho^{(p-1)+ap} f(\rho^a x, \rho^{(p-2)+ap}t).$$

We choose $a > 0$ such that $a < 2/(n+p)$ and

$$((p-1)+ap)\,r - a(n+p) - (p-2) > 0$$

which is always possible since the second inequality holds true for $a = 0$ because $r \geq 1$, and so we can use it's continuity with respect to a; we also choose $0 < \rho < 1$ so small that

$$||v||^p_{p,avg,G_1} \leq \rho^{2-a(n+p)}||u||^p_{p,avg,G_1} \leq 1$$

and

$$||\tilde{f}||^r_{L^{q,r}(G_1)} = \rho^{((p-1)+ap)r - a(n+p) - (p-2)}||f||^r_{L^{q,r}(G_1)} \leq \epsilon^r.$$

Given an arbitrary $0 < r < 1/2$, there exists a non-negative integer k such that $\lambda^{k+1} < r \leq \lambda^k$, where λ is given by Lemma 13. Then, for any non-negative integer s, it holds

$$|G_r|^{-1}\int_{G_r}|u - c_s|^p dxdt \leq |G_{\lambda^{k+1}}|^{-1}\int_{G_{\lambda^k}}|u - c_s|^p dxdt. \qquad (3.3.18)$$

Now, if $s < k$, then $\lambda^s > \lambda^k$ and

$$\begin{aligned} |G_{\lambda^{k+1}}|^{-1} \int_{G_{\lambda^k}} |u - c_s|^p dxdt \leq & |G_{\lambda^{k+1}}|^{-1} \int_{G_{\lambda^k}} |u - c_s|^p dxdt \leq \\ & \leq \lambda^{s-(k+1)} |G_{\lambda^s}|^{-1} \int_{G_{\lambda^s}} |u - c_s|^p dxdt \leq \\ & \leq \lambda^{-(k+1)} (\lambda^s)^\alpha \leq Cr^\alpha. \end{aligned}$$

If $s > k$, then $\lambda^s \leq \lambda^{k+1} < \lambda^k$, so

$$\begin{aligned} |G_{\lambda^{k+1}}|^{-1} \int_{G_{\lambda^k}} |u - c_s|^p dxdt \leq & |G_{\lambda^{k+1}}|^{-1} \int_{G_{\lambda^k}} |u - c_k|^p dxdt + \sum_{i=1}^{s-k} |c_{k+i} - c_{k+i-1}| \leq \\ & \leq \left(\lambda^k\right)^\alpha + C(\lambda^{\alpha k} - \lambda^{\alpha s}) \leq Cr^\alpha. \end{aligned}$$

We can finally take $s \to \infty$ in (3.3.18), getting

$$\frac{1}{|G_r|} \int_{G_r} |u - \overline{c}|^p dxdt \leq C' r^{p\alpha}.$$

where

$$\overline{c} := \lim_{s \to \infty} c_s.$$

Noting that $|G_r| = Kr^{n+\theta}$ for some K depending on n, it holds

$$\frac{1}{|G_r|^\lambda} \int_{G_r} |u - \overline{c}|^p dxdt \leq C. \tag{3.3.19}$$

with $\lambda = \frac{p\alpha}{n+\theta} + 1$.

Now, noting that for all $\gamma \in \mathbb{R}$ we have

$$\begin{aligned} & \Big(\frac{1}{|G_r|} \int_{G_r} \Big|u - \frac{1}{|G_r|} \int_{G_r} udy\Big|^p dx\Big)^{\frac{1}{p}} = \\ & = \Big(\frac{1}{|G_r|} \int_{G_r} \Big|(u - \gamma) - \frac{1}{|G_r|} \int_{G_r} (u - \gamma)dy\Big|^p dx\Big)^{\frac{1}{p}} \leq \\ & \leq \Big(\frac{1}{|G_r|} \int_{G_r} |u - \gamma|^p\Big)^{\frac{1}{p}} + \Big(\frac{1}{|G_r|} \int_{G_r} \Big|\frac{1}{|G_r|} \int_{G_r} (u - \gamma)dy\Big|^p dx\Big)^{\frac{1}{p}} = \\ & = \frac{1}{|G_r|^{\frac{1}{p}}} ||u - \gamma||_p + \frac{1}{|G_r|} ||u - \gamma||_1. \end{aligned}$$

By Hölder inequality, we get

$$||u - \gamma||_1 \leq |G_r|^{\frac{1}{p'}} ||u - \gamma||_p.$$

Therefore we immediately get

$$\Big(\frac{1}{|G_r|} \int_{G_r} \Big|u - \frac{1}{|G_r|} \int_{G_r} udy\Big|^p dx\Big)^{\frac{1}{p}} \leq \frac{2}{|G_r|^{\frac{1}{p}}} ||u - \gamma||_p. \tag{3.3.20}$$

By (3.3.19) and (3.3.20), using notation of chapter 2, we get that for every $0 < r < \frac{1}{2}$,

$$|G_r|^{-\lambda} \int_{G_r} |u - u_{\xi,r}|^p dxdt \leq C, \tag{3.3.21}$$

where $\xi = (0, t_0)$ is the center of G_r and $\lambda = \frac{p\alpha}{n+\theta} + 1$. Using standard covering arguments, we conclude for every $y \in \Omega$, $t_0 \in [-T, 0)$. Therefore $u \in \mathcal{L}^{(p,\lambda)}(\Omega, \delta)$, which implies that $u \in C^{0,\beta}(\overline{\Omega}, \delta)$, with $\beta = \frac{m}{p}(\lambda - 1)$. Noting that, for this metric, $m = n + \theta$, we get $\beta = \alpha$. Finally, recalling the definition of the metric δ, we can conclude the local Hölder continuity of u with exponents α in space and α/θ in time, as intended.

$\square$

Chapter 4

Application to the degenerate inhomogeneous PME

In this chapter, we will apply a similar technique as that of the previous chapter to study the regularity of weak solutions to the porous medium equation (PME)

$$u_t - \operatorname{div}(m|u|^{m-1}\nabla u) = f, \quad m > 1. \tag{4.0.1}$$

Let $U \subset \mathbb{R}^N$, $T > 0$ and $U_T = U \times (0,T)$ as before. We consider the source term $f \in L^{q,r}(U_T)$, where

$$\frac{1}{r} + \frac{N}{2q} < 1. \tag{4.0.2}$$

We will show that bounded weak solutions of (4.0.1) are locally of class $C^{0,\gamma,\frac{\gamma}{\theta}}$, with

$$\gamma = \frac{\alpha}{m}, \qquad \alpha = \min\left\{\alpha_0^-, \frac{m[(2q-n)r-2q]}{q[mr-(m-1)]}\right\},$$

where $0 < \alpha_0 \leq 1$ denotes the optimal Hölder exponent of (4.0.1) with $f \equiv 0$ and

$$\theta = \alpha\left(1 + \frac{1}{m}\right) + (1-\alpha)2. \tag{4.0.3}$$

The regularity class is to be understood in the following sense: if

$$\frac{m[(2q-n)r-2q]}{q[mr-(m-1)]} < \alpha_0$$

then solutions are in $C^{0,\gamma,\frac{\gamma}{\theta}}$, with

$$\gamma = \frac{m[(2q-n)r-2q]}{q[mr-(m-1)]}.$$

Otherwise, solutions are in $C^{0,\gamma,\frac{\gamma}{\theta}}$ for every $0 < \gamma < \frac{\alpha_0}{m}$.

4.1 Weak solutions and Caccioppoli's estimate

We start, as before, by defining weak solutions to the equation (4.0.1) and by stating an essential energy estimate for these weak solutions.

Definition 6. *We say a locally bounded function*

$$u \in C_{loc}(0,T;L^2(U)), \quad \text{with } |u|^{\frac{m+1}{2}} \in L^2_{loc}\left(0,T;W^{1,2}_{loc}(U)\right)$$

is a weak solution of (4.0.1) *if, for every compact set* $K \subset U$ *and every subinterval* $[t_1,t_2] \subset (0,T]$, *we have*

$$\int_K u\phi \, dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \left(-u\phi_t + m|u|^{m-1}\nabla u \nabla \phi\right) dxdt = \int_{t_1}^{t_2}\int_K f\phi \, dxdt, \qquad (4.1.1)$$

for all test functions

$$\phi \in W^{1,2}_{loc}\left(0,T;L^2(K)\right) \cap L^2_{loc}\left(0,T;W^{1,2}_0(K)\right).$$

Noting that, formally,

$$|u|^{m-1}\nabla u = \frac{2}{m+1}\operatorname{sign}(u)|u|^{\frac{m-1}{2}}\nabla|u|^{\frac{m+1}{2}},$$

and since $\nabla|u|^{\frac{m+1}{2}}$ lives in $L^2(U_T)$, in addition to u being locally bounded, it is now clear that these integrals converge.

The following is a similar energy result to Lemma 7 of the previous chapter, with a similar proof which shall be omitted.

Lemma 14 (Caccioppoli's estimate)**.** *Let u be a local weak solution of* (3.1.1). *Given* $K \times [t_1,t_2] \subset U \times (0,T]$, *there exists a constant C, depending only on n, m and $K \times [t_1,t_2]$, such that*

$$\begin{aligned} &\sup_{t_1<t<t_2}\int_K u^2\xi^2 \, dx + \int_{t_1}^{t_2}\int_K |u|^{m-1}|\nabla u|^2\xi^2 \, dxdt \leq \qquad (4.1.2)\\ &\leq C\int_{t_1}^{t_2}\int_K u^2\xi|\xi_t| \, dxdt + \int_{t_1}^{t_2}\int_K |u|^{m+1}(|\nabla\xi|^2+\xi^2) \, dxdt + C||f||^2_{q,r}. \end{aligned}$$

for all $\xi \in C_0^\infty(K \times [t_1,t_2])$ *such that* $0 \leq \xi \leq 1$.

4.2 Towards Hölder continuity of weak solutions of the PME

In order to obtain the desired explicit expression for the Hölder exponent, we first need to prove the general Hölder continuity of weak solutions of equation (4.0.1). More explicitly, we want to prove the following theorem.

Theorem 4. *Let u be a weak solution to the degenerate PME* (4.2.2). *Then u is locally Hölder continuous in U_T, and there exist constants $\gamma > 1$ and $\alpha \in (0,1)$, that depend only on the data, such that for every compact set $K \subset U_T$, it holds*

$$|u(x_1,t_1) - u(x_2,t_2)| \leq \gamma||u||_{\infty,U_T}\left(\frac{|x_1-x_2| + ||u||^{\frac{m-1}{2}}_{\infty,U_T}|t_1-t_2|}{m - dist(K;\Gamma)}\right)^{\alpha} \qquad (4.2.1)$$

for every $(x_1,t_1),(x_2,t_2) \in K$.

Note that this theorem is not exactly what we want to prove in this chapter as it does not give an explicit expression for α. However, it will prove useful later on.

In this chapter however, we will only prove such a result for the homogeneous case, redirecting the reader to the article [12] for the more general case. We will henceforth consider the following equation

$$u_t - \mathrm{div} A(x,t,u,Du) = 0, \tag{4.2.2}$$

with $A(x,t,u,Du) = m|u|^{m-1}\nabla u$.

We say a function u is a weak sub(super)-solution to (4.2.2) if it satisfies Definition 6 with $f \equiv 0$ and $\leq$ $(\geq)$ instead of equality. u is a weak solution if it is both a sub-solution and a super-solution.

We start with some useful definitions beforehand.

Definition 7. *Introduce the cylinders $Q_\rho^\pm(\theta)$ and their translated $(y,s)+Q_\rho^\pm(\theta)$ by*

$$Q_\rho^-(\theta) = K_\rho \times (\theta\rho^2, 0], \qquad Q_\rho^+(\theta) = K_\rho \times (0, \theta\rho^2]. \tag{4.2.3}$$

For $\theta = 1$, write $Q_\rho^\pm(1) = Q_\rho^\pm$.

For a fixed $(y,s) \in \mathbb{R}^{N+1}$ denote by

$$(y,s) + Q_\rho^-(\theta) = K_\rho(y) \times (s - \theta\rho^2, s], \tag{4.2.4}$$

$$(y,s) + Q_\rho^+(\theta) = K_\rho(y) \times (s, s + \theta\rho^2], \tag{4.2.5}$$

where $K_\rho(y)$ is the cube with edge ρ centered in y and we write $K_\rho(0) = K_\rho$.

Finally, define the truncated functions $(v-k)_\pm$ by

$$(v-k)_+ = \max\{(v-k), 0\};$$

$$(v-k)_- = \max\{-(v-k), 0\}.$$

4.2.1 Energy estimate

As always, the study needs to start with an energy estimate. In the following, we present an energy estimate for the functions $(u-k)_\pm$ in the cylinders $(y,s)+Q_\rho^\pm(\theta)$.

Proposition 3. *Let u be a local weak solution to* (4.2.2)*. There exists a positive constant γ depending only on the data such that, for every cylinder $(y,s)+Q_\rho^\pm(\theta) \subset U_T$, every $k \in \mathbb{R}$ and every piecewise smooth function ζ vanishing on*

$\partial K_\rho(y)$ and such that $0 \le \zeta \le 1$, it holds

$$\begin{aligned}
&\operatorname*{ess\,sup}_{s-\theta\rho^2<t\le s}\int_{K_\rho(y)}(u-k)_\pm^2\zeta^2(x,t)\,dx-\int_{K_\rho(y)}(u-k)_\pm\zeta^2(x,s-\theta\rho^2)\,dx+\\
&+\int\int_{(y,s)+Q_\rho^-(\theta)}|u|^{m-1}|D(u-k)_\pm|^2\zeta^2\,dxdt\le \qquad (4.2.6)\\
\le&\gamma\int\int_{(y,s)+Q_\rho^-(\theta)}(u-k)_\pm^2\zeta|\zeta_t|\,dxdt+\\
&+\gamma\int\int_{(y,s)+Q_\rho^-(\theta)}|u|^{m-1}(u-k)_\pm^2|D\zeta|^2\,dxdt.
\end{aligned}$$

Analogous estimates hold in the "forward" cylinder $(y,s)+Q_\rho^+(\theta)$.

Proof. After a translation, we may assume $(y,s)=(0,0)$. In (4.1.1) set $f\equiv 0$. We wish to take the testing function

$$\varphi_\pm=\pm(u-k)_\pm\zeta^2,\quad \text{over } K_\rho\times(-\theta\rho^2,t],\text{ for } -\theta\rho^2<t\le 0,$$

however the time derivative $(u-k)_{\pm t}$ is not well-defined. Therefore, we start by considering the Steklov average. Using an equivalent formulation (see Definition 4 for a general idea), we get the following

$$\int_{K\times\{t\}}\left(u_{h,t}\phi+\left(A(x,\tau,u,Du)\right)_h D\phi\right)\,dx=0.$$

Clearly

$$(u_h-k)=\pm(u_h-k)_\pm\mp(u_h-k)\mp$$

so that

$$u_{h,\tau}=\pm(u_h-k)_{\pm,\tau}\mp(u_h-k)_{\mp,\tau}.$$

Since it holds

$$(u_h-k)_{\pm,\tau}(u_h-k)_\pm=0,$$

it follows

$$\begin{aligned}
&\pm\int\int_{K_\rho\times(-\theta\rho^2,t]} u_{h,\tau}(u_h-k)_\pm\zeta^2\,dxd\tau =\\
=&\pm\int\int_{K_\rho\times(-\theta\rho^2,t]} \pm(u_h-k)_{\pm,\tau}(u_h-k)_\pm\zeta^2\,dxd\tau\\
&\pm\int\int_{K_\rho\times(-\theta\rho^2,t]} \mp(u_h-k)_{\mp,\tau}(u_h-k)_\pm\zeta^2\,dxd\tau =\\
=&\int\int_{K_\rho\times(-\theta\rho^2,t]}(u_h-k)_{\pm,\tau}(u_h-k)_\pm\zeta^2\,dxd\tau =\\
=&\frac{1}{2}\int\int_{K_\rho\times(-\theta\rho^2,t]}[(u_h-k)_\pm^2]_\tau\zeta^2\,dxd\tau =\\
=&\frac{1}{2}\int_{K_\rho\times\{t\}}(u_h-k)_\pm^2\zeta^2\,dx-\frac{1}{2}\int_{k_\rho\times\{\theta\rho^2\}}(u_h-k)_\pm^2\zeta^2\,dx-\\
&-\int\int_{Q_\rho^-}(u_h-k)^2\zeta\zeta_t.
\end{aligned}$$

Since we have freed u_h from its time derivative, we can now take $h\to 0$ getting

$$\begin{aligned}
&\frac{1}{2}\int_{K_\rho\times\{t\}}(u-k)_\pm^2\zeta^2\,dx-\frac{1}{2}\int_{K_\rho\times\{-\theta\rho^2\}}(u-k)_\pm^2\zeta^2\,dx\\
&\pm\int\int_{K_\rho\times(-\theta\rho^2,t]} A(x,\tau,u,Du)D(u-k)_\pm\zeta^2\,dxd\tau\\
&\pm 2\int\int_{K_\rho\times(-\theta\rho^2,t]}(u-k)_\pm A(x,\tau,u,Du)D\zeta\zeta\,dxd\tau =\\
=&\int\int_{Q_\rho^-(\theta)}(u-k)_\pm^2\zeta\zeta_\tau\,dxd\tau.
\end{aligned}$$

Noting that $D(u-k)_\pm=\pm Du\chi[(u-k)_\pm>0]$, so

$$\begin{aligned}
\pm A(x,\tau,u,Du)D(u-k)_\pm &= m|u|^{m-1}|Du|^2\chi[(u-k)_\pm>0]=\\
&= m|u|^{m-1}|D(u-k)_\pm|^2.
\end{aligned}$$

Also, by Young's inequality,

$$\begin{aligned}
&2\left|\int\int_{K_\rho\times(-\theta\rho^2,t]}(u-k)_\pm A(x,\tau,u,Du)D\zeta\zeta\,dxd\tau\right|\le\\
\le&2m\int\int_{K_\rho\times(-\theta\rho^2,t]}(u-k)_\pm|u|^{m-1}|Du||D\zeta|\zeta\,dxd\tau\le\\
\le&2m\int\int_{K_\rho\times(-\theta\rho^2,t]}|u|^{m-1}\left(|D(u-k)_\pm\zeta\right)\left((u-k)_\pm|D\zeta|\right)\,dxd\tau\le\\
\le&\epsilon^{-1}m\int\int_{K_\rho\times(-\theta\rho^2,t]}|u|^{m-1}|D(u-k)|^2\zeta^2\,dxd\tau+\\
&+m\epsilon\int\int_{Q_\rho^-(\theta)}|u|^{m-1}(u-k)_\pm^2|D\zeta|^2\,dxd\tau,
\end{aligned}$$

for every $\epsilon>0$.

Combining these estimates, we get

$$\frac{1}{2}\int_{K_\rho\times\{t\}}(u-k)_\pm^2\zeta^2\,dx-\frac{1}{2}\int_{K_\rho\times\{-\theta\rho^2\}}(u-k)_\pm^2\zeta^2\,dx$$
$$+\,m(1-\epsilon^{-1})\int\int_{K_\rho\times(-\theta\rho^2,t]}|u|^{m-1}|D(u-k)_\pm|^2\zeta^2\,dxd\tau\leq$$
$$\leq\int\int_{Q_\rho^-(\theta)}(u-k)_\pm^2\zeta|\zeta_\tau|\,dxd\tau+m\epsilon\int\int_{Q_\rho^-(\theta)}|u|^{m-1}(u-k)_\pm^2|D\zeta|^2\,dxd\tau.$$

Finally, we choose $\epsilon=\frac{2m}{2m-1}>0$ and take the supremum over $t\in(-\theta\rho^2,0]$ which proves the proposition for $\gamma=\max\{2,\frac{2m^2}{2m-1}\}=\frac{2m^2}{2m-1}$ since $m>1$.

□

4.2.2 A DeGiorgi-Type Lemma

It is known that sub(super)-solutions to (4.2.2) in U_T are locally bounded above (below) in U_T.

For a cylinder $(y,s)+Q_{2\rho}^\pm(\theta)\subset U_T$, denote by $\mu_\pm$ and ω numbers satisfying

$$\mu_+\geq\operatorname*{ess\,sup}_{(y,s)+Q_{2\rho}^\pm(\theta)}u,\qquad \mu_-\leq\operatorname*{ess\,inf}_{(y,s)+Q_{2\rho}^\pm(\theta)}u,$$
$$\omega=\mu_+-\mu_-.$$

Since the equation becomes degenerated at $u=0$, we will assume at the outset that $\mu_-=0$ so that $\omega=\mu_+$.

Fix numbers ξ and a in $(0,1)$. In the following we show that if there is a cylinder in which u is essentially away from its infimum or supremum then, going down to a smaller cylinder, the oscillation decreases by a factor of $(1-a\xi)$.

Lemma 15. *Let u be a nonnegative, locally bounded, local weak supersolution to* (4.2.2) *in U_T. There exists a positive number ν_- depending on θ, ω, ξ, a and the data such that if*

$$|[u\leq\xi\omega]\cap[(y,s)+Q_{2\rho}^-(\theta)]|\leq\nu_-|Q_{2_\rho}^-(\theta)|,\tag{4.2.7}$$

then

$$u\geq a\xi\omega\qquad a.e.\ in\ (y,s)+Q_\rho^-(\theta).\tag{4.2.8}$$

Likewise, if u is a nonnegative, locally bounded, local weak subsolution to (4.2.2) *in U_T, then there exists a positive number ν_+ depending on θ, ω, ξ, a and the data such that if*

$$|[u\geq\mu_+-\xi\omega]\cap[(y,s)+Q_{2\rho}^-(\theta)]|\leq\nu_+|Q_{2\rho}^-(\theta)|,$$

then

$$u \le \mu_+ - a\xi\omega \qquad a.e.\ in\ (y,s) + Q_\rho^-(\theta).$$

Proof. Assume $(y,s) = (0,0)$. For $n = 0, 1, ...$ set

$$\rho_n = \rho + \frac{\rho}{2^n}; \quad K_n = K_{\rho_n}; \quad Q_n = K_n \times (-\theta\rho_n^2, 0], \tag{4.2.9}$$

and consider the cutoff function $\zeta(x,t) = \zeta_1(x)\zeta_2(t)$ such that

$$\zeta_1 = \begin{cases} 1 \text{ in } K_{n+1} \\ 0 \text{ in } \mathbb{R}^N - K_n \end{cases}, |D\zeta_1| \le \frac{2^{n+1}}{\rho} \tag{4.2.10}$$

$$\zeta_2 = \begin{cases} 1 \text{ for } t \ge -\theta\rho_{n+1}^2 \\ 0 \text{ for } t < \theta\rho_n^2 \end{cases}, 0 \le \zeta_{2,t} \le \frac{1}{\theta(\rho_n^2 - \rho_{n+1}^2)} \le \frac{2^{2(n+1)}}{\theta\rho^2} \tag{4.2.11}$$

We will only prove the first case, since the proof for the second case is similar.

Define the truncating levels by $k_n = \xi_n\omega$ where $\xi_n = a\xi + \frac{1-a}{2^n}\xi$.

The energy estimate (4.2.6) on Q_n for $(u-k_n)_-$, gives

$$\begin{aligned} &\operatorname*{ess\,sup}_{-\theta\rho_n^2 < t \le 0} \int_{K_n} (u-k_n)_-^2 \zeta^2(x,t)\, dx + \int\int_{Q_n} |u|^{m-1} |D(u-k_n)_-|^2 \zeta^2 \, dx d\tau \le \\ &\le \int_{K_n} (u-k_n)_-^2 \zeta(x, -\theta\rho_n^2)\, dx + \gamma \int\int_{Q_n} (u-k_n)_-^2 \zeta |\zeta_\tau| \, dx d\tau + \\ &\quad + \gamma \int\int_{Q_n} |u|^{m-1} (u-k_n)_-^2 |D\zeta|^2 \, dx d\tau \le \\ &\le \gamma \int\int_{Q_n} (u-k_n)_-^2 \zeta \frac{2^{2(n+1)}}{\theta\rho^2} \, dx d\tau + \gamma \int\int_{Q_n} |u|^{m-1} (u-k_n)_-^2 |D\zeta|^2 \, dx d\tau. \end{aligned}$$

Noting that

$$\begin{aligned} |D[(u-k_n)_-\zeta]|^2 &= |D(u-k_n)_-\zeta + (u-k_n)_- D\zeta|^2 \le \\ &\le 2|D(u-k_n)_-|^2\zeta^2 + 2(u-k_n)_-^2 |D\zeta|^2, \end{aligned}$$

we get

$$\begin{aligned} &\operatorname*{ess\,sup}_{\theta\rho_n^2 < t \le 0} \int_{K_n} (u-k_n)_-^2 \zeta^2 \, dx d\tau + \int\int_{Q_n} |u|^{m-1} |D[(u-k_n)_-\zeta]|^2 \le \\ &\le \gamma \int\int_{Q_n} (u-k_n)^2 \zeta \frac{2^{2(n+1)}}{\theta\rho^2} \chi_{[u<k_n]} \, dx d\tau + \\ &\quad + \gamma \int\int_{Q_n} |u|^{m-1} (u-k_n)^2 \chi_{[u<k_n]} \frac{2^{2(n+1)}}{\rho^2} \, dx d\tau. \end{aligned}$$

Since either $(u-k_n)_- = 0$ or

$$\begin{aligned} (u-k_n)_- = (\xi_n\omega - u) &= -u + a\xi\omega + \frac{1-a}{2^n}\xi\omega \le \\ &\le \xi\omega\left(a + \frac{1-a}{2^n}\right) \le \xi\omega \end{aligned}$$

we get

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\theta\rho_n^2<t\le 0}\int_{K_n}(u-k_n)_-^2\zeta^2\,dxd\tau+\int\int_{Q_n}|u|^{m-1}|D[(u-k_n)_-\zeta]|^2\le\\
\le&\gamma\frac{2^{2n}}{\theta\rho^2}(\xi\omega)^2|[u<k_n]\cap Q_n|+\gamma(\xi_n\omega)^{m-1}(\xi\omega)^2\frac{2^{2n}}{\rho^2}|[u<k_n]\cap Q_n]\le\\
\le&\frac{\gamma 2^{2n}}{\rho^2}(\xi\omega)^{m+1}\left(1+\frac{1}{\theta(\xi\omega)^{m-1}}\right)|[u<k_n]\cap Q_n|.
\end{aligned}$$

To estimate the left-hand side, we keep u away from zero by introducing the function

$$v=\max\left\{u,\frac{1}{2}a\xi\omega\right\}. \tag{4.2.12}$$

Clearly $v\ge u$ and so $(u-k_n)_-\ge(v-k_n)_-$. Hence

$$\int_{K_n}(u-k_n)_-^2\zeta(x,t)\,dx\ge\int_{K_n}(v-k_n)_-^2\zeta(x,t)\,dx.$$

Also, we estimate below

$$\begin{aligned}
&\left(\frac{1}{2}a\xi\omega\right)^{m-1}\int\int_{Q_n}|D[(v-k_n)_-\zeta]|^2\,dxd\tau\le\\
\le&\int\int v^{m-1}|D[(v-k_n)_-\zeta]|^2\,dxd\tau\le\\
\le&\int\int_{Q_n\cap[u=v]}u^{m-1}|D[(u-k_n)_-\zeta]|^2\,dxd\tau+\\
&+\left(\frac{1}{2}a\xi\omega\right)^{m-1}\int\int Q_n\cap[u<v](v-k_n)_-^2|D\zeta|^2\,dxd\tau\le\\
\le&\int\int_{Q_n}u^{m-1}|D[(u-k_n)_-\zeta]|^2\,dxd\tau+\gamma 2^{2n}(\xi\omega)^{m+1}|[u<k_n]\cap Q_n|.
\end{aligned}$$

By the definition 4.2.12 of the function v, the two sets $[v<k_n]$ and $[u<k_n]$ coincide since $\frac{1}{2}a\xi\omega<k_n$. Then, setting $A_n=[v<k_n]\cap Q_n$ and $Y_n=\frac{|A_n|}{|Q_n|}$, combining these estimates gives

$$\begin{aligned}
&\operatorname*{ess\,sup}_{-\theta\rho_n^2<t\le 0}\int_{K_n}(v-k_n)_-^2\zeta^2\,dx+(\xi\omega)^{m-1}\int\int_{Q_n}|D[(v-k_n)\zeta]|^2\,dxd\tau\le\\
\le&\gamma\Lambda(a)\frac{2^{2n}}{\rho^2}(\xi\omega)^{m+1}\left(1+\frac{1}{\theta(\xi\omega)^{m-1}}\right)|A_n|
\end{aligned}$$

with $\Lambda(a)=\frac{1}{\left(\frac{1}{2}a\right)^{m-1}}$.

Noting that in $[v<k_{n+1}]\subset[u<k_n]$,

$$(v-k_n)_-^2\ge(k_n-k_{n+1})^2=\left(\frac{1-a}{2^{n+1}}\xi\omega\right)^2,$$

by Hölder inequality, it holds

$$\begin{aligned}
\left(\frac{1-a}{2^{n+1}}\xi\omega\right)^2|A_{n+1}|\le&\int\int_{Q_{n+1}}(v-k_n)_-^2\zeta\,dxd\tau\le\\
&\le\left(\int\int_{Q_n}[(v-k_n)_-\zeta]^{2\frac{N+2}{N}}\,dxd\tau\right)^{\frac{N}{N+2}}|A_n|^{\frac{2}{N+2}}.
\end{aligned}$$

Now we invoke Lemma 5 which guarantees that $(v-k_n)_- \in V^2(U_T)$ ensuring that we satisfy the conditions to use the embedding in Proposition 1, choosing $m=p=2$ and $q = 2\frac{N+2}{N}$, which gives

$$\begin{aligned}
&\left(\int\int_{Q_n} [(v-k_n)_-\zeta]^{2\frac{N+2}{N}}\, dx d\tau\right)^{\frac{N}{N+2}} |A_n|^{\frac{2}{N+2}} \\
\leq &\gamma \left(\int\int_{Q_n} |D[(v-k_n)_-\zeta]|^2\, dx d\tau\right)^{\frac{N}{N+2}} \cdot \\
&\cdot \left(\operatorname*{ess\,sup}_{-\theta\rho_m^2 < t \leq 0} \int_{K_n} [(v-k_n)_-\zeta]^2(x,t)\, dx\right)^{\frac{2}{N+2}} |A_n|^{\frac{2}{N+2}},
\end{aligned}$$

for a constant γ depending only on N.

Recall that, by Young's inequality, for every $a,b>0$, $\epsilon>0$ and every $p,q>0$ such that $\frac{1}{p}+\frac{1}{q}=1$, it holds

$$ab \leq \frac{1}{\epsilon^p}\left(\frac{a^p}{p} + \epsilon^{p+q}\frac{b^q}{q}\right).$$

Then for $p = \frac{N+2}{2}$, $q = \frac{N+2}{N}$ and

$$\epsilon = (\xi\omega)^{\frac{2N(m-1)}{(N+2)^2}},$$

we get that

$$\begin{aligned}
&\gamma \left(\int\int_{Q_n} |D[(v-k_n)_-\zeta]|^2\, dx d\tau\right)^{\frac{N}{N+2}} \cdot \\
&\cdot \left(\operatorname*{ess\,sup}_{-\theta\rho_m^2 < t \leq 0} \int_{K_n} [(v-k_n)_-\zeta]^2(x,t)\, dx\right)^{\frac{2}{N+2}} |A_n|^{\frac{2}{N+2}} \leq \\
\leq &\gamma(\xi\omega)^{\frac{N(m-1)}{N+2}} \Big(\operatorname*{ess\,sup}_{-\theta\rho_n^2 < t \leq 0} \int_{K_n} (v-k_n)_-^2 \zeta^2\, dx + \\
&+ (\xi\omega)^{m-1} \int\int_{Q_n} |D[(v-k_n)\zeta]|^2\, dx d\tau\Big).
\end{aligned}$$

Combine these estimates to get

$$|A_{n+1}| \leq \gamma \frac{2^{4n}\Lambda(a)}{(1-a)^2\rho^2} (\xi\omega)^{\frac{2(m-1)}{N+2}} \left(1 + \frac{1}{\theta(\xi\omega)^{m-1}}\right) |A_n|^{1+\frac{2}{N+2}},$$

which in terms of Y_n can be written as

$$Y_{n+1} \leq \gamma \frac{2^{4n}\Lambda(a)}{(1-a)^2} \frac{(1+\theta(\xi\omega)^{m-1})}{(\theta(\xi\omega)^{m-1})^{\frac{N}{N+2}}} Y_n^{1+\frac{2}{N+2}}.$$

which we rewrite as

$$Y_{n+1} \leq C b^n Y_n^{1+\alpha},$$

with $b = 2^4$, $\alpha = \frac{2}{N+2}$ and

$$C = \gamma \frac{\Lambda(a)}{(1-a)^2} \left(\frac{\theta(\xi\omega)^{m-1} + 1}{((\xi\omega)^{m-1}\theta)^{\frac{N}{N+2}}} \right).$$

By Lemma 6, $Y_n \to 0$ as $n \to \infty$ as long as

$$Y_0 \le \gamma^{-\frac{N+2}{N}} \left[\frac{(1-a)^2}{\Lambda(a)} \right]^{\frac{N+2}{2}} 2^{-(n+2)^2} \frac{(\theta(\xi\omega)^{m-1})^{\frac{N}{2}}}{(1+\theta(\xi\omega)^{m-1})^{\frac{N+2}{2}}} =: \nu_-.$$

$\frac{|A_n|}{|Q_n|} \to 0$ means $|A_n| \to 0$ since $|Q_n|$ is bounded from above. Therefore

$$|[u < k_n] \cap Q_n| \to 0 \text{ as } n \to \infty.$$

Noting that $k_n \to a\xi\omega$ and

$$\lim_{n\to\infty} Q_n = \bigcap_n Q_n = Q_\rho^-(\theta),$$

it then holds

$$u \ge a\xi\omega \text{ a.e. in } Q_\rho^-(\theta),$$

which proves (4.2.8).

In a similar way, we prove second case with

$$\nu_+ = \left[\frac{(1-a)^2}{\gamma\Lambda(\xi)} \right]^{\frac{N+2}{2}} 2^{-(N+2)^2} \left(\frac{\xi\omega}{\mu_+} \right)^{N+m+1} \frac{(\theta(\xi\omega)^{m-1})^{\frac{N}{2}}}{(1+\theta(\xi\omega)^{m-1})^{\frac{N+2}{2}}}.$$

□

Next we state a variant of the previous lemma. Assume for the following lemma that some information is available on the "initial data" relative to the cylinder $(y,s) + Q_{2\rho}^+(\theta) \subset U_T$. Then

Lemma 16. *Let u be a nonnegative, locally bounded, local weak solution to (4.2.2) for $m > 1$, in U_T. Let $a \in (0,1)$ and suppose it holds*

$$u(x,s) \ge \xi M \text{ a.e. in } K_{2\rho}(y) \tag{4.2.13}$$

for some $M > 0$ and $\xi \in (0,1]$. Then there exists $\nu_0 \in (0,1)$ depending only on a and the data, such that, if

$$|[u \le \xi M] \cap Q_{2\rho}^+(\theta)| \le \frac{\nu_0}{\theta(\xi M)^{m-1}} |Q_{2\rho}^+(\theta)|$$

then

$$u \ge a\xi M \text{ a.e. in } K_\rho(y) \times (s, s+\theta(2\rho)^2].$$

Proof. Assume $(y,s)=(0,0)$ and construct sequences of cubes $\{K_n\}$ and levels $\{\xi_n\}$ as before, and the cylinders by

$$Q_n^+ = K_n \times (0, \theta(2\rho)^2].$$

Define now the cutoff function $\zeta(x,t)=\zeta(x)$ independent of t and satisfying (4.2.10). Now, the same reasoning as before, with the obvious changes in symbolism, proves this result. $\square$

4.2.3 Expansion of positivity

Throughout this section let u be a nonnegative, local weak supersolution to (4.2.2) in U_T for $m>1$. We start with a time propagation of positivity.

Lemma 17. *Assume that, for some $(y,s)\in U_T$ and some $\rho>0$*

$$|[u(.,s)\geq M]\cap K_\rho(y)| \geq \alpha |K_\rho(y)|$$

for some $M>0$ and some $\alpha\in(0,1)$. There exist δ and ϵ in $(0,1)$, depending only on the data and α, and independent of M, such that

$$|[u(.,t)>\epsilon M]\cap K_\rho(y)|, \qquad \textit{for all } t \in \left(s, s+\frac{\delta\rho^2}{M^{m-1}}\right].$$

Proof. Assume $(y,s)=(0,0)$. For $k>0, t\geq 0$ set

$$A_{k,\rho}(t) = [u(.,t)<k]\cap K_\rho.$$

The assumption implies

$$|A_{M,\rho}(0)| \leq (1-\alpha)|K_\rho|. \tag{4.2.14}$$

Recall the energy estimates (4.2.6) for the truncated function $(u-M)_-$, over the forward cylinder $K_\rho\times(0,\theta^2]$, where $\theta>0$ is to be chosen. The cutoff function ζ is to be taken independent of t, nonnegative, such that $\zeta=1$ on $K_{(1-\sigma)\rho}$ and $|D\zeta|\leq\frac{1}{\sigma\rho}$, where $\sigma\in(0,1)$ is to be chosen. We have

$$\begin{aligned}
&\operatorname*{ess\,sup}_{0<t\leq\theta\rho^2}\int_{K_r(y)}(u-M)_-\zeta(x,t)\,dx - \int_{K_\rho}(u-M)_-\zeta(x,0)\,dx \leq \\
\leq&\gamma\int_0^{\theta\rho^2}\int_{K_\rho}|u|^{m-1}(u-M)_-^2\left|\frac{1}{\sigma\rho}\right|^2 dxd\tau.
\end{aligned}$$

So that, for every $t\in(0,\theta\rho^2]$,

$$\begin{aligned}
&\int_{K_{(1-\sigma)\rho}}(u-M)_-^2(x,t)\,dx \leq \\
\leq&\int_{K_\rho}(u-M)_-^2(x,0)\,dx + \frac{\gamma}{\sigma^2\rho^2}\int_0^{\theta\rho^2}\int_{K_\rho}|u|^{m-1}(u-M)_-^2\,dxd\tau.
\end{aligned}$$

Recalling that

$$(u-M)_- \leq M\chi_{[u<M]},$$

we get

$$\begin{aligned}\int_{K_{(1-\sigma)\rho}} (u-M)_-^2(x,t)\,dx \leq & M^2|A_{M,\rho}(0)| + \frac{\gamma}{\sigma^2}M^{m+1}\theta|K_\rho| \leq \\ \leq & M^2\left[(1-\alpha)+\gamma\frac{M^{m-1}\theta}{\sigma^2}\right]|K_\rho|.\end{aligned}$$

The left-hand side is estimated below by

$$\begin{aligned}\int_{K_{(1-\sigma)\rho}} (u-M)_-^2(x,t)\,dx \geq & \int_{K_{(1-\sigma)\rho}\cap[u<\epsilon M]} (u-M)_-^2(x,t)\,dx \geq \\ \geq & M^2(1-\epsilon)^2|A_{\epsilon M,(1-\sigma)\rho}(t)|.\end{aligned}$$

Next we estimate

$$|A_{\epsilon M,\rho}(t)| = \left|A_{\epsilon M,(1-\sigma)\rho}(t) \cup (A_{\epsilon M,\rho}(t) - A_{\epsilon M,(1-\sigma)\rho}(t))\right|.$$

By definition,

$$A_{\epsilon M,\rho}(t) - A_{\epsilon M,(1-\sigma)\rho}(t) = A_{\epsilon M,\rho}(t) \cap (K_\rho - K_{(1-\sigma)\rho}) \subset K_\rho - K_{(1-\sigma)\rho}.$$

Furthermore,

$$\begin{aligned}|K_\rho - K_{(1-\sigma)\rho}| = & \rho^N(1-(1-\sigma)^N) \leq \\ \leq & N\sigma\rho^N.\end{aligned}$$

In fact, for $x \in (0,1)$, the function

$$f(x) = 1 - x^N - N(1-x)$$

is decreasing for $N \geq 1$ and $f(0) = 1 - N \leq 0$.

Therefore, we have

$$\begin{aligned}|A_{\epsilon M,\rho}(t)| \leq & \frac{1}{(1-\epsilon)^2}\left[(1-\alpha)+\gamma\frac{M^{m-1}\theta}{\sigma^2} + (1-\epsilon)^2N\sigma\right]|K_\rho| \leq \\ \leq & \frac{1}{(1-\epsilon)^2}\left[(1-\alpha)+\gamma\frac{M^{m-1}\theta}{\sigma^2} + N\sigma\right]|K_\rho|.\end{aligned}$$

We must finally fix the constants accordingly. Let $\sigma = \frac{\alpha}{8N}$, $\theta = \delta M^{1-m}$,

$$\delta = \frac{\alpha^3}{\gamma 2^{10} N^2}, \qquad \epsilon = 1 - \frac{\sqrt{1-\frac{3}{4}\alpha}}{\sqrt{1-\frac{1}{2}\alpha}}.$$

The conclusion follows immediately.

$\square$

The following proposition will be stated without a proof. For an idea of the proof, see for example [9].

Proposition 4. *Assume that for some* $(y,s) \in U_T$ *and some* $\rho > 0$*,*

$$|[u(.,s) \geq M] \cap K_\rho(y)| \geq \alpha |K_\rho(y)| \tag{4.2.15}$$

for some $M > 0$ *and some* $\alpha \in (0,1)$*. There exists* $b > 1$*,* $\delta, \eta \in (0,1)$ *depending on the data and* α *and independent of* $(y,s), \rho, M$ *such that*

$$u(.,t) \geq \eta M \text{ in } K_{2\rho}(y) \tag{4.2.16}$$

for all times

$$s + \frac{b^{m-1}}{(\eta M)^{m-1}} \frac{1}{2} \delta \rho^2 \leq t \leq s + \frac{b^{m-1}}{(\eta M)^{m-1}} \delta \rho^2.$$

Note that this proposition is powerful in the sense that it transforms the measure-theoretical information (4.2.15) into a pointwise expansion of positivity (4.2.16) in a future cylinder.

4.2.4 The intrinsic Harnack inequality

Assume now, for simplicity, that u is a continuous, nonnegative, local weak solution to (4.2.2). Fix $(x_0,t_0) \in U_T$ such that $u(x_0,t_0) > 0$ and construct the intrinsic cylinders

$$(x_0,t_0) + Q_\rho^\pm(\theta), \qquad \text{where } \theta = \left(\frac{c}{u(x_0,t_0)}\right)^{m-1}. \tag{4.2.17}$$

These cylinders are intrinsic to the solution in the sense that their length depends on the solution itself.

Theorem 5. *Let* u *be a continuous, nonnegative, local weak solution to* (4.2.2)*. There exist positive constants* C *and* γ*, depending only on the data, such that for all intrinsic cylinders* $(x_0,t_0) + Q_{4\rho}^\pm(\theta)$ *contained in* U_T*, it holds*

$$\gamma^{-1} \sup_{K_\rho(x_0)} u(.,t_0 - \theta\rho^2) \leq u(x_0,t_0) \leq \gamma \inf_{K_\rho(x_0)} u(.,t_0 + \theta\rho^2). \tag{4.2.18}$$

Proof. Only the forward inequality (i.e. the second inequality) will be proved. For the backward inequality, we direct the reader to, for example, [9].

Fix $(x_0,t_0) \in U_T$, assume $u(x_0,t_0) > 0$ and construct cylinders $(x_0,t_0) + Q_{4\rho}^\pm(\theta)$ as in (4.2.17), where $c \geq 1$ is to be determined. Apply the change of variables

$$x \to \frac{x - x_0}{\rho}, \quad t \to u(x_0,t_0)^{m-1} \frac{t - t_0}{\rho^2}$$

mapping these cylinders into $Q^{\pm}$, where

$$Q^{+} = K_4 \times (0, 4^2 c^{m-1}], \quad Q^{-} = K_4 \times (-4^2 c^{m-1}, 0].$$

Denoting again by (x,t) the transformed variables, the rescaled function

$$v(x,t) = \frac{1}{u(x_0,t_0)} u\left(x_0 + \rho x, t_0 + \frac{t\rho^2}{u(x_0,t_0)^{m-1}}\right)$$

is still a solution to (4.0.1) with $v(0,0) = 1$.

The theorem is then a consequence of the following.

Proposition 5. *There exists constants $\gamma_0 \in (0,1)$, $\gamma_1, \gamma_2 > 1$, that can be quantitatively determined a priori only in terms of the data, and independent of $u(x_0,t_0)$ such that*

$$v(.,\gamma_1) \geq \gamma_0.$$

Proof of Proposition 5. For $\tau \in [0,1)$, introduce the family of nested cylinders $\{Q_\tau\}$ with the same vertex at $(0,0)$, and the families of nonnegative numbers $\{M_\tau\}$, $\{N_\tau\}$ defined by

$$Q_\tau^{-} = K_\tau \times (-\tau^2, 0], \quad M_\tau = \sup_{Q_\tau^{-}} v, \quad N_\tau = (1-\tau)^{-\beta},$$

where β is to be chosen. For $\tau \in [0,1)$, the two functions M_τ and N_τ are increasing and $M_0 = N_0 = 1$. Moreover, $N_\tau \to \infty$ as $\tau \to 1$ whereas M_τ is bounded since v is locally bounded. Therefore, the equation $M_\tau = N_\tau$ has a closed set of solutions and we can denote by τ_* the largest one.

By the continuity of v, there exists $(y,s) \in \overline{Q_{\tau_*}}$ such that

$$v(y,s) = M_{\tau_*} = N_{\tau_*} = (1-\tau_*)^{-\beta} =: M. \tag{4.2.19}$$

Moreover,

$$(y,s) + Q_r^{-} \subset Q^{-}_{\frac{1+\tau_*}{2}} \subset Q_1, \quad \text{where } r := \frac{1}{2}(1-\tau_*). \tag{4.2.20}$$

Therefore, by the definition of M_τ and N_τ,

$$\sup_{(y,s)+Q_r^{-}} v \leq \sup_{Q^{-}_{\frac{1+\tau_*}{2}}} v \leq 2^\beta (1-\tau_*)^{-\beta} =: M_*,$$

since $\frac{1+\tau_*}{2} > \tau_*$ and $N_\tau > M_\tau, \forall \tau > \tau_*$.

Local largeness of v near (y,s).

The largeness condition (4.2.19) can be expanded, although in a weaker sense, to a space-time neighborhood of (y,s). To make this quantitative, set

$$\xi = 1 - \frac{1}{2^{\beta+1}}, \quad a = \frac{1 - \frac{3}{2}\frac{1}{2^{\beta+1}}}{1 - \frac{1}{2^{\beta+1}}},$$

noting that with these choices, the conditions

$$v > (1-\xi)M_*, \quad v \le (1-a\xi)M_*,$$

become, respectively,

$$v > \frac{1}{2}M, \quad v \le \frac{3}{4}M.$$

Lemma 18. *Fixing*

$$\nu = \left[\frac{(1-a)^2}{\gamma\Lambda(\xi)}\right]^{\frac{N+2}{2}} 2^{-(N+2)^2}\xi^{N+m+1}\frac{\xi^{(m-1)\frac{N}{2}}}{(1+\xi^{m-1})^{\frac{N+2}{2}}},$$

it holds

$$|[v > \frac{1}{2}M] \cap [(y,s) + Q_r^-(M_*^{1-m})]| \ge \nu|Q_r^-(M_*^{1-m})|. \tag{4.2.21}$$

Proof. If (4.2.21) does not hold, apply DeGiorgi's Lemma 15 over the cylinder

$$(y,s) + Q_r^-(M_*^{1-m}) = K_r(y) \times (s - M_*^{1-m}r^2, s],$$

for the choices $\mu_+ = \omega = M_*$ and $\theta = M_*^{1-m}$ to conclude that

$$v(y,s) \le \frac{3}{4}M$$

contradicting (4.2.19). □

Corollary 1. *There exists a time level*

$$s - M_*^{m-1}r^2 \le \overline{s} \le s$$

such that

$$|[v(.,\overline{s}) > \frac{1}{2}M] \cap K_r(y)| > \nu|K_r|. \tag{4.2.22}$$

This is immediate from the lemma, for if it isn't true and there is '$\le$' for every $\overline{s}$ in this interval, this would clearly contradict (4.2.21).

Expanding the positivity of v

Our next goal is to expand the positivity of v in such a way that we obtain constants that are independent of β which will give us enough free room to iterate this expansion and fix the free constants in such a way that they become quantitative.

Starting from (4.2.22), apply the expansion of positivity of Proposition 4, to v with $\frac{1}{2}M$ and r given by (4.2.19)-(4.2.20) and $\alpha = \nu$, yielding

$$v(.,t) \ge \eta_* M, \quad \text{in } K_{2r}(y), \tag{4.2.23}$$

for all t in the range

$$\bar{s} + \frac{b_*^{m-1}}{(\eta_* M)^{m-1}} \frac{1}{2} \delta_* r^2 \leq t \leq \bar{s} + \frac{b_*^{m-1}}{(\eta_* M)^{m-1}} \delta_* r^2 =: s_* \tag{4.2.24}$$

The expansion of positivity implies in particular that

$$|[v(., s_*) > \eta_* M] \cap K_{2r}(y)| = |K_{2r}|. \tag{4.2.25}$$

Therefore, the expansion of positivity can be applied again, starting at time level s_*, with M replaced by $(\eta_* M)$, $\rho = 2r$, $\alpha = 1$. We get

$$v(., t) \geq \eta(\eta_* M) \quad \text{in } K_{4r}(y) \tag{4.2.23(1)}$$

for all t in the range

$$s_* + \frac{b^{m-1}}{[\eta(\eta_* M)]^{m-1}} \frac{1}{2} \delta (2r)^2 \leq t \leq s_* + \frac{b^{m-1}}{[\eta(\eta_* M)]^{m-1}} \delta (2r)^2 =: s_1. \tag{4.2.24(1)}$$

It is important to keep track of the dependence of the constants provided by each iteration. First we see that the constants $\{\gamma_*, b_*, \delta_*, \eta_*\}$ in (4.2.23)-(4.2.24) depend on the data and β through the constant $\nu = \nu(\beta)$. However, they are independent of M and r. Moreover, the constants $\{\gamma, b, \delta, \eta\}$ in (4.2.23(1))-(4.2.24(1)) are different from $\{\gamma_*, b_*, \delta_*, \eta_*\}$ in (4.2.23)-(4.2.24). They depend on the data but not on β, since we were able to place the β-dependent constant η_* inside the definition of the set in (4.2.25), and set $\alpha = 1$. β is still to be chosen.

Picking up from (4.2.23(1)), we can apply the expansion of positivity once again at $s = s_1$ with M and ρ replaced by $\eta(\eta_* M)$ and $4r$, and $\alpha = 1$. In fact, this can be iterated, yielding

$$v(., t) \geq \eta^n(\eta_* M) \quad \text{in } K_{2^{n+1}r}(y), \tag{4.2.23(n)}$$

for all t in the range

$$\begin{aligned} s_{n-1} + \frac{b^{m-1}}{[\eta^n(\eta_* M)]^{m-1}} \frac{1}{2} \delta (2^n r)^2 \leq t \leq \\ \leq s_{n-1} + \frac{b^{m-1}}{[\eta^n(\eta_* M)]^{m-1}} \delta (2^n r)^2 =: s_n. \end{aligned} \tag{4.2.24(n)}$$

Proof of the proposition concluded

Without loss of generality, we may assume that $(1 - \tau_*)$ is a negative, integral power of 2. Then, choosing n such that $2^{n+1} r = 2$, the cube $K_2(y)$ covers the cube K_1 centered at $x = 0$ and from (4.2.23(n)),

$$v(., t) \geq \eta^n(\eta_* M) \quad \text{in } K_1$$

for all t in the interval (4.2.24(n)). For the indicated choice of n, and the values of M and r given by (4.2.19)-(4.2.20),

$$\eta^n(\eta_* M) = \eta^n \frac{2^{-\beta}\eta_*(\beta)}{r^\beta} = (2^\beta)^n \gamma_0$$

where $\gamma_0 = 2^{-\beta}\eta_*(\beta)$.

Now we choose β such that $2^\beta \eta = 1$ which is always possible because η is independent of β. This removes the qualitative knowledge of τ_* and hence n, and makes $\gamma_0 = \eta_*(\beta)$ quantitative.

The time level s_n is computed from

$$s_n = s_* + \frac{b^{m-1}}{(\eta_* M)^{m-1}} \delta r^2 \sum_{j=1}^{n} \left(\frac{2^2}{\eta^{m-1}} \right)^j .$$

Therefore, from the previous choices, the range of t for which (4.2.23(n)) holds can be estimated as

$$s_* + \overline{\gamma_1} \leq t \leq s_* + 4\overline{\gamma_1}, \quad \text{with} \quad \overline{\gamma_1} = \left(\frac{b}{\gamma_0} \right)^{m-1} \frac{\delta}{2}.$$

Next we note that we have freedom to choose a smaller η_* and (4.2.23) will still hold. So, if necessary, we can choose η_* even smaller to ensure that $\overline{\gamma_1} \geq 1$ so that $s_* + \overline{\gamma_1} \geq 0$ and hence $\gamma_1 = 3\overline{\gamma_1}$ is included in the times for which (4.2.23(n)) holds. Note that b and η do not depend on η_* and hence taking η_* smaller will not pose a problem. □

Proving the theorem

Finally, from the proposition, we can take the infimum over K_ρ to conclude the proof.

□

4.2.5 From Harnack inequality to Hölder continuity

The main purpose of this section was to obtain the Hölder continuity of weak solutions to (4.2.2). As we pointed out in the beginning of this section, these solutions are locally bounded. For the following proof, we will assume $u \in L^\infty(U_T)$.

Theorem 6. *Let u be a bounded, local weak solution to* (4.2.2). *Then u is locally Hölder continuous in U_T, and there exist constants $\gamma > 1$ and $\alpha \in (0,1)$ that can be determined a priori only in terms of the data, such that for every compact set $K \subset U_T$,*

$$|u(x_1,t_1) - u(x_2,t_2)| \leq \gamma C^{-\alpha} ||u||_{\infty,U_T} \left(|x_1 - x_2| + ||u||_{\infty,U_T}^{\frac{m-1}{2}} |t_1 - t_2|^{\frac{1}{2}} \right)^\alpha \tag{4.2.26}$$

for every $(x_1,t_1),(x_2,t_2) \in K$, where $C = C(K,U_T)$.

Proof. Fix a point in U_T which up to a translation, we can take the origin of $\mathbb{R}^{N+1}$. For $\rho > 0$, consider the cylinder

$$Q_\epsilon = K_\rho \times (-\rho^{2-(m-1)\epsilon}, 0]$$

where $\epsilon \in (0, 1)$ is to be determined, and set

$$\mu_0^+ = \operatorname*{ess\,sup}_{Q_\epsilon} u, \quad \mu_0^- = \operatorname*{ess\,inf}_{Q_\epsilon} u, \quad \omega_0 = \operatorname*{ess\,osc}_{Q_\epsilon} u = \mu_0^+ - \mu_0^-.$$

with ω_0 at hand, we construct the cylinder of intrinsic geometry

$$Q_0 = K_\rho \times (-\omega_0^{1-m}\rho^2, 0].$$

If $\omega_0 \geq \rho^\epsilon$, then $Q_0 \subset Q_\epsilon$. The theorem is then a consequence of the following proposition.

Proposition 6. *There exist constants $\gamma > 1$, ε, ϵ, $\delta \in (0, 1)$ and independent of u, ρ, such that if $\omega_0 \geq \rho^\varepsilon$, setting $\rho_0 = \rho$ and $\omega_n = \delta\omega_{n-1}$, $\rho_n = \epsilon\rho_{n-1}$, $Q_n = Q^-_{\rho_n}(\omega_n^{1-m})$, we have $Q_{n-1} \subset Q_n$ and*

$$\operatorname*{ess\,osc}_{Q_n} \leq \omega_n$$

Proof. The proof is done by induction. Clearly it holds for $n = 0$. Suppose it holds true up to n.

Since u being a solution to (4.2.2), does not imply that $(\mu_n^+ - u)$ or $(u - \mu^-)$ are also solutions to (4.2.2), they need not satisfy the Harnack inequality. However, it can be shown (cfr. [14]), that at least one of them does satisfy the intrinsic Harnack inequality with respect to the point $P_n = (0, -s_n)$, $s_n = \frac{1}{2}\omega_n^{1-m}\rho_n^2$, as long as their intrinsic waiting time is of the order $\theta_n\rho_n^p$. Suppose, for example, that the first one does. By the induction hypothesis,

$$\operatorname*{ess\,osc}_{Q_n} u \leq \omega_n.$$

Let

$$\mu_n^+ = \operatorname*{ess\,sup}_{Q_n} u, \quad \mu_n^- = \operatorname*{ess\,inf}_{Q_n} u.$$

We may assume at least one of the following holds

$$\mu_n^+ - u(P_n) > \frac{1}{4}\omega_n, \qquad u(P_n) - \mu_n^- > \frac{1}{4}\omega$$

for if both fail to hold then

$$\operatorname*{ess\,osc}_{Q_{n+1}} u \leq \operatorname*{ess\,osc}_{Q_n} u = \mu_n^+ - \mu_n^- \leq \frac{1}{2}\omega_n$$

and the proof by induction is trivial, with $\delta = \frac{1}{2}$. Therefore, we shall assume the first one holds. By the intrinsic, forward Harnack inequality, we get

$$\mu_n^+ - u(P_n) \leq \gamma \operatorname*{ess\,inf}_{K_\rho} \left(\mu_n^+ - u\left(., -\frac{1}{2}\omega_n^{1-m}\rho_n^2 + \left(\frac{c}{\mu_n^+ - u(P_n)}\right)^{m-1} \rho_n^2 \right)\right).$$

Start by considering the time variable.

$$-\frac{1}{2}\omega_n^{1-m}\rho_n^2 + \left(\frac{c}{\mu_n^+ - u(P_n)}\right)^{m-1} \rho_n^2 = \rho_n^2\left(c^{m-1}\left(\mu_n^+ - u(P_n)\right)^{1-m} - \frac{1}{2}\omega_n^{1-m}\right)$$

and $\left(\mu_n^+ - u(P_n)\right)^{1-m} < \frac{1}{4^{1-m}}\omega_n^{1-m}$. Since we have freedom in choosing the constant $c \geq 1$ that determines the waiting time , we will vary it so that

$$c^{m-1}\left(\mu_n^+ - u(P_n)\right)^{1-m} - \frac{1}{2}\omega_n^{1-m} \in \left(-\frac{1}{4}\omega_n^{1-m}, 0\right).$$

Then it holds

$$\operatorname*{ess\,inf}_{Q_{\frac{1}{4}\rho_n}(\omega_n^{1-m})} \left(\mu_n^+ - u\right) \geq \frac{1}{\gamma}\left(\mu_n^+ - u(P_n)\right) > \frac{1}{4\gamma}\omega_n.$$

Now we fix

$$\delta = \left(1 - \frac{1}{4\gamma}\right) < 1, \quad \epsilon = \frac{1}{4}\delta^{\frac{m-1}{2}} \leq \frac{1}{4},$$

so that $Q_{n+1} \subset Q^-_{\frac{1}{4}\rho_n}(\omega_n^{1-m}) \subset Q_n$. Then

$$\mu_n^+ \geq \operatorname*{ess\,sup}_{Q_{n+1}} u + \frac{1}{4\gamma}\omega_n.$$

Subtracting $\operatorname{ess\,inf}_{Q_{n+1}} \geq \mu_n^-$ from both sides and using the induction hypothesis,

$$\omega_n \geq \operatorname*{ess\,osc}_{Q_{n+1}} u + \frac{1}{4\gamma}\omega_n$$

which is the same as

$$\operatorname*{ess\,osc}_{Q_{n+1}} u \leq \delta\omega_n = \omega_{n+1}.$$

□

Proof of the theorem

Following from the proposition, we have

$$\operatorname*{ess\,osc}_{Q_n} u \leq \omega_n = \delta^n\omega_0.$$

Fix $0 < r < \rho$. Since $\epsilon \in (0,1)$, there exists n such that $\epsilon^{n+1}\rho \leq r \leq \epsilon^n\rho$, which means

$$(n+1) \geq ln\left(\frac{r}{\rho}\right)^{\frac{1}{\ln\epsilon}}.$$

Since $\delta \in (0,1)$,

$$\delta^{n+1} \leq \delta^{\ln\left(\frac{r}{\rho}\right)^{\frac{1}{\ln \epsilon}}} = \left(\frac{r}{\rho}\right)^{\frac{|\ln \delta|}{|\ln \epsilon|}}$$

Fixing n,

$$\operatorname*{ess\,osc}_{Q_n} u \leq \frac{1}{\delta}\omega_0 \left(\frac{r}{\rho}\right)^{\alpha_1}$$

with $\alpha_1 = \frac{|\ln \delta|}{|\ln \epsilon|}$.

Observing that since $\omega_n \leq \omega_0$, the cylinder $Q_r(\omega_0^{1-m})$ is included in Q_n and therefore, for every $(x_1,t_1),(x_2,t_2) \in Q_r(\omega_0^{1-m})$,

$$|u(x_1,t_1) - u(x_2,t_2)| \leq \operatorname*{ess\,osc}_{Q_r(\omega_0^{1-m})} u \leq \frac{1}{\delta}\omega_0 \left(\frac{r}{\rho}\right)^{\alpha_1}.$$

To conclude the proof, we cover U_T with finitely many cylinders $Q_r(\omega_0^{1-m})$ by compactness, and iterate this result for each cylinder.

□

4.3 Sharp regularity for the inhomogeneous PME

With the Caccioppoli's estimate and general Hölder continuity at hand, we are ready to prove the sharp regularity of the weak solutions to equation (4.0.1).

We start by defining the intrinsic θ-parabolic cylinder as

$$G_\rho := (-\rho^\theta, 0) \times B_\rho(0).$$

As in the previous chapter, we use a compactness property to approximate u to a solution of the homogeneous equation, under a smallness regime.

Lemma 19. *For every $\delta > 0$, there exists $0 < \epsilon \ll 1$ such that if $||f||_{L^{q,r}(G_1)} \leq \epsilon$ and u is a local weak solution of the homogeneous equation* (4.0.1) *in G_1, with $||u||_{\infty,G_1} \leq 1$, then there exists ϕ solution of* (4.2.2) *such that*

$$||u - \phi||_{\infty, G_{1/2}} \leq \delta. \tag{4.3.1}$$

Proof. The proof is done by contradiction. Suppose that, for some $\delta_0 > 0$ there exists a sequence of functions $(u^j)_j$ and $(f^j)_j$ that satisfy the hypothesis but not the thesis. That is, for every j

$$u_j \in C_{\text{loc}}(-1,0; L^2_{\text{loc}}(B_1)), \quad |u^j|^{\frac{m+1}{2}} \in L^2_{\text{loc}}(-1,0; W_{\text{loc}}(B_1))$$

and $f^j \in L^{q,r}(G_1)$ such that

$$u_t^j - \operatorname{div}\left(m|u^j|^{m-1}\nabla u^j\right) = f^j \quad \text{in } g_1 \tag{4.3.2}$$

$$||u^j||_{\infty,G_1} \leq 1, \tag{4.3.3}$$

$$||f^j||_{L^{q,r}(G_1)} \leq \frac{1}{j}, \tag{4.3.4}$$

but still, for any j and any solution ϕ of (4.2.2) in $G_{1/2}$,

$$||u^j - \phi||_{\infty,G_{1/2}} > \delta_0. \tag{4.3.5}$$

Since u^j is a solution to (4.0.1) in G_1, we can use the Caccioppoli estimate, choosing the cutoff function $\xi \in C_0^\infty(G_1)$ such that $0 \leq \xi \leq 1$, $\xi \equiv 1$ in $G_{1/2}$ and $\xi \equiv 0$ near ∂G_1. Comparing to Lemma 11, since we have (4.3.3) it is now trivial to obtain an upper bound for the right-hand side.

$$\begin{aligned}
&\sup_{-1<t<0} \int_{B_1} (u^j)^2 \xi^2 \, dx + \int_{-1}^0 \int_{B_1} |\nabla u^j|^2 \xi^2 \, dxdt \leq \\
\leq & C \int_{-1}^0 \int_{B_1} (u^j)^2 \xi |\xi_t| \, dxdt + \int_{-1}^0 \int_{B_1} |u^j|^{m+1} (|\nabla \xi|^2 + |\xi|^2) \, dxdt + ||f^j||^2_{L^{q,r}} \leq \tilde{c}.
\end{aligned}$$

Setting $v^j := |u^j|^{\frac{m+1}{2}} \in L^2_{\text{loc}}(-1,0; W^{1,2}_{\text{loc}}(B_1))$, we compute

$$|\nabla v^j|^2 = \left(\frac{m+1}{2}\right)^2 |u^j|^{m-1} |\nabla u^j|^2,$$

and we get the uniform bound

$$||\nabla v^j||^2_{2,G_{1/2}} \leq \int_{-1}^0 \int_{B_1} |\nabla v^j|^2 \xi^2 \, dxdt \leq \left(\frac{m+1}{2}\right)^2 \tilde{c},$$

which, since L^2 is reflexive, implies that up to a subsequence,

$$\nabla v^j \rightharpoonup \psi$$

weakly in $L^2(G_{1/2})$. Moreover, by Theorem 6, noting the smallness regime (4.3.3), the equibounded sequence $(u^j)_j$ is also equicontinuous. Therefore, by Arzelà-Ascoli theorem implies the uniform pointwise convergence

$$u^j \to \phi$$

in $G_{1/2}$, for yet another relabeled subsequence. This implies the following pointwise convergence

$$v^j = |u^j|^{\frac{m+1}{2}} \to |\phi|^{\frac{m+1}{2}} =: v.$$

Since weak limits must be unique, we can identify

$$\psi = \nabla v.$$

Considering the limit in (4.3.2), we find that ϕ solves (4.2.2) which contradicts (4.3.5) for j large enough.

□

We now begin the geometric iteration, exploiting the intrinsic scaling of the pme. The following result is the first step in this iteration. Recall that $\gamma = \frac{\alpha}{m}$ with

$$0 < \alpha = \min\left\{\alpha_0^-, \frac{m[(2q-n)r-2q]}{q[mr-(m-1)]}\right\} < \alpha_0 \le \min\left\{1, \frac{1}{m-1}\right\} \le 1. \tag{4.3.6}$$

Lemma 20. *There exists $\epsilon > 0$, and $0 < \lambda \ll \frac{1}{2}$, depending only on m, n and α, such that if $||f||_{L^{q,r}(G_1)} \le \epsilon$ and u is a local weak solution of (4.0.1) in G_1, with $||u||_{\infty,G_1} \le 1$, then*

$$||u||_{\infty,G_\lambda} \le \lambda^\gamma$$

provided

$$|u(0,0)| \le \frac{1}{4}\lambda^\gamma. \tag{4.3.7}$$

Proof. Take $0 < \delta < 1$ to be chosen later and apply Lemma 19 to obtain $0 < \epsilon \ll 1$ such that

$$||u-\phi||_{\infty,G_{1/2}} \le \delta.$$

Since ϕ solves (4.2.2), by Theorem 6, ϕ is locally $C^{\alpha_0,\alpha_0/2}$ for $0 < \alpha_0 < 1$. Thus we obtain

$$\sup_{G_\lambda} |\phi - \phi(0,0)| \le C\lambda^{\frac{\alpha_0}{m}}$$

for $\lambda \ll \frac{1}{2}$ to be chosen soon, and $C > 1$ universal. In fact, for $(x,t) \in G_\lambda$, it holds

$$\begin{aligned}|\phi(x,t) - \phi(0,0)| \le& C_1|x|^{\alpha_0} + C_2|t|^{\frac{\alpha_0}{2}} \le \\ \le& C_1\lambda^{\alpha_0} + C_2\lambda^{\frac{\theta}{2}\alpha_0} \le \\ \le& C\lambda^{\frac{\alpha_0}{m}},\end{aligned}$$

since $\theta \ge 1 + \frac{1}{m} > \frac{2}{m}$. We can now estimate

$$\begin{aligned}\sup_{G_\lambda} |u| \le& \sup_{G_{1/2}} |u-\phi| + \sup_{G_\lambda} |\phi - \phi(0,0)| + |\phi(0,0) - u(0,0)| + |u(0,0)| \le \\ \le& 2\delta + C\lambda^{\frac{\alpha_0}{m}} + \frac{1}{4}\lambda^\gamma,\end{aligned}$$

since $G_\lambda \subset G_{1/2}$. It is worth comparing this estimate to that of Lemma 12. As we will see later, it seems it would make sense to estimate $||u - u(0,0)||_{\infty, G_{1/2}}$ instead of $||u||_{\infty, G_{1/2}}$. However, this would make the constant c_0 not universal which would pose a problem at the iterations. Also, we can't consider $c_0 = \phi(0,0)$ as in Lemma 12 because it may not be universally bounded.

We finally fix the constants. Choose

$$\lambda = \left(\frac{1}{4C}\right)^{\frac{m}{\alpha-\alpha_0}} \quad \text{and} \quad \delta = \frac{1}{4}\lambda^\gamma,$$

fixing also $\epsilon > 0$ via Lemma 19.

□

We now iterate the previous result in the appropriate geometric setting.

Lemma 21. *There exists $\epsilon > 0$ and $0 < \lambda \ll \frac{1}{2}$, depending only on m, n and α, such that if $||f||_{L^{q,r}(G_1)} \leq \epsilon$ and u is a local weak solution of* (4.0.1) *in G_1, with $||u||_{\infty, G_1} \leq 1$, then*

$$||u||_{\infty, G_{\lambda^k}} \leq \left(\lambda^k\right)^\gamma \tag{4.3.8}$$

provided

$$|u(0,0)| \leq \frac{1}{4}\left(\lambda^k\right)^\gamma.$$

Proof. The proof is done by induction on k. If $k = 1$, (4.3.8) holds by Lemma 20. Suppose it holds up to k. Consider the functions

$$v(x,t) = \frac{u(\lambda^k x, \lambda^{k\theta} t)}{\lambda^{k\gamma}}, \quad \tilde{f}(x,t) = \lambda^{k(2-\alpha)} f\left(\lambda^k x, \lambda^{k\theta} t\right)$$

defined over G_1. A computation similar to Lemma 13 shows that

$$v_t - \operatorname{div}(m|v|^{m-1}\nabla v) = \tilde{f}(x,t)$$

with v and $\tilde{f}$ satisfying the smallness regime and

$$|v(0,0)| = \left|\frac{u(0,0)}{(\lambda^k)^\gamma}\right| \leq \frac{1}{4}\lambda^\gamma,$$

which entitled v to Lemma 20 yielding

$$||v||_{\infty, G_\lambda} \leq \lambda^\gamma,$$

which is the same as

$$||u||_{\infty, G_{\lambda^{k+1}}} \leq \lambda^{\gamma(k+1)},$$

concluding the induction.

□

In the following we show that the smallness regime required for the previous results are not restrictive and they can be generalized to an arbitrary small radius.

Theorem 7. *If u is a local weak solution of* (4.0.1) *in G_1 then, for every $0 < r < \lambda$, we have*

$$||u||_{\infty,G_r} \leq Cr^{\gamma}$$

provided

$$|u(0,0)| \leq \frac{1}{4}r^{\gamma}.$$

This result follows from the same reasoning as in the p-Laplace equation, so the proof will be omitted.

Finally, we prove the main result of this section.

Theorem 8. *Let u be a locally bounded weak solution of* (4.0.1) *in G_1, with $f \in L^{q,r}$ satistying* (4.0.2)*. Then u is locally of class $C^{0,\gamma,\frac{\gamma}{\theta}}$ with*

$$\gamma = \frac{\alpha}{m}, \qquad \alpha = \min\left\{\alpha_0^-, \frac{m[(2q-n)r-2q]}{q[mr-(m-1)]}\right\}.$$

Here $0 < \alpha_0 \leq 1$ denotes the optimal Hölder exponent for solutions of the homogeneous case and θ is given in (4.0.3).

Proof. We prove the Hölder continuity at the origin, proving there is a uniform constant K such that for every $r > 0$,

$$||u - u(0,0)||_{\infty,G_r} \leq Kr^{\gamma}. \tag{4.3.9}$$

This is clearly enough. In fact, choosing an arbitrary $(x,t) \in G_\lambda$, let

$$r = \delta\left((x,t),(0,0)\right) \leq \lambda$$

where the metric δ is defined in (3.0.1). Then

$$|u(x,t) - u(0,0)| \leq ||u - u(0,0)||_{\infty,G_r} \leq K\left(\max\{|x|, |t|^{\frac{1}{\theta}}\}\right)^{\gamma}.$$

Hölder continuity at the origin follows from standard covering arguments, and the main result follows since everything is translation invariant.

Since we know, by Theorem 4.2.26, that solutions to (4.0.1) are continuous, it makes sense to define

$$\mu := (4|u(0,0)|)^{-\gamma} \geq 0.$$

We choose an arbitrary $0 < r < \lambda$ and consider three alternative cases, exhausting all possibilities.

- If $\mu \le r < \lambda$ it follows immediately by Theorem 7

$$\sup_{(x,t)\in G_r} |u(x,t) - u(0,0)| \le Cr^\gamma + |u(0,0)| \le \left(C + \frac{1}{4}\right) r^\gamma.$$

- If $0 < r < \mu$, we consider the function

$$\omega(x,t) := \frac{u(\mu x, \mu^\theta t)}{\mu^\gamma}$$

which clearly satisfies $|\omega(0,0)| = \frac{1}{4}$ and solves in G_1 the pme

$$\omega_t - \mathrm{div}\left(m|\omega|^{m-1}\nabla\omega\right) = \mu^{2-\alpha} f(\mu x, \mu^\theta t). \tag{4.3.10}$$

Therefore, by Theorem 7, it follows that

$$||\omega||_{\infty,G_1} = \mu^{-\gamma}||u||_{\infty,G_\mu} \le C.$$

With this universal estimate in hand, we invoke Theorem 4.2.1 yet again, which ensures that there exists a radius ρ_0, depending only on the data, such that

$$|\omega(x,t)| \ge \frac{1}{8}, \qquad \text{in } G_{\rho_0}.$$

Looking back at (4.3.10), we find that ω solves, in G_{ρ_0}, a uniformly parabolic equation of the form

$$\omega_t - \mathrm{div}\,(a(x,t)\nabla\omega) = f \in L^{q,r},$$

with coefficients satisfying the bounds $0 < c_1 \le a(x,t) \le c_2$, from which we fall into the framework in which the regularity results from Theorem 3 still apply (see [12]). So we let $p = 2$ to get the continuity

$$\omega \in C^{0,\beta,\beta/2} \quad \text{with } \beta = 1 - \left(\frac{2}{r} + \frac{n}{q} - 1\right).$$

Since for $m = 1$ we have $\beta = \gamma$ and γ is increasing with m, it holds $\beta > \gamma$. Hence

$$\sup_{(x,t)\in G_r} |\omega(x,t) - \omega(0,0)| \le Cr^\beta, \quad \forall 0 < r < \frac{\rho_0}{2},$$

which is the same as

$$\sup_{(x,t)\in G_r} \left|\frac{u(\mu x, \mu^\theta t)}{\mu^\gamma} - \frac{u(0,0)}{\mu^\gamma}\right| \le Cr^\beta, \quad \forall 0 < r < \frac{\rho_0}{2}.$$

Since $\gamma < \beta$,

$$\sup_{(x,t)\in G_{\mu r}} |u(x,t) - u(0,0)| \le Cr^\beta\mu^\gamma \le C(r\mu)^\gamma,$$

for every $0 < r\mu < \mu\frac{\rho_0}{2}$. Relabeling, we get

$$\sup_{(x,t)\in G_r} |u(x,t) - u(0,0)| \le Cr^\gamma, \forall 0 < r < \mu\frac{\rho_0}{2}.$$

- Finally, for $\mu\frac{\rho_0}{2} \leq r < \mu$, we have

$$\begin{aligned}\sup_{(x,t)\in G_r} |u(x,t) - u(0,0)| \leq \sup_{(x,t)\in G_\mu} |u(x,t) - u(0,0)| \leq \\ \leq C\mu^\gamma \leq C\left(\frac{2r}{\rho_0}\right)^\gamma = \tilde{C}r^\gamma.\end{aligned}$$

We conclude the proof, setting $K = \max\left\{C + \frac{1}{4}, \tilde{C}\right\}$ and combining the previous estimates. □

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