The principle of relativity and the De Broglie relation

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Abstract

The De Broglie relation is revisited in connection with an *ab initio* relativistic description of particles and waves, which was the same treatment that historically led to that famous relation. In the same context of the Minkowski four-vectors formalism, we also discuss the phase and the group velocity of a matter wave, explicitly showing that, under a Lorentz transformation, both transform as ordinary velocities. We show that such a transformation rule is a necessary condition for the covariance of the De Broglie relation, and stress the pedagogical value of the Einstein-Minkowski-Lorentz relativistic context in the presentation of the De Broglie relation.

I. INTRODUCTION

The motivation for this paper is to emphasize the advantage of discussing the De Broglie relation in the framework of special relativity, formulated with Minkowski's four-vectors, as actually was done by De Broglie himself.[1] The De Broglie relation, usually written as

$$\lambda = \frac{h}{p}\,, \tag{1}$$

which introduces the concept of a wavelength λ, for a "material wave" associated with a massive particle (such as an electron), with linear momentum p and h being Planck's constant. Equation (1) is typically presented in the first or the second semester of a physics major, in a curricular unit of introduction to modern physics. At that stage, students are generally not yet acquainted with the relativistic formalism (or even with relativity at all). Therefore we accept the way in which the De Broglie relation is usually taught, but we claim that, in a later stage, the relation should be revisited in a full relativistic framework, such as the one pedagogically presented in this paper. For instance, in a second course in electromagnetism or in quantum mechanics (wherein special relativity may appear again, this quantum example can be presented). In this way, students may get a new physical insight on the interrelations in physics, in particular between relativity and quantum mechanics.

The supposed unfamiliarity with a relativistic formalism is probably the reason why typical university introductory physics textbooks do not stress that the De Broglie relation connecting wave to particle properties, (expressed either by $\vec{p} = \hbar\vec{k}$ or $E = \hbar\omega$), are co-related intrinsically relativistic expressions.[2] (In these expressions, $\vec{p}$ is the particle momentum, E is its energy, while $\vec{k}$ and ω are its wavenumber and its angular frequency, and $\hbar = h/(2\pi)$.) In quantum mechanics textbooks, the De Broglie relation is usually presented in its most simplified form,[3] our Eq. (1). On the other hand, basic textbooks on special relativity typically do not present De Broglie relations (although exceptions can be found[4]).

The "wave-particle duality" concept was actually introduced by Planck when he related the energy of a stationary electromagnetic wave, of frequency ν, in thermal equilibrium with the walls of a cavity (blackbody), with the quantum of energy $E = h\nu$ times an integer number. Such an idea was reinforced by Einstein when he assigned the linear momentum

$$p_{\text{photon}} = \frac{h\nu}{c}\,, \tag{2}$$

to a photon with frequency ν and energy $E_{\text{photon}} = h\nu$,[5] though the fact that light carries linear momentum (besides energy) was already established by the Maxwell's theory of electromagnetism.

Now, using the relativistic expressions for the energy and linear momentum of a particle with inertia m, moving with velocity $\vec{v}$ in a certain inertial frame S, namely $E = \gamma mc^2$ and $\vec{p} = \gamma m\vec{v}$, where $\gamma = (1 - \beta^2)^{-1/2}$ and $\beta = v/c$, one obtains $\vec{p} = E\vec{v}/c^2$. For a photon, $v = c$ and, using Planck's relation $E = h\nu$, one arrives at Eq. (2). Several experiments using light, such as diffraction and interference experiments, showed that light behaves as a wave of wavelength $\lambda = c/\nu$. In other experiments, such as those involving Compton scattering, the photoelectric effect, etc. could only be interpreted using the concept of a photon,[6] a particle with linear momentum $\vec{p}_{\text{photon}}$.

Hence, at the beginning of the twentieth century the dual nature of light was unquestionable. Louis De Broglie, in the second decade of last century, was inspired to generalize this idea, introducing the "wave-particle duality" for massive particles,[2] relating the wavelength of the matter wave and the linear momentum of the particle as expressed by Eq. (1). The validity of that equation was later on confirmed by Davisson and Germer in their famous diffraction experiment of electrons on a nickel crystal. Equation (1) is also consistent with the resonance condition for the Bohr's atom (quantization of angular momentum).[7]

The wavelength and the frequency of a plane wave are related through $\lambda\nu = v_{\text{phase}}$, where v_{phase} is the so-called phase velocity (i.e. the velocity at which a fixed phase point in a wave moves through space). The same concept of phase velocity applies to a De Broglie wave. The energy of a particle of mass m, moving with velocity v, is $E = \gamma mc^2$. Using Planck's equation $E = h\nu$, the frequency for this material wave is $\nu = \gamma mc^2/h$. Finally, writing down the wavelength as $\lambda = h/(\gamma mv)$, by using the same expression $\lambda\nu = v_{\text{phase}}$ one recognizes that

$$v_{\text{phase}} = \frac{c^2}{v} \,. \tag{3}$$

The way this result was obtained makes it clear that the phase velocity for a De Broglie material wave is an intrinsically relativistic concept. One notes that the De Broglie phase velocity is larger than c,[8] but this is not an issue, because there is no observable associated with the wave's phase, which does not carry any physical information.[9] Rather, we know that information cannot be transferred faster than the so-called signal velocity, which is the speed of the front of a truncated wave or disturbance. In relativity, this is always less than or equal to c.

The wavenumber and the wavelength are related through $k = 2\pi/\lambda$, and the angular frequency and the frequency are related through $\omega = 2\pi\nu$. The so-called group velocity, defined as the velocity at which the maximum of a wave packet moves through space, can be found from

$$v_{\text{group}} = \frac{\mathrm{d}\omega}{\mathrm{d}k}, \tag{4}$$

and as we shall see, this velocity is equal to the velocity of the particle, $v_{\text{group}} = v$. Thus, for De Broglie material waves, $v_{\text{phase}} v_{\text{group}} = c^2$.[4]

In the next section we apply a full covariant formalism, based on the Minkowski relativistic four-vector description of particles and waves, to show that 'wave-particle duality' in relativity naturally leads to the De Broglie relation. The compliance of the De Broglie equation with the principle of relativity is analyzed in Sect. III, where we also explicitly show that the phase and group velocities actually do transform as ordinary velocities under Lorentz boosts. In fact, the way the phase and the group velocities actually transform is crucial to ensure the De Broglie hypothesis covariance, and this is shown explicitly. In Sect. IV we draw our conclusions, stressing the genuine relativistic character of the De Broglie relation.

II. RELATIVISTIC DESCRIPTION FOR PARTICLES AND FOR WAVES

The wave-particle duality is an intrinsically relativistic concept, with no correspondence in classical non-relativistic physics.[2] In this section we explore the four-vector description of a massive particle and of a plane wave, to conclude that the De Broglie relation naturally emerges from such a formalism.

A. Four-vector description of a particle

Let us consider a free particle with inertia m, moving with velocity $\vec{v} = (v_x, v_y, v_z)$ in reference frame S. The linear momentum, $\vec{p} = (p_x, p_y, p_z)$, and the energy, E, are $\vec{p} = \gamma m \vec{v}$, and $E = \gamma m c^2$, respectively, and for this particle, relations hold that

$$\vec{v} = \frac{c^2}{E}\,\vec{p}, \tag{5}$$

and

$$E^2 = c^2\,\vec{p}\cdot\vec{p} + m^2 c^4. \tag{6}$$

The momentum-energy four-vector (in energy units), is (the time-like component is real and it is the fourth component of the four-vector)

$$(E^\mu) = \begin{pmatrix} c\,p_x \\ c\,p_y \\ c\,p_z \\ E \end{pmatrix} = (E) = \begin{pmatrix} c\,\gamma\,m\,v_x \\ c\,\gamma\,m\,v_y \\ c\,\gamma\,m\,v_z \\ \gamma\,m\,c^2 \end{pmatrix}. \tag{7}$$

The norm of this four-vector is a relativistic invariant,[10] expressing the information already provided by Eq. (6), we obtain $E_\mu E^\mu = E^2 - c^2 p^2 = m^2 c^4$ (we are using a diagonal metric tensor with $g_{44} = -g_{ii} = 1$). From Eq. (6) we may write $E\,\mathrm{d}E = c^2\,\vec{p}\cdot\mathrm{d}\vec{p}$, or using (5),

$$\mathrm{d}E = \vec{v}\cdot\mathrm{d}\vec{p}, \tag{8}$$

which is equivalent to $\mathrm{d}\,(\gamma c^2) = \vec{v}\cdot\mathrm{d}(\gamma\vec{v})$. From Eq. (8), the components of the velocity are $v_i = \partial E/\partial p_i$, exactly as in non-relativistic mechanics for a free particle.

B. Four-vector description of a wave

A harmonic plane wave of amplitude A is described by

$$\psi(x,y,z,t) = A \exp\left[\mathrm{i}\left(\vec{k}\cdot\vec{r} - \omega t\right)\right], \tag{9}$$

where $\vec{k}$ is the wavenumber and ω the angular frequency, both already introduced in Sect. 1. For this wave one usually considers the four-vector k^μ (with wavenumber units),[11] which is explicitly written below, together with the contravariant position-time four-vector as

$$(k^\mu) = \begin{pmatrix} k_x \\ k_y \\ k_z \\ \omega/c \end{pmatrix} \quad \text{and} \quad (x^\mu) = \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}. \tag{10}$$

One recognizes that the exponent in Eq. (9) is the internal product in Minkowski space of these four-vectors. Hence it is a relativistic invariant that[12]

$$x_\mu k^\mu = -x k_x - y k_y - z k_z + \omega t = \phi, \tag{11}$$

with ϕ being the phase of the wave, which is the same for all inertial observers. For a given wavenumber $\vec{k}$ (hence, also a given ω) the internal product $\mathrm{d}x_\mu k^\mu$ vanishes or, $\omega \mathrm{d}t - \vec{k}\cdot\mathrm{d}\vec{r} = 0$.

This is the condition for constant phase , or $\phi = \text{constant}$, meaning $\mathrm{d}\phi = 0$ (which is a frame independent equation). From this condition, the phase velocity in frame S calculated from $\vec{v}_{\text{phase}} = \mathrm{d}\vec{r}/\mathrm{d}t$ for the wave Eq. (9), can be related to the wavenumber and to the angular frequency in that frame or

$$\omega = \vec{v}_{\text{phase}} \cdot \vec{k}\,. \tag{12}$$

This relation generalizes the more familiar expression $\omega/k = \lambda\nu = v_{\text{phase}}$.

C. Relativistic De Broglie equation

According to Einstein's wave-particle duality hypothesis for the light, one may consider the four-vector $(E_k^\mu) = c\hbar(k^\mu)$ for the wave (in energy units) and the four-vector of Eq. (7), (E^μ), for the particle with the linear momentum given by Eq. (2) (i.e. with the inertia given by $h\nu/c^2$). Then, the "wave–particle duality," expressed by $E_k^\mu = E^\mu$, states that

$$\begin{pmatrix} c\,\hbar\,k_x \\ c\,\hbar\,k_y \\ c\,\hbar\,k_y \\ \hbar\,\omega \end{pmatrix} = \begin{pmatrix} c\,(h\nu/c)\,\ell_x \\ c\,(h\nu/c)\,\ell_y \\ c\,(h\nu/c)\,\ell_z \\ h\,\nu \end{pmatrix}\,, \tag{13}$$

where $\vec{\ell} = (\ell_x, \ell_y, \ell_z)$ is the unit vector pointing in the photon's direction.

The De Broglie hypothesis for a particle of mass m consists in expressing the "wave–particle duality" again by $E_k^\mu = E^\mu$, which now implies[13]

$$\begin{pmatrix} c\,\hbar\,k_x \\ c\,\hbar\,k_y \\ c\,\hbar\,k_y \\ \hbar\,\omega \end{pmatrix} = \begin{pmatrix} c\,\gamma\,m\,v_x \\ c\,\gamma\,m\,v_y \\ c\,\gamma\,m\,v_z \\ \gamma\,m\,c^2 \end{pmatrix}\,. \tag{14}$$

This four-vector equation can be regarded as the relativistic generalization[14] of the simpler De Broglie relation Eq. (1). Indeed, the four-vector Eq. (14) provides vector and scalar De Broglie relations,[15] namely

$$\hbar\vec{k} = \vec{p} \;\; = \;\; \gamma\,m\,\vec{v} \tag{15}$$

and

$$\hbar\omega = E \;\; = \;\; \gamma\,m\,c^2\,. \tag{16}$$

Taking the modulus of (15), one may write this pair of relations as

$$\lambda = \frac{h}{p} \quad = \quad \frac{h}{\gamma\, m\, v} \tag{17}$$

and

$$\nu = \frac{E}{h} \quad = \quad \frac{\gamma\, m\, c^2}{h} \,. \tag{18}$$

From Eq. (12) and from the De Broglie four-vector relation Eq. (14), the phase velocity is such that

$$\vec{v}_{\text{phase}} \cdot \vec{v} = c^2 \,, \tag{19}$$

an expression that generalizes Eq. (3).

A dispersion relation (i.e. a connection between a wave's frequency and its wavenumber) for matter waves[16] follows from Eq. (6) and from the previous De Broglie relations Eq. (14). In fact, the norm of (E_k^μ) allows us to write

$$\omega^2 = c^2 \vec{k} \cdot \vec{k} + \frac{m^2\, c^4}{\hbar^2} \,. \tag{20}$$

From this dispersion relation one obtains $\omega\, \mathrm{d}\omega = c^2\, \vec{k} \cdot \mathrm{d}\vec{k}$ and, after expressing $\vec{k}$ and ω as functions of the velocity of the particle, as given by (14), results in

$$\mathrm{d}\omega = \vec{v} \cdot \mathrm{d}\vec{k} \,. \tag{21}$$

The group velocity is found from Eq. (4) and, more generally, satisfies $\mathrm{d}\omega = \vec{v}_{\text{group}} \cdot \mathrm{d}\vec{k}$. Therefore, one concludes that the group velocity associated with the matter wave is the particle velocity $\vec{v}_{\text{group}} = \vec{v}$, at least when the group velocity is aligned with $\vec{k}$. Using the energy and the linear momentum, one may also write down $\mathrm{d}E = \vec{v}_{\text{group}} \cdot \mathrm{d}\vec{p}$ and a direct comparison with Eq. (8) allows us to conclude that the group velocity is the particle velocity indeed. For an electromagnetic wave (or photon), the phase and the group velocities are the same, as in $v_{\text{phase}} = v_{\text{group}} = c$.

A remark on the wave equation for a relativistic massive particle, corresponding to Eq. (9), is pertinent here. From that equation and the De Broglie relations one has

$$\psi(x, y, z, t) = A \exp\left[-\frac{\mathrm{i}}{\hbar} \left(Et - \vec{p} \cdot \vec{r} \right) \right] . \tag{22}$$

The exponent in the exponential is reference frame invariant since it corresponds to an internal product in Minkowski space, $x_\mu p^\mu / \hbar = \phi$ (where $p^\mu = c^{-1} E^\mu$). The condition for obtaining the phase velocity, or the condition for constant phase, is also frame invariant, as per $\mathrm{d}x_\mu p^\mu = 0$. Equation (22) is the plane wave solution of the Klein-Gordon equation for a free particle.[3] The Klein-Gordon equation is the relativistic quantum mechanical equation that applies to wave components of particles, and was established from the relativistic connection between E and $\vec{p}$, whereas the Schrödinger equation was established from the non-relativistic connection between energy and linear momentum.

III. RELATIVISTIC TRANSFORMATIONS

One of the advantages of the Minkowski's four-vector formalism is the equation covariance under Lorentz transformations. This means given a four-vector equation in reference frame S, such as $E_k^\mu = E^\mu$, in reference frame S', the same equation holds, or $E_k'^\mu = E'^\mu$. In particular, the four-vector De Broglie relation of Eq. (14) is valid in any reference frame[13] and, in S', is explicitly given by

$$\begin{pmatrix} c\,\hbar\,k'_x \\ c\,\hbar\,k'_y \\ c\,\hbar\,k'_y \\ \hbar\,\omega' \end{pmatrix} = \begin{pmatrix} c\,\gamma'\,m\,v'_x \\ c\,\gamma'\,m\,v'_y \\ c\,\gamma'\,m\,v'_z \\ \gamma'\,m\,c^2 \end{pmatrix}, \tag{23}$$

where the primed quantities refer to S$'$, in particular $\gamma' = \gamma(v')$.

In the asynchronous formulation of relativity[17] the four-vectors in S and S$'$ are related through the Lorentz transformation matrix. For the standard configuration (S$'$ moving with respect to S with velocity V along the common x-x' direction) this matrix is

$$\mathcal{L}(V) = \begin{pmatrix} \gamma_V & 0 & 0 & -\beta_V\gamma_V \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta_V\gamma_V & 0 & 0 & \gamma_V \end{pmatrix}. \tag{24}$$

where $\gamma_V = (1-\beta_V^2)^{-1/2}$ and $\beta_V = V/c$. The contravariant four-vector transformation rule is

$$E'^\mu = \mathcal{L}^\mu_\nu(V)\,E^\nu, \tag{25}$$

and it applies to any other four-vector. For a particle moving in S with 3-momentum $\vec{p}$, the momentum-energy transformation is

$$
\begin{aligned}
{p'}_x &= \gamma_V \left(p_x - V c^{-2} E\right), \\
{p'}_y &= p_y\,, \qquad {p'}_z = p_z \text{ and} \\
E' &= \gamma_V \left(E - V p_x\right).
\end{aligned}
\tag{26}
$$

A similar set of relations holds for the wavenumber components and the angular frequency (with the replacements $p'_i, \rightarrow k'_i$, $p_i \rightarrow, k_i$ and $E' \rightarrow \omega'$, $E \rightarrow \omega$).

For the sake of simplicity we shall restrict ourselves to the case of a particle moving along the x (or the x') direction, though the generalization to the more general case is straightforward (but tedious). Hence, for a particle with $\vec{p} = (p_x, 0, 0)$ the following expressions, relating the wavelength and the frequency in S$'$ with the wavelength in S,

$$
\lambda' = \frac{h}{p'} = \frac{h}{\gamma_V \left(p_x - c^{-2} E\, V\right)} = \frac{\gamma_V^{-1} \lambda}{1 - v_{\text{phase},x} V/c^2} \tag{27}
$$

$$
\nu' = \frac{E'}{h} = \frac{\gamma_V \left(E - p_x V\right)}{h} = \gamma_V \lambda^{-1} (v_{\text{phase},x} - V)\,, \tag{28}
$$

then hold, where use has been made of the relations Eqs. (3), (5) and (26). Since the phase velocity in S$'$ should be given by ${v'}_{\text{phase},x} = \lambda' \nu'$, one concludes that

$$
{v'}_{\text{phase},x} = \frac{v_{\text{phase},x} - V}{1 - v_{\text{phase},x} V/c^2}, \tag{29}
$$

which is the ordinary $x-$component velocity transformation[18] for the standard configuration. This result is not surprising because the phase velocity in S$'$ is defined as ${\vec{v}\,'}_{\text{phase}} = \mathrm{d}\vec{r}\,'/\mathrm{d}t'$, and both $\mathrm{d}\vec{r}\,'$ and $\mathrm{d}t'$ transform in the standard way under Lorentz transformations as given by Eq. (25), which is like Eq. (26), with the replacements $p'_i \rightarrow \mathrm{d}x'_i, p_i \rightarrow \mathrm{d}x_i$ and $E'/c \rightarrow c\mathrm{d}t'$, $E/c \rightarrow c\mathrm{d}t$.

Should one consider a particle moving in any direction, the usual expressions for the y and z components of the transformed phase velocity would also be obtained, namely

$$
{v'}_{\text{phase},i} = \frac{\gamma_V^{-1}\, v_{\text{phase},i}}{1 - v_{\text{phase},x} V/c^2} \quad \text{for} \quad i = y,\, z\,. \tag{30}
$$

One should note that the phase velocity can be found from the frame independent equation $\omega' \mathrm{d}t' - \vec{k}\,' \cdot \mathrm{d}\vec{r}\,' = 0$, giving ${\vec{v}\,'}_{\text{phase}} = \mathrm{d}\vec{r}\,'/\mathrm{d}t'$ is the phase velocity in S$'$. Therefore, the phase velocity transforms like a normal velocity under Lorentz transformations, as also shown in ref.[19].

The group velocity is equal to the particle velocity. Therefore the group velocity also transforms in the same way, like an ordinary velocity, or

$$v'_{\mathrm{group},x} = \frac{v_{\mathrm{group},x} - V}{1 - v_{\mathrm{group},x} V/c^2} \quad v'_{\mathrm{group},i} = \frac{\gamma_V^{-1}\, v_{\mathrm{group},i}}{1 - v_{\mathrm{group},x} V/c^2} \quad \text{for} \quad i = y,\, z\,. \tag{31}$$

Now, it can be directly checked that, provided Eq. (12) holds in S, $\omega' = \vec{v}\,'_{\mathrm{phase}} \cdot \vec{k}'$ must hold as well in S′; and, provided Eq. (21) holds in S′, with $\vec{v} = \vec{v}_{\mathrm{group}}$, $\mathrm{d}\omega' = \vec{v}\,'_{\mathrm{group}} \cdot \mathrm{d}\vec{k}'$ must also hold in S′. Moreover, from Eqs. (29)–(31) we may explicitly show that Eq. (19) is also valid in S′:

$$\vec{v}\,'_{\mathrm{phase}} \cdot \vec{v}\,'_{\mathrm{group}} = c^2\,, \tag{32}$$

provided $\vec{v}_{\mathrm{phase}} \cdot \vec{v}_{\mathrm{group}} = c^2$ in S, as is actually the case for a De Broglie matter wave. The dot product of two 3-vectors is not a Lorentz invariant, in general, but remarkably the dot product of the phase and the group velocity for a De Broglie wave is a frame independent result. In fact, a Lorentz transformation (along the $x-$axis) acting on two velocities $\vec{u}$ and $\vec{v}$ gives

$$\begin{aligned} \vec{u}\,' \cdot \vec{v}\,' &= u'_x v'_x + u'_y v'_y + u'_z v'_z \\ &= \frac{(u_x - V)(v_x - V) + (\vec{u} \cdot \vec{v} - u_x v_x)\left[1 - (V/c)^2\right]}{(1 - u_x V/c^2)(1 - v_x V/c^2)}\,. \end{aligned} \tag{33}$$

If and only if, in frame S, $\vec{u} \cdot \vec{v} = c^2$, will the above become $\vec{u}\,' \cdot \vec{v}\,' = c^2$, making this dot product a Lorentz invariant. We stress that, in order to obtain this important result, the phase and the group velocities should transform exactly in the same way, as ordinary velocities. This ensures the covariance of Eqs. (12) and (21), and the frame invariance of Eq. (19). In sum, only the proper velocity transformations of both the phase and the group velocity lead to the frame independent result $\vec{v}_{\mathrm{phase}} \cdot \vec{v}_{\mathrm{group}} = \vec{v}\,'_{\mathrm{phase}} \cdot \vec{v}\,'_{\mathrm{group}} = c^2$.

We should note that, while both the phase and group velocities transform as ordinary velocities for this special case, in general they do not transform in the same way for all wave phenomena.[18]

IV. CONCLUSIONS

The De Broglie relation is usually presented in university courses before the students get acquainted with relativity and particularly with the four-vector formalism in Minkowski space. But when students are familiarized with this more advanced relativity formalism, it is a good opportunity to re-visit the De Broglie relations, as we propose in this article. In our view, the relativistic formalism brings more physical insight to that basic quantum-mechanical relation. Note that we do not criticize the way the De Broglie relation is usually introduced, rather we claim that it is worthwhile to revisit the matter after the students are familiarized with advanced relativistic topics, including the covariance of four-vector equations under Lorentz transformations. As we have shown in this paper, relativity puts, in a better perspective, the scalar and the vector content of the four-vector equation that expresses the De Broglie's idea. On the other hand, relativity also allows for a more clear understanding of the wave-particle duality both when we refer to light or to matter.

Moreover, in this paper we present in vector form certain relations that are usually presented as scalar ones, such as the relation between linear momentum and the wavenumber. This is almost automatically accomplished by working with four-vectors. On the other hand, the four-vector formalism embodies covariance under Lorentz transformations, and this facilitates finding the De Broglie relations in different inertial reference frames. In the framework of the four-vector formalism, the way the De Broglie relation is expressed in different frames is self-evident. This is not the case in the usual presentation of the De Broglie relation in textbooks.

The relativistic context is also an opportunity to discuss the phase and the group velocity (this is the particle's velocity) as presented in this paper. We showed that $\vec{v}_{\text{phase}} \cdot \vec{v}_{\text{group}} = c^2$ is frame independent only if the phase and the group velocities transform under Lorentz transformations as normal velocities. The expression can be generalized to any two velocities whose dot product is c^2 in a certain inertial frame.

We deem that there is an unquestionable pedagogical advantage in discussing the De Broglie relation in a relativistic framework and this can be done, with great advantage to the students, in an intermediate course on relativity (sometimes integrated in a second Electromagnetism curricular unit) or even in an advanced quantum mechanics course, where the interrelations between quantum mechanics and relativity are presented and explored. According to own experience as instructors, by placing the De Broglie relation in a four-vector perspective, when the students are becoming familiar with the formalism of special relativity, brings them a new physical insight both on the powerfulness of the relativistic four-vector formalism and on the significance of the De Broglie relation.

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