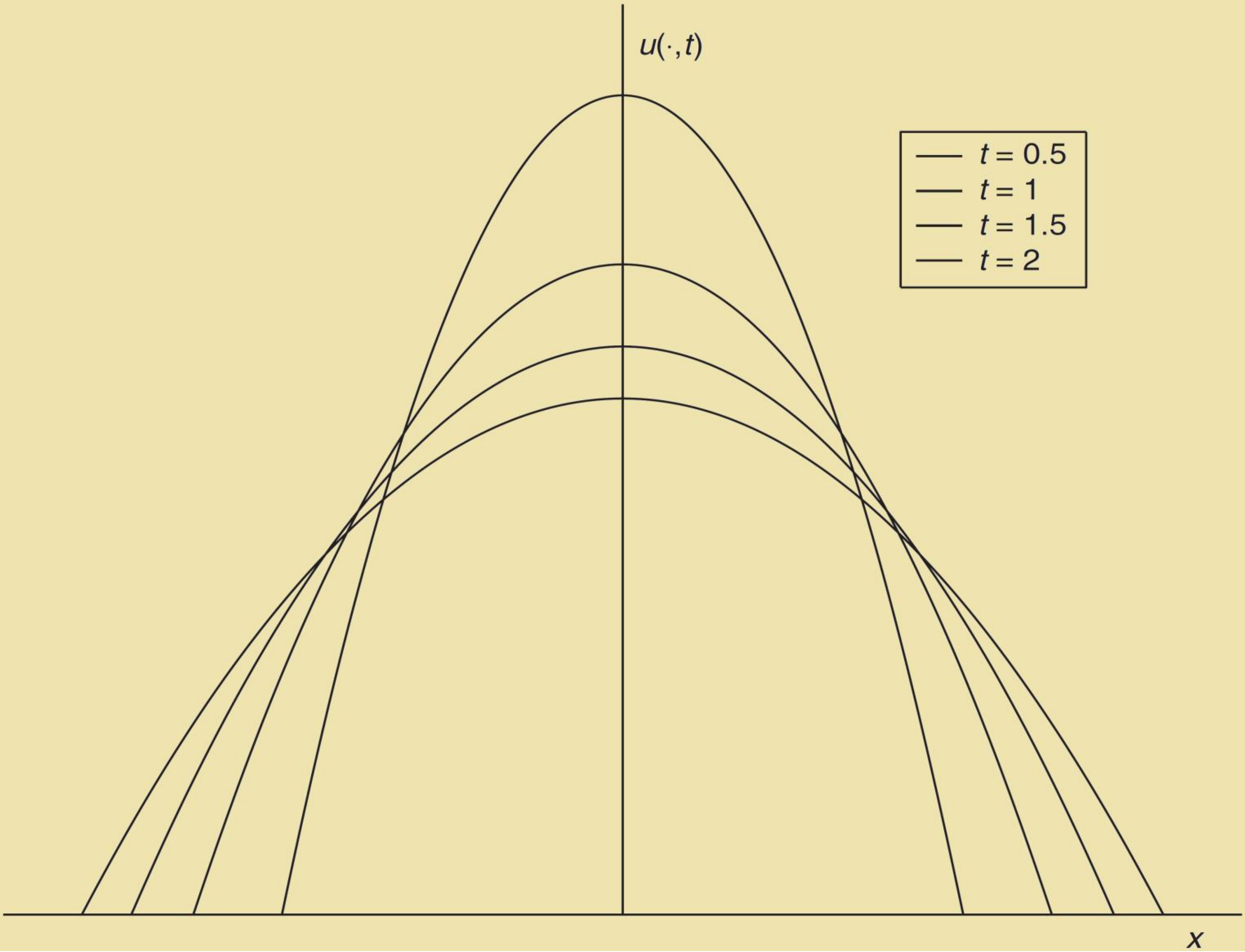


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# SHARP REGULARITY FOR THE INHOMOGENEOUS POROUS MEDIUM EQUATION

Tese de Doutoramento do Programa Inter-Universitário de Doutoramento em Matemática, orientada pelo Doutor José Miguel Urbano e apresentada ao Departamento de Matemática da Faculdade de Ciências e Tecnologia da Universidade de Coimbra.

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# Sharp regularity for the inhomogeneous porous medium equation

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Tese apresentada à Faculdade de Ciências e Tecnologia da Universidade de Coimbra, para a obtenção do grau de Doutor - Programa Inter-Universitário de Doutoramento em Matemática.

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# Abstract


The porous medium equation

$$u_t - \operatorname{div}\left(mu^{m-1}\nabla u\right) = f$$

is one of the most relevant parabolic equations of degenerate type. The modulus of ellipticity of its principal part vanishes at points where $u = 0$ and the pde looses its uniform parabolic nature. Despite this fact, some of the regularity properties of its solutions survive, like, for example, their Hölder continuity. This is a celebrated result mainly due, in its most general form, to DiBenedetto and Friedman.

On the other hand, the quest for precise, quantitative derivations of the Hölder exponent has hitherto eluded the community, the only exception being the one-dimensional case. This type of quantitative information, apart from its own intrinsic value, plays an important role in the analysis of a number of qualitative issues for parabolic pdes, such as blow-up analysis, Liouville type results, free boundary problems, and so forth.

In this thesis, we first revisit the local Hölder continuity of weak solutions of the porous medium equation using the method of intrinsic scaling. The continuity of a solution at a point follows from measuring its oscillation in a sequence of nested and shrinking cylinders, with vertex at that point, and showing that the oscillation converges to zero as the cylinders shrink to the point. The idea behind the method of intrinsic scaling is to perform this iterative process in cylinders that reflect the structure of the equation. Although the results are well-known, the proofs are scattered in the literature and we provide here a self-contained approach to the issue.

Most importantly, on the second part of the thesis, we show that locally bounded solutions of the inhomogeneous porous medium equation are locally Hölder continuous, with precise exponent

$$\gamma = \min\left\{\frac{\alpha_0^-}{m}, \frac{[(2q-n)r-2q]}{q[(mr-(m-1)]}\right\},$$

where $\alpha_0$ denotes the optimal Hölder exponent for solutions of the homogeneous case. The proof relies on an approximation lemma and geometric iteration in the appropriate intrinsic scaling.

Although regularity estimates for degenerate evolution equations have been successfully obtained in great generality, explicit expressions for the Hölder exponent of continuity for weak solutions have only been known in the linear setting. For nonlinear equations, the classical tools from harmonic analysis, such as singular integrals, are precluded from being used and an entirely new approach is needed. The estimates we obtain are striking in their simplicity and follow by applying a method based on the notion of geometric tangential equations, which explores the intrinsic scaling of the operator and the integrability of the forcing term.

**Keywords:** degenerate parabolic equations, porous medium equation, sharp Hölder regularity, intrinsic scaling.

# Resumo

A equação dos meios porosos

$$u_t - \operatorname{div}\left(mu^{m-1}\nabla u\right) = f$$

é uma das mais importantes equações parabólicas de tipo degenerado. O módulo de elipticidade da sua parte principal anula-se em pontos onde $u = 0$ e a equação perde a sua natureza parabólica uniforme. Apesar disso, algumas das propriedades de regularidade das suas soluções sobrevivem como, por exemplo, a continuidade Hölderiana. Este é um resultado célebre devido, na sua forma mais geral, a DiBenedetto e Friedman.

Por outro lado, a procura de uma expressão quantitativa precisa para o expoente de Hölder permaneceu um problema em aberto, sendo a única excepção o caso unidimensional. Este tipo de informação quantitativa, além do seu próprio valor intrínseco, desempenha um papel importante na análise de uma série de questões qualitativas para equações com derivadas parciais parabólicas, como a análise de explosão, resultados do tipo Liouville e problemas com fronteira livre, entre outros.

Nesta tese, revisitamos a continuidade Hölderiana local das soluções fracas da equação dos meios porosos usando o método da mudança intrínseca de escala. A continuidade de uma solução num ponto obtém-se medindo a sua oscilação numa sucessão de cilindros encaixados, com vértice nesse ponto, e mostrando que a oscilação converge para zero à medida que os cilindros colapsam no ponto. A idéia por detrás do método da mudança intrínseca de escala é realizar este processo iterativo em cilindros que refletem a estrutura da equação. Embora os resultados sejam bem conhecidos, as demonstrações estão espalhadas na literatura e fornecemos aqui uma abordagem auto-contida do problema.

Na segunda parte da tese, obtemos o principal resultado, que consiste em mostrar que soluções localmente limitadas da equação não-homogénea dos meios porosos são localmente contínuas à Hölder, com expoente exactamente igual a

$$\gamma = \min\left\{\frac{\alpha_0^-}{m}, \frac{[(2q-n)r-2q]}{q[(mr-(m-1)]}\right\},$$

onde $\alpha_0$ é o expoente de Hölder óptimo para soluções do caso homogéneo. A prova é baseada num lema de aproximação e num processo geométrico iterativo usando a escala intrínseca apropriada.

Embora as estimativas de regularidade para as equações de evolução degeneradas tenham sido obtidas com sucesso em grande generalidade, expressões explícitas para o expoente de continuidade Hölderiana para soluções fracas só eram conhecidas no caso linear. Para equações não-lineares, as ferramentas clássicas da análise harmónica, tais como os integrais singulares, não podem ser utilizadas e é necessária uma abordagem totalmente nova. As estimativas que obtemos são surpreendentes na sua simplicidade e resultam da aplicação de um método baseado na noção geométrica de equação tangencial, que explora a mudança intrínseca de escala para o operador e a integrabilidade do termo fonte.

**Palavras-chave:** equações parabólicas degeneradas, equação dos meios porosos, regularidade Hölder óptima, escalonamento intrínseco.

To my love, Rafaela Martins.

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# Notation

| | |
|---|---|
| $(x,t) = (x_1, ..., x_N, t)$ | generic point in $\mathbb{R}^{N+1}$ |
| $\|A\|$ | Lebesgue measure of a set $A$ |
| $\chi_A$ | characteristic function of a set $A$ |
| a.e in $A$ | holds everywhere except on a subset of $A$ with Lebesgue measure zero |
| $\|u\| = \sqrt{\sum_{i=1}^N u_i^2}$ | Euclidian norm of $u$ |
| $u_t$, $\frac{\partial}{\partial t}u$ | partial derivative of $u$ with respect to $t$ |
| $u_{x_i}$, $\frac{\partial}{\partial x_i}u$ | partial derivative of $u$ with respect to $x_i, i = 1, ..., N$ |
| $\nabla u$ | gradient of $u$ |
| $\operatorname{div} u$ | divergent of $u$ |
| $\Delta u = \operatorname{div}(\nabla u)$ | Laplacian of $u$ |
| $\Delta_p u = \operatorname{div}(\|\nabla u\|^{p-2}\nabla u)$ | $p$- Laplacian of $u$ |
| $E$ | bounded open subset of $\mathbb{R}^N$ |
| $\partial E$ | boundary of $E$ |
| $E_T = E \times (0,T]$ | cylindrical space-time domain |
| $\Sigma = \partial E \times (0,T)$ | lateral boundary of $E_T$ |
| $\partial_p E_T = \Sigma \cup (E \times \{0\})$ | parabolic boundary of $E_T$ |
| $\operatorname{supp}(f)$ | support of a function $f$ |
| $f_+$, $f_-$ | $\max(f,0)$, $\max(-f,0)$ |
| $f \wedge g$, $f \vee g$ | $\inf(f,g)$, $\sup(f,g)$ |
| $B_r(x)$ | ball in $\mathbb{R}^N$ of centre $x$ and radius $r$ |
| $\rightarrow$ | strong convergence |
| $\rightharpoonup$ | weak convergence |

# Contents

# Introduction

The quest for obtaining sharp, optimal regularity results is one of the most exciting current trends in the study of nonlinear pdes. Degenerate parabolic equations are known to have Hölder continuous solutions (cf. [19, 51]) under quite general structure assumptions, corresponding to the archetypal $p$-Laplace equation, $u_t - \mathrm{div}\,(|\nabla u|^{p-2}\nabla u) = 0$, and porous medium equation (PME), $u_t - \mathrm{div}\,(mu^{m-1}\nabla u) = 0$. The main difference between these two extensively studied pdes is that the first degenerates at points where the gradient of a solution vanishes and the second at points where this happens for the solution itself. The regularity theory for both equations has evolved in parallel and results for one normally have a counterpart for the other. Recently, in [50], the sharp Hölder exponent

$$\frac{(pq-n)r-pq}{q[(p-1)r-(p-2)]}$$

for weak solutions of the inhomogeneous $p$-Laplace equation was determined precisely only in terms of $p$, the space dimension $n$ and the $L^{q,r}$-integrability of the source. Inspired by the recent breakthroughs in [4, 5, 6], our goal in this thesis is to do the same for the porous medium equation (cf. [54]).

Let $U \subset \mathbb{R}^n$ be open and bounded, $T > 0$ and $U_T = U \times (0,T)$. We consider the prototype inhomogeneous equation

$$u_t - \mathrm{div}\left(mu^{m-1}\nabla u\right) = f, \quad m > 1, \tag{0.1}$$

with a source term $f \in L^{q,r}(U_T) \equiv L^r(0,T;L^q(U))$, where

$$\frac{1}{r} + \frac{n}{2q} < 1, \tag{0.2}$$

which is the standard minimal integrability condition that guarantees the existence of bounded weak solutions and their Hölder regularity.

We will show that bounded weak solutions of (0.1) are locally of class $C^{0,\gamma}$ in space, with

$$\gamma = \frac{\alpha}{m}, \qquad \alpha = \min\left\{\alpha_0^-, \frac{m[(2q-n)r-2q]}{q[mr-(m-1)]}\right\},$$

where $0 < \alpha_0 \leq 1$ denotes the optimal Hölder exponent for solutions of (0.1) with $f \equiv 0$. The regularity class is to be interpreted in the following sense: if

$$\frac{m[(2q-n)r-2q]}{q[mr-(m-1)]} < \alpha_0,$$

then solutions are in $C^{0,\gamma}$, with

$$\gamma = \frac{(2q-n)r-2q}{q[mr-(m-1)]};$$

if, alternatively,

$$\frac{m[(2q-n)r-2q]}{q[mr-(m-1)]} \geq \alpha_0,$$

then solutions are in $C^{0,\gamma}$, for any $0 < \gamma < \frac{\alpha_0}{m}$.

We also obtain the $C^{0,\frac{\gamma}{\theta}}$ regularity in time, where

$$\theta = 2 - \left(1 - \frac{1}{m}\right)\alpha = \alpha\left(1 + \frac{1}{m}\right) + (1-\alpha)\,2$$

is the $\alpha-$interpolation between $1 + \frac{1}{m}$ and 2. It is worth stressing that, as in the case of the $p$-Laplace equation, the integrability in time (respectively, in space) of the source affects the regularity in space (respectively, in time) of the solution.

We remark that for $m = 1$ we obtain

$$\gamma = 1 - \left(\frac{2}{r} + \frac{n}{q} - 1\right) \qquad \text{and} \qquad \theta = 2,$$

recovering the optimal Hölder regularity for the non-homogeneous heat equation, in accordance with estimates obtained by energy considerations.

For $n = 1$, it is proven in [9] that

$$\alpha_0 = \min\left\{1, \frac{1}{m-1}\right\},$$

but this is not the case in higher dimensions as corroborated by the celebrated counter-example in [10]. The question of the sharp regularity for the homogeneous PME was recently addressed in [30], where it is shown that in the case $m \geq 2$ (see also [35] for $1 < m < 2$) a solution achieves the optimal modulus of continuity $C^{0,\frac{1}{m-1}}$ of the Barenblatt fundamental solution after a precise time lag, which is quantified in the paper. This optimal regularity issue is strongly intertwined with the regularity of the free boundary (cf. [15]).

Observe that

$$\frac{m[(2q-n)r-2q]}{q[mr-(m-1)]} = \frac{2m\left(1-\dfrac{1}{r}-\dfrac{n}{2q}\right)}{m\left(1-\dfrac{1}{r}\right)+\dfrac{1}{r}} > 0$$

and so indeed $\gamma > 0$. Note also that

$$\frac{m[(2q-n)r-2q]}{q[mr-(m-1)]} > 1$$

if

$$\left(1+\frac{1}{m}\right)\frac{1}{r}+\frac{n}{q} < 1,$$

and, as $q, r \to \infty$,

$$\frac{m[(2q-n)r-2q]}{q[mr-(m-1)]} \longrightarrow 2,$$

which means that after a certain integrability threshold it is the optimal regularity exponent of the homogeneous case that prevails, with

$$\alpha = \alpha_0^- \qquad \text{and} \qquad \gamma < \frac{\alpha_0}{m} < 1.$$

The relevance of the porous medium equations and degenerate partial differential equations in general stems from their central role in the modeling of a host of nonlinear phenomena, such as thin film dynamics, non-Newtonian fluid mechanics, flow in porous media, heat transfer, population dynamics, plasma radiation, biomathematics and biophysics. Some of these applications can be found in [11, 44, 52, 53].

**Outline of the thesis**

The 1st chapter deals with the presentation of the porous medium equation (PME). We will present the theory and the fundamental results that will be crucial for the understanding of the research developed in this work. We will start by recalling some definitions and properties of function spaces, classical results of convergence and compactness. Then we will present Hölder spaces, with some characterizations of Hölder continuity. Next, we will present the formulation of the equation in its inhomogeneous parabolic version, which is the version used in this work, and the precise definition of weak solution is introduced for the model problem. Then, we present another version of the fundamental energy estimate, and some estimates for local solutions of degenerate parabolic equations. In the final part of the chapter we will present the theory of sharp regularity developed in the literature on Hölder continuity for the solutions of PME, established by Aronson in [7] (see also [9]) for the case in one dimension.

In chapter 2 we will present the intrinsic scaling method ([23], [51]), which analyzes the equation in a geometry given by its singular/degenerate structure to the study of regularity of solutions of quasi-linear equations of type $u_t - \text{div}\, \mathbf{a}(x,t,u,\nabla u) = 0$. This method allows us, in a heuristic way, to say that the equation $u_t - \text{div}(mu^{m-1}\nabla u) = 0$ behaves, in its own geometry, as the Heat Equation. Next, we will present the celebrated result obtained by DiBenedetto and Friedman, which establishes Hölder continuity of nonnegative solutions of the PME using cylinders suitably scaled to reflect in a precise quantitative way the power-like degeneracy of the equation, which is the approach provided by the intrinsic scaling method.

Chapter 3 will be divided into three parts. The first part will be dedicated to the presentation of an approximation technique, specifically, we will develop a new approximation result by homogeneous functions for the solutions of the porous medium equation, which is similar to $p$-caloric approximation. In the second part we start our fine regularity analysis by fixing the intrinsic geometric setting for our problem. Next, we will construct a geometric iteration using the theory of approximation by functions developed in the first part of the chapter, and the fact that the solutions of the porous medium equation are $C_x^{\alpha_0} \cap C_t^{\alpha_0/2}$, for $0 < \alpha_0 < 1$. Lastly, we will conclude the chapter using the geometric iteration constructed in the appropriate intrinsic scaling and the characterization of Hölder continuity to establish a new estimate for determination of the exact Hölder exponent for the solutions of the porous medium equation considered in this work, which is the main goal of the thesis.

# 1. The porous medium equation

## Contents



In this chapter, we start with the background and the auxiliary results needed for the understanding of the research developed in the thesis, specifically, definitions and properties of function spaces, results of convergence and compactness, Hölder spaces. Next, we introduce a formulation of the porous medium equation in its inhomogeneous parabolic version, emphasizing its properties, exhibiting the concept of weak solution, and presenting another version of the fundamental energy estimate, which is commonly known as Cacciopoli estimate. Then, we present $L^{\infty}_{\text{loc}}$ estimates for local solutions. At last, we present one result of great relevance in the regularity theory of solutions for PME, which is the sharp regularity for solutions of PME in the one-dimensional case due to Aronson [7]. The results that we do not demonstrate will contain references to where the proofs may be found. Standard references for the material presented here are [8, 19, 22, 54].

## 1.1 Auxiliary results

In this section, we recall some definitions and properties of function spaces, classical results of convergence, an auxiliary lemma of fast geometric convergence and a classical result of compactness. Next, we will present a brief introduction of Hölder spaces, the definition of locally Hölder continuity and some characterizations of Hölder continuity.

### 1.1.1 Basic notions

We start by presenting some basic concepts that will be used in the course of our work, emphasizing the function space that we will use.

Given a point $x_0 \in \mathbb{R}^N$ and a real number $\rho > 0$, we define the open ball with center $x_0$ and radius $\rho$ by

$$B_\rho(x_0) := \{x \in \mathbb{R}^N : |x - x_0| < \rho\} \tag{1.1}$$

and cube with centre at $x_0$ and wedge $2\rho$ by

$$K_\rho(x_0) := \left\{ x \in \mathbb{R}^N : \max_{1 \leq i \leq N} |x_i - x_{0_i}| < \rho \right\}. \tag{1.2}$$

Given a point $(x_0, t_0) \in \mathbb{R}^N$ we define the cylinder of radius $\rho$ and height $\tau > 0$ with vertex at $(x_0, t_0)$ by

$$(x_0, t_0) + Q(\tau, \rho) := K_\rho(x_0) \times (t_0 - \tau, t_0). \tag{1.3}$$

Let $u$ be a function defined on $(x_0, t_0) + Q(\tau, \rho)$; for $t \in (t_0 - \tau, t_0)$, we define the sets

$$A^{\pm}_{k,\rho} := \{x \in K_\rho(x_0) : u(x,t) \lesseqgtr k\}. \tag{1.4}$$

Given a continuous function $u : E \to \mathbb{R}$ and two real numbers $k_1, k_2$, we define

$$[u > k_2] := \{x \in E \mid u(x) > k_2\},$$

$$[u < k_1] := \{x \in E \mid u(x) < k_1\},$$

$$[k_1 < u < k_2] := \{x \in E \mid k_1 < u(x) < k_2\}.$$

Now we introduce a logarithmic function for which we obtain additional local estimates in the next chapter.

Given constants $a, b, c$, with $0 < c < a$, define the nonnegative function

$$
\begin{aligned}
\psi^{\pm}_{\{a,b,c\}}(s) &:= \left( ln \left\{ \frac{a}{(a+c)-(s-b)_{\pm}} \right\} \right)_{+} \\
&= \begin{cases} \ln \left\{ \frac{a}{(a+c)\pm(b-s)} \right\} & if\ b \pm c \lesseqgtr s \lesseqgtr b \pm (a+c) \\ 0 & if\ s \lesseqgtr b \pm c, \end{cases}
\end{aligned}
$$

whose first derivative is

$$
\left( \psi^{\pm}_{\{a,b,c\}} \right)'(s) = \begin{cases} \ln \left\{ \frac{a}{(a+c)\pm(b-s)} \right\} & if\ b \pm c \lesseqgtr s \lesseqgtr b \pm (a+c) \\ 0 & if\ s \lesseqgtr b \pm c \end{cases} \qquad \lesseqgtr 0,
$$

and second derivative, off $s = b \pm c$, is

$$
\left( \psi^{\pm}_{\{a,b,c\}} \right)'' = \left\{ \left( \psi^{\pm}_{\{a,b,c\}} \right)' \right\}^{2} \geq 0.
$$

Let $E$ be a bounded domain in $\mathbb{R}^N$ with boundary $\partial E$. For $1 \leq p \leq \infty$, we denote by $L^p(E)$ the space of Lebesgue measurable functions $u : E \to \mathbb{R}$ such that, if $p < \infty$,

$$
\|u\|_{L^p(E)} = \|u\|_p = \left( \int_E |u|^p dx \right)^{1/p} < \infty,
$$

and, for $p = \infty$,

$$
\|u\|_{L^\infty(E)} = \|u\|_\infty = \operatorname*{ess\,sup}_E |u| < \infty.
$$

We recall that, for $1 < p < \infty$, the dual space of $L^p(E)$ is identified with $L^{p'}(E)$, where $p' = \frac{p}{p-1}$ is the conjugate of $p$.

Let us denote by $L^p_{loc}(E)$ the space of Lebesgue measurable functions $u$ such that $\|u\|_{L^p(K)} < \infty$, for all compact subsets $K \subset E$. For $u \in C^1(E)$, denote by $\frac{\partial u}{\partial x_i}$ (or simply $u_{x_i}$), its partial derivative and by $\nabla u = (u_{x_1}, \cdots, u_{x_N})$ its gradient.

Let $C_c^\infty$ denote the space of infinitely differentiable functions $\phi : E \to \mathbb{R}$, with compact support in $E$. We will sometimes call a function $\phi$ belonging to $C_c^\infty$ as a test function.

The Sobolev space $W^{1,p}(E)$, with $1 \leq p \leq \infty$, is the space of functions $u \in L^p(E)$, whose generalized derivatives or derivatives in the distribution sense $u_{x_i}$ belong to $L^p(E)$ for all $i = 1, \cdots, N$, namely $\nabla u \in (L^p(E))^N$, endowed with the natural norm

$$
\|u\|_{W^{1,p}(E)} = \|u\|_{1,p} = \|u\|_{L^p(E)} + \|\nabla u\|_{L^p(E)}.
$$

$W_0^{1,p}(E)$ denotes the closure of $C_0^\infty(E)$ under this norm. A function $u$ belongs to $W^{1,p}_{loc}(E)$ if $\|u\|_{W^{1,p}(K)} < \infty$, for every compact subset $K \subset E$.

A basic result of Sobolev's space theory is the following lemma.

**Lemma 1.1.1.** *(De Giorgi) Let $v \in W^{1,1}(B_\rho(x_0)) \cap C(B_\rho(x_0))$, with $\rho > 0$ and $x_0 \in \mathbb{R}^N$, and let $k_1 < k_2 \in \mathbb{R}$. There exists a constant $C$, depending only on $N$ (and thus independent of $\rho$, $x_0$, $v$, $k_1$ and $k_2$), such that*

$$(k_2 - k_1)|[v > k_2]| \leq C \frac{\rho^{N+1}}{|[v < k_1]|} \int_{[k_1 < v < k_2]} |\nabla v| dx.$$

*Proof.* See [16]. □

**Remark 1.1.2.** *The conclusion of the lemma continues to hold for functions $v \in W^{1,1}(E) \cap C(E)$ provided that $E$ is convex.*

We recall the following Sobolev embedding for functions in $W_0^{1,p}$, $1 \leq p < N$.

Let $V_0^p(E_T)$ denote the space

$$V_0^p(E_T) = L^\infty(0, T; L^p(E)) \cap L^p(0, T; W_0^{1,p}(\Omega))$$

endowed with the norm

$$||u||^p_{V^p(E_T)} = \operatorname*{ess\,sup}_{0 \leq t \leq T} ||u(\cdot, t)||^p_{p,E} + ||\nabla u||^p_{p,E_T}.$$

The following embedding theorem holds.

**Theorem 1.1.3.** *Let $p > 1$. There exists a constant $\gamma$, depending only on $N$ and $p$, such that for every $v \in V_0^p(E_T)$,*

$$||v||^p_{p,E_T} \leq \gamma \,|\, |v| > 0 \,|^{\frac{p}{N+p}} ||v||^p_{V^p(E_T)}.$$

*Proof.* See [19, Chapter 1-Section 1.3]. □

For $0 < T < \infty$, let us denote by $E_T$ the cylindrical domain $E \times (0, T]$. Also let,

$$\Sigma = \partial E \times (0, T) \quad \text{and} \quad \partial_p E_T = \Sigma \cup (E \times \{0\})$$

denote the lateral boundary and the parabolic boundary of $E_T$ respectively. The space $L^r(0, T; L^q(E))$ for $r, q \geq 1$ is the collection of functions $u(x, t)$ defined and measurable in $E_T$ such that for almost every $t$, $0 < t < T$, the function $u$ belongs to $L^q(E)$ and

$$\|u\|_{r,q,E_T} = \left( \int_0^T \left( \int_E |u(x,t)|^q dx \right)^{r/q} dt \right)^{1/r} < \infty.$$

Also, $u$ belongs to $L^r_{loc}(0,T;L^q_{loc}(E))$ if for every compact subset $K \subset E$ and every subinterval $[t_1,t_2] \subset (0,T]$, we have

$$\int_{t_1}^{t_2} \left( \int_K |u|^q dx \right)^{r/q} dt < \infty.$$

Whenever $r = q$, we set

$$L^q(0,T;L^q(E)) = L^q(E_T), \quad L^q_{loc}(0,T;L^q_{loc}(E)) = L^q_{loc}(E_T)$$

and $\|u\|_{q,q,E_t} = \|u\|_{q,E_t}$ . These definitions are extended in the obvious way when either $q$ or $r$ are infinity.

From now on we shall denote

$$L^{q,r}(E_T) := L^r(0,T;L^q(E)).$$

The parabolic Sobolev space $L^r(0,T;W^{1,p}(E))$ for $r,p \geq 1$ is the space of functions $u(x,t)$, such that for almost every $t$, $0 < t < T$, the functions $u$ belongs to $W^{1,p}(E)$ and

$$\int_0^T \left( \int_E |u|^p + |\nabla u|^p \right)^{r/p} dt < \infty.$$

The space $C(0,T;L^p(E))$ is defined as the space of all measurable functions $u$ on $E_T$ such that for all $t \in [0,T]$, $u(t,\cdot) \in L^p(E)$ and $u(t,\cdot)$ is a continuous function from $[0,T]$ to $L^q(E)$, that is

$$\lim_{h \to 0} \|u(t+h,\cdot) - u(t,\cdot)\|_{p,E} = 0.$$

### 1.1.2 Convergence and Hölder spaces

In this section we present an auxiliary lemma of geometric convergence and the Steklov average, that will be used in the local energy and logarithmic estimates developed in the next chapter. At last, we will present a classical result, the Arzelà-Ascoli theorem, and we will also make a brief presentation of Hölder spaces.

The following lemma concerns the geometric convergence of sequences and it is instrumental in the iterative schemes that will be derived along the proofs in chapter 2.

**Lemma 1.1.4.** *Let $(X_n)$ for $n = 0,1,2,...,$ be a sequence of positive real numbers satisfying the recurrence relation*

$$X_{n+1} \leq Cb^n X_n^{1+\alpha}, \tag{1.5}$$

*where $C, b > 1$ and $\alpha > 0$ are given numbers. If*

$$X_0 \le Cb^{-1/\alpha}b^{-1/\alpha^2}, \tag{1.6}$$

*then*

$$(X_n) \to 0 \text{ as } n \to \infty.$$

*Proof.* By direct verification by applying (1.6) recursively. □

Now, we will present the Steklov average and a result from the theory of $L^p$ spaces, which will be used to reformulate the definition of a weak solution.

Let $v \in L^1(E_T)$ and let $0 < h < T$. The Steklov average $v_h(\cdot, t)$ is defined by

$$v_h := \begin{cases} \frac{1}{h}\int_t^{t+h} v(\cdot,\tau)d\tau, & if \quad t \in (0, T-h], \\ 0 & if \quad t \in (T-h, T]. \end{cases} \tag{1.7}$$

**Proposition 1.1.5.** *Let $v \in L^{q,r}(E_T)$. Then, as $h \to 0$, $v_h \to v in L^{q,r}(E_{T-\varepsilon})$ for every $\varepsilon \in (0,T)$. If $v \in C(0,T; L^q(E))$, then $v_h(\cdot,t) \to v(\cdot,t)$ in $L^q(E)$ for every $t \in (0, T-\varepsilon)$ for all $\varepsilon \in (0,T)$.*

*Proof.* See [29, Chapter 1-Section 1.4]. □

**Theorem 1.1.6.** *(Arzelà-Ascoli)If a sequence $\{f_n\}$ in $C(X)$ is bounded and equicontinuous then it has a uniformly convergent subsequence.*
*In this statement,*

*(a) "$\mathcal{F} \subset C(X)$ is bounded" means that there exists a positive constant $M < \infty$ such that $|f(x)| \le M$ for each $x \in X$ and each $f \in \mathcal{F}$, and*

*(b) "$\mathcal{F} \subset C(X)$ is equicontinuous" means that: for every $\epsilon > 0$ there exists $\delta > 0$ (which depends only on $\epsilon$) such that for $x, y \in X$:*

$$d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon \qquad \forall f \in \mathcal{F}.$$

*Proof.* See [46, Appendix $A$]. □

Now, we will make a brief introduction of Hölder spaces. We will start by defining Hölder continuous functions with the exponent $\alpha$.

**Definition 1.1.7.** *Let $0 < \alpha < 1$. A function $u : E \to \mathbb{R}$ is said to be Hölder continuous with exponent $\alpha$ at a point $x_0$ if there exists a constant $C > 0$ such that*

$$|u(x) - u(x_0)| \le c|x - x_0|^\alpha,$$

*for all $x \in E, x \ne x_0$. If this property is satisfied for every point $x_0 \in E$ we say that $u$ is Hölder continuous with exponent $\alpha$ in $E$, and we write $u \in C^\alpha(E)$.*

**Definition 1.1.8.** *The spaces $C^{k,\alpha}(E)$ are subspaces $C^{\alpha}(E)$ consisting of functions whose partial derivatives up to the order $k$ are Hölder continuous with exponent $\alpha$ in $E$, that is,*

$$C^{k,\alpha}(E) = \left\{ u \in C^k(E) : D^\beta u \in C^\alpha(E) \quad \text{forall} \quad |\beta| \leq k \right\}.$$

We also define $C^{k,\alpha}(\overline{E})$ as the space given by all the functions $u \in C^k(\overline{E})$ for which the norm

$$\|u\|_{C^{k,\alpha}(\overline{E})} := \sum_{|\beta|\leq k} \|D^\beta u\|_{C(\overline{E})} + \sum_{|\beta|=k} \|D^\beta u\|_{C^{0,\alpha}(\overline{E})}$$

is finite, where

$$[u]_{C^{0,\alpha}} := \sup_{\substack{x,y\in E \\ x\neq y}} \left\{ \frac{|u(x)-u(y)|}{|x-y|^\alpha} \right\}.$$

Now, we will present locally Hölder continuity in $E_T$.

**Definition 1.1.9.** *A function $u$ is locally Hölder continuous in $E_T$ if there exist constants $C$ and $\beta \in (0,1)$, depending only on the data, such that, for every compact subset $K$ of $E_T$,*

$$|u(x_1,t_1) - u(x_2,t_2)| \leq C \left( \frac{|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}}}{\text{dist}(K; \partial_p E_T)} \right)^\beta,$$

*for every pair of points $(x_i,t_i) \in K$, $i = 1,2$, where*

$$dist(K; \partial_p E_T) := \inf_{\substack{(x,y)\in K \\ (y,s)\in\partial_p E_T}} \left( |x-y| + |t-s|^{\frac{1}{2}} \right)$$

*is the degenerate intrinsic parabolic distance from a compact set $K \subset E_T$ to $\partial_p E_T$.*

## 1.2 Equations of porous medium type

In this section, we will present the formulation of the equation of porous media of the parabolic type, which will be the main goal of this work. Next, the precise definition of weak solution is introduced for the model problem.

### 1.2.1 Motivation and physical background

The porous media equation stands out not only for its mathematical theory, but also due to the set of applications in physics theory. Taking this into account we will deduce the porous medium equation through a problem of the theory of fluid mechanics.

Let an ideal gas flowing in a homogeneous porous medium. The flow is governed by the following three laws [40].

(I) Equation of state:

$$p = p_0 \eta^{\beta}, \tag{1.8}$$

where $\eta$ is the density of the gas at any point, $p$ the pressure, and $p_0$ a constant. If the flow is isothermic then $\beta = 1$, while if it is adiabatic then $\beta > 1$.

(II) Conservation of mass:

$$div\ \eta \mathbf{v} = -\rho \frac{\partial \eta}{\partial t}, \tag{1.9}$$

where $\mathbf{v}$ is the velocity vector and $\rho$ is the porosity of the medium (i.e., the volume fraction available to the gas).

III Darcy's Law:

$$\mathbf{v} = -\frac{\kappa}{\mu} \nabla p, \tag{1.10}$$

where $\mu$ is the viscosity of the gas and $\kappa$ is the permeability of the medium.

If we combine the above equations, we obtain

$$\begin{aligned} \rho \eta_t = -div(\eta \mathbf{v}) &= div\left(\frac{\kappa}{\mu} \eta \nabla p\right) \\ &= div\left(\frac{\kappa}{\mu} \eta \nabla p_0 \eta^{\beta}\right) \\ &= \frac{k p_0}{\mu} div(\eta \nabla \eta^{\beta}) \\ &= \frac{k \beta p_0}{\mu(\beta+1)} \Delta(\eta^{\beta+1}). \end{aligned}$$

Thus,

$$\eta_t = c \Delta \eta^m,$$

with exponent $m = 1 + \beta$ and

$$c = \frac{k \beta p_0}{\rho \mu (\beta + 1)}.$$

The constant $c$ can be scaled out(define for instance a new time $t' = ct$). Now, adapting the notation, we will use the letter $u$ instead of $\eta$ and so we get the equation

$$u_t = \Delta u^m,$$

that is commonly known as the classical porous medium equation $(PME)$. Note that if we use the letter $v$ for the pressure, we have the following expression

$$v = \frac{m}{m-1} u^{m-1}.$$

### 1.2.2 The porous medium equation

Let $E$ be an open set in $\mathbb{R}^N$ and for $T > 0$ let $E_T$ denote the cylindrical domain $E \times (0, T]$. Consider nonlinear, degenerate or singular parabolic partial differential equations of the form

$$u_t - \mathrm{div}\mathbf{A}(x, t, u, \nabla u) = B(x, t, u, \nabla u) \text{ weakly in } E_T \tag{1.11}$$

where the functions $\mathbf{A} : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ and $B : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ are only assumed to be measurable and subject to the structure conditions

$$\begin{cases} \mathbf{A}(x, t, u, \nabla u) \cdot \nabla u \geq C_0 m |u|^{m-1} |\nabla u|^2 - C^2 |u|^{m+1} \\ |\mathbf{A}(x, t, u, \nabla u)| \leq C_1 m |u|^{m-1} |\nabla u| + C |u|^m & \text{a.e. in } E_T \\ |B(x, t, u, \nabla u)| \leq C m |u|^{m-1} |\nabla u| + C^2 |u|^m, \end{cases} \tag{1.12}$$

where $m > 0, C_0$ and $C_1$ are given positive constants, and $C$ is a given nonnegative constant. When $C = 0$ the equation is homogeneous.

The inhomogeneous prototype of this class of parabolic equations that we will consider is

$$u_t - \mathrm{div}(m u^{m-1} \nabla u) = f, \quad m > 1, \quad \text{weakly in } E_T, \tag{1.13}$$

with a source term $f \in L^{q,r}(U_T) \equiv L^r(0, T; L^q(U))$ satisfying

$$\frac{1}{r} + \frac{n}{2q} < 1, \tag{1.14}$$

which is the standard minimal integrability condition that guarantees the existence of bounded weak solutions and their Hölder continuity regularity.

The first two estimates of the structural conditions presented in (1.12) originate from Moser's work [39] when in the mid-1960 he sought to extend the results of Hölder continuity of weak solutions of De Giorgi's [16] to parabolic equations and proved that the weak solutions of

$$u_t - (a_{ij}(x,t)u_{x_j})_{x_i} = 0 \tag{1.15}$$

are locally Hölder continuous in $E_T$, with bounded and measurable coefficients $a_{ij}$. The third structural condition becomes necessary when we are working with an inhomogeneous equation. The structural conditions of (1.12), in addition to a scaling method that we will present in the next chapter, have allowed DiBenedetto in [21] to show that the solutions of general quasilinear equations of the type (1.11) are locally Hölder continuous, the main result of the next chapter. Such structural conditions have a strong role in the development of energy estimates. The conditions presented here follow the proposed version in [22] considering the case in which we have a forcing term.

The partial differential equation (1.11) is degenerate when $m > 1$ and singular when $m < 1$, since the modulus of ellipticity $|u|^{m-1}$ respectively tends to 0 or to $\infty$ as $|u| \to 0$.

When $m = 1$, the equation is nondegenerate, and becomes the heat equation.

The importance of these classes of degenerate partial differential equations stems from their intrinsic mathematical interest and its various applications, especially, their central role in the modeling of a host of nonlinear phenomena, such as non-Newtonian fluid mechanics, flow in porous media, heat transfer, population dynamics, biomathematics and biophysics. For example, Barenblatt and Pattle independently found an explicit formula for the solution of

$$u_t = \Delta u^m \tag{1.16}$$

beginning from a delta function of integral $\Gamma$ (positive constant) at the origin:

$$u(|x|,t) = \max\left\{0, t^{-\alpha}\left[\Gamma - \frac{\alpha(m-1)}{2nm}\frac{|x|^2}{t^{2\alpha/n}}\right]^{1/(m-1)}\right\}, \tag{1.17}$$

where n is the number of space dimensions and $\alpha = (m - 1 + 2/n)^{-1}$. This solutions is radially symmetric and has compact support. The figures below give an idea of a typical spreading drop in one and two dimensions, respectively.

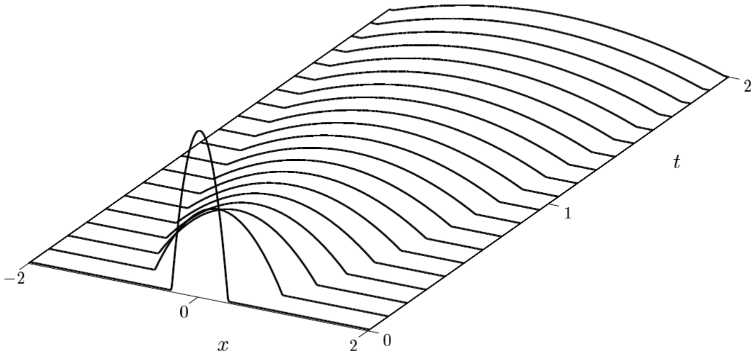


Figure 1.1: 1D Barenblatt-Pattle solution $(\Gamma = 0.2, m = 2)$

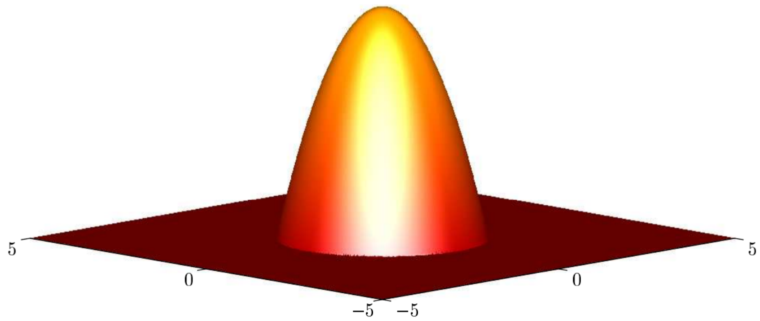


Figure 1.2: 2D Barenblatt-Pattle solution $(t = 2, \Gamma = 0.2, m = 2)$

### 1.2.3 Weak solution

In this section we will present the concept of weak solution for (1.13). Next, using the Steklov average, we will present another formulation of weak solution.

**Definition 1.2.1.** *(Weak solution.) A nonnegative locally bounded function*

$$u \in C_{loc}(0,T; L^2_{loc}(U)), \; u^{\frac{m+1}{2}} \in L^2_{loc}(0,T; W^{1,2}_{loc}(U))$$

*is a local weak solution of* (1.13) *if, for every compact set* $K \subset E$ *and every subinterval* $[t_1, t_2] \subset (0,T]$*, we have*

$$\int_K u\varphi dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_K \{-u\varphi_t + mu^{m-1}\nabla u.\nabla\varphi\} dxdt = \int_{t_1}^{t_2} \int_K f\varphi dxdt \qquad (1.18)$$

*for all nonnegative test functions*

$$\varphi \in W^{1,2}_{loc}(0,T; L^2(K)) \cap L^2_{loc}(0,T; W^{1,2}_0(K)). \qquad (1.19)$$

It is clear that all integrals in the above definition are convergent, interpreting the symbol $\nabla u$ in the sense of the following definition:

$$\nabla u := \frac{2}{m+1} u^{\frac{1-m}{2}} \nabla u^{\frac{m+1}{2}}$$

and the gradient term as

$$u^{m-1}\nabla u := \frac{2}{m+1} u^{\frac{m-1}{2}} \nabla u^{\frac{m+1}{2}}.$$

It would be technically convenient to have a formulation of weak solutions that involves $u_t$. Unfortunately, weak solutions to (1.13), whenever they exist, possess a modest degree of regularity in the time variable and, in general, $u_t$ has a meaning only in the sense of distributions. Therefore, an alternative definition makes use of the Steklov average of a function $v \in L^1(E_T)$.

**Definition 1.2.2.** *(Weak solution-Steklov average)A nonnegative locally bounded function*

$$u \in C_{loc}(0,T; L^2_{loc}(U)),\ u^{\frac{m+1}{2}} \in L^2_{loc}(0,T; W^{1,2}_{loc}(U))$$

*is a local weak solution of* (1.13) *if, for every compact set $K \subset E$ and every $0 < t < T - h$, we have*

$$\int_{K\times\{t\}} \{(u_h)_t \varphi + (mu^{m-1}\nabla u)_h . \nabla\varphi\} dx = \int_{t_1}^{t_2} \int_{K\times t} f_h \varphi dx \tag{1.20}$$

*and all nonnegative $\varphi \in W^{1,2}_0$.*

**Remark 1.2.3.** *Note that the definition of local weak solution previously introduced is equivalent to the definition given above. First, fix a subinterval $[t_1, t_2] \subset (0, T)$, and then choose $h$ sufficiently small such that $t_2 + h \leq T$, and in* (1.20) *take a test function as in* (1.19)*. Such a choice is admissible, since the test functions in* (1.20) *are independent of the variable $\tau \in (t, t+h)$ but may be dependent on $t$. Thus, integrating over $[t_1, t_2]$ and letting $h \to 0$ using the proposition* 1.1.5 *follows the equivalence.*

**Remark 1.2.4.** *The porous medium equation has a number of different notions of solutions proposed in the literature. A general overview and comparison among the different definitions can be found in [54].*

## 1.3 Energy estimate

Now we introduce a fundamental energy estimate for weak solutions of (1.13), commonly known as Cacciopoli estimate. A presentation of this estimate in its most general form can be found in [22, Chapter 3; Section 6].

**Proposition 1.3.1.** *Let $u$ be a local weak solution to (1.13) and $K \times [t_1, t_2] \subset U \times [0, T]$. There exists a constant $C$, depending only on $n, m, K \times [t_1, t_2]$, such that*

$$\begin{aligned} \sup_{t_1<t<t_2} \int_K u^2\xi^2 dx + \int_{t_1}^{t_2}\int_K u^{m-1}|\nabla u|^2\xi^2 dxdt \quad &\leq \quad C\int_{t_1}^{t_2}\int_K u^2\xi|\xi_t|dxdt \\ &+ \quad \int_{t_1}^{t_2}\int_K u^{m+1}|\nabla\xi|^2 dxdt + C\|f\|^2_{L^{q,r}} \end{aligned}$$

*for all $\xi \in C_0^\infty(K \times (t_1, t_2))$ such that $\xi \in [0, 1]$.*

*Proof.* In (1.20) take the test function

$$\varphi = u_h\xi^2$$

over $K \times (t_1, t)$ for $t \in (t_1, t_2)$. This gives, after integrating in time,

$$\begin{aligned} \int_{t_1}^{t}\int_K (u_h)_t u_h\xi^2 dxd\tau \quad &+ \quad \int_{t_1}^{t}\int_K (mu^{m-1}\nabla u)_h \cdot \nabla u_h\xi^2 dxd\tau \\ &+ \quad 2\int_{t_1}^{t}\int_K u_h(mu^{m-1}\nabla u)_h \cdot \nabla\xi\xi dxd\tau \\ &= \quad \int_{t_1}^{t}\int_K f_h u_h\xi^2 dxd\tau. \end{aligned}$$

Transform and estimate these integrals, to get

$$\begin{aligned} \int_{t_1}^{t}\int_K (u_h)_t u_h\xi^2 dxd\tau \quad &= \quad \frac{1}{2}\int_{t_1}^{t}\int_K (u_h^2)_t\xi^2 dxd\tau \\ &\longrightarrow \quad \frac{1}{2}\int_K (u^2)\xi^2(x,t)dx \\ &- \quad \frac{1}{2}\int_K (u^2)\xi^2(x,t_1)dx \\ &- \quad \int_{t_1}^{t}\int_K (u^2)\xi\xi_t dxd\tau, \end{aligned}$$

after integrating by parts and passing to the limit in $h \to 0$ (using the Lemma 1.1.5). In relation to the other term, letting first $h \to 0$, we obtain

$$\begin{aligned} \int_{t_1}^{t}\int_K (mu^{m-1}\nabla u)_h \cdot \nabla u_h\xi^2 dxd\tau \quad &\longrightarrow \quad \int_{t_1}^{t}\int_K mu^{m-1}\nabla u \cdot \nabla u\xi^2 dxd\tau \\ &= \quad m\int_{t_1}^{t}\int_K |u|^{m-1}|\nabla u|^2\xi^2 dxd\tau. \end{aligned}$$

In the third term, letting first $h \to 0$ and using Youngs inequality it follows that

$$\begin{aligned}
2\Big|\int_{t_1}^{t}\int_K u_h(mu^{m-1}\nabla u)_h\cdot\nabla\xi\xi dxd\tau\Big| \quad &\longrightarrow \quad 2\Big|\int_{t_1}^{t}\int_K u(mu^{m-1}\nabla u)\cdot\nabla\xi\xi dxd\tau\Big| \\
&\leq \quad 2m\int_{t_1}^{t}\int_K |u|^{m-1}|\nabla u|\xi u|\nabla\xi|dxd\tau \\
&\leq \quad m\int_{t_1}^{t}\int_K |u|^{m-1}|\nabla u|^2\xi^2 dxd\tau \\
&+ \quad m\int_{t_1}^{t}\int_K |u|^{m+1}|\nabla\xi|^2 dxd\tau.
\end{aligned}$$

Moreover, by Hölder inequality, we have

$$\begin{aligned}
\int_K |f_h u_h\xi^2|dx \quad &\leq \quad \|u_h\xi^2\|_{\frac{q}{q-1},K}\|f_h\|_{q,K} \\
&\leq \quad C\,(K,q)\,\|u_h\xi^2\|_{2,K}\|f_h\|_{q,K} \\
&\leq \quad C\,(K,q)\left(\int_K u_h^2\xi^2 dx\right)^{\frac{1}{2}}\|f_h\|_{q,K}, \qquad (1.21)
\end{aligned}$$

using in the last inequality that $\xi^4 \leq \xi^2$. Thus, integrating (1.21) and passing to the limit in $h \to 0$ we estimate

$$\int_{t_1}^{t}\int_K |fu\xi^2|dxd\tau \quad \leq \quad |t-t_1|^{1-\frac{1}{r}}C\,(K,q)\left(\int_K u_h^2\xi^2 dx\right)^{\frac{1}{2}}\|f_h\|_{L^{q,r}}.$$

Finally, using the Youngs inequality, we obtain

$$\left|\int_{t_1}^{t}\int_K fu\xi^2 dxd\tau\right| \quad \leq \quad \frac{1}{2}\int_K u_h^2\xi^2 dx + C\,(t_1,t,K,q,r)\,\|f\|_{L^{q,r}}^2.$$

Combining these estimates, and taking the supremum over $t \in (t_1, t_2]$ the proposition is proved. □

## 1.4 Estimates for local solutions of degenerate parabolic equations

Now, we present some $L^\infty_{\mathrm{loc}}$ estimates for local solutions of degenerate parabolic equations. The estimates will be used in the geometric interaction constructed in the last chapter of this thesis.

There are several local estimates for local solutions of degenerate parabolic equations, but generally involving dependence on the initial and boundary data. The next estimate is a very interesting result, because Daniele Andreucci presented in [1] local estimates that do not involve any dependence on the initial and boundary data, but rather, provide a bound for the solution in a given domain, only in terms of some integral norm of the solution in a larger domain.

Let $\phi : [0, \infty) \to [0, \infty)$ be a locally absolutely continuous function satisfying, for some $\Gamma > 1$,

$$1 < \frac{\phi'(s)s}{\phi(s)} \leq \Gamma \quad \text{a.e.} \quad s > 0, \tag{1.22}$$

and consider

$$\Phi(s) = \phi(s)s^{-1}, \quad s > 0;$$
$$Q_\infty = B_\rho(x_0) \times \left(\frac{t}{2}, t\right);$$
$$Q_0 = B_{(1+\sigma)\rho}(x_0) \times \left(\frac{t}{2} - \frac{\sigma}{2}t, t\right),$$

where $\rho, t > 0, \sigma \in (0,1), x_0 \in \mathbb{R}^N$ are given.

We will consider nonnegative local subsolutions of the equation

$$u_t - \mathrm{div}\mathbf{A}(x, t, u, \nabla\phi(u)) = 0, \tag{1.23}$$

assuming that the function

$$(x, t) \to \mathbf{A}(x, t, u(x, t), \nabla\phi(u(x, t))) \tag{1.24}$$

is measurable and satisfies

$$\begin{cases} \mathbf{A}(x, t, u, \nabla\phi(u)) \cdot \nabla u \geq \Gamma^{-1}\Phi(u)|\nabla u|^2 \\ |\mathbf{A}(x, t, u, \nabla\phi(u))| \leq \Gamma\Phi(u)|\nabla u|. \end{cases} \tag{1.25}$$

Let us define a local subsolution $u$ to (1.23) in $Q_0$ as a function

$$u \in C(t_0, t; L^1((B_{1+\sigma)\rho})), \qquad t_0 \equiv \frac{t}{2} - \frac{\sigma}{2}t, \qquad \nabla\phi(u) \in L^2(Q_0),$$

satisfying

$$u_t - \mathrm{div}\mathbf{A}(x, t, u, \nabla\phi(u)) \leq 0, \tag{1.26}$$

in the usual weak sense in $Q_0$.

We will use the following notation

$$⨏\!\!⨏_{Q_0} f dxdt = |Q_0|^{-1} \int\int_{Q_0} f dxdt.$$

Note that from (1.22) we have

$$\frac{1}{s} < \frac{\phi'(s)s}{\phi(s)} \leq \frac{\Gamma}{s}.$$

Taking $\phi(s) = s^m, m > 1$, we obtain the following inequalities

$$h^\Gamma s^m \leq (hs)^m \quad \leq \quad hs^m, \quad s > 0 \quad h \in \quad (0,1), \tag{1.27}$$

$$h^{\Gamma-1} s^{m-1} \leq (hs)^{m-1} \leq s^{m-1}, \quad s > 0 \quad h \in \quad (0,1) \tag{1.28}$$

**Lemma 1.4.1.** *Let $z, z_0 \geq 0$, and $\alpha > 2$ such that*

$$F(z, z_0) = \left( \int_{z_0}^{z} \left( \int_{z_0}^{r} s^{m\alpha-2} ds \right)_+ dr \right)^{1/2}.$$

*Then we have for all $z > 0, z \neq z_0$,*

$$z^{m\alpha-2} \geq c \left| \frac{\partial F}{\partial z}(z, z_0) \right|^2, \tag{1.29}$$

*where $c = c(\alpha, \Gamma)$.*

*Proof.* If $z > z_0$, we have

$$\left| \frac{\partial F}{\partial z}(z, z_0) \right|^2 = \frac{\left( \int_{z_0}^{z} s^{m\alpha-2} ds \right)^2}{4 \left( \int_{z_0}^{z} dr \int_{z_0}^{r} s^{m\alpha-2} ds \right)} = \frac{\left( \frac{z^{m\alpha-1} - z_0^{m\alpha-1}}{m\alpha-1} \right)^2}{4 \left( \int_{z_0}^{z} dr \int_{z_0}^{r} s^{m\alpha-2} ds \right)}. \tag{1.30}$$

Define $g(t) = t^{m\alpha-1}, t \in [z_0, z]$, as

$$\begin{aligned} \frac{g(z) - g(z_0)}{z - z_0} &= g'(\tilde{z}), \qquad \tilde{z} \in [z_0, z] \\ &= \tilde{z}^{m\alpha-2}(m\alpha - 1). \end{aligned}$$

Thus,

$$\begin{aligned} \left( \frac{g(z) - g(z_0)}{m\alpha - 1} \right)^2 &= \tilde{z}^{m\alpha-2}(m\alpha - 1)(z - z_0)^2 \\ &\leq z^{2m\alpha} z^{-4} (z - z_0)^2, \end{aligned}$$

since $m\alpha - 2 > 0$ and $\tilde{z} \leq z$.

Using the last inequality in (1.30) , obtain

$$4 \left| \frac{\partial F}{\partial z}(z, z_0) \right|^2 \leq z^{2m\alpha} z^{-4} (z - z_0)^2 \left( z^{-2} \int_{z_0}^{z} dr \int_{z_0}^{r} s^{m\alpha} ds \right)^{-1}.$$

We will now analyze two cases

1. $z_0 \geq z/2$

$$\begin{aligned}\int_{z_0}^{z} dr \int_{z_0}^{r} s^{m\alpha} ds &\geq \int_{z_0}^{z} dr \int_{z_0}^{r} \left(\frac{z}{2}\right)^{m\alpha} ds \\ &\geq 2^{-\Gamma\alpha-1} z^{m\alpha}(z-z_0)^2.\end{aligned}$$

2. $z_0 < z/2$

$$\begin{aligned}\int_{z_0}^{z} dr \int_{z_0}^{r} s^{m\alpha} ds &\geq \int_{z_0}^{z} dr \int_{z/}^{r} \left(\frac{z}{2}\right)^{m\alpha} ds \\ &\geq 2^{-\Gamma\alpha-3} z^{m\alpha+2}.\end{aligned}$$

In the two above cases we use the fact that $s^m$ and $s^{m\alpha-2}$ are increasing functions of $s$ and the inequality (1.27). Thus, taking $c = 2^{-\Gamma\alpha-1}$ the result follows. □

Note that (1.29) implies that $F(z,)$ is nonincreasing on $\mathbb{R}^+$. Given all of the above considerations we have Andreucci's estimate.

**Theorem 1.4.2.** *Let $u$ be a bounded nonnegative local subsolution of* 1.23 *in $Q_0$. Then for all $\eta, \varepsilon > 0$*

$$\begin{aligned}\phi(\|u\|_{\infty,Q_\infty}) &\leq \gamma\left(\frac{t}{\rho^2}\right)^{1/\varepsilon}\left(1+\frac{1}{\eta}\right)^{(N+2)/2\varepsilon}\left(\iint_{Q_0}\Phi(u)\phi(u)^\varepsilon dx d\tau\right)^{1/\varepsilon} \\ &+ \eta\frac{\rho^2}{t}\Phi^{-1}\left(\eta\frac{\rho^2}{t}\right),\end{aligned}$$

*where $\gamma = \gamma(\varepsilon, \sigma, \Gamma, N) : \gamma$ becomes unbounded as $\varepsilon$ (or $\sigma$, or $\Gamma^{-1}$) tends to zero.*

*Proof.* See [1]. □

**Remark 1.4.3.** *The inequality shown in the above theorem is also valid for any nonnegative subsolution which can be approximated locally by bounded subsolutions.*

Andreucci's estimate has the following corollaries. The proof of the next corollary will be given succinctly prioritizing the main steps, for more details see [1, 2].

**Corollary 1.4.4.** *If $\phi(u) = u^m, m > 1$, taking $\varepsilon = \lambda/m, \lambda > 0$ the inequality shown in the above theorem becomes*

$$\begin{aligned}\|u\|_{\infty,Q_\infty} &\leq \gamma\left(\frac{t}{\rho^2}\right)^{1/\lambda}\left(1+\frac{1}{\eta}\right)^{(N+2)/2\lambda}\left(\iint_{Q_0} u^{m-1+\lambda} dx d\tau\right)^{1/\lambda} \\ &+ \left(\eta\frac{\rho^2}{t}\right)^{1/(m-1)}, \qquad (1.31)\end{aligned}$$

*for all $\eta, \lambda > 0$.*

*Proof.* Let $\rho, t > 0, \sigma \in (0,1)$ be fixed. Consider for all $n \geq 0$ the sequences

$$t_n = \frac{t}{2} - \frac{\sigma t}{2^{n+1}}, \quad \rho_n = \rho + \frac{\sigma}{2^n}\rho, \quad k_n = k - \frac{k}{2^{n+1}} \qquad k > 0,$$

and let

$$B_n = B_{\rho_n}, \quad Q_n = B_{\rho_n} \times (t_n, t), B_n(\tau) = B_n \times \{\tau\}.$$

Now, consider a cutoff function $\xi_n$ sucht that

$$\xi_n(x,\tau) = 0, \quad (x,\tau) \notin Q_n, \quad \xi_n(x,\tau) = 1, (x,\tau) \in Q_{n+1},$$

$$|\nabla \xi_n| \leq \frac{2^{n+1}}{\sigma\rho}, \quad 0 \leq \xi_{n\tau} \leq \frac{2^{n+2}}{\sigma t},$$

and in the weak formulation of (1.26) take the test function

$$f_n(x,\tau) = \left(\int_{k_{n+1}}^{u(x,\tau)} s^{m\alpha-2} ds\right)_+ \xi_n(x,\tau)^2, \quad \alpha > 2.$$

By standard calculations similar to section 1.3 and using the fact that

$$F(z,z_0)^{-2}\left(\int_{z_0}^{z)} s^{m\alpha-2} ds\right)^2 = 4\left|\frac{\partial F}{\partial z_0}(z,z_0)\right|^2,$$

using also that $u \geq k_{n+1} > k_0 = k/2$, and (1.28) we obtain

$$\begin{aligned}
&\sup_{t_n \leq \tau \leq t} \int_{B_n(\tau)} F(u,k_{n+1})^2 \xi_n^2 dx + k^{m-1} \int\int_{Q_n} |\nabla F(u,k_{n+1})\xi_n|^2 dx d\tau \\
\leq\ & \gamma_1 \frac{2^{2n}}{\sigma^2 t}\left(1 + \frac{t}{\rho^2}(\|u\|_{\infty,Q_0})^{m-1}\right)\int\int_{Q_n} F(u,k_n)^2 dx d\tau, \qquad (1.32)
\end{aligned}$$

with $\gamma_1 = 2^{20\alpha\Gamma}$. Then fix $\eta > 0$, and consider the case

$$\mu := \frac{t}{\rho^2}(\|u\|_{\infty,Q_0})^{m-1} \geq \eta.$$

Thus, (1.32) becomes

$$\begin{aligned}
&\sup_{t_n \leq \tau \leq t} \int_{B_n(\tau)} F(u,k_{n+1})^2 \xi_n^2 dx + k^{m-1} \int\int_{Q_n} |\nabla F(u,k_{n+1})\xi_n|^2 dx d\tau \\
\leq\ & \gamma_1 \frac{2^{2n}}{\sigma^2 t}\left(1 + \frac{1}{\eta}\right)\mu \int\int_{Q_n} F(u,k_n)^2 dx d\tau. \qquad (1.33)
\end{aligned}$$

Now, we will see that $\int\int_{Q_n} F(u,k_n)^2 dxd\tau \to 0$ as $n \to \infty$. First, note that

$$\begin{aligned}\int\int_{Q_n} F(u,k_n)^2 dxd\tau &\geq |A_{n+1}| \int_{k_n}^{k_{n+1}} dr \int_{k_n}^{r} s^{m\alpha-2} \\ &\geq |A_{n+1}| k_n^{m\alpha-2} \frac{(k_{n+1}) - k_n)^2}{2} \\ &\geq 2^{-\alpha\Gamma(n+2)} |A_{n+1}| k^{m\alpha}, \end{aligned} \tag{1.34}$$

where

$$|A_{n+1}| = meas\{(x,\tau) \in Q_n | u(x,\tau) > k_{n+1}\}.$$

By the standard results of embedding in [38] (see pp. $74 - 75; (3.1)$) applied to the last two inequalities, we get

$$\begin{aligned} &\int\int_{Q_{n+1}} F(u,k_{n+1})^2 dxd\tau \leq \int\int_{Q_n} F(u,k_{n+1})^2 \xi_n^2 dxd\tau \\ \leq\ & |A_{n+1}|^{2/(N+2)} \left( \int\int_{Q_n} \left[ F(u,k_{n+1})^2 \xi_n^2 \right]^{(N+2)/N} dxd\tau \right)^{N/(N+2)} \\ \leq\ & C(N) |A_{n+1}|^{2/(N+2)} \left( \int\int_{Q_n} |\nabla F(u,k_{n+1}) \xi|^2 dxd\tau \right)^{N/(N+2)} \\ \cdot\ & \left( \sup_{t_n \leq \tau \leq t} \int_{B_n(\tau)} F(u,k_{n+1})^2 \xi_n^2 dx \right)^{2/(N+2)} \\ \leq\ & DB^n \left(1 + \frac{1}{\eta}\right) \mu(\sigma^2 t)^{-1} k^{-m\alpha(2/(N+2))} k^{(m-1)(-N/(N+2))} \\ \cdot\ & \left( \int\int_{Q_0} F(u,k_n) dxd\tau \right)^{1+(2/N+2)} \\ \leq\ & DB^n \left(1 + \frac{1}{\eta}\right) \mu(\sigma^2 t)^{-1} k^{-m\alpha(2/(N+2)) - (m-1)(N/(N+2))} \\ \cdot\ & \left( \int\int_{Q_0} F(u,k_n)^2 dxd\tau \right)^{1+(2/N+2)}, \end{aligned} \tag{1.35}$$

with

$$D = C(N) 2^{25\alpha\Gamma}, \qquad B = 2^{2\alpha\Gamma},$$

where $C(N)$ is a constant depending only on $N$.

Now, if $k$ is such that

$$\begin{aligned}\int\int_{Q_0} F(u,k_0)^2 dxd\tau &\leq \int\int_{Q_0} F(u,0)^2 dxd\tau \\ &= D^{-(N+2)/2} B^{-((N+2)/2)^2} \left[ \left(1 + \frac{1}{\eta}\right) \mu(\sigma^2 t)^{-1} \right]^{-(N+2)/2} k^{m\alpha+(m-1)(N/2)}, \end{aligned}$$

we have (by Lemma 1.1.4) that $\int\int_{Q_n} F(u,k_n)^2 dx d\tau \to 0$ as $n \to \infty$.

Then, using that

$$F(u,0)^2 \le u^{m\alpha}, \quad \Phi(u) = u^{m-1} \quad \text{and} \quad \mu = \frac{t}{\rho^2}(\|u\|_{\infty,Q_0})^{m-1},$$

we get

$$\begin{aligned}(\|u\|_{\infty,Q_\infty})^{m(\alpha+(N/2))}(\|u\|_{\infty,Q_\infty})^{-(N/2)} &\le \gamma_2(\sigma\rho)^{-(2+N)}\left(1+\frac{1}{\eta}\right)^{(N+2)/2}(\|u\|_{\infty,Q_0})^{m(\alpha+(N/2)-\varepsilon)} \\ &\cdot (\|u\|_{\infty,Q_0})^{-1-(N/2)}\int\int_{Q_0} u^{m(1+\varepsilon)} dx d\tau,\end{aligned}$$

with $\gamma_2 = C(N)2^{60\alpha\Gamma N^2}$ and consequently

$$\begin{aligned}(\|u\|_{\infty,Q_\infty})^{m(\alpha+(N/2))} &\le \gamma_2(\sigma\rho)^{-(2+N)}\left(1+\frac{1}{\eta}\right)^{(N+2)/2}(\|u\|_{\infty,Q_0})^{m(\alpha+(N/2)-\varepsilon)} \\ &\cdot \int\int_{Q_0} u^{(m-1)+m\varepsilon} dx d\tau. \qquad (1.36)\end{aligned}$$

Now, consider

$$\beta = 1 - \frac{\varepsilon}{\alpha+(N/2)}$$

$$Q(s,\tau) + \{(x,\theta)\big||x-x_0 < s, \tau < \theta < t\}$$

$$U(s,\tau) = (\|u\|_{\infty,Q_{(s,\tau)}})^{m(\alpha+(N/2))}.$$

Then applying Young's inequality to (1.36) and an iteration process considering

$$s_0 = \rho; \qquad s_{i+1} - s_i = (1-\sigma)\sigma^i(\sigma\rho)$$

$$\tau_0 = \rho; \qquad \tau_i - \tau_{i+1} = (1-\sigma)\sigma^i(\sigma\frac{t}{2})$$

we get

$$U(s_0,\tau_0) \le \delta^n U(s_n,\tau_n) + \gamma_3((1-\sigma)\sigma\rho)^{-(N+2)/(1-\beta)} M_0 \sum_{i=0}^{n}[\delta\sigma^{-(N+2)/(1-\beta)}]^i,$$

for any $\delta \in (0,1)$, where

$$M_0 = \delta^{\beta/(1-\beta)}\left(1+\frac{1}{\eta}\right)^{(N+2)/2(1-\beta)}\left(\int\int_{Q_0} u^{(m-1)+m\varepsilon} dx d\tau\right)^{1/(1-\beta)}.$$

Lastly, we choose $\delta = \frac{1}{2}\sigma^{(N+2)/(1-\beta)}$ and let $n \to \infty$. Then, taking the $(\alpha + (N/2))th$ root of both sides of the inequality and taking $\varepsilon = \lambda/m$, we obtain

$$\|u\|_{\infty,Q_\infty} \leq \gamma \left(\frac{t}{\rho^2}\right)^{1/\lambda} \left(1 + \frac{1}{\eta}\right)^{(N+2)/2\lambda} \left(\iint_{Q_0} u^{m-1+\lambda} dx d\tau\right)^{1/\lambda}.$$

On the other hand, if $\mu \leq \eta$, we have

$$\|u\|_{\infty,Q_\infty} \leq \left(\eta\frac{\rho^2}{t}\right)^{1/(m-1)}.$$

Combining the last two inequalities we get

$$\begin{aligned}\|u\|_{\infty,Q_\infty} &\leq \gamma \left(\frac{t}{\rho^2}\right)^{1/\lambda} \left(1 + \frac{1}{\eta}\right)^{(N+2)/2\lambda} \left(\iint_{Q_0} u^{m-1+\lambda} dx d\tau\right)^{1/\lambda} \\ &+ \left(\eta\frac{\rho^2}{t}\right)^{1/(m-1)}.\end{aligned}$$

□

For a solution $u_t - \Delta u^m = 0$ we have

$$\begin{aligned}\|u\|_{\infty,Q_\infty} &\leq \left[1 + \left(\gamma\left(\frac{t}{\rho^2}\right)^{1/\lambda}\left(1+\frac{1}{\eta}\right)^{(N+2)/2\lambda}\frac{1}{|Q_0|^{1/\lambda}}\|u\|_{L_{m-1+\lambda}}^{\frac{m-1+\lambda}{\lambda}}\right)\Big/\left(\eta\frac{\rho^2}{t}\right)^{1/(m-1)}\right] \\ &\cdot \left(\eta\frac{\rho^2}{t}\right)^{1/(m-1)}. \end{aligned} \tag{1.37}$$

We shall now prove that we can not substitute the norm $\|u\|_{L_{m-1+\lambda}}$ for a norm $\|u\|_{L_p}$ with $p > m - 1 + \lambda$. In this sense, the estimate (1.37) is sharp in the norm $L_{m-1+\lambda}$ on the right side of (1.37). In fact,

$$V(x,t) = \alpha|x|^{2/m-1}\left(1 - \frac{t}{T^*}\right)^{-1/(m-1)} \quad t \in (0,T^*), \quad x \in \mathbb{R}^n,$$

is solution and satisfies

$$\|v\|_{\infty,Q_\infty} \leq c\left(\frac{\rho^2}{t}\right)^{1/(m-1)}, \text{ for all } t \in \left(0, \frac{T^*}{2}\right).$$

Now suppose that (1.37) holds for some $p > m - 1 + \lambda$ (in place of $m - 1 + \lambda$). So, for the estimate to work, we must have

$$c \geq \frac{\gamma\left(\frac{t}{\rho^2}\right)^{1/\lambda}\left(1+\frac{1}{\eta}\right)^{(N+2)/2\lambda}\left(\frac{1}{|Q_0|^{1/\lambda}}\|v\|_{L_p}^{\frac{p}{\lambda}}\right)}{\left(\eta\frac{\rho^2}{t}\right)^{1/(m-1)}}$$

where $c$ does not depend on $t$ and $\rho$. In fact,

$$\begin{aligned}\int\int_{Q_0} v^p dxdt &= \alpha \int\int_{Q_0} |x|^{\frac{2p}{(m-1)}} \left(1-\frac{t}{T^*}\right)^{\frac{-p}{m-1}} dxdt \\ &= \left(\int_{B_{\rho(1+\sigma)}} |x|^{\frac{2p}{(m-1)}} dx\right) \cdot \left(\int_{\frac{t}{2}(1-\sigma)}^{t} \left(1-\frac{t}{T^*}\right)^{\frac{-p}{m-1}} dt\right),\end{aligned}$$

where

$$\begin{aligned}\int_{B_{\rho(1+\sigma)}} |x|^{\frac{2p}{(m-1)}} dx &= \int_0^{\rho(1+\sigma)} \left(\int_{B_1} r^{\frac{2p}{(m-1)}} r^{n-1} dx\right) dr \\ &= |B_1| \int_0^{\rho(1+\sigma)} r^{\frac{2p}{(m-1)}+n-1} dr \\ &= |B_1| \frac{1}{\frac{2p}{(m-1)}+n} \left[\rho(1+\sigma)\right]^{\frac{2p}{(m-1)}+n},\end{aligned}$$

and taking $\mu = 1-\frac{t}{T^*}$

$$\begin{aligned}\int_{\frac{t}{2}(1-\sigma)}^{t} \left(1-\frac{t}{T^*}\right)^{\frac{-p}{(m-1)}} dt &= (-T^*) \int_{1-\frac{t}{2}(1-\sigma)\frac{1}{T^*}}^{1-\frac{t}{T^*}} \mu^{\frac{-p}{(m-1)}} d\mu \\ &= T^* \int_{1-\frac{t}{T^*}}^{1-\frac{t}{2}(1-\sigma)\frac{1}{T^*}} \mu^{\frac{-p}{(m-1)}} d\mu \\ &= \left[\left(1-\frac{t}{2}(1-\sigma)\frac{1}{T^*}\right)^{\frac{-p}{m-1}+1} - \left(1-\frac{t}{T^*}\right)^{\frac{-p}{m-1}+1}\right] \\ &\cdot \quad T^* \left(\frac{-p}{m-1}+1\right)^{-1}.\end{aligned}$$

Define $g(z) = z^{\frac{-p}{m-1}+1}, z \in \left[1-\frac{t}{T^*}, 1-\frac{t}{2}(1-\sigma)\frac{1}{T^*}\right]$. Thus, we have by the mean value theorem

$$\begin{aligned}\left(\frac{-p}{m-1}+1\right)^{-1} \left[g\left(1-\frac{t}{2}(1-\sigma)\frac{1}{T^*}\right) - g\left(1-\frac{t}{T^*}\right)\right] &= g'(\tilde{z}) \cdot t \cdot c \left(\frac{-p}{m-1}+1\right)^{-1} \\ &= \left(\frac{-p}{m-1}+1\right)^{1-1} (\tilde{z})^{\frac{-p}{m-1}} tc \\ &\overset{\frac{-p}{m-1}<0}{\leq} \left(1-\frac{t}{T^*}\right)^{\frac{-p}{m-1}} tc \\ &\overset{1-\frac{t}{T^*}\geq\frac{t}{T^*}}{\leq} (T^*)^{\frac{p}{m-1}} c(t)^{\frac{-p}{m-1}+1},\end{aligned}$$

where $c = c(T^*, \sigma)$, and consequently

$$\int_{\frac{t}{2}(1-\sigma)}^{t} \left(1-\frac{t}{T^*}\right)^{\frac{-p}{(m-1)}} dt \leq c(T^*, \sigma) t^{\frac{-p}{m-1}+1}.$$

Thus, we obtain

$$\frac{1}{|Q_0|^{1/\lambda}}\|v\|_{L_p}^{p/\lambda} \leq \left[c(\lambda,p,m,n,\sigma)\cdot\rho^{\left(\frac{2p}{m-1}\right)\frac{1}{\lambda}}\right]\cdot[c(T^*,\sigma)]\,t^{\frac{-p}{m-1}\frac{1}{\lambda}}.$$

Therefore, we find the following estimate

$$\begin{aligned}\left(\frac{t}{\rho^2}\right)^{1/\lambda}\left(\frac{1}{|Q_0|^{1/\lambda}}\|v\|_{L_p(Q_0)}^{p/\lambda}\right) &\leq C(\lambda,m,n,\gamma,T^*,p)\left[\rho^{\left(\frac{2p}{(m-1)}-2\right)\frac{1}{\lambda}}\right]\\ &\cdot\ \left[t^{\left(\frac{-p}{(m-1)}+1\right)\frac{1}{\lambda}}\right].\end{aligned}$$

Finally, we get

$$\begin{aligned}\frac{\left(\frac{t}{\rho^2}\right)^{1/\lambda}\left(\frac{1}{|Q_0|^{1/\lambda}}\|v\|_{L_p}^{\frac{p}{\lambda}}\right)}{\left(\eta\frac{\rho^2}{t}\right)^{1/(m-1)}} &\leq C(\eta,\lambda,m,\sigma,T^*,p,n)\\ &\cdot\ \left[\rho^{\left(\frac{2p}{(m-1)}-2\right)\frac{1}{\lambda}-\frac{2}{(m-1)}}\right]\cdot\left[t^{\left(\frac{-p}{(m-1)}+1\right)\frac{1}{\lambda}+\frac{1}{(m-1)}}\right],\end{aligned}$$

and to get the desired bounded we must have

$$\left(\frac{2p}{(m-1)}-2\right)\frac{1}{\lambda}-\frac{2}{(m-1)}\geq 0 \tag{1.38}$$

and

$$\left(\frac{-p}{(m-1)}+1\right)\frac{1}{\lambda}+\frac{1}{(m-1)}\geq 0, \tag{1.39}$$

that is,

$$p\geq m-1+\lambda \quad \text{e} \quad p\leq m-1+\lambda.$$

Therefore,

$$p=m-1+\lambda$$

is the best choice, that is, the sharp choice.

**Corollary 1.4.5.** *In the linear case $\phi(u)=u,\Phi\equiv 1$, we can take $\eta=t/\rho^2$, so that our estimate takes the form*

$$\|u\|_{\infty,Q_\infty} \leq \gamma_1\left(\frac{t}{\rho^2}\right)^{1/\varepsilon}\left(1+\frac{1}{\eta}\right)^{(N+2)/2\varepsilon}\left(⨌_{Q_0}u^\varepsilon dxd\tau\right)^{1/\varepsilon}, \tag{1.40}$$

*$\gamma_1=\gamma_1(\varepsilon,\sigma,\Gamma,N)$, which in turn reduces to Moser's sup-estimate presented in [39].*

## 1.5 Sharp regularity for solutions of the PME in the one-dimensional case

The main objective of this work is to study the optimal regularity for the porous medium equation solutions in the $n$-dimensional case. However, the whole theory behind this problem for the $n = 1$ case was completely established in [43, 7]. In this section, we will address the sharp regularity for solutions of PME in the one-dimensional case, through the results developed by Aronson in [7]. Due to its technical nature, complexity and the fact that the tools used in this case are completely different from those used in our case, we will approach the main results superficially, and some statements will be omitted. The main objective of this section is to present the case complementary to the one developed in our work, so that the reader has an understanding and knowledge of all cases.

We will study the regularity properties of a class of generalized solutions of the Cauchy problem for (1.16). We will see that with respect to the space variables, the velocity potential is Lipschitz continuous, the flux is continuous, and the density is Hölder continuous with exponent

$$\alpha = \min\left\{1, \frac{1}{m-1}\right\}.$$

Furthermore, we will see that this exponent is the best possible.

Let $S := (-\infty, +\infty) \times (0, T]$ in the two-dimensional $x, t-$space for some fixed $T > 0$ and consider the Cauchy problem

$$\begin{cases} (u^m)_{xx} = u_t, & for \quad (x,t) \in S, \quad m > 1 \\ u(x,0) = u_0(x) & for \quad -\infty < x < \infty, \end{cases} \tag{1.41}$$

where $u_0$ is a given bounded, continuous, nonnegative function on the real line.

Now, consider $u$ be a smooth positive classical solution of

$$(u^m)_{xx} = u_t \tag{1.42}$$

in a rectangle

$$R = (a, b) \times (0, T],$$

and take $M = \max_{\overline{R}} u$. Note that if $v = u^{m-1}$(velocity potential), then $v$ satisfies the nonlinear degenerate parabolic equation

$$v_t = mvv_{xx} + \frac{m}{m-1} v_x^2 \tag{1.43}$$

in $R$.

**Remark 1.5.1.** *With the expression smooth we mean at least* $u \in C^2(R) \cap C^0(\overline{R})$ *and* $u_{xxx} \in C^0(R)$.

Now, we will present a result in which $|v_x|$ is independent of the lower bound for $u$.

**Lemma 1.5.2.** *Let* $u$ *be a smooth positive classical solution of* (1.42) *in* $R$ *and let*

$$R^* = (a_1, b_1) \times (\tau, T]$$

*be any proper subrectangle of* $R$. *Then*

$$|v_x(x,t)| \leq \mathscr{C}$$

*in* $\overline{R}^*$, *where* $\mathscr{C}$ *is a positive constant which depends only on* $m, M, a_1 - a, b - b_1$ *and* $\tau$.

*Proof.* See [7]. $\square$

**Remark 1.5.3.** *In the lemma above, if*

$$M_1 = \max_{[a,b]} \left| \frac{\partial}{\partial x} u^{m-1}(x,0) \right| < \infty$$

*then the same conclusion holds for*

$$R^* = (a_1, b_1) \times (0, T],$$

*where now* $\mathscr{C}$ *depends on* $M_1$ *instead of* $\tau$.

**Remark 1.5.4.** *The weak solution* $u$ *of de Cauchy problem is the pointwise limit of a decreasing sequence of positive functions* $(w_n)$. *This result was shown in [42].*

Note that given any rectangle $R$, if n is sufficiently large, then the $w_n$ satisfy the hypothesis of the lemma in $R$. Besides that,

$$w_n \leq 1 + \sup u_0$$

for $n$ at every point of $S$.

Given the above considerations we have the following result:

**Theorem 1.5.5.** *Let* $u$ *be the weak solution of the Cauchy problem* (1.41) *in* $S$, *where it is assumed that* $u_0^m$ *is Lipschitz continuous.*

*(i) If $\tau > 0$, then*

$$|u^{m-1}(x_1,t) - u^{m-1}(x_2,t)| \leq \mathscr{C}_1|x_2 - x_1| \tag{1.44}$$

*and*

$$|u(x_1,t) - u(x_2,t)| \leq \mathscr{C}_2|x_1 - x_2|^{\nu} \tag{1.45}$$

*hold for all $(x_1,t),(x_2,t) \in (-\infty,\infty) \times [\tau,T]$ where $\nu = min\{1,(m-1)^{-1}\}$ and the $\mathscr{C}_i$ are positive constants which depend only on $m,\tau$ and $\sup u_0$ If $u^{m-1}$ is Lipschitz continuous, then the same conclusions hold for all $(x_1,t),(x_2,t) \in \overline{S}$ where now the $\mathscr{C}_i$ depend on the Lipschitz constant for $u_0^{m-1}$ instead of $\tau$.*

*(ii) The derivate $\partial u^m/\partial x$ exists and is continuous as a function of $x$ everywhere in $S$, and, in particular,*

$$\partial u^m(x,t)/\partial x = 0 \quad \text{if} \quad u(x,t) = 0.$$

*(iii) If $1 < m < 2$, then $\partial u/\partial x$ exists and is continuous as a function of $x$ everywhere in $S$, and, in particular,*

$$\partial u(x,t)/\partial x = 0 \quad \text{if} \quad u(x,t) = 0.$$

*Proof.* First, we will present the proof for item $(i)$ and we also will give a sketch for item item $(ii)$ and item $(iii)$.

In fact, applying the previous lemma to $w_n^{m-1}$ and using the convergence of $w_n^{m-1}$ to $u^{m-1}$ we get the estimate (1.44). In the case $m > 2$, we have that (1.45) is a consequence of (1.44) and the observation that

$$|u(x_1,t) - u(x_2,t)|^{m-1} \leq |u^{m-1}(x_1,t) - u^{m-1}(x_2,t)|.$$

Now suppose $m \leq 2$. Since

$$\frac{\partial}{\partial x} w_n^{m-1} = (m-1)w_n^{m-2}\frac{\partial}{\partial x}w_n$$

and

$$0 < w_n \leq 1 + \sup u_0 = M_1,$$

then it follows from the previous lemma that

$$\left|\frac{\partial w_n}{\partial x}\right| \leq \frac{M_1^{2-m}}{m-1}\mathscr{C} = \mathscr{C}'.$$

Therefore,

$$|w_n(x_1,t) - w_n(x_2,t)|\left|\frac{\partial}{\partial x}w_n(\xi,t)d\xi\right| \leq \mathscr{C}'|x_1 - x_2|$$

and letting $n \to \infty$ it follows the result.

To prove the items $(ii)$ and $(iii)$, first consider $u(x,t) > 0$ and $t > 0$, using the results established by Kalashnikov in [42] we have that $\partial u/\partial x$ exists and is continuous in a neighborhood of $(x,t)$ and the same is true for $\partial u^m/\partial x$ , then to complete the proof, we need to verify only in the points $(x,t_0)$ for which $u(x,t_0) = 0$ (This part is completely developed in [7]). □

Now, through an example of explicit solution due to Pattle[43] of (1.42) we will see that

$$\alpha = \min\left\{1, \frac{1}{m-1}\right\}$$

is the best possible exponent for Hölder continuity of $u$.

Consider

$$\lambda(t) = \left\{\frac{2m(m+1)}{m-1}(t+1)\right\}^{1/(m+1)}, \qquad t \geq 0.$$

Thus,

$$u(x,t) = \begin{cases} \frac{1}{\lambda(t)}\left[1-\left(\frac{x}{\lambda(t)}\right)^2\right]^{1/(m-1)} & for \quad |x| \leq \lambda(t), \quad t \geq 0, \\ 0 & for \quad |x| > \lambda(t), \quad t \geq 0, \end{cases} \tag{1.46}$$

is a weak solution of the Cauchy problem (1.41) with initial data

$$u_0(x) = \begin{cases} \frac{1}{\lambda(0)}\left[1-\left(\frac{x}{\lambda(0)}\right)^2\right]^{1/(m-1)} & for \quad |x| \leq \lambda(0), \\ 0 & for \quad |x| > \lambda(0). \end{cases} \tag{1.47}$$

Note that in $x = \pm\lambda(t)$ the $\partial u^{m-1}/\partial x$ is discontinuous. Then the Lipschitz continuity of $u^{m-1}$ is the best possible global result. Furthermore, since

$$u(x,t) - u(\lambda(t),t) = (\lambda(t))^{(m+1)/(1-m)}[(\lambda(t)+x)(\lambda(t)+x)]^{1/(m-1)}$$

the exponent

$$\min\left\{1, \frac{1}{m-1}\right\}$$

in (1.45) cannot be increased.

**Remark 1.5.6.** *These are several examples of explicit solutions of* (1.42) *can be found in the works of Oleinik [41] and Kalashnikov [42].*

The reading of the article discussed above is necessary in the first contact with the theory of regularity for the porous medium equation, and can be seen as the first motivation. However we will see in the next chapters that the approach used here is not repeated in the case $n$-dimensional, with $n > 1$, one of the main reasons being that we do not have examples of explicit solutions like Pattle's[43] in our case, where we can do a similar analysis. However, the theory for our case begun to be developed by Cafarelli and Friedman in the study of the continuity of the density/and regularity of the free boundary of a gas flow in an $n$-dimensional porous medium[13, 14]. The main approach is based on an idea using scaling (or similarity of transformation) of solutions. These results, in addition to [20], culminate in the famous result proposed by DiBenedetto and Friedman in [21] and will be the focus of our next chapter.

# 2. Regularity $C^{0,\beta}$ for solutions of the PME in the N-dimensional case

## Contents



In this chapter we will present the celebrated result obtained by DiBenedetto and Friedman in [21] about the Hölder continuity for the N-dimensional case of the homogeneous porous medium equation. First, we will present the method of intrinsic scaling, more precisely we will see that the results on the continuity of solutions, at a point $(x_0, t_0)$, can be obtained by measuring the oscilation of the solution in a sequence of nested and shrinking cylinders with vertex at that point (see Figure 3.1),

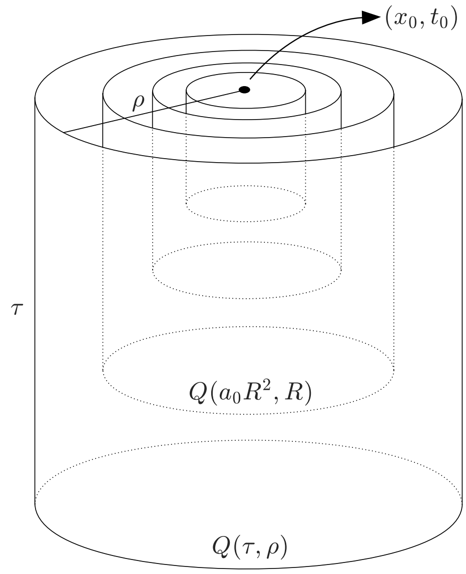


Figure 2.1: $(x_0, t_0)$ := vertex; $\rho$ := radius; $\tau$ := height; $a_0$ :=scaling factor

and in showing that the essential oscillation of the solution in those cylinders converges to zero as the cylinders shrink to the point. In this regard, it is usual to prove that the oscillation is reduced by a factor $\sigma \in (0,1)$ in each iteration, i.e., $\omega_{n+1} \leq \sigma\omega_n$, where $\omega$ is the oscillation in the nth cylinder, which is a method called reduction of oscillation. Lastly, through the approach mentioned above we establishes Hölder continuity of nonnegative solutions of the degenerate parabolic equation, PME. The results that we do not demonstrate will contain references where the proofs may be found. Standard references for the material presented here are [21], [45] and [51].

## 2.1 Intrinsic scaling

In the late $1950's$, De Giorgi [16] developed a method to study the regularity of uniformly elliptic linear equations, which late was adapted by the Russian school in [36] and [37] to study the regularity in the linear parabolic case. It was only in the mid $1980's$ that DiBenedetto [21], generalizing the method developed by De Giorgi, introduced the Intrinsic Scaling Method to the study of regularity of solutions of quasi-linear equations of type

$$u_t - \text{div } \mathbf{a}(x,t,u,\nabla u) = 0.$$

We will consider in this section the equation

$$u_t - \text{div}(mu^{m-1}\nabla u) = 0, \quad m > 1. \tag{2.1}$$

The equation above is the homogeneous prototype of the porous medium equation, and its modulus of ellipticity is $u^{m-1}$. If $m > 1$ such a modulus vanishes whenever $u = 0$ and at such points the equation is degenerate. Note that in the case $m < 1$ the modulus of ellipticity becomes infinity whenever $u = 0$ and the equation is singular at these points.

The idea behind the method of intrinsic scaling is to perform this iterative process in cylinders that reflect the structure of the equation. That is to say, we consider cylinders presenting a change of scale, with respect to the standard parabolic cylinders, that are redefined in terms of scaling factors that take into account the nature of the degeneracy or singularity, and depend on the oscillation of the solution itself (thus, the term intrinsic). This method allows us to show the Hölder continuity of the weak solutions of the equation.

### 2.1.1 Local energy and logarithmic estimates

In this subsection we will consider the equation (2.1), which is the homogeneous prototype of the porous medium equation presented in the previous chapter (section 2.2), and then deduce the local energy and logarithmic estimates. The main goal with these estimates, which we should do in the next subsection, is to show that every bounded weak solution of the equation is locally Hölder continuous in $E_T$. This is a consequence of the following fact:

If, for every $(x_0, t_0) \in E_T$, it is possible to define a decreasing sequence of nested cylinders $(x_0, t_0) + Q(\tau_n, \rho_n)$ such that the central oscillation $\omega_n$ of the weak solution tends to zero in the cylinders when the cylinders converge to $(x_0, t_0)$, then the function $(x, t) \mapsto u(x, t)$ can be modified in a set with measure zero in such a way that we can take a continuous representative in the equivalence class.

We start by making some considerations.

Let $u$ be a bounded weak solution in $E_T$ and let

$$M := \|u\|_{L^\infty(E_T)}.$$

For $(x_0, t_0) \in E_T$, we consider the cylinder

$$(x_0, t_0) + Q(\tau, \rho)$$

where $\tau$ and $\rho$ are chosen in such a way that

$$(x_0, t_0) + Q(\tau, \rho) \subset E_T$$

and denote the cylinders with vertex at the origin $(0, 0)$ by

$$Q(\tau, \rho) := (0, 0) + Q(\tau, \rho),$$

and put $K_\rho := K_\rho(0)$. Let $0 \leq \xi \leq 1$ be a piecewise smooth cutoff function in $Q(\tau, \rho)$ such that

$$|\nabla \xi| < \infty \quad \text{and} \quad \xi(x, t) = 0, \; x \in K_\rho. \tag{2.2}$$

We use the usual notations for the positive and negative parts of a function:

$$f_+ = \max(f, 0) \qquad \text{and} \qquad f_- = \max(-f, 0),$$

and we will consider the auxiliary function $u_\pm^l = \pm\min\{\pm u, \pm l\}$.

From the above considerations, we have the following estimate.

**Proposition 2.1.1.** *Let $u$ be a local weak solution of* (2.1) *in $E_T$ and $k, l \in \mathbb{R}^+$. There exists a constant $C \equiv C(m) > 0$ such that, for every cylinder $Q(\tau,\rho) \subset E_T$,*

$$\begin{aligned}
& \operatorname*{ess\,sup}_{-\tau<t<0} \int_{K_\rho \times \{t\}} (u^l_\pm - k)^2_\pm \xi^2 dx \\
+ & \int_{-\tau}^0 \int_{K_\rho} (u^l_\pm)^{m-1} |\nabla (u^l_\pm - k)_\pm \xi|^2 dxdt \\
\leq & \; C \int_{-\tau}^0 \int_{K_\rho} (u^l_\pm)^{m-1} (u^l_\pm - k)^2_\pm |\nabla \xi|^2 dxdt \\
+ & \; 2 \int_{-\tau}^0 \int_{K_\rho} \left( (u^l_\pm - k)^2_\pm + 2(l-k)_\pm (u-l)_\pm \right) \xi \xi_t dxdt \\
+ & \; C(l-k)_\pm \int_{-\tau}^0 \int_{K_\rho} \left( \int_l^u s^{m-1} ds \right) (|\nabla \xi|^2 + \xi \Delta \xi) \chi_{\left[u \gtreqless l\right]} dxdt. \qquad (2.3)
\end{aligned}$$

*Proof.* Firstly, consider

$$\phi = \pm((u^l_\pm)_h - k)_\pm \xi^2$$

in (1.20) and integrate in time over $(-\tau, t)$, for $t \in (-\tau, 0)$. Then in the first term of (1.20) we obtain

$$\begin{aligned}
\int_{-\tau}^t \int_{K_\rho} \pm(u_h)_t ((u^l_\pm)_h - k)_\pm \xi^2 dxd\theta \;=\; & \int_{-\tau}^t \int_{K_\rho} \pm(u_h)_t ((u^l_\pm)_h - k)_\pm \xi^2 \chi_{\left[u^l_\pm = u\right]} dxd\theta \\
+ & \int_{-\tau}^t \int_{K_\rho} \pm(u_h)_t ((u^l_\pm)_h - k)_\pm \xi^2 \chi_{\left[u^l_\pm = l\right]} dxd\theta \\
=: & \; A + B.
\end{aligned}$$

Now, to evaluate the term $A$ of the above equality, we define $S^h_\pm = \operatorname{supp}(u_h - k)_\pm$.Then, note that in $S^h_\pm$, $((u_h - k)_\pm)_t = (u_h)_t$, and outside the integrand function is 0. Thus,

$$\int_{-\tau}^t \int_{K_\rho} \frac{1}{2} \left( ((u^l_\pm)_h - k)^2_\pm \right)_t \xi^2 \chi_{\left[u^l_\pm = u\right]} dxd\theta = \int_{-\tau}^t \int_{K_\rho} \frac{1}{2} \left( ((u^l_\pm)_h - k)^2_\pm \right)_t \xi^2 dxd\theta$$

and so

$$\begin{aligned}
\int_{-\tau}^t \int_{K_\rho} \pm(u_h)_t ((u^l_\pm)_h - k)_\pm \xi^2 dxd\theta \;\to\; & \frac{1}{2} \int_{K_\rho \times \{t\}} (u^l_\pm - k)^2_\pm \xi^2 dx \\
- & \int_{-\tau}^0 \int_{K_\rho} (u^l_\pm - k)^2_\pm \xi \xi_t dxdt,
\end{aligned}$$

after integrating by parts and passing to the limit in $h \to 0$(by Lemma 1.1.5). Note that just one term related to the boundary appears because $\xi = 0$ in $K_\rho \times \{-\tau\}$ by definition.

For the term $B$, we have that

$$\int_{-\tau}^{t}\int_{K_\rho} \pm(u_h)_t((u_\pm^l)_h - k)_\pm \xi^2 \chi_{[u_\pm^l = l]} dxd\theta = (l-k)_\pm \int_{-\tau}^{t}\int_{K_\rho} ((u_h - l)_\pm)_t \xi^2 dxd\theta$$

since

$$u_h \chi_{[u_\pm^l = l]} = u_h \chi_{[u_\pm^l \gtreqless l]} = \pm(u_h - l)_\pm + l\chi_{[u_\pm^l = l]}.$$

Thus, integrating by parts and passing to the limit in $h \to 0$ again, we get that

$$\begin{aligned}
& \int_{-\tau}^{t}\int_{K_\rho} \pm(u_h)_t((u_\pm^l)_h - k)_\pm \xi^2 \chi_{[u_\pm^l = l]} dxd\theta \\
\to \quad & (l-k)_\pm \int_{K_\rho \times \{t\}} (u-l)_\pm \xi^2 dx - 2(l-k)_\pm \int_{-\tau}^{0}\int_{K_\rho} (u-l)_\pm \xi\xi_t dxdt \\
\geq \quad & -2(l-k)_\pm \int_{-\tau}^{0}\int_{K_\rho} (u-l)_\pm \xi\xi_t dxdt.
\end{aligned}$$

Now, we evaluate the second term of (1.20), we first let $h \to 0$ and the we divide it in two integrals as before,

$$\begin{aligned}
mD + mE \quad = \quad & m\int_{-\tau}^{t}\int_{K_\rho} \pm u^{m-1}\nabla u \cdot (\nabla(u_\pm^l - k)_\pm \xi^2 + 2(u_\pm^l - k)_\pm \xi\nabla\xi)\chi_{[u_\pm^l = u]} dxd\theta \\
+ \quad & m\int_{-\tau}^{t}\int_{K_\rho} \pm u^{m-1}\nabla u \cdot (\nabla(u_\pm^l - k)_\pm \xi^2 + 2(u_\pm^l - k)_\pm \xi\nabla\xi)\chi_{[u_\pm^l = l]} dxd\theta.
\end{aligned}$$

Now, to evaluate the term $D$, we note that $\nabla u = \nabla(u-k) = \pm\nabla(u-k)_\pm$, in $S_\pm = \operatorname{supp}(u-k)_\pm$ , and we use Cauchy-Schwarz and Young's inequality ($ab \leq a^2/4 + b^2$), for $a = |\nabla(u_\pm^l - k)_\pm|\xi$ and $b = (u_\pm^l - k)_\pm|\nabla\xi|$. Thus, we get

$$\begin{aligned}
D \quad = \quad & \int_{-\tau}^{t}\int_{K_\rho} u^{m-1}\nabla(u-k)_\pm \cdot (\nabla(u_\pm^l - k)_\pm \xi^2 + 2(u_\pm^l - k)_\pm \xi\nabla\xi)\chi_{[u_\pm^l = u]} dxd\theta \\
\geq \quad & \int_{-\tau}^{t}\int_{K_\rho} (u_\pm^l)^{m-1}|\nabla(u_\pm^l - k)_\pm|^2 \xi^2 \\
- \quad & 2\int_{-\tau}^{t}\int_{K_\rho} (u_\pm^l)^{m-1}|\nabla(u_\pm^l - k)_\pm|(u_\pm^l - k)_\pm \xi|\nabla\xi| dxd\theta \\
\geq \quad & \int_{-\tau}^{t}\int_{K_\rho} (u_\pm^l)^{m-1}|\nabla(u_\pm^l - k)_\pm \xi|^2 \\
- \quad & 2\int_{-\tau}^{t}\int_{K_\rho} (u_\pm^l)^{m-1}\left(\frac{|\nabla(u_\pm^l - k)_\pm \xi|^2}{4} + (u_\pm^l - k)_\pm^2 |\nabla\xi|^2\right) dxd\theta \\
= \quad & \frac{1}{2}\int_{-\tau}^{t}\int_{K_\rho} (u_\pm^l)^{m-1}|\nabla(u_\pm^l - k)_\pm \xi|^2 - 2\int_{-\tau}^{t}\int_{K_\rho} (u_\pm^l)^{m-1}(u_\pm^l - k)_\pm^2 |\nabla\xi|^2 dxd\theta.
\end{aligned}$$

For the term $E$, we have that

$$\begin{aligned} E &= \pm\int_{-\tau}^{t}\int_{K_\rho} u^{m-1}\nabla u\cdot(\nabla(u^l_\pm-k)_\pm\xi^2+2(u^l_\pm-k)_\pm\xi\nabla\xi)\chi_{[u^l_\pm=l]}dxd\theta \\ &= \pm 2(l-k)_\pm\int_{-\tau}^{t}\int_{K_\rho}\nabla\left(\int_l^u s^{m-1}ds\right)\cdot\xi\nabla\xi\chi_{[u\gtreqless l]}dxd\theta \end{aligned}$$

since

$$\nabla\left(\int_l^u s^{m-1}ds\right)\chi_{[u\geq l]}=u^{m-1}\nabla u\chi_{[u\geq l]}$$

and

$$\nabla\left(\int_l^u s^{m-1}ds\right)\chi_{[u\leq l]}=-\nabla\left(\int_u^l s^{m-1}ds\right)\chi_{[u\leq l]}=u^{m-1}\nabla u\chi_{[u\leq l]}.$$

Thus,

$$E=\mp 2(l-k)_\pm\int_{-\tau}^{t}\int_{K_\rho}\left(\int_l^u s^{m-1}ds\right)(|\nabla\xi|^2+\xi\Delta\xi)\chi_{[u\gtreqless l]}dxd\theta$$

after integrating by parts.

Since $t\in(\tau,0)$ is arbitrary, we can combine estimates $A, B, D$ and $E$ to obtain

$$\begin{aligned} &\frac{1}{2}\int_{K_\rho\times\{t\}}(u^l_\pm-k)^2_\pm\xi^2dx-\int_{-\tau}^{0}\int_{K_\rho}(u^l_\pm-k)^2_\pm\xi\xi_tdxdt \\ -\ & 2(l-k)_\pm\int_{-\tau}^{0}\int_{K_\rho}(u-l)_\pm\xi\xi_tdxdt+\frac{m}{2}\int_{-\tau}^{t}\int_{K_\rho}(u^l_\pm)^{m-1}|\nabla(u^l_\pm-k)_\pm\xi|^2 \\ -\ & 2m\int_{-\tau}^{t}\int_{K_\rho}(u^l_\pm)^{m-1}(u^l_\pm-k)^2_\pm|\nabla\xi|^2dxd\theta \\ \mp\ & 2(l-k)_\pm\int_{-\tau}^{t}\int_{K_\rho}\left(\int_l^u s^{m-1}ds\right)(|\nabla\xi|^2+\xi\Delta\xi)\chi_{[u\gtreqless l]}dxd\theta, \end{aligned}$$

and, from this, follows (2.3). □

**Remark 2.1.2.** *Note that there is a certain difference between the energy estimates presented in [19, 51] and the estimate presented in our work, where the term $u^l_\pm$ appears. This follows from the regularity proof requiring a double truncation, above and below certain levels. This is due to the fact that the equation is degenerate at points where $u = 0$, which does not happen for the p-Laplacian, where the degeneracy is at the points where the gradient cancels out and not where the function vanishes.*

Now we will present the logarithmic estimates in cylinders $Q(\tau,\rho)$ with vertex at the origin.

Let $u$ be a bounded function in a cylinder and a number k, define the constant

$$H_{u,k}^{\pm} := \operatorname*{ess\ sup}_{Q(\tau,\rho)} |(u-k)_{\pm}|.$$

Now consider the following function

$$\Psi^{\pm}\left(H_{u,k}^{\pm},(u-k)_{\pm},c\right) \equiv \psi_{H_{u,k}^{\pm},k,c}(u), \quad 0<c<H_{u,k}^{\pm}, \tag{2.4}$$

where $\psi_{H_{u,k}^{\pm},k,c}$ is the logarithmic function introduced in the section 1.1. For simplify the notation of the function in (2.4) we will write from now on such function as $\psi^{\pm}(u)$.

Let $\xi$ be a time-independent cutoff function in $K_\rho$ satisfying (2.2).

From the above considerations, we have the following logarithmic estimate.

**Proposition 2.1.3.** *Let $u$ be a local weak solution of* 2.1 *and $k\in\mathbb{R}$. There exists a constant $C\equiv C(m)>0$ such that, for every cylinder $Q(\tau,\rho)\subset E_T$,*

$$\begin{aligned} \sup_{-\tau\le t\le 0}\int_{K_\rho\times\{t\}}\left[\psi^{\pm}(u)\right]^2\xi^2dx &\le \int_{K_\rho\times\{-\tau\}}\left[\psi^{\pm}(u)\right]^2\xi^2dx \\ &+ C\int_{-\tau}^{0}\int_{K_\rho} u^{m-1}\psi^{\pm}(u)|\nabla\xi|^2dxdt. \end{aligned} \tag{2.5}$$

*Proof.* Firstly, consider

$$\phi = 2\psi^{\pm}(u_h)\left[(\psi^{\pm})'(u_h)\right]\xi^2$$

in (1.20) and integrate in time over $(-\tau,t)$, for $t\in(-\tau,0)$. Since $\xi_t\equiv 0$, then in the first term of (1.20) we obtain

$$\begin{aligned} \int_{-\tau}^{t}\int_{K_\rho}(u_h)_t\varphi dxd\theta &= \int_{-\tau}^{t}\int_{K_\rho}(u_h)_t 2\psi^{\pm}(u_h)\left[(\psi^{\pm})'(u_h)\right]\xi^2dxd\theta \\ &= \int_{-\tau}^{t}\int_{K_\rho}(u_h)_t\left(\left[\psi^{\pm}(u_h)\right]^2\right)'\xi^2dxd\theta \\ &= \int_{-\tau}^{t}\int_{K_\rho}\left(\left[\psi^{\pm}(u_h)\right]^2\right)_t\xi^2dxd\theta \\ &= \int_{K_\rho\times\{t\}}\left[\psi^{\pm}(u_h)\right]^2\xi^2 - \int_{K_\rho\times\{-\tau\}}\left[\psi^{\pm}(u_h)\right]^2\xi^2dx. \end{aligned}$$

From this, letting $h\to 0$,

$$\begin{aligned} \int_{-\tau}^{t}\int_{K_\rho}(u_h)_t\varphi dxd\theta &= \int_{-\tau}^{t}\int_{K_\rho}(u_h)_t 2\psi^{\pm}(u_h)\left[(\psi^{\pm})'(u_h)\right]\xi^2dxd\theta \\ &\to \int_{K_\rho\times\{t\}}\left[\psi^{\pm}(u)\right]^2\xi^2 - \int_{K_\rho\times\{-\tau\}}\left[\psi^{\pm}(u)\right]^2\xi^2. \end{aligned}$$

We now will estimate the second term, we first let $h \to 0$, and then we use Cauchy Schwarz,

$$
\begin{aligned}
& m \int_{-\tau}^{t} \int_{K_\rho} u^{m-1} \nabla u \cdot \nabla (2\psi^{\pm}(u) \left[(\psi^{\pm})'(u)\right] \xi^2) dx d\theta \\
= \;& m \int_{-\tau}^{t} \int_{K_\rho} u^{m-1} \nabla u \cdot \nabla u 2 \left( \left[(\psi^{\pm})'(u)\right]^2 + \psi^{\pm}(u) \left[(\psi^{\pm})'(u)\right]^2 \right) \xi^2 dx d\theta \\
+ \;& m \int_{-\tau}^{t} \int_{K_\rho} u^{m-1} \nabla u \cdot 4\psi^{\pm}(u) \left[(\psi^{\pm})'(u)\right] \xi \nabla \xi dx d\theta \\
\geq \;& m \int_{-\tau}^{t} \int_{K_\rho} u^{m-1} |\nabla u|^2 \left( 2(1 + \psi^{\pm}(u)) \left[(\psi^{\pm})'(u)\right]^2 \xi^2 \right) \\
- \;& 2m \int_{-\tau}^{t} \int_{K_\rho} u^{m-1} |\nabla u| |\nabla \xi| 2\psi^{\pm}(u) |(\psi^{\pm})'(u)| \xi dx d\theta \\
\geq \;& m \int_{-\tau}^{t} \int_{K_\rho} u^{m-1} |\nabla u|^2 \left( 2(1 + \psi^{\pm}(u) - \psi^{\pm}(u)) \left[(\psi^{\pm})'(u)\right]^2 \xi^2 \right) dx d\theta \\
- \;& m \int_{-\tau}^{t} \int_{K_\rho} u^{m-1} \psi^{\pm}(u) |\nabla \xi|^2 dx d\theta.
\end{aligned}
$$

In the last inequality, the Young's inequality was used,

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2},$$

with $a = |\nabla \xi| |(\psi^{\pm})'(u)|^{-1}$, and $b = |\nabla u \xi|$.

Combining the estimates for the two terms and using that $u^{m-1} \geq 0$ we obtain

$$
\begin{aligned}
\int_{K_\rho \times \{t\}} \left[\psi^{\pm}(u)\right]^2 \xi^2 dx \;&-\; \int_{K_\rho \times \{-\tau\}} \left[\psi^{\pm}(u)\right]^2 \xi^2 dx \\
&-\; m \int_{-\tau}^{t} \int_{K_\rho} u^{m-1} \psi^{\pm}(u) |\nabla \xi|^2 dx d\theta \\
&\leq\; 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\underset{-\tau < t < 0}{sup} \int_{K_\rho \times \{t\}} \left[\psi^{\pm}(u)\right]^2 \xi^2 dx \;&\leq\; \int_{K_\rho \times \{-\tau\}} \left[\psi^{\pm}(u)\right]^2 \xi^2 dx \\
&+\; C \int_{-\tau}^{0} \int_{K_\rho} u^{m-1} \psi^{\pm}(u) |\nabla \xi|^2 dx dt,
\end{aligned}
$$

where $C = C(m) > 0$. □

**Remark 2.1.4.** *The function* (2.4) *has been used as a recurrent tool in the proof of results concerning the local behaviour of solutions of degenerate and singular equations for more details on this feature see [18], work on which it was presented for the first time.*

**Remark 2.1.5.** *We admit during the results presented in this section that the solution $u$ is non negative, but this restriction is not necessary. In the case where we do not require this restriction for solution, we have the so-called signed $PME$, and the energy and logarithmic estimates are still valid considering the constants $l$ and $k$, possibly negative and by changing $(u_{\pm}^{l})^{m-1}$ to $|(u_{\pm}^{l})|^{m-1}$ in energy estimate, and $u^{m-1}$ to $|u|^{m-1}$ in logarithmic estimate.*

### 2.1.2 Reduction of the oscillation

In this subsection, we present the Intrinsic Scaling Method for the degenerate case. This method allows us, in a heuristic way, to say that the equation (2.1) behaves, in its own geometry, as the Heat Equation.

The standard parabolic cylinders

$$(x_0, t_0) + Q\left(R^2, R\right)$$

reflect the natural homogeneity between the space and time variables for the heat equation,

$$u_t - \Delta u = 0.$$

In fact, if $u(x,t)$ is a solution, then $u(\lambda x, \lambda 2t), \lambda \in \mathbb{R}$, is also a solution, i.e., the equation remains invariant through a similarity transformation of the space-time variables that leaves constant the ratio

$$\frac{|x|^2}{t}.$$

For solutions of most degenerate or singular equations, the energy and logarithmic estimates are not homogeneous because they involve integral norms corresponding to different powers, namely $u^{m-1}$, in our case. For overcome about this difficulty, the equation has to be analyzed in a geometry dictated by its own degenerate structure. This amounts to rescale the standard parabolic cylinders by a factor that depends on the oscillation of the solution. This procedure of intrinsic scaling, which can be seen as an accommodation of the degeneracy, allows for the restoration of the homogeneity in the energy estimates, when written over the rescaled cylinders. Let's make this idea precise.

**Remark 2.1.6.** *Note that we can recast the equation,*

$$u_t - \Delta u^m = 0,$$

*in the form*

$$\frac{u^{1-m}}{m} u_t - \Delta u = 0, \quad u > 0.$$

*Basically, this equation is saying to us that it is possible, by using a scaling factor, to recover the homogeneity of the function. Although this scaling factor is dependent of the solution itself, we still have an idea of how to construct the rescaled cylinders.*

Firstly, we fix a point $(x_0, t_0) \in E_T$, without loss of generality we will take $(x_0, t_0) = (0, 0)$, as was assumed before, and consider the cylinder

$$Q\left(4R^{2-\epsilon}, 2R\right) \subset E_T, \quad 0 < \epsilon < 1,$$

where $0 < R < 1$ is taken such that the inclusion holds, and define the essential oscillation of the solution $u$ within this cylinder

$$\omega := \operatorname*{ess\,osc}_{Q(4R^{2-\epsilon},2R)} u = \mu^+ - \mu^-,$$

where

$$\mu^+ := \operatorname*{ess\,sup}_{Q(4R^{2-\epsilon},2R)} u \quad \text{and} \quad \mu^- := \operatorname*{ess\,inf}_{Q(4R^{2-\epsilon},2R)} u.$$

Next, construct the rescaled cylinder

$$Q\left(\omega^{1-m}R^2, R\right) = K_R(0) \times (-\omega^{1-m}R^2, 0), \tag{2.6}$$

we will assume, without loss of generality, that

$$\omega^{m-1} > R^{\epsilon}, \tag{2.7}$$

which implies the inclusion of cylinders

$$Q\left(\omega^{1-m}R^2, R\right) \subset Q\left(4R^{2-\epsilon}, 2R\right)$$

and the relation

$$\operatorname*{ess\,osc}_{Q(\omega^{1-m}R^2,R)} u \leq \omega. \tag{2.8}$$

**Remark 2.1.7.** *Note that if* (2.7) *does not hold, i.e.,*

$$\omega \leq R^{\frac{\epsilon}{m-1}}$$

*then, there is nothing to prove since the oscillation is comparable to the radius.*

**Remark 2.1.8.** *In the case $m = 1$, i.e., in the non-degenerate case, these are the standard parabolic cylinders reflecting the natural homogeneity between the space and time variables.*

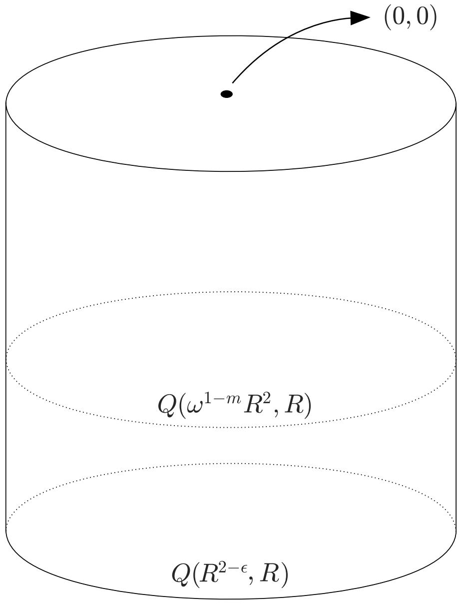


Figure 2.2: $0 < R < 1$; $0 < \epsilon < 1$; $\omega^{m-1} > R^{\epsilon}$

**Remark 2.1.9.** *We note that we could have introduced a scaling with different parameters in the space and the time variables, that is to say that the geometry chosen is not the only possible one.*

Now, we will prove the reduction of oscillation, for this will be necessary to consider two alternatives, precisely, the proof will be divided in two complementary cases: In the first case, $u$ is essentially away from its infimum in $Q\left(\omega^{1-m}R^2, R\right)$, or in second case, $u$ is essentially away from its supremum. We state this in a precise way.

For a constant $\nu_0 \in (0,1)$, that will be determined later, either

**The First Alternative:**

$$\frac{\left|\left\{(x,t) \in Q\left(\omega^{1-m}R^2, R\right) : u(x,t) < \mu^- + \frac{\omega}{2}\right\}\right|}{\left|Q\left(\omega^{1-m}R^2, R\right)\right|} \leq \nu_0 \tag{2.9}$$

or this does not hold, i.e,

$$\frac{\left|\left\{(x,t) \in Q\left(\omega^{1-m}R^2, R\right) : u(x,t) \geq \mu^- + \frac{\omega}{2}\right\}\right|}{\left|Q\left(\omega^{1-m}R^2, R\right)\right|} < 1 - \nu_0.$$

Then, since $\mu^+ - \frac{\omega}{2} = \mu^- + \frac{\omega}{2}$, it holds

**The Second Alternative:**

$$\frac{\left|\left\{(x,t) \in Q\left(\omega^{1-m}R^2, R\right) : u(x,t) > \mu^+ - \frac{\omega}{2}\right\}\right|}{\left|Q\left(\omega^{1-m}R^2, R\right)\right|} < 1 - \nu_0. \tag{2.10}$$

We start the analysis assuming the first alternative holds.

**Proposition 2.1.10.** *Assume*

$$\mu^- < \frac{\omega}{4}$$

*is true. If the first alternative* (2.9) *holds, then*

$$u(x,t) > \mu^- + \frac{\omega}{4} \quad a.e. \quad in \quad Q\left(\omega^{1-m}\left(\frac{R}{2}\right)^2, \frac{R}{2}\right). \tag{2.11}$$

*Proof.* First, consider the sequence

$$R_n = \frac{R}{2} + \frac{R}{2^{n+1}}, \quad n = 0, 1, ..., \tag{2.12}$$

that converges to $R/2$, and construct the family of nested and shrinking cylinders $Q\left(\omega^{1-m}R_n^2, R_n\right)$. Next, take piecewise smooth cutoff functions $0 \le \xi_n \le 1$, defined in these cylinders, and satisfying the following set of assumptions:

$$\begin{aligned} &\xi_n = 1 \quad in \quad Q\left(\omega^{1-m}R_{n+1}^2, R_{n+1}\right); \quad \xi_n = 0 \quad on \quad \partial Q\left(\omega^{1-m}R_n^2, R_n\right); \\ &|\nabla \xi_n| \le \frac{2^{n-1}}{R}; \qquad 0 \le (\xi_n)_t \le \frac{2^{2n-2}}{R^2}\omega^{m-1}; \qquad \Delta \xi_n \le \frac{2^{2n-2}}{R^2}. \end{aligned}$$

Now, we will write the energy inequality, developed in the previous section, over the cylinders $Q\left(\omega^{1-m}R_n^2, R_n\right)$, for functions $(u_-^l - k_n)_-$, with

$$l = \mu^- + \frac{\omega}{4}, \quad k_n = \mu^- + \frac{\omega}{4} + \frac{\omega}{2^{n+2}}, \quad n = 0, 1, ...,$$

and $\xi = \xi_n$. Since

$$u_-^l = \max\left\{u, \mu^- + \frac{\omega}{4}\right\} \ge \mu^- + \frac{\omega}{4} \ge \frac{\omega}{4},$$

and the following properties

(i) $0 \le \mu^- \le \frac{\omega}{4}$, so $u \le \frac{5\omega}{4}$ and $l \le \frac{\omega}{2}$, which implies that $u_-^l \le \frac{5\omega}{4}$;

(ii) $l = \mu^- + \frac{\omega}{4} < k_n$, so $\chi_{[u \le l]} \le \chi_{[u < k_n]} = \chi_{[(u-k_n)_- > 0]}$;

(iii) Where $u_-^l = u$, we have $\chi_{[(u-k_n)_- > 0]} = \chi_{\left[(u_-^l - k_n)_- > 0\right]}$, but even when $u_-^l = l$, i.e., $u \le l < k_n$, the same result holds: $\chi_{[(u-k_n)_- > 0]} = 0 = \chi_{[(l-k_n)_- > 0]}$. Then, we have that $\chi_{[(u-k_n)_- > 0]} = \chi_{\left[(u_-^l - k_n)_- > 0\right]}$ for all $(x,t)$;

(iv) $(l - k_n)_- = \frac{\omega}{2^{n+2}} \le \frac{\omega}{2}, (u_-^l - k_n)_- \le \frac{\omega}{2^{n+2}} \le \frac{\omega}{2}$ and $(u - l)_- \le \frac{\omega}{4}$

are satisfied for our functions and constants, with $\chi_E$ denotes the characteristic function of the set $E$, then, using the energy estimate with the above considerations, we obtain the following inequalities

(I)

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{-\frac{R_n^2}{\omega^{m-1}}<t<0} \int_{K_{R_n}\times\{t\}} (u_-^l - k_n)_-^2 \xi_n^2 dx \\
&+ \int_{-\frac{R_n^2}{\omega^{m-1}}}^0 \int_{K_{R_n}} (u_-^l)^{m-1} |\nabla (u_-^l - k_n)_- \xi_n|^2 dxdt \\
&\geq 4^{1-m} \Bigg( \operatorname*{ess\,sup}_{-\frac{R_n^2}{\omega^{m-1}}<t<0} \int_{K_{R_n}\times\{t\}} (u_-^l - k_n)_-^2 \xi_n^2 dx \\
&+ \omega^{m-1} \int_{-\frac{R_n^2}{\omega^{m-1}}}^0 \int_{K_{R_n}} |\nabla (u_-^l - k_n)_- \xi_n|^2 dxdt \Bigg);
\end{aligned}
$$

(II)

$$
\begin{aligned}
&\int_{-\frac{R_n^2}{\omega^{m-1}}}^0 \int_{K_{R_n}} (u_-^l)^{m-1} (u_-^l - k_n)_-^2 |\nabla \xi_n|^2 dxdt \\
&\leq C\omega^{m-1} \left(\frac{\omega}{2}\right)^2 \frac{2^{2n-2}}{R^2} \int_{-\frac{R_n^2}{\omega^{m-1}}}^0 \int_{K_{R_n}} \chi_{[(u_-^l - k_n)_- > 0]} dxdt;
\end{aligned}
$$

(III)

$$
\begin{aligned}
&2\int_{-\frac{R_n^2}{\omega^{m-1}}}^0 \int_{K_{R_n}} \left( \left(u_-^l - k_n\right)_-^2 + 2\left(l - k_n\right)_- \left(u - l\right)_- \right) \xi_n (\xi_n)_t dxdt \\
&\leq C\omega^{m-1} \left(\frac{\omega}{2}\right)^2 \frac{2^{2n-2}}{R^2} \int_{-\frac{R_n^2}{\omega^{m-1}}}^0 \int_{K_{R_n}} \chi_{[(u_-^l - k_n)_- > 0]} dxdt;
\end{aligned}
$$

(IV)

$$
\begin{aligned}
&-C(l-k_n)_- \int_{-\frac{R_n^2}{\omega^{m-1}}}^0 \int_{K_{R_n}} \left( \int_l^u s^{m-1} ds \right) \left( |\nabla \xi_n|^2 + \xi_n \Delta \xi_n \right) \chi_{[u\leq l]} dxdt \\
&\leq C\omega^{m-1} \left(\frac{\omega}{2}\right)^2 \frac{2^{2n-1}}{R^2} \int_{-\frac{R_n^2}{\omega^{m-1}}}^0 \int_{K_{R_n}} \chi_{[(u_-^l - k_n)_- > 0]} dxdt.
\end{aligned}
$$

Now, joining the above estimates, we obtain

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{-\frac{R_n^2}{\omega^{m-1}}<t<0} \int_{K_{R_n}\times\{t\}} (u_-^l - k_n)_-^2 \xi_n^2 dx + \omega^{m-1} \int_{-\frac{R_n^2}{\omega^{m-1}}}^0 \int_{K_{R_n}} |\nabla (u_-^l - k)_- \xi_n|^2 dxdt \\
&\leq \omega^{m-1} \left(\frac{\omega}{2}\right)^2 \left( C\frac{2^{2n-2}}{R^2} + \frac{2^{2n-2}}{R^2} + C\frac{2^{2n-1}}{R^2} \right) \int_{-\frac{R_n^2}{\omega^{m-1}}}^0 \int_{K_{R_n}} \chi_{\left[(u_-^l - k_n)_- > 0\right]} dxdt \\
&\leq C\omega^{m-1} \left(\frac{\omega}{2}\right)^2 \frac{2^{2n}}{R^2} \int_{-\frac{R_n^2}{\omega^{m-1}}}^0 \int_{K_{R_n}} \chi_{\left[(u_-^l - k_n)_- > 0\right]} dxdt.
\end{aligned}
$$

In the sequel, putting $\bar{t} = \omega^{m-1}t$, and defining

$$\overline{u}_-^l(\cdot,\bar{t}) := u_-^l(\cdot,t), \qquad \overline{\xi_n}(\cdot,\bar{t}) := \xi_n(\cdot,t), \tag{2.13}$$

we perform a change in the time variable, where the intrinsic geometric framework place an important rule.

Now, define, for each $n$

$$A_n := \int_{-R_n^2}^0 \int_{K_{R_n}} \chi_{\left[(\overline{u}_-^l - k_n)_- > 0\right]} dx d\bar{t}.$$

Thus, due to (2.13) we obtain the last simplified inequality

$$\|(\overline{u}_-^l - k_n)_- \overline{\xi_n}\|^2_{V^2(Q(R_n^2, R_n))} \leq C \frac{2^{2n}}{R^2} \left(\frac{\omega}{2}\right)^2 A_n. \tag{2.14}$$

Furthermore,

$$\frac{1}{2^{2(n+2)}} \left(\frac{\omega}{2}\right)^2 A_{n+1} \leq C \frac{2^{2n}}{R^2} \left(\frac{\omega}{2}\right)^2 A_n^{1+\frac{2}{N+2}}. \tag{2.15}$$

Indeed, from the definition of $A_n$, the fact that $k_{n+1} < k_n$ and of the Theorem 1.1.3 (for $p = 2$), we have that

$$\begin{aligned}
\frac{1}{2^{2(n+2)}} \left(\frac{\omega}{2}\right)^2 A_{n+1} &\leq |k_n - k_{n+1}|^2 A_{n+1} \\
&= \int_{-R_{n+1}^2} \int_{K_{R_{n+1}}} (k_n - k_{n+1})^2 \chi_{\left[(\overline{u}_-^l - k_{n+1})_- > 0\right]} dx dt \\
&\leq \int_{-R_{n+1}^2} \int_{K_{R_{n+1}}} (k_n - \overline{u}_-^l)^2 \chi_{\left[(\overline{u}_-^l - k_n)_- > 0\right]} dx dt \\
&\leq \|(\overline{u}_-^l - k_n)_-\|^2_{2,Q(R_{n+1}^2, R_{n+1})} \\
&\leq \|(\overline{u}_-^l - k_n)_- \overline{\xi_n}\|^2_{2,Q(R_n^2, R_n)} \\
&\leq C \|(\overline{u}_-^l - k_n)_- \overline{\xi_n}\|^2_{V^2(Q(R_n^2, R_n))} A_n^{\frac{2}{N+2}} \\
&\leq C \frac{2^{2n}}{R^2} \left(\frac{\omega}{2}\right)^2 A_n^{1+\frac{2}{N+2}},
\end{aligned}$$

the last equality is due to (2.14).

To conclude, define the numbers

$$X_n = \frac{A_n}{|Q(R_n^2, R_n)|}. \tag{2.16}$$

Then, dividing (2.15) by $|Q(R_{n+1}^2, R_{n+1})|$, $|Q(R_{n+1}^2, R_{n+1})| = R_{n+1}^{N+2} < R^{N+2}$, we obtain

$$
\begin{aligned}
\frac{A_{n+1}}{|Q(R_{n+1}^2, R_{n+1})|} &\leq C\frac{2^{4n+4}}{R^2}|Q(R_{n+1}^2, R_{n+1})|^{\frac{2}{N+2}}\left(\frac{A_n}{|Q(R_{n+1}^2, R_{n+1})|}\right)^{1+\frac{2}{N+2}} \\
&\leq C\frac{2^{4n+4}}{R^2}R^2\left(\frac{|Q(R_n^2, R_n)|}{|Q(R_{n+1}^2, R_{n+1})|}\frac{A_n}{|Q(R_n^2, R_n)|}\right)^{1+\frac{2}{N+2}} \\
&\leq C4^{2n}\left(\frac{A_n}{|Q(R_n^2, R_n)|}\right)^{1+\frac{2}{N+2}},
\end{aligned}
$$

and therefore, we have the following recursive relation (by (2.16))

$$X_{n+1} \leq C4^{2n}X_n^{1+\frac{2}{N+2}},$$

for a constant $C$ depending only upon $N$ and $m$. If,

$$X_0 \leq C^{-\frac{N+2}{2}}4^{-\frac{(N+2)^2}{2}} := \nu_0, \tag{2.17}$$

we have by the Lemma 1.1.4 on fast geometric convergence that

$$X_n \to 0. \tag{2.18}$$

But (2.17) is precisely our hypothesis (2.9), for the indicated choice of $\nu_0$, and from (2.18) we immediately obtain, returning to the original variables,

$$\left|\left\{(x,t) \in Q\left(\omega^{1-m}\left(\frac{R}{2}\right)^2, \frac{R}{2}\right) : u_-^l(x,t) \leq \mu^- + \frac{\omega}{4}\right\}\right| = 0$$

which is equivalent to

$$u_-^l > \mu^- + \frac{\omega}{4} \quad \text{a.e. in} \quad Q\left(\omega^{1-m}\left(\frac{R}{2}\right)^2, \frac{R}{2}\right). \tag{2.19}$$

Note that if $u_-^l > \mu^- + \frac{\omega}{4}$, then, $u_-^l = u$, and thus follows the result. $\square$

**Remark 2.1.11.** *The constant $\nu_0$, that appears in the formulation of the alternative, is now fixed by* (2.17)*.*

We finally reach the conclusion of the first alternative, namely the reduction of the oscillation.

**Corollary 2.1.12.** *Assume*

$$\mu^- < \frac{\omega}{4}$$

*is true. If the first alternative* (2.9) *holds, then there exists a constant $\sigma_0 \in (0,1)$, depending only on the data, such that*

$$\operatorname*{ess\,osc}_{Q\left(\omega^{1-m}\left(\frac{R}{2}\right)^2, \frac{R}{2}\right)} u \leq \sigma_0\omega. \tag{2.20}$$

*Proof.* By Proposition 2.1.10,

$$\operatorname*{ess\,inf}_{Q\left(\omega^{1-m}\left(\frac{R}{2}\right)^2,\frac{R}{2}\right)} u \geq \mu^- + \frac{\omega}{4},$$

and thus

$$\begin{aligned}
\operatorname*{ess\ osc}_{Q\left(\omega^{1-m}\left(\frac{R}{2}\right)^2,\frac{R}{2}\right))} u &= \operatorname*{ess\,sup}_{Q\left(\omega^{1-m}\left(\frac{R}{2}\right)^2,\frac{R}{2}\right)} u - \operatorname*{ess\,inf}_{Q\left(\omega^{1-m}\left(\frac{R}{2}\right)^2,\frac{R}{2}\right)} u \\
&\leq \mu^+ - \mu^- - \frac{\omega}{4} \\
&= \frac{3}{4}\omega.
\end{aligned}$$

This way, the corollary follows with

$$\sigma_0 = \frac{3}{4}.$$

□

Now, we will make an analysis to the case in which the second alternative happens.

The second alternative is somehow the unfavorable case but we will show that a conclusion similar to the corollary 2.1.12 can still be reached. Recall that the constant $\nu_0$ has already been quantitatively determined by (2.17), and it is now fixed.

First, note that, if (2.10) holds, i.e.,

$$\frac{\left|\left\{(x,t) \in Q\left(\omega^{1-m}R^2, R\right) : u(x,t) > \mu^+ - \frac{\omega}{2}\right\}\right|}{\left|Q\left(\omega^{1-m}R^2, R\right)\right|} < 1 - \nu_0. \tag{2.21}$$

Then there exists a time level

$$t_0 \in \left[-\omega^{1-m}R^2, -\frac{\nu_0}{2}\omega^{1-m}R^2\right]$$

such that

$$\left|\left\{x \in K_R : u(x,t_0) > \mu^+ - \frac{\omega}{2}\right\}\right| \leq \left(\frac{1-\nu_0}{1-\nu_0/2}\right)|K_R|. \tag{2.22}$$

Otherwise, for all $t \in \left[-\omega^{1-m}R^2, -\frac{\nu_0}{2}\omega^{1-m}R^2\right]$, we would have

$$\begin{aligned}
&\left|\left\{(x,t) \in Q\left(\omega^{1-m}R^2, R\right) : u(x,t) > \mu^+ - \frac{\omega}{2}\right\}\right| \\
\geq\ & \int_{-\frac{R^2}{\omega^{m-1}}}^{-\frac{\nu_0}{2}\frac{R^2}{\omega^{m-1}}} \left|\left\{x \in K_R : u(x,\tau) > \mu^+ - \frac{\omega}{2}\right\}\right| d\tau \\
>\ & \left(-\frac{\nu_0}{2}\frac{R^2}{\omega^{m-1}} + \frac{R^2}{\omega^{m-1}}\right)\left(\frac{1-\nu_0}{1-\nu_0/2}\right)|K_R| \\
=\ & (1-\nu_0)\left|Q\left(\omega^{1-m}R^2, R\right)\right|,
\end{aligned}$$

which contradicts (2.10).

Now, we will present a lemma that extends what was observed above, specifically asserts that the set where $u(\cdot,t)$ is close to its supremum is small, not only at the specific time level $t_0$, but in $\left[-\frac{\nu_0}{2}\omega^{1-m}R^2,0\right]$.

**Lemma 2.1.13.** *Assume* (2.10) *and suppose that*

$$\mu^- < \frac{\omega}{4}$$

*holds. There exists* $s\in\mathbb{R}$*, depending only on the data, such that*

$$\left|\left\{x\in K_R : u(x,t) > \mu^+ - \frac{\omega}{2^q}\right\}\right| \leq \left(1-\left(\frac{\nu_0}{2}\right)^2\right)|K_R|,$$

*for all* $t\in\left[-\frac{\nu_0}{2}\omega^{1-m}R^2,0\right]$.

*Proof.* The proof this lemma consists in using the logarithmic inequalities (2.5) applied to the function $(u-k)_+$ in the cylinder $Q(-t_0,R)$, with

$$k=\mu^+-\frac{\omega}{2} \qquad \text{and} \qquad c=\frac{\omega}{2^{n+1}},$$

where $n\in\mathbb{N}$ will be chosen later. Since we can assume that

$$H^+_{u,k} := \operatorname*{ess\,sup}_{x\in Q(R,t_0)}\left|\left(u-\mu^++\frac{\omega}{2}\right)_+\right| > \frac{w}{4} \geq \frac{\omega}{2^{n+1}};$$

we can apply the logarithmic estimate with this constants. Otherwise, by choosing $q=2$, the lemma would be trivial.

Now, note that by the logarithmic inequality (2.5) we have

$$\begin{aligned}\sup_{t_0\leq t\leq 0}\int_{K_R\times\{t\}}[\psi^+(u)]^2\xi^2dx &\leq \int_{K_R\times\{t_0\}}[\psi^+(u)]^2\xi^2dx\\ &+ C\int_{t_0}^0\int_{K_R}\omega^{m-1}\psi^+(u)|\nabla\xi|^2dxdt, \qquad (2.23)\end{aligned}$$

since $u\leq\frac{5}{4}\omega$.

Let us now recall that $\psi^+(u)$ is defined in the whole cylinder $Q(-t_0,R)$, and it is given by

$$\psi^+_{\{H^+_{u,k},k,\frac{\omega}{2^{n+1}}\}}(u) = \begin{cases}\ln\left(\frac{H^+_{u,k}}{H^+_{u,k}-u+k+\frac{\omega}{2^{n+1}}}\right) & \text{if } u>k+\frac{\omega}{2^{n+1}},\\ 0 & \text{if } u\leq k+\frac{\omega}{2^{n+1}}.\end{cases}$$

Moreover, in this cylinder, we have that we have that

$$u-k\leq H^+_{u,k}\leq\frac{\omega}{2}, \qquad (2.24)$$

which implies that

$$\begin{aligned}
\psi^+ &\leq \ln\left(\frac{H^+_{u,k}}{H^+_{u,k} - u + k + \frac{\omega}{2^{n+1}}}\right) \\
&\leq \ln\left(\frac{\frac{\omega}{2}}{\frac{\omega}{2^{n+1}}}\right) \\
&= n\ln(2). \qquad (2.25)
\end{aligned}$$

To conclude, we will choose our cutoff function $0 \leq \xi(x) \leq 1$ defined in $K_R$ such that, for some $\delta \in (0,1)$,

$$\xi(x) = 1, \text{ for } x \in K_{(1-\delta)R} \quad \text{and} \quad |\nabla\xi| \leq (\delta R)^{-1}.$$

Given the above considerations, we will now to bound the terms on the right side of inequality(2.23). The first term of the right hand side can be bounded above using (2.22) and taking into account that $\psi^+(u) = 0$ on the set

$$\left\{x \in K_R : u(x,\cdot) \leq \mu^+ - \frac{\omega}{2}\right\}.$$

This gives, using also (2.25),which first term is bounded by

$$\int_{K_R\times\{t_0\}} [\psi^+]^2\xi^2 dx \leq n^2\ln(2)^2\left(\frac{1-\nu_0}{1-\nu_0/2}\right)|K_R|.$$

To bound the second term of the right hand side we use the fact that $-t_0 \leq \frac{R^2}{\omega^{m-1}}$ and (2.25),then, we obtain

$$\begin{aligned}
C\int_{t_0}^0\int_{K_R} \omega^{m-1}\psi^+(u)|\nabla\xi|^2 dxdt &\leq Cn\ln(2)\omega^{m-1}(\delta R)^{-2}(-t_0)|K_R| \\
&\leq Cn\omega^{m-1}\frac{1}{\delta^2R^2}\frac{R^2}{\omega^{m-1}}|K_R| \\
&\leq Cn\frac{1}{\delta^2}|K_R|.
\end{aligned}$$

The left hand side of the inequality is estimated below by integrating over the smaller set,

$$S_t = \left\{x \in K_{(1-\delta)R} : u(x,t) > \mu^+ - \frac{\omega}{2^{n+1}}\right\} \subset K_R, \quad t \in (t_0, 0).$$

In this set, we have that $\xi = 1$ and, since

$$-u + k + \frac{\omega}{2^{n+1}} < \frac{\omega}{2^{n+1}} - \frac{\omega}{2} + \frac{\omega}{2^{n+1}} < 0,$$

we have that

$$\frac{H_{u,k}^{+}}{H_{u,k}^{+}-u+k+\frac{\omega}{2^{n+1}}}$$

is a decreasing function of $H_{u,k}^{+}$. Thus, from (2.24),

$$\begin{aligned}
\frac{H_{u,k}^{+}}{H_{u,k}^{+}-u+k+\frac{\omega}{2^{n+1}}} &\geq \frac{\frac{\omega}{2}}{\frac{\omega}{2}-u+k+\frac{\omega}{2^{n+1}}} \\
&> \frac{\frac{\omega}{2}}{\frac{\omega}{2^{n+1}}+\frac{\omega}{2^{n+1}}} \\
&= 2^{n-1}.
\end{aligned}$$

Thus, in $S_t$,

$$[\psi^{+}(u)]^2 \geq [\ln(2^{n-1})]^2 = (n-1)^2(\ln(2))^2,$$

and, from this,

$$\sup_{t_0\leq t\leq 0}\int_{K_R\times\{t\}}[\psi^{+}]^2\xi^2 dx \geq (n-1)^2(\ln(2))^2|S_t|.$$

Combining these three estimates, we arrive at

$$\begin{aligned}
|S_t| &\leq \left(\frac{n}{n-1}\right)^2\left(\frac{1-\nu_0}{1-\nu_0/2}\right)|K_R| + C\frac{n}{(n-1)^2}\frac{1}{\delta^2}|K_R| \\
&\leq \left(\left(\frac{n}{n-1}\right)^2\left(\frac{1-\nu_0}{1-\nu_0/2}\right)+C\frac{1}{n\delta^2}\right)|K_R|.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left|\left\{x\in K_R : u(x,t) > \mu^{+}-\frac{\omega}{2^{n+1}}\right\}\right| &\leq |S_t| + |K_R\backslash K_{(1-\delta)R}| \\
&= |S_t| + 1-(1-\delta)^N|K_R|,
\end{aligned}$$

and therefore

$$\begin{aligned}
&\left|\left\{x\in K_R : u(x,t) > \mu^{+}-\frac{\omega}{2^{n+1}}\right\}\right| \\
\leq\ &\left(\left(\frac{n}{n-1}\right)^2\left(\frac{1-\nu_0}{1-\nu_0/2}\right)+C\frac{1}{n\delta^2}+1-(1-\delta)^N\right)|K_R|,
\end{aligned}$$

for all $t\in(t_0,0)\supset[-\frac{\nu_0}{2}\omega^{1-m}R^2,0]$. Lastly, if we choose $N\delta\leq\frac{3}{8}\nu_0^2$ and $n$ so large that

$$\left(\frac{n}{n-1}\right)^2 \leq \left(1-\frac{\nu_0}{2}\right)(1+\nu_0) \quad \text{and} \quad \frac{C}{n\delta^2}\leq\frac{3}{8}\nu_0^2,$$

our lemma follows with $q=n+1$. $\square$

In fact, we would also state that $u$ is strictly bellow its supremum in a smaller cylinder, $Q\left(\frac{\nu_0}{2}\omega^{1-m}\left(\frac{R}{2}\right)^2, \frac{R}{2}\right)$.

The next result is equivalent to the proposition (2.1.10) for the second alternative.

**Proposition 2.1.14.** *Assume*

$$\mu^- < \frac{\omega}{4}$$

*is true. If the second alternative* (2.10) *holds, then there exists a number* $s_0 > 1$, *independent of* $\omega$, *such that*

$$u(x,t) \leq \mu^+ - \frac{\omega}{2^{s_0}} \quad \forall (x,t) \in Q\left(\frac{\nu_0}{2}\omega^{1-m}\left(\frac{R}{2}\right)^2, \frac{R}{2}\right). \tag{2.26}$$

*Proof.* We start by defining a sequence

$$R_n = \frac{R}{2} + \frac{R}{2^{n+1}}, \quad n = 0, 1, 2, ...,$$

and to construct the family of nested and shrinking cylinders $Q(\omega^{1-m}R_n^2, R_n)$. Next, consider piecewise smooth cutoff functions $0 \leq \xi_n \leq 1$ defined in these cylinders, and satisfying the following set of assumptions:

$$\xi_n = 1 \text{ in } Q\left(\nu_0 \frac{R_{n+1}^2}{2\omega^{m-1}}, R_{n+1}\right); \quad \xi_n = 0 \text{ on } \partial Q\left(\nu_0 \frac{R_n^2}{2\omega^{m-1}}, R_n\right);$$

$$|\nabla \xi_n| \leq \frac{2^{n-1}}{R}; \quad 0 \leq (\xi_n)_t \leq \frac{2^{2n-2}}{R^2}\omega^{m-1}; \quad \Delta \xi_n \leq \frac{2^{2n-2}}{R^2}.$$

Similarly to the proof of the first alternative, we will write the energy inequality, developed in the previous section, over the cylinders $Q\left(\nu_0 \frac{R^2}{2\omega^{m-1}}, R_n\right)$ for functions $(u_+^l - k_n)_+$, with

$$l = \mu^+ - \frac{\omega}{2^{s_0}}, \qquad k_n = \mu^+ - \frac{\omega}{2^{s_0}} - \frac{\omega}{2^{s_0+n}},$$

and $\xi = \xi_n$, where $s_0$ will be determined later in the proof.

Since

$$u_+^l > k_n \geq \mu^+ - \frac{\omega}{2} \geq \omega - \frac{\omega}{2} = \frac{\omega}{2},$$

and the following properties

(i) $0 \leq \mu^- \leq \frac{\omega}{4}$, so $u \leq \frac{5\omega}{4}$ and consequently $u_+^l = \min\{u, l\} \leq \frac{5\omega}{4}$;

(ii) $l = \mu^+ - \frac{\omega}{2^{s_0}} > k_n$, so $\chi_{[u \geq l]} \leq \chi_{[u > k_n]} = \chi_{[(u-k_n)_+ > 0]}$;

(iii) $\chi_{[(u-k_n)_+ > 0]} = \chi_{[(u_+^l - k_n)_+ > 0]}$ for all $(x,t)$;

(iv) $(l-k_n)_+ = \frac{\omega}{2^{s_0+n}} \leq \frac{\omega}{2^{s_0-1}}, (u^l_+ - k_n)_+ \leq \frac{\omega}{2^{s_0+n}} \leq \frac{\omega}{2^{s_0-1}}$ and $(u-l)_+ \leq \frac{\omega}{2^{s_0-1}}$,

are satisfied for our functions and constants, with $\chi_E$ denotes the characteristic function of the set $E$, then, using the energy estimate with the above considerations, we obtain the following inequalities

(I)

$$\begin{aligned}
& \operatorname*{ess\,sup}_{-\frac{\nu_0 R_n^2}{2\omega^{m-1}}<t<0} \int_{K_{R_n}\times\{t\}} (u^l_+ - k_n)^2_+ \xi_n^2 dx \\
+ & \int_{-\frac{\nu_0 R_n^2}{2\omega^{m-1}}}^{0} \int_{K_{R_n}} (u^l_+)^{m-1} |\nabla (u^l_+ - k_n)_+ \xi_n|^2 dxdt \\
\geq & \; 2^{1-m} \Bigg( \operatorname*{ess\,sup}_{-\frac{\nu_0 R_n^2}{2\omega^{m-1}}<t<0} \int_{K_{R_n}\times\{t\}} (u^l_+ - k_n)^2_+ \xi_n^2 dx \\
+ & \; \omega^{m-1} \int_{-\frac{\nu_0 R_n^2}{2\omega^{m-1}}}^{0} \int_{K_{R_n}} |\nabla (u^l_+ - k_n)_+ \xi_n|^2 dxdt \Bigg);
\end{aligned}$$

(II)

$$\begin{aligned}
& \int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} (u^l_+)^{m-1} (u^l_+ - k_n)^2_+ |\nabla \xi_n|^2 dxdt \\
\leq & \; C\omega^{m-1} \left(\frac{\omega}{2^{s_0-1}}\right)^2 \frac{2^{2n-2}}{R^2} \int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} \chi_{[(u^l_+ - k_n)_+ > 0]} dxdt;
\end{aligned}$$

(III)

$$\begin{aligned}
& 2 \int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} \left( \left(u^l_+ - k_n\right)^2_+ + 2\left(l-k_n\right)_+ \left(u-l\right)_+ \right) \xi_n (\xi_n)_t dxdt \\
\leq & \; C\omega^{m-1} \left(\frac{\omega}{2^{s_0-1}}\right)^2 \frac{2^{2n-2}}{R^2} \int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} \chi_{[(u^l_+ - k_n)_+ > 0]} dxdt;
\end{aligned}$$

(IV)

$$\begin{aligned}
& C(l-k_n)_+ \int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} \left( \int_l^u s^{m-1} ds \right) \left( |\nabla \xi_n|^2 + \xi_n \Delta \xi_n \right) \chi_{[u \geq l]} dxdt \\
\leq & \; C\omega^{m-1} \left(\frac{\omega}{2^{s_0-1}}\right)^2 \frac{2^{2n-1}}{R^2} \int_{-\frac{R_n^2}{\omega^{m-1}}}^{0} \int_{K_{R_n}} \chi_{[(u^l_+ - k_n)_+ > 0]} dxdt.
\end{aligned}$$

Now, joining the above estimates, we obtain

$$\operatorname*{ess\,sup}_{-\frac{R_n^2}{\omega^{m-1}}<t<0} \int_{K_{R_n}\times\{t\}} (u_+^l - k_n)_+^2 \xi_n^2 dx + \omega^{m-1} \int_{-\frac{R_n^2}{\omega^{m-1}}}^0 \int_{K_{R_n}} |\nabla (u_+^l - k)_+ \xi_n|^2 dxdt$$
$$\leq \; C\omega^{m-1} \left(\frac{\omega}{2^{s_0-1}}\right)^2 \frac{2^{2n}}{R^2} \int_{-\frac{R_n^2}{\omega^{m-1}}}^0 \int_{K_{R_n}} \chi_{\left[(u_+^l - k_n)_+ > 0\right]} dxdt.$$

Next, perform a change in the time variable, putting

$$\bar{t} = \frac{2\omega^{m-1}}{\nu_0} t$$

and defining

$$\overline{u}_+^l(\cdot, \bar{t}) := u_+^l(\cdot, t), \qquad \overline{\xi_n}(\cdot, \bar{t}) := \xi_n(\cdot, t). \tag{2.27}$$

Then, the last inequality can be rewritten as

$$\operatorname*{ess\,sup}_{-R_n^2<t<0} \int_{K_{R_n}\times\{\bar{t}\}} (\overline{u}_+^l - k_n)_+^2 \overline{\xi_n}^2 dx + \frac{\nu_0}{2} \int_{-R_n^2}^0 \int_{K_{R_n}} |\nabla (\overline{u}_+^l - k)_+ \overline{\xi_n}|^2 dxdt$$
$$\leq \; C\frac{\nu_0}{2} \left(\frac{\omega}{2^{s_0-1}}\right)^2 \frac{2^{2n}}{R^2} \int_{-R_n^2}^0 \int_{K_{R_n}} \chi_{\left[(\overline{u}_+^l - k_n)_+ > 0\right]} dxdt. \tag{2.28}$$

Now, define, for each $n$

$$A_n := \int_{-R_n^2}^0 \int_{K_{R_n}} \chi_{\left[(\overline{u}_+^l - k_n)_- > 0\right]} dxd\bar{t}.$$

Thus, if we multiply the inequality (2.28) by $\frac{2}{\nu_0} > 1$ we obtain

$$\|(\overline{u}_+^l - k_n)_- \overline{\xi_n}\|_{V^2(Q(R_n^2, R_n))} \leq C\frac{2^{2n}}{R^2} \left(\frac{\omega}{2^{s_0-1}}\right)^2 A_n. \tag{2.29}$$

As before, we use definition of $A_n$, the fact that $k_n < k_{n+1}, \overline{\xi_n} = 1 \mathrm{in} Q(R_{n+1}^2, R_{n+1})$ and the Theorem 1.1.3 (for $p = 2$) to obtain that

$$\begin{aligned}
\frac{1}{2^{2(n+2)}} \left(\frac{\omega}{2^{s_0-1}}\right)^2 A_{n+1} \;&\leq\; |k_{n+1} - k_n|^2 A_{n+1} &\quad (2.30)\\
&\leq\; \|(\overline{u}_+^l - k_n)_+\|_{2,Q(R_{n+1}^2, R_{n+1})}^2 \\
&\leq\; \|(\overline{u}_+^l - k_n)_+ \overline{\xi_n}\|_{2,Q(R_n^2, R_n)}^2 \\
&\leq\; C\|(\overline{u}_+^l - k_n)_+ \overline{\xi_n}\|_{V^2(Q(R_n^2, R_n))}^2 A_n^{\frac{2}{N+2}} \\
&\leq\; C\frac{2^{2n}}{R^2} \left(\frac{\omega}{2^{s_0-1}}\right)^2 A_n^{1+\frac{2}{N+2}}, &\quad (2.31)
\end{aligned}$$

the last equality is due to (2.29).

To conclude, define the numbers

$$X_n = \frac{A_n}{|Q(R_n^2, R_n)|}.$$

Next, divide (2.30) by $|Q(R_{n+1}^2, R_{n+1})|$, then, we obtain the recursive relation

$$X_{n+1} \leq C 4^{2n} X_n^{1+\frac{2}{N+2}},$$

for a constant $C$ depending only upon $N$ and $m$. If,

$$X_0 \leq C^{-\frac{N+2}{2}} 4^{-\frac{(N+2)^2}{2}} := \nu_0^*, \tag{2.32}$$

we have by the Lemma 1.1.4 on fast geometric convergence that

$$X_n \to 0, \tag{2.33}$$

which implies that

$$\left|\left\{(x,t) \in Q\left(\frac{\nu_0}{2\omega^{m-1}}\left(\frac{R}{2}\right)^2, \frac{R}{2}\right) : u_+^l(x,t) > \mu^+ - \frac{\omega}{2^{s_0}}\right\}\right| = 0.$$

Similarly to the proof of the first alternative, since $u_+^l \leq \mu^+ - \frac{\omega}{2^{s_0}} = l$ implies that $u_+^l = u$, the final result follows immediately. So to complete this proof, it just remain to prove (2.32).

To prove (2.32), we begin by simplifying the notation by introducing the sets

$$B_\sigma(t) = \left\{x \in K_R : u(x,t) > \mu^+ - \frac{\omega}{2^\sigma}\right\}$$

and

$$B_\sigma = \left\{(x,t) \in Q\left(\frac{\nu_0}{2\omega^{m-1}} R^2, R\right) : u(x,t) > \mu^+ - \frac{\omega}{2^\sigma}\right\}.$$

Thus, with this notation, (2.32) reads

$$A_0 \leq |B_{s_0-1}| \leq \nu_0^* \left|Q\left(\frac{\nu_0}{2\omega^{m-1}} R^2, R\right)\right|. \tag{2.34}$$

The inequality (2.34) means that the subset of the cylinder $Q\Big(\frac{\nu_0}{2\omega^{m-1}} R^2, R\Big)$ where $u$ is close to its supremum $\mu^+$ can be made arbitrarily small. To check this, first consider the energy estimate (2.3) for the function $(u_+^l - k)_+ = (u-k)_+$ in the cylinders $Q\Big(\frac{\nu_0}{\omega^{m-1}} R^2, 2R\Big)$ with

$$k = \mu^+ - \frac{\omega}{2^s},$$

where $s$ will be chosen later satisfying $q < s < s_0$, recall that $q$ was chosen in Lemma (2.1.13). Take a piecewise smooth cutoff function $0 \le \xi \le 1$ is defined in $Q\Big(\frac{\nu_0}{\omega^{m-1}}R^2, 2R\Big)$ with the following assumptions,

$$\xi = 1 \text{ in } Q\Big(\frac{\nu_0}{2\omega^{m-1}}R^2, R\Big), \qquad \xi = 0 \text{ on } \partial Q\Big(\frac{\nu_0}{\omega^{m-1}}R^2, 2R\Big),$$

$$|\nabla \xi| \le \frac{1}{R}, \qquad 0 \le \xi_t \le \frac{\omega^{m-1}}{R^2}.$$

Disregarding the first term on the left hand side of the energy estimate (2.3), because it is nonnegative, and integrate the second one in a smaller set, $Q\Big(\frac{\nu_0}{2\omega^{m-1}}R^2, R\Big)$. Next, repeating the same argument, when $|\nabla(u-k)_+\xi| \neq 0$, $u > k = \mu^+ - \frac{\omega}{2^s} > \frac{\omega}{2}$, we arrive at

$$\Big(\frac{\omega}{2}\Big)^{m-1}\int_{-\frac{\nu_0 R^2}{2\omega^{m-1}}}^{0}\int_{K_R} |\nabla(u-k)_+\xi|^2 dxdt \le \operatorname*{ess\,sup}_{-\frac{\nu_0 R_n^2}{\omega^{m-1}}<t<0}\int_{K_{2R}\times\{t\}} (u-k)_+^2\xi^2 dx + \int_{-\frac{\nu_0 R^2}{\omega^{m-1}}}^{0}\int_{K_{2R}} u^{m-1}|\nabla(u-k)_+\xi|^2 dxdt.$$

Now, we estimate the two terms on the right hand side of the energy estimate (2.3). First, note that the third term is now 0, because $\chi_{\{u\ge\mu^+\}} = 0$. For the others terms, using similar properties, we can obtain the following upper bounds:
1st term:

$$\int_{-\frac{\nu_0 R^2}{\omega^{m-1}}}^{0}\int_{K_{2R}} u^{m-1}(u-k)_+^2|\nabla\xi|^2 dxdt \le C\omega^{m-1}\Big(\frac{\omega}{2^s}\Big)^2\frac{1}{R^2}\left|Q\Big(\frac{\nu_0}{\omega^{m-1}}R^2, 2R\Big)\right|;$$

2nd term:

$$2\int_{-\frac{\nu_0 R^2}{\omega^{m-1}}}^{0}\int_{K_{2R}}\Big((u-k)_+^2 + 2(\mu^+-k)_+(u-\mu^+)_+\Big)\xi\xi_t dxdt \le C\omega^{m-1}\Big(\frac{\omega}{2^s}\Big)^2\frac{1}{R^2}\left|Q\Big(\frac{\nu_0}{\omega^{m-1}}R^2, 2R\Big)\right|.$$

Next, joining these inequalities obtained for the terms of the energy estimate(2.3) and multiplying both sides by $\Big(\frac{\omega}{2}\Big)^{1-m}$, we conclude that

$$\int_{-\frac{\nu_0 R^2}{2\omega^{m-1}}}^{0}\int_{K_R} |\nabla(u-k)_+\xi|^2 dxdt \le \frac{C}{R^2}\Big(\frac{\omega}{2^s}\Big)^2\left|Q\Big(\frac{\nu_0}{\omega^{m-1}}R^2, 2R\Big)\right|.$$

In addition, we also have that

$$\left|Q\Big(\frac{\nu_0}{\omega^{m-1}}R^2, 2R\Big)\right| = 2^{N+1}\left|Q\Big(\frac{\nu_0}{2\omega^{m-1}}R^2, R\Big)\right|,$$

and $\xi = 1$ in $Q\Big(\frac{\nu_0}{2\omega^{m-1}}R^2, R\Big)$, therefore

$$\int_{-\frac{\nu_0 R^2}{2\omega^{m-1}}}^{0}\int_{K_R}|\nabla(u-k)_+|^2dxdt \le \frac{C}{R^2}\Big(\frac{\omega}{2^s}\Big)^2\Big|Q\Big(\frac{\nu_0}{2\omega^{m-1}}R^2, R\Big)\Big|.$$

Now, note that $B_s \subset Q\left(\frac{\nu_0}{2\omega^{m-1}}R^2, R\right)$ and, that in $B_s$ we have

$$|\nabla(u-k)_+| = |\nabla(u-k)| = |\nabla u|,$$

and consequently

$$\int\int_{B_s}|\nabla u|^2dxdt \le \frac{C}{R^2}\left(\frac{\omega}{2^s}\right)^2\Big|Q\Big(\frac{\nu_0}{2\omega^{m-1}}R^2, R\Big)\Big|. \tag{2.35}$$

We next apply Lemma (1.1.1) to the function $u(\cdot,t)$, for all $t \in \Big(-\frac{\nu_0 R^2}{2\omega^{m-1}}, 0\Big)$, and with

$$k_1 = \mu^+ - \frac{\omega}{2^s}, \qquad k_2 = \mu^+ - \frac{\omega}{2^{s+1}}.$$

So the lemma gives us

$$\frac{\omega}{2^{s+1}}\Big|\Big[u(.,t) > \mu^+ - \frac{\omega}{2^{s+1}}\Big]\Big| \le C\frac{R^{N+1}}{\Big|\Big[u(.,t) < \mu^+ - \dfrac{\omega}{2^s}\Big]\Big|}\int_{[-\frac{\omega}{2^s}<u(.,t)-\mu^+<-\frac{\omega}{2^{s+1}}]}|\nabla u|dx. \tag{2.36}$$

In addition, since $q \le s-1$, by the Lemma (2.1.13) we have that

$$|B_{s-1}(t)| \le |B_q(t)| \le \left(1-\left(\frac{\nu_0}{2}\right)^2\right)|K_R|, \tag{2.37}$$

for all $t \in (-\frac{\nu_0}{2\omega^{m-1}}, 0)$. Using (2.37) we deduce that

$$\begin{aligned}\Big|\Big\{x\in K_R : u(x,t) < \mu^+ - \frac{\omega}{2^s}\Big\}\Big| &\ge \Big|\Big\{x\in K_R : u(x,t) \le \mu^+ - \frac{\omega}{2^{s-1}}\Big\}\Big| \\ &= |K_R| - |B_{s-1}(t)| \ge |K_R| - |B_q(t)| \ge \Big(\frac{\nu_0}{2}\Big)^2|K_R|.\end{aligned}$$

Thus, using our notation in (2.36), we obtain

$$\frac{\omega}{2^{s+1}}\big|B_{s+1}(t)\big| \le \frac{CR^{N+1}}{\nu_0^2|K_R|}\int_{B_s(t)\setminus B_{s+1}(t)}|\nabla u|dx,$$

for $t \in \Big[-\frac{\nu_0 R^2}{2\omega^{m-1}}, 0\Big]$. Integrating over this interval, and use (2.35),we conclude that

$$\begin{aligned}\frac{\omega}{2^{s+1}}\big|B_{s+1}(t)\big| &\le \frac{CR}{\nu_0^2}\int\int_{B_s\setminus B_{s+1}}|\nabla u|dxdt \\ &\le \frac{CR}{\nu_0^2}\Big(\int\int_{B_s\setminus B_{s+1}}|\nabla u|^2dxdt\Big)^{\frac{1}{2}}|B_s\setminus B_{s+1}|^{\frac{1}{2}} \\ &\le \frac{CR}{\nu_0^2}\Big(\int\int_{B_s}|\nabla u|^2dxdt\Big)^{\frac{1}{2}}|B_s\setminus B_{s+1}|^{\frac{1}{2}} \\ &\le \frac{C}{\nu_0^2}\frac{\omega}{2^s}\Big|Q\left(\frac{\nu_0}{2\omega^{m-1}}R^2, R\right)\Big|^{\frac{1}{2}}|B_s\setminus B_{s+1}|^{\frac{1}{2}}.\end{aligned}$$

Simplifying and taking the 2 power, we obtain

$$|B_{s+1}|^2 \le \frac{C}{\nu_0^4}\left|Q\left(\frac{\nu_0}{2\omega^{m-1}}R^2, R\right)\right||B_s \setminus B_{s+1}|.$$

Since the above inequality is valid for $q < s < s_0$, we can add them for

$$s = q+1, q+2, ..., s_0 - 2,$$

i.e.,

$$\sum_{s=q+1}^{s_0-2} |B_{s+1}| \le \frac{C}{\nu_0^4}\left|Q\left(\frac{\nu_0}{2\omega^{m-1}}R^2, R\right)\right|\sum_{s=q+1}^{s_0-2} |B_s \setminus B_{s+1}|.$$

Then, as

$$\sum_{s=q+1}^{s_0-2} |B_s \setminus B_{s+1}| \le \left|Q\left(\frac{\nu_0}{2\omega^{m-1}}R^2, R\right)\right|,$$

and $B_{s_0-1} \subset B_{s+1}$ for $s = q+1, q+2, ..., s_0-2$, we deduce that

$$(s_0 - q)|B_{s_0-1}|^2 \le \frac{C}{\nu_0^4}\left|Q\left(\frac{\nu_0}{2\omega^{m-1}}R^2, R\right)\right|^2,$$

that is,

$$|B_{s_0-1}| \le \frac{C}{\nu_0^2(s_0-q)^{1/2}}\left|Q\left(\frac{\nu_0}{2\omega^{m-1}}R^2, R\right)\right|.$$

If we choose $s_0$ so large that

$$\frac{C}{\nu_0^2(s_0-q)^{1/2}} < \nu_0^*$$

we prove (2.32) and consequently we conclude the proof of the proposition. $\square$

Now, using the previous proposition, we obtain again the reduction of the oscillation, in the case where the second alternative happens.

**Corollary 2.1.15.** *Assume*

$$\mu^- < \frac{\omega}{4}$$

*is true. If the second alternative* (2.10) *holds, then there exists a constant* $\sigma_1 \in (0,1)$, *depending only on the data, such that*

$$\operatorname*{ess\,osc}_{Q\left(\frac{\nu_0}{2}\omega^{1-m}\left(\frac{R}{2}\right)^2, \frac{R}{2}\right)} u \le \sigma_1\omega. \tag{2.38}$$

*Proof.* By Proposition 2.1.14, there exists $s_0 \in \mathbb{N}$ such that

$$\operatorname*{ess\,sup}_{Q\left(\frac{\nu_0}{2}\omega^{1-m}\left(\frac{R}{2}\right)^2,\frac{R}{2}\right)} u \leq \mu^+ - \frac{\omega}{2^{s_0}},$$

and thus

$$\begin{aligned}
\operatorname*{ess\,osc}_{Q\left(\omega^{1-m}\left(\frac{R}{2}\right)^2,\frac{R}{2}\right)} u &= \operatorname*{ess\,sup}_{Q\left(\omega^{1-m}\left(\frac{R}{2}\right)^2,\frac{R}{2}\right)} u - \operatorname*{ess\,inf}_{Q\left(\omega^{1-m}\left(\frac{R}{2}\right)^2,\frac{R}{2}\right)} u \\
&\leq \mu^+ - \frac{\omega}{2^{s_0}} - \mu^- \\
&= \left(1 - \frac{1}{2^{s_0}}\right)\omega.
\end{aligned}$$

This way, the corollary follows with

$$\sigma_1 = \left(1 - \frac{1}{2^{s_0}}\right).$$

□

## 2.2 Hölder continuity of weak solutions of the PME

The purpose of this part of the chapter is to present the celebrated result obtained by DiBenedetto and Friedman developed in [21], which establishes Hölder continuity of nonnegative solutions of the degenerate parabolic equation

$$\frac{\partial}{\partial t}u - \frac{\partial}{\partial x_k}\left(a^{k,l}(x,t,\nabla u)\frac{\partial}{\partial x_l}u^m\right) = f(x,t,u,\nabla u), \tag{2.39}$$

where $m > 1$ and

$$\begin{cases} a^{k,l}\xi_k\xi_l \geq c_0|\xi|^2 \\ |a^{k,l}| \leq c_1 \\ f(x,t,u,\nabla u) \leq c_2|\nabla u^m| + c_3 \quad , \end{cases} \tag{2.40}$$

where $c_i$ are positive constants. Let's make this idea precise.

The above result was obtained by working on cylinders suitably scaled to reflect in a precise quantitative way the power-like degeneracy of the equation, i.e., precisely using the approach presented in the previous section.

**Remark 2.2.1.** *To describe the main idea of proof of the well-celebrated result due to DiBenedetto and Friedman, we restrict ourselves in the previous section and in this section to a particular case of PME and, as the proof is based on integral estimates and not on special form, the general case follows similarly.*

We begin the process to prove the Hölder continuity of weak solutions of the porous media equation presenting an immediate consequence of Corollaries 2.1.12 and 2.1.15 which is the following.

**Proposition 2.2.2.** *There exists a constant* $\sigma \in (0,1)$*, that depends only on the data, such that*

$$\operatorname*{ess\ osc}_{Q\left(\frac{\nu_0}{2}\omega^{1-m}\left(\frac{R}{2}\right)^2,\frac{R}{2}\right)} u \leq \sigma\omega. \tag{2.41}$$

*Proof.* Assume

$$\mu^- < \frac{\omega}{4}$$

is true. Since $\nu_0/2 < 1$,

$$Q\left(\frac{\nu_0}{2}\omega^{1-m}\left(\frac{R}{2}\right)^2,\frac{R}{2}\right) \subset Q\left(\omega^{1-m}\left(\frac{R}{2}\right)^2,\frac{R}{2}\right),$$

and then, by Corollaries 2.1.12 and 2.1.15,

$$\operatorname*{ess\ osc}_{Q\left(\frac{\nu_0}{2}\omega^{1-m}\left(\frac{R}{2}\right)^2,\frac{R}{2}\right))} u \leq \sigma\omega,$$

where $\sigma = \max\{\sigma_0, \sigma_1\}$. $\square$

The second step of the process is through an iterative scheme designed from all previous results.

**Lemma 2.2.3.** *There exist constants* $\eta > 1$ *and* $\beta \in (0,1)$*, that can be determined a priori in terms of the data, such that for all the cylinders*

$$Q\left(\omega^{1-m}r^2, r\right), \qquad \textit{with} \qquad 0 < r \leq R,$$

$$\operatorname*{ess\ osc}_{Q(\omega^{1-m}r^2,r)} u \leq \eta\omega\left(\frac{r}{R}\right)^{\beta}.$$

*Proof.* We start by defining

$$R_k = c_0^k R, \quad c_0 = \frac{1}{2}\left(\frac{\nu_0}{2}\right)^{\frac{1}{2}} < \frac{1}{2},$$

for $k \in \mathbb{N}$, where $\sigma$ is given by Proposition 2.2.2 and $\nu_0$ is the constant defined in Proposition 2.1.10 Since we are assuming that

$$\omega^{m-1} > R^{\epsilon},$$

for $k = 0$, we have that our initial condition

$$\begin{aligned}
\operatorname*{ess\,osc}_{Q(\omega^{1-m}R_0^2, R_0)} u &\le \operatorname*{ess\,osc}_{Q(R_0^{2-\epsilon}, R_0)} u \\
&\le \omega
\end{aligned}$$

is verified.

For $k = 1$, by (2.41), we have

$$\begin{aligned}
\operatorname*{ess\,osc}_{Q(\omega^{1-m}R_1^2, R_1)} u &\le \operatorname*{ess\,osc}_{Q(\omega^{1-m}R_1^2, \frac{R}{2})} u \\
&\le \sigma\omega \\
&= \omega_1.
\end{aligned}$$

Repeating the process again starting in $Q(\omega^{1-m}R_1^2, R_1)$, with $\omega_1 = \sigma\omega$, we can, inductively, deduce that

$$\begin{aligned}
\operatorname*{ess\,osc}_{Q(\omega^{1-m}R_k^2, R_k)} u &\le \omega_k \\
&= \sigma^k\omega,
\end{aligned}$$

for all $k = 0, 1, 2\dots$ .

Moreover, for $0 < r \le R$, there exists some $k$ such that

$$Rc_0^{(k+1)} \le r \le Rc_0^k.$$

Then, choosing $\beta = \dfrac{\log\sigma}{\log c_0} > 0$, we get

$$\sigma^{k+1} \le \left(\frac{r}{R}\right)^{\beta},$$

which means that

$$\begin{aligned}
\operatorname*{ess\,osc}_{Q(\omega^{1-m}r^2, r)} u &\le \operatorname*{ess\,osc}_{Q(\omega^{1-m}R_k^2, R_k} u \\
&\le \sigma^k\omega \\
&\le \eta\omega\left(\frac{r}{R}\right)^{\beta},
\end{aligned}$$

with $\eta = \sigma^{-1}$.

We can suppose, without loss of generality, that $\sigma > \frac{1}{2}$. Hence, $\beta \in (0, 1)$.

$\square$

Now we are able to prove the first part of the main theorem.

**Theorem 2.2.4.** *Let $u$ be a bounded local weak solution of* (2.1) *in $E_T$ and $M = ||u||_{\infty,E_T}$. Then $u$ is locally Hölder continuous in $E_T$, i.e., there exist constants $\eta > 1$ and $\beta \in (0,1)$, depending only on the data, such that, for every compact subset $K$ of $E_T$,*

$$|u(x_1,t_1) - u(x_2,t_2)| \leq \eta M \left( \frac{|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}}}{\operatorname{dist}(K;\partial_p E_T)} \right)^{\beta},$$

*for every pair of points $(x_i,t_i) \in K$, $i = 1,2$.*

*Proof.* We start by fixing $(x,t_i) \in K$, $i = 1,2$, with $t_2 > t_1$ and constructing the cylinder

$$S = (x,t_2) + Q(R^2,R).$$

There exists a constant $l$ such that $\operatorname{dist}(K;\partial_p E_T) < l$ for every compact set $K$. Then, if we choose

$$R = \frac{1}{2l}\operatorname{dist}(K;\partial_p E_T),$$

we have that $S \subset E_T$ and $R < 1$.

Also, supposing that

$$t_2 - t_1 < R^2,$$

it is possible to choose

$$r = |t_2 - t_1|^{\frac{1}{2}} \in (0,R).$$

Hence, we can apply Lemma 2.2.3 to

$$(x,t_2) + Q(\omega^{1-m}r^2, r)$$

and conclude that

$$\begin{aligned} |u(x,t_1) - u(x,t_2)| &\leq \eta\omega(2l)^{\beta} \left( \frac{|t_2 - t_1|^{\frac{1}{2}}}{\operatorname{dist}(K;\partial_p E_T)} \right)^{\beta} \\ &= C \left( \frac{|t_2 - t_1|^{\frac{1}{2}}}{\operatorname{dist}(K;\partial_p E_T)} \right)^{\beta}. \end{aligned}$$

On the other hand, if $t_2 - t_1 \geq R^2$, then we have

$$\frac{2l|t_2 - t_1|^{\frac{1}{2}}}{\operatorname{dist}(K;\partial_p E_T)} \geq 1,$$

which leads us to conclude that

$$
\begin{aligned}
|u(x,t_1)-u(x,t_2)| &\leq 2M \\
&\leq 2M(2l)^\beta \left( \frac{|t_2-t_1|^{\frac{1}{2}}}{\operatorname{dist}(K;\partial_p E_T)} \right)^\beta \\
&= C \left( \frac{|t_2-t_1|^{\frac{1}{2}}}{\operatorname{dist}(K;\partial_p E_T)} \right)^\beta .
\end{aligned}
$$

Similarly, we can prove the Hölder continuity in the space variables, i.e., for all $(x_i,t) \in K$, $i=1,2$, we get

$$
|u(x_1,t)-u(x_2,t)| \leq C \left( \frac{|x_2-x_1|}{\operatorname{dist}(K;\partial_p E_T)} \right)^\beta .
$$

Now, using both inequalities, we get that

$$
\begin{aligned}
|u(x_1,t_1)-u(x_2,t_2)| &= |u(x_1,t_1)-u(x_2,t_1)+u(x_2,t_1)-u(x_2,t_2)| \\
&\leq C \left( \left( \frac{|x_2-x_1|}{\operatorname{dist}(K;\partial_p E_T)} \right)^\beta + \left( \frac{|t_2-t_1|^{\frac{1}{2}}}{\operatorname{dist}(K;\partial_p E_T)} \right)^\beta \right) \\
&\leq C \left( \frac{|x_2-x_1|+|t_2-t_1|^{\frac{1}{2}}}{\operatorname{dist}(K;\partial_p E_T)} \right)^\beta ,
\end{aligned}
$$

for all $(x_i,t) \in K$, $i=1,2$. □

**Remark 2.2.5.** *We admit in the previous results that*

$$
\mu^- < \frac{\omega}{4}.
$$

*In the case where the infimum is not comparatively small, i. e,*

$$
\mu^- \geq \frac{\omega}{4},
$$

*since, $\omega^{m-1} > R^\epsilon$, we then have*

$$
\inf_{Q(4R^{2-\epsilon},2R)} v \geq \frac{1}{4} R^{\frac{\epsilon}{m-1}}, \qquad \textit{with} \qquad v = u^m.
$$

*We may then rescale the equation in the $x$ or $t$ direction so as to obtain a uniformly parabolic operator, to which we may apply standard local estimates. Going back to the original coordinates we easily get the dimensional form*

$$
|v(x_1,t_1)-v(x_2,t_2)| \leq C \left( \frac{|x_2-x_1|+|t_2-t_1|^{\frac{1}{2}}}{\textit{dist}\,(K;\partial_p E_T)} \right)^\beta
$$

*for suitable $\sigma > 0$ and $\beta > 0$.*

# 3. Sharp regularity for the porous medium equation with m > 1

## Contents



In this chapter we will establish a new result in the regularity theory for solutions of the porous medium equation looking for sharp regularity. First, we will construct a geometric iteration, more precisely by means of the approximation lemma obtained in the first part of the chapter, which is a mechanism linking solutions of the inhomogeneous PME and solutions of the homogeneous equation, and the celebrated result obtained by DiBenedetto and Friedman (cf. Chapter 3 and [21]), the Hölder continuity for the n-dimensional case, we get the first step of the iterative process, then iterating the result obtained we conclude the process of geometric iteration. Next, we show that the smallness regime required in the iterative process is not restrictive. Finally, we will present the main result of this work, that bounded weak solutions of (1.13) are locally of class $C^{0,\gamma}$ in space, with

$$\gamma = \frac{\alpha}{m}, \qquad \alpha = \min\left\{\alpha_0^-, \frac{m[(2q-n)r-2q]}{q[mr-(m-1)]}\right\},$$

where $0 < \alpha_0 \leq 1$ denotes the optimal Hölder exponent for solutions of (1.13) with $f \equiv 0$. The regularity class is to be interpreted in the following sense: if

$$\frac{m[(2q-n)r-2q]}{q[mr-(m-1)]} < \alpha_0$$

then solutions are in $C^{0,\gamma}$, with

$$\gamma = \frac{(2q-n)r-2q}{q[mr-(m-1)]};$$

if, alternatively,

$$\frac{m[(2q-n)r-2q]}{q[mr-(m-1)]} \geq \alpha_0,$$

then solutions are in $C^{0,\gamma}$, for any $0<\gamma<\frac{\alpha_0}{m}$.

We also obtain the $C^{0,\frac{\gamma}{\theta}}$ regularity in time, where

$$\theta = 2-\left(1-\frac{1}{m}\right)\alpha = \alpha\left(1+\frac{1}{m}\right)+(1-\alpha)\,2$$

is the $\alpha-$interpolation between $1+\frac{1}{m}$ and 2.

## 3.1 Approximation by homogeneous functions

We will start fixing the intrinsic geometric setting for our problem. Then we will use the available compactness to derive a mechanism linking solutions of the inhomogeneous PME and solutions of the homogeneous equation. This result will be fundamental to construct the geometric iteration that will allow us to approach the optimal regularity theory for the porous medium equation.

Given, $0<\alpha<1$, let

$$\theta := 2-\left(1-\frac{1}{m}\right)\alpha. \tag{3.1}$$

which clearly satisfies the bounds

$$1+\frac{1}{m}<\theta<2.$$

For such $\theta$, define the intrinsic $\theta$-parabolic cylinder

$$G_\rho := (-\rho^\theta, 0)\times B_\rho(0), \quad \rho>0.$$

Now, we present the result of approximation for the equation (1.13); this result is to be compared with a similar statement for the $p$-Laplace equation in [50] (see also [25, 12]).

**Lemma 3.1.1.** *(Approximation by homogeneous functions). Given $\delta>0$, there exists $0<\epsilon\ll 1$ such that if $\|f\|_{L^{q,r}(G_1)}\leq\epsilon$ and $u$ is a local weak solution of* (1.13) *in $G_1$, with $\|u\|_{\infty,G_1}\leq 1$, then there exists $\phi$ such that*

$$\phi_t - \operatorname{div}\left(m\phi^{m-1}\nabla\phi\right) = 0 \quad \text{in } G_{1/2} \tag{3.2}$$

*and*

$$\|u-\phi\|_{\infty,G_{1/2}}\leq\delta.$$

*Proof.* Suppose, for the sake of contradiction, that, for some $\delta_0 > 0$, there exist sequences $(u^j)_j$ and $(f^j)_j$, with

$$u^j \in C_{\text{loc}}\left(-1,0;L^2_{\text{loc}}(B_1)\right), \quad (u^j)^{\frac{m+1}{2}} \in L^2_{\text{loc}}\left(-1,0;W^{1,2}_{\text{loc}}(B_1)\right)$$

and $f^j \in L^{q,r}(G_1)$, such that

$$u^j_t - \operatorname{div}\left(m(u^j)^{m-1}\nabla u^j\right) = f^j \quad \text{in } G_1 \tag{3.3}$$

$$\|u^j\|_{\infty,G_1} \leq 1, \tag{3.4}$$

$$\|f^j\|_{L^{q,r}(G_1)} \leq 1/j, \tag{3.5}$$

but still, for any $j$ and any solution $\phi$ of the homogeneous equation in $G_{1/2}$,

$$\|u^j - \phi\|_{\infty,G_{1/2}} > \delta_0. \tag{3.6}$$

Consider a cutoff function $\xi \in C^\infty_0(G_1)$ such that $\xi \in [0,1], \xi \equiv 1$ in $G_{1/2}$ and $\xi \equiv 0$ near $\partial_p G_1$. Thus, since $u^j$ is a solution of (1.13), we can apply the Caccioppoli estimate of Proposition 1.3.1 to get

$$\begin{aligned}
\int_{-1}^0\int_{B_1}(u^j)^{m-1}|\nabla u^j|^2\xi^2 &\leq \sup_{-1<t<0}\int_{B_1}(u^j)^2\xi^2 + \int_{-1}^0\int_{B_1}(u^j)^{m-1}|\nabla u^j|^2\xi^2 \\
&\leq C\int_{-1}^0\int_{B_1}(u^j)^2\xi|\xi_t| + \int_{-1}^0\int_{B_1}(u^j)^{m+1}|\nabla\xi|^2 + C\|f^j\|^2_{L^{q,r}} \\
&\leq c\|u\|^2_{2,G_1} + c'\|u\|^{m+1}_{m+1,G_1} + c''\frac{1}{j} \\
&\leq \tilde{c},
\end{aligned} \tag{3.7}$$

using (3.4) and (3.5).

Let us now define $v^j := (u^j)^{\frac{m+1}{2}}$. Observing that

$$|\nabla v^j|^2 = \left(\frac{m+1}{2}\right)^2 (u^j)^{m-1}|\nabla u^j|^2,$$

we obtain, due to (3.7),

$$\begin{aligned}
\|\nabla v^j\|^2_{2,G_{1/2}} &\leq \int_{-1}^0\int_{B_1}|\nabla v^j|^2\xi^2 dxdt \\
&\leq \left(\frac{m+1}{2}\right)^2 \tilde{c}
\end{aligned}$$

and then, for a subsequence,

$$\nabla v^j \rightharpoonup \psi$$

weakly in $L^2(G_{1/2})$.

Moreover, owing to Hölder continuity of the solutions of PME (chapter 2; for more details cf. [21, 19]), the equibounded sequence $(u^j)_j$ is also equicontinuous and, by Arzelà–Ascoli theorem 1.1.6,

$$u^j \longrightarrow \phi$$

uniformly in $G_{1/2}$, for yet another (relabelled) subsequence. Since we also have the pointwise convergence

$$v^j := (u^j)^{\frac{m+1}{2}} \longrightarrow \phi^{\frac{m+1}{2}} =: v,$$

we can identify $\psi = \nabla v$.

Passing to the limit in (3.3), we find that $\phi$ solves (3.2) which contradicts (3.6) for $j \gg 1$. □

## 3.2 Geometric iteration

We now set up a geometric iteration, exploring the intrinsic scaling of the PME that will be crucial in obtaining the sharp Hölder exponent. The next result is the first step in this iteration. Let $\alpha_0$ denote the sharp Hölder continuity exponent for solutions of (1.13) in the homogeneous case and

$$\gamma = \frac{\alpha}{m},$$

with

$$0 < \alpha = \min\left\{\alpha_0^-, \frac{m[(2q-n)r-2q]}{q[mr-(m-1)]}\right\} < \alpha_0 \le \min\left\{1, \frac{1}{m-1}\right\} \le 1. \tag{3.8}$$

**Lemma 3.2.1.** *There exists $\epsilon > 0$, and $0 < \lambda \ll 1/2$, depending only on $m, n$ and $\alpha$, such that if $\|f\|_{L^{q,r}(G_1)} \le \epsilon$ and $u$ is a local weak solution of (1.13) in $G_1$, with $\|u\|_{\infty,G_1} \le 1$, then*

$$\|u\|_{\infty,G_\lambda} \le \lambda^\gamma$$

*provided*

$$|u(0,0)| \le \frac{1}{4}\lambda^\gamma.$$

*Proof.* Take $0 < \delta < 1$, to be chosen later, and apply Lemma 3.1.1 to obtain $0 < \epsilon \ll 1$ and a solution $\phi$ of (3.2) in $G_{1/2}$ such that

$$\|u - \phi\|_{\infty,G_{1/2}} \le \delta.$$

Since $\phi$ solves (3.2), it follows from the available regularity theory (chapter 2; for more details cf. [21]) that $\phi$ is locally $C_x^{\alpha_0} \cap C_t^{\alpha_0/2}$, for $0 < \alpha_0 < 1$. Thus we obtain

$$\sup_{(x,t)\in G_\lambda} |\phi(x,t) - \phi(0,0)| \le C\lambda^{\frac{\alpha_0}{m}},$$

for $\lambda \ll 1$, to be chosen soon, and $C > 1$ universal. In fact, for $(x,t) \in G_\lambda$,

$$\begin{aligned}
|\phi(x,t) - \phi(0,0)| &\le |\phi(x,t) - \phi(0,t)| + |\phi(0,t) - \phi(0,0)| \\
&\le c_1|x-0|^{\alpha_0} + c_2|t-0|^{\alpha_0/2} \\
&\le c_1\lambda^{\alpha_0} + c_2\lambda^{\frac{\theta}{2}\alpha_0} \\
&\le c_1\lambda^{\frac{\alpha_0}{m}} + c_2\lambda^{\frac{\alpha_0}{m}} \\
&\le C\lambda^{\frac{\alpha_0}{m}}
\end{aligned}$$

since $\theta \ge 1 + \frac{1}{m} > \frac{2}{m}$. We can therefore estimate

$$\begin{aligned}
\sup_{G_\lambda} |u| &\le \sup_{G_{1/2}} |u-\phi| + \sup_{G_\lambda} |\phi| && (3.9)\\
&\le \sup_{G_{1/2}} |u-\phi| + \sup_{G_\lambda} |\phi - \phi(0,0)| + |\phi(0,0)| && (3.10)\\
&\le \sup_{G_{1/2}} |u-\phi| + \sup_{G_\lambda} |\phi - \phi(0,0)| + |\phi(0,0) - u(0,0)| + |u(0,0)| \\
&\le 2\delta + C\lambda^{\frac{\alpha_0}{m}} + \frac{1}{4}\lambda^\gamma. && (3.11)
\end{aligned}$$

Note that we will choose $\lambda \ll 1/2$ and thus

$$G_\lambda := (-\lambda^\theta, 0) \times B_\lambda \subset (-(1/2)^\theta, 0) \times B_{1/2} = G_{1/2}.$$

We finally fix the constants, choosing

$$\lambda = \left(\frac{1}{4C}\right)^{\frac{m}{\alpha_0 - \alpha}} \qquad \text{and} \qquad \delta = \frac{1}{4}\lambda^\gamma,$$

and fixing also $\epsilon > 0$, through Lemma 3.1.1. The result follows from estimate (3.11) with the indicated choices. □

We now iterate the previous result in the appropriate geometric setting.

**Theorem 3.2.2.** *There exists $\epsilon > 0$, and $0 < \lambda \ll 1/2$, depending only on $m, n$ and $\alpha$, such that if $\|f\|_{L^{q,r}(G_1)} \le \epsilon$ and $u$ is a local weak solution of (1.13) in $G_1$, with $\|u\|_{\infty,G_1} \le 1$, then*

$$\|u\|_{\infty,G_{\lambda^k}} \le (\lambda^k)^\gamma \tag{3.12}$$

*provided*

$$|u(0,0)| \le \frac{1}{4}\left(\lambda^k\right)^\gamma.$$

*Proof.* The proof is by induction on $k \in \mathbb{N}$. If $k = 1$, (3.12) holds due to Lemma 3.2.1. Now suppose the conclusion holds for $k$ and let's show it also holds for $k+1$. Consider the function $v : G_1 \to \mathbb{R}$ defined by

$$v(x,t) = \frac{u(\lambda^k x, \lambda^{k\theta} t)}{\lambda^{\gamma k}}.$$

We have

$$v_t(x,t) = \lambda^{k\theta - \gamma k} u_t(\lambda^k x, \lambda^{k\theta} t),$$

$$\nabla v(x,t) = \lambda^{k-\gamma k} \nabla u(\lambda^k x, \lambda^{k\theta} t)$$

and,

$$\begin{aligned} m v^{m-1} \nabla v(x,t) &= m[\ \lambda^{-\frac{\alpha}{m}k} (u(\lambda^k x, \lambda^{k\theta} t))\ ]^{m-1} \lambda^{k - \frac{\alpha}{m}k} \nabla u(\lambda^k x, \lambda^{k\theta} t) \\ &= \lambda^{k(1-\alpha)} m[u(\lambda^k x, \lambda^{k\theta} t)\ ]^{m-1} \nabla u(\lambda^k x, \lambda^{k\theta} t). \end{aligned}$$

Thus,

$$\begin{aligned} &\operatorname{div}\left(m(v(x,t))^{m-1} \nabla v(x,t)\right) \\ &= \lambda^{k(2-\alpha)} \operatorname{div}\left(m(u(\lambda^k x, \lambda^{k\theta} t))^{m-1} \nabla u(\lambda^k x, \lambda^{k\theta} t)\right). \end{aligned}$$

Recalling (3.1), we conclude, since $u$ is a local weak solution of (1.13) in $G_1$, that

$$v_t - \operatorname{div}\left(m v^{m-1} \nabla v\right) = \lambda^{k(2-\alpha)} f(\lambda^k x, \lambda^{k\theta} t) = \tilde{f}(x,t).$$

We now compute

$$\begin{aligned} \|\tilde{f}\|^r_{L^{q,r}(G_1)} &= \int_{-1}^{0} \left( \int_{B_1} \left| \tilde{f}(x,t) \right|^q dx \right)^{r/q} dt \\ &= \int_{-1}^{0} \left( \int_{B_1} \lambda^{k(2-\alpha)q} \left| f(\lambda^k x, \lambda^{k\theta} t) \right|^q dx \right)^{r/q} dt \\ &= \int_{-1}^{0} \left( \int_{B_{\lambda^k}} \lambda^{k(2-\alpha)q - kn} \left| f(x, \lambda^{k\theta} t) \right|^q dx \right)^{r/q} dt \\ &= \lambda^{[k(2-\alpha)q - kn]\frac{r}{q}} \int_{-1}^{0} \left( \int_{B_{\lambda^k}} \left| f(x, \lambda^{k\theta} t) \right|^q dx \right)^{r/q} dt \\ &= \lambda^{[k(2-\alpha)q - kn]\frac{r}{q} - k\theta} \int_{-\lambda^{k\theta}}^{0} \left( \int_{B_{\lambda^k}} |f(x,t)|^q dx \right)^{r/q} dt. \end{aligned}$$

Because of the crucial and optimal choice of $\alpha$ in (3.8), we have

$$[k(2-\alpha)q - kn]\,\frac{r}{q} - k\theta \geq 0$$

and thus

$$\|\tilde{f}\|_{L^{q,r}(G_1)} \leq \|f\|_{L^{q,r}((-\lambda^{\theta k},0)\times B_{\lambda^k})} \leq \|f\|_{L^{q,r}(G_1)} \leq \epsilon,$$

which entitles $v$ to Lemma 3.2.1. Note that $\|v\|_{\infty,G_1} \leq 1$, due to the induction hypothesis, i.e.,

$$\begin{aligned}
\sup_{G_1} |v| &= \sup_{G_{\lambda^k}} \frac{|u(\lambda^k x, \lambda^{k\theta} t)|}{\lambda^{\gamma k}} \\
&\leq \frac{\lambda^{\gamma k}}{\lambda^{\gamma k}} \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
|v(0,0)| &= \left| \frac{u(0,0)}{(\lambda^k)^\gamma} \right| \\
&\leq \left| \frac{\frac{1}{4}\left(\lambda^{k+1}\right)^\gamma}{(\lambda^k)^\gamma} \right| \\
&\leq \frac{1}{4}\lambda^\gamma.
\end{aligned}$$

It then follows that

$$\|v\|_{\infty,G_\lambda} \leq \lambda^\gamma,$$

and, therefore,

$$\begin{aligned}
\sup_{(x,t)\in G_{\lambda^{k+1}}} \frac{|u(x,t)|}{\lambda^{\frac{\alpha}{m}k}} &\leq \sup_{(x,t)\in G_\lambda} \frac{|u(\lambda^k x, \lambda^{k\theta} t)|}{\lambda^{\frac{\alpha}{m}k}} \\
&= \sup_{(x,t)\in G_\lambda} |v(x,t)| \\
&\leq \lambda^{\frac{\alpha}{m}}.
\end{aligned}$$

which is the same as

$$\|u\|_{\infty,G_{\lambda^{k+1}}} \leq \lambda^{\gamma(k+1)}.$$

The induction is complete. $\square$

## 3.3 Sharp regularity

In this section we will present the main result of this work, a new result about sharp regularity theory for the solutions of the porous medium equation in the n-dimensional case.

We next show the smallness regime required in the previous theorem is not restrictive and generalize it to cover the case of any small radius.

**Theorem 3.3.1.** *If $u$ is a local weak solution of* (1.13) *in $G_1$ then, for every $0 < r < \lambda$, we have*

$$\|u\|_{\infty, G_r} \leq C\, r^{\gamma}$$

*provided*

$$|u(0,0)| \leq \frac{1}{4} r^{\gamma}.$$

*Proof.* Take

$$v(x,t) = \rho u \left(\rho^a x, \rho^{(m-1)+2a} t\right)$$

with $\rho, a$ to be fixed, which solves

$$v_t - \operatorname{div}(m v^{m-1} \nabla v) = \rho^{m+2a} f(\rho^a x, \rho^{(m-1)+2a)} t) = \tilde{f}(x,t).$$

In fact, let

$$v(x,t) = \rho u(\rho^a x, \rho^b t)$$

We have

$$v_t(x,t) = \rho^{1+b} u_t(\rho^a x, \rho^b t)$$

and

$$\partial x_i v(x,t) = \rho^{1+a} u_{x_i}(\rho^a x, \rho^b t).$$

Since

$$\nabla v(x,t) = \rho^{1+a} \nabla u(\rho^a x, \rho^b t)$$

and

$$\begin{aligned} m v^{m-1} \nabla v(x,t) &= m[\rho u(\rho^a x, \rho^b t)]^{m-1} \rho^{1+a} \nabla u(\rho^a x, \rho^b t) \\ &= \rho^{(m-1)+1+a} m[u(\rho^a x, \rho^b t)]^{m-1} \nabla u(\rho^a x, \rho^b t). \end{aligned}$$

we obtain

$$\mathrm{div}(mv^{m-1}\nabla v(x,t)) = \rho^{(m-1)+1+2a}\mathrm{div}(m(u(\rho^a x, \rho^b t))^{m-1}\nabla u(\rho^a x, \rho^b t)).$$

Now we choose $b$ such that

$$1 + b = m + 2a.$$

Therefore, we have

$$\begin{aligned} v_t - \mathrm{div}(mv^{m-1}\nabla v) &= \rho^{(m+2a)}u_t(\rho^a x, \rho^{(m-1)+2a}t)) \\ &- \rho^{(m+2a)}\mathrm{div}(m(u(\rho^a x, \rho^{(m-1)+2a)}t))^{m-1}\nabla u(\rho^a x, \rho^{(m-1)+2a}t) \\ &= \rho^{(m+2a)}f(\rho^a x, \rho^{(m-1)+2a)}t) = \tilde{f}(x,t). \end{aligned}$$

Furthermore, we have

$$\|v\|_{\infty,G_1} \le \rho\|u\|_{\infty,G_1}, \tag{3.13}$$

due to the definition of $v$, and

$$\|\tilde{f}\|^r_{L^{q,r}(G_1)} = \rho^{(m+2a)r-a(n\frac{r}{q}+2)-(m-1)}\|f\|^r_{L^{q,r}(G_1)}. \tag{3.14}$$

In fact,

$$\begin{aligned} \|\tilde{f}\|^r_{L^{q,r}(G_1)} &= \int_{-1}^{0}\left(\int_{B_1}|\tilde{f}(x,t)|^q dx\right)^{r/q} dt \\ &= \int_{-1}^{0}\left(\int_{B_1}\rho^{(m+2a)q}|f(\rho^a x, \rho^{(m-1)+2a}t)|^q dx\right)^{r/q} dt \\ &= \int_{-1}^{0}\left(\int_{B_{\rho^a}}\rho^{(m+2a)q-an}|f(x, \rho^{(m-1)+2a}t)|^q dx\right)^{r/q} dt \\ &= \rho^{[(m+2a)q-an]\frac{r}{q}}\int_{-1}^{0}\left(\int_{B_{\rho^a}}|f(x, \rho^{(m-1)+2a}t)|^q dx\right)^{r/q} dt \\ &= \rho^{[(m+2a)q-an]\frac{r}{q}-[(m-1)+2a]}\int_{-\rho^{(m-1)+2a}}^{0}\left(\int_{B_{\rho^a}}|f(x,t)|^q dx\right)^{r/q} dt \\ &= \rho^{(m+2a)r-a(n\frac{r}{q}+2)-(m-1)}\|f\|^r_{L^{q,r}(G_1)} \end{aligned}$$

Now, choosing $a > 0$ such that

$$(m+2a)r - a\left(\frac{nr}{q}+2\right) - (m-1) > 0,$$

which is always possible (observe that the condition holds for $a = 0$, so by continuity with respect to $a$ there is a neighborhood of zero where the condition is still valid), and $0 < \rho \ll 1$, we enter the smallness regime required by Theorem 3.2.2, i.e.,

$$\|v\|_{\infty, G_1} \leq 1 \qquad \text{and} \qquad \|\tilde{f}\|_{L^{q,r}(G_1)} \leq \epsilon.$$

Now, given $0 < r < \lambda$, there exists $k \in \mathbb{N}$ such that

$$\lambda^{k+1} < r \leq \lambda^k.$$

Since

$$|u(0,0)| \leq \frac{1}{4} r^\gamma \leq \frac{1}{4} (\lambda^k)^\gamma,$$

it follows from Theorem 3.2.2 that

$$\|u\|_{\infty, G_{\lambda^k}} \leq (\lambda^k)^\gamma.$$

Then, for $C = \lambda^{-\gamma}$,

$$\|u\|_{\infty, G_r} \leq \|u\|_{\infty, G_{\lambda^k}} \leq (\lambda^k)^\gamma < \left(\frac{r}{\lambda}\right)^\gamma = C\, r^\gamma.$$

$\square$

We now complete our study, with the main result of the thesis.

**Theorem 3.3.2.** *Let $u$ be a locally bounded weak solution of* (1.13) *in $G_1$, with $f \in L^{q,r}$ satisfying* (1.14). *Then $u$ is locally of class $C^{0,\gamma}$ in space and $C^{0,\frac{\gamma}{\theta}}$ in time, with*

$$\gamma = \frac{\alpha}{m}, \qquad \alpha = \min\left\{ \alpha_0^-, \frac{m[(2q-n)r - 2q]}{q[mr - (m-1)]} \right\}.$$

*Here $0 < \alpha_0 \leq 1$ denotes the optimal Hölder exponent for solutions of the homogeneous case and $\theta$ is given in* (3.1).

*Proof.* We study the Hölder continuity at the origin, proving there is a uniform constant $K$ such that

$$\|u - u(0,0)\|_{\infty, G_r} \leq K r^\gamma. \tag{3.15}$$

We know, *a priori*, that $u$ is continuous so we can define

$$\mu := (4|u(0,0)|)^{-\gamma} \geq 0.$$

Take any radius $0 < r < \lambda$. We analyse three alternative cases, exhausting all possibilities.

- If $\mu \leq r < \lambda$ then, by Theorem 3.3.1,

$$\sup_{G_r} |u(x,t) - u(0,0)| \leq C\, r^\gamma + |u(0,0)| \leq \left( C + \frac{1}{4} \right) r^\gamma. \tag{3.16}$$

- If $0 < r < \mu$, we consider the function

$$w(x,t) := \frac{u(\mu x, \mu^\theta t)}{\mu^\gamma}.$$

Note that,

$$\begin{aligned} |w(0,0)| &= \frac{u(0,0)}{\mu^\gamma} &(3.17)\\ &= \frac{1}{4}, &(3.18) \end{aligned}$$

since $\mu = (4|u(0,0)|)^{-\gamma}$, and $w$ solves in $G_1$ the PME

$$w_t - \mathrm{div}\left( m w^{m-1} \nabla w \right) = \mu^{2-\alpha} f(\mu x, \mu^\theta t).$$

Indeed, we have that

$$w_t(x,t) = \mu^{\theta - \frac{\alpha}{m}} u_t(\mu x, \mu^\theta t),$$

$$\nabla w(x,t) = \mu^{1-\frac{\alpha}{m}} \nabla u(\mu x, \mu^\theta t)$$

and,

$$\begin{aligned} m w^{m-1} \nabla w(x,t) &= m[\ \mu^{-\frac{\alpha}{m}} (u(\mu x, \mu^\theta t))\ ]^{m-1} \mu^{1-\frac{\alpha}{m}} \nabla u(\mu x, \mu^\theta t) \\ &= \mu^{(1-\alpha)} m [u(\mu x, \mu^\theta t)\ ]^{m-1} \nabla u(\mu x, \mu^\theta t). \end{aligned}$$

Thus,

$$\mathrm{div}\left( m(w(x,t))^{m-1} \nabla w(x,t) \right)$$
$$= \mu^{2-\alpha} \mathrm{div}\left( m(u(\mu x, \mu^\theta t))^{m-1} \nabla u(\mu x, \mu^\theta t) \right),$$

and, then,

$$w_t - \mathrm{div}\left( m w^{m-1} \nabla w \right) = \mu^{2-\alpha} f(\mu x, \mu^\theta t) = \tilde{f}(x,t).$$

Moreover, again using Theorem 3.3.1, it follows that

$$\|w\|_{\infty, G_1} = \mu^{-\gamma} \|u\|_{\infty, G_\mu} \leq C,$$

since $|u(0,0)| = \frac{1}{4}\mu^\gamma$. With this uniform estimate in hand, and using local $C^{0,\alpha}$ regularity estimates (cf. section 1.3 and for more details [1]), we find that there exists a radius $\rho_0$, depending only on the data, such that

$$|w(x,t)| \geq \frac{1}{8}, \quad \forall\, (x,t) \in G_{\rho_0}.$$

This implies that, in $G_{\rho_0}$, $w$ solves a uniformly parabolic equation of the form

$$w_t - \operatorname{div}\left(a(x,t)\nabla w\right) = f \in L^{q,r},$$

with continuous coefficients satisfying the bounds $0 < c_1 \leq a(x,t) \leq c_2$. In particular, we have (see [50])

$$w \in C^{0,\beta}(G_{\rho_0}), \quad \text{with } \ \beta = 1 - \left(\frac{2}{r} + \frac{n}{q} - 1\right) > \gamma,$$

which is the optimal Hölder regularity for solutions of the heat equation with a source in $L^{q,r}$, for exponents satisfying (1.14). As an immediate consequence,

$$\sup_{(x,t)\in G_r} |w(x,t) - w(0,0)| \leq C\, r^\beta, \quad \forall\, 0 < r < \frac{\rho_0}{2},$$

which, in terms of $u$, reads

$$\sup_{(x,t)\in G_r} \left| \frac{u(\mu x, \mu^\theta t)}{\mu^\gamma} - \frac{u(0,0)}{\mu^\gamma} \right| \leq C\, r^\beta, \quad \forall\, 0 < r < \frac{\rho_0}{2}.$$

Since $\gamma < \beta$, we conclude

$$\sup_{(x,t)\in G_{\mu r}} |u(x,t) - u(0,0)| \leq C\, (\mu r)^\gamma, \quad \forall\, 0 < \mu r < \mu\frac{\rho_0}{2},$$

and, relabelling, we obtain

$$\sup_{(x,t)\in G_r} |u(x,t) - u(0,0)| \leq C\, r^\gamma, \quad \forall\, 0 < r < \mu\frac{\rho_0}{2}. \tag{3.19}$$

- Finally, for $\mu\frac{\rho_0}{2} \leq r < \mu$, we have

$$\begin{aligned} \sup_{(x,t)\in G_r} |u(x,t) - u(0,0)| \ &\leq \ \sup_{(x,t)\in G_\mu} |u(x,t) - u(0,0)| \\ &\leq \ C\, \mu^\gamma \leq C \left(\frac{2r}{\rho_0}\right)^\gamma = \tilde{C} r^\gamma. \end{aligned} \tag{3.20}$$

Putting $K = \max\left\{C + \frac{1}{4}, \tilde{C}\right\}$ and combining (3.16)–(3.20), we obtain (3.15), for every $0 < r < \lambda$, and the proof is complete. □

# Final Considerations

In this thesis we investigate the optimal regularity for the porous medium equation(PME) and a new result is established on the optimal Hölder continuity for solutions of the non-homogeneous parabolic equation $u_t - div(mu^{m-1}\nabla u) = f$, when $m > 1$, via a new version of Caccioppoli estimate for weak solutions of the PME in the non-homogeneous case, which was used together with some results of functional analysis to develop the approximation lemma, and a geometric iteration using the intrinsic scaling method.

The research developed in this work is a finding in the regularity theory of PME and a contribution in determining the exact Hölder exponent for the solutions of the PME in any dimension, a problem of certain relevance in the literature. The techniques (approximation theory, intrinsic scaling and the iterative geometric process) which were used in this thesis are a recent and innovative approach in the study of the regularity theory for parabolic equations that explores the degenerate structure of the operator. Although some of these tools are well used in the theory of regularity in previous works, the approach used here followed the construction developed in [50]. The proofs can be adapted to more general degenerate parabolic equations of type

$$u_t - div\ A(x,t,u,\nabla u) = f \in L^{r,q}$$

satisfying the usual structure assumptions similar to those adopted in this thesis. In the background, the heuristic is to interpret the homogeneous problem as the geometric tangential equation of its inhomogeneous counterpart, for small perturbations in $f \in L^{r,q}, \|f\|_{r,q} < 1$. However, it is important to emphasize that every equation has its own identity, and that the techniques are just a direction toward the study of the Hölder exponent, and the adaptations are the most crucial and complex part in determining the exponent. This is clearly seen when we compare the approach used to study the $p$-Laplacian equation in [50] with that of the porous media developed in our work.

The results developed here have become an article [3], which was submitted recently.

For possible future work in this theory, it would be interesting, through the same tools and techniques used in the thesis, to develop the research established here for the doubly nonlinear degenerate parabolic equation

$$(u^{p-1})_t - \nabla \cdot (|\nabla u|^{p-2} \nabla u) = f \in L^{q,r}, \quad p \geq 2.$$

It would also be interesting to use the results obtained on the optimal regularity in [50] and here to study the equivalence between the two popular models of nonlinear diffusion, the porous medium equation and the p-Laplacian equation, and thus extend the equivalence presented by Iagar and Vázquez in [34]. They have established exact correspondence formulas between these solutions, showing precisely that for $0 < n < 2$ the radially symmetric solutions $u$ and $\tilde{u}$ of the porous medium equation and the $p$-Laplacian equation are related through the following transformation:

$$\tilde{u}_{\tilde{r}}(\tilde{r}, t) = D_1 r^{\frac{2n-2}{m+1}} u(r, t), \quad D_1 = \left( \frac{(mn - n + 2)^2}{m(m+1)^2} \right)^{\frac{1}{m-1}}, \quad p = m + 1.$$

If we succeed in this generalization, we would be establishing an unprecedented approach, in which we could find the optimal regularity of the porous medium equation directly from the $p$-Laplacian equation. For more details on this equivalence and its progress see ([32, 33, 49]).

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