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Superconvergence of Piecewise Linear Semi-Discretizations for Parabolic Equations with Nonuniform Triangulations

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Abstract. In this paper we study the convergence properties of semi-discrete approximations for parabolic problems defined on two dimensional polygonal domains. These approximations are constructed using a nonstandard piecewise linear finite element method based on nonuniform triangulations of the domain and considering a variational formulation with a sesquilinear form which can be no strongly coercive. In order to increase accuracy a post-process procedure is studied.

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1. Introduction

Advection-diffusion-reaction equations are usually used on the description of the behaviour of reactive flows. In the literature an approach widely used for the computation of numerical approximations to the solution of those equations is the so called method of lines. This approach is based on the spatial discretization of the spatial operator involving only partial derivatives with respect to spatial variables, for instance by finite differences or finite element methods (FEM). Using this approach an ordinary differential system is obtained and the numerical approximation to the solution of the time dependent problem is computed considering an efficient time integrator.

The study of such approximation was largely considered in the literature during the eighties. Without being exhaustive we mention [32] as the state of the art in the 80's for semi-discrete approximations defined using FEMs and for the semi-discrete approximations defined using finite difference operators we mention [31]. Nevertheless nowadays the study of semi-discrete approximations defined using FEMs under smooth assumptions on the spatial grid remains subject of research as we can see in [1] and [6].

Our aim is to study the convergence properties of the semi-discrete approximations for the class of time dependent problems

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au = g & \text{in } (0,T] \times \Omega, \\ u(0,.) = u_0 & \text{in } \overline{\Omega}, \\ u = 0 & \text{on } [0,T] \times \partial\Omega, \end{cases} \tag{1}$$

where Ω is a polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and A is the elliptic operator

$$Au = -\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial x}\left(b\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(b\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y}\right) + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu,$$

where a, b, c, d, e and f are coefficient functions that are smooth enough.

The spatial discretization is defined using the nonstandard piecewise linear FEM introduced in [10] and [11] and considered in [2] and [3]. In the mentioned papers the method studied is based on nonuniform triangulations and the variational formulation of the elliptic problem is defined using a sesquilinear form which can be no strongly coercive. Those assumptions are less restrictive then those considered in the literature (for instance [5], [26], [33]).

We are interested in carrying over to the semi-discrete approximation for the solution of (1) the recent superconvergence results proved in the mentioned papers ([2], [3], [10], [11]). Attending to that, in Section 2 we define the semi-discrete approximation introducing the nonstandard piecewise linear FEM. In Section 3 for the elliptic equation associated with (1) we present superconvergence results for the nonstandard linear piecewise linear FE approximation established in [10] when smooth assumptions are assumed to the solution and analogously to those established in [11] when the solution satisfies weaker conditions. In these results the error estimated is the difference between the interpolation of the solution and the FE approximation. In order to compare the FE approximation with the solution we introduce in this section a way how to post-process the gradient of the solution such that it becomes of higher order accuracy. In order to prove the accuracy of the post-process procedure some smoothness assumptions on the triangulation are assumed.

Using the results presented in Section 4 we prove superconvergent upper bounds to the error of the nonstandard semi-discrete piecewise linear FE approximation under the assumptions mentioned before.

In Section 5 the semi-discrete approximation studied is related with the semi-discrete approximation defined using a certain finite differences method and so in the language of finite differences our superconvergence results are supraconvergence results (see for example [12]–[14], [16], [18], [23], [24], [28]). Numerical results illustrating the performance of the methods are also presented.

Finally, we remark that following this paper similar results can be obtained for parabolic systems of equations using the estimates proved in [2] and [3].

2. The piecewise linear semi-discrete approximation

In what follows we introduce the discrete variational problem which allows us to compute the semi-discrete FE approximation to the solution of (1).

Let $h = (h_j)_{\mathbb{Z}}$ and $k = (k_\ell)_{\mathbb{Z}}$ be two sequences of positive numbers. We define the grid

$$\mathbb{R}_1 := \{x_j \in \mathbb{R} : x_{j+1} = x_j + h_j,\, j \in \mathbb{Z}\}$$

with $x_0 \in \mathbb{R}$ given and a corresponding grid $\mathbb{R}_2$ with the mesh-size vector k in place of h. Let

$$\mathbb{R}_H := \mathbb{R}_1 \times \mathbb{R}_2 \subset \mathbb{R}^2.$$

Define also

$$\Omega_H := \Omega \cap \mathbb{R}_H, \qquad \partial\Omega_H := \partial\Omega \cap \mathbb{R}_H, \qquad \overline{\Omega}_H := \overline{\Omega} \cap \mathbb{R}_H.$$

The grid $\overline{\Omega}_H$ is assumed to satisfy the following regularity condition with respect to the region Ω.

(Geo) Let $\square$ be any rectangle $(x_j, x_{j+1}) \times (y_\ell, y_{\ell+1})$ formed by the grid $\mathbb{R}_H$. Then either $\square \cap \partial\Omega$ is empty or it is a diagonal of $\square$.

Let $\mathcal{T}_H$ be any triangulation of $\overline{\Omega}$ such that the nodes of $\mathcal{T}_H$ coincide with $\overline{\Omega}_H$. By $P_H v_H$ we denote the continuous piecewise linear interpolation of a grid function v_H with respect to $\mathcal{T}_H$. By $\overset{\circ}{W}_H$ we represent the space of grid functions defined in $\overline{\Omega}_H$ and vanishing on $\partial\Omega_H$.

Considering the sesquilinear form

$$a(v,w) = \Big(a\frac{\partial v}{\partial x}, \frac{\partial w}{\partial x}\Big) + \Big(c\frac{\partial v}{\partial y}, \frac{\partial w}{\partial y}\Big) + \Big(b\frac{\partial v}{\partial x}, \frac{\partial w}{\partial y}\Big) + \Big(b\frac{\partial v}{\partial y}, \frac{\partial w}{\partial x}\Big)$$
$$+ \Big(d\frac{\partial v}{\partial x}, w\Big) + \Big(e\frac{\partial v}{\partial y}, w\Big) + (fv, w), \qquad v, w \in H_0^1(\Omega)$$

and the finite dimensional subspace $\mathcal{S}_H$ of $H_0^1(\Omega)$, $\mathcal{S}_H = \{P_H w_H, w_H \in \overset{\circ}{W}_H\}$, the standard semi-discrete Galerkin approximation $P_H u_H(t) := P_H u_H(t,.) \in \mathcal{S}_H$, is introduced as the solution of the initial variational problem

$$\begin{aligned} &\Big(\frac{\partial}{\partial t} P_H u_H(t), P_H v_H\Big) + a(P_H u_H(t), P_H v_H) = (g(t), P_H v_H), \quad t > 0, \\ &P_H u_H(0) = w, \end{aligned} \tag{2}$$

for $P_H v_H \in \mathcal{S}_H$, where $(\cdot\,,\cdot)$ is the L^2 inner product, $g(t) \in L^2(\Omega)$ is such that $g(t)(x,y) = g(t,x,y), (x,y) \in \Omega$ and $w \in \mathcal{S}_H$ is an approximation of u_0 in $\mathcal{S}_H$.

It is known that for certain kind of domains, if the family of triangulations $(\mathcal{T}_H)_H$ is regular and the sesquilinear form $a(\cdot\,,\cdot)$ is strongly coercive then $P_H u_H(t)$ is second order convergent to $u(t,.)$ with respect to the L^2-norm. By regular we mean that

$$\exists C \in \mathbb{R},\, \forall \mathcal{T}_H \in (\mathcal{T}_H)_H,\, \forall \Delta \in \mathcal{T}_H,\, |\Delta| \geq C(\operatorname{diam}\Delta)^2.$$

We would like to warn the reader that different authors have different naming conventions when it comes to this. For example, in his book [32], Thomée calls this condition *quasi-uniformity* of the mesh.

In what follows we define a semi-discrete approximation to the solution u which has the latter convergence property under weaker assumptions:

(H1) nonuniform triangulations such that the nodes coincide with $\overline{\Omega}_H$;

(H2) sesquilinear forms $a(.,.)$ such that the variational problem

$$\text{find } v \in H_0^1(\Omega) \text{ such that } a(v,w) = 0, \quad w \in H_0^1(\Omega),$$

has only the solution $v = 0$.

We remark that if the sesquilinear form is strongly coercive then (H2) holds. Otherwise (H2) does not imply that $a(.,.)$ is strongly coercive.

Let us consider two special triangulations related to the set $\overline{\Omega}_H$, which we call $\mathcal{T}_H^{(1)}$ and $\mathcal{T}_H^{(2)}$. They are obtained from the disjoint decomposition

$$\mathbb{R}_H = \mathbb{R}_H^{(1)} \dot{\cup} \mathbb{R}_H^{(2)},$$

where the sum $j + \ell$ of the indices of the points (x_j, y_ℓ) in $\mathbb{R}_H^{(1)}$ and in $\mathbb{R}_H^{(2)}$ is even or odd, respectively. To simplify the following definition we introduce $\mathbb{R}_H^{(3)} := \mathbb{R}_H^{(1)}$. With each point $(x_j, y_\ell) \in \mathbb{R}_H$ we associate the triangles $\Delta_{j,\ell}^{(i)}$, $i = 1, 2, 3, 4$, which have a right angle at (x_j, y_ℓ) and two of the four closest neighbour grid points of (x_j, y_ℓ) as further vertices. We then define the triangulations

$$\begin{aligned}
\mathcal{T}_{H,1}^{(s)} &:= \{\Delta_{j,\ell}^{(i)} \subset \bar{\Omega}, \;\; (x_j, y_\ell) \in \mathbb{R}_H^{(s)}, \;\; i \in \{1,2,3,4\}\} \\
\mathcal{T}_{H,2}^{(s)} &:= \left\{\Delta_{j,\ell}^{(i)} \subset \left(\bar{\Omega} \backslash \bigcup_{\Delta \in \mathcal{T}_{H,1}^{(s)}} \mathring{\Delta}\right), \;\; (x_j, y_\ell) \in \mathbb{R}_H^{(s+1)}, \;\; i \in \{1,2,3,4\}\right\} \qquad (3) \\
\mathcal{T}_H^{(s)} &:= \mathcal{T}_{H,1}^{(s)} \cup \mathcal{T}_{H,2}^{(s)}, \quad s = 1, 2,
\end{aligned}$$

of $\overline{\Omega}$ ($\mathring{\Delta}$ denotes the interior of Δ). Figure A shows an example of one of these triangulations.

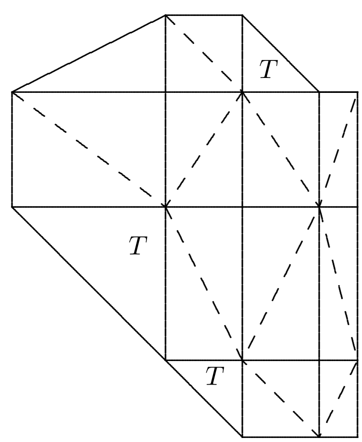

FIG. A. Triangulation $\mathcal{T}_H^{(s)}$. T indicates triangles of $\mathcal{T}_{H,2}^{(s)}$.

With respect to these triangulations, the continuous piecewise linear interpolations $P_H^{(s)} v_H$ of $v_H \in \mathring{W}_H$, $s = 1, 2$, are well-defined.

Now we define the sesquilinear form $a_H(.\,,.)$ as the arithmetical mean

$$a_H = \frac{1}{2}(a_H^{(1)} + a_H^{(2)}) \tag{4}$$

of two sesquilinear forms, each of which has the form

$$a_H^{(s)} = a^{(s)} + b^{(s)} + c^{(s)} + d^{(s)} + e^{(s)} + f^{(s)}, \; s = 1, 2. \tag{5}$$

The sesquilinear forms on the right-hand side of (5) are all constructed in a similar way by summing particular approximations of the "energy" related to each corresponding differential term over the triangles of $\mathcal{T}_H^{(s)}$. So let $\Delta \in \mathcal{T}_H^{(s)}$. We define a_Δ to be the value of a at the midpoint of the side of Δ parallel to the x-axis. Then let

$$a^{(s)}(v_H, w_H) := \sum_{\Delta \in \mathcal{T}_H^{(s)}} a_\Delta \int_\Delta \frac{\partial}{\partial x} P_H^{(s)} v_H \frac{\partial}{\partial x} P_H^{(s)} w_H \, dx \, dy. \tag{6}$$

Similarly, let c_Δ be the value of c at the midpoint of the side of Δ parallel to the y-axis and

$$c^{(s)}(v_H, w_H) := \sum_{\Delta \in \mathcal{T}_H^{(s)}} c_\Delta \int_\Delta \frac{\partial}{\partial y} P_H^{(s)} v_H \frac{\partial}{\partial y} P_H^{(s)} w_H \, dx \, dy. \tag{7}$$

In the approximation of the mixed derivative terms we need

$$b_\Delta := b(x_\Delta, y_\Delta), \tag{8}$$

where (x_Δ, y_Δ) is the vertex of Δ associated with the angle $\frac{\pi}{2}$ of Δ. Then

$$b^{(s)}(v_H, w_H) := \sum_{\Delta \in \mathcal{T}_H^{(s)}} b_\Delta \int_\Delta \left[\frac{\partial}{\partial x} P_H^{(s)} v_H \frac{\partial}{\partial y} P_H^{(s)} w_H + \frac{\partial}{\partial y} P_H^{(s)} v_H \frac{\partial}{\partial x} P_H^{(s)} w_H \right] dx \, dy. \tag{9}$$

For approximating the first order terms let

$$(P_H^{(s)} v_H)_{\Delta,x} := P_H^{(s)} v_H(x_\Delta, y_\Delta), \quad \Delta \in \mathcal{T}_H^{(s)}, \tag{10}$$

where (x_Δ, y_Δ) is the midpoint of the side of Δ parallel to the x-axis. Correspondingly, we introduce $(P_H^{(s)} v_H)_{\Delta,y}$, where in this case (x_Δ, y_Δ) is taken to be the midpoint of the side of Δ parallel to the y-axis. Then we define

$$d^{(s)}(v_H, w_H) := \sum_{\Delta \in \mathcal{T}_H^{(s)}} [P_H^{(s)}(d w_H)]_{\Delta,x} \int_\Delta \frac{\partial}{\partial x} P_H^{(s)} v_H \, dx \, dy, \tag{11}$$

$$e^{(s)}(v_H, w_H) := \sum_{\Delta \in \mathcal{T}_H^{(s)}} [P_H^{(s)}(e w_H)]_{\Delta,y} \int_\Delta \frac{\partial}{\partial y} P_H^{(s)} v_H \, dx \, dy. \tag{12}$$

Let $\square_{j,\ell}$ be $[x_{j-1/2}, x_{j+1/2}] \times [y_{\ell-1/2}, y_{\ell+1/2}] \cap \Omega$ with $x_{j-1/2} = x_j - h_{j-1}/2$, $x_{j+1/2} = x_j + h_j/2$ being $y_{\ell-1/2}$ and $y_{\ell+1/2}$ defined analogously and let $|\square_{j,\ell}|$ be the area of $\square_{j,\ell}$. Finally

$$f^{(s)}(v_H, w_H) := \sum_{(x_j,y_\ell)\in\Omega_H} |\square_{j,\ell}| f(x_j, y_\ell) v_{j,\ell} w_{j,\ell}. \tag{13}$$

In the following we consider the discrete L^2 inner product

$$(v_H, w_H)_H = \sum_{(x_j,y_\ell)\in\bar{\Omega}_H} |\square_{j,\ell}| v_H(x_j, y_\ell) w_H(x_j, y_\ell) \tag{14}$$

for $v_H, w_H \in \overset{\circ}{W}_H$.

Replacing, in (2), $a(P_H u_H(t), P_H v_H)$ by $a_H(u_H(t), v_H)$ and the continuous L^2 inner product by its discrete version $(.,.)_H$ we obtain a semi-discretization,

$$\begin{aligned} &\Big(\frac{\partial u_H}{\partial t}(t), v_H\Big)_H + a_H(u_H(t), v_H) = (g_H(t), v_H)_H, \quad t > 0, \\ &u_H(0) = R_H u_0, \end{aligned} \tag{15}$$

for $v_H \in \overset{\circ}{W}_H$, where

$$g_H(t) = R_H g(t,.) \tag{16}$$

and R_H denotes the restriction operator.

If g does not allow us to compute its value at each point of the grid then we can't use the last method. In this cases another method is defined replacing $g_H(t)$ by $\widetilde{g}_H(t)$,

$$\widetilde{g}_H(t)(x_j, y_\ell) = \frac{1}{|\square_{j,\ell}|} \int_{\square_{j,\ell}} g(t, x, y)\, dx\, dy, \tag{17}$$

and another discrete variational problem is obtained

$$\begin{aligned} &\Big(\frac{\partial u_H}{\partial t}(t), v_H\Big)_H + a_H(u_H(t), v_H) = (\widetilde{g}_H(t), v_H)_H, \\ &u_H(0) = R_H u_0, \end{aligned} \tag{18}$$

for $v_H \in \overset{\circ}{W}_H$.

The estimate for the error $\|R_H u(t,.) - u_H(t)\|_H$ where u_H is the semidiscrete solution computed with (15) or (18) is obtained using the corresponding estimate for the stationary case. Attending to that, in the next section we summarize some superconvergence results obtained for this case. We also develop a post-process procedure which enable us to compute a second order approximation to the gradient of the solution in the stationary case.

3. The stationary case

3.1. Some superconvergence results for the nonstandard linear FE solution

Let us consider in what follows the elliptic boundary value problem

$$\begin{cases} Av = g^* & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{19}$$

Let $v_H \in \overset{\circ}{W}_H$ be the approximation to v computed using the following two nonstandard piecewise linear FEMs

$$a_H(v_H, w_H) = (g_H^*, w_H)_H \tag{20}$$

for $w_H \in \overset{\circ}{W}_H$, being the first one defined by (20) with $g_H^* = R_H g^*$ and the second one defined by (20) with g_H^* given by (17) with g replaced by g^*.

We now consider a sequence of grids $\mathbb{R}_H$ such that the maximal mesh-size $H_{\max}$ tends to zero. We use the symbol "$H \in \Lambda$" to indicate the sequence of discretizations considered and write "$(H \in \Lambda)$" for the convergence with respect to H running through this sequence. By $H_{\max}$ we denote the maximal mesh-size in both x and y directions. We write $\|\cdot\|_r$ for the standard norm in the Sobolev space $H^r(\Omega)$, $r \in \mathbb{N}_0$, and $\|\cdot\|_{r,D}$ if the underlying region is the domain D. The notation $\|\cdot\|_{r,\infty}$ is used for the standard norm in $W^{r,\infty}(\Omega)$.

We start by noting that for the sesquilinear form $a(.,.)$ coercivity holds ([17]). In [10] was proved that $|a_H(v_H, w_H) - a(P_H v_H, P_H w_H)| \to 0$ $(H \in \Lambda)$, $v_H, w_H \in \overset{\circ}{W}_H$. So the following result holds:

Proposition 1. *For $H_{\max}$ small enough*

$$a_H(w_H, w_H) \geq C_E \|P_H w_H\|_1^2 - C_K \|w_H\|_H^2, \tag{21}$$

for all $w_H \in \overset{\circ}{W}_H$ where $C_E > 0$ and C_K denote constants depending on the coefficients of A but not on the triangulation $\mathcal{T}_H$ or w_H.

This proposition means that $a_H(.,.)$ is "coercive" in $\overset{\circ}{W}_H \times \overset{\circ}{W}_H$. One main ingredient for convergence analysis is the inverse stability of $a_H(.,.)$ which follows from the inequality above. So, for the nonstandard linear FE solution defined by the described first method we have the following result ([10], Theorem 1):

Theorem 1. *Assume that the grids $\overline{\Omega}_H$ satisfy condition* (Geo) *and that* (H2) *holds. If the solution v of* (19) *is in $C^4(\bar{\Omega})$ then, for $H_{\max}$ small enough, the vari-*

ational problem (20) *with* $g_H^* = R_H g^*$ *has a unique solution* $v_H \in \overset{\circ}{W}_H$ *satisfying*

$$\|P_H v_H - P_H R_H v\|_1 \le C \left(\sum_{\Delta \in \mathcal{T}_H} |\Delta| (\operatorname{diam} \Delta)^4 \, \|v\|^2_{4,\infty,\Delta} \right)^{1/2}, \tag{22}$$

where C *denotes a positive constant depending on the coefficients of* A *but not on the triangulation* $\mathcal{T}_H$ *or* v.

Remark 1. In [10], it was observed that if the boundary $\partial\Omega$ contains a straight line segment which is not parallel to a coordinate axis then in (22) arises also the term $\sum_{\Delta \in \mathcal{T}_{H,2}} |\Delta| (\operatorname{diam} \Delta)^2 \|v\|^2_{4,\infty,\Delta}$.

Remark 2. The condition $v \in C^4(\bar{\Omega})$ was weakened in [11] for a nonstandard piecewise linear FE solution analogous to the one defined by (20) with g_H^* given by (17) with g replaced by g^* and considering general boundary conditions. Following the procedure introduced in that paper it can be proved that

1. if Ω is a union of rectangles then

$$\|P_H v_H - P_H R_H v\|_1 \le C \left(\sum_{\Delta \in \mathcal{T}_H} (\operatorname{diam} \Delta)^4 \, \|v\|^2_{3,\Delta} \right)^{1/2}, \tag{23}$$

   assuming that $v \in H^3(\Omega)$;
2. if $\partial\Omega$ has a straight line segment not parallel to a coordinate axis and $b = 0$ then in (23) arises also the term

$$\sum_{\Delta \in \mathcal{T}_{H,2}} (\operatorname{diam} \Delta)^2 |\Delta| \|u\|^2_{W^2_\infty(\Delta)}.$$

3.2. Post-processing

In this section we will present a way how to post-process the gradient of the discrete solution in such a way, that it becomes of higher order accuracy and we assume for simplicity that Ω is the union of rectangles. The reason to be interested in post-processing is the following. We know that the gradient of the standard linear FE solution satisfies the following bound,

$$\|\nabla(u - P_H u_H)\|_0 \le C H_{\max} \|u\|_2. \tag{24}$$

Apart from that, one of the main results from [11] is that for nonuniform triangulations, we have

$$\|\nabla P_H (R_H u - u_H)\|_0 \le C \left(\sum_{\Delta \in \mathcal{T}_H} (\operatorname{diam} \Delta)^4 \|u\|^2_{3,\Delta} \right)^{1/2}. \tag{25}$$

This bound reduces to

$$\|\nabla P_H(R_H u - u_H)\|_0 \leq C H_{\max}^2 \|u\|_3. \tag{26}$$

This result, which is well-known for the standard FEM on uniform triangulations of the domain ([30]), is merely a comparison between the gradients of the nonstandard FE solution and the continuous piecewise linear interpolant of the exact solution. In spite of the fact that it constitutes an $\mathcal{O}(H_{\max}^2)$ bound, it is not automatically clear that it is the key to construct a better approximation of ∇u than $\nabla P_H u_H$. As a matter of fact, since $\nabla P_H u_H$ is piecewise constant, the bound (24) is of optimal order and cannot be improved by performing a better analysis. Only after a suitable *post-processing* step, that we will explain now, it is indeed possible to find a global $\mathcal{O}(H_{\max}^2)$ approximation of ∇u, which necessarily lives in a space that allows $\mathcal{O}(H_{\max}^2)$ approximations. The procedure, of which the computational costs are negligible compared to the computation of $P_H u_H$, combines several standard ideas from the superconvergence community (see for example [15, 22, 25]), but applied to nonuniform meshes.

3.2.1. Main idea and implementation of the post-processing

The main observation is not very difficult. Basically, it states that the top of a parabola is located at the average of its zeros.

Proposition 2. *Let $\Delta \in \mathcal{T}_H$ be given, and q in the space of the second order polynomials over Δ, $\mathcal{P}_2(\Delta)$. Then with z denoting either x or y, we have*

$$\frac{\partial}{\partial z}(P_H R_H q)(M_z) = \frac{\partial}{\partial z} q(M_z) \tag{27}$$

at the midpoint M_z of the edge of Δ parallel to the z-axis.

The consequence is that a vector field that is locally equal to the gradient of a quadratic polynomial q, can be recovered from its interpolant $P_H R_H q$ by sampling derivatives at the proper points. Moreover, those points are situated within a patch of neighbouring elements, and the number of elements in such a patch is uniformly bounded. We will now give an example how this reconstruction may take place.

Example of the reconstruction process

In Figure B, part of a mesh is shown. Suppose we are given the piecewise constant gradient of the interpolant $P_H R_H q$ of some unknown quadratic polynomial q that is defined on this part. We will now recover the linear vector field ∇q on the triangle with vertices P, Q and R. The first step is to observe, that it is sufficient to find the exact (vector)-values of ∇q at the points P, Q and R. Linear interpolation between those values will then result in ∇q itself.

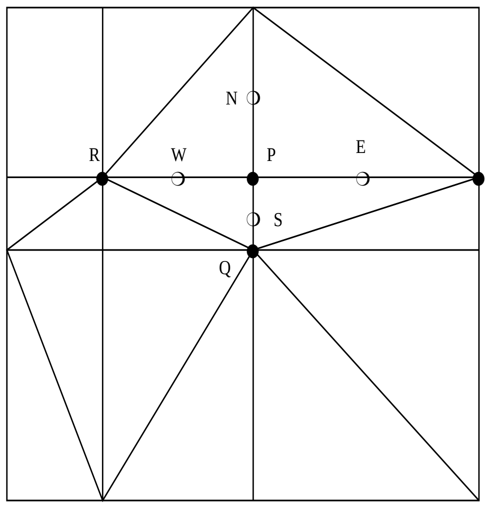

FIG. B. Post-processing of the gradient of the continuous piecewise linear interpolant of a quadratic function.

So, we will concentrate on finding the exact value of ∇q at the point P. Around P, we see four midpoints of edges, denoted by N, E, S and W. Since q is quadratic, by Proposition 2 we know that

$$\frac{\partial}{\partial x}(P_H R_H q)(E) = \frac{\partial}{\partial x} q(E) \quad \text{and} \quad \frac{\partial}{\partial x}(P_H R_H q)(W) = \frac{\partial}{\partial x} q(W). \tag{28}$$

But since $\frac{\partial}{\partial x} q$ is a linear function on the line through the points E and W, we can easily compute its value at P in terms of its values at E and W, which results in

$$\frac{\partial}{\partial x} q(P) = \frac{E-P}{E-W} \frac{\partial}{\partial x}(P_H R_H q)(W) + \frac{P-W}{E-W} \frac{\partial}{\partial x}(P_H R_H q)(E). \tag{29}$$

Moreover, since the x-derivatives of the interpolant are piecewise constant, for later purposes we may choose to employ only nodal values of $P_H R_H q$ and write (29) accordingly as

$$\frac{\partial}{\partial x} q(P) = \frac{T-P}{P-R} \frac{q(P)-q(R)}{T-R} + \frac{P-R}{T-P} \frac{q(T)-q(P)}{T-R}, \tag{30}$$

which is in fact a reconstruction in the form of a so-called *long difference quotient*. Clearly, the y-derivative at the point P can be computed similarly, using the values of the y-derivative of $P_H R_H q$ at the points N and S in the form (29), or the closest-by nodal values of $P_H R_H q$ in the y-direction in the form (30). Finally, by doing the same at the points Q and R, we obtain the five scalar values that are necessary to find ∇q on the triangle PQR.

Remark 3. In case that P would happen to be a node at the boundary, we compute the exact value of ∇q at P by extrapolation of the nearest midpoints. For example, supposing that in Figure B the point T is at the boundary, we compute $\dfrac{\partial}{\partial x}q(T)$ by extrapolating the x-derivatives of $P_H R_H q$ at the points W and E. An alternative is to firstly compute the reconstructed values at P and R and to do an extrapolation on those two values.

3.2.2. Post-processing of the interpolant

We will now study the reconstruction process first applied to gradients of interpolants of arbitrary functions $w \in H^3(\Omega)$. Then, in Section 3.2.3, we will make the step towards the nonstandard FE approximations.

Definition 1. Let w_H be a grid function. We denote by K_H the linear operator that maps the function $\nabla P_H w_H$ onto the continuous piecewise linear vector function $K_H \nabla P_H w_H$ obtained by means of the reconstruction process explained above. Moreover, for each Δ, we denote the convex hull of the patch of elements that is needed to obtain the reconstructed function on Δ by $P(\Delta)$.

The following technical lemma will be needed. Note that it necessarily exploits an L^∞ setting, since the corresponding result does not hold in L^2.

Lemma 1. *Let $w \in H^3(\Omega)$. Then for all $\Delta \in \mathcal{T}_H$,*

$$\|K_H \nabla P_H R_H w\|_{\infty,\Delta} \leq 2\|\nabla w\|_{\infty,P(\Delta)}. \tag{31}$$

Proof. Let $w \in H^3(\Omega)$ be given. First, assume that Δ is a triangle whose nodes are not on $\partial\Omega$. The maximum of a linear function over a triangle is taken at one of the nodes. Hence, since the nodal values of $K_H \nabla P_H R_H w$ on Δ are convex combinations of values of partial derivatives of $P_H R_H w$ on neighbouring triangles, we get

$$\|K_H \nabla P_H R_H w\|_{\infty,\Delta} \leq \|\nabla P_H R_H w\|_{\infty,P(\Delta)}. \tag{32}$$

Second, if Δ is a triangle such that one of its nodes lies on $\partial\Omega$, then the reconstructed value is obtained by linear extrapolation, as mentioned in Remark 3. So, the reconstructed value is not a convex combination of neighbouring values anymore, so (32) does not hold. However, since the extrapolation does not go over a longer distance than half the edge length (i.e., the length between the points E and T in Figure B), it holds that

$$\|K_H \nabla P_H R_H w\|_{\infty,\Delta} \leq 2\|\nabla P_H R_H w\|_{\infty,P(\Delta)}. \tag{33}$$

Finally, since $w \in H^3(\Omega)$, by the Mean Value Theorem we have that

$$\|\nabla P_H R_H w\|_{\infty,P(\Delta)} \leq \|\nabla w\|_{\infty,P(\Delta)}, \tag{34}$$

and the lemma is proved. □

Theorem 2. *Let $\Delta \in \mathcal{T}_H$ be given. Denote by $\rho(P(\Delta))$ the radius of the largest ball that is included in $P(\Delta)$. Then for all $w \in H^3(\Omega)$,*

$$\|\nabla w - K_H \nabla P_H R_H w\|_{0,\Delta} \leq C \sqrt{\frac{|\Delta|}{|P(\Delta)|}} \frac{(\operatorname{diam} P(\Delta))^3}{\rho(P(\Delta))} |w|_{3,P(\Delta)}. \tag{35}$$

Proof. Switching from the L^2-norm to the supremum-norm gives, using a crude triangle inequality, that for arbitrary $w \in H^3(\Omega)$,

$$\begin{aligned} \|\nabla w - K_H \nabla P_H R_H w\|_{0,\Delta} &\leq C|\Delta|^{1/2} \|\nabla w - K_H \nabla P_H R_H w\|_{\infty,\Delta} \\ &\leq C|\Delta|^{1/2} \left(\|\nabla w\|_{\infty,\Delta} + \|K_H \nabla P_H R_H w\|_{\infty,\Delta}\right) \\ &\leq C|\Delta|^{1/2} \|\nabla w\|_{\infty,P(\Delta)}, \end{aligned} \tag{36}$$

where in the latter bound we have used Lemma 1. Since the constant C in (36) does not depend on w, the following holds for all polynomials q that are quadratic on $P(\Delta)$, since on Δ we have that $\nabla q = K_H \nabla P_H R_H q$ by construction, so

$$\begin{aligned} \|\nabla w - K_H \nabla P_H R_H w\|_{0,\Delta} &\leq \|\nabla (w-q) - K_H \nabla P_H R_H (w-q)\|_{0,\Delta} \\ &\leq C|\Delta|^{1/2} \|\nabla (w-q)\|_{\infty,P(\Delta)}. \end{aligned} \tag{37}$$

Now, due to the continuous embedding of $H^3(\Omega)$ in $W^{1,\infty}(\Omega)$, interpolation theory in Sobolev spaces (see Ciarlet [7], Chapter 3) yields that by choosing for q the best approximation for w on $P(\Delta)$ in the $W^{1,\infty}$ sense,

$$\|\nabla (w-q)\|_{\infty,P(\Delta)} \leq C|P(\Delta)|^{-1/2} \frac{(\operatorname{diam} P(\Delta))^3}{\rho(P(\Delta))} \|w\|_{3,P(\Delta)}. \tag{38}$$

Combining (37) and (38), this Bramble–Hilbert approach proves the theorem. □

Clearly, in order for this bound to be of interest, we need certain relations between the geometry of $P(\Delta)$ in comparison with Δ. From now on, we will therefore assume that the family of meshes is regular. We will also assume that the grids are *patch-regular*, in the sense that

$$\exists C \in \mathbb{R},\, \forall \mathcal{T}_H \in (\mathcal{T}_H)_H,\, \forall \Delta \in \mathcal{T}_H,\, |P(\Delta)| \geq C(\operatorname{diam} P(\Delta))^2.$$

This last assumption guarantees that the diameter of the largest ball included in the patch is of the same order of magnitude as the diameter of the patch.

Corollary 1. *Under the additional assumption of patch-regularity, the bound* (23) *can be written as*

$$\|\nabla w - K_H \nabla P_H R_H w\|_{0,\Delta} \leq C(\operatorname{diam} P(\Delta))^2 \|w\|_{3,P(\Delta)}. \tag{39}$$

3.2.3. Post-processing of the nonstandard FE solution

We are now able to prove, that our post-processing operator K_H is also successfully applicable to the gradient of the nonstandard FE approximation in the elliptic setting. One extra assumption is needed on the meshes, and that is that the quotient of the smallest and the longest diameter of elements within a patch $P(\Delta)$ is bounded from both sides, uniformly over all triangulations. We will assume this from now on.

Lemma 2. *Under the above assumption on the mesh within a patch $P(\Delta)$, we have for all grid functions w_H that,*

$$\|K_H \nabla P_H w_H\|_{0,\Delta} \leq C \|\nabla P_H w_H\|_{0,P(\Delta)}. \tag{40}$$

Proof. Again, we work through the supremum-norm. This gives

$$\begin{aligned} \|K_H \nabla P_H w_H\|_{0,\Delta} &\leq |\Delta|^{1/2} \|K_H \nabla P_H w_H\|_{\infty,\Delta} \\ &\leq 2|\Delta|^{1/2} \|\nabla P_H w_H\|_{\infty,P(\Delta)}, \end{aligned} \tag{41}$$

where the latter inequality is borrowed from the proof of Lemma 1. The proof is now completed by the discrete inverse inequality (Ciarlet [7]) for continuous piecewise linear FE functions,

$$\|\nabla P_H w_H\|_{\infty,P(\Delta)} \leq C|\Delta|^{-1/2} \|\nabla P_H w_H\|_{0,P(\Delta)}, \tag{42}$$

for which we have used the above additional assumption on the mesh. □

Theorem 3. *Under all previous assumptions on the meshes and if the solution u is in $H^3(\Omega)$, we have*

$$\|\nabla u - K_H \nabla P_H u_H\|_0 \leq C \left(\sum_{\Delta \in \mathcal{T}_H} (\operatorname{diam} \Delta)^4 \|u\|^2_{3,P(\Delta)} \right)^{1/2} + C\|P_H(R_H u - u_H)\|_1. \tag{43}$$

Proof. We start off with a simple triangle inequality,

$$\|\nabla u - K_H \nabla P_H u_H\|_0 \leq \|\nabla u - K_H \nabla P_H R_H u\|_0 + \|K_H \nabla P_H (R_H u - u_H)\|_0. \tag{44}$$

The first term in the right-hand side of (44) can be bounded by splitting it into contributions over each triangle, and applying Corollary 1. Note that by the final assumption on the mesh made in this section, the diameter of the patch is a constant times the diameter of the element itself. To the second term in the right-hand side of (44) we apply Lemma 2. By adding the two contributions, we arrive at the bound (43). □

4. The time-dependent problem

Using the main results of Section 3 we are able to present the convergence results for the time dependent problem.

Theorem 4. *Assume that $\partial\Omega$ is the union of straight line segments parallel to the coordinate axes, the grids $\overline{\Omega}_H$ satisfy condition* (Geo) *and that* (H2) *holds. Then, for $H_{\max}$ small enough, the solution $u_H(t)$ of* (15) *satisfies*

$$\begin{aligned}
&\|u_H(t) - R_H u(t,.)\|_H \\
&\le C\Bigg(\Big(\sum_{\Delta\in\mathcal{T}_H} |\Delta|(\operatorname{diam}\Delta)^4\,\|u(t,.)\|^2_{4,\infty,\Delta}\Big)^{1/2} + \exp(C_K t)\Bigg(C_{\mathcal{T}_H,4}(u,|\Delta|^2) \\
&\quad + \int_0^t \exp(-C_K\tau)\Big(\sum_{\Delta\in\mathcal{T}_H} |\Delta|(\operatorname{diam}\Delta)^4\,\|\frac{\partial u}{\partial t}(\tau,.)\|^2_{4,\infty,\Delta}\Big)^{1/2} d\tau\Bigg)\Bigg),
\end{aligned} \tag{45}$$

where $C_{\mathcal{T}_H,4}(u,|\Delta|^2) = \Big[\sum_{\Delta\in\mathcal{T}_H} |\Delta|(\operatorname{diam}\Delta)^4\,\|u(0,.)\|^2_{4,\infty,\Delta}\Big]^{1/2}$ *and provided that* $\dfrac{\partial u}{\partial t}(t,.) \in C^4(\bar{\Omega})$.

Proof. For each $t\in[0,T]$ let us define $u_{H,t}$ as the solution of (20) with $g_H^*(t) = g(t,.) - \frac{\partial u}{\partial t}(t,.)$ in Ω_H. According to Theorem 1 and with $\theta_H(t) := u_{H,t} - R_H u(t,.) \in \overset{\circ}{W}_H$, we have

$$\|\theta_H(t)\|_H \le C\|P_H\theta_H(t)\|_1 \le C\left(\sum_{\Delta\in\mathcal{T}_H} |\Delta|(\operatorname{diam}\Delta)^4\,\|u(t,.)\|^2_{4,\infty,\Delta}\right)^{1/2}, \tag{46}$$

$C>0$. Furthermore we also have

$$\Big\|\frac{\partial\theta_H}{\partial t}(t)\Big\|_H \le C\left(\sum_{\Delta\in\mathcal{T}_H} |\Delta|(\operatorname{diam}\Delta)^4\,\Big\|\frac{\partial u}{\partial t}(t,.)\Big\|^2_{4,\infty,\Delta}\right)^{1/2}. \tag{47}$$

By $e_H(t) \in \overset{\circ}{W}_H$ we represent the error $u_H(t) - u_{H,t}$. It can be shown that

$$\Big(\frac{\partial e_H}{\partial t}(t), w_H\Big)_H + a_H(e_H(t), w_H) = -\Big(\frac{\partial\theta_H}{\partial t}(t), w_H\Big)_H \tag{48}$$

for $w_H \in \overset{\circ}{W}_H$. Choosing $w_H = e_H(t)$ and attending to Proposition 1 we get

$$\Big(\frac{\partial e_H}{\partial t}(t), e_H(t)\Big)_H + C_E\|P_H e_H(t)\|_1^2 - C_k\|e_H(t)\|_H^2 \le \|\frac{\partial\theta_H}{\partial t}(t)\|_H\|e_H(t)\|_H.$$

Since we have

$$\big(\|e_H(t)\|_H^2 + \varepsilon^2\big)^{1/2}\frac{d}{dt}\big(\|e_H(t)\|_H^2 + \varepsilon^2\big)^{1/2} = \Big(\frac{\partial e_H}{\partial t}(t), e_H(t)\Big)_H$$

attending to (21) we obtain

$$\frac{d}{dt}\big(\|e_H(t)\|_H^2 + \varepsilon^2\big)^{1/2} - C_K\|e_H(t)\|_H \le \Big\|\frac{\partial\theta_H}{\partial t}(t)\Big\|_H. \tag{49}$$

Considering Gronwall's Lemma in (49) and letting $\varepsilon \mapsto 0$ we obtain

$$\|e_H(t)\|_H \leq \exp(C_K t)\left(\|e_H(0)\|_H + \int_0^t \exp(-C_K\tau)\Big\|\frac{\partial \theta_H}{\partial t}(\tau,.)\Big\|_H \, d\tau\right). \tag{50}$$

Finally, from (46), (47) and (50) we conclude the proof. □

Remark 4. Attending to Remark 1, if the boundary $\partial\Omega$ contains a straight line segment which is not parallel to a coordinate axis then (45) holds with an additional term obtained considering a summation over triangles $\Delta \in \mathcal{T}_{H,2}$.

Let us consider now the case that A is the Laplace operator. It is known that the semi-discrete approximation defined by using the standard piecewise linear FEM, has second convergence order with respect to norm $\|.\|_0$, if the triangulations are regular. For the Laplace operator we observe that the nonstandard piecewise linear FEM coincides with the classical piecewise linear FEM combined with a special quadrature formula. In the next result we establish that $\|P_H u_H(t) - P_H R_H u(t,.)\|_1$ has second convergence order when nonuniform grids are considered.

Theorem 5. *Let Ω be such that $\partial\Omega$ is the union of straight line segments. Assume that the grids $\overline{\Omega}_H$ satisfy condition* (Geo). *Then, for $H_{\max}$ small enough, the solution $u_H(t)$ of* (15) *with the Laplace operator satisfies the following*

$$\begin{aligned}\|P_H u_H(t) - P_H R_H u(t,.)\|_1 \leq C \Bigg(\Big(\sum_{\Delta\in\mathcal{T}_H} |\Delta|(\operatorname{diam}\Delta)^4 \, \|u(t,.)\|^2_{4,\infty,\Delta}\Big)^{1/2} \\ + \Big(C_{\mathcal{T}_H,4}(u,|\Delta|^2) + \sum_{\Delta\in\mathcal{T}_H} |\Delta|(\operatorname{diam}\Delta)^4 \int_0^t \|\frac{\partial u}{\partial t}(\tau,.)\|^2_{4,\infty,\Delta} \, d\tau\Big)^{1/2}\Bigg)\end{aligned} \tag{51}$$

where $C_{\mathcal{T}_H,4}(u,|\Delta|^2) = \left[\sum_{\Delta\in\mathcal{T}_H} |\Delta|(\operatorname{diam}\Delta)^4 \, \|u(0,.)\|^2_{4,\infty,\Delta}\right]$ *and provided that* $\dfrac{\partial u}{\partial t}(t,.) \in C^4(\bar{\Omega})$.

Proof. As before we write $u_H(t) - R_H u(t,.) = e_H(t) + \theta_H(t)$. According to Theorem 1 we have

$$\|P_H\theta_H(t)\|_1 \leq C\left(\sum_{\Delta\in\mathcal{T}_H} |\Delta|(\operatorname{diam}\Delta)^4 \, \|\frac{\partial u}{\partial t}(t,.)\|^2_{4,\infty,\Delta}\right)^{1/2}. \tag{52}$$

In order to estimate $\|\nabla P_H e_H(t)\|_0$, we use (48) with $w_H = \dfrac{\partial e_H}{\partial t}(t)$. Since

$$a_H\left(e_H(t), \frac{\partial e_H}{\partial t}(t)\right) = \left(\nabla P_H e_H(t), \nabla P_H \frac{\partial e_H}{\partial t}(t)\right)$$

and $C_2\|P_H w_H\|_0 \le \|w_H\|_H \le C_1\|P_H w_H\|_0$, we have

$$C_2^2\Big\|P_H\frac{\partial e_H}{\partial t}(t)\Big\|_0^2+\frac{1}{2}\frac{d}{dt}\|\nabla P_H e_H(t)\|_0^2 \le C_1^2\xi^2\Big\|P_H\frac{\partial e_H}{\partial t}(t)\Big\|_0^2+\frac{1}{4\xi^2}\Big\|P_H\frac{\partial \theta_H}{\partial t}(t)\Big\|_0^2$$

for arbitrary positive constant ξ. Choosing ξ such that $\xi^2 < \frac{C_2^2}{C_1^2}$ we get

$$\|\nabla P_H e_H(t)\|_0 \le \left(\|\nabla P_H e_H(0)\|_0^2+\frac{1}{4\xi^2}\int_0^t\Big\|P_H\frac{\partial \theta_H}{\partial t}(\tau,.)\Big\|_0^2 d\tau\right)^{1/2}. \tag{53}$$

We conclude the proof using Theorem 1 and Remark 4. □

For the semi-discrete approximation defined by (18) we have:

Theorem 6. *Let Ω be a union of rectangles. Assume that the grids $\overline{\Omega}_H$ satisfy condition* (Geo) *and that* (H2) *holds. Then, for $H_{\max}$ small enough, the solution $u_H(t)$ of* (18) *satisfies the following*

$$\begin{aligned}&\|u_H(t)-R_H u(t,.)\|_H\\ &\le C\Bigg(\Big(\sum_{\Delta\in\mathcal{T}_H}(\operatorname{diam}\Delta)^4\,\|u(t,.)\|_{3,\Delta}^2\Big)^{1/2}+\exp(C_K t)\Big(C_{\mathcal{T}_H,3}(u,|\Delta|^2)\\ &\quad+\int_0^t\exp(-C_K\tau)\Big(\Big(\sum_{\Delta\in\mathcal{T}_H}(\operatorname{diam}\Delta)^4\Big\|\frac{\partial u}{\partial t}(\tau,.)\Big\|_{3,\Delta}^2\Big)^{1/2}\\ &\qquad+\Big\|\frac{\partial u}{\partial t}(\tau,.)-\widetilde{\frac{\partial u}{\partial t}}(\tau,.)\Big\|_H\Big)d\tau\Big)\Bigg)\end{aligned} \tag{54}$$

with $\widetilde{\frac{\partial u}{\partial t}}(t,.)$ *defined by* (17) *replacing g by* $\frac{\partial u}{\partial t}(t,.)$,

$$C_{\mathcal{T}_H,3}(u,|\Delta|^2)=\Big[\sum_{\Delta\in\mathcal{T}_H}(\operatorname{diam}\Delta)^4\,\|u(0,.)\|_{3,\Delta}^2\Big]^{1/2}$$

and provided that $\frac{\partial u}{\partial t}\in H^3(\Omega)$.

Proof. The proof follows the proof of Theorem 4. We only remark that $u_{H,t}$ must be considered as the solution of the discrete variational problem

$$a_H(u_{H,t},w_H)=\Big(\tilde{g}_H(t)-\widetilde{\frac{\partial u}{\partial t}}(t),w_H\Big)_H \tag{55}$$

for $w_H \in \overset{\circ}{W}_H$. □

Remark 5. If the domain contains a section which is not parallel to a coordinate axis then on the estimation of $\|u_H(t)-R_H u(t,.)\|_H$ we must take into account Remark 2.

Remark 6. For simplicity we assume in what follows that Ω is the union of rectangles. Attending that for functions w with bounded second order derivatives in Ω we have

$$\begin{aligned}
&|\widetilde{w}(x_j, y_\ell) - w(x_j, y_\ell)| \\
&\leq \frac{1}{4}\left|\frac{\partial w}{\partial x}(x_j, y_\ell)\right| |h_j - h_{j-1}| + \frac{1}{4}\left|\frac{\partial w}{\partial y}(x_j, y_\ell)\right| |k_\ell - k_{\ell-1}| \\
&\quad + \|w\|_{2,\infty,\square_{j,\ell}}\Bigg(\frac{1}{24}\left(|h_j^2 - h_j h_{j-1} + h_{j-1}^2| + |k_\ell^2 - k_\ell k_{\ell-1} + k_{\ell-1}^2|\right) \\
&\qquad\qquad + \frac{1}{16}|h_j - h_{j-1}||k_\ell - k_{\ell-1}|\Bigg)
\end{aligned} \tag{56}$$

we conclude from Theorem 6 a first order estimate for the error $\|u_H(t) - R_H u(t,.)\|_H$.

Considering in above inequality w replaced by $\dfrac{\partial u}{\partial t}$ and assuming that $\overline{\Omega}_H$ is uniform in both directions x and y, from (54) we conclude that the solution of (18) is second order accurate. The same order can be obtained for nonuniform grids $\bar{\Omega}_H$ which are the image of uniform grids in x and y directions and which are widely used on the computation of numerical approximations to solutions of partial differential equations. In fact if $(x_j, y_\ell) = (\phi(\xi_j), \psi(\beta_\ell))$ with $\{\xi_j\}$, $\{\beta_\ell\}$ uniform grids and ϕ and ψ with bounded second order derivatives we conclude from (56) and (54) that the solution of (18) is second order accurate.

Another procedure widely used on the computation of numerical approximations for the solution of a partial differential equation with nonuniform grids is the equidistribution principle ([9], [21], [27]). We consider in what follows $\Omega = (0,1)\times(0,1)$. Let us assume that spatial grid $\{(x_j, y_\ell), j = 0, \ldots, N, \ell = 1, \ldots, M\}$ is computed by equidistributing a monitor function M, that is $x_0 = y_0 = 0$, $x_N = y_M = 1$ and

$$\int_{x_j}^{x_{j+1}} \int_{y_\ell}^{y_{\ell+1}} M(t,x,y)\,dy\,dx = \frac{1}{NM}\int_\Omega M(t,x,y)\,dy\,dx. \tag{57}$$

If M is smooth enough then it can be proved the following

$$|h_j - h_{j-1}| \leq \frac{1}{N}\left|\int_\Omega M(t,x,y)\,dy\,dx \frac{\int_0^1 M_x(t,\xi,y)\,dy}{\left(\int_0^1 M(t,\xi,y)\,dy\right)^2}\right|(h_j + h_{j-1}),$$
$$\xi \in (x_{j-1}, x_{j+1})$$

and

$$|k_\ell - k_{\ell-1}| \leq \frac{1}{M}\left|\int_\Omega M(t,x,y)\,dy\,dx \frac{\int_0^1 M_y(t,x,\eta)\,dx}{\left(\int_0^1 M(t,x,\eta)\,dx\right)^2}\right|(k_\ell + k_{\ell-1}),$$
$$\eta \in (y_{\ell-1}, y_{\ell+1}).$$

Then, from (54) we conclude that the solution of (18) is second order accurate.

Remark 7. The post-processing procedure described before in the stationary case can be applied for time dependent problems. We note that

$$\begin{aligned}\|\nabla u(t,\cdot) - K_H \nabla P_H u_H(t)\|_0 \le \|\nabla u(t,\cdot) - K_H \nabla P_H R_H u(t,\cdot)\|_0 \\ + \|K_H \nabla P_H (R_H u(t,\cdot) - u_H(t))\|_0.\end{aligned} \tag{58}$$

This shows that the post-processing is successfully applicable to the parabolic problem at time t if $u(\cdot,t) \in H^3(\Omega)$. In that case, the left term in (58) can be bounded as in Theorem 3, and using the fact that the reconstruction operator K_H is bounded, the right-hand side term in (58) can be bounded by

$$\|K_H \nabla P_H (R_H u(t,\cdot) - u_H(t))\|_0 \le C\|\nabla P_H (R_H u(t,\cdot) - u_H(t))\|_0, \tag{59}$$

of which the right-hand side was successfully bounded in Section 3.2 at least for Laplace operator.

5. Numerical results

Let us define a finite difference scheme which allows us to compute the semi-discrete approximation $u_H(t)$. For each grid point $(x_j, y_\ell) \in \mathbb{R}_H$ we define the central finite difference quotients

$$\delta_x^{(1/2)} w_{j,\ell} = \frac{w_{j+1/2,\ell} - w_{j-1/2,\ell}}{x_{j+1/2} - x_{j-1/2}}, \quad \delta_x^{(1/2)} w_{j+1/2,\ell} = \frac{w_{j+1,\ell} - w_{j,\ell}}{x_{j+1} - x_j},$$

$$\delta_x w_{j,\ell} = \frac{w_{j+1,\ell} - w_{j-1,\ell}}{x_{j+1} - x_{j-1}}.$$

Correspondingly, the central finite difference quotients with respect to the variable y are defined. Let A_H be defined by

$$\begin{aligned}A_H u_H := -\delta_x^{(1/2)}(a\delta_x^{(1/2)} u_H) - \delta_x(b\delta_y u_H) - \delta_y(b\delta_x u_H) - \delta_y^{(1/2)}(c\delta_y^{(1/2)} u_H) \\ + d\delta_x u_H + e\delta_y u_H + f u_H \qquad \text{in} \ \ \Omega_H.\end{aligned} \tag{60}$$

Attending that the following equality holds

$$a_H(v_H, w_H) = (A_H v_H, w_H)_H, \quad v_H, w_H \in \overset{\circ}{W}_H,$$

choosing in (15) a grid function v_H to vanish in all but one single point in Ω_H, it is easy to verify that (15) is equivalent to

$$\begin{aligned}\frac{d}{dt} u_H(t, x_j, y_\ell) + A_H u_H(t, x_j, y_\ell) &= g(t, x_j, y_\ell), \quad (x_j, y_\ell) \in \Omega_H, t \in (0, T], \\ u_H(t, x_j, y_\ell) &= 0, \quad (x_j, y_\ell) \in \partial\Omega_H, t \in [0, T], \\ u_H(0, x_j, y_\ell) &= u_0(x_j, y_\ell), \quad (x_j, y_\ell) \in \Omega_H.\end{aligned} \tag{61}$$

Integrating the last initial boundary value problem we get $u_H(t)$ for $t > 0$.

Analogously the computation of the solution of the discrete variational problem (18) is made solving numerically the ordinary differential system (61) with $g(t, x_j, y_\ell)$ replaced by $\tilde{g}(t, x_j, y_\ell)$.

Let $u_{j,\ell}^n$ be the numerical solution obtained combining (15) or (18) with the Crank–Nicolson method at time level t_n and at the grid point (x_j, y_ℓ). By u_H^n we denote the grid function $u_H^n(x_j, y_\ell) = u_{j,\ell}^n$. Attending to the behaviour of the Crank–Nicolson method, the error

$$u(t_n, x_j, y_\ell) - u_{j,\ell}^n = u(t_n, x_j, y_\ell) - u_H(t_n, x_j, y_\ell) + u_H(t_n, x_j, y_\ell) - u_{j,\ell}^n$$

is dominated by the space discretization error $u_H(t_n, x_j, y_\ell) - u_{j,\ell}^n$. So, in all numerical experiments we took $\log \|R_H u(t_n, .) - u_H^n\|$ as an estimate to $\log \|R_H u(t_n, .) - u_H(t_n, .)\|$ being the time step equal to 0.025.

Example 1 – Method (15). Let us consider the boundary problem (1) with the Laplace operator, defined in the rectangle $\Omega = (0, 1) \times (0, 1)$, with solution $u(t, x, y) = t \sin(\pi x) \sin(\pi y)$, and corresponding right-hand side g and initial condition u_0.

In Figure 1 we plot the logarithm of the error $R_H u(t_n, .) - u_H^n$ (with $t_n = 1$) against the logarithm of square of the maximum step-size, where '$\star$' corresponds to the $\|.\|_1$-norm and '$\bullet$' corresponds to the $\|.\|_H$-norm. Attending that $\|R_H u(t_n, .) - u_H^n\|_H / H_{\max}^2 \simeq$ Constant we conclude that the results are satisfactory. The same happens when we use the norm $\|.\|_1$ which is according to Theorem 5.

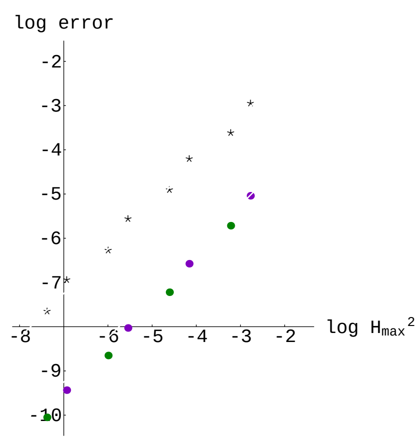

FIG. 1. Numerical results obtained with method (15)

Example 2 – Method (18). For the problem from Example 1 we obtain the numerical results plotted in Figure 2. We use '$\star$' for $\log \|.\|_1$ and '$\bullet$' for $\log \|.\|_H$-norm.

We remark that this method is recommended when the right-hand side of the partial differential equation does not allow us to compute its value at all grid points. This implies an increasing of the error comparing with the previous example.

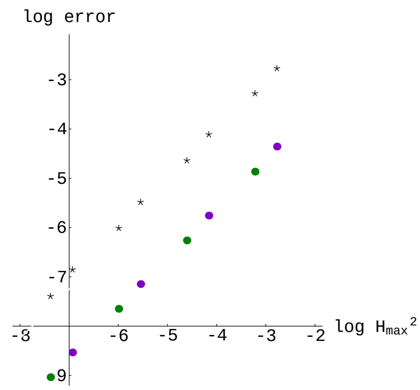

FIG. 2. Numerical results obtained with method (18).

Example 3. We consider problem (1) with the Laplace operator and with solution $u(t,x,y) = 10tx\sin(\pi y)(y+2x-2)$ defined in the polygonal domain $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < x, y < 1,\ y < -0.5x + 1\}$.

In Figure 3 we plot the numerical results obtained with method (15).

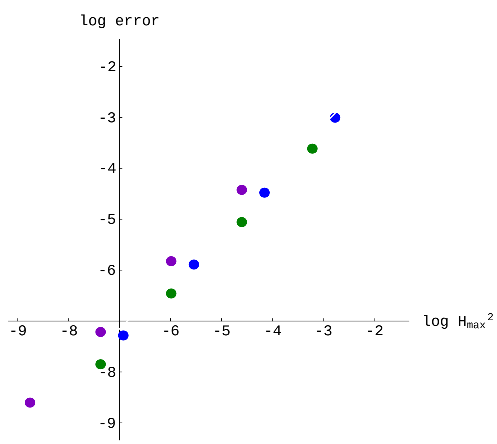

FIG. 3. Numerical results obtained with method (15) and $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < x, y < 1,\ y < -0.5x + 1\}$.

Example 4. On the proof of the convergence results was only assumed that the bilinear form $a(.,.)$ need to be coercive. In the following example we illustrate the behaviour of method (15) in this case (when $a(.,.)$ is not elliptic). We consider

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u - u = g & \text{in } \Omega \times (0,T], \\ \qquad u(0,x) = u_0(x) & \text{in } \overline{\Omega}, \\ \qquad u(t,x) = 0 & \text{on } \partial\Omega \times [0,T], \end{cases}$$

with $\Omega = (0,1) \times (0,1)$. The initial condition and g are such that this problem as the solution $u(t,x,y) = 10t \sin x \sin y (x-1)^2(y-1)^2$. In Figure 4 we plot the numerical results obtained.

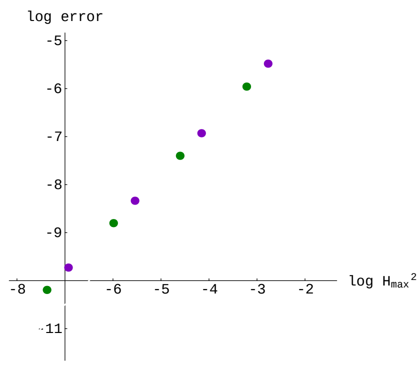

FIG. 4. Numerical results obtained with method (15) with a non coercive bilinear form.

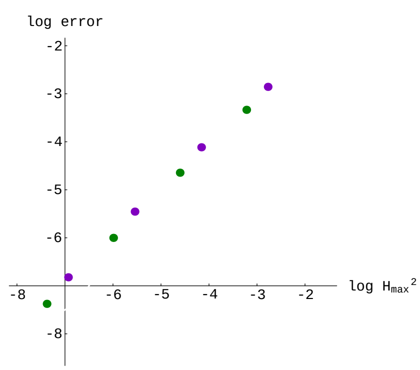

FIG. 5. Numerical results illustrating the performance of the post-processing mechanism.

Example 5 – Post-processing mechanism. We consider the same problem as in Example 1, the same discretization and then we apply the reconstruction process. In Figure 5 we plot $\log \|\nabla u(t_n,.) - K_H \nabla P_H u_H^n\|_0$ with $t_n = 1$ (which is an estimate to $\log \|\nabla u(1,.) - K_H \nabla P_H u_H(1)\|_0$) against $\log H_{\max}^2$.

6. Conclusions

The convergence properties of nonstandard semi-discrete piecewise linear FE approximations defined by (15) and (18) are studied. The convergence results were established under weaker assumptions on the regularity of the triangulation than those usually considered in the literature. Attending that the convergence analysis is performed comparing $P_H u_H(t,.)$ with $P_H R_H u(t,.)$, in Section 3.2 a post-processing procedure is introduced for the stationary case wish allows us to compare $\nabla P_H u_H$ with ∇u. An estimate for the last error was obtained under stronger regularity conditions for the mesh than those assumed in Section 3.1. Several numerical experiments were presented illustrating the convergence studies.

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