Letters in Mathematical Physics (2007) 80:155–169
DOI 10.1007/s11005-007-0151-5

On Quasi-Jacobi and Jacobi-Quasi Bialgebroids

J. M. NUNES DA COSTA[1] and F. PETALIDOU[2]
[1] *Departamento de Matemática, Universidade de Coimbra, Apartado 3008, 3001-454 Coimbra, Portugal. e-mail: jmcosta@mat.uc.pt*
[2] *Faculty of Sciences and Technology, University of Peloponnese, 22100 Tripoli, Greece. e-mail: petalido@uop.gr*

Received: 5 December 2006; revised version: 26 February 2007; accepted: 27 February 2007
Published online: 5 April 2007

Abstract. We study quasi-Jacobi and Jacobi-quasi bialgebroids and their relationships with twisted Jacobi and quasi Jacobi manifolds. We show that we can construct quasi-Lie bialgebroids from quasi-Jacobi bialgebroids, and conversely, and also that the structures induced on their base manifolds are related via a "quasi Poissonization".

Mathematics Subject Classification (2000). 17B62, 17B66, 53D10, 53D17.

Keywords. quasi-Jacobi bialgebroid, Jacobi-quasi bialgebroid, quasi-Jacobi bialgebra, twisted Jacobi manifold, quasi Jacobi manifold.

1. Introduction

The notion of *quasi-Lie bialgebroid* was introduced in [15]. It is a structure on a pair (A, A^*) of vector bundles, in duality, over a differentiable manifold M that is defined by a Lie algebroid structure on A^*, a skew-symmetric bracket on the space of smooth sections of A and a bundle map $a : A \to TM$, satisfying some compatibility conditions. These conditions are expressed in terms of a section φ of $\bigwedge^3 A^*$, which turns out to be an obstruction to the Lie bialgebroid structure on (A, A^*). A quasi-Lie bialgebroid will be denoted by (A, A^*, φ). In the case where A is a Lie algebroid, its dual vector bundle A^* is equipped with a skew-symmetric bracket on its space of smooth sections and a bundle map $a_* : A^* \to TM$ and the compatibility conditions are expressed in terms of a section Q of $\bigwedge^3 A$, the triple (A, A^*, Q) is called a *Lie-quasi bialgebroid* [9]. When $\varphi = 0$ and $Q = 0$, quasi-Lie and Lie-quasi bialgebroids are just Lie bialgebroids. We note that, while the dual of a Lie bialgebroid is itself a Lie bialgebroid, the dual of a quasi-Lie bialgebroid is a Lie-quasi bialgebroid, and conversely [9]. The quasi-Lie and Lie-quasi bialgebroids are particular cases of proto-bialgebroids [9]. As in the case of a Lie bialgebroid, the doubles $A \oplus A^*$ of a quasi-Lie and of a Lie-quasi bialgebroid are endowed with a Courant algebroid structure [9, 15].

It was shown in [16] that the theory of quasi-Lie bialgebroids is the natural framework in which we can treat *twisted Poisson manifolds*. These structures were introduced in [17], under the name of Poisson manifolds with a closed 3-form background, motivated by problems of string theory [14] and of topological field theory [8].

The notion of *Jacobi bialgebroid* and the equivalent one of *generalized Lie bialgebroid* were introduced, respectively, in [3] and [5], in such a way that a Jacobi bialgebroid is canonically associated to a Jacobi manifold and conversely. A Jacobi bialgebroid over M is a pair $((A,\phi),(A^*,W))$ of Lie algebroids over M, in duality, endowed with 1-cocycles $\phi\in\Gamma(A^*)$ and $W\in\Gamma(A)$ in their Lie algebroid cohomology complexes with trivial coefficients, respectively, that satisfy a compatibility condition. Also, its double $(A\oplus A^*,\phi+W)$ is endowed with a Courant–Jacobi algebroid structure [4,11].

In order to adapt to the framework of Jacobi manifolds the concepts of twisted Poisson manifold and quasi-Lie bialgebroid, we have recently introduced in [12] the notions of *twisted Jacobi manifold* and *quasi-Jacobi bialgebroid*. The purpose of the present paper is to develop the theory of quasi-Jacobi bialgebroids, as well as of its dual concept of *Jacobi-quasi bialgebroids*, and to establish a very close relationship between quasi-Jacobi and quasi-Lie bialgebroids.

The paper contains four sections, besides Section 1, and one Appendix. In Section 2 we recall the definition of quasi-Jacobi bialgebroid, we present some basic results established in [12], we develop the examples of quasi-Jacobi and Jacobi-quasi bialgebroids associated to twisted Jacobi manifolds and to quasi Jacobi manifolds, and, finally, we study the triangular quasi-Jacobi bialgebroids. Section 3 is devoted to the study of the structures induced on the base manifolds of quasi-Jacobi and Jacobi-quasi bialgebroids. Several examples are presented. In Section 4 we establish a one-to-one correspondence between quasi-Jacobi bialgebroid structures $((A,\phi),(A^*,W),\varphi)$ over a manifold M and quasi-Lie bialgebroid structures $(\tilde{A},\tilde{A}^*,\tilde{\varphi})$ over $\tilde{M}=M\times\mathbb{R}$. Also, we prove that the structure induced on $\tilde{M}=M\times\mathbb{R}$ by $(\tilde{A},\tilde{A}^*,\tilde{\varphi})$ is the "quasi Poissonization" of the structure induced on M by $((A,\phi),(A^*,W),\varphi)$. The dual version of these results is also presented. Finally, in the Appendix, we define the action of a Lie algebroid with 1-cocycle on a differentiable manifold, a concept that is used in the paper.

Notation: If (A,ϕ) is a Lie algebroid with 1-cocycle ϕ, we denote by d^ϕ the differential operator d of A modified by ϕ, i.e., $d^\phi\alpha=d\alpha+\phi\wedge\alpha$, for any $\alpha\in\Gamma(\bigwedge^k A^*)$. Moreover, we denote by δ the usual de Rham differential operator on a manifold M and by d the differential operator of the Lie algebroid $TM\times\mathbb{R}$, $\mathrm{d}(\alpha,\beta)=(\delta\alpha,-\delta\beta)$, for $(\alpha,\beta)\in\Gamma(\bigwedge^k(T^*M\times\mathbb{R}))\equiv\Gamma(\bigwedge^k T^*M)\times\Gamma(\bigwedge^{k-1}T^*M)$. We also consider the identification $\Gamma(\bigwedge^k(TM\times\mathbb{R}))\equiv\Gamma(\bigwedge^k TM)\times\Gamma(\bigwedge^{k-1}TM)$.

2. Quasi-Jacobi and Jacobi-Quasi Bialgebroids

Let $((A,\phi),(A^*,W))$ be a pair of dual vector bundles over a differentiable manifold M endowed with a 1-form ϕ and W, respectively, and φ a 3-form of A.

DEFINITION 2.1. A *quasi-Jacobi bialgebroid* structure on $((A,\phi),(A^*,W),\varphi)$ consists of a Lie algebroid structure with 1-cocycle $([\cdot,\cdot]_*,a_*,W)$ on A^*, a bundle map $a: A\to TM$ and a skew-symmetric operation $[\cdot,\cdot]$ on $\Gamma(A)$ satisfying, for all $X,Y,Z\in\Gamma(A)$ and $f\in C^\infty(M,\mathbb{R})$, the following conditions:

1. $[X,fY]=f[X,Y]+(a(X)f)Y$;
2. $a([X,Y])=[a(X),a(Y)]-a_*(\varphi(X,Y,\cdot))$;
3. $[[X,Y],Z]+\text{c.p.}=-d_*^W(\varphi(X,Y,Z))-((i_{\varphi(X,Y,\cdot)}d_*^W Z)+\text{c.p.})$;
4. $d\phi-\varphi(W,\cdot,\cdot)=0$, where d is the quasi-differential operator on $\Gamma(\bigwedge A^*)$ determined by the structure $([\cdot,\cdot],a)$ on A;
5. $d^\phi\varphi=0$, d^ϕ being the quasi-differential operator modified by ϕ;
6. $d_*^W[P,Q]^\phi=[d_*^W P,Q]^\phi+(-1)^{p+1}[P,d_*^W Q]^\phi$, where $[\cdot,\cdot]^\phi$ is the bracket on $\Gamma(\bigwedge A)$ modified by ϕ, $P\in\Gamma(\bigwedge^p A)$ and $Q\in\Gamma(\bigwedge A)$.

As in the case of quasi-Lie and Lie-quasi bialgebroids, by interchanging the roles of (A,ϕ) and (A^*,W) in the above definition, we obtain the notion of *Jacobi-quasi bialgebroid* over a differentiable manifold M. We have: *If* $((A,\phi),(A^*,W),\varphi)$ *is a quasi-Jacobi bialgebroid over a differentiable manifold* M*, then* $((A^*,W),(A,\phi),\varphi)$ *is a Jacobi-quasi bialgebroid over* M*, and conversely.*

In the case where both 1-cocycles ϕ and W are zero, we recover, from Definition 2.1, the notion of quasi-Lie bialgebroid. On the other hand, if $\varphi=0$, then $((A,\phi),(A^*,W),0)\equiv((A,\phi),(A^*,W))$ is a Jacobi bialgebroid over M.

Remark 2.2. In [12], we proved that the double of a quasi-Jacobi bialgebroid is a Courant–Jacobi algebroid ([4, 11]). By a similar computation, we may conclude that the double of a Jacobi-quasi bialgebroid is also a Courant–Jacobi algebroid.

The rest of this section is devoted to some important examples of quasi-Jacobi and Jacobi-quasi bialgebroids.

2.1. QUASI- JACOBI AND JACOBI- QUASI BIALGEBRAS

A *quasi-Jacobi bialgebra* is a quasi-Jacobi bialgebroid over a point, that is a triple $((\mathcal{G},\phi),(\mathcal{G}^*,W),\varphi)$, where $(\mathcal{G}^*,[\cdot,\cdot]_*,W)$ is a real Lie algebra of finite dimension with 1-cocycle $W\in\mathcal{G}$ in its Chevalley–Eilenberg cohomology, $(\mathcal{G},\phi)$ is the dual space of $\mathcal{G}^*$ endowed with a bilinear skew-symmetric bracket $[\cdot,\cdot]$ and an element $\phi\in\mathcal{G}^*$ and $\varphi\in\bigwedge^3\mathcal{G}^*$, such that conditions 3–6 of Definition 2.1 are satisfied.

By dualizing the above notion, we get a *Jacobi-quasi bialgebra*, i.e. a Jacobi-quasi bialgebroid over a point.

In the particular case where $\varphi=0$, we recover the concept of Jacobi bialgebra [5]. When $\phi=0$ and $W=0$, we recover the notion of quasi-Lie bialgebra due to Drinfeld [2].

We postpone the study of quasi-Jacobi bialgebras to a future paper. We believe that they can be considered as the infinitesimal invariants of Lie groups endowed with a certain type of twisted Jacobi structures that can be constructed from the solutions of a twisted Yang–Baxter equation.

2.2. THE QUASI-JACOBI AND THE JACOBI-QUASI BIALGEBROIDS OF A TWISTED JACOBI MANIFOLD

We recall that a *twisted Jacobi manifold* [12] is a differentiable manifold M equipped with a section (Λ, E) of $\bigwedge^2(TM\times\mathbb{R})$ and a 2-form ω such that

$$\frac{1}{2}[(\Lambda, E), (\Lambda, E)]^{(0,1)} = (\Lambda, E)^{\#}(\delta\omega, \omega),^{1} \tag{1}$$

where $[\cdot,\cdot]^{(0,1)}$ denotes the Schouten bracket of the Lie algebroid $(TM\times\mathbb{R}, [\cdot,\cdot], \pi)$ over M modified by the 1-cocycle $(0, 1)$ and $(\Lambda, E)^{\#}$ is the natural extension of the homomorphism of $C^\infty(M,\mathbb{R})$-modules $(\Lambda, E)^{\#}:\Gamma(T^*M\times\mathbb{R})\to\Gamma(TM\times\mathbb{R})$, $(\Lambda, E)^{\#}(\alpha, f) = (\Lambda^{\#}(\alpha) + fE, -\langle\alpha, E\rangle)$, to a homomorphism from $\Gamma(\bigwedge^k(T^*M\times\mathbb{R}))$ to $\Gamma(\bigwedge^k(TM\times\mathbb{R}))$, $k\in\mathbb{N}$, given, for any $(\eta,\xi)\in\Gamma(\bigwedge^k(T^*M\times\mathbb{R}))$ and $(\alpha_1, f_1),\ldots,(\alpha_k, f_k)\in\Gamma(T^*M\times\mathbb{R})$, by

$$\begin{aligned}&(\Lambda, E)^{\#}(\eta,\xi)((\alpha_1, f_1),\ldots,(\alpha_k, f_k)) =\\ &\quad = (-1)^k(\eta,\xi)((\Lambda, E)^{\#}(\alpha_1, f_1),\ldots,(\Lambda, E)^{\#}(\alpha_k, f_k))\end{aligned}$$

and, for all $f\in C^\infty(M,\mathbb{R})$, by $(\Lambda, E)^{\#}(f) = f$.

Examples of twisted Jacobi manifolds are *twisted exact Poisson manifolds* and *twisted locally conformal symplectic manifolds*, both of them presented in [12], and also *twisted contact Jacobi manifolds* described in [13].

If $(M, (\Lambda, E), \omega)$ is a twisted Jacobi manifold and f a function on M that never vanishes, we can define a new twisted Jacobi structure $((\Lambda^f, E^f), \omega^f)$ on M, which is said to be *f-conformal* to $((\Lambda, E), \omega)$, by setting

$$\Lambda^f = f\Lambda; \quad E^f = \Lambda^{\#}(\delta f) + fE; \quad \omega^f = \frac{1}{f}\omega.$$

In the sequel, let $(M, (\Lambda, E), \omega)$ be a twisted Jacobi manifold and $(T^*M\times\mathbb{R}, [\cdot,\cdot]^{\omega}_{(\Lambda,E)}, \pi\circ(\Lambda, E)^{\#}, (-E, 0))$ its canonically associated Lie algebroid with 1-cocycle, [12]. The Lie bracket $[\cdot,\cdot]^{\omega}_{(\Lambda,E)}$ on $\Gamma(T^*M\times\mathbb{R})$ is given, for all $(\alpha, f), (\beta, g)\in\Gamma(T^*M\times\mathbb{R})$, by

$$\begin{aligned}[(\alpha, f), (\beta, g)]^{\omega}_{(\Lambda,E)} &= [(\alpha, f), (\beta, g)]_{(\Lambda,E)} +\\ &\quad + (\delta\omega, \omega)((\Lambda, E)^{\#}(\alpha, f), (\Lambda, E)^{\#}(\beta, g), \cdot),\end{aligned}$$

[1] Since, for any $(\varphi,\omega)\in\Gamma(\bigwedge^3(T^*M\times\mathbb{R}))$, $\mathrm{d}^{(0,1)}(\varphi,\omega) = (\delta\varphi, \varphi-\delta\omega)$ and $\mathrm{d}^{(0,1)}(\varphi,\omega) = (0,0) \Leftrightarrow \varphi = \delta\omega$, Equation (1) means that $\frac{1}{2}[(\Lambda, E), (\Lambda, E)]^{(0,1)}$ is the image by $(\Lambda, E)^{\#}$ of a $\mathrm{d}^{(0,1)}$-closed 3-form of $TM\times\mathbb{R}$.

where $[\cdot,\cdot]_{(\Lambda,E)}$ is the usual bracket on $\Gamma(T^*M\times\mathbb{R})$ associated to a section (Λ, E) of $\bigwedge^2(TM\times\mathbb{R})$ ([7], [5]):

$$[(\alpha, f), (\beta, g)]_{(\Lambda,E)} = \mathcal{L}^{(0,1)}_{(\Lambda,E)^\#(\alpha,f)}(\beta, g) - \mathcal{L}^{(0,1)}_{(\Lambda,E)^\#(\beta,g)}(\alpha, f) - \\ - \mathrm{d}^{(0,1)}((\Lambda, E)((\alpha, f), (\beta, g))). \tag{2}$$

We consider, on the vector bundle $TM\times\mathbb{R}\to\mathbb{R}$, the Lie algebroid structure over M with 1-cocycle $([\cdot,\cdot],\pi,(0,1))$ and also a new bracket $[\cdot,\cdot]'$ on the space of its smooth sections given, for all $(X, f), (Y, g)\in\Gamma(TM\times\mathbb{R})$, by

$$[(X, f), (Y, g)]' = [(X, f), (Y, g)] - (\Lambda, E)^\#((\delta\omega, \omega)((X, f), (Y, g), \cdot)).$$

We have shown in [12] that the triple $((TM\times\mathbb{R}, [\cdot,\cdot]', \pi, (0,1)), (T^*M\times\mathbb{R}, [\cdot,\cdot]^\omega_{(\Lambda,E)}, \pi\circ(\Lambda, E)^\#, (-E, 0)), (\delta\omega, \omega))$ is a quasi-Jacobi bialgebroid over M. Furthermore, we have

THEOREM 2.3. *The triple* $((TM\times\mathbb{R}, [\cdot,\cdot], \pi, (0,1)), (T^*M\times\mathbb{R}, [\cdot,\cdot]_{(\Lambda,E)}, \pi\circ(\Lambda, E)^\#, (-E, 0)), (\Lambda, E)^\#(\delta\omega, \omega))$ *is a Jacobi-quasi bialgebroid over* M.

Proof. It suffices to check that the dual versions of all conditions of Definition 2.1 are satisfied. Condition (1) can be checked directly, using (2). For Condition (2), we take into account that $((\Lambda, E), \omega)$ is a twisted Jacobi structure, hence (1) holds, and we apply the general formula

$$(\Lambda, E)^\#([(\alpha, f), (\beta, g)]_{(\Lambda,E)}) = [(\Lambda, E)^\#(\alpha, f), (\Lambda, E)^\#(\beta, g)] - \\ -\frac{1}{2}[(\Lambda, E), (\Lambda, E)]^{(0,1)}((\alpha, f), (\beta, g), \cdot). \tag{3}$$

By projection, we obtain Condition (2). Condition (3) can be checked directly, after a long computation. In order to prove Condition (4), we remark that the quasi-differential operator d_* determined by $([\cdot,\cdot]_{(\Lambda,E)}, \pi\circ(\Lambda, E)^\#)$ is given [5], for all $(R, S)\in\Gamma(\bigwedge^k(TM\times\mathbb{R}))$, by

$$d_*(R, S) = ([\Lambda, R] + kE\wedge R + \Lambda\wedge S, -[\Lambda, R] + (1-k)E\wedge S + [E, R]).$$

So, $d_*(-E, 0) = ([E, \Lambda], 0)$, and since $(M, (\Lambda, E), \omega)$ is a twisted Jacobi manifold,

$$d_*(-E, 0) = ([E, \Lambda], 0) = \frac{1}{2}[(\Lambda, E), (\Lambda, E)]^{(0,1)}((0,1), \cdot, \cdot) = \\ = ((\Lambda, E)^\#(\delta\omega, \omega))((0,1), \cdot, \cdot).$$

On the other hand, since $d_*^{(-E,0)}(R, S) = [(\Lambda, E), (R, S)]^{(0,1)}$, we have

$$d_*^{(-E,0)}((\Lambda, E)^\#(\delta\omega, \omega)) = [(\Lambda, E), (\Lambda, E)^\#(\delta\omega, \omega)]^{(0,1)} = \\ = \frac{1}{2}[(\Lambda, E), [(\Lambda, E), (\Lambda, E)]^{(0,1)}]^{(0,1)} = (0, 0),$$

whence we get Condition (5). Finally, Condition (6) can be established, as in the proof of Theorem 8.2 in [12], by a straightforward but long computation. □

In the case of twisted Poisson manifolds, the previous results were treated in [16] and [9].

2.3. THE JACOBI- QUASI BIALGEBROID OF A QUASI JACOBI MANIFOLD

Let $(\mathcal{G}, [\cdot,\cdot])$ be a Lie algebra, ϕ a 1-cocycle in its Chevalley–Eilenberg cohomology and $(\cdot,\cdot)$ a nondegenerate symmetric bilinear invariant form on $\mathcal{G}$. We denote by ψ the canonical 3-form on $\mathcal{G}$ defined by $\psi(X,Y,Z)=\frac{1}{2}(X,[Y,Z])$, for all $X,Y,Z\in\mathcal{G}$, and by $Q_\psi\in\bigwedge^3\mathcal{G}$ its dual trivector that is given, for all $\mu,\nu,\xi\in\mathcal{G}^*$, by

$$Q_\psi(\mu,\nu,\xi)=\psi(X_\mu,X_\nu,X_\xi),$$

where X_μ, X_ν, X_ξ are, respectively, dual to μ,ν,ξ via $(\cdot,\cdot)$.

A $(\mathcal{G},\phi)$-manifold M is a differentiable manifold on which $(\mathcal{G},\phi)$ acts infinitesimally by $a^\phi:\mathcal{G}\to TM\times\mathbb{R}$, $a^\phi(X)=a(X)+\langle\phi,X\rangle$, for all $X\in\mathcal{G}$ (see Appendix). We keep the same notation a^ϕ for the induced maps on exterior algebras.

Let M be a $(\mathcal{G},\phi)$-manifold. A section $(\Lambda,E)\in\Gamma(\bigwedge^2(TM\times\mathbb{R}))$ is said to be *invariant* (under the infinitesimal action a^ϕ) if, for any $X\in\mathcal{G}$,

$$\mathcal{L}^{(0,1)}_{a^\phi(X)}(\Lambda,E)=(0,0).$$

A natural generalization of the notion of quasi Poisson manifold, given in [1], is the concept of $(\mathcal{G},\phi)$-*quasi Jacobi manifold*, that we introduce as follows.

DEFINITION 2.4. A $(\mathcal{G},\phi)$-*quasi Jacobi manifold* is a $(\mathcal{G},\phi)$-manifold M equipped with an invariant section $(\Lambda,E)\in\Gamma(\bigwedge^2(TM\times\mathbb{R}))$ such that

$$\frac{1}{2}[(\Lambda,E),(\Lambda,E)]^{(0,1)}=a^\phi(Q_\psi).$$

A long, but not difficult computation, leads us to the following:

THEOREM 2.5. *Let (M,Λ,E) be a $(\mathcal{G},\phi)$-quasi Jacobi manifold. Then, $((TM\times\mathbb{R},[\cdot,\cdot],\pi,(0,1)),(T^*M\times\mathbb{R},[\cdot,\cdot]_{(\Lambda,E)},\pi\circ(\Lambda,E)^\#,(-E,0)),a^\phi(Q_\psi))$ is a Jacobi-quasi bialgebroid over M.*

Remark 2.6. If M is a $\mathcal{G}$-manifold equipped with a quasi Poisson structure, i.e an invariant bivector field Λ on M such that $[\Lambda,\Lambda]=2a(Q_\psi)$, a similar result holds: *The triple $((TM,[\cdot,\cdot],id),(T^*M,[\cdot,\cdot]_\Lambda,\Lambda^\#),a(Q_\psi))$ is a Lie-quasi bialgebroid over M, $[\cdot,\cdot]_\Lambda$ being the Koszul bracket associated to Λ.*

2.4. TRIANGULAR QUASI- JACOBI AND JACOBI- QUASI BIALGEBROIDS

Let $(A,[\cdot,\cdot],a,\phi)$ be a Lie algebroid with 1-cocycle over a differentiable manifold M, Π a section of $\bigwedge^2 A$ and Q a trivector on A such that

$$\frac{1}{2}[\Pi,\Pi]^\phi=Q. \tag{4}$$

We shall discuss what happens on the dual vector bundle A^* of A when we consider the vector bundle map $a_*: A^* \to TM$, $a_* = a \circ \Pi^\#$, $\Pi^\#: A^* \to A$ being the bundle map associated to Π, and the Koszul bracket $[\cdot,\cdot]_\Pi$ on the space $\Gamma(A^*)$ of its smooth sections given, for all $\alpha, \beta \in \Gamma(A^*)$, by

$$[\alpha, \beta]_\Pi = \mathcal{L}^{\phi}_{\Pi^\#(\alpha)}\beta - \mathcal{L}^{\phi}_{\Pi^\#(\beta)}\alpha - d^\phi(\Pi(\alpha, \beta)). \tag{5}$$

Let us set $W = -\Pi^\#(\phi)$. Taking into account that, for all $\alpha, \beta, \gamma \in \Gamma(A^*)$,

$$[[\alpha, \beta]_\Pi, \gamma]_\Pi + \text{c.p.} = -d^\phi(Q(\alpha, \beta, \gamma)) - ((i_{Q(\alpha,\beta,\cdot)}d^\phi\gamma) + \text{c.p.}),$$

we can directly prove that

THEOREM 2.7. *The triple* $((A, [\cdot,\cdot], a, \phi), (A^*, [\cdot,\cdot]_\Pi, a_*, W), Q)$ *is a Jacobi-quasi bialgebroid over M, which is called a* triangular Jacobi-quasi bialgebroid.

Clearly, the Lie-quasi bialgebroid associated to a twisted Poisson manifold [16] and the Jacobi-quasi bialgebroid associated to a twisted Jacobi manifold (see Theorem 2.3) are special cases of triangular Jacobi-quasi bialgebroids. Another important type of triangular quasi-Jacobi bialgebroid is the triangular quasi-Jacobi bialgebra, where Π is a solution of a Yang-Baxter's type equation.

Now, we consider the particular case where $Q = \Pi^\#(\varphi)$, with φ a d^ϕ-closed 3-form on A, and the spaces $\Gamma(A^*)$ and $\Gamma(A)$ are equipped, respectively, with the brackets

$$[\alpha, \beta]^\varphi_\Pi = [\alpha, \beta]_\Pi + \varphi(\Pi^\#(\alpha), \Pi^\#(\beta), \cdot), \quad \text{for all } \alpha, \beta \in \Gamma(A^*),$$

$[\cdot,\cdot]_\Pi$ being the Koszul bracket (5), and

$$[X, Y]' = [X, Y] - \Pi^\#(\varphi(X, Y, \cdot)), \quad \text{for all } X, Y \in \Gamma(A).$$

Under the above assumptions, by a straightforward calculation, we get

PROPOSITION 2.8. *The vector bundle $A^* \to M$ endowed with the structure* $([\cdot,\cdot]^\varphi_\Pi, a_*)$ *is a Lie algebroid over M with* 1*-cocycle* $W = -\Pi^\#(\phi)$.

Also, we have

THEOREM 2.9. *The triple* $((A, [\cdot,\cdot]', a, \phi), (A^*, [\cdot,\cdot]^\varphi_\Pi, a_*, W), \varphi)$ *is a triangular quasi-Jacobi bialgebroid over M.*

Proof. The proof is analogous to that of Theorem 8.2 in [12] and so it is omitted. □

Remark 2.10. Obviously, if A is $TM\times\mathbb{R}$ equipped with the usual Lie algebroid structure with 1-cocycle, $([\cdot,\cdot],\pi,(0,1))$, and $\Pi=(\Lambda,E)\in\Gamma(\bigwedge^2(TM\times\mathbb{R}))$ satisfies (4), then the manifold M is endowed with a twisted Jacobi structure. The Lie algebroid structure on $A^*=T^*M\times\mathbb{R}$ given by Proposition 2.8, is the Lie algebroid structure canonically associated with the twisted Jacobi structure on M.

3. The Structure Induced on the Base Manifold of a Quasi-Jacobi Bialgebroid

In this section we will investigate the structure induced on the base manifold of a quasi-Jacobi bialgebroid. Similar results hold for a Jacobi-quasi bialgebroid.

Let $((A,\phi),(A^*,W),\varphi)$ be a quasi-Jacobi bialgebroid over M. In [12], we have already considered the bracket $\{\cdot,\cdot\}$ on $C^\infty(M,\mathbb{R})$ defined, for all $f,g\in C^\infty(M,\mathbb{R})$, by

$$\{f,g\}=\langle d^\phi f, d_*^W g\rangle. \tag{6}$$

We have proved that it is $\mathbb{R}$-bilinear, skew-symmetric and a first-order differential operator on each argument [12]. On the other hand, the quasi-differential operator d on $\Gamma(\bigwedge A^*)$ determined by $(a,[\cdot,\cdot])$ is a derivation with respect to the usual product of functions. Therefore, the map $(f,g)\mapsto\langle df, d_*g\rangle$ is a derivation on each argument and so, there exists a bivector field Λ on M such that, for all $f,g\in C^\infty(M,\mathbb{R})$,

$$\Lambda(\delta f,\delta g)=\langle df, d_*g\rangle=-\langle dg, d_*f\rangle.$$

If E is the vector field $a_*(\phi)=-a(W)$ on M then, from (6) and because $\langle\phi,W\rangle=0$ holds [12], we get

$$\{f,g\}=\langle \mathrm{d}^{(0,1)}g,(\Lambda,E)^\#(\mathrm{d}^{(0,1)}f)\rangle. \tag{7}$$

Since, for all $f\in C^\infty(M,\mathbb{R})$, $d^\phi f=(a^\phi)^*(\mathrm{d}^{(0,1)}f)$ and $d_*^W f=(a_*^W)^*(\mathrm{d}^{(0,1)}f)$ [12], where $(a^\phi)^*$ and $(a_*^W)^*$ denote, respectively, the transpose of a^ϕ and a_*^W, we obtain

$$(\Lambda,E)^\#=-a^\phi\circ(a_*^W)^*=a_*^W\circ(a^\phi)^*. \tag{8}$$

It is well known that any bracket of type (7) satisfies the following relation:

$$\{f,\{g,h\}\}+\text{c.p.}=\frac{1}{2}[(\Lambda,E),(\Lambda,E)]^{(0,1)}(\mathrm{d}^{(0,1)}f,\mathrm{d}^{(0,1)}g,\mathrm{d}^{(0,1)}h). \tag{9}$$

Therefore, for the bracket defined by (6), in general, the Jacobi identity does not hold.

PROPOSITION 3.1. *Let $((A,\phi),(A^*,W),\varphi)$ be a quasi-Jacobi bialgebroid over M. Then, the bracket (6) satisfies, for all $f,g,h\in C^\infty(M,\mathbb{R})$, the following identity:*

$$\{f,\{g,h\}\}+c.p.=a_*^W(\varphi)(\mathrm{d}^{(0,1)}f,\mathrm{d}^{(0,1)}g,\mathrm{d}^{(0,1)}h). \tag{10}$$

In (10), a_*^W denotes the natural extension of $a_*^W:\Gamma(A^*)\to\Gamma(TM\times\mathbb{R})$ to a bundle map from $\Gamma(\bigwedge^3 A^*)$ to $\Gamma(\bigwedge^3(TM\times\mathbb{R}))$.

Proof. Let f,g and h be any three functions on $C^\infty(M,\mathbb{R})$. Taking into account the formulæ $d_*^W\{f,g\}=[d_*^W g,d_*^W f]$ (see [12]), (8) and (3), and the properties of a quasi-Jacobi bialgebroid, after a simple computation, we get

$$\{h,\{f,g\}\}=\{\{g,f\},h\}-\frac{1}{2}[(\Lambda,E),(\Lambda,E)]^{(0,1)}(\mathrm{d}^{(0,1)}f,\mathrm{d}^{(0,1)}g,\mathrm{d}^{(0,1)}h)+ \\ +a_*^W(\varphi)(\mathrm{d}^{(0,1)}f,\mathrm{d}^{(0,1)}g,\mathrm{d}^{(0,1)}h).$$

Consequently,

$$\frac{1}{2}[(\Lambda,E),(\Lambda,E)]^{(0,1)}(\mathrm{d}^{(0,1)}f,\mathrm{d}^{(0,1)}g,\mathrm{d}^{(0,1)}h)=a_*^W(\varphi)(\mathrm{d}^{(0,1)}f,\mathrm{d}^{(0,1)}g,\mathrm{d}^{(0,1)}h). \tag{11}$$

Hence, from (9) and (11), we obtain (10). □

Looking at Equation (11), we remark that the obstruction for (M,Λ,E) to be a Jacobi manifold, i.e. to have $[(\Lambda,E),(\Lambda,E)]^{(0,1)}=(0,0)$, is the image by a_*^W of the element φ in $\Gamma(\bigwedge^3 A^*)$. This obstruction can also be viewed as the image of φ under the infinitesimal action of the Lie algebroid with 1-cocycle (A^*,W) on M (see Appendix). Thus, inspired by the analogous terms of *quasi Poisson $\mathcal{G}$-manifold* ([1,9]) and of *$(\mathcal{G},\phi)$-quasi Jacobi manifold* (see Section 2.3), we say that the pair (Λ,E) defines on M a *(A^*,W)-quasi Jacobi structure*.

Thus, we have proved:

THEOREM 3.2. *Let $((A,\phi),(A^*,W),\varphi)$ be a quasi-Jacobi bialgebroid over M. Then, the bracket $\{\cdot,\cdot\}:C^\infty(M,\mathbb{R})\times C^\infty(M,\mathbb{R})\to C^\infty(M,\mathbb{R})$ given by*

$$\{f,g\}=\langle d^\phi f,d_*^W g\rangle,\quad for\ f,g\in C^\infty(M,\mathbb{R}),$$

defines a (A^*,W)-quasi Jacobi structure on M.

Remark 3.3. In the case where $((A,\phi),(A^*,W),Q)$ is a Jacobi-quasi bialgebroid over M, we can easily prove that the Jacobi identity of the bracket defined by (6) is violated by the image of Q under a^ϕ. For this reason, we shall call the structure (Λ,E) induced on M, an *(A,ϕ)-quasi Jacobi structure.* We note that, for the proof of this result, we use the relation $[d^\phi f,d^\phi g]_*=d^\phi\{f,g\}$, $f,g\in C^\infty(M,\mathbb{R})$, which leads to

$$(\Lambda,E)^{\#}=a^\phi\circ(a_*^W)^*=-a_*^W\circ(a^\phi)^*. \tag{12}$$

Examples 3.4. 1. *A^*-Quasi Poisson structures:* If $((A,\phi),(A^*,W),\varphi)$ is a quasi-Lie bialgebroid over M, i.e. both 1-cocycles ϕ and W are zero, Theorem 3.2 establishes the existence of a structure on M, defined by the bracket

$$\{f,g\}=\langle df,d_*g\rangle,\quad f,g\in C^\infty(M,\mathbb{R}),$$

on $C^\infty(M,\mathbb{R})$, which is associated to a bivector filed Λ on M satisfying $[\Lambda,\Lambda]=2a_*(\varphi)$. In our terminology, Λ endows M with a *A^*-quasi Poisson structure*. We remark that this result was obtained in [6] by different techniques.

2. *Jacobi structures:* When $\varphi=0$, i.e. $((A,\phi),(A^*,W))$ is a Jacobi bialgebroid over M, the structure (Λ,E) on M determined by Theorem 3.2 is a Jacobi structure, and we recover the well-known result of [5].

3. *Twisted Jacobi structures:* When φ is the image of an element $(\varphi_M,\omega_M)\in\Gamma(\bigwedge^3(T^*M\times\mathbb{R}))$ by the transpose map $(a^\phi)^*:\Gamma(\bigwedge^3(T^*M\times\mathbb{R}))\to\Gamma(\bigwedge^3 A^*)$ of a^ϕ, i.e. $\varphi=(a^\phi)^*(\varphi_M,\omega_M)$, then,

$$\frac{1}{2}[(\Lambda,E),(\Lambda,E)]^{(0,1)}=a_*^W(\varphi)=a_*^W((a^\phi)^*(\varphi_M,\omega_M))\overset{(8)}{=}(\Lambda,E)^\#(\varphi_M,\omega_M).$$

Also, we have

$$d^\phi(\underbrace{(a^\phi)^*(\varphi_M,\omega_M)}_{=\varphi})=0\Leftrightarrow(a^\phi)^*(\mathrm{d}^{(0,1)}(\varphi_M,\omega_M))=0,$$

which means that (φ_M,ω_M) is $\mathrm{d}^{(0,1)}$-closed on the distribution $\mathrm{Im}(a^\phi)$. This distribution is not, in general, involutive due to Condition (2) of Definition 2.1. However, when $\mathrm{Im}(a^\phi)$ is involutive, as in the case where a^ϕ is surjective, $((\Lambda,E),\omega_M)$ defines a twisted Jacobi structure on the leaves of $\mathrm{Im}(a^\phi)$.

4. *The case of the quasi-Jacobi bialgebroid associated to a twisted Jacobi manifold:* Let $(M,(\Lambda_1,E_1),\omega)$ be a twisted Jacobi manifold and let $((TM\times\mathbb{R},[\cdot,\cdot]',\pi,(0,1)),(T^*M\times\mathbb{R},[\cdot,\cdot]^\omega_{(\Lambda_1,E_1)},\pi\circ(\Lambda_1,E_1)^\#,(-E_1,0)),(\delta\omega,\omega))$ be its associated quasi-Jacobi bialgebroid. Then, the $(T^*M\times\mathbb{R},(-E_1,0))$-quasi Jacobi structure induced on M coincides with the initial one (Λ_1,E_1). In fact, for any $f,g\in C^\infty(M,\mathbb{R})$ and taking into account that $\mathrm{d}'^{(0,1)}f=\mathrm{d}^{(0,1)}f$, d' being the quasi-differential of $TM\times\mathbb{R}$ determined by the structure $([\cdot,\cdot]',\pi)$, and that $(d_*^\omega)^{(-E_1,0)}g=-(\Lambda_1,E_1)^\#(\mathrm{d}^{(0,1)}g)$, we have

$$\{f,g\}=\langle\mathrm{d}'^{(0,1)}f,(d_*^\omega)^{(-E_1,0)}g\rangle=\langle\mathrm{d}^{(0,1)}f,-(\Lambda_1,E_1)^\#(\mathrm{d}^{(0,1)}g)\rangle=\{f,g\}_1,$$

where $\{\cdot,\cdot\}_1$ denotes the bracket associated to (Λ_1,E_1).

Moreover, if we consider the Jacobi-quasi bialgebroid $((TM\times\mathbb{R},[\cdot,\cdot],\pi,\ (0,1)),(T^*M\times\mathbb{R},[\cdot,\cdot]_{(\Lambda_1,E_1)},\pi\circ(\Lambda_1,E_1)^\#,(-E_1,0)),(\Lambda_1,E_1)^\#(\delta\omega,\omega))$ associated to the twisted Jacobi manifold $(M,(\Lambda_1,E_1),\omega)$, we get that the $(TM\times\mathbb{R},(0,1))$-quasi Jacobi structure (Λ,E) induced on M is the opposite of (Λ_1,E_1). It suffices to remark that

$$(\Lambda,E)^{\#}\overset{(12)}{=}\pi^{(0,1)}\circ((\pi\circ(\Lambda_1,E_1)^{\#})^{(-E_1,0)})^{*}=$$
$$=\pi^{(0,1)}\circ((\Lambda_1,E_1)^{\#})^{*}\circ(\pi^{(0,1)})^{*}=-(\Lambda_1,E_1)^{\#}. \tag{13}$$

5. *The induced structure on a quasi Jacobi manifold:* We consider the Jacobi-quasi bialgebroid $((TM\times\mathbb{R},[\cdot,\cdot],\pi,(0,1)),(T^{*}M\times\mathbb{R},[\cdot,\cdot]_{(\Lambda_1,E_1)},\pi\circ(\Lambda_1,E_1)^{\#},(-E_1,0)),a^{\phi}(Q_{\psi}))$ associated to a $(\mathcal{G},\phi)$-quasi Jacobi manifold (M,Λ_1,E_1). Then, repeating the computation (13), we conclude that, as in the previous case, the $(TM\times\mathbb{R},(0,1))$-quasi Jacobi structure (Λ,E) induced on M is the opposite of (Λ_1,E_1).

6. *The case of a triangular quasi-Jacobi bialgebroid:* If we consider a triangular quasi-Jacobi bialgebroid over M of type $((A,[\cdot,\cdot]',a,\phi),(A^{*},[\cdot,\cdot]^{\varphi}_{\Pi},a_{*},W),\varphi)$, presented in Theorem 2.9, then, for all $f\in C^{\infty}(M,\mathbb{R})$,

$$d_{*}^{W}f=-\Pi^{\#}(d^{\phi}f)=-(\Pi^{\#}\circ(a^{\phi})^{*})(\mathrm{d}^{(0,1)}f).$$

So, the bracket (6) in $C^{\infty}(M,\mathbb{R})$ is given by

$$\{f,g\}=\langle d^{\phi}f,d_{*}^{W}g\rangle=\langle \mathrm{d}^{(0,1)}g,(a^{\phi}\circ\Pi^{\#}\circ(a^{\phi})^{*})\mathrm{d}^{(0,1)}f\rangle.$$

On the other hand, considering the (A^{*},W)-quasi Jacobi structure (Λ,E) on M, we also have

$$\{f,g\}=\langle \mathrm{d}^{(0,1)}g,(\Lambda,E)^{\#}(\mathrm{d}^{(0,1)}f)\rangle.$$

Hence,

$$(\Lambda,E)^{\#}=a^{\phi}\circ\Pi^{\#}\circ(a^{\phi})^{*},$$

which means that (Λ,E) is the image by a^{ϕ} of Π and that a^{ϕ} is a type of "twisted Jacobi morphism" between (A,ϕ,Π) and $(TM\times\mathbb{R},(0,1),(\Lambda,E))$.

4. Quasi-Lie Bialgebroids Associated to Quasi-Jacobi Bialgebroids

Given a Lie algebroid $(A,[\cdot,\cdot],a)$ over M, we can endow the vector bundle $\tilde{A}=A\times\mathbb{R}\to M\times\mathbb{R}$ with a Lie algebroid structure over $M\times\mathbb{R}$ as follows. The sections of $\tilde{A}$ can be identified with the t-dependent sections of A, t being the canonical coordinate on $\mathbb{R}$, i.e., for any $\tilde{X}\in\Gamma(\tilde{A})$ and $(x,t)\in M\times\mathbb{R}$, $\tilde{X}(x,t)=\tilde{X}_t(x)$, where $\tilde{X}_t\in\Gamma(A)$. This identification induces, in a natural way, a Lie bracket on $\Gamma(\tilde{A})$, also denoted by $[\cdot,\cdot]$:

$$[\tilde{X},\tilde{Y}](x,t)=[\tilde{X}_t,\tilde{Y}_t](x),\qquad \tilde{X},\tilde{Y}\in\Gamma(\tilde{A}),\ (x,t)\in M\times\mathbb{R},$$

and a bundle map, also denoted by a, $a:\tilde{A}\to T(M\times\mathbb{R})\equiv TM\oplus T\mathbb{R}$ with $a(\tilde{X})=a(\tilde{X}_t)$, in such a way that $(\tilde{A},[\cdot,\cdot],a)$ becomes a Lie algebroid over $M\times\mathbb{R}$. If ϕ is a 1-cocycle of the Lie algebroid A, we know from [5] that $\tilde{A}$ can be equipped with two other Lie algebroid structures over $M\times\mathbb{R}$, $([\cdot,\cdot]^{\widetilde{\phi}},\widetilde{a}^{\phi})$ and $([\cdot,\cdot]^{\widehat{\phi}},\widehat{a}^{\phi})$ given, for all $\tilde{X},\tilde{Y}\in\Gamma(\tilde{A})$, by

$$[\tilde{X},\tilde{Y}]^{\widetilde{\phi}}=[\tilde{X}_t,\tilde{Y}_t]+\langle\phi,\tilde{X}_t\rangle\frac{\partial\tilde{Y}}{\partial t}-\langle\phi,\tilde{Y}_t\rangle\frac{\partial\tilde{X}}{\partial t}; \tag{14}$$

$$\widetilde{a}^{\phi}(\tilde{X})=a(\tilde{X}_t)+\langle\phi,\tilde{X}\rangle\frac{\partial}{\partial t}; \tag{15}$$

and

$$[\tilde{X},\tilde{Y}]^{\widehat{\phi}}=\mathrm{e}^{-t}\left([\tilde{X}_t,\tilde{Y}_t]+\langle\phi,\tilde{X}_t\rangle(\frac{\partial\tilde{Y}}{\partial t}-\tilde{Y})-\langle\phi,\tilde{Y}_t\rangle(\frac{\partial\tilde{X}}{\partial t}-\tilde{X})\right); \tag{16}$$

$$\widehat{a}^{\phi}(\tilde{X})=\mathrm{e}^{-t}(\widetilde{a}^{\phi}(\tilde{X})). \tag{17}$$

Let $((A,\phi),(A^*,W),\varphi)$ be a quasi-Jacobi bialgebroid over M. Then, $(A^*,[\cdot,\cdot]_*,a_*,W)$ is a Lie algebroid with 1-cocycle and we can consider on $\tilde{A}^*$ the Lie algebroid structure $([\cdot,\cdot]_*^{\widehat{W}},\widehat{a}_*^{W})$ defined by (16) and (17). Although A is not endowed with a Lie algebroid structure, we can still consider on $\Gamma(\tilde{A})$ a bracket $[\cdot,\cdot]^{\widetilde{\phi}}$ and a bundle map $\widetilde{a}^{\phi}$ given by (14) and (15), respectively. We set $\tilde{\varphi}=\mathrm{e}^t\varphi$.

THEOREM 4.1. *Under the above assumptions, we have*

1. *The triple $((A,\phi),(A^*,W),\varphi)$ is a quasi-Jacobi bialgebroid over M if and only if $(\tilde{A},\tilde{A}^*,\tilde{\varphi})$ is a quasi-Lie bialgebroid over $M\times\mathbb{R}$.*
2. *If $\tilde{\Lambda}$ is the induced $\tilde{A}^*$-quasi Poisson structure on $M\times\mathbb{R}$, then it is the "quasi Poissonization" of the induced (A^*,W)-quasi Jacobi structure (Λ,E) on M.*

Proof. (1) Let us suppose that $((A,\phi),(A^*,W),\varphi)$ is a quasi-Jacobi bialgebroid over M and let $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ be three arbitrary sections in $\Gamma(\tilde{A})$ and $\tilde{f}\in C^\infty(M\times\mathbb{R},\mathbb{R})$. A straightforward computation gives

$$[\tilde{X},\tilde{f}\tilde{Y}]^{\widetilde{\phi}}=\tilde{f}[\tilde{X},\tilde{Y}]^{\widetilde{\phi}}+(\widetilde{a}^{\phi}(\tilde{X})\tilde{f})\tilde{Y}. \tag{18}$$

Moreover,

$$\begin{aligned}\widetilde{a}^{\phi}([\tilde{X},\tilde{Y}]^{\widetilde{\phi}})&=[a(\tilde{X}_t),a(\tilde{Y}_t)]-a_*(\varphi(\tilde{X}_t,\tilde{Y}_t,\cdot))+\langle\phi,[\tilde{X}_t,\tilde{Y}_t]\rangle\frac{\partial}{\partial t}+\\&\quad+\langle\phi,\tilde{X}\rangle\left(a(\frac{\partial\tilde{Y}}{\partial t})+\langle\phi,\frac{\partial\tilde{Y}}{\partial t}\rangle\frac{\partial}{\partial t}\right)-\langle\phi,\tilde{Y}\rangle\left(a(\frac{\partial\tilde{X}}{\partial t})+\langle\phi,\frac{\partial\tilde{X}}{\partial t}\rangle\frac{\partial}{\partial t}\right)=\\&=[\widetilde{a}^{\phi}(\tilde{X}),\widetilde{a}^{\phi}(\tilde{Y})]-\widehat{a}_*^{W}(\tilde{\varphi}(\tilde{X},\tilde{Y},\cdot)),\end{aligned}$$

where, in the last equality, we have used $d\phi - \varphi(W, \cdot, \cdot) = 0$. On the other hand,

$$
\begin{aligned}
[[\tilde{X}, \tilde{Y}]^{\widetilde{\phi}}, \tilde{Z}]^{\widetilde{\phi}} + \text{c.p.} &= \left([[\tilde{X}_t, \tilde{Y}_t], \tilde{Z}_t] - d\phi(\tilde{X}_t, \tilde{Y}_t) \frac{\partial \tilde{Z}_t}{\partial t} \right) + \text{c.p.} = \\
&= -d_*^W (\varphi(\tilde{X}_t, \tilde{Y}_t, \tilde{Z}_t)) \\
&\quad - \left(\left(i_{\varphi(\tilde{X}_t, \tilde{Y}_t, \cdot)} d_*^W \tilde{Z}_t + \varphi(W, \tilde{X}_t, \tilde{Y}_t) \frac{\partial \tilde{Z}_t}{\partial t} \right) + \text{c.p.} \right) = \\
&= -\widehat{d}_*^W (\tilde{\varphi}(\tilde{X}, \tilde{Y}, \tilde{Z})) - \left(i_{\tilde{\varphi}(\tilde{X}, \tilde{Y}, \cdot)} \widehat{d}_*^W \tilde{Z} + \text{c.p.} \right),
\end{aligned} \tag{19}
$$

where $\widehat{d}_*^W$ is the differential operator of $(\tilde{A}^*, \widehat{[\cdot, \cdot]}_*^W, \widehat{a}_*^W)$. Because $d^\phi \varphi = 0$, we get

$$
\widetilde{d}^\phi \tilde{\varphi} = 0, \tag{20}
$$

where $\widetilde{d}^\phi$ is the quasi-differential operator determined by the structure $([\cdot, \cdot]^{\widetilde{\phi}}, \widetilde{a}^\phi)$ on $\tilde{A}$. Finally, after a very long computation we obtain

$$
\widehat{d}_*^W [\tilde{P}, \tilde{Q}]^{\widetilde{\phi}} = [\widehat{d}_*^W \tilde{P}, \tilde{Q}]^{\widetilde{\phi}} + (-1)^{p+1} [\tilde{P}, \widehat{d}_*^W \tilde{Q}]^{\widetilde{\phi}}, \tag{21}
$$

for $\tilde{P} \in \Gamma(\bigwedge^p \tilde{A})$ and $\tilde{Q} \in \Gamma(\bigwedge \tilde{A})$. From relations (18)–(21), we conclude that $(\tilde{A}, \tilde{A}^*, \tilde{\varphi})$ is a quasi-Lie bialgebroid over $M \times \mathbb{R}$.

Now, let us suppose that $(\tilde{A}, \tilde{A}^*, \tilde{\varphi})$ is a quasi-Lie bialgebroid over $M \times \mathbb{R}$ and take three sections X, Y and Z of A and $f \in C^\infty(M, \mathbb{R})$. These sections can be viewed as sections of $\tilde{A}$ that do not depend on t, as well as the function f can also be viewed as a function on $C^\infty(M \times \mathbb{R}, \mathbb{R})$. Condition (1) of Definition 2.1 is immediate from $[X, fY]^{\widetilde{\phi}} = f[X, Y]^{\widetilde{\phi}} + (\widetilde{a}^\phi(X) f) Y$. The condition $\widetilde{a}^\phi([X, Y]^{\widetilde{\phi}}) = [\widetilde{a}^\phi(X), \widetilde{a}^\phi(Y)] - \widehat{a}_*^W(\tilde{\varphi}(X, Y, \cdot))$ is equivalent to conditions (2) and (4) of Definition 2.1. From $\widetilde{d}^\phi \tilde{\varphi} = 0$ we deduce $d^\phi \varphi = 0$. Finally, by similar computations, we obtain the two remaining conditions that lead to the conclusion that $((A, \phi), (A^*, W), \varphi)$ is a quasi-Jacobi bialgebroid over M.

(2) Let $\tilde{\Lambda}$ be the $\tilde{A}^*$-quasi Poisson structure induced by $(\tilde{A}, \tilde{A}^*, \tilde{\varphi})$ on $M \times \mathbb{R}$. For all $\tilde{f}, \tilde{g} \in C^\infty(M \times \mathbb{R}, \mathbb{R})$, we have

$$
\{\tilde{f}, \tilde{g}\} = \tilde{\Lambda}(\delta \tilde{f}, \delta \tilde{g})
$$

and, on the other hand,

$$
\{\tilde{f}, \tilde{g}\} = \langle \tilde{d}^\phi \tilde{f}, \hat{d}_*^W \tilde{g} \rangle = \mathrm{e}^{-t} \left(\langle d\tilde{f}, d_* \tilde{g} \rangle + \frac{\partial \tilde{f}}{\partial t} a_*(\phi)(\tilde{g}) + \frac{\partial \tilde{g}}{\partial t} a(W)(\tilde{f}) \right).
$$

If (Λ, E) is the (A^*, W)-quasi Jacobi structure induced by $((A, \phi), (A^*, W), \varphi)$ on M, since $E = a_*(\phi) = -a(W)$ and $\Lambda(\delta \tilde{f}, \delta \tilde{g}) = \langle d\tilde{f}, d_* \tilde{g} \rangle$, we get that $\tilde{\Lambda} = \mathrm{e}^{-t}(\Lambda + \frac{\partial}{\partial t} \wedge E)$. □

For the case of Jacobi-quasi bialgebroids we can prove a similar result. Let $((A, \phi), (A^*, W), Q)$ be a Jacobi-quasi bialgebroid over M. We consider on $\tilde{A}$ the Lie algebroid structure $([\cdot, \cdot]^{\widehat{\phi}}, \widehat{a}^\phi)$ defined by (16) and (17), on $\tilde{A}^*$ the structure $(\widetilde{[\cdot, \cdot]}_*^W, \widetilde{a}_*^W)$ defined by (14) and (15), and we set $\tilde{Q} = e^t Q$.

THEOREM 4.2. *Under the above assumptions, we have*

1. *The triple* $((A, \phi), (A^*, W), Q)$ *is a Jacobi-quasi bialgebroid over* M *if and only if* $(\tilde{A}, \tilde{A}^*, \tilde{Q})$ *is a Lie-quasi bialgebroid over* $M \times \mathbb{R}$.
2. *If* $\tilde{\Lambda}$ *is the* $\tilde{A}$*-quasi Poisson structure induced on* $M \times \mathbb{R}$, *then it is the "quasi Poissonization" of the* (A, ϕ)*-quasi Jacobi structure* (Λ, E) *induced on* M.

5. Appendix: Action of a Lie Algebroid with 1-Cocycle

In this Appendix, we extend the definition of Lie algebroid action [10] to that of Lie algebroid with 1-cocycle action.

Let $(A, [\cdot, \cdot], a)$ be a Lie algebroid on M and $\varpi : F \to M$ a fibered manifold with base M, i.e. $\varpi : F \to M$ is a surjective submersion onto M. We recall that an *infinitesimal action of* A *on* F ([10]) is a $\mathbb{R}$-linear map $\mathbf{ac} : \Gamma(A) \to \Gamma(TF)$ such that

1. for each $X \in \Gamma(A)$, $\mathbf{ac}(X)$ is projectable to $a(X)$,
2. the map $\mathbf{ac}$ preserves brackets,
3. $\mathbf{ac}(fX) = (f \circ \varpi)\mathbf{ac}(X)$, for all $f \in C^\infty(M, \mathbb{R})$ and $X \in \Gamma(A)$.

DEFINITION 5.1. *An* infinitesimal action of (A, ϕ) on F *is a* $\mathbb{R}$*-linear map* $\mathbf{ac}^\phi : \Gamma(A) \to \Gamma(TF \times \mathbb{R})$ *given, for each* $X \in \Gamma(A)$, *by*

$$\mathbf{ac}^\phi(X) = \mathbf{ac}(X) + \langle \phi, X \rangle,$$

where $\mathbf{ac}$ *is an infinitesimal action of* A *on* F.

In the particular case where M is a point and therefore A is a Lie algebra, we obtain, from Definition 5.1, the notion of infinitesimal action of a Lie algebra with 1-cocycle on a manifold F, used in the definition of quasi Jacobi structures, in section 2.3.

If, in the Definition 5.1, $F = M$ and $\varpi : M \to M$ is the identity, we get the concept of infinitesimal action of (A, ϕ) on the base manifold M that we have used to characterize the structure induced on the base manifold of a quasi-Jacobi bialgebroid.

Acknowledgments

Research of J. M. Nunes da Costa supported by CMUC-FCT and POCI/MAT/58452/2004.

References

1. Alekseev, A., Kosmann-Schwarzbach, Y., Meinrenken, E.: Quasi-Poisson manifolds. Can. J. Math. **54**, 3–29 (2002)
2. Drinfeld, V.G.: Quasi-Hopf algebras. Leningrad Math. J. **1**, 1419–1457 (1990)

3. Grabowski, J., Marmo, G.: Jacobi structures revisited. J. Phys. A Math. Gen. **34**, 10975–10990 (2001)
4. Grabowski, J., Marmo, G.: The graded Jacobi algebras and (co)homology. J. Phys. A Math. Gen. **36**, 161–181 (2003)
5. Iglesias, D., Marrero, J.C.: Generalized Lie bialgebroids and Jacobi structures. J. Geom. Phys. **40**, 176–200 (2001)
6. Iglesias Ponte, D., Laurent-Gengoux, C., Xu P.: Universal lifting theorem and quasi-Poisson groupoids. ArXiv: math.DG/0507396
7. Kerbrat, Y., Souici-Benhammadi, Z.: Variétés de Jacobi et groupoïdes de contact. C. R. Acad. Sci. Paris Série I **317**, 81–86 (1993)
8. Klimčik, C., Strobl, T.: WZW-Poisson manifolds. J. Geom. Phys. **43**, 341 (2002)
9. Kosmann-Schwarzbach, Y.: Quasi, twisted, and all that … in Poisson geometry and Lie algebroid theory. In: Marsden, J.E., Ratiu, T.S. (eds.) The Breadth of Symplectic and Poisson Geometry. Progress in Mathematics, vol. 232, pp. 363–389. Birkhauser, Boston (2005)
10. Kosmann-Schwarzbach, Y., Mackenzie, K.: Differential operators and actions of Lie algebroids. Contemp. Math. **315**, 213–233 (2002)
11. Nunes da Costa, J.M., Clemente–Gallardo, J.: Dirac structures for generalized Lie bialgebroids. J. Phys. A Math. Gen. **37**, 2671–2692 (2004)
12. Nunes da Costa, J.M., Petalidou, F.: Twisted Jacobi manifolds, twisted Dirac–Jacobi structures and quasi-Jacobi bialgebroids. J. Phys. A Math. Gen. **39**, 10449–10475 (2006)
13. Nunes da Costa, J.M., Petalidou, F.: Characteristic foliation of twisted Jacobi manifolds. ArXiv: math.DG/0612145.
14. Park, J.S.: Topological open p-branes. In: Fukaya, K., Oh, Y.G., Ono, K., Tian, G. (eds.) Symplectic geometry and Mirror Symmetry (Seoul, 2000), pp. 311–384. World Scientific River Edge, NJ (2001)
15. Roytenberg, D.: Courant algebroids, derived brackets and even symplectic supermanifolds. Ph.D. Thesis, University of California, Berkeley (1999) (arXiv: math.DG/9910078).
16. Roytenberg, D.: Quasi-Lie Bialgebroids and Twisted Poisson Manifolds. Lett. Math. Phys. **61**, 123–137 (2002)
17. Ševera P., Weinstein A: Poisson geometry with a 3-form background. In: Proceedings of the International Workshop on Noncommutative Geometry and String Theory. Progress Theoretical Physics, Suppl. 144, pp. 145–154 (2001)