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GRAPHS AND THEIR PARALLEL GROUPS

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Given an immersion of a manifold $f : M \to R^{n+k}$, dimension $M = n$, the parallel group $P(f)$ of f is formed by the diffeomorphisms of M such that the normal k-planes at points of each orbit are parallel.

In [3] we studied the parallel group of a plane closed curve. Here we concentrate on immersions $f : R^n \to R^{n+1}$, special attention being paid to graphs of smooth maps from R to R. Graphs of smooth maps $f : S^n \to R^m$ are also dealt with and we characterise those maps of which the graph has nontrivial parallel group. To end up we find a sufficient condition for the triviality of the tangent group.

1. Codimension 1 Immersions of $R^n, n = 1, 2$.

Let $f : R^n \to R^{n+1}$ be a smooth ($= C^\infty$) immersion which we shall assume to have nonvanishing Gaussian curvature. The *parallel group* $P(f)$ consists of the diffeomorphisms $\delta : R^n \to R^n$ such that, for any $x \in R^n$, f has parallel normal lines at x and $\delta(x)$. There is a natural action of $P(f)$ on R^n as a subgroup of $Diff R^n$ and such an action is properly discontinuous, (see [3]).

PROPOSITION 1. *Let $f : R^n \to R^{n+1}, n = 1, 2$, be an immersion with nonvanishing Gaussian curvature. Then $P(f)$ is either trivial or not finite.*

Proof. Define $N : R^n \to R^{n+1}_n \approx R^{n+1}_1$, where R^k_r denotes the Grassmannian of r-dimensional vector subspaces in R^k, by $N(x) = f_{*x}(T_x R^n)$. Here, as usual, f_{*x} is the linear map induced by f. Since f has nonvanishing Gaussian curvature N is an immersion and this leads to the action of $P(f)$ being properly discontinuous [**3**]. Basic results in the theory of covering spaces [**5**] imply that $\pi_1(R^n/P(f))$ is isomorphic to $P(f)$.

If $n = 1$ and $P(f)$ is finite then the quotient manifold $R/P(f)$ is Hausdorff, second-countable and, consequently, diffeomorphic to either R or S^1. Since $P(f)$ is finite it follows that it must be trivial.

For $n = 2$ and $P(f)$ finite we can conclude in a similar way that $R^2/P(f)$ is a Hausdorff, second-countable, connected surface. Since the only such surface with nontrivial, finite fundamental group is the projective plane [**1,6**], which is not covered by R^2, the result follows. $\bowtie$

The logarithmic spiral given by $f(t) = a(e^{bt}cost, e^{bt}sint)$ is an example of a simple curve with $P(f) \approx Z$. Also the immersion $f : R^2 \to R^3$ given by $f(x, y) = (e^y \cos x, e^y \sin x, x)$ has nonvanishing Gaussian curvature and $P(f) \approx Z$.

In the compact case, that is, with a compact manifold M replacing R^n, the nonvanishing Gaussian curvature assumption leads to $M \approx S^n$ and $P(f) \approx Z_2$ which is not very interesting.

In the noncompact case it is also possible to have $P(f)$ nontrivial and finite as immersing $S^1 \times R$ as a round 2-sphere with a pair of antipodal points removed or as a hyperboloid of revolution shows.

It would be interesting to know if Z_2 is the only nontrivial group which can occur for $S^1 \times R$, assuming injectivity of the immersion and, again, nonvanishing Gaussian curvature.

We remark that, with or without the curvature assumption of Proposition 1, the parallel group acts transitively on R^n if and only if the immersion is not substantial.

2. Graphs.

Let $f : R \to R$ be a smooth function and let us denote by F the graph of f, $F(t) = (t, f(t))$. In this case the elements of $P(F)$ are the diffeomorphisms $\delta : R \to R$ such that, for all $x \in R$, $f'(x) = f'(\delta(x))$.

The result of § 1 is not of interest in the present case since the nonvanishing curvature assumption implies that f'' never vanishes and hence that $P(F)$ is trivial. Otherwise we should have a diffeomorphism δ and $x \in R$ with $\delta(x) \neq x$ and $f'(x) = f'(\delta(x))$. Thus f'' would vanish between x and $\delta(x)$, by Rolle's theorem.

The following examples illustrate the great variety of possibilities for $P(F)$.

a) At the extremes, we have $P(F)$ trivial for any function whose second derivative is nowhere zero, and $P(F) = Diff R$ for any constant function.

b) The sine function, for instance, has graph with the infinite dihedral group D_∞ as parallel group.

c) Let g be the function whose graph is shown in Figure 1 and let $f' = g$. Then $P(F)$ contains a subgroup of order 2 and another isomorphic to the group of all diffeomorphims δ of $[-1, 1]$ such that $\delta(\pm 1) = \pm 1$, $\delta'(\pm 1) = 1$, and for all $r > 1$, $\delta^{(r)}(\pm 1) = 0$.

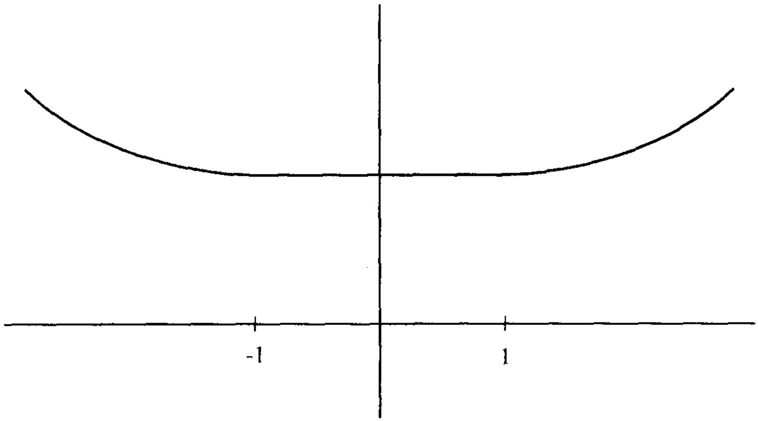

Figure 1

PROPOSITION 1. *Let $f : R \to R$ be a smooth function such that f' vanishes only at finitely many points. If $\delta : R \to R$ is an orientation preserving diffeomorphism and $f(x) = f(\delta(x))$, for $x \in R$, then δ is the identity map id_R.*

Proof. Denote by x_i, $i = 1, \ldots, n$, the points where f' vanishes. Observe now that each restriction

$$f|(-\infty, x_1), f|(x_1, x_2), \ldots, f|(x_n, +\infty)$$

is injective and that the repeated use of the mean value theorem leads us to conclude that $\delta(x_i) = x_i$, $i = 1, \ldots, n$. ⋈

PROPOSITION 2. *Let $f : R \to R$ be a polynomial function of degree greater than one.*

If f has even degree then $P(F)$ is trivial.

If f has odd degree then $P(F)$ is either trivial or Z_2.

Proof. Assume that f has even degree and that $\delta \in P(F)$. Then $f' o \delta = f'$. Since $\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} f'(\delta(x))$ and f' has odd degree it follows that δ must be orientation preserving. By proposition 1, $\delta = id_R$.

Assume now that f has odd degree. If $\delta \in P(F)$ then, by proposition 1, δ is either the identity map or orientation reversing. If δ_1, δ_2 are orientation reversing it follows that $\delta_1 o \delta_2 = id_R$. Therefore, for $\delta \in P(F)$, $\delta = \delta^{-1}$ and consequently $\delta_1 = \delta_2$. $\bowtie$

We do not have a simple criterion enabling us to deduce whether the graph of a given polynomial function of odd degree has non-trivial parallel group.

If the degree is 3 then obviously the group is Z_2. In fact, for $f(x) = ax^3 + bx^2 + cx + d$, δ given by $\delta(x) = -x - 2b/(3a)$ is the non-trivial element of $P(f)$.

For higher degrees it is possible to have trivial as well as non - trivial parallel group. These cases are exemplified by $f(x) = x^5/5 - x^4/4 - x^3/3 + x^2/2$ and $f(x) = x^5$, respectively.

3. Maps on spheres

Let $f : S^n \to R^m$ be a smooth map and let us consider its graph $F : S^n \to R^{n+m+1}$.

PROPOSITION 1. *If $P(F)$ is non-trivial then it is isomorphic to Z_2 and that happens if and only if, for some $c \in R^m$ and every $x \in S^n$, $f(x) = -f(-x) + c$.*

Proof. If, for some $c \in R^m$ and every $x \in S^n$, $f(x) = -f(-x) + c$ then the antipodal map of S^n is an element of $P(F)$.

Assume now that $\delta \neq id_{S^n}$ is an element of $P(F)$. The linear maps induced by F between the tangent spaces, F_{*x} and $F_{*\delta(x)}$, have the same image if we identify the tangent spaces to R^{n+m+1} with itself. Since $F = (inclusion, f)$, $\delta(x) = -x$, for every $x \in S^n$. This shows that $P(F) \approx Z_2$.

As to the extra condition that f must satisfy one only has to argue as in the proof of Theorem 1 of **[2]** where the full strength of the notion of transnormality is not needed. $\bowtie$

4. Further examples.

Graphs are obviously a very particular type of immersions. Some other examples of parallel groups can arise as follows.

Take two immersions f and g of manifolds into Euclidean spaces. Then the direct product $f \times g$ is such that $G(f) \times G(g) \subset G(f \times g)$. Now using results and examples from [3] one sees, for instance, that $S^1 \times S^1$ can be immersed in R^4 with highly non-trivial parallel group.

On the other hand the diagonal product of two immersions f and g of a manifold is such that $G(f, g) \subset G(f) \cap G(g)$.

Also if f is an immersion of a manifold M into some R^k then M can be immersed into $R^k \times R^s \equiv R^{k+s}$ with $G(f)$ as parallel group. We just take $(f, 0)$, where 0 is the zero map into R^s.

The self - parallelism group of an immersion [4, 7] is a subgroup of its parallel group. Theorem 3 in [4] then allows us to deduce that any orientable surface can be immersed in R^3 with nontrivial parallel group. We can complement this piece of information by observing that the parallel group of any embedding of a surface in R^3 symmetrical with respect to a point contains an element of order 2.

5. The tangent group.

In this paragraph we deal with immersions from R^n into R^{n+1}, with $n \geq 1$.

A subgroup of $P(f)$ is the *Tangent Group* $T(f)$ which consists of the diffeomorphisms $\delta : R^n \to R^n$ such that, for $x \in R^n$, at x and $\delta(x)$, f has the same tangent affine n-plane. As proposition 1 below shows it is not a very interesting subgroup though if the curvature is allowed to vanish non-trivial goups can occur as can be seen from some of the examples in § 2.

We now say that $y \in R^n$ is **simple** if $f^{-1}(f(y)) = \{y\}$.

PROPOSITION 1. *Let $f : R^n \to R^{n+1}$ be an immersion with nonvanishing Gaussian curvature and at least one simple point. Then $T(f)$ is trivial.*

Proof. Let $\delta \in T(f)$. The map $h : R^n \to R$ given by $h(x) = (f(x) - f(\delta(x)) \mid N(x))$, where $(\ldots \mid \ldots)$ stands for the ordinary inner product and N is a normal unit vector field, is constant and equal to 0. Since

$$h_{*x} = (f(x) - f(\delta(x)) \mid N_{*x}) + (f_{*x} - f_{*\delta(x)} \circ \delta_{*x} \mid N(x))$$

and N_{*x} is an isomorphism from $T_x R^n$ to $f_{*x}(T_x R^n)$, for every x, it follows that, for $x \in R^n$, $f(x) = f(\delta(x))$. If y is a simple point then $\delta(y) = y$ and since the action of $P(f)$ is properly discontinuous δ is the identity. $\bowtie$

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Added in Proof. The referee of this paper made the following suggestion for a suitable criterion which is necessary and sufficient for the graph of a polynomial function of odd degree to have a non-trivial parallel group:

Let $x_0 < x_1 < \ldots < x_k$ denote the real roots of the second derivative P'' of the polynomial function P, let $rg(x_i)$ denote the order of x_i as a root of P''. Then the following has to be satisfied by P for all $i = 0, \ldots, k$:

1) $rg(x_i) = rg(x_{k-i})$.
2) $P'(x_i) = P'(x_{k-i})$.

The proof is a little technical but not difficult. The criterion allows to derive a complete classification of the polynomials of degree 5 whose graph has parallel group Z_2 immediately. The proof of the sufficiency of the conditions above also indicates the validity of another interesting phenomenon. The condition on the order of the roots is only needed to get the differentiability of the involutory map δ. If we are content with having homeomorphisms as members of the parallel group, then the condition on the values of the roots will be necessary and sufficient only.

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