Some applications of renormalized RPA in bosonic field theories

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Abstract. We present some applications of the renormalized RPA in bosonic field theories. We first present some developments for the explicit calculation of the total energy in Φ^4 theory and discuss its phase structure in $1 + 1$ dimensions. We also demonstrate that the Goldstone theorem is satisfied in the $O(N)$ model within the renormalized RPA.

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1 Introduction

The application to quantum field theories of non-perturbative methods used in the nuclear many-body problem [1] has recently given rise to numerous promising works. One central motivation is to obtain tools to describe matter made of strongly interacting hadrons in the presence of broken symmetries such as chiral symmetry. In particular, the well-known RPA method, which has been originally developed in the context of condensed matter physics, has been recently applied to study a bosonic $O(N)$ model (*i.e.* the linear sigma model). It has been demonstrated [2] that the standard RPA is able to restore the Goldstone theorem which is violated at the level of the usual variational Gaussian approximation [3–5]. Although this result can be seen as a major success, the RPA method in its standard form possesses some weak points. In particular it has the tendency to overestimate the attractive correlation energy, at least in examples of nuclear physics. In a recent paper [6], hereafter referred as I, we have developed the formalism of a superior version of the RPA, namely the renormalized RPA (r-RPA) in the particular context of $\lambda\Phi^4$ theory. As shown in I, one important merit of the r-RPA is to cure the instability problem appearing in the standard RPA in $1 + 1$ dimensions. The purpose of this paper is to present a more detailed study of the phase structure of this theory within the r-RPA approach. Besides, we introduce improvements of the former calculation and obtain a second-order phase transition for this model. We also present the r-RPA method for the $O(N)$ model, demonstrating that the Goldstone theorem (massless pions) is also satisfied at this level. Since all the details of

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the approach have been given in I, we limit ourselves to the strict minimum for what concerns the formalism.

2 The r-RPA in Φ^4 theory

We consider the Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\,\partial^\mu\Phi(x,t)\,\partial_\mu\Phi(x,t) - \frac{1}{2}\,\mu_0^2\,\Phi^2(x,t) - \frac{b}{24}\Phi^4(x,t), \tag{1}$$

where μ_0^2 is a constant and the bare coupling constant $b = \lambda/6$ is positive for reasons of stability. We decompose the scalar field $\Phi(x,t)$ into a classical part or condensate s and a fluctuating piece $\phi(x,t)$:

$$\Phi(x,t) = \phi(x,t) + s, \qquad s = \langle\Phi(x,t)\rangle. \tag{2}$$

The presence of the condensate s indicates a spontaneous breaking of the underlying $\Phi \to -\Phi$ symmetry. Introducing the conjugate field $\Pi(x)$, one obtains for the Hamiltonian (in $d + 1$ dimensions)

$$
\begin{aligned}
H = \int \mathrm{d}^d x \, \Bigg\{ & \frac{1}{2}\mu_0^2 s^2 + \frac{b}{24}\,s^4 + \left(\mu_0\,s + \frac{b}{6}\,s^3\right)\phi(x) \\
& + \frac{1}{2}\left[\Pi^2(x) + (\partial_i\phi)^2\,(x) + \left(\mu_0^2 + \frac{b}{2}\,s^2\right)\phi^2(x)\right] \\
& + \frac{b\,s}{6}\,\phi^3(x) + \frac{b}{24}\,\phi^4(x) \Bigg\} .
\end{aligned}
\tag{3}
$$

Putting the system in a large box of volume $V = L^d$, it is convenient to work in momentum space and to expand

the fields according to

$$\phi(x) = \frac{1}{\sqrt{V}} \sum_{\vec{q}} e^{i\vec{q}\cdot\vec{x}} \phi_{\vec{q}}(t) \,,$$

$$\Pi(x) = -\frac{i}{\sqrt{V}} \sum_{\vec{q}} e^{i\vec{q}\cdot\vec{x}} \Pi_{\vec{q}}(t) \,, \qquad (4)$$

i.e., in terms of creation and annihilation operators obeying the standard canonical commutation relations:

$$\phi_{\vec{q}} = \sqrt{\frac{1}{2\kappa_{\vec{q}}}} \left(b_{\vec{q}} + b_{-\vec{q}}^{\dagger} \right), \qquad \Pi_{\vec{q}} = \sqrt{\frac{\kappa_{\vec{q}}}{2}} \left(b_{\vec{q}} - b_{-\vec{q}}^{\dagger} \right) \,. \qquad (5)$$

The choice of the basis (*i.e.* the choice of the $\kappa_{\vec{q}}$) will come out as a part of the RPA solution.

In I [6] we have explicitly solved the r-RPA problem, using the Green's function method, taking into account one-particle $b_{\vec{q}}^{\dagger}$ and two-particle $b_{\vec{q}}^{\dagger}b_{\vec{q}'}^{\dagger}, b_{\vec{q}}^{\dagger}b_{-\vec{q}'}$ excitation operators. We refer the reader to I for the detailed derivation and we only quote here the main results (we also give in the appendix the results for the 1-1 1-2 and 2-2 Green's functions).

It is convenient to choose the basis $\kappa_{\vec{q}} = \varepsilon_{\vec{q}}$ where $\varepsilon_{\vec{q}}$ is the generalized mean-field energy, solution of the gap equation $\langle [H, b_{\vec{q}} b_{-\vec{q}}] \rangle = 0$. This gap equation reads

$$\varepsilon_{\vec{q}}^{2} = \vec{q}^{2} + \mu_{0}^{2} + \frac{b}{2} s^{2} + \frac{b}{2} \langle \phi^{2} \rangle_{R} \equiv \vec{q}^{2} + m^{2} \,, \qquad (6)$$

where $\langle \phi^{2} \rangle_{R} = (1/V) \sum_{\vec{q}} \langle 1 + 2b_{\vec{q}}^{\dagger} b_{\vec{q}} \rangle / 2\varepsilon_{\vec{q}}$ is the self-consistent scalar density. To obtain the standard RPA case, one simply has to replace the self-consistent scalar density $\langle \phi^{2} \rangle_{R}$ by the Gaussian one $\langle \phi^{2} \rangle_{\varepsilon}$, *i.e.* the expectation value is calculated on the vacuum of particles having the energy $\varepsilon_{\vec{q}}$ and such that $\langle b_{\vec{q}}^{\dagger} b_{\vec{q}} \rangle_{\varepsilon} = 0$. In one spatial dimension, the generalized mean-field mass m is rendered finite by a simple mass renormalization:

$$m^{2} = \mu^{2} + \frac{b}{2} s^{2} + \frac{b}{2} \left(\langle \phi^{2} \rangle_{R} - \int_{-\Lambda}^{+\Lambda} \frac{dq}{2\pi} \frac{1}{2\sqrt{q^{2} + \mu^{2}}} \right) \,, \qquad (7)$$

where μ is the renormalized bare mass of the theory.

The r-RPA single-particle propagator has been derived in I and is given by:

$$G(E, \vec{P}) = \left(E^{2} - \varepsilon_{\vec{P}}^{2} - \Sigma(E, \vec{P}) \right)^{-1} \quad \text{with}$$

$$\Sigma(E, \vec{P}) = \frac{b^{2} s^{2}}{2} \frac{I(E, \vec{P})}{1 - \frac{b}{2} I(E, \vec{P})}. \qquad (8)$$

The two-particle loop $I(E, \vec{P})$ has the explicit form given by eq. (A.3) in the appendix. It explicitly depends on the "occupation number" $\mathcal{N}_{\vec{q}} = \langle \phi_{\vec{q}} \phi_{\vec{q}}^{\dagger} \rangle_{R}$ which constitutes the remaining problem to solve. One serious difficulty is that covariance is lost in the r-RPA in the sense that the loop integral $I(E, \vec{P})$ and, consequently, the mass operator

$\Sigma(E, \vec{P})$, depends separately on E and $\vec{P}$ due to the presence of the density $\mathcal{N}$ in its expression. This is certainly a weakness of the present approach (see discussion in I). One natural possibility to recover covariance consists in imposing that the correct $I(E, \vec{P})$ is obtained through its center-of-mass (CM) expression according to

$$I(E, \vec{P}) \equiv I(E^{2} - \vec{P}^{2}) =$$

$$\int \frac{d\vec{t}}{(2\pi)^{d}} \frac{2\mathcal{N}_{t}}{E^{2} - \vec{P}^{2} - 4\varepsilon_{t}^{2} + i\eta} \,. \qquad (9)$$

The densities $\mathcal{N}_{t}$ can be calculated self-consistently using the spectral theorem:

$$\mathcal{N}_{\vec{P}} = \int \frac{i dE}{2\pi} e^{iE\eta^{+}} G(E, \vec{P}). \qquad (10)$$

In the quasi-particle approximation used in I, the solution is $\mathcal{N}_{P} = 1/2\Omega_{P}$ where $\Omega_{P} = \sqrt{M^{2} + P^{2}}$ is the energy of the one-particle RPA mode. As explained in detail in I, the problem reduces to determining the pole of the one-particle propagator. One important result of I is linked to the fact that the instability present in $1 + 1$ dimensions at the level of the standard RPA (imaginary solution for the mass M of the RPA mode) just disappears in r-RPA.

3 r-RPA correlation energy in Φ^{4} theory

The energy of the system is calculated from the various Green's functions using the spectral theorem. However, as is well-known, this cannot be done directly from the RPA results since important correlations would be lacking at the level of the expectation value of the kinetic energy. To solve this problem we have generalized the so-called "charging formula" method [7] to the r-RPA case. We decompose the Hamiltonian in two pieces H_{0} and H_{int}:

$$H = V \left(\frac{1}{2} \mu_{0}^{2} s^{2} + \frac{b}{24} s^{4} \right)$$

$$+ \sum_{1} \frac{1}{2} \left(\Pi_{1} \Pi_{1}^{\dagger} + \mathcal{O}_{1}^{2} \phi_{1} \phi_{1}^{\dagger} \right) + H_{3} + H_{4}$$

$$\equiv H_{0} + H_{int} \,, \qquad (11)$$

where H_{3} and H_{4} are the 3- and 4-particle pieces of the Hamiltonian (last line of eq. (3)). H_{0} has a form of a free Hamiltonian for quasi-particles with the generalized mean-field mass m (eq. (7)):

$$H_{0} = E_{0} + \sum_{1} \frac{1}{2} \left(: \Pi_{1} \Pi_{1}^{\dagger} :_{\varepsilon} + \varepsilon_{1}^{2} : \phi_{1} \phi_{1}^{\dagger} :_{\varepsilon} \right) \,. \qquad (12)$$

where E_{0} is the generalized mean-field vacuum energy given in eq. (61) of I. As explained in subsect. 4.2 of I, the interacting Hamiltonian is thus

$$H_{int} = H_{3} + H_{4} - \frac{b}{4} \langle \phi^{2} \rangle_{R} \sum_{1} : \phi_{1} \phi_{1}^{\dagger} :_{\varepsilon} - V \frac{b}{8} \langle \phi^{2} \rangle_{\varepsilon}^{2} \,. \qquad (13)$$

We also introduce an auxiliary Hamiltonian:

$$H'(\rho) = H_0 + \rho H_{\text{int}}, \qquad H'(\rho = 1) = H . \qquad (14)$$

The price to pay is to solve the r-RPA problem (in practice the calculation of the commutators and double commutators entering the RPA equations) for the $H'(\rho)$ Hamiltonian. As explained in I, this can be done by making the following modifications in the corresponding r-RPA problem for H:

$$H \rightarrow H'(\rho),$$
$$\varepsilon_1^2 \rightarrow \varepsilon_{1\rho}^2 = \varepsilon_1^2 + \frac{b}{2} \rho \left(\langle \phi^2 \rangle_{R\rho} - \langle \phi^2 \rangle_R \right) ,$$
$$H_3 + H_4 \rightarrow \rho (H_3 + H_4), \qquad (15)$$

where $\langle \phi^2 \rangle_{R\rho}$ is the self-consistent scalar density in the correlated RPA ground state of $H'(\rho)$. We now employ the charging formula to calculate the ground-state energy as a function of the condensate s, *i.e.*, the effective potential needed to study the phase structure of the theory:

$$E_{\text{RPA}} = E_0 + \int_0^1 \frac{\mathrm{d}\rho}{\rho} \langle \rho H_{\text{int}} \rangle_\rho. \qquad (16)$$

Using the Wick theorem with respect to the vacuum of the quasi-particle with energies ε_ρ, the correlated part can be rewritten as

$$\langle \rho H_{\text{int}} \rangle_\rho = \langle \rho H_3 \rangle_\rho + \langle \rho : H_4 :_{\varepsilon_\rho} \rangle_\rho$$
$$- V \frac{\rho b}{8} \left(\langle \phi^2 \rangle_{\varepsilon_\rho} - \langle \phi^2 \rangle_\varepsilon \right)^2$$
$$- V \frac{\rho b}{4} \left(\langle \phi^2 \rangle_R - \langle \phi^2 \rangle_{\varepsilon_\rho} \right) \left(\langle \phi^2 \rangle_{R\rho} - \langle \phi^2 \rangle_\varepsilon \right) . \qquad (17)$$

In this formula $\langle \phi^2 \rangle_R$ is as before the self-consistent scalar density of the original H, whereas $\langle \phi^2 \rangle_{R\rho}$ corresponds to the same quantity in the $H'(\rho)$ problem. $\langle \phi^2 \rangle_\varepsilon$ is the scalar density in the generalized mean-field vacuum (vacuum of quasi-particles with energy $\varepsilon_{\vec{q}}$) in the H problem and $\langle \phi^2 \rangle_{\varepsilon_\rho}$ corresponds to the equivalent quantity for the $H'(\rho)$ Hamiltonian. The expectation values $\langle \rho H_3 \rangle_\rho$ and $\langle \rho : H_4 :_{\varepsilon_\rho} \rangle_\rho$ are calculated by using the spectral theorem applied to the 1-2 and 2-2 Green's functions relative to the r-RPA $H'(\rho)$ problem. These are actually the main contributions noted $E^{(3)}{}_{\text{corr}}$ and $E^{(4)}{}_{\text{corr}}$ of the correlation energy. However, at this level, we would like to precise one point. In the previous article I, we have taken into account the term

$$FT = \int \frac{\mathrm{d}\rho}{\rho} \left[- V \frac{\rho b}{8} \left(\langle \phi^2 \rangle_{\varepsilon_\rho} - \langle \phi^2 \rangle_\varepsilon \right)^2 \right.$$
$$\left. - V \frac{\rho b}{4} \left(\langle \phi^2 \rangle_R - \langle \phi^2 \rangle_{\varepsilon_\rho} \right) \left(\langle \phi^2 \rangle_{R\rho} - \langle \phi^2 \rangle_\varepsilon \right) \right] \quad (18)$$

(called factorized term thereafter) in the following manner. The self-consistent scalar density in the ρ problem was calculated by using the spectral theorem:

$$\langle \phi^2 \rangle_{R\rho} = \int \frac{\mathrm{d}p}{2\pi} \int \frac{i\,\mathrm{d}E}{2\pi} e^{iE\eta^+} G_\rho(E, p). \qquad (19)$$

In this article, we calculate the difference of scalar densities for the various masses with the following formula ($\Lambda \rightarrow +\infty$):

$$\langle \Delta\phi^2 \rangle_{m_1, m_2} = \langle \phi^2 \rangle_{m_2} - \langle \phi^2 \rangle_{m_1}$$
$$= \int_{-\Lambda}^{\Lambda} \frac{\mathrm{d}p}{2\pi} \left(\frac{1}{2\sqrt{m_2^2 + p^2}} - \frac{1}{2\sqrt{m_1^2 + p^2}} \right)$$
$$= -\frac{1}{4\pi} \ln \left(\frac{m_2^2}{m_1^2} \right) .$$

We will show in the section about numerical results that this method gives better results. In particular, with the non-covariant terms (see discussion below) we obtain this very important new result: the Simmon-Griffith theorem [8] (which states that the order of the transition in the Φ^4 model cannot be of the first order) is satisfied.

Two problems arise in the calculation of the effective potential. First, there is an ambiguity in the calculation of the expectation value $\langle \rho H_3 \rangle_\rho$ which was not addressed in I and second, non-covariant terms appear, even when the covariance is forced in the loop integral $I_\rho(E, P)$. We will show below that the first ambiguity gives a numerically negligible effect and concentrate mainly on the non-covariance problem. We notice that we have combinations of Green's function (see eqs. (A.9) and (A.10) in the appendix) which are explicitly covariant if the two-particle loop integral is covariant. Hence, we can decompose the 3-particle and 4-particle correlation energies into covariant pieces (which correspond to expressions (A.9) and (A.10)) and non-covariant pieces (the remaining terms) according to

$$E_{\text{corr}} = E_{\text{corr}}^{(3c)} + E_{\text{corr}}^{(3nc)} + E_{\text{corr}}^{(4c)} + E_{\text{corr}}^{(4nc)}. \qquad (20)$$

The 4-body pieces have been given in I with the result

$$\frac{E_{\text{corr}}^{(4c)}}{V} = \int_0^1 \frac{\mathrm{d}\rho}{\rho} \int \frac{\mathrm{d}\vec{P}}{(2\pi)^d} \int \frac{i\,\mathrm{d}E}{(2\pi)} e^{iE\eta^+}$$
$$\times I_\rho^2(E, \vec{P}) F_\rho(E, \vec{P}), \qquad (21)$$

$$\frac{E_{\text{corr}}^{(4nc)}}{V} = \int_0^1 \frac{\mathrm{d}\rho}{\rho} \int \frac{\mathrm{d}\vec{P}}{(2\pi)^d} \int \frac{i\,\mathrm{d}E}{(2\pi)} e^{iE\eta^+}$$
$$\times 4 I_\rho^{(1)2}(E, \vec{P}) F_\rho(E, \vec{P}) \qquad (22)$$

with $$F_\rho(E, \vec{P}) = \frac{1}{24} \left(\frac{\rho^2 b^2}{1 - \frac{\rho b}{2} I_\rho(E, \vec{P})} \right.$$
$$\left. + \frac{\rho^2 b^3 s^3 G_\rho(E, \vec{P})}{\left(1 - \frac{\rho b}{2} I_\rho(E, \vec{P}) \right)^2} \right), \qquad (23)$$

where the indices ρ mean that the quantities are related to the $H'(\rho)$ problem. The non covariance comes from the presence of the loop integral:

$$I^{(1)}(E, \vec{P}) \int \frac{\mathrm{d}\vec{k}_1\,\mathrm{d}\vec{k}_2}{(2\pi)^d} \frac{\delta^{(d)}(\vec{P} - \vec{k}_1 - \vec{k}_2)}{2\,\varepsilon_1\,\varepsilon_2}$$
$$\times \frac{\varepsilon_1 \mathcal{N}_1 + \varepsilon_2 \mathcal{N}_2}{E - \varepsilon_1 - \varepsilon_2 + i\eta} \qquad (24)$$

which is not covariant (*i.e.,* it depends separately on E and $\vec{P}$) even if covariance is forced by taking as before the CM expression. We have neglected $E_{\text{corr}}^{(4\text{nc})}$ (which vanishes to leading order in the interaction) in our previous numerical estimate in I. However, this contribution, although relatively small, turns out to be very important for the precise nature of the phase transition. In other words, $E^{(4\text{nc})}$ is very important to reproduce good numerical results. Let us come to the calculation of the expectation value of H_3:

$$\langle H_3 \rangle = \frac{b\,s}{6\,\sqrt{V\Pi_i 2\varepsilon_i}}\,\delta_{1+2+3}\left(\langle (b_1^\dagger\,b_2^\dagger + b_{-1}\,b_{-2})\,(b_{-3} + b_3^\dagger)\right.$$
$$\left. + 2\,(b_1^\dagger\,b_2^\dagger\,b_{-3} + b_1^\dagger\,b_{-2}\,b_{-3})\rangle\right). \tag{25}$$

The first line generates the covariant contribution:

$$\frac{E_{\text{corr}}^{(3\text{c})}}{V} = \int_0^1 \frac{\mathrm{d}\rho}{\rho} \int \frac{\mathrm{d}\vec{P}}{(2\pi)^d} \int \frac{i\,\mathrm{d}E}{(2\pi)}\,e^{i\,E\,\eta^+}$$
$$\times \frac{\rho^2\,b^2\,s^2}{6}\,\frac{G_\rho(E,\vec{P})\,I_\rho(E,\vec{P})}{1 - \frac{\rho\,b}{2}\,I_\rho(E,\vec{P})} \tag{26}$$

which was already considered in I. The second line of eq. (25) generates the already mentioned ambiguity: it is not uniquely defined since two different combinations of Green's functions can be used:

$$\delta_{1+2+3}\left\langle b_1^\dagger b_{-2} b_{-3}\right\rangle = \delta_{1+2+3}\int \frac{i\,\mathrm{d}E}{2\pi} e^{\imath E\eta^+} G^{(1)}_{-2-3,1^\dagger}(E) \tag{27}$$

or

$$\delta_{1+2+3}\left\langle b_1^\dagger b_{-2} b_{-3}\right\rangle = \delta_{1+2-3}\int \frac{i\,\mathrm{d}E}{2\pi} e^{\imath E\eta^+} G^{(2)}_{-1-2,3}(E)\,, \tag{28}$$

where the notations of the appendix (eq. (A.7)) have been used. The second form gives identically zero in standard RPA and in r-RPA when covariance is forced in $I^{(2)}$ (see appendix for its definition), whereas the first one gives a finite contribution. If eq. (27) is used, we obtain for the non-covariant correlation energy:

$$\frac{E_{\text{corr}}^{(3\text{nc})}}{V} = \int_0^1 \frac{\mathrm{d}\rho}{\rho} \int \frac{\mathrm{d}\vec{P}}{(2\pi)^d} \int \frac{i\,\mathrm{d}E}{(2\pi)}\,e^{i\,E\,\eta^+}$$
$$\times \frac{\rho^2\,b^2\,s^2}{6}\left(\frac{E + \varepsilon_{\rho P}}{2\varepsilon_{\rho P}}\right)\frac{2\,I_\rho^{(1)}(E,\vec{P})\,G_\rho(E,\vec{P})}{1 - \frac{\rho\,b}{2}I_\rho(E,\vec{P})}\,. \tag{29}$$

In $1+1$ dimensions all the various contributions can be calculated using a Wick rotation $(E^2 - P^2 \to -\mathcal{S})$ according to the method explained in I.

$$\frac{E_{\text{corr}}^{(4\text{c})}}{V} = -\int_0^1 \frac{\mathrm{d}\rho}{\rho} \int_0^\infty \frac{\mathrm{d}\mathcal{S}}{4\pi}\,I_\rho^2(-\mathcal{S})\,F_\rho(-\mathcal{S})\,, \tag{30}$$

$$\frac{E_{\text{corr}}^{(3\text{c})}}{V} = -\int_0^1 \frac{\mathrm{d}\rho}{\rho} \int_0^\infty \frac{\mathrm{d}\mathcal{S}}{4\pi}\,\frac{\rho^2\,b^2\,s^2}{6}\,\frac{I_\rho(-\mathcal{S})\,G_\rho(-\mathcal{S})}{1 - \frac{\rho\,b}{2}\,I_\rho(-\mathcal{S}))}\,, \tag{31}$$
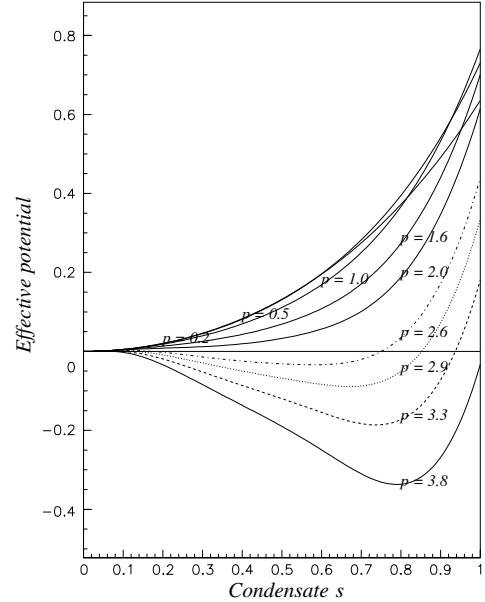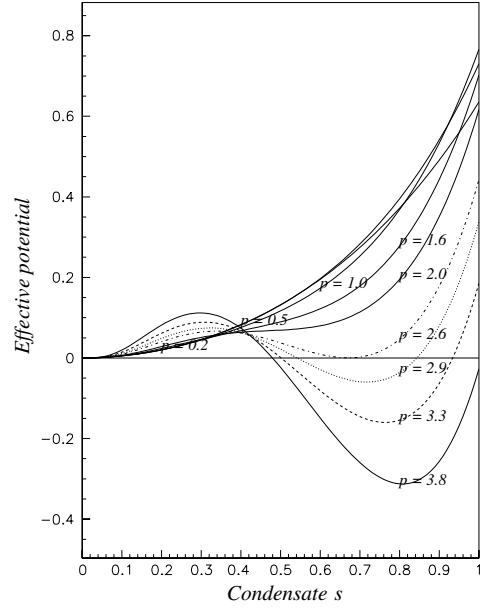

Fig. 1. Generalized mean-field effective potential (top panel); generalized mean-field effective potential plus the factorized term FT (bottom panel), for different values of the reduced coupling constant p, as a function of s.

$$\frac{E_{\text{corr}}^{(4\text{nc})}}{V} = \int_0^1 \frac{\mathrm{d}\rho}{\rho} \int_0^\infty \frac{\mathrm{d}\mathcal{S}}{2\pi^2} \int_0^{\pi/2} \mathrm{d}\theta$$
$$\times \left(\mathcal{S}\cos^2\theta\,J^2(\mathcal{S},\theta) - I^2(-\mathcal{S})\right)F_\rho(-\mathcal{S})\,, \tag{32}$$

$$\frac{E_{\text{corr}}^{(3\text{nc})}}{V} = -\int_0^1 \frac{\mathrm{d}\rho}{\rho}\,\frac{\rho^2 b^2 s^2}{3} \int \frac{\mathrm{d}\mathcal{S}}{2\pi^2} \int_0^{\frac{\pi}{2}} \mathrm{d}\theta$$
$$\times \left(\frac{\mathcal{S}\cos^2\theta}{4\sqrt{\varepsilon_{t\rho}^2 + \mathcal{S}\sin^2\theta}}J(\mathcal{S},\theta) - \frac{1}{4}I(-\mathcal{S})\right)\frac{G_\rho(-\mathcal{S})}{1 - \frac{\rho b}{2}I_\rho(-\mathcal{S})} \tag{33}$$

with

$$J(\mathcal{S}\,,\,\theta) = -\int \frac{\mathrm{d}t}{2\pi} \, \frac{1}{\sqrt{4\varepsilon_{t\rho}^2 + \mathcal{S}\sin^2\theta}} \, \frac{2\mathcal{N}_{t\rho}}{\mathcal{S} + 4\,\varepsilon_{t\rho}^2} \; . \quad (34)$$

4 Numerical results

Before going to the discussion of the correlation energy, let us mention the following result. The generalized mean-field effective potential $E_0(s)$ (vacuum energy as a function of the condensate s) presents a strong first-order phase transition as shown in fig. 1, top panel (all numerical results are obtained with the reduced coupling constant $p = b/24\mu^2$ and $\mu = 1$). When we add the factorized term FT, this strong potential barrier disappears (fig. 1, bottom panel). The transition is practically, up to an extremely small potential barrier, of second-order nature with a critical coupling constant $p_c \simeq 2.3$. It is thus tempting to consider the sum $E_0 + FT$ as the true mean-field energy for the r-RPA calculation because it contains all factorisable or reducible parts of the total energy. The fact that this term has a second-order phase transition is very important in the following. The full energy is obtained by adding the pure interaction terms $\langle H_3 + : H_4 : \rangle$ (diagrammatically, irreducible terms) which we will discuss below.

We first consider the ambiguous term $E^{(3\mathrm{nc})}$ term using eq. (33) and we obtain the results shown in fig. 2 for standard and renormalized RPA. We can verify that this term is negligible at least in r-RPA. The maximum of this term is about, respectively, 10 and 100 times in s-RPA and r-RPA, smaller than the covariant contribution in the correlation energy shown in fig. 3. This demonstrates that the ambiguity linked to the non-covariance problem is not so serious in r-RPA. Hence, in the following we take the reasonable option of putting this contribution to zero. The other term not considered in I, $E^{(4\mathrm{nc})}$, is shown in fig. 3 in comparison with the covariant correlation energy $E^{(3\mathrm{c})} + E^{(4\mathrm{c})}$. We clearly see that $E^{(4\mathrm{nc})}$ which was ignored in I is sizeable especially for low values of the condensate s. The covariant effective potential ($E_{\mathrm{cov}} = E_0 + FT + E^{(3\mathrm{c})} + E^{(4\mathrm{c})}$) and the total effective potential ($E_{\mathrm{tot}} = E_0 + FT + E^{(3\mathrm{c})} + E^{(4\mathrm{c})} + E^{(4\mathrm{nc})}$) in r-RPA are shown in fig. 4. For completeness we also show in fig. 5 the results obtained in standard RPA which were already given in I (in this calculation the RPA single-particle propagator is replaced by the mean-field one to avoid the divergence associated with instability). One sees that the RPA fluctuations in the standard RPA case are able to transform the strong first-order transition of the Gaussian approximation into a second-order phase transition. This comes from the fact that the attractive s-RPA correlations in the metastable region of the Gaussian approximation ($p \in [0.2, 0.5]$) strongly reduce the potential barrier of the Gaussian effective potential. We see in fig. 5 the evolution of the global minimum of the effective potential with increasing p. The fact that the transition is of second order is demonstrated by the fact that the condensate s is a continuous function of p and this is in agreement with
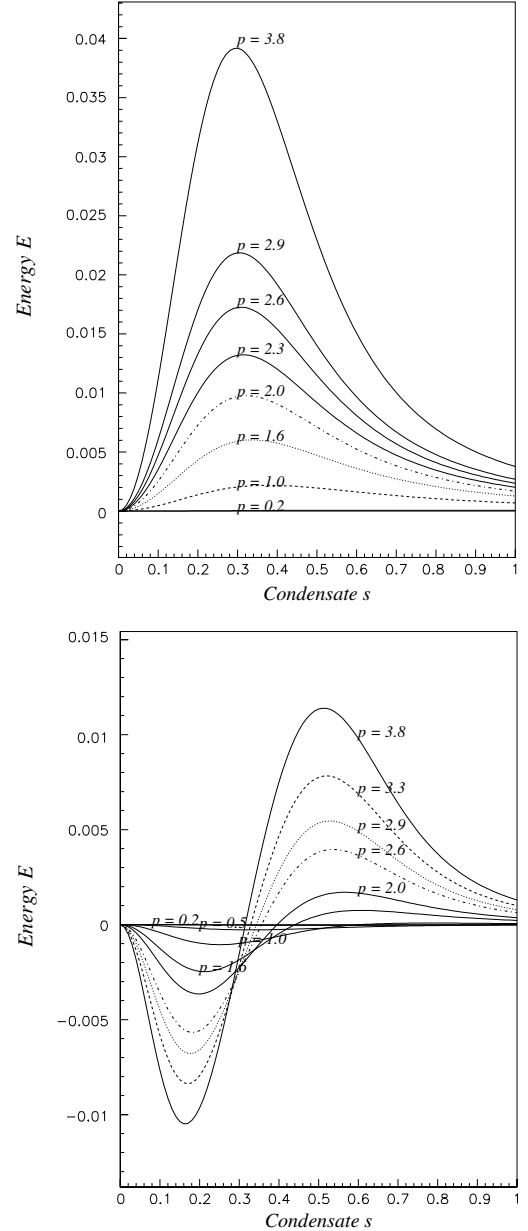

Fig. 2. Non-covariant 3-body correlation energy $E^{(3\mathrm{nc})}$ calculated with eq. (33) (top panel: standard RPA; bottom panel: r-RPA), for different values of the reduced coupling constant p, as a function of s.

the Simon-Griffith theorem [8] which states that the phase transition in the $\lambda\phi^4$ cannot be of the first order. However the critical coupling $p_c = 1.8$ is different from both the lattice result [9–11], $p_c = 2.55$, and the cluster expansion technique result [12], $p_c = 2.45$. Although this result is not so bad, the standard RPA result cannot be really trusted since this method is spoiled by the instability problem. We obtained in I a preliminary result in r-RPA but keeping only the covariant pieces in the correlation energy. Here, the result is shown in the top panel of fig. 4 with the explicit incorporation of the FT term as explained in sect. 3.
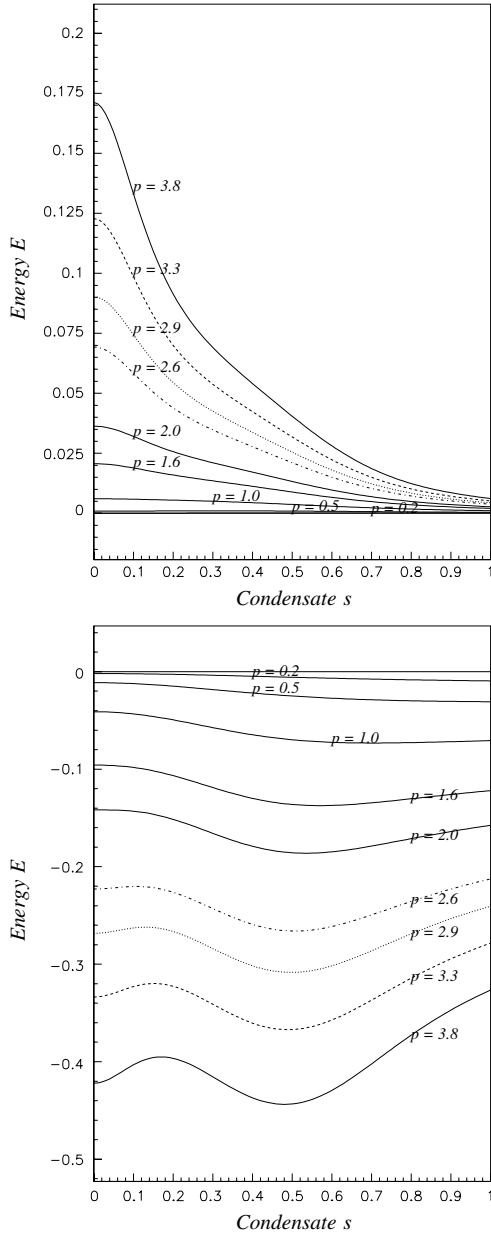
Fig. 3. Top panel: non-covariant 4-particle correlation energy $E^{(4nc)}$. Bottom panel: covariant correlation energy $E^{(3c)} + E^{(4c)}$ (in r-RPA, for different values of p, as a function of s).

Fig. 4. Effective potential in r-RPA. Top panel: covariant potential $E_{\text{tot}}^{(c)} = E_0 + E^{(3c)} + E^{(4c)} + FT$. Bottom panel: total potential $E_{\text{tot}} = E_0 + E^{(3c)} + E^{(4c)} + E^{(4nc)} + FT$.

Although much less marked than in the Gaussian case, one again obtains a first-order transition. There is nevertheless a slight progress with respect to the result obtained in I. The potential barrier is smaller and the critical coupling constant, $p_c = 1.9$, is much closer to the lattice and cluster results. When the non-covariant contribution $E^{(4nc)}$ is added, the transition is of second-order nature (see fig. 4, bottom panel). Indeed, this repulsive $E^{(4nc)}$ decreases the correlation energy in the metastable region *i.e.*, around the local maximum. Consequently, the potential barrier is so weakened that it becomes negligible (about 10^{-3}). Although the critical parameter is $p_c = 1.6$, this consti-
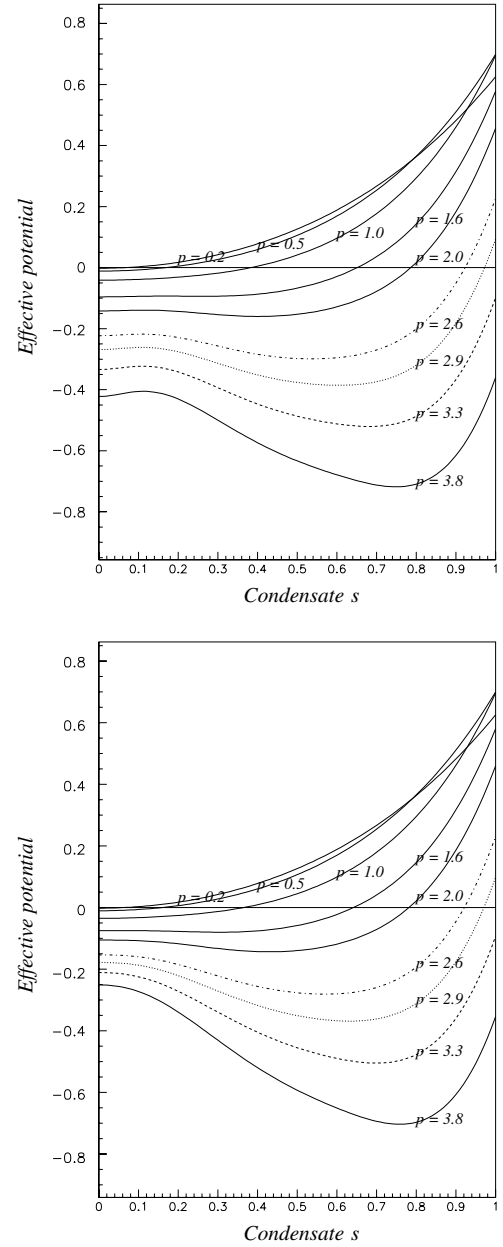
tutes an important result of this paper. The restoration of Simmon-Griffith theorem indicates that the r-RPA describes correctly the phase transition region. In addition, the absolute value of the effective potential becomes very similar to the cluster effective potential. As an example for $s = 0$ and $p = 2$, one gets the following results for the total energy. Ignoring the non-covariant contribution one gets -0.14 to be compared with -0.11 in the cluster calculation. When $E^{(4nc)}$ is added, one gets -0.105. This is a very encouraging result and certainly further work is needed in the direction of the fully self-consistent RPA. The problem of the covariance is certainly a key issue in
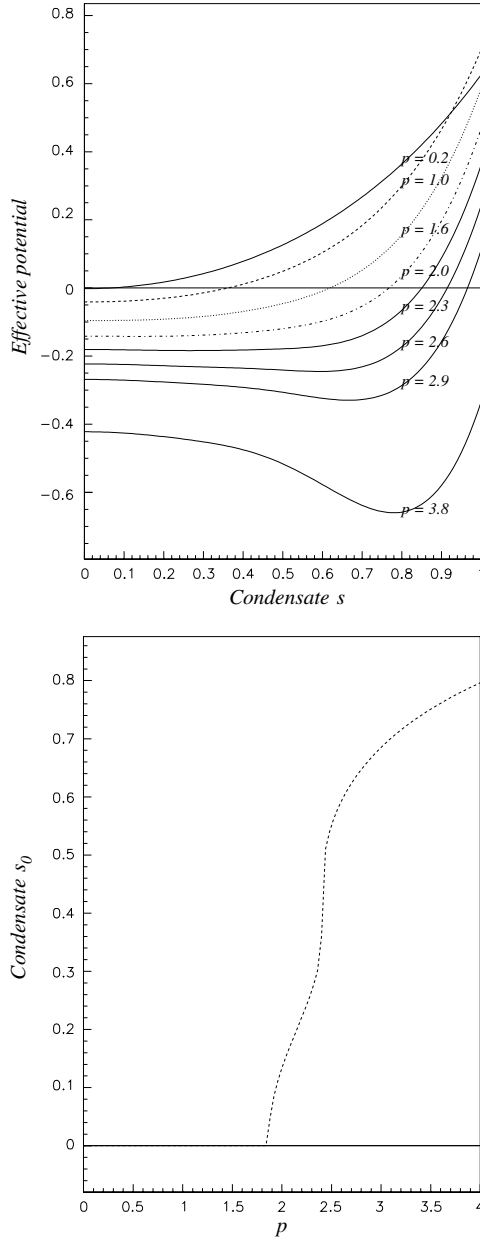
The Lagrangian reads

$$\mathcal{L}_\sigma = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\vec{\pi}\partial^\mu\vec{\pi}$$
$$-\frac{m^2}{2}\left(\sigma^2 + \pi^2\right) - \frac{\lambda}{4}\left(\sigma^2 + \pi^2\right)^2 + c\sigma\,, \qquad (35)$$

where $\vec{\pi}$ is a pseudo-scalar isovector field and σ a scalar iso-scalar field. $\vec{\pi}$ corresponds to the physical pion and its chiral partner σ may describe a mode associated with the amplitude fluctuation of the chiral condensate.

This model can formally be seen as a generalization of the Φ^4 model for a $N+1$ dimensional multiplet $(\sigma, \vec{\pi})$. It possesses an exact $O(N+1)$ invariance if the parameter c is zero. The $c\sigma$ piece of the Lagrangian describes the amount of explicit breaking of chiral symmetry in QCD.

For the application of r-RPA we use notations similar to the Φ^4 model. We introduce s, the condensate in σ field direction (the chiral symmetry being broken in one direction) and the fluctuating field σ': $\sigma = \sigma' + s$ with $\langle\sigma\rangle = s$ (hence $\langle\sigma'\rangle = 0$) and we omit the prime thereafter. We note Π_σ and $\vec{\Pi}_\pi$ the conjugate momenta of the fields σ and $\vec{\pi}$ with usual commutation relations.

The Hamiltonian reads

$$H(\sigma, \pi) = \int dx \left\{ \frac{1}{2}\left(\vec{\Pi}_\pi^2 + (\vec{\nabla}\vec{\pi})^2 + (\mu_0^2 + \lambda s^2)\vec{\pi}^2\right) \right.$$
$$+ \frac{1}{2}\left(\Pi_\sigma^2 + (\nabla\sigma)^2 + (\mu_0^2 + 3\lambda s^2)\sigma^2\right)$$
$$+ \lambda s\sigma(\sigma^2 + \vec{\pi}^2) + \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2)^2$$
$$\left. + \sigma(\mu_0^2 s + \lambda s^3 - c) + \frac{\mu_0^2}{2}s^2 + \frac{\lambda}{4}s^4 - cs \right\}. \quad (36)$$

We define quasi-particle operators for quasi-pion and quasi-sigma:

$$b_{\pi\beta} = \sqrt{\frac{\kappa_{\pi\beta}}{2}}\,\pi_\beta + \sqrt{\frac{1}{2\kappa_{\pi\beta}}}\Pi_{\pi\beta} \quad \text{and}$$

$$b_{\sigma\beta} = \sqrt{\frac{\kappa_{\sigma\beta}}{2}}\,\sigma_\beta + \sqrt{\frac{1}{2\kappa_{\sigma\beta}}}\Pi_{\sigma\beta}$$

and we introduce the scalar densities $\mathcal{N}_\alpha = \langle\pi_\alpha\pi_\alpha^\dagger\rangle$ and $N_\alpha = \langle\sigma_\alpha\sigma_\alpha^\dagger\rangle$. For what concerns the sigma operators, the index α represents a momentum state whereas for the pionic operators it represents a momentum and isospin state.

As in ref. [2] we can introduce the r-RPA excitation operators in the pionic channel according to

$$Q_\nu^\dagger = X_\alpha\,b_{\pi,\alpha}^\dagger - Y_{-\alpha}\,b_{\pi,-\alpha}$$
$$+ X_{\alpha\beta}\,b_{\pi,\alpha}^\dagger b_{\sigma,\beta}^\dagger - Y_{-\alpha-\beta}\,b_{\pi,-\alpha}b_{\sigma,-\beta}\,.$$

This is equivalent, in the Green's function approach that we really use at variance with [2], to calculate the RPA correction to the pion mass operator (Σ_π) originating from the $\pi\sigma$ RPA bubbles.

Fig. 5. Top panel: covariant effective potential in s-RPA $E_{\text{tot}}^{(c)} = E_0 + E^{(3c)} + E^{(4c)}$. Bottom panel: value of the condensate at the global minimum of $E_{\text{tot}}^{(c)}$ in s-RPA. The transition is of the second order, because this curve is continuous.

that respect. One interesting possibility is to incorporate three-body excitations in the line of the work of ref. [13].

5 The linear-σ model

We now discuss another model, the linear-σ one. It can be seen as a Ginsburg-Landau effective Lagrangian for the $SU(2) \times SU(2)$ chiral symmetry (spontaneously broken in one direction). This model can give us physical insight into the chiral phase transition and may also describe far from the transition the dynamics of pionic systems.

With the same techniques used in the Φ^4 theory, we derive the generalized mean-field equations for pion and sigma modes. For $N = 3$ they read

$$\varepsilon_\alpha^2 = \mu_0^2 + k_\alpha^2 + \lambda \sum_\alpha (N_\alpha + 5\mathcal{N}_\alpha) + \lambda s^2 \,, \tag{37}$$

$$E_\alpha^2 = \mu_0^2 + k_\alpha^2 + 3\lambda \sum_\alpha (N_\alpha + \mathcal{N}_\alpha) + 3\lambda s^2 \,. \tag{38}$$

The r-RPA inverse pion propagator is obtained as

$$G^{-1}(\omega, k_\alpha) = \omega^2 - \varepsilon_\alpha^2(\omega) - \Sigma_\alpha^\pi(\omega) \,, \tag{39}$$

with the pion mass operator given by

$$\Sigma_\alpha^\pi(\omega) = 4\lambda^2 s^2 \frac{I_\alpha^{\pi\sigma}(\omega)}{1 - 2\lambda I_\alpha^{\pi\sigma}(\omega)}.$$

It contains an iteration of the $\pi\sigma$ bubble $I^{\pi\sigma}$ which is given by

$$I_\alpha^{\pi\sigma}(\omega) = \sum_{\beta\beta'} 2\delta_{\alpha-\beta-\beta'}$$
$$\times \frac{(\mathcal{N}_\beta + N_{\beta'})\omega^2 + (N_{\beta'} - \mathcal{N}_\beta)(E_{\beta'}^2 - \varepsilon_\beta^2)}{[\omega^2 - (\varepsilon_\beta + E_{\beta'})^2][\omega^2 - (\varepsilon_\beta - E_{\beta'})^2]} \tag{40}$$

or, in a form analogous to the RPA loop in the Φ^4 model

$$I_\alpha^{\pi\sigma}(\omega) = \sum_{\beta\beta'} \delta_{\alpha-\beta-\beta'} \frac{\varepsilon_\beta + E_{\beta'}}{2\varepsilon_\beta E_{\beta'}} \frac{\mathcal{N}_\beta \varepsilon_\beta + N_{\beta'} E_{\beta'}}{\omega^2 - (\varepsilon_\beta + E_{\beta'})^2}$$
$$- \frac{\varepsilon_\beta - E_{\beta'}}{2\varepsilon_\beta E_{\beta'}} \frac{\mathcal{N}_\beta \varepsilon_\beta - N_{\beta'} E_{\beta'}}{\omega^2 - (\varepsilon_\beta - E_{\beta'})^2}. \tag{41}$$

Our preliminary goal was to show that the r-RPA fluctuations restore the Goldstone theorem. With the use of the generalized mean-field equations and using the expression of eq. (39) for $\omega = 0$, one can obtain the following result:

$$G^{-1}(\omega, k_\alpha) = \omega^2 - \left[\mu_0^2 + \lambda s^2 + 3\lambda \sum_\alpha (N_\alpha + \mathcal{N}_\alpha) \right]$$
$$- \left(\Sigma_\alpha^\pi(\omega) - \Sigma_0^\pi(\omega = 0) \right) \tag{42}$$

$$= \omega^2 - \frac{c}{s} - \left(\Sigma_\alpha^\pi(\omega) - \Sigma_0^\pi(\omega = 0) \right) \tag{43}$$

(the last result (43) is obtained by using the gap equation $\partial E/\partial s = 0$ which shows that the term in square brackets in eq. (42) is just c/s). These expressions clearly show that the Goldstone theorem is satisfied in the chiral limit because the spurious mode $\omega = 0$ is allowed.

As a preliminary conclusion, we underline this encouraging result: the r-RPA fluctuations can correctly treat the spontaneously broken symmetry and the spurious mode is obtained even if the covariance is lost. In particular it restores the Goldstone theorem violated at the level of the mean-field or Gaussian approximation [5]. This generalizes in r-RPA the result already obtained in the s-RPA formalism in [2].

6 Conclusion

We have discussed in this article some problems encountered in the calculation of the effective potential in r-RPA and we have also presented some new numerical results for the $\lambda\Phi^4$ theory in $1 + 1$ dimensions. We have shown that the ambiguity in the calculation of the 3-particle energy is only apparent in the sense that it is numerically very small. We have also shown that the incorporation of the so-called non-covariant contributions in the effective potential significantly improves the description of the phase transition in the direction of lattice and cluster expansion results. The most important result of this work is that we found a way to take into account the different contributions of the RPA correlations that give us a second-order phase transition.

For what concerns the linear-σ model we have shown that the Goldstone theorem is explicitly satisfied despite the covariance problem. A further work of interest is evidently to calculate the effective potential possibly at finite temperature to study the chiral phase transition.

Appendix A.

In this appendix we list the explicit expressions for the Green's functions. Here the momentum indices are represented by Greek letters α, β, γ. For this purpose we introduce various quantities:

$$I_{\beta\beta'}^{(1)}(E) = \frac{1}{2\varepsilon_\beta} \frac{1}{2\varepsilon_{\beta'}} \frac{\varepsilon_\beta \mathcal{N}_\beta + \varepsilon_{\beta'} \mathcal{N}_{\beta'}}{E - \varepsilon_\beta - \varepsilon_{\beta'} + i\eta} \,,$$

$$I_{\beta\beta'}^{(2)}(E) = \frac{\varepsilon_\beta - \varepsilon_{\beta'}}{2\,\varepsilon_\beta\,\varepsilon_{\beta'}} \frac{\varepsilon_\beta \mathcal{N}_\beta - \varepsilon_{\beta'} \mathcal{N}_{\beta'}}{E^2 - (\varepsilon_\beta - \varepsilon_{\beta'})^2 + i\eta} \,,$$

$$I_{\beta\beta'}^{(3)}(E) = -\frac{1}{2\varepsilon_\beta} \frac{1}{2\varepsilon_{\beta'}} \frac{\varepsilon_\beta \mathcal{N}_\beta + \varepsilon_\beta' \mathcal{N}_{\beta'}}{E + \varepsilon_\beta + \varepsilon_{\beta'} - i\eta} \,. \tag{A.1}$$

We also introduce the loop integrals

$$I_\alpha^{(i)}(E) = \frac{1}{V} \sum_{\beta\beta'} \delta_{\alpha-\beta-\beta'} I_{\beta\beta'}^{(i)}(E) \,,$$

$$I_\alpha(E) = I_\alpha^{(1)}(E) + I_\alpha^{(2)}(E) + I_\alpha^{(3)}(E) \,. \tag{A.2}$$

In particular, for α corresponding to the momentum $\vec{P}$, one has the explicit expression

$$I(E, \vec{P}) \equiv I_{\alpha=\vec{P}}(E)$$
$$= \int \frac{\mathrm{d}\vec{k}_1 \, \mathrm{d}\vec{k}_2}{(2\pi)^d} \delta^{(d)}\left(\vec{P} - \vec{k}_1 - \vec{k}_2\right)$$
$$\times \left[\frac{\varepsilon_1 + \varepsilon_2}{2\,\varepsilon_1\,\varepsilon_2} \frac{\varepsilon_1 \mathcal{N}_1 + \varepsilon_2 \mathcal{N}_2}{E^2 - (\varepsilon_1 + \varepsilon_2)^2 + i\eta} \right.$$
$$\left. - \frac{\varepsilon_1 - \varepsilon_2}{2\,\varepsilon_1\,\varepsilon_2} \frac{\varepsilon_1 \mathcal{N}_1 - \varepsilon_2 \mathcal{N}_2}{E^2 - (\varepsilon_1 - \varepsilon_2)^2 + i\eta} \right]. \tag{A.3}$$

For the one-particle Green's functions one obtains

$$G_{\alpha\alpha'^\dagger}(E) = \delta_{\alpha,\alpha'} \frac{E + \varepsilon_\alpha + \frac{\Sigma_\alpha(E)}{2\varepsilon_\alpha}}{2\varepsilon_\alpha} G_\alpha(E)\,,$$

$$G_{-\alpha^\dagger-\alpha'}(E) = \delta_{\alpha,\alpha'} \frac{-E + \varepsilon_\alpha + \frac{\Sigma_\alpha(E)}{2\varepsilon_\alpha}}{2\varepsilon_\alpha} G_\alpha(E)\,,$$

$$G_{-\alpha^\dagger\alpha'^\dagger}(E) = G_{\alpha-\alpha'}(E) = \delta_{\alpha,\alpha'} \frac{\Sigma_\alpha(E)}{4\varepsilon_\alpha^2} G_\alpha(E)\,, \text{ (A.4)}$$

where the full propagator is

$$G_{\phi_\alpha\phi_{\alpha'}^\dagger}(E) = \delta_{\alpha,\alpha'} G_\alpha(E) = \delta_{\alpha,\alpha'} \left(E^2 - \varepsilon_\alpha^2 - \Sigma_\alpha(E)\right)^{-1}. \tag{A.5}$$

The mass operator being given by

$$\Sigma_\alpha(E) = \frac{b^2 s^2}{2} \frac{I_\alpha(E)}{1 - \frac{b}{2} I_\alpha(E)}\,. \tag{A.6}$$

For what concerns the 2p-1h and 2p-2p Green's functions, we introduce indices i to label the destruction (creation) operators: $1 = \beta, \beta'(\beta^\dagger, \beta'^\dagger)$, $2 = (\beta, -\beta'^\dagger)_\text{sym}((\beta^\dagger, -\beta')_\text{sym})$ and $3 = -\beta^\dagger, -\beta'^\dagger(-\beta, -\beta')$. The results are:

$$G_{\beta\beta',\alpha^\dagger}^{(i)}(E) = G_{\alpha^\dagger,\beta\beta'}^{(i)}(E)$$
$$= \frac{b\,s}{\sqrt{V}} \delta_{\alpha-\beta-\beta'} \frac{I_{\beta\beta'}^{(i)}(E)}{1 - \frac{b}{2} I_\alpha(E)} \left(\frac{E + \varepsilon_\alpha}{2\varepsilon_\alpha}\right) G_{\phi_\alpha\phi_\alpha^\dagger}(E),$$

$$G_{\beta\beta',-\alpha}^{(i)}(E) = G_{-\alpha,\beta\beta'}^{(i)}(E)$$
$$= \frac{b\,s}{\sqrt{V}} \delta_{\alpha-\beta-\beta'} \frac{I_{\beta\beta'}^{(i)}(E)}{1 - \frac{b}{2} I_\alpha(E)} \left(\frac{-E + \varepsilon_\alpha}{2\varepsilon_\alpha}\right) G_{\phi_\alpha\phi_\alpha^\dagger}(E),$$
$$\tag{A.7}$$

$$G_{\beta\beta',\gamma\gamma'}^{(ij)}(E) = I_{\beta\beta'}^{(i)}(E)\,\delta_{i,j}\,(\delta_{\beta\gamma}\,\delta_{\beta'\gamma'} + \delta_{\beta\gamma'}\,\delta_{\beta'\gamma})$$
$$+ \frac{b}{V} \sum_\alpha \frac{\delta_{\alpha-\beta-\beta'} I_{\beta\beta'}^{(i)}(E) \quad \delta_{\alpha-\gamma-\gamma'} I_{\gamma\gamma'}^{(j)}(E)}{1 - \frac{b}{2} I_\alpha(E)}$$
$$+ \frac{b^2 s^2}{V} \sum_\alpha \frac{\delta_{\alpha-\beta-\beta'} I_{\beta\beta'}^{(i)}(E)\,\delta_{\alpha-\gamma-\gamma'} I_{\gamma\gamma'}^{(j)}(E)}{\left(1 - \frac{b}{2} I_\alpha(E)\right)^2} G_{\phi_\alpha\phi_\alpha^\dagger}(E).$$
$$\tag{A.8}$$

Finally, we give the useful relations

$$\sum_{i=1,2,3} \sum_{\beta\beta'} \frac{1}{\sqrt{V}} b\,s\,\left(G_{\beta\beta',\bar\alpha}^{(i)}(E) + G_{\beta\beta',-\alpha}^{(i)}(E)\right) =$$
$$2\,\Sigma_\alpha(E)\,G_{\phi_\alpha\phi_\alpha^\dagger}(E) \tag{A.9}$$

and

$$\sum_{i,j\in 1,2,3} \sum_{\beta\beta'\gamma\gamma'} G_{\beta\beta',\gamma\gamma'}^{(ij)}(E) = 2 \sum_\alpha I_\alpha(E)$$

$$+ b \sum_\alpha \frac{(I_\alpha(E))^2}{1 - \frac{b}{2} I_\alpha(E)}$$

$$+ b^2 s^2 \sum_\alpha \frac{(I_\alpha(E))^2}{\left(1 - \frac{b}{2} I_\alpha(E)\right)^2} G_{\phi_\alpha\phi_\alpha^\dagger}(E). \tag{A.10}$$

These particular combinations are explicitly covariant if we forced the covariance in the loop $I(E,p) = I(E^2 - p^2)$.

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