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Cost optimization of singly and doubly reinforced concrete beams with EC2-2001

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Abstract A model for the optimal design of rectangular reinforced concrete sections is presented considering the stress–strain diagrams described in EC2-2001 and MC90. The following expressions are developed: economic bending moment; optimal area of steel and optimal steel ratio between upper and lower steel. All the expressions are in nondimensional form. The present model is applied to four different classes of concrete described in MC90. It is concluded that in nondimensional form the equations are nearly coincident for both singly and doubly reinforcement. It is also concluded that the ultimate strain for concrete in the compression zone, ε_{cm}, lies between the strain for peak stress ε_{c1} and the ultimate strain ε_{cu}. This result is relevant once that the maximum moment is obtained for this value, and not the value ε_{cu}, as defined in EC2-2001. Cost optimization is implemented in the code and compared with other optimum models based on the ultimate design of ACI.

Keywords Reinforced concrete · Ultimate design · MC90 equation · Cost optimization · Optimization of reinforcement · Ultimate concrete strain

1 Introduction

The principles for the ultimate design of reinforced concrete structures are established in design codes, namely EC2 (2001). According to these, the evaluation of the area of steel reinforcement is based on the ultimate conditions of the section that can occur either in concrete or in steel. The solution for the minimum area of steel reinforcement in rectangular section under bending moment is described in CEB (1982). In this model the parabola rectangle stress diagram in compressed concrete is used. More recently MC90 (1993) suggests for concrete a more elaborate equation dependent on the concrete class that is included in the recent version of EC2 (2001). This work considers the nonlinear MC90 equation in compressed concrete implemented in a mathematical manipulation program described in Barros et al. (2004), where the use of Heaviside functions allows strains and stresses to be defined by a single equation.

In the work regarding the optimization of reinforced concrete structures by Kanagasundaram et al. (1991) the cost optimization is formulated as a nonlinear programming problem. The ultimate bending moment of reinforced concrete is evaluated by a design expression. Restrictions in terms of serviceability, strength, durability and fire resistance as well as geometry, fire resistance, minimum flexural strength and ductility are considered. In the work by Adamu et al. (1994) a method based on a continuum-type optimality criteria is used, while Han et al. (1996) uses a discretized continuum-type optimality criteria. Other works, such as Leps et al. (2003), use genetic algorithms. In all these models either the parabola rectangle law or other design methods for compressed concrete are used.

In the present work the optimization process is developed by the use of the Lagrange multiplier method where the objective function is the bending moment equation taking the equilibrium load equation as a restriction. For the cost optimization process the global cost including concrete steel and form work is the objective function and the equilibrium equations are restrictions. The optimum reinforcement related to the cost ratio and strength ratio of the materials is obtained and compared with the results of Ceranic et al. (2000).

2 Ultimate design of reinforced concrete sections

The ultimate design of a doubly reinforced concrete beam (DRB) section, represented in Fig. 1a, is considered in this

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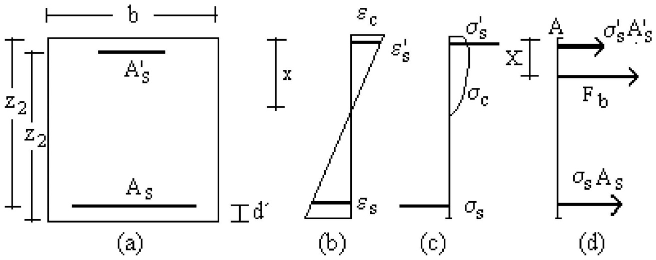

Fig. 1 **a** Section, **b** strains, **c** stresses, and **d** resulting force in concrete and steel

work. The following symbols are used: A_s = tension steel area; A'_s = compression steel area; z_2 = distance of the centroid of steel area from the opposite surface; d' = concrete cover; x = position of the neutral axis; ε_c = maximum concrete deformation in compression; ε'_s = strain in the compression steel; ε_s = strain in the tension steel, σ'_s = stress in the compression steel and σ_s = stress in the tension steel.

The ultimate design of a reinforced concrete section, according to design codes such as EC2 (2001), occurs if the concrete strain ε_c attains the ultimate value ε_{cu} or if the steel strain ε_s equals 0.01 in tension, as represented in Fig. 1b.

2.1 Constitutive relation for compressed concrete

The constitutive equation given in EC2 (2001) for concrete under compression is defined by

$$\sigma_c = \left(\frac{E_{cc1}\varepsilon}{\varepsilon_{c1}} + \frac{\varepsilon^2}{\varepsilon_{c1}^2}\right) f_{cd} \Bigg/ \left(1 - \frac{(E_{cc1}-2)\varepsilon}{\varepsilon_{c1}}\right) \tag{1}$$

where σ_c and ε represent, respectively, the current stress and strain in concrete; ε is negative, as it corresponds to compression; f_{cd} is the maximum stress for strain equal to ε_{c1}; $\varepsilon_{c1} = 0.7 f_{cm}^{0.31}$ is considered in modulus; E_{cc1} is the ratio $E_{cc1} = E_c/E_{c1}$; E_c is the tangent elasticity modulus at the origin and E_{c1} is the secant elasticity modulus for the peak stress. According to EC2 (2001) this equation is valid up to a maximum strain of ε_{cu}.

2.2 Compression force in the concrete

The compression force in the concrete, F_b, located at a distance X from the upper fibers, is obtained by integrating the stresses (1) in the compressed concrete limited by the neutral axis, Fig. 1c. Denoting by F_{bred}, the nondimensional form, this becomes

$$F_{bred} = F_b/(b\,z_2 f_{cd}) = k_1\alpha \tag{2}$$

with $\alpha = x/d$ and k_1 given by

$$\begin{aligned} k_1 = {} & -\frac{1}{2}\frac{1}{(E_{cc1}-2)^3\,\varepsilon_c\varepsilon_{c1}} \Big\{\varepsilon_c\varepsilon_{c1}\left(2E_{cc1}^3 - 8E_{cc1}^2\right. \\ & \left. +10E_{cc1} - 4\right) + \left(E_{cc1}^2 - 4E_{cc1} + 4\right)\varepsilon_c^2 - 2\left(2E_{cc1}\right. \\ & \left. -E_{cc1}^2 - 1\right)\varepsilon_{c1}^2 \left[\ln\left(\varepsilon_{c1} - E_{cc1}\varepsilon_c + 2\varepsilon_c\right) - \ln\left(\varepsilon_{c1}\right)\right]\Big\} \end{aligned} \tag{3}$$

The nondimensional form, XF_{bred}, of the bending moment evaluated in the upper concrete fibers, (point A in Fig. 1c), is

$$XF_{bred} = XF_b/(b\,z_2 f_{cd} x) = k_2\alpha \tag{4}$$

with k_2 given by

$$\begin{aligned} k_2 = {} & -\frac{1}{6}\frac{1}{(E_{cc1}-2)^4\,\varepsilon_{c1}\varepsilon_c^2} \Big\{3\Big[\varepsilon_c^2\left(E_{cc1}^4 - 12E_{cc1} + 4\right) \\ & +4\varepsilon_c\varepsilon_{c1}\Big]\varepsilon_{c1} - \varepsilon_c^3\left(8 + 6E_{cc1}^2\right) + 6\varepsilon_{c1}^2 \\ & \Big[(5\varepsilon_c + 2\varepsilon_{c1})\,E_{cc1} - E_{cc1}^2\,(\varepsilon_{c1} + 4\varepsilon_c) + \varepsilon_c\left(E_{cc1}^3 - 2\right) - \varepsilon_{c1}\Big] \\ & \times\left[\ln\left(\varepsilon_{c1} - E_{cc1}\varepsilon_c + 2\varepsilon_c\right) - \ln\left(\varepsilon_{c1}\right)\right] \\ & + E_{cc1}^2\varepsilon_c\varepsilon_{c1}\,(39\varepsilon_c + 24\varepsilon_{c1}) + E_{cc1}\left(12\varepsilon_c^3 - 30\varepsilon_c\varepsilon_{c1}^2\right) \\ & +E_{cc1}^3\left(\varepsilon_c^3 - 18\varepsilon_{c1}\varepsilon_c^2 - 6\varepsilon_{c1}^2\varepsilon_c\right)\Big\} \end{aligned} \tag{5}$$

2.3 Equilibrium equations

The equilibrium load equation in pure bending is given by

$$F_{bred} b z_2 f_{cd} + \sigma_s A_s + \sigma'_s A'_s = 0\,. \tag{6}$$

Defining the lower steel percentile $\omega = (A_s f_{syd})/(b z_2 f_{cd})$ this becomes

$$F_{bred} + \frac{\sigma_s}{f_{syd}}\omega + \frac{\sigma'_s}{f_{syd}}\omega\frac{A'_s}{A_s} = 0\,. \tag{7}$$

The bending moment equilibrium equation, calculated in the upper concrete fibers, (point A in Fig. 1d), is the following

$$XF_{bred} x b z_2 f_{cd} + \sigma_s A_s z_2 + \sigma'_s A'_s d' + M_{sd} = 0\,. \tag{8}$$

Considering the definitions of α and ω, this equation becomes

$$\mu = -XF_{bred}\alpha - \omega\frac{\sigma_s}{f_{syd}} - \omega\frac{A'_s}{A_s}\frac{\sigma'_s}{f_{syd}}\frac{d'}{z_2}\,, \tag{9}$$

where μ is the reduced bending moment equal to $\mu = M_{sd}/(b z_2^2 f_{cd})$.

3 Reinforcing steel optimization

3.1 Introduction

Reinforced concrete rectangular sections under flexural bending, without axial force, have necessarily tensile steel and eventually compression steel. It is known that for small bending moment single reinforcement is more economic, but for higher bending moments double reinforcement results in smaller total steel area. Optimization of the design can be performed only in terms of the steel area or with a more elaborate design including other costs, such as concrete or formwork. The optimization in the present work is achieved by using the Lagrange multiplier method (LMM).

3.2 Singly reinforced beam (SRB)

The objective function considered in the optimization of an SRB is the bending moment equation, which must be maximized for a given reinforcement ratio, ω. The bending moment, μ, defined by (9), is simplified by considering that in the ultimate design the tension steel is equal to the maximum stress, $\sigma_s = f_{syd}$. Since in SRB there is no compression steel ($A'_s = 0$), (9) becomes

$$\mu = -\left(XF_{bred}\alpha + \omega\right) = -\left(k_2\alpha^2 + \omega\right) \tag{10}$$

The constraint equation is the axial force equilibrium equation (7), that with similar simplifications is

$$F_{bred} + \omega = 0 \Leftrightarrow k_1\alpha + \omega = 0 \tag{11}$$

There are two design variables in the present analysis: the position of the neutral axis, α, and the maximum compressive strain in the concrete, ε_c, appearing in the definition of k_1 and k_2.

3.2.1 Optimal design variables

Applying the LMM the optimal design values, α^* and ε_{cm}, are obtained

$$\alpha^* = -\omega/K_1 \text{ and} \tag{12a}$$

$$\varepsilon_{cm} = \left(-e^{root(h)} + \varepsilon_{c1}\right) / \left(E_{cc1} - 2\right) \tag{12b}$$

with h being defined by

$$\begin{aligned} h = {} & \varepsilon_{c1}^5 \left(-68E_{cc1}^2 + 40E_{cc1} - 8 - 12E_{cc1}^4 + 48E_{cc1}^3\right) \\ & + \varepsilon_{c1}^4 e^{\beta} \left(12\beta + 12\beta^2 + \ln\left(\varepsilon_{c1}\right)^2 \left(12E_{cc1}^4 - 48E_{cc1}^3 + 72E_{cc1}^2\right.\right. \\ & \left. -48E_{cc1} + 12\right) + \ln\left(\varepsilon_{c1}\right) \left(-24E_{cc1}^4\beta - 144E_{cc1}^2\beta\right. \\ & \left. +96E_{cc1}\beta + 96E_{cc1}^3\beta + 24E_{cc1} - 24\beta - 12 - 12E_{cc1}^2\right) \\ & - 24E_{cc1}\beta - 112E_{cc1} + 152E_{cc1}^2 + 12E_{cc1}^2\beta - 48E_{cc1}^3\beta^2 \\ & + 12E_{cc1}^4\beta^2 - 48E_{cc1}\beta^2 + 72E_{cc1}^2\beta^2 + 31 - 96E_{cc1} \\ & \left. +24E_{cc1}^4\right) + \varepsilon_{c1}^3 e^{2\beta} \left(-32 - 96E_{cc1}^2 + 96E_{cc1} + 48E_{cc1}^3\right. \\ & \left. -12E_{cc1}^4\right) + \varepsilon_{c1}^2 e^{3\beta} \left(\ln\left(\varepsilon_{c1}\right)\left(12E_{cc1}^2 + 12 - 24E_{cc1}\right)\right. \\ & \left. +8E_{cc1}^2 + 2 + 24E_{cc1}\beta - 12\beta - 12E_{cc1}^2\beta - 16E_{cc1}\right) \\ & + \varepsilon_{c1} e^{4\beta}\left(8 + 4E_{cc1}^2 - 8E_{cc1}\right) - e^{5\beta} = 0 \end{aligned} \tag{13}$$

Since h has more than one solution, β, the one to be considered will lead to ε_{cm} within the following limits:

$$-\varepsilon_{cu} \le \varepsilon_{cm} \le 0 \tag{14}$$

due to the fact that strains in concrete must be negative and cannot exceed the maximum ε_{cu}. In the present formulation the ultimate strain, ε_{cm}, which corresponds to the optimal bending moment, is only a function of the class of the concrete, as can be concluded from (12b) and (13). The ultimate strain, ε_{cm}, is indicated in Table 1 together with other properties of the concrete classes C16/20, C25/30, C40/50 and C50/60 taken from EC2 (2001).

Note that this value of ε_{cm} should be used in the calculation of the ultimate bending moment instead of the value of ε_{cu} as defined in EC2 (Table 3.1). This means that this formulation gives higher values for the bending moment than those obtained with ε_{cu}, which lie in the descending branch of the diagram moment versus curvature.

Table 1 summarizes relevant properties of concrete C16/20, C25/30, C40/50 and C50/60 taken from EC2 (2001, Table 3.1).

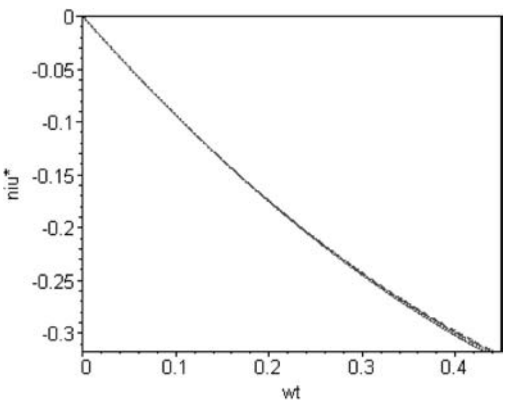

Fig. 2 Bending moment for single reinforcement in the four classes of concrete

Table 1 Mechanical properties for concretes C16/20, C25/30, C40/50 and C50/60

	C16/20	C25/30	C40/50	C50/60
$E_{c,nom}$ (GPa)	27.50	30.50	35.00	37.00
f_{cd} (MPa)	10.67	16.67	26.67	33.33
$E_c = E_{c,nom}/1.5$ (GPa)	18.33	20.33	23.33	24.66
ε_{c1}	0.001875	0.002069	0.002324	0.002465
ε_{cu}	0.0035	0.0035	0.0035	0.0035
ε_{cm}	0.00255965	0.00274712	0.00302707	0.0035
$E_c/E_{c1} = 1.1E_c\varepsilon_{c1}/f_{cd}$	3.5431	2.7762	2.2366	2.0059

3.2.2 Optimal bending moment

The optimal bending moment for an SRB, denoted by μ_s^*, is obtained by substituting α^* and ε_{cm} into the bending moment equation (10)

$$\mu_s^* = \left(-K_2 \alpha^{*^2} + \omega\right) \tag{15}$$

This equation is plotted in Fig. 2 considering the four classes of concrete summarized in Table 1. The four curves nearly coincide.

3.3 Doubly reinforced beam (DRB)

The objective function in the DRB section is again the bending moment, μ, written in terms of the steel ratio, A_s'/A_s, and the total reinforcement, $\omega_t = \omega(1 + A_s'/A_s)$. Considering the compression steel also in the plastic domain, that is $\sigma_s' = -f_{syd}$, the bending moment equation (9) becomes

$$\mu = -\left(XF_{bred}\alpha + \omega - \omega \frac{A_s'}{A_s}\frac{d'}{z_2}\right)$$
$$= -\left(k_2\alpha^2 + \frac{\omega_t}{1 + A_s'/A_s} - \frac{\omega_t}{1 + A_s'/A_s}\frac{A_s'}{A_s}\frac{d'}{z_2}\right) \tag{16}$$

The design variables are now the position of the neutral axis, α, the maximum compression strain in concrete, ε_c, and the ratio A_s'/A_s.

The constraint equation is as before defined by the axial force equation

$$F_{bred} + \omega - \omega \frac{A_s'}{A_s}$$
$$= 0 \Leftrightarrow k_1\alpha + \frac{\omega_t}{1 + A_s'/A_s} - \frac{\omega_t}{1 + A_s'/A_s}\frac{A_s'}{A_s} = 0 \tag{17}$$

3.3.1 Optimal design variables

Applying the LMM the following optimal α^* and ε_{cm} are obtained

$$\alpha_d^* = \frac{1}{4}\left(1 + \frac{d'}{z_2}\right)\frac{K_1}{K_2} \text{ and} \tag{18a}$$

$$\varepsilon_{cm} = \left(-e^{root(h)} + \varepsilon_{c1}\right) / \left(E_{cc1} - 2\right) \tag{18b}$$

α_d^* now depends also on the ratio of reinforcement cover d' to the effective depth, z_2. It can be noted that the concrete optimal strain ε_{cm} is the same as in the case of single reinforcement. The optimal ratio, $(A_s'/A_s)^*$, is also obtained:

$$\left(\frac{A_s'}{A_s}\right)^* = \left[1 + \frac{1}{4}\frac{K_1^2}{K_2\omega_t}\left(1 + \frac{d'}{z_2}\right)\right]$$
$$\times \left[1 - \frac{1}{4}\frac{K_1^2}{K_2\omega_t}\left(1 + \frac{d'}{z_2}\right)\right]^{-1} \tag{19}$$

3.3.2 Optimal bending moment

Substituting the optimal values α_d^* and $(A_s'/A_s)^*$ into the bending moment equation (16) the optimal reduced bending moment for double reinforcement, μ_d^*, is also found:

$$\mu_d^* = \frac{K_1^2 + 2K_1^2\frac{d'}{z_2} + K_1^2\left(\frac{d'}{z_2}\right)^2 - 8\omega_t K_2 + 8\frac{d'}{z_2}\omega_t K_2}{16K_2} \tag{20}$$

It must be noted that the optimal equations are dependent on the concrete class through K_1 and K_2. These values are a function of the strain ε_c, which in the optimal situation is given by ε_{cm}.

3.3.3 Economic bending moment

The bending moment beyond which it is more economic to use double reinforcement is known as the economic bending moment, μ_{eco}, and corresponds to zero area A_s', that is $A_s'/A_s = 0$. The percentile of the lower steel is then equal to the total percentile, or $\omega_t = \omega$. Considering this relation, (11) becomes:

$$\omega_t = \omega = -\alpha K_1 \tag{21}$$

and substituting into (20) the economic bending moment, μ_{eco}, is obtained:

$$\mu_{eco} = -K_2\left(\frac{1}{4}\frac{K_1}{K_2}\left(1 + \frac{d'}{z_2}\right)\right)^2 + \frac{1}{4}\frac{K_1}{K_2}\left(1 + \frac{d'}{z_2}\right)K_1\,. \tag{22}$$

The economic reinforcement, ω_{eco}, can be written as the ratio of the steel and concrete areas, ρ_{eco}, by

$$\rho_{eco} = \omega_{eco} f_{cd}/f_{syd} \tag{23}$$

3.3.4 Comparison of results

The optimal value α_d^* obtained in the present model is summarized in Table 2, for $d'/z_2 = 0.05$ and the same concrete

Table 2 Optimal values for concretes C16/20, C25/30, C40/50 and C50/60

	C16/20 CEB	C16/20	C25/30	C40/50	C50/60
α_d^* $d'/z_2 = 0.05$	0.6310	0.618614	0.635801	0.652850	0.6800186633
μ_{eco}	−0.3202	−0.370567	−0.366734	−0.362273	−0.3025241165
μ_d^*	$-0.0921 - 0.475\omega_t$	$-0.131897 - 0.475\omega_t$	$-0.130533 - 0.475\omega_t$	$-0.128945 - 0.475\omega_t$	$-0.107678 - 0.475\omega_t$
$(A_s'/A_s)^*$	$\frac{\omega_t - 0.4342}{\omega_t + 0.4342}$	$\frac{\omega_t - 0.502464}{\omega_t + 0.502464}$	$\frac{\omega_t - 0.497267}{\omega_t + 0.497267}$	$\frac{\omega_t - 0.491218}{\omega_t + 0.491218}$	$\frac{\omega_t - 0.410202}{\omega_t + 0.410202}$

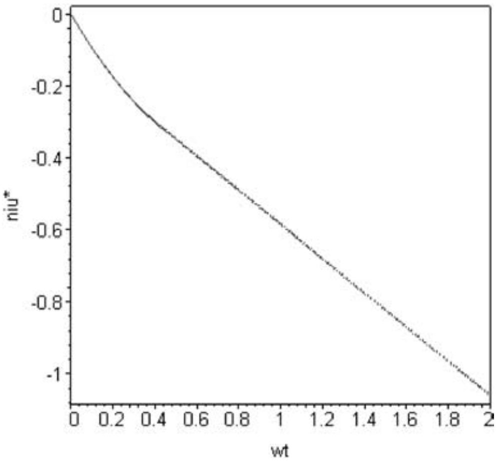

Fig. 3 Bending moment versus total reinforcement in the four classes of concrete (for $d'/z_2 = 0.05$)

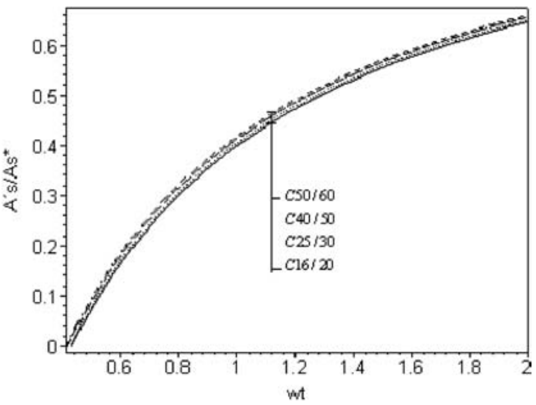

Fig. 4 Relation $(A'_s/A_s)^*$ (for $d'/z_2 = 0.05$)

classes. This table also contains the corresponding economic bending moment μ_{eco}, optimal bending moment μ_d^* and optimal reinforcement ratio $(A'_s/A_s)^*$.

Figure 3 plots the optimal bending moment versus the total reinforcement for the four classes of concrete considered and $d'/z_2 = 0.05$. Note that, if the bending moment is less than the economic bending moment μ_{eco}, only single reinforcement is used. After this value the curve gives the total reinforcement and the way it is distributed is obtained from the relation $(A'_s/A_s)^*$, which is plotted in the Fig. 4.

4 Cost optimization with the Lagrange multiplier method

4.1 Cost in a singly reinforced beam (SRB)

The cost objective function in singly reinforced concrete SRB section, denoted by C, is expressed as

$$C = C_c b \left[\omega \frac{f_{cd}}{f_{syd}} q z_2 + \left(1 + \frac{d'}{z_2}\right) z_2 \right] \tag{24}$$

where $q = C_s/C_c$ is the ratio of the materials costs, C_c is the cost of concrete per unit volume and C_s is the cost of steel. The bending moment equilibrium equation and axial force equation are restrictions.

The maximum concrete extension ε_{cm} (12b) is considered as a prescribed variable. As a matter of fact this value is obtained from the optimization of the steel at the section level, giving the maximum bending moment, and is not considered here as a design variable. Since the concrete cover ratio, $dl = d'/z_2$, in the objective function (24) is a constant, the design variables are only α, z_2 and ω.

Constructing the augmented Lagrangian and making the corresponding differentiations, the optimum reinforcement ω_s^* and optimum depth $z_{2_s}^*$ are obtained

$$\omega_s^* = -\frac{k_1^2(1+dl)}{-qk_1^2 f_{cd}/f_{syd} + 2k_2\,(1+dl)}$$

$$z_{2s}^* = \sqrt{\frac{-2qM_{sd}k_1}{b f_{syd}\left(k_1^2 + 2k_2\omega\right)\left(1 + dl + q\omega\, f_{cd}/f_{syd}\right)}} \tag{25}$$

The optimum reinforcement, ω_s^*, (25), is plotted in Fig. 5 for $dl = d'/z_2 = 0.15$ and concrete C16/20 in terms of the material stress ratio, f_{syd}/f_{cd}. Variable cost ratios q ($q =$ 25; 50; 75 and 100) are considered. The reinforcement ratio, obtained by solving to the order of ω the economic bending moment μ_{eco} (22), is also plotted. This equation is the upper limit of optimum single reinforcement, meaning that single reinforcement is only used in the lower zone, delimited by μ_{eco} in Fig. 5.

In the work by Ceranic et al. (2000) the constant stress diagram from ACI (1995) is used and the optimum reinforcement percentile $\rho_c = A_s/bz_2 = 1/(\frac{q}{1+d'/z_2} + 1.96\frac{f_{syd}}{f_{cd}})$ and the economic reinforcement ratio $\rho_{c_{eco}} =$

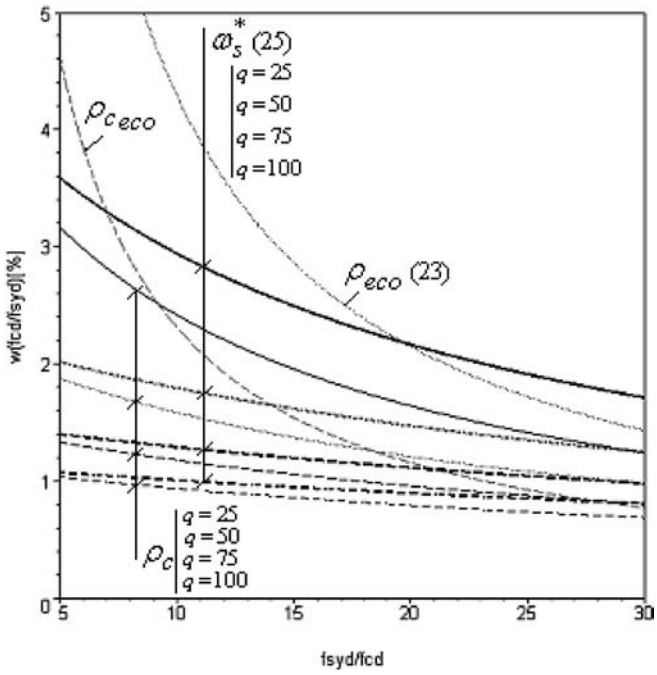

Fig. 5 Optimum reinforcement versus material stress ratio for single reinforcement

$0.2314 f_{cd}/(f_{syd})$ are derived. Figure 5 also represents these equations.

In Fig. 5 it is observed that the optimum reinforcement in the present model is greater than in the constant stress diagram. The limit for the singly reinforcement is also greater in the present model than in Ceranic et al. (2000).

4.2 Cost in a doubly reinforced beam (DRB)

The cost objective function, C, in the doubly reinforced concrete section is similar to the SRB with the replacement of ω with ω_t, and is expressed as

$$C = C_c b \left[\omega_t \frac{f_{cd}}{f_{syd}} q z_2 + \left(1 + \frac{d'}{z_2}\right) z_2 \right] \tag{26}$$

The constraint equations are the bending moment and axial force. The maximum concrete extension ε_{cm} is also defined by (12b) and the concrete cover ratio $dl = d'/z_2$ is a constant. The design variables are α, z_2, ω_t and A'_s/A_s.

Constructing the augmented Lagrangian, the optimization process gives the following optimum values, ω_t^*, $z_{2_d}^*$ and $(A'_s/A_s)^*$, which are

$$\omega_t^* = \Big(k_1^2 f_{syd}/f_{cd} \Big(dl^2 \left(A'_s/A_s\right)^* + dl^2 \left(A'_s/A_s\right)^{*^2} + dl \left(A'_s/A_s\right)^{*^2} - dl - \left(A'_s/A_s\right)^* - 1\Big)\Big) \Big/ \Big(2k_2 f_{syd}/f_{cd} (1+dl) - qk_1^2 + \left(A'_s/A_s\right)^* \times \Big(qk_1^2 (dl-1) - 4k_2 f_{syd}/f_{cd} (dl+1)\Big) + \left(A'_s/A_s\right)^{*^2} \Big(qdl\,k_1^2 + 2k_2 f_{syd}/f_{cd} (1+dl)\Big)\Big) \tag{27}$$

$$z_{2_d}^* = \sqrt{\frac{-M_{sd} f_{syd}/f_{cd}}{\omega_t^* b f_{syd}\Big[\left(1+\left(A'_s/A_s\right)^*\right) k_1^2 \left(1 - dl \left(A'_s/A_s\right)^*\right) + k_2 \omega_t^* \left(1 - \left(A'_s/A_s\right)^*\right)^2\Big]} \times k_1 \left(1 + \left(A'_s/A_s\right)^*\right)} \tag{28}$$

$$(A'_s/A_s)^* = \frac{-2\,dl\,k_2\,f_{syd}/f_{cd} + 2\,k_2\,f_{syd}/f_{cd} + qk_1^2}{-2\,dl\,k_2\,f_{syd}/f_{cd} + q\,dl\,k_1^2 + 2\,k_2\,f_{syd}/f_{cd}} \tag{29}$$

Eliminating f_{syd}/f_{cd} by the use of (37) this becomes:

$$(A'_s/A_s)^* = \frac{dl^2 k_1^2 + 4\omega_t^* k_2 dl - 4\omega_t^* k_2 - k_1^2}{-dl^2 k_1^2 + 4\omega_t^* k_2 dl + k_1^2 - 4\omega_t^* k_2} \tag{30}$$

The optimum reinforcement, ω_t^*, (27) multiplied by f_{syd}/f_{cd}, with the substitution of $(A'_s/A_s)^*$, is plotted in Fig. 6. The parameters considered are $dl = d'/z_2 = 0.15$, $q = 25$; 30; 35 and 40 and concrete C16/20.

In the figure the optimum reinforcement percentile, ρ_{cd}, derived in Ceranic et al. (2000), resulting from the application of the constant stress diagram in ACI (1995), is also plotted. The optimum reinforcement percentile ρ_{cd}, Ceranic et al. (2000), is:

$$\rho_{cd} = 0.3445 \frac{f_{cd}}{f_{syd}} - 0.3585 \frac{f_{cd}}{f_{syd}} \frac{1}{1 - d'/z_2} + \frac{1 + d'/z_2}{2q} \tag{31}$$

The optimum reinforcement ratio, $(A'_s/A_s)^*$, (30) is dependent on the concrete, the cover ratio and the total reinforcement. This equation is plotted in Fig. 7 for $dl = 0.15$ and C16/20.

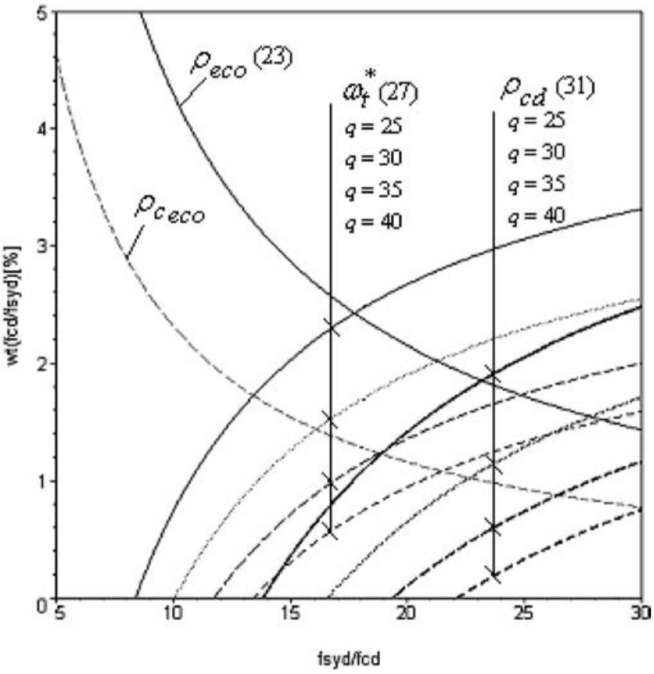

Fig. 6 Optimum reinforcement versus material stress ratio for double reinforcement

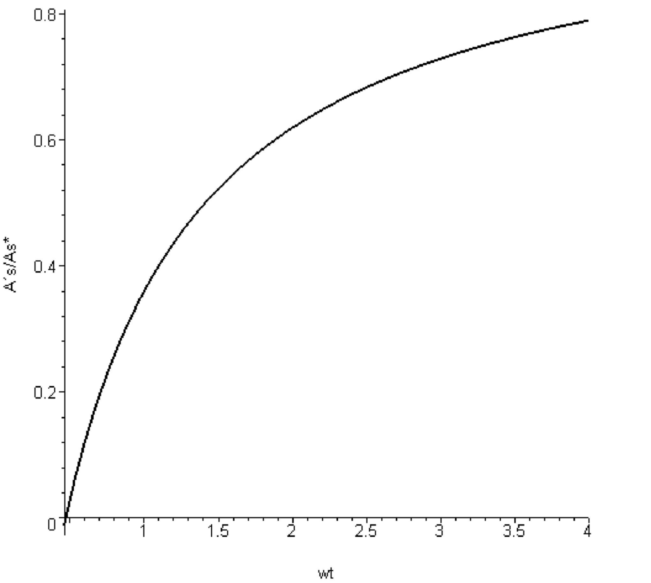

Fig. 7 Optimum reinforcement ratio $(A'_s/A_s)^*$ for double reinforcement

5 Conclusions

A model for the optimal design of rectangular reinforced concrete sections is presented considering the stress–strain diagrams indicated in EC-2001 and MC90. The following expressions are developed: economic bending moment; optimal area of steel and optimal steel ratio between upper and lower steel. All the expressions are in nondimensional form.

The present model is applied to four different classes of concrete described in EC-2001. It is concluded that in nondimensional form the equations for the bending moment versus reinforcement are nearly coincident, for both single or double reinforcement. It is also concluded that the ultimate strain for concrete in the compression zone, ε_{cm}, lies between the strain for peak stress, ε_{c1}, and the ultimate strain, ε_{cu}, defined in EC2 (Table 3.1). It is concluded from Table 3 that the CEB model is more conservative than EC-2001 since it gives a smaller bending moment for the same steel area. Considering that the MC90 relation more closely approximates the real behavior of concrete under compression than the parabola–rectangle law from CEB, it can be concluded that EC-2001 gives a more economic design.

The model is implemented with the costs of the materials, the cost optimization is performed and the results are compared to Ceranic et al. (2000).

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