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Theory and Methodology

Using cutting planes in an interactive reference point approach for multiobjective integer linear programming problems

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Abstract

We propose an interactive approach for multiple objective integer linear programming (MOILP) problems that combines the use of the Tchebycheff metric with cutting plane techniques. At each interaction, the method computes the nondominated solution for the MOILP problem that is closest to a reference point according to the Tchebycheff metric. The information provided by the decision maker in each dialogue phase is used to adjust the next reference point through a sensitivity analysis stage. Cutting plane techniques enable the method to take advantage of computations performed at previous iterations to solve the next scalarizing integer program. We address both theoretical issues and the computational implementation.

Keywords: Multicriteria analysis; Integer programming; Linear programming; Cutting planes

1. Introduction

Over the last decades, various interactive methods and decision support systems have been developed to deal with multicriteria problems. However, most research efforts have been so far concentrated on multiobjective linear programs and multiattribute models. Research in developing methods, even noninteractive, to tackle multiobjective combinatorial problems is very restricted. This is well recognized in Ulungu and Teghem's survey (Ulungu and Teghem, 1994) where they refer to the multiobjective spirit (in combinatorial optimization) that is not yet prevalent and much progress remains to be made in this direction. They also point out two main reasons that probably explain why multiobjective combinatorial optimization has been substantially ignored compared with the vast literature on single objective problems: first, the "multiobjective paradigm" is not really implemented among the circle of research workers and, second, the inherent difficulties which are not easy to tackle.

Multiobjective problems with discrete variables arise naturally in many practical applications on several different areas, such as transportation and location-allocation problems, capital budgeting, project selection, among others. However, the development of methods for these problems faces

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two relevant types of difficulties, those caused by the existence of more than one objective function (criterion), and the computational complexity of the scalarizing (single-objective) combinatorial problems to produce nondominated (efficient) solutions. It is, therefore, imperative to pay special attention to the issues concerning mathematical programming as well as the decision process in order to build an effective method that helps the decision maker (DM) to compromise and choose an acceptable solution. A survey of techniques for finding nondominated solutions in MOILP can be found in (Teghem and Kunsch, 1986a). Also, (Evans, 1984) and (Teghem and Kunsch, 1986b) provide overviews of approaches for MOILP problems.

Interactive methods offer a more attractive way to deal with multiobjective problems than the methods designed to find all the nondominated solutions (generating methods). Interactive methods try to overcome the main difficulties of generating methods that usually require a large amount of computational resources, both in time and storage space. This is particularly relevant in hard problems. Interactive methods should however be aware of the DM's capacities for processing information, not demanding too much about the DM's preferences at each interaction.

Another important issue is the way nondominated solutions are computed. Since the set of nondominated solutions for problems with discrete variables is not convex, weighted sums of the objective functions do not provide a way of reaching every nondominated solution. Besides *supported* there exist *unsupported* nondominated solutions – solutions that are dominated by convex combinations of other nondominated solutions. Tchebycheff metric-based scalarizing programs have the advantage over weighted-sums programs of being able to reach, not only supported, but also unsupported nondominated solutions. A general characterization for the nondominated solution set based on the Tchebycheff metric was first proposed by Bowman (1976). A Tchebycheff scalarizing program computes the (weakly) nondominated solution closest to a *reference point* (e.g. the ideal criterion point) according to a (weighted) Tchebycheff (L_∞) metric.

In general, *reference point* approaches for multiobjective problems (considering discrete variables or not) rely on the definition of an *achievement* scalarizing function – as suggested by Wierzbicki (1980) – by means of aspiration levels (reference point) for the objective functions. The *achievement* scalarizing function projects the reference point onto the nondominated set, for instance, through the minimization of a (weighted) Tchebycheff distance to the reference point (assuming the reference point larger than all nondominated criterion points). General information about aspiration based decision support systems can be found in Lewandowski and Wierzbicki (1989, 1988). Interactive methods using Tchebycheff or other *achievement* scalarizing functions have been developed for MOILP problems by Steuer and Choo (1983), Steuer (1986), Karaivanova et al. (1993), Karaivanova et al. (1995), Vassilev and Narula (1993), Narula and Vassilev (1994), among others.

While many researchers use augmented weighted Tchebycheff scalarizing programs keeping constant the reference point (usually the ideal criterion point) and varying the weights (e.g. Steuer and Choo, 1983), we have opted for augmented non-weighted Tchebycheff programs parameterized on the reference point. Besides being able to reach all the nondominated solutions, two main advantages can be drawn from using reference points instead of weights as the controlling parameters: (i) nondominated solutions that improve one criterion in relation to a previous nondominated solution can be obtained by changing only the corresponding component of the reference point leaving the others unchanged; (ii) concerning sensitivity analysis of the scalarizing programs, it is easier to deal with changes in the reference point than in the weight vector because, while the former is placed on the right-hand side of constraints, the latter appears as coefficients of variables in the constraints.

Following in form the theory presented in Steuer and Choo (1983) and Steuer (1986) for weighted Tchebycheff programs, we present in Section 2 some new propositions concerning the use in MOILP of non-weighted Tchebycheff programs controlled by the reference points.

In this paper we propose an interactive method for multiobjective all-integer linear programming (MOILP) problems that combines Tchebycheff scalarizing programs, cutting planes and sensitivity analysis. This combination promotes local and directional searches. The sensitivity analysis procedure aims to adjust the reference point for the next computational phase in a directional search. This feature avoids considering reference points that would lead to the previous nondominated solution. In different contexts, other research work on sensitivity analysis in multiobjective decision making has been conducted. Interested readers may refer to Insua (1990) for an extensive study on this field.

The main issues of the interactive multiobjective method we propose are the following:

(i) simple protocol to interact with the DM, which is based on the specification of reference points (aspiration levels), or just the indication of an objective function that the DM wants to improve in respect to a previous nondominated solution – (local or directional search);

(ii) identification of ranges for the components of the reference points that lead to the same nondominated solution – this feature aims to avoid wasting computational effort;

(iii) taking advantage of computations previously performed for producing other nondominated solutions – this means that sequential scalarizing programs are not solved independently.

We have adopted cutting plane techniques to solve the scalarizing programs. Although cutting planes have shown great limitations from a practical point of view, they facilitate the incorporation of the sensitivity analysis procedure we have developed. Since the nondominated solution set of a MOILP problem is discrete, it is desirable that the multiobjective approach is able to early identify reference points that lead to the same result when performing local or directional searches. The sensitivity analysis procedure is intended to satisfy this requirement (issue (ii)). Moreover, cutting planes enable to profit from previous computations to solve the next scalarizing programs (issue (iii)). These are the main reasons that encouraged us to go ahead with this work using cutting planes in spite of their practical limitations.

The current version of the interactive method we propose herein uses Gomory's cutting planes (Gomory, 1963) and *minimum cover inequalities* of Crowder et al. (1983) if all the decision variables of the multiobjective problem are 0–1. These techniques possess the usual feature of cutting off fractional solutions without eliminating any feasible integer solution. Recent research on cutting planes has been developed for certain classes of pure integer or mixed 0–1 problems (e.g. Roy and Wolsey, 1987; Cook et al., 1990; Boyd, 1994; Balas et al., 1993, 1996a, b; Ceria et al., 1998). Most of them focus on the integration of cutting planes within a branch-and-bound framework, which cannot apply to our approach in a straightforward manner. Nevertheless, our approach is not confined to the use of Gomory's cutting planes and *minimum cover inequalities*. Other cutting plane techniques devoted to pure integer programs could also be included.

Remainder sections of this paper are organized as follows: In Section 2 the problem is defined and some theoretical results (whose proofs are in Appendix A) are stated. Section 3 describes the interactive method and discusses the sensitivity analysis issues. An illustrative numerical example is presented in Section 4. Section 5 is devoted to the computational implementation. The paper closes with some concluding remarks in Section 6.

2. Problem definition and theoretical results

The multiobjective integer linear programming problem can be formulated as follows:

(MOILP)

$$\begin{aligned}
\max \quad & z_1 = c^1 x, \\
& \ldots \\
\max \quad & z_k = c^k x, \\
\text{s.t.} \quad & x \in S, \\
& S = \{x | Ax = b, x \geqslant 0, x \text{ integer}\},
\end{aligned}$$

where k is the number of objective functions (criteria), n is the number of variables, A is a $m \times n$ matrix, b is a column m-vector and $c^i, i = 1, \ldots, k$ are row n-vectors.

It is assumed that S is bounded and A, b, and $c^i, i=1,\dots,k$ have integer components. MOILP is thus a multiobjective pure integer program.

Let Z be the set of images of all $x \in S$ under the z_i. We define *efficiency* and *nondominance* in the usual manner:

$\tilde{x} \in S$ is an *efficient* solution iff there does not exist another $x \in S$ such that $c^i x \geqslant c^i \tilde{x}$ for all i and $c^j x > c^j \tilde{x}$ for at least one j. $\bar{x} \in S$ is a *weakly-efficient* solution iff there does not exist another $x \in S$ such that $c^i x > c^i \bar{x}$ for all i.

The criterion points corresponding to (weakly) efficient solutions are called (weakly) *nondominated* points/solutions.

Let z^+ denote a criterion reference point. Without loss of generality we shall assume that z^+ satisfies $z_i^+ \geqslant z_i', i=1,\dots,k, \forall z' \in Z$. This can be assured by considering $z^+ \geqslant z^*$, where z^* denotes the ideal criterion point.

The following program determines a solution $z \in Z$ closest to the reference point z^+ according to the Tchebycheff metric:

$$
(\mathrm{P}_1, z^+) \qquad \begin{aligned} &\min \quad \{\alpha\} \\ &\text{s.t.} \quad z_i^+ - c^i x \leqslant \alpha, \quad i=1,\dots,k, \\ &\qquad\quad x \in S, \alpha \geqslant 0. \end{aligned}
$$

Although (P_1, z^+) may have optimal solutions that are weakly-nondominated for the (MOILP) problem, among the alternative optima there always exists at least one nondominated solution (Proposition 1).

Proposition 1. *Let $M \subseteq Z$ be the set of criterion points z with $z_i = c^i x, i=1,\dots,k$, such that x minimizes* (P_1, z^+)*. Then, there exists $\tilde{z} \in M$ such that $\tilde{z}$ is nondominated.*

Proposition 2. *$\tilde{z}$ is a nondominated criterion point with $\tilde{z}_i = c^i \tilde{x}, i=1,\dots,k$ ($\tilde{x}$ being an efficient solution)*

$$\Longleftrightarrow$$

There exists $z^+ \geqslant z^*$, such that $\tilde{x}$ minimizes the augmented Tchebycheff program:

$$
(\mathrm{P}_2, z^+) \qquad \begin{aligned} &\min \quad \left\{\alpha - \rho \sum_{i=1}^{k} c^i x\right\} \\ &\text{s.t.} \quad z_i^+ - c^i x \leqslant \alpha, \quad i=1,\dots,k, \\ &\qquad\quad x \in S, \alpha \geqslant 0, \end{aligned}
$$

with ρ positive small enough.

From Proposition 2 we conclude that non-weighted Tchebycheff scalarizing programs controlled by reference points, (P_2, z^+), or even (P_1, z^+), allow to reach all the nondominated solutions. Propositions 1 and 2 are adaptations of the Tchebycheff theory presented in Steuer and Choo (1983) and Steuer (1986). Besides, we state other results for MOILP:

(i) to obtain nondominated solutions that improve a specific objective function, say z_p, in relation to a previous nondominated solution, one may increase the pth component of the reference point used in the Tchebycheff scalarizing program leaving the other components unchanged – (a similar result was stated by Metev and Yordanova (1993) for MOLP);

(ii) one may consider only integer reference points and integer increments without loosing intermediate nondominated solutions – this allows the variables in Tchebycheff scalarizing programs to be all-integer;

(iii) under the circumstances of (i) and (ii), Gomory's cutting planes introduced in the resolution of a Tchebycheff scalarizing program are still valid for the next scalarizing program. In spite of their validity, these inequalities become weaker than before.

Besides Gomory's cutting planes, we also use *minimum cover inequalities* for 0–1 MOILP problems. Since the Tchebycheff scalarizing programs are never pure 0–1, these inequalities are always introduced in the MOILP problem, i.e., only the constraints $Ax=b$ are used to generate *minimum cover inequalities*.

Statements (i), (ii) and (iii) translate, respectively, Propositions 3–5 that follow (the latter is stated in a more global sense). By reasons of simplicity and clearness of the proofs (see Appendix A), these propositions are established for programs (P_1, z^+) instead of (P_2, z^+).

Proposition 3. *Let x^a be an efficient solution that minimizes* (P_1, z^{a+}) *and $z^{b+} = (z_1^{a+}, \dots, z_p^{a+} + \theta_p, \dots, z_k^{a+})$ with $\theta_p > 0$. Then, either* (i) *x^a minimizes* (P_1, z^{b+}) *or* (ii) *there exists x^b that minimizes*

$(P_1,\ z^{b+})$ and improves the p^{th} objective function in relation to x^a (i.e. $z_p^b > z_p^a$).

Proposition 4. *Let $z^{a+} = (z_1^{a+}, \ldots, z_p^{a+}, \ldots, z_k^{a+})$ be an integer reference point, $z^{b+} = (z_1^{a+}, \ldots, z_p^{a+} + 1, \ldots, z_k^{a+})$ and $z^{c+} = (z_1^{a+}, \ldots, z_p^{a+} + \theta_p, \ldots, z_k^{a+})$ with $0 < \theta_p < 1$. Then, there does not exist any efficient solution that minimizes $(P_1,\ z^{c+})$ without minimizing $(P_1,\ z^{a+})$ or $(P_1,\ z^{b+})$.*

Proposition 5. *If the integer reference points, z^{a+} and z^{b+} satisfy $z^{b+} \geqslant z^{a+}$ then all the cutting planes introduced during the resolution of $(P_1,\ z^{a+})$ are still valid for $(P_1,\ z^{b+})$.*

Propositions 1–5 state some of the theoretical foundations of the interactive method we propose in Section 3.

3. The interactive method

The flowchart in Fig. 1 presents the algorithm of the interactive MOILP method.

Notice that the DM can choose any integer reference point, attainable or not (step 2). Step 3 is just a technical step which ensures that $z^+ \geqslant z^*$. This conversion does not modify the outcome of the original z^+ and allows to restrict α to be non-negative in the scalarizing programs. Otherwise, the unique difference would be α free.

The flowchart presents a proposal of a protocol to interact with the DM. However, the core of the algorithm is the iterative process of step 7 that embodies sensitivity analysis and computing phases. Other interactive protocols with this kind of tools could be studied.

3.1. Sensitivity analysis

Whenever the DM specifies an objective function z_p to be improved, it would be desirable to know the maximum integer increment of the pth component of the reference point, say $\theta_p^{\max}$, that would still lead to the current nondominated solution. Once this value has been determined, we would consider $\bar{\theta}_p = \theta_p^{\max} + 1$ having the guarantee of getting a "nearby" but different nondominated solution that improves z_p. Unfortunately, there is not so far any sensitivity analysis procedure that answers this question in an easy computational way. Therefore, we propose a stepwise process that approaches iteratively the value of $\theta_p^{\max}$, and consequently $\bar{\theta}_p$, until a new nondominated solution is obtained.

For reasons of clearness, and without losing generality, we consider programs $(P_1,\ z^+)$ instead of $(P_2,\ z^+)$. Recall that the objective function of $(P_1,\ z^+)$ is solely the minimization of α, the L_∞ distance to the reference point.

Notation: Let s_i be the surplus variable associated with the ith constraint $c^i x + \alpha \geqslant z_i^+, i \in \{1, \ldots, k\}$ of (P_1, z^+). Since z^+ and $c^i, i = 1, \ldots, k$, are integer vectors, all s_i variables are also integer.

Let us suppose that the current reference point is z^{a+} and x^a is the current efficient solution (with z^a nondominated) corresponding to (x^a, α^a), the optimal solution of (P_1, z^{a+}); $s_i = s_i^a, i = 1, \ldots, k$.

Let $z^{a+} + \theta_p$ denote $(z_1^{a+}, \ldots, z_p^{a+} + \theta_p, \ldots, z_k^{a+})$. Further, let $(P_1, z^{a+})^{LR+CP}$ be the program obtained from the introduction of cutting planes into the linear relaxation of $(P_1,\ z^{a+})$ such that (x^a, α^a) minimizes $(P_1, z^{a+})^{LR+CP}$ too. "$\lceil\ \rceil$" will denote the operation of taking the smallest integer larger than or equal to the relevant quantity.

In the sensitivity analysis iterative process there are two different situations to be considered: the *entrance*, i.e., the first iteration of the process that starts with an integer solution; the *return* while the current solution is non-integer.

Entrance: If $s_p^a > 0$ $(s_p^a = \alpha^a - z_p^{a+} + z_p^a)$ then (x^a, α^a) minimizes $(P_1, z^{a+} + \theta_p)$ for θ_p (at least) between 0 and s_p^a, (according to the proof of Proposition 3 – *case II*). Hence, we consider $\theta_p^{\max} = s_p^a, \bar{\theta}_p = \theta_p^{\max} + 1$ and $z^{b+} = z^{a+} + \bar{\theta}_p$. Since the L_∞ distance between z^a and z^{b+} is $\alpha^a + 1$, then the α optimal value of $(P_1,\ z^{b+})$, say α^b, will satisfy $\alpha^a \leqslant \alpha^b \leqslant \alpha^a + 1 \iff \alpha^b = \alpha^a \vee \alpha^b = \alpha^a + 1$.

If $s_p^a = 0$ then the α optimal value of $(P_1, z^{a+} + \theta_p)^{LR+CP}$ is $\alpha^a + \pi_p^a \cdot \theta_p$ for θ_p values that keep feasible the current basis. π_p^a denotes the shadow price of constraint p in $(P_1, z^{a+})^{LR+CP}$. For any value of $\theta_p, \alpha^a + \pi_p^a \cdot \theta_p$ is a lower bound for α in the optimal solution of $(P_1, z^{a+} + \theta_p)$. Since α is integer, the

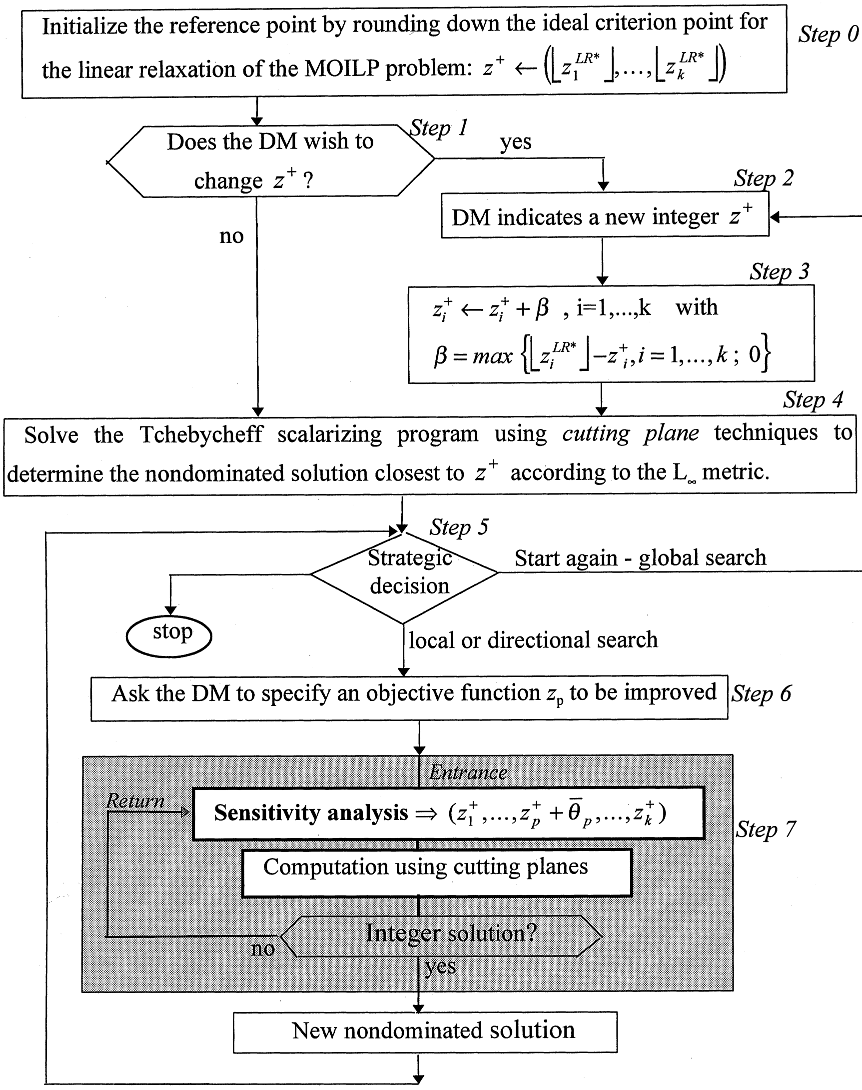


Fig. 1. Flowchart of the method.

lower bound can be adjusted to $\alpha^a + \lceil \pi_p^a \cdot \theta_p \rceil$. On the other hand, $(x^a, \alpha^a + \theta_p)$ is feasible for $(\mathrm{P}_1, z^{a+} + \theta_p)$. Therefore, x^a minimizes $(\mathrm{P}_1, z^{a+} + \theta_p)$ for (at least) the values of θ_p that satisfy $\lceil \pi_p^a \cdot \theta_p \rceil = \theta_p$. Hence,

$$\theta_p^{\max} = \max\{\theta_p | \lceil \pi_p^a \cdot \theta_p \rceil = \theta_p\}.$$

$$\bar{\theta}_p = \theta_p^{\max} + 1, z^{b+} = z^{a+} + \bar{\theta}_p$$

and

$$(\alpha^b = \alpha^a + \theta_p^{\max} \vee \alpha^b = \alpha^a + \bar{\theta}_p).$$

A particular case occurs when $\pi_p^a = 1$ in which case $\theta_p^{\max} = \infty$ and x^a optimizes the objective function p of the MOILP problem.

Conclusion: In both cases, a $\bar{\theta}_p$ value is determined such that reference points $z^{a+} + \theta_p$, with θ_p integer lower than $\bar{\theta}_p$, would lead to the previous nondominated solution. Since the result is not yet known with respect to $z^{b+} = z^{a+} + \bar{\theta}_p$, this will be the next reference point. This analysis also provides an upper bound for the α optimal value of (P_1, z^{b+}) which is $\alpha^a + 1$ when $s_p^a = 0$ and $\alpha^a + \bar{\theta}_p$ when $s_p^a > 0$. The upper bound represents the L_∞ distance from the previous nondominated solution, z^a, to z^{b+}.

After determining the new reference point, z^{b+}, the current basis and the corresponding solution are updated for z^{b+} yielding $(\bar{x}, \bar{\alpha})$. If $\bar{a}$ is fractional, the procedure returns to the sensitivity analysis phase; if $\bar{\alpha}$ is integer but $\bar{\alpha}$ fractional, a new round of cutting planes is introduced discarding all the inactive cutting planes.

Return: At this time, an upper bound U for α^b is known such that $\alpha^b = U - 1 \vee \alpha^b = U$. Note that this information has not been included in the simplex tableau, it just supports the sensitivity analysis.

Since $\bar{\alpha}$ is fractional, $\lceil \bar{\alpha} \rceil = U$. Hence, x^a still minimizes (P_1, z^{b+}) and $\theta_p^{\max}$ may be increased.

Let δ_p denote an integer increment in $\theta_p^{\max}$. A lower and an upper bound for the α optimal value of $(\mathrm{P}_1,\ z^{b+} + \delta_p)$ can be stated. They are $\lceil \bar{\alpha} + \bar{\pi}_p \cdot \delta_p \rceil$ for the lower bound and $U + \delta_p$ for the upper bound. Therefore, while δ_p is such that $\lceil \bar{\alpha} + \bar{\pi}_p \cdot \delta_p \rceil = U + \delta_p$, x^a is guaranteed to minimize $(\mathrm{P}_1,\ z^{b+} + \delta_p)$. For that reason, $\theta_p^{\max}$, $\bar{\theta}_p$ and, consequently, the pth component of z^{b+} are increased by

$$\delta_p^{\max} = \max\{\delta_p \text{ integer} \mid \lceil \bar{\alpha} + \bar{\pi}_p \cdot \delta_p \rceil = U + \delta_p\}.$$

Likewise, $U \leftarrow U + \delta_p^{\max}$.

Note: When $(\mathrm{P}_2,\ z^+)$ is used instead of $(\mathrm{P}_1,\ z^+)$, the shadow price $\bar{\pi}_p$ must be replaced by a slightly different value $(-\bar{a}_{\alpha, s_p})$ – the symmetric value of the α-row, s_p-column entry in the updated simplex tableau. The sensitivity analysis is always made in relation to α, rather than to the objective function of the scalarizing problem.

4. An illustrative example

Let us consider the MOILP problem:

$$\begin{array}{ll} \max & z_1 = x_1 - x_2, \\ \max & z_2 = x_1 + 2x_2, \\ \text{s.t.} & x_1 + 6x_2 \leqslant 21, \\ & 4x_1 + 6x_2 \leqslant 63, \\ & x_1, x_2 \geqslant 0 \text{ and integer.} \end{array}$$

Fig. 2 shows the efficient solutions on the decision variable space (a) and the corresponding nondominated points on the criterion space (b).

A preprocessing phase (for more details see Section 5), has included the bounds $x_1 \leqslant 4$ and $x_2 \leqslant 3$ into the problem.

Let us consider the scalarizing program $(\mathrm{P}_2,\ z^+)$ (considering $\rho = 0.001$) and start with $z^+ = z^{a+} = (6,\ 10)$. Denoting by x_3, x_4 the slack variables corresponding to the constraints of the MOILP problem and s_1, s_2 the surplus variables of the two first constraints in $(\mathrm{P}_2,\ z^+)$, the optimal solution of the linear relaxation of $(\mathrm{P}_2,\ z^{a+})$ is: $x_1 = 3.929$, $x_2 = 1.333$, $x_3 = 9.071$, $x_4 = 0$, $s_1 = 0$, $s_2 = 0$, $\alpha = 3.405$. Three Gomory's cutting planes have been generated, from the rows of α, x_2 and x_1, and added to the tableau. After two iterations of the dual-simplex, an integer solution is obtained. It is the point B on Fig. 2: $x^a = (4,\ 1,\ 11,\ 1)$; $s^a = (1,\ 0)$; $\alpha^a = 4$; $z^a = (3,\ 6)$. Only the third cut is active, so the other two may be discarded.

Let us suppose that the DM wishes to improve z_1. Since $s_1^a = 1$, then $\theta_1^{\max} = s_1^a = 1$ and $\bar{\theta}_1 = 2$. Therefore, the reference point is changed to $z^{b+} = (8{,}10)$ knowing a priori the possible optimal values for α: $\alpha^b = 4 \vee \alpha^b = 5$. The simplex tableau is updated by changing the basis that has become unfeasible, yielding a fractional solution where $\alpha = 4.66(6)$ and $(-\bar{a}_{\alpha, s_1}) = 0.66(6)$ (π_1, the first shadow price, is 0.667). We conclude that $\delta_1^{\max} = 1$ by testing the values for δ_1 such that $\lceil \alpha + (-\bar{a}_{\alpha, s_1}) \cdot \delta_1 \rceil = 5 + \delta_1$. In other words, z^a is the closest point (according to the Tchebycheff metric) to the reference points (8, 10) and (9, 10) – corresponding to $\delta_1 = 0$ and $\delta_1 = 1$, respectively. z^{b+} is thus adjusted to (10, 10) and the possible optimal values for α^b are now 6 or 7. An integer solution is then obtained (by updating the simplex tableau): $x^b = (4,\ 0,\ 17,\ 7)$; $s^b = (0{,}0)$; $\alpha^b = 6$; $z^b = (4,\ 4)$ – point A on the picture.

Assuming that the DM wants to continue improving z_1, the sensitivity analysis leads to the reference point $z^{c+} = (13,\ 10)$ (due to $(-\bar{a}_{\alpha, s_1}) = 0.66(6)$). This forces x_2 – a degenerate basic variable for $z^{b+} = (10,\ 10)$ – to become negative; x_2 is therefore replaced by s_2 in the basis. The result is the same efficient solution – x^b – but now $(-\bar{a}_{\alpha, s_1}) = 1$ enabling to conclude that x^b is the

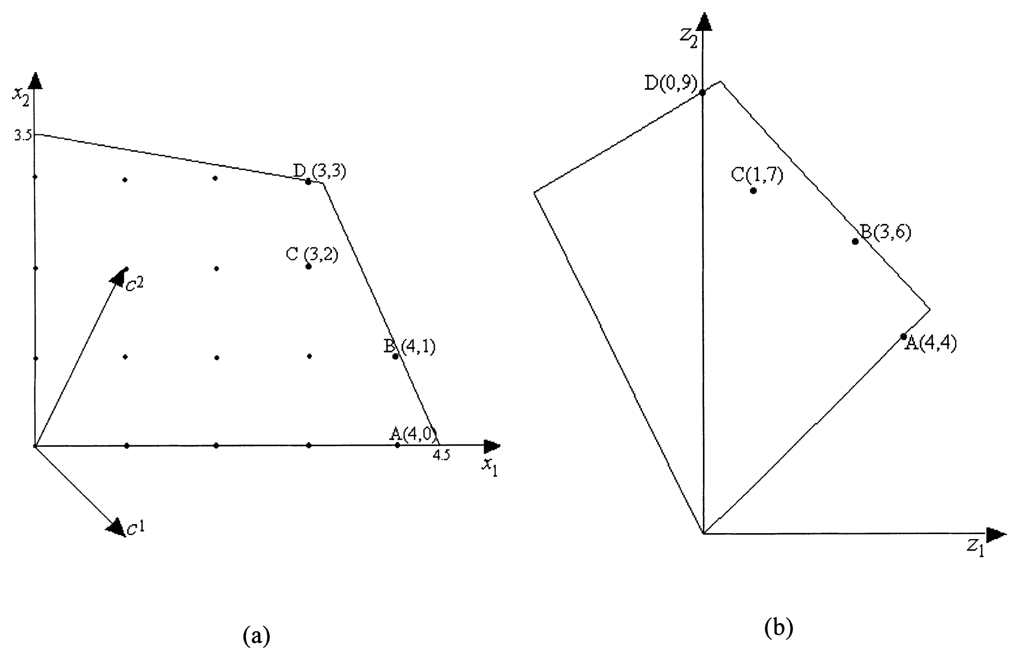


Fig. 2. Decision variable space and criterion space of the example.

best efficient solution for the first objective function, z_1.

5. Computational implementation

The multiobjective interactive method has been implemented for Windows 95 (in a PC – pentium, 166 MHz) using the DELPHI developer. This implementation contains a spreadsheet-based problem editor, graphical procedures to interact with the DM and computing routines. The latter includes routines for problem preprocessing, sensitivity analysis, cutting plane generation and a solver that rely on the dual simplex method for bounded-variable linear programs. Problem preprocessing attempts to reduce the problem, namely by (i) removing redundant constraints, (ii) replacing constraints by bounds on variables, (iii) fixing variables or tightening bounds, and (iv) reducing coefficients. Items (i), (ii) and (iii) follow Brearly et al. (1975) with some modifications in (iii) and adaptations to the multiobjective case. Item (iv) follows Johnson et al. (1985) and is exclusively designed for 0–1 problems.

We have tested about forty randomly generated problems of different types (0–1 knapsack, integer knapsack, 0–1 multidimensional knapsack, set covering and set packing). They have 10–100 variables, 1–40 constraints and 2 or 3 objective functions. For each problem we made several tests considering either reference points explicitly given or following directional searches. Since the computing routines rely solely on cutting planes, this approach is very sensitive to numerical errors: it failed in about 6% of the tests within our set of problems. Anyway, we have found out that this approach is very useful for local or directional searches because it allows the DM to skip reference points that lead to an already known result, thus enabling to save computational effort. The sensitivity analysis embedded in this process does not seem to cause additional numerical difficulties.

In order to better illustrate the computational application, let us consider a facility location model. It concerns the selection of the sites, among n regions, for waste processing facilities. Other regions must be situated within a pre-defined distance, say d, from its nearest facility. All the n regions that define the network are potential sites to locate a facility. Three objective functions are defined; while the first concerns the cost – to minimize the total cost of building the facilities – the two other concern the risk – to minimize the total population directly affected by the facilities and to minimize the total number of facilities that cover each region. This problem, whose constraints follow a set-covering formulation, may be stated as follows:

$$\begin{aligned} \min \quad & f_1 = \sum_{j=1}^{n} c_j x_j, \\ \min \quad & f_2 = \sum_{j=1}^{n} a_j x_j, \\ \min \quad & f_3 = \sum_{i=1}^{n} \sum_{j \in F_i} x_j, \\ \text{s.t.} \quad & \sum_{j \in F_i} x_j \geqslant 1, \quad i = 1, \ldots, n, \\ & x_j = 0 \text{ or } 1, \quad j = 1, \ldots, n, \end{aligned}$$

where c_j is the cost of building a facility in region j, a_j is the population in region j and F_i the set of regions that are within d from i (F_i always contains i). We consider herein a network with 40 regions. Although being consistent, the data is not real (it is available with the authors).

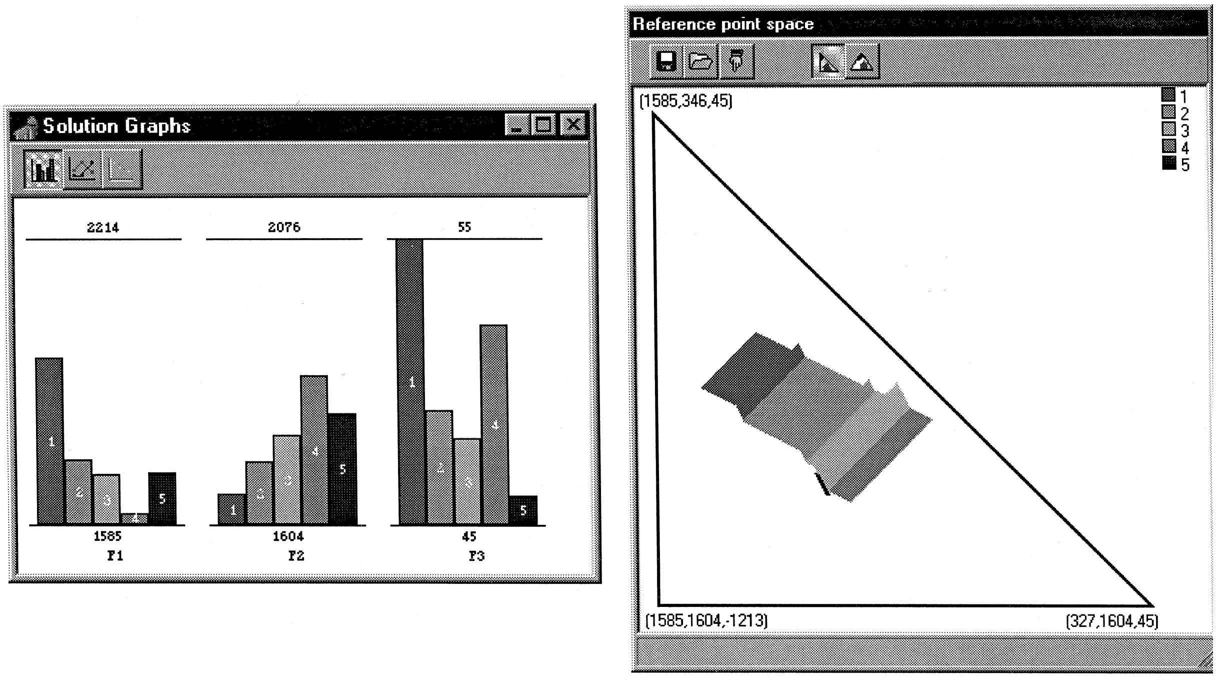
Knowing the ideal criterion point, $(1585 \times 10^3, 1604 \times 10^3, 45)$, the reference point $z^+ = (1650 \times 10^3, 1220 \times 10^3, 45)$ was chosen to start the search, leading to a first nondominated solution. Suppose that a direction of decreasing the cost (f_1) was first selected and then the direction of motion was changed to improve f_3. Table 1 shows the results obtained by this search. Notice that all the reference points other than the first one were automatically changed. As we mentioned before, "nearby" nondominated solutions are obtained throughout a directional search. The designation of "nearby" is in the sense of the L_∞ distance to the reference point. Apart from the biobjective case, this does not surely mean the "closest" solution for the criterion chosen to be improved, as we can see from the results in Table 1 (sol. 5 is closer to sol. 2 in f_1 than sol. 3). Fig. 3 shows windows with graphical displays for these solutions: on the left (a), bar graphs with the criterion values; on the right (b), a representation of the reference point space where regions of reference points that lead to the same nondominated solution, translated to a plane $\sum_{i=1}^{3} z_i^+ = K$ (K being constant), are iteratively appended. We shall omit explanations about the way these regions are computed because this is not within the main scope of the paper and also due to space reasons.

6. Concluding remarks and future research

We have proposed an interactive method designed for multiobjective pure integer linear problems. By taking advantage of *cutting plane* techniques and Tchebycheff theory, we have developed a sensitivity analysis tool intended to identify ranges for the reference points that lead to the same nondominated solution. For instance, if the DM wants to look for nondominated solutions "nearby" the current one, he/she does not need to point "in the dark" to reference points that would probably lead to the same solution. This tool is specially useful to perform a local search around the nondominated solutions most preferred by the

Table 1
Some results of the facility location problem

Strategy	Reference point $(z_1^+ \times 10^3, z_2^+ \times 10^3, z_3^+)$	Solution		time (s)
		$(f_1 \times 10^3, f_2 \times 10^3, f_3)$	Open facilities	
Start	(1650, 1220, 45)	sol. 1: (1950, 1655, 55)	3, 7, 14, 16, 24, 26, 31, 35, 37	6.3
Decrease f_1	⇒(1460, 1220, 45)	sol. 2: (1728, 1709, 49)	2, 7, 11, 24, 26, 31, 35, 37	14.7
Decrease f_1	⇒(1194, 1220, 45)	sol. 3: (1695, 1753, 48)	7, 12, 14, 24, 26, 31, 35, 37	9.8
Decrease f_1	⇒(1064, 1220, 45)	sol. 4: (1610, 1850, 52)	7, 11, 12, 24, 26, 31, 35, 37	30.9
Decrease f_3	⇒(1064, 1220, −580)	back to sol. 3		0.2
Decrease f_3	⇒(1064, 1220, −589)	sol. 5: (1700, 1788, 46)	7, 12, 20, 24, 26, 34, 37	0.9



(a) (b)

Fig. 3. Graphical displays for the facility location problem.

DM, a directional search to improve one objective function or a more strategic search in problems with scattered nondominated sets. Computational probes have highlighted how such a search tool may be of value as an element of an interactive MOILP method.

The current version of our method only uses some well known *cutting planes* that make it very limited in practice. However, further cutting plane techniques for pure integer programs could also be included. In order to make this multiobjective cutting plane approach more reliable in practice, we have been exploiting its combination with a multiobjective *Branch and Bound* approach. This seems to us a promising avenue. Besides, we have been improving the computational implementation by building an integrated MOILP decision support system. This system makes the most of graphical displays, namely the representation of the reference point space (like Fig. 3(b)).

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Appendix A

Proof of Proposition 1. This proposition is proved in a similar way as Theorem 3.1 in Steuer and Choo (1983). The difference between Theorem 3.1 and this proposition is that, while the former refers to a weighted Tchebycheff program where the reference point is always the ideal criterion point, the latter refers to a non-weighted Tchebycheff program considering any reference point. □

Proof of Proposition 2. This proposition can be proved by following steps similar to the proof of Theorem 3.7 in Steuer and Choo (1983). We recall that Theorem 3.7 refers to an augmented weighted Tchebycheff program where the reference point is fixed to the ideal criterion point, z^*, and the controlling parameters are the weights λ in $(\lambda_i(z_i^* - c^i x))$.

$\Leftarrow$ follows directly from Theorem 3.7 in Steuer and Choo (1983) considering $\lambda_i = 1$, $i = 1,\ldots,k$, and $z^+ = z^*$.

$\Rightarrow$ considering $z_i^+ = \tilde{z}_i + \theta$, $i = 1,\ldots,k$, where $\theta = \max_{i=1,\ldots,k}(z_i^* - \tilde{z}_i)$, this implication is easily proved by simple adaptations of Lemmas 3.2 and 3.3 and Theorem 3.4 of Steuer and Choo (1983). □

Proof of Proposition 3. Let α^a be the α optimal value for (P_1, z^{a+}) and z^a the image of x^a on the objective function space.

Case I: If $z_p^{a+} - z_p^a = \alpha^a$ then $(x^a, \alpha^a + \theta_p)$ is a feasible solution for (P_1, z^{b+}). If it is not an optimal solution (this means that (i) does not hold) then there exists another solution (x^b, α^b) that optimizes (P_1, z^{b+}). Let z^b be the image of x^b on the objective function space. Since $\alpha^b < \alpha^a + \theta_p$ and $z_p^{b+} - z_p^b \leqslant \alpha^b \iff z_p^{a+} + \theta_p - z_p^b \leqslant \alpha^b$, then $z_p^{a+} - z_p^b < \alpha^a$. By the assumption of *case I*, $z_p^{a+} - z_p^a = \alpha^a$, thus $z_p^b > z_p^\alpha$.

Case II: $z_p^{a+} - z_p^a < \alpha^a$. Consider two situations:

II.1: If $\theta_p \leqslant \alpha^a - z_p^{a+} + z_p^a$ then

$$z_p^{b+} - z_p^a \leqslant z_p^{a+} + (\alpha^a - z_p^{a+} + z_p^a) - z_p^a = \alpha^a$$

$$\iff \quad z_p^{b+} - z_p^a \leqslant \alpha^a. \tag{A.1}$$

Also,

$$z_i^{b+} - z_i^a \leqslant \alpha^a \quad \text{for all } i \neq p$$

$$\text{because } z_i^{b+} = z_i^{a+}, i \neq p. \tag{A.2}$$

By A.1 and A.2, (x^a, α^a) is a feasible solution for (P_1, z^{b+}). On the other hand, any feasible solution ($\tilde{x}$, $\tilde{\alpha}$) for (P_1, z^{b+}) is also feasible to (P_1, z^{a+}). Since (x^a, α^a) is optimal for (P_1, z^{a+}), then $\alpha^a \leqslant \tilde{\alpha}$. Hence ($x^a$, α^a) is also optimal for (P_1, z^{b+}).

II.2: If $\theta_p > \alpha^a - z_p^{a+} + z_p^a$ then

$$\theta_p = (\alpha^a - z_p^{a+} + z_p^a) + \delta_p \text{ with } \delta_p > 0.$$

Define

$$z^{c+} = (z_1^{a+}, \ldots, z_p^{a+} + (\alpha^a - z_p^{a+} + z_p^a), \ldots, z_k^{a+}).$$

According to II.1, (x^a, α^a) is an optimal solution for (P_1, z^{c+}) and $z_p^{c+} - z_p^a = \alpha^a$ holds. The result produced by

$$z^{b+} = (z_1^{c+}, \ldots, z_p^{c+} + \delta_p, \ldots, z_k^{c+})$$

in relation to z^{c+} follows *case I*. □

Proof of Proposition 4. Suppose that there exists x^c efficient that minimizes (P_1, z^{c+}) and does not minimize (P_1, z^{a+}) neither (P_1, z^{b+}). Let z^c be the image of x^c on the objective function space. Let α^a, α^b and α^c the optimal values of α for (P_1, z^{a+}), (P_1, z^{b+}) and (P_1, z^{c+}), respectively.

Case I: If $z_p^{c+} - z_p^c = \alpha^c$ then α^c is non-integer; Let $\lfloor \alpha^c \rfloor$ be the integer part and $\langle \alpha^c \rangle$ the fractional part of α^c, i.e. $\alpha^c = \lfloor \alpha^c \rfloor + \langle \alpha^c \rangle$.

For all $i \neq p$, since z_i^{c+} is integer,

$$z_i^{c+} - z_i^c \leqslant \lfloor \alpha^c \rfloor. \tag{A.3}$$

$$\text{For } p,\ z_p^c = z_p^{c+} - \alpha^c = \underbrace{z_p^{a+} - \lfloor \alpha^c \rfloor}_{\text{integer}} + \underbrace{\theta_p - \langle \alpha^c \rangle}_{\text{integer}},$$

$\Rightarrow$

$0 < \theta_p < 1$ and $0 < \langle \alpha^c \rangle < 1$ implying that

$\theta_p - \langle \alpha^c \rangle = 0$

$$\text{and} \quad z_p^c = z_p^{a+} - \lfloor \alpha^c \rfloor. \tag{A.4}$$

By A.3 and A.4, (x^c, $\lfloor \alpha^c \rfloor$) is feasible for (P_1, z^{a+}). Let (x^a, α^a) be an optimal solution for (P_1, z^{a+}). Then (x^a, $\alpha^a + \theta_p$) is feasible for (P_1, z^{c+}). $\alpha^a < \lfloor \alpha^c \rfloor$ because x^c does not minimize (P_1, z^{a+}). So, $\alpha^a + \theta_p < \lfloor \alpha^c \rfloor + \theta_p = \lfloor \alpha^c \rfloor + \langle \alpha^c \rangle = \alpha^c$ which contradicts the hypothesis that (x^c, α^c) minimizes (P_1, z^{c+}).

Case II: If $z_p^{c+} - z_p^c < \alpha^c$ then there exists $I_j \subseteq \{1, \ldots, k\} \backslash \{p\}$ such that $z_j^{c+} - z_j^c = \alpha^c, j \in I_j$ and α^c is integer.

$$z_j^{c+} - z_j^c = \alpha^c \iff z_j^{b+} - z_j^c = \alpha^c,\ j \in I_j, \tag{A.5}$$

$$\text{for all } i \notin I_j \cup \{p\},\ z_i^{b+} - z_i^c < \alpha^c, \tag{A.6}$$

and for p,

$$z_p^{c+} - z_p^c < \alpha^c \Longleftrightarrow z_p^{b+} - z_p^c < \alpha^c - \theta_p + 1.$$

The left-hand side is integer, which implies that

$$z_p^{b+} - z_p^c \leqslant \alpha^c. \tag{A.7}$$

By A.5, A.6 and A.7, (x^c, α^c) is feasible for (P_1, z^{b+}). Let (x^b, α^b) be an optimal solution for (P_1, z^{b+}). (x^b, α^b) is feasible for (P_1, z^{c+}) and $\alpha^b < \alpha^c$ because (x^c, α^c) does not minimize (P_1, z^{b+}) which contradicts the hypothesis that (x^c, α^c) minimizes (P_1, z^{c+}). □

Proof of Proposition 5. Every integer feasible solution ($\tilde{x}$, $\tilde{\alpha}$) for (P_1, z^{b+}) satisfies $A.\ \tilde{x} = b$ and $c^i \cdot \tilde{x} + \tilde{\alpha} \geqslant z_i^{b+}, i = 1, \ldots, k$. As $z_i^{b+} \geqslant z_i^{a+}$ for all i, ($\tilde{x}$, $\tilde{\alpha}$) is also feasible to (P_1, z^{a+}). Hence, the (P_1, z^{b+}) feasible set is a subset of the (P_1, z^{a+}) feasible set and the proposition holds. □

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