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TOPOLOGY OF 3-COSYMPLECTIC MANIFOLDS

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Abstract. We continue the program of Chinea, De León and Marrero who studied the topology of cosymplectic manifolds. We study 3-cosymplectic manifolds which are the closest odd-dimensional analogue of hyper-Kähler structures. We show that there is an action of the Lie algebra $so(4,1)$ on the basic cohomology spaces of a compact 3-cosymplectic manifold with respect to the Reeb foliation. This implies some topological obstructions to the existence of such structures which are expressed by bounds on the Betti numbers. It is known that every 3-cosymplectic manifold is a local Riemannian product of a hyper-Kähler factor and an abelian three dimensional Lie group. Nevertheless, we present a nontrivial example of compact 3-cosymplectic manifold which is not the global product of a hyper-Kähler manifold and a flat 3-torus.

1. Introduction

Cosymplectic geometry is considered to be the closest odd-dimensional analogue of Kähler geometry (see e.g. [2, Section 6.5], [9, Section 14.5]). This becomes even more evident when one passes to the setting of 3-structures. Indeed, while both cosymplectic and Sasakian manifolds admit a transversal Kähler structure, only 3-cosymplectic manifolds admit a transversal hyper-Kähler structure (cf. [6]).

In the fundamental paper [8], Chinea, De León and Marrero studied the topology of cosymplectic manifolds, refining the previous results of Blair and Goldberg ([3]). They proved a monotonicity result for the Betti numbers of a compact cosymplectic manifold M^{2n+1} up to the middle dimension. Next, the differences $b_{2p+1} - b_{2p}$ (with $0 \leq p \leq n$) were shown to be even integers (in particular, b_1 is odd). Moreover, they found an example of a compact cosymplectic manifold which is not the global product of a Kähler manifold and the circle. Later on, other nontrivial examples were provided (cf. [19, 11]). More recently, Li ([18]) gave an alternative proof of the monotonicity property of the Betti numbers of cosymplectic manifolds (which he prefers to call co-Kähler) by using topological techniques.

A 3-cosymplectic manifold (see e.g. [5, Section 13.1]) is a smooth manifold M of dimension $4n+3$ endowed with an almost contact metric 3-structure such that each structure is cosymplectic. This class of Riemannian manifolds is contained in the wider class of 3-quasi Sasakian manifolds. Every 3-cosymplectic manifold is in particular cosymplectic hence all the previously mentioned results still hold. A natural problem is whether the quaternionic-like conditions which relate the structure tensors of 3-cosymplectic manifolds can induce additional

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rigidity to the underlying topological structure. The aim of this paper is to give an answer to this question.

Every 3-cosymplectic manifold M admits the canonical Reeb foliation $\mathcal{F}_3$ of dimension three. We denote by $H^*_B(M)$ the basic cohomology with respect to this foliation. The first result we prove in Section 3 can be restated in the form

$$H^*_{dR}(M) \cong H^*_B(M) \otimes H^*_{dR}\left(\mathbb{T}^3\right) \tag{1.1}$$

for any compact 3-cosymplectic manifold M. This shows that the Betti numbers of M are completely determined by the basic Betti numbers $b^h_p := \dim H^p_B(M)$, namely

$$b_p = b^h_p + 3b^h_{p-1} + 3b^h_{p-2} + b^h_{p-3}. \tag{1.2}$$

When $\mathcal{F}_3$ is a regular foliation we can identify $H^*_B(M)$ with $H^*_{dR}(M/\mathcal{F}_3)$. In this case $M/\mathcal{F}_3$ is a hyper-Kähler manifold. There are known several results which give restrictions on possible values of Betti numbers of compact hyper-Kähler manifolds. This suggests to look for the similar results about $H^*_B(M)$. The results on Betti numbers of compact hyper-Kähler manifolds can be divided into two families. In one family there are results that can be obtained from the existence of the $so(4,1)$ action on the cohomology ring of a hyper-Kähler manifold discovered by Verbitsky in [28]. In the other family there are the equations derived from the Riemann-Roch theorem by Salamon in [24]. There is no hope at the moment to get an extension of the Salamon's result for $H^*_B(M)$ for a case when $\mathcal{F}_3$ is non-regular, as the theory of transversally hyper-Kähler foliations is not developed enough.

In Section 4 we show the existence of an $so(4,1)$ action on $H^*_B(M)$. From representation theory of $so(4,1)$ it follows that the basic Betti numbers b^h_{2p+1} are divisible by four, and that

$$b^h_{2p} \geq \binom{p+2}{2},\ 0 \leq p \leq n.$$

We show these results in Section 5 and Section 6 by more elementary arguments to make the article accessible to a wider audience. As consequences, we will obtain that for a compact 3-cosymplectic manifold $b_{2p} + b_{2p+1}$ are multiples of four and that

$$b_p \geq \binom{p+2}{2} \qquad \text{for } 0 \leq p \leq 2n+1.$$

From the above considerations one can see that there are strong obstructions to the existence of compact 3-cosymplectic manifolds. On the other hand, every 3-cosymplectic manifold is a local Riemannian product of a hyper-Kähler factor and an abelian three dimensional Lie group. Moreover, the formula (1.1) could suggest that every compact 3-cosymplectic manifold is the total space of a toric bundle over a hyper-Kähler manifold. We disprove this by an example in Section 7. Namely, we construct a compact 3-cosymplectic seven dimensional manifold M^7 such that $H^*_B\left(M^7\right)$ cannot coincide with the cohomology ring of a hyper-Kähler manifold. Note that in particular M^7 is not a global product of a hyper-Kähler manifold and $\mathbb{T}^3$, which answers the open question about the existence of non-trivial examples of such manifolds.

2. Preliminaries

An *almost contact manifold* is an odd-dimensional manifold M which carries a field ϕ of endomorphisms of the tangent spaces, a vector field ξ, called *characteristic* or *Reeb vector field*, and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1,$$

where $I \colon TM \to TM$ is the identity mapping. From the definition it follows that $\phi\xi = 0$, $\eta \circ \phi = 0$ and that the $(1,1)$-tensor field ϕ has constant rank $2n$ (cf. [2]). An almost contact manifold (M, ϕ, ξ, η) is said to be *normal* when the tensor field $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$ vanishes identically, where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ. It is known (see e.g. [2, page 44]) that any almost contact manifold (M, ϕ, ξ, η) admits a Riemannian metric g such that

$$g(\phi E, \phi F) = g(E, F) - \eta(E)\eta(F) \tag{2.1}$$

holds for all $E, F \in \Gamma(TM)$. This metric g is called a *compatible metric* and the manifold M together with the structure (ϕ, ξ, η, g) is called an *almost contact metric manifold*. As an immediate consequence of (2.1), one has $\eta = g(\cdot, \xi)$ and $g(\phi E, F) = -g(E, \phi F)$. Hence $\Phi(E, F) = g(E, \phi F)$ defines a 2-form, which is called the *fundamental* 2-*form* of M. Almost contact metric manifolds such that both η and Φ are closed are called *almost cosymplectic manifolds* and those for which $d\eta = \Phi$ are called *contact metric manifolds*. Finally, a normal almost cosymplectic manifold is called a *cosymplectic manifold*, and a normal contact metric manifold is said to be a *Sasakian manifold*. In terms of the covariant derivative of ϕ, the cosymplectic and the Sasakian conditions can be expressed respectively by

$$\nabla\phi = 0$$

and

$$(\nabla_E \phi) F = g(E, F)\xi - \eta(F) E,$$

for all $E, F \in \Gamma(TM)$.

It should be noted that both in Sasakian and in cosymplectic manifolds ξ is a Killing vector field. The Sasakian and the cosymplectic manifolds represent the two extremal cases of the larger class of quasi-Sasakian manifolds (cf. [1]).

An *almost contact* 3-*structure* on a $(4n+3)$-dimensional smooth manifold M is given by three almost contact structures (ϕ_1, ξ_1, η_1), (ϕ_2, ξ_2, η_2), (ϕ_3, ξ_3, η_3) satisfying the following relations, for every $\alpha, \beta \in \{1, 2, 3\}$,

$$\phi_\alpha \phi_\beta - \eta_\beta \otimes \xi_\alpha = \sum_{\gamma=1}^{3} \epsilon_{\alpha\beta\gamma} \phi_\gamma - \delta_{\alpha\beta} I, \tag{2.2}$$

$$\phi_\alpha \xi_\beta = \sum_{\gamma=1}^{3} \epsilon_{\alpha\beta\gamma} \xi_\gamma, \quad \eta_\alpha \circ \phi_\beta = \sum_{\gamma=1}^{3} \epsilon_{\alpha\beta\gamma} \eta_\gamma, \tag{2.3}$$

where $\epsilon_{\alpha\beta\gamma}$ is the totally antisymmetric symbol. This notion was introduced by Kuo ([17]) and, independently, by Udriste ([27]). In [17] Kuo proved that given an almost contact 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$, $\alpha \in \{1, 2, 3\}$, there exists a Riemannian metric g compatible with each of the structures and hence we can speak of *almost contact metric* 3-*structure*. It is well known that in any almost 3-contact manifold the Reeb vector fields ξ_1, ξ_2, ξ_3 are orthonormal with respect to any compatible metric g and that the structural group of the tangent bundle is reducible to $Sp(n) \times \{I_3\}$. Moreover, the tangent bundle of any almost 3-contact metric manifold splits up as the orthogonal sum $TM = \mathcal{H} \oplus \mathcal{V}$, where the $4n$-dimensional subbundle $\mathcal{H} = \bigcap_{\alpha=1}^{3} \ker(\eta_\alpha)$ is called the *horizontal distribution* and $\mathcal{V} = \langle \xi_1, \xi_2, \xi_3 \rangle$ is called the *vertical* (or *Reeb*) *distribution*. An almost 3-contact manifold M is said to be *normal* if each almost contact structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$ is normal.

Let $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be an almost contact metric 3-structure. When each structure is Sasakian M is called a 3-*Sasakian manifold*.

By an *almost 3-cosymplectic manifold* we mean an almost 3-contact metric manifold M such that each almost contact metric structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is almost cosymplectic. The almost cosymplectic 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is called *cosymplectic* if it is normal. In this case M is said to be a 3-*cosymplectic manifold*. However it has been proved recently in [10, Theorem 4.13] that these two notions are the same, i.e. every almost 3-cosymplectic manifold is 3-cosymplectic.

Just as in the case of a single structure, the 3-Sasakian and the 3-cosymplectic manifolds represents the two extremal cases of the larger class of 3-quasi-Sasakian manifolds (cf. [7]).

In any 3-cosymplectic manifold the forms η_α and Φ_α are harmonic ([12, Lemma 3]). Moreover, we have that ξ_α, η_α, ϕ_α and Φ_α are ∇-parallel. In particular

$$[\xi_\alpha, \xi_\beta] = \nabla_{\xi_\alpha}\xi_\beta - \nabla_{\xi_\beta}\xi_\alpha = 0 \tag{2.4}$$

for all $\alpha, \beta \in \{1,2,3\}$, so that $\mathcal{V}$ defines a 3-dimensional foliation $\mathcal{F}_3$ of M^{4n+3}. Since each Reeb vector field is Killing and is parallel, such a foliation turns out to be Riemannian with totally geodesic leaves.

Recall that a foliation $\mathcal{F}$ is *regular* (in the sense of Palais [23]) if each point $p \in M$ has a foliated coordinate chart (U, p) such that each leaf of $\mathcal{F}$ passes through U at most once.

Theorem 2.1. ([6, Corollary 3.10]) *Let $\left(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g\right)$ be a 3-cosymplectic manifold. If the foliation $\mathcal{F}_3$ is regular, then the space of leaves $M^{4n+3}/\mathcal{F}_3$ is a hyper-Kähler manifold of dimension $4n$. Consequently, every 3-cosymplectic manifold is Ricci-flat.*

Remark 2.2. If we drop the assumption of regularity in Theorem 2.1 and we assume instead that the vertical foliation has compact leaves, then the space of leaves is a hyper-Kähler orbifold, i.e. a second countable Hausdorff space locally modeled on finite quotients of $\mathbb{R}^m$. We refer to [22] for the formal definition and properties of orbifolds and to [25] for the generalization of geometric objects to the orbifold category.

Concerning the horizontal subbundle, note that — unlike the case of 3-Sasakian geometry — in any 3-cosymplectic manifold $\mathcal{H}$ is integrable. Indeed, for all $X, Y \in \Gamma(\mathcal{H})$, $\eta_\alpha([X,Y]) = -2d\eta_\alpha(X,Y) = 0$ since $d\eta_\alpha = 0$.

3. Decomposition of the cohomology of 3-cosymplectic manifolds

Unless otherwise stated, in the remaining of the paper we will assume that all manifolds are compact. In this section we investigate some algebraic properties of the de Rham cohomology $H^*_{dR}(M)$ of a 3-cosymplectic manifold M^{4n+3}. By the Hodge-de Rham theory the vector space $H^k_{dR}(M)$ can be identified with the vector space $\Omega^k_H(M)$ of harmonic k-forms on M.

For each $\alpha \in \{1,2,3\}$ we define linear operators λ_α and l_α by

$$l_\alpha \colon \Omega^k(M) \to \Omega^{k+1}(M) \qquad\qquad \lambda_\alpha \colon \Omega^{k+1}(M) \to \Omega^k(M)$$
$$\omega \mapsto \eta_\alpha \wedge \omega \qquad\qquad \omega \mapsto i_{\xi_\alpha}\omega.$$

We denote by $\{A, B\}$ the anticommutator $AB + BA$ of two linear operators A and B. From $\eta_\alpha(\xi_\beta) = \delta_{\alpha\beta}$ it follows that

$$\{\lambda_\alpha, l_\beta\} = \delta_{\alpha\beta}. \tag{3.1}$$

Moreover

$$\{\lambda_\alpha, \lambda_\beta\} = \{l_\alpha, l_\beta\} = 0. \tag{3.2}$$

Define $e_\alpha = l_\alpha \lambda_\alpha$. Then it follows from (3.1) that e_α are idempotents. In fact

$$e_\alpha e_\alpha = l_\alpha \lambda_\alpha l_\alpha \lambda_\alpha = -l_\alpha l_\alpha \lambda_\alpha \lambda_\alpha + l_\alpha \lambda_\alpha = e_\alpha .$$

Moreover from (3.1) and (3.2) it follows that $[e_\alpha, e_\beta] = 0$, for $\alpha \neq \beta$. Thus $\{e_1, e_2, e_3\}$ are pairwise commuting idempotents.

By [8, Proposition 1] all operators l_α, λ_α, and thus e_α, preserve harmonic forms. Now we fix $k \in \{0, \ldots, 4n+3\}$ and consider the restrictions of the operators e_α to $\Omega^k_H(M)$, $\alpha \in \{1,2,3\}$. Note that $\Omega^k_H(M)$ is a finite dimensional vector space over $\mathbb{R}$. As e_α is idempotent, its minimal polynomial $m_\alpha(x)$ is a divisor of $x(x-1)$. Therefore the only possible eigenvalues of e_α are 0 and 1. Moreover, since $m_\alpha(x)$ does not have multiple roots, the operator e_α is diagonalizable with 0 and 1 on the diagonal. As the operators $\{e_1, e_2, e_3\}$ commute with each other, by Bourbaki [4, Proposition VII.13] they can be simultaneously diagonalized. Define for all triples ε_1, ε_2, $\varepsilon_3 \in \{0,1\}$

$$\Omega^k_{H,\varepsilon_1\varepsilon_2\varepsilon_3}(M) = \left\{ \omega \in \Omega^k_H(M) \,\middle|\, e_\alpha \omega = \varepsilon_\alpha \omega,\ \alpha = 1,2,3 \right\}.$$

Since e_1, e_2, e_3 can be simultaneously diagonalized on $\Omega^k_H(M)$ we get that

$$\Omega^k_H(M) = \bigoplus_{\varepsilon_1,\varepsilon_2,\varepsilon_3 \in \{0,1\}} \Omega^k_{H,\varepsilon_1\varepsilon_2\varepsilon_3}(M). \tag{3.3}$$

Now let $\omega \in \Omega^k_{H,0\varepsilon_2\varepsilon_3}(M)$. Then $l_1\omega \in \Omega^{k+1}_{H,1\varepsilon_2\varepsilon_3}$. In fact

$$e_1 l_1 \omega = l_1 \lambda_1 l_1 \omega = -\lambda_1 l_1 l_1 \omega + l_1 \omega = l_1 \omega$$
$$e_\alpha l_1 \omega = l_1 e_\alpha \omega = \varepsilon_\alpha l_1 \omega, \qquad \alpha = 2,3.$$

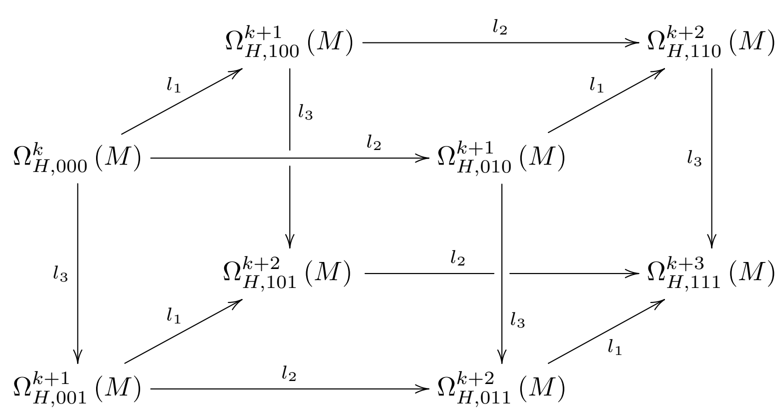
Similarly if $\omega \in \Omega^{k+1}_{H,1\varepsilon_2\varepsilon_3}(M)$, then $\lambda_1\omega \in \Omega^k_{H,0\varepsilon_2\varepsilon_3}(M)$. Therefore, we get maps of vector spaces

$$l_1^{\varepsilon_2\varepsilon_3} \colon \Omega^k_{H,0\varepsilon_2\varepsilon_3}(M) \to \Omega^{k+1}_{H,1\varepsilon_2\varepsilon_3}(M), \qquad \lambda_1^{\varepsilon_2\varepsilon_3} \colon \Omega^{k+1}_{H,1\varepsilon_2\varepsilon_3}(M) \to \Omega^k_{H,0\varepsilon_2\varepsilon_3}(M).$$

Now $l_1^{\varepsilon_2\varepsilon_3}\lambda_1^{\varepsilon_2\varepsilon_3}$ is the restriction of e_1 to $\Omega^{k+1}_{H,1\varepsilon_2\varepsilon_3}(M)$ and thus $l_1^{\varepsilon_2\varepsilon_3}\lambda_1^{\varepsilon_2\varepsilon_3} = \mathrm{id}$. Analogously the composition $\lambda_1^{\varepsilon_2\varepsilon_3} l_1^{\varepsilon_2\varepsilon_3}$ is the restriction of

$$\lambda_1 l_1 = \mathrm{id} - l_1\lambda_1 = \mathrm{id} - e_1$$

to $\Omega^k_{H,0\varepsilon_2\varepsilon_3}(M)$ and thus $\lambda_1^{\varepsilon_2\varepsilon_3} l_1^{\varepsilon_2\varepsilon_3} = \mathrm{id}$. Thus $\lambda_1^{\varepsilon_2\varepsilon_3}$ and $l_1^{\varepsilon_2\varepsilon_3}$ are inverse isomorphisms between the vector spaces $\Omega^k_{H,0\varepsilon_2\varepsilon_3}(M)$ and $\Omega^{k+1}_{H,1\varepsilon_2\varepsilon_3}(M)$. Replacing 1 with 2, 3, and putting all together we get for every $0 \le k \le 4n$ the cube



whose faces are anti-commutative and edge arrows are isomorphisms of vector spaces. Therefore the whole information about the cohomology groups of M is contained in the vector spaces $\Omega^k_{H,000}(M)$, $0 \le k \le 4n$.

Denote by b^h_k the dimension of $\Omega^k_{H,000}(M)$. Then

$$\begin{aligned}
&\dim \Omega^k_{H,100} = \dim \Omega^k_{H,010} = \dim \Omega^k_{H,001} = \dim \Omega^{k-1}_{H,000} = b^h_{k-1} && k \ge 1\\
&\dim \Omega^k_{H,110} = \dim \Omega^k_{H,101} = \dim \Omega^k_{H,011} = \dim \Omega^{k-2}_{H,000} = b^h_{k-2} && k \ge 2\\
&\dim \Omega^k_{H,111} = \dim \Omega^{k-3}_{H,000} = b^h_{k-3} && k \ge 3.
\end{aligned}$$

Therefore, from the decomposition (3.3) we get

$$\begin{aligned}
b_0 &= b^h_0\\
b_1 &= b^h_1 + 3b^h_0\\
b_2 &= b^h_2 + 3b^h_1 + 3b^h_0\\
b_k &= b^h_k + 3b^h_{k-1} + 3b^h_{k-2} + b^h_{k-3} \qquad 3 \le k \le 4n+3.
\end{aligned} \tag{3.4}$$

Now we will identify the vector spaces $\Omega^k_{H,000}(M)$ with the basic cohomology of the Reeb foliation generated by ξ_α, $\alpha \in \{1,2,3\}$, on M. In our case the spaces of basic forms are given by

$$\Omega^k_B(M) := \left\{ \omega \in \Omega^k(M) \,\middle|\, i_{\xi_\alpha}\omega = 0,\ i_{\xi_\alpha} d\omega = 0 \text{ for each } \alpha = 1,2,3 \right\}.$$

The basic differential d_B is the restriction of the exterior derivative d to $\Omega^*_B(M)$. The basic cohomology spaces are defined as cohomology spaces of the complex $(\Omega^*_B(M), d_B)$. In our case the mean curvature of the Reeb foliation $\mathcal{F}_3$ is zero since the foliation is totally geodesic, therefore we can use the transversal de Rham-Hodge theory developed in [15]. By this theory, the basic cohomology spaces can be identified with the kernel of the basic Laplacian

$$\triangle_B := d_B\delta_B + \delta_B d_B,$$

where δ_B is the codifferential δ followed by the orthogonal projection of $\Omega^*(M)$ onto $\Omega^*_B(M)$. We denote by $\Omega^*_{BH}(M)$ the kernel of $\triangle_B$.

Proposition 3.1. *Let M^{4n+3} be a compact 3-cosymplectic manifold. Then $\Omega^*_{BH}(M) = \Omega^*_{H,000}(M)$. In particular, the numbers b^h_k coincide with the basic Betti numbers of the Reeb foliation on M.*

Proof. First we show that $\Omega^*_{H,000}(M) \subset \Omega^*_B(M)$. Let $\omega \in \Omega^k_{H,000}(M)$. Then

$$i_{\xi_\alpha}\omega = \lambda_\alpha\omega = \left(\lambda_\alpha - l_\alpha\lambda_\alpha^2\right)\omega = \lambda_\alpha l_\alpha \lambda_\alpha \omega = \lambda_\alpha e_\alpha \omega = 0, \quad \alpha \in \{1,2,3\}.$$

Moreover, $d\omega = 0$ therefore $\omega \in \Omega^k_B(M)$. Thus we have to show that $\Omega^*_{H,000}(M)$ is the kernel of $\triangle_B$. We know that $\Omega^*_{H,000}(M)$ is the kernel of $\triangle$. Thus it is enough to show that $\triangle_B = \triangle$ on $\Omega^*_B(M)$. From the definitions of $\triangle$ and $\triangle_B$ we see that it is enough to check that $\delta = \delta_B$ on $\Omega^*_B(M)$. Recall, that δ_B is the restriction of δ to $\Omega^*_B(M)$ followed by the orthogonal projection from $\Omega^*(M)$ to $\Omega^*_B(M)$. Therefore, the map δ_B coincides with the restriction of δ to $\Omega^*_B(M)$ if and only if $\delta(\Omega^*_B(M)) \subset \Omega^*_B(M)$.

Let $\omega \in \Omega^k_B(M)$. The operators l_α and d anticommute in our case, since l_α is the wedge product with a closed 1-form. As shown in [13, pages 97-98], on a Riemannian manifold the usual operator of interior product i_X, where X is a vector field, can be defined as the Hodge dual of the operator $g(X,-)\wedge -$. Thus $\lambda_\alpha = i_{\xi_\alpha}$ is the Hodge dual of $l_\alpha = \eta_\alpha \wedge -$. Since δ is the Hodge dual of d we get that δ and λ_α anticommute, which implies

$$i_{\xi_\alpha}\delta\omega = -\delta i_{\xi_\alpha}\omega = 0. \tag{3.5}$$

Now we use the fact that the anticommutator of i_{ξ_α} and d is the Lie derivative $\mathcal{L}_{\xi_\alpha}$. In the last paragraph of page 109 of [13], it is shown that for a Killing vector field X

$$\mathcal{L}_X + \{\delta, g(X,-)\wedge -\} = 0.$$

Since ξ_α is a Killing vector field, we get

$$\mathcal{L}_{\xi_\alpha} + \{\delta, l_\alpha\} = 0.$$

Therefore, δ and $\mathcal{L}_{\xi_\alpha}$ commute

$$[\delta, \mathcal{L}_{\xi_\alpha}] = -[\delta, \{\delta, l_\alpha\}] = -\delta^2 l_\alpha - \delta l_\alpha \delta + \delta l_\alpha \delta + l_\alpha \delta^2 = 0$$

and by (3.5) we get

$$i_{\xi_\alpha} d\delta\omega = \mathcal{L}_{\xi_\alpha}\delta\omega - d i_{\xi_\alpha}\delta\omega = \delta\mathcal{L}_{\xi_\alpha}\omega + 0 = \delta\left(d i_{\xi_\alpha} + i_{\xi_\alpha} d\right)\omega = 0.$$

In the last step we use that ω is basic. Thus if $\omega \in \Omega^*_B(M)$ then $\delta\omega \in \Omega^*_B(M)$. This concludes the proof.

□

4. Action of $so(4,1)$ on the cohomology of 3-cosymplectic manifolds

In this section we will show that $\Omega^k_{H,000}(M)$ admits an action of the Lie algebra $so(4,1)$. This result is the odd-dimensional analogue of the one obtained by Verbitsky in [28] about the action of $so(4,1)$ on the cohomology groups of a hyper-Kähler manifold M^{4n}. In fact, intuitively the space $\bigoplus_{k=0}^{4n}\Omega^k_{H,000}(M)$ can be thought of as a cohomology ring of the hyper-Kähler orbifold obtained from M^{4n+3} by taking the quotient under the action of the three Reeb vector fields.

For every cyclic permutation (α,β,γ) of $(1,2,3)$ we denote by Ξ_α the 2-form

$$\Xi_\alpha := \frac{1}{2}\left(\Phi_\alpha + 2\eta_\beta\wedge\eta_\gamma\right). \tag{4.1}$$

Define the operators $L_\alpha\colon \Omega^k(M)\to\Omega^{k+2}(M)$ and $\Lambda_\alpha\colon \Omega^{k+2}(M)\to\Omega^k(M)$ by $L_\alpha\omega = \Xi_\alpha\wedge\omega$ and $\Lambda_\alpha := *L_\alpha *$.

We will give now a local description of these operators. Let

$$\{X_1, \phi_1X_1, \phi_2X_1, \phi_3X_1, \dots, X_n, \phi_1X_n, \phi_2X_n, \phi_3X_n, \xi_1, \xi_2, \xi_3\}$$

be an orthonormal basis of vector fields in some open subset U of M. Denote by ζ_s the 1-form dual to X_s, that is $\zeta_s = g(X_s,-)$. Then

$$i_{\phi_\alpha X_s}\left(\phi^*_\alpha\zeta_t\right) = g\left(X_s, \phi_\alpha(\phi_\alpha X_t)\right) = g\left(X_s, \phi^2_\alpha X_t\right) = -\delta_{st}, \qquad 1\le s,t\le n. \tag{4.2}$$

Therefore the set

$$\{\zeta_1, \phi^*_1\zeta_1, \phi^*_2\zeta_1, \phi^*_3\zeta_1, \dots, \zeta_n, \phi^*_1\zeta_n, \phi^*_2\zeta_n, \phi^*_3\zeta_n, \eta_1, \eta_2, \eta_3\} \tag{4.3}$$

is a basis of 1-forms on U.

Proposition 4.1. *Let (α, β, γ) be a cyclic permutation of $(1,2,3)$. Then*

$$\Phi_\alpha = 2\sum_{s=1}^{n}\left(\zeta_s \wedge \phi_\alpha^* \zeta_s - \phi_\beta^* \zeta_s \wedge \phi_\gamma^* \zeta_s\right) - 2\eta_\beta \wedge \eta_\gamma \tag{4.4}$$

and therefore

$$\Xi_\alpha = \sum_{s=1}^{n}\left(\zeta_s \wedge \phi_\alpha^* \zeta_s - \phi_\beta^* \zeta_s \wedge \phi_\gamma^* \zeta_s\right). \tag{4.5}$$

Proof. Let us denote by $\langle\ ,\ \rangle$ the natural pairing between k-forms and k-vector fields. By definition of Φ_α we have

$$\begin{aligned}\langle \Phi_\alpha, X_s \wedge \phi_\alpha X_s\rangle &= g\left(X_s, \phi_\alpha^2 X_s\right) = -1\\ \langle \Phi_\alpha, \phi_\beta X_s \wedge \phi_\gamma X_s\rangle &= g\left(\phi_\beta X_s, \phi_\alpha \phi_\gamma X_s\right) = g\left(\phi_\beta X_s, -\phi_\beta X_s\right) = -1\\ \langle \Phi_\alpha, \eta_\beta \wedge \eta_\gamma\rangle &= g\left(\eta_\beta, \phi_\alpha \eta_\gamma\right) = g\left(\eta_\beta, -\eta_\beta\right) = -1,\end{aligned}$$

and $\langle \Phi_\alpha, V\rangle = 0$ for any other element V of the basis of the space of bivector fields on U. On the other hand,

$$\begin{aligned}\langle \zeta_s \wedge \phi_\alpha^* \zeta_s, X_s \wedge \phi_\alpha X_s\rangle &= \frac{1}{2}\zeta_s\left(X_s\right)\phi_\alpha^*\zeta_s\left(\phi_\alpha X_s\right) = -\frac{1}{2}\\ \left\langle \phi_\beta^* \zeta_s \wedge \phi_\gamma^* \zeta_s, \phi_\beta X_s \wedge \phi_\gamma X_s\right\rangle &= \frac{1}{2}\phi_\beta^*\zeta_s\left(\phi_\beta X_s\right)\phi_\gamma^*\zeta_s\left(\phi_\gamma X_s\right) = \frac{1}{2}\\ \langle \eta_\beta \wedge \eta_\gamma, \xi_\beta \wedge \xi_\gamma\rangle &= \frac{1}{2}\eta_\beta\left(\xi_\beta\right)\eta_\gamma\left(\xi_\gamma\right) = \frac{1}{2}.\end{aligned}$$

□

Note that for any k-form ω on M, any vector field Y of unit norm, and ρ the dual 1-form such that $\rho(Y) = 1$, we have

$$*\left(\rho \wedge *\omega\right) = (-1)^{(4n+3-k)(k-1)}\, i_Y \omega. \tag{4.6}$$

From (4.2), (4.5), (4.6), and the fact that $*^2 = \mathrm{id}$ for odd dimensional manifolds, it is easy to obtain the formula

$$\Lambda_\alpha = \sum_{s=1}^{n}\left(i_{X_s} i_{\phi_\alpha X_s} + i_{\phi_\beta X_s} i_{\phi_\gamma X_s}\right). \tag{4.7}$$

Remark 4.2. From [3, Lemma 2.3] it follows that the operators $\omega \mapsto \Phi_\alpha \wedge \omega$ preserve harmonic forms. Since the operator $\omega \mapsto \eta_\beta \wedge \eta_\gamma \wedge \omega$ is equal to $l_\beta l_\gamma$, it also preserves harmonicity. Then, by definition of the operators L_α, they preserve harmonicity as well. Since the Hodge star $*$ preserves harmonic forms we get that also Λ_α preserves them.

Now we verify that $L_\alpha\left(\Omega_{H,000}^*(M)\right) \subset \Omega_{H,000}^*(M)$ and $\Lambda_\alpha\left(\Omega_{H,000}^*(M)\right) \subset \Omega_{H,000}^*(M)$. For this it is enough to show that L_α and Λ_α commute with e_μ for any pair $1 \leq \alpha, \mu \leq 3$. Since Λ_α is the Hodge dual of L_α and $\mathrm{id} - e_\mu$ is the Hodge dual of e_μ, it is enough to check that L_α commute with e_μ. We know that $e_\mu = l_\mu \lambda_\mu$. Since l_μ is the wedge product with a 1-form and L_α is the wedge product with a 2-form, they commute. Now, let $\omega \in \Omega^k(M)$, then

$$\lambda_\mu L_\alpha \omega = i_{\xi_\mu}\left(\Xi_\alpha \wedge \omega\right) = \left(i_{\xi_\mu}\Xi_\alpha\right) \wedge \omega + \Xi_\alpha \wedge \left(i_{\xi_\mu}\omega\right) = \left(i_{\xi_\mu}\Xi_\alpha\right) \wedge \omega + L_\alpha \lambda_\mu \omega$$

and by (4.5)

$$i_{\xi_\mu}\Xi_\alpha = \sum_{s=1}^{n}\left(i_{\xi_\mu}\zeta_s \wedge \phi^*_\alpha\zeta_s - \zeta_s \wedge i_{\xi_\mu}\phi^*_\alpha\zeta_s - i_{\xi_\mu}\phi^*_\beta\zeta_s \wedge \phi^*_\gamma\zeta_s + \phi^*_\beta\zeta_s \wedge i_{\xi_\mu}\phi^*_\gamma\zeta_s\right) = 0.$$

As consequence, we can restrict the operators L_α and Λ_α to $\Omega^*_{H,000}(M)$. From now on, we will consider L_α and Λ_α as endomorphisms of $\Omega^*_{H,000}(M)$.

Define the operator $H\colon \Omega^k_{H,000}(M) \to \Omega^k_{H,000}(M)$ by $H\omega = (2n-k)\,\omega$.

Proposition 4.3. *We have $[L_\alpha, \Lambda_\alpha] = -H$ on $\Omega^*_{H,000}(M)$.*

Proof. Every element of $\Omega^k_{H,000}$ can be locally written as a linear combination of wedges of elements in

$$\{\zeta_1, \phi^*_1\zeta_1, \phi^*_2\zeta_1, \phi^*_3\zeta_1, \dots, \zeta_n, \phi^*_1\zeta_n, \phi^*_2\zeta_n, \phi^*_3\zeta_n\}. \tag{4.8}$$

Note that for any $1 \le s, t \le n$ and any cyclic permutation (α, β, γ) of $(1, 2, 3)$

$$\left[\zeta_s \wedge \phi^*_\alpha\zeta_s \wedge -, i_{\phi_\beta X_t} i_{\phi_\gamma X_t}\right] = 0 \qquad \left[\phi^*_\beta\zeta_s \wedge \phi^*_\gamma\zeta_s \wedge -, i_{X_t} i_{\phi_\alpha X_t}\right] = 0$$

and for $s \neq t$

$$\left[\zeta_s \wedge \phi^*_\alpha\zeta_s \wedge -, i_{X_t} i_{\phi_\alpha X_t}\right] = 0 \qquad \left[\phi^*_\beta\zeta_s \wedge \phi^*_\gamma\zeta_s \wedge -, i_{\phi_\beta X_t} i_{\phi_\gamma X_t}\right] = 0.$$

Therefore by (4.5) and (4.7), we get

$$[L_\alpha, \Lambda_\alpha] = \sum_{s=1}^{n}\left(\left[\zeta_s \wedge \phi^*_\alpha\zeta_s \wedge -, i_{X_s} i_{\phi_\alpha X_s}\right] - \left[\phi^*_\beta\zeta_s \wedge \phi^*_\gamma\zeta_s \wedge -, i_{\phi_\beta X_s} i_{\phi_\gamma X_s}\right]\right). \tag{4.9}$$

Now for any linear operators a, b, c, d, we have

$$\begin{aligned}
[ab, cd] &= [ab, c]\,d + c\,[ab, d] \\
&= (a\{b,c\} - \{a,c\}\,b)\,d + c\,(a\{b,d\} - \{a,d\}\,b) \\
&= a\{b,c\}\,d - \{a,c\}\,bd + ca\{b,d\} - c\{a,d\}\,b \\
&= a\{b,c\}\,d - \{a,c\}\,bd - ac\{b,d\} - c\{a,d\}\,b + \{a,c\}\{b,d\}.
\end{aligned} \tag{4.10}$$

It is also obvious that for arbitrary α, $\beta \neq \gamma$:

$$\begin{aligned}
&\{\zeta_s \wedge -, i_{\phi_\alpha X_s}\} = 0 && \{\zeta_s \wedge -, i_{X_s}\} = 1 \\
&\{\phi^*_\beta\zeta_s \wedge -, i_{\phi_\gamma X_s}\} = 0 && \{\phi^*_\beta\zeta_s \wedge -, i_{\phi_\beta X_s}\} = -1 \\
&\{\phi^*_\alpha\zeta_s \wedge -, i_{X_s}\} = 0.
\end{aligned} \tag{4.11}$$

Therefore, using (4.10) we get

$$\begin{aligned}
[L_\alpha, \Lambda_\alpha] &= \sum_{s=1}^{n}\left(-\phi^*_\alpha\zeta_s \wedge i_{\phi_\alpha X_s} + \zeta_s \wedge i_{X_s} - 1 - \left(\phi^*_\gamma\zeta_s \wedge i_{\phi_\gamma X_s} + \phi^*_\beta\zeta_s \wedge i_{\phi_\beta X_s} + 1\right)\right) \\
&= -2n + \sum_{s=1}^{n}\left(\zeta_s \wedge i_{X_s} - \phi^*_\alpha\zeta_s \wedge i_{\phi_\alpha X_s} - \phi^*_\beta\zeta_s \wedge i_{\phi_\beta X_s} - \phi^*_\gamma\zeta_s \wedge i_{\phi_\gamma X_s}\right).
\end{aligned}$$

Now the sum in the last row operates on any fixed-degree form involving only elements in (4.8) by multiplying the form by its degree. Hence

$$[L_\alpha, \Lambda_\alpha]\,\omega = -H\omega$$

for all $\omega \in \Omega^*_{H,000}$. □

For every cyclic permutation (α, β, γ) of $(1,2,3)$ we define the operator

$$K_\alpha = \sum_{s=1}^{n} \left(\phi_\alpha^* \zeta_s \wedge i_{X_s} + \zeta_s \wedge i_{\phi_\alpha X_s} + \phi_\gamma^* \zeta_s \wedge i_{\phi_\beta X_s} - \phi_\beta^* \zeta_s \wedge i_{\phi_\gamma X_s} \right).$$

Let $\rho_1, \ldots, \rho_k$ be a sequence of elements in (4.8). Then from (4.2) and

$$\phi_\alpha^* \phi_\beta^* = -\phi_\gamma^*, \qquad \phi_\beta^* \phi_\alpha^* = \phi_\gamma^*,$$

it follows that

$$K_\alpha (\rho_1 \wedge \cdots \wedge \rho_k) = \sum_{j=1}^{k} (-1)^{j+1} \rho_1 \wedge \cdots \wedge \phi_\alpha^* \rho_j \wedge \cdots \wedge \rho_k.$$

Proposition 4.4. *For any cyclic permutation (α, β, γ) of $(1,2,3)$ we have on $\Omega^*_{H,000}(M)$*

$$[L_\alpha, \Lambda_\beta] = K_\gamma \tag{4.12}$$

$$[L_\alpha, \Lambda_\gamma] = -K_\beta. \tag{4.13}$$

In particular K_α is globally defined, for each $\alpha \in \{1,2,3\}$.

Proof. We have

$$[L_\alpha, \Lambda_\beta] = \sum_{s=1}^{n} \Big(\left[\zeta_s \wedge \phi_\alpha^* \zeta_s \wedge -, i_{X_s} i_{\phi_\beta X_s} \right] + \left[\zeta_s \wedge \phi_\alpha^* \zeta_s \wedge -, i_{\phi_\gamma X_s} i_{\phi_\alpha X_s} \right]$$
$$- \left[\phi_\beta^* \zeta_s \wedge \phi_\gamma^* \zeta_s \wedge -, i_{X_s} i_{\phi_\beta X_s} \right] - \left[\phi_\beta^* \zeta_s \wedge \phi_\gamma^* \zeta_s \wedge -, i_{\phi_\gamma X_s} i_{\phi_\alpha X_s} \right] \Big).$$

Now, by (4.10) and (4.11) we get

$$\begin{aligned} [L_\alpha, \Lambda_\beta] &= \sum_{s=1}^{n} \left(-\phi_\alpha^* \zeta_s \wedge i_{\phi_\beta X_s} + \zeta_s \wedge i_{\phi_\gamma X_s} - i_{X_s} \left(\phi_\gamma^* \zeta_s \wedge - \right) + \phi_\beta^* \zeta_s \wedge i_{\phi_\alpha X_s} \right) \\ &= \sum_{s=1}^{n} \left(\zeta_s \wedge i_{\phi_\gamma X_s} + \phi_\gamma^* \zeta_s \wedge i_{X_s} + \phi_\beta^* \zeta_s \wedge i_{\phi_\alpha X_s} - \phi_\alpha^* \zeta_s \wedge i_{\phi_\beta X_s} \right) \\ &= K_\gamma. \end{aligned}$$

Equation (4.13) is proved as follows. We have

$$[L_\alpha, \Lambda_\gamma] = \sum_{s=1}^{n} \Big(\left[\zeta_s \wedge \phi_\alpha^* \zeta_s \wedge -, i_{X_s} i_{\phi_\gamma X_s} \right] + \left[\zeta_s \wedge \phi_\alpha^* \zeta_s \wedge -, i_{\phi_\alpha X_s} i_{\phi_\beta X_s} \right]$$
$$- \left[\phi_\beta^* \zeta_s \wedge \phi_\gamma^* \zeta_s \wedge -, i_{X_s} i_{\phi_\gamma X_s} \right] - \left[\phi_\beta^* \zeta_s \wedge \phi_\gamma^* \zeta_s \wedge -, i_{\phi_\alpha X_s} i_{\phi_\beta X_s} \right] \Big).$$

Again by (4.10) we get

$$\begin{aligned} [L_\alpha, \Lambda_\gamma] &= \sum_{s=1}^{n} \left(-\phi_\alpha^* \zeta_s \wedge i_{\phi_\gamma X_s} - \zeta_s \wedge i_{\phi_\beta X_s} - \phi_\beta^* \zeta_s \wedge i_{X_s} - i_{\phi_\alpha X_s} \left(\phi_\gamma^* \zeta_s \wedge - \right) \right) \\ &= -\sum_{s=1}^{n} \left(\zeta_s \wedge i_{\phi_\beta X_s} + \phi_\beta^* \zeta_s \wedge i_{X_s} + \phi_\alpha^* \zeta_s \wedge i_{\phi_\gamma X_s} - \phi_\gamma^* \zeta_s \wedge i_{\phi_\alpha X_s} \right) \\ &= -K_\beta. \end{aligned}$$

□

Theorem 4.5. *The linear span* $\mathfrak{g}$ *of the operators* $\{ L_\alpha, \Lambda_\alpha, K_\alpha, H \mid \alpha = 1,2,3\}$ *on* $\Omega^*_{H,000}(M)$ *is a Lie algebra.*

Proof. We have to check that $\mathfrak{g}$ is closed under taking commutators. Clearly it is enough to check that the commutator of any two operators from the set $\{ L_\alpha, \Lambda_\alpha, K_\alpha, H \mid \alpha = 1,2,3\}$ lies in $\mathfrak{g}$. It is obvious that $[L_\alpha, L_\beta] = 0$ and $[\Lambda_\alpha, \Lambda_\beta] = 0$ for any pair $1 \le \alpha, \beta \le 3$. Since K_α does not change the degree of forms, L_α raises the degree by 2 and Λ_α decreases the degree by 2, we get

$$[K_\alpha, H] = 0 \qquad [L_\alpha, H] = 2L_\alpha \qquad [\Lambda_\alpha, H] = -2\Lambda_\alpha. \tag{4.14}$$

Furthermore, by Proposition 4.3 we know that $[L_\alpha, \Lambda_\alpha] = -H$, and by Proposition 4.4 that $[L_\alpha, \Lambda_\beta] = K_\gamma$ for any cyclic permutation (α, β, γ) of $(1,2,3)$. Therefore it is left to check that the commutators $[K_\alpha, L_\alpha]$, $[K_\alpha, L_\beta]$, $[K_\alpha, \Lambda_\alpha]$, $[K_\alpha, L_\beta]$ and $[K_\alpha, K_\beta]$ for all pairs $1 \le \alpha, \beta \le 3$ lie in $\mathfrak{g}$.

For any cyclic permutation (α, β, γ) of $(1,2,3)$ we have

$$\begin{aligned}[K_\alpha, L_\alpha] &\overset{(4.13)}{=\!=} [[L_\beta, \Lambda_\gamma], L_\alpha] = [[L_\beta, L_\alpha], \Lambda_\gamma] + [L_\beta, [\Lambda_\gamma, L_\alpha]] = [L_\beta, K_\beta] \\ &= -[K_\beta, L_\beta].\end{aligned}$$

As (α, β, γ) is an arbitrary cyclic permutation of $(1,2,3)$ we get also

$$[K_\beta, L_\beta] = -[K_\gamma, L_\gamma] \qquad [K_\gamma, L_\gamma] = -[K_\alpha, L_\alpha]$$

and combining we obtain $[K_\alpha, L_\alpha] = -[K_\alpha, L_\alpha]$, which implies $[K_\alpha, L_\alpha] = 0$ for all $1 \le \alpha \le 3$. Similarly, we have $[K_\alpha, \Lambda_\alpha] = 0$.

Now for any cyclical permutation (α, β, γ) of $(1,2,3)$ we have

$$\begin{aligned}
[K_\alpha, L_\beta] &= -[[L_\gamma, \Lambda_\beta], L_\beta] = -[L_\gamma, [\Lambda_\beta, L_\beta]] = -[L_\gamma, H] = -2L_\gamma, \\
[K_\alpha, L_\gamma] &= [[L_\beta, \Lambda_\gamma], L_\gamma] = [L_\beta, [\Lambda_\gamma, L_\gamma]] = [L_\beta, H] = 2L_\beta, \\
[K_\alpha, \Lambda_\beta] &= [[L_\beta, \Lambda_\gamma], \Lambda_\beta] = [[L_\beta, \Lambda_\beta], \Lambda_\gamma] = [-H, \Lambda_\gamma] = -2\Lambda_\gamma, \\
[K_\alpha, \Lambda_\gamma] &= -[[L_\gamma, \Lambda_\beta], \Lambda_\gamma] = -[[L_\gamma, \Lambda_\gamma], \Lambda_\beta] = [H, \Lambda_\beta] = 2\Lambda_\beta, \\
[K_\alpha, K_\beta] &= [[L_\beta, \Lambda_\gamma], K_\beta] = [L_\beta, [\Lambda_\gamma, K_\beta]] = [L_\beta, 2\Lambda_\alpha] = -2K_\gamma.
\end{aligned}$$

□

Now we prove that the Lie algebra $\mathfrak{g}$ can be identified with the Lie algebra $so(4,1)$. Let us recall the definition of $so(4,1)$. We denote by E_1 the matrix

$$\operatorname{diag}(1,1,1,1,-1).$$

Then

$$so(4,1) := \left\{ A \in M_5(\mathbb{R}) \,\middle|\, AE_1 = -E_1 A^t \right\}$$

as a set. The Lie bracket on $so(4,1)$ is given by the usual commutator of matrices. We denote by e_{ij} the matrix with 1 at the place (i,j) and zeros elsewhere. Define for $1 \le i < j \le 5$

$$t_{ij} = \begin{cases} e_{i5} + e_{5i} & j = 5 \\ e_{ij} - e_{ji} & \text{otherwise.} \end{cases}$$

Then the set $\{ t_{ij} \,|\, 1 \le i < j \le 5\}$ is a basis of $so\,(4,1)$. A direct computation shows that

$$\begin{array}{llll} [t_{ij}, t_{ik}] = -t_{jk} & [t_{ij}, t_{jk}] = t_{ik} & [t_{ik}, t_{jk}] = -t_{ij} & i < j < k < 5 \\ [t_{ij}, t_{i5}] = -t_{j5} & [t_{ij}, t_{j5}] = t_{i5} & [t_{i5}, t_{j5}] = t_{ij} & i < j < 5 \end{array}$$

We will also use t_{ji} to denote $-t_{ij}$ for $1 \le i < j \le 4$. Now for any cyclic permutation (α, β, γ) of $(1,2,3)$ we have

$$\begin{aligned} [t_{\alpha 5} + t_{\alpha 4}, t_{\alpha 5} - t_{\alpha 4}] &= [t_{\alpha 5}, -t_{\alpha 4}] + [t_{\alpha 4}, t_{\alpha 5}] = -2t_{45} \\ [t_{\alpha 5} + t_{\alpha 4}, 2t_{45}] &= 2\,(t_{\alpha 4} + t_{\alpha 5}) \\ [t_{\alpha 5} - t_{\alpha 4}, 2t_{45}] &= 2\,(t_{\alpha 4} - t_{\alpha 5}) = -2\,(t_{\alpha 5} - t_{\alpha 4}) \\ [t_{\alpha 5} + t_{\alpha 4}, t_{\beta 5} + t_{\beta 4}] &= t_{\alpha\beta} - t_{\alpha\beta} = 0 \\ [t_{\alpha 5} + t_{\alpha 4}, t_{\beta 5} - t_{\beta 4}] &= t_{\alpha\beta} + t_{\alpha\beta} = 2t_{\alpha\beta} \\ [t_{\alpha 5} + t_{\alpha 4}, t_{\gamma 5} - t_{\gamma 4}] &= -2t_{\gamma,\alpha} \\ [2t_{\beta\gamma}, t_{\beta 5} + t_{\beta 4}] &= -2\left(t_{\gamma 5 + t_{\gamma 4}}\right) \\ [2t_{\beta\gamma}, t_{\gamma 5} + t_{\gamma 4}] &= 2\,(t_{\beta 5} + t_{\beta 4}) \\ [2t_{\beta\gamma}, t_{\beta 5} - t_{\beta 4}] &= -2\,(t_{\gamma 5} - t_{\gamma 4}) \\ [2t_{\beta\gamma}, t_{\gamma 5} - t_{\gamma 4}] &= 2\,(t_{\beta 5} - t_{\beta 4})\,. \end{aligned}$$

Therefore the assignment

$$H \mapsto 2t_{45} \qquad L_\alpha \mapsto t_{\alpha 5} + t_{\alpha 4} \qquad \Lambda_\alpha \mapsto t_{\alpha 5} - t_{\alpha 4} \qquad K_\alpha \mapsto 2t_{\beta\gamma}$$

induces an isomorphism of Lie algebras $so\,(4,1) \to \mathfrak{g}$. Thus we have proved the following result.

Theorem 4.6. *The operators L_α, Λ_α, $\alpha \in \{1,2,3\}$, give a structure of $so\,(4,1)$-module on $\Omega^*_{H,000}\,(M)$.*

5. Action of $\mathbb{H}$ on $\Omega^{2k+1}_{H,000}\,(M)$ and Betti numbers of compact 3-cosymplectic manifolds

Let $U \subset M$ be an open subset and

$$\{\zeta_1, \phi_1^*\zeta_1, \phi_2^*\zeta_1, \phi_3^*\zeta_1, \dots, \zeta_n, \phi_1^*\zeta_n, \phi_2^*\zeta_n, \phi_3^*\zeta_n, \eta_1, \eta_2, \eta_3\}$$

an orthonormal basis of 1-forms on U. Define $\Omega^*_{000}\,(U)$ as a linear span with coefficients in $C^\infty\,(U)$ of the set

$$Y := \{\zeta_1, \phi_1^*\zeta_1, \phi_2^*\zeta_1, \phi_3^*\zeta_1, \dots, \zeta_n, \phi_1^*\zeta_n, \phi_2^*\zeta_n, \phi_3^*\zeta_n\}\,.$$

Then $\Omega^*_{H,000}\,(U)$ is a subspace of $\Omega^*_{000}\,(U)$. Define the operator I_α on $\Omega^*_{000}\,(U)$ extending by linearity the map

$$\rho_1 \wedge \cdots \wedge \rho_k \mapsto \phi_\alpha^*\rho_1 \wedge \cdots \wedge \phi_\alpha^*\rho_k \qquad\qquad \rho_1, \dots, \rho_k \in Y.$$

Proposition 5.1. *The operators I_α, $\alpha \in \{1,2,3\}$, are well-defined on $\Omega^*_{000}\,(M)$. Moreover, they preserve harmonic forms. In particular, we can consider I_α as an endomorphism of $\Omega^*_{H,000}\,(M)$.*

Proof. For $1 \le s \le k$, we define the operators $K_{\alpha,s}$ on $\Omega^k_{000}(U)$ extending by linearity the map

$$\rho_1 \wedge \cdots \wedge \rho_k \mapsto \sum_{1 \le j_1 < \cdots < j_s \le k} (-1)^{j_1 + \cdots + j_s + s} \rho_1 \wedge \cdots \wedge \phi^*_\alpha \rho_{j_1} \wedge \cdots \wedge \phi^*_\alpha \rho_{j_s} \wedge \cdots \wedge \rho_k,$$

where $\rho_1, \ldots, \rho_k \in Y$. We also denote the identity operator by $K_{\alpha,0}$. Then $K_{\alpha,1} = K_\alpha$ and $K_{\alpha,k} = (-1)^{\binom{k+1}{2}} I_\alpha$. It is easy to check in local coordinates that

$$K_\alpha K_{\alpha,s} = (s+1)K_{\alpha,s+1} - (k-s+1)K_{\alpha,s-1}.$$

These formulae can be used to show that $K_{\alpha,s}$ is a polynomial in K_α with constant coefficients which do not depend on the used local chart. Since K_α are globally defined and preserve harmonic forms we get that the operators $K_{\alpha,s}$ are globally defined and preserve harmonic forms for all s. In particular, I_α is a well-defined operator on $\Omega^*_{000}(M)$ and preserves harmonic forms. □

It is straightforward to see that the operators I_α, $\alpha \in \{1,2,3\}$, restricted to $\Omega^{odd}_{H,000}(M)$ satisfy the same relations as the imaginary units of the quaternion algebra $\mathbb{H}$. Therefore we get

Theorem 5.2. *Let k be odd. Then $\Omega^k_{H,000}(M)$ is an $\mathbb{H}$-module.*

Corollary 5.3. *Let k be odd. Then b^h_k is divisible by 4.*

Proof. Every finite dimensional module over $\mathbb{H}$ is a direct sum of regular modules. As the dimension of the regular module is 4, the result follows. □

We denote by (d) the principal ideal in $\mathbb{Z}$ generated by d. In other words, (d) will be the set of the integers divisible by d.

Corollary 5.4. *Let M be a compact 3-cosymplectic manifold. For any odd k we have*

$$b_{k-1} + b_k \in (4).$$

Proof. Using (3.4) we get for $k = 1$

$$b_0 + b_1 = b^h_0 + b^h_1 + 3b^h_0 = b^h_1 + 4b^h_0 \in (4).$$

Similarly, for $k = 3$ we get

$$b_2 + b_3 = b^h_2 + 3b^h_1 + 3b^h_0 + b^h_3 + 3b^h_2 + 3b^h_1 + b^h_0 = b^h_3 + 4b^h_2 + 6b^h_1 + 4b^h_0 \in (4).$$

Finally, for odd $k \ge 5$ we have

$$\begin{aligned} b_{k-1} + b_k &= b^h_{k-1} + 3b^h_{k-2} + 3b^h_{k-3} + b^h_{k-4} + b^h_k + 3b^h_{k-1} + 3b^h_{k-2} + b^h_{k-3} \\ &= b^h_k + 4b^h_{k-1} + 6b^h_{k-2} + 4b^h_{k-3} + b^h_{k-4} \in (4). \end{aligned}$$

□

6. Inequalities on Betti numbers

In this section we give a bound from below on the Betti numbers of a compact 3-cosymplectic manifold. We start with the following statement about horizontal Betti numbers, which is a generalization of Wakakuwa's Theorem 9.1 in [29].

Proposition 6.1. *Let M be a compact 3-cosymplectic manifold of dimension $4n+3$. Then for $0 \leq k \leq n$*

$$b_{2k}^h \geq \binom{k+2}{2}.$$

Proof. Recall the definition (4.1) of the 2-forms Ξ_α. Let us fix $0 \leq k \leq n$. We consider the set

$$S_k := \left\{ \Xi_1^{k_1} \wedge \Xi_2^{k_2} \wedge \Xi_3^{k_3} \,\middle|\, k_1 + k_2 + k_3 = k \right\}.$$

All the elements of S_k can be obtained from the constant 0-form 1 on M by successive applications of operators L_α, $\alpha \in \{1,2,3\}$. Therefore by Remark 4.2 we get $S_k \subset \Omega_{H,000}^{2k}(M)$. Thus, to prove the proposition it is enough to show that S_k contains $\binom{k+2}{2}$ linearly independent elements. This can be checked locally. Let U be a trivializing neighbourhood like in Section 4. We order the elements of the basis (4.3) of 1-forms on U by

$$\zeta_1 < \zeta_2 < \cdots < \zeta_n < \phi_1^*\zeta_1 < \phi_1^*\zeta_2 < \cdots < \phi_1^*\zeta_n \\ < \phi_2^*\zeta_1 < \cdots < \phi_2^*\zeta_n < \phi_3^*\zeta_1 < \cdots < \phi_3^*\zeta_n < \eta_1 < \eta_2 < \eta_3.$$

Then we get an induced lexicographical ordering on the basis of $\Omega^k(U)$. By using the local expression (4.5) of Ξ_α, $\alpha \in \{1,2,3\}$, we see that the first basis element with respect to this ordering that enters in $\Xi_1^{k_1} \wedge \Xi_2^{k_2} \wedge \Xi_3^{k_3}$ with a non-zero coefficient is

$$\zeta_1 \wedge \phi_1^*\zeta_1 \wedge \zeta_2 \wedge \phi_1^*\zeta_2 \wedge \cdots \wedge \zeta_{k_1} \wedge \phi_1^*\zeta_{k_1} \wedge \zeta_{k_1+1} \wedge \phi_2^*\zeta_{k_1+1} \wedge \cdots \wedge \zeta_{k_1+k_2} \wedge \phi_2^*\zeta_{k_1+k_2} \\ \wedge \zeta_{k_1+k_2+1} \wedge \phi_3^*\zeta_{k_1+k_2+1} \wedge \cdots \wedge \zeta_{k_1+k_2+k_3} \wedge \phi_3^*\zeta_{k_1+k_2+k_3}.$$

Since for different triples (k_1, k_2, k_3) such that $k_1 + k_2 + k_3 = k$ the above basis elements are different, we get that S_k contains $\binom{k+2}{2}$ elements and they are linearly independent. ☐

As a consequence we get the following lower bound on the Betti numbers of a compact 3-cosymplectic manifold.

Theorem 6.2. *Let M be a compact 3-cosymplectic manifold of dimension $4n+3$. Then for $0 \leq k \leq 2n+1$*

$$b_k \geq \binom{k+2}{2}.$$

Proof. For $k=0$ we have obviously $b_0 = 1 = \binom{2}{2}$. First we consider the case $k = 2l$, $1 \leq l \leq n$. Then by (3.4) and Proposition 6.1

$$\begin{aligned} b_k &= b_{2l}^h + 3b_{2l-1}^h + 3b_{2l-2}^h + b_{2l-3}^h \geq \binom{l+2}{2} + 3\cdot 0 + 3\binom{l-1+2}{2} + 0 \\ &= \frac{(l+2)(l+1)}{2} + 3\frac{(l+1)\,l}{2} = \frac{(l+1)(l+2+3l)}{2} = \frac{(2l+2)(2l+1)}{2} \\ &= \binom{k+2}{2}. \end{aligned}$$

Now, suppose that $k = 2l + 1$, $0 \leq l \leq n$. Then, again by (3.4) and Proposition 6.1

$$\begin{aligned} b_k &= b^h_{2l+1} + 3b^h_{2l} + 3b^h_{2l-1} + b^h_{2l-2} \geq 0 + 3\binom{l+2}{2} + 3 \cdot 0 + \binom{l-1+2}{2} \\ &= 3\frac{(l+2)(l+1)}{2} + \frac{(l+1)\,l}{2} = \frac{(l+1)(3l+6+l)}{2} = \frac{(2l+2)(2l+3)}{2} \\ &= \binom{2l+3}{2} = \binom{k+2}{2}. \end{aligned}$$

□

7. Nontrivial examples of compact 3-cosymplectic manifolds

The standard example of a compact 3-cosymplectic manifold is given by the torus $\mathbb{T}^{4n+3}$ with the following structure (cf. [20, page 561]). Let $\{\theta_1, \ldots, \theta_{4n+3}\}$ be a basis of 1-forms such that each θ_i is integral and closed. Let us define a Riemannian metric g on $\mathbb{T}^{4n+3}$ by

$$g := \sum_{i=1}^{4n+3} \theta_i \otimes \theta_i.$$

For each $\alpha \in \{1, 2, 3\}$ we define a tensor field ϕ_α of type $(1, 1)$ by

$$\begin{aligned} \phi_\alpha = \sum_{i=1}^{n} (&E_{\alpha n+i} \otimes \theta_i - E_i \otimes \theta_{\alpha+i} + E_{\gamma n+i} \otimes \theta_{\beta n+i} \\ &- E_{\beta n+i} \otimes \theta_{\gamma n+i}) + E_{4n+\gamma} \otimes \theta_{4n+\beta} - E_{4n+\beta} \otimes \theta_{4n+\gamma}, \end{aligned}$$

where $\{E_1, \ldots, E_{4n+3}\}$ is the dual (orthonormal) basis of $\{\theta_1, \ldots, \theta_{4n+3}\}$ and (α, β, γ) is a cyclic permutation of $\{1, 2, 3\}$. Setting, for each $\alpha \in \{1, 2, 3\}$, $\xi_\alpha := E_{4n+\alpha}$ and $\eta_\alpha := \theta_{4n+\alpha}$, one can easily check that the torus $\mathbb{T}^{4n+3}$ endowed with the structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is 3-cosymplectic.

On the other hand, the standard example of a noncompact 3-cosymplectic manifold is given by $\mathbb{R}^{4n+3}$ with the structure described in [6, Theorem 4.4].

Both the above examples are the global product of a hyper-Kähler manifold with a 3-dimensional flat abelian Lie group. In fact, locally this is always true.

Proposition 7.1. *Any 3-cosymplectic manifold M^{4n+3} is locally the Riemannian product of a hyper-Kähler manifold N^{4n} and a 3-dimensional flat abelian Lie group G^3.*

Proof. The tangent bundle of M^{4n+3} splits up as the orthogonal sum of the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, which define Riemannian foliations with totally geodesic leaves. Therefore, by the de Rham decomposition theorem the manifold M is locally the Riemannian product of a leaf N^{4n} of $\mathcal{H}$ and a leaf G^3 of $\mathcal{V}$. The structure tensors ϕ_1, ϕ_2, ϕ_3 induce an almost hyper-complex structure (J_1, J_2, J_3) on N^{4n}. Furthermore, for each $\alpha \in \{1, 2, 3\}$ and for all $X, X' \in \Gamma(TN^{4n}) = \Gamma(\mathcal{H})$,

$$[J_\alpha, J_\alpha](X, X') = N_{\phi_\alpha}(X, X') - 2d\eta_\alpha(X, X')\xi_\alpha = 0,$$

as M^{4n+3} is normal and η_α is closed. Consequently, the structure is hyper-complex. Finally, the induced metric is clearly compatible with such a hyper-complex structure, so that N^{4n} is hyper-Kähler. On the other hand, from Lie group theory (see e.g. [26, page 10]) it follows that G^3 is an abelian Lie group. Since the Reeb vector fields are parallel, we get

$$R(\xi_\alpha, \xi_\beta)\xi_\gamma = \nabla_{\xi_\alpha}\nabla_{\xi_\beta}\xi_\gamma - \nabla_{\xi_\beta}\nabla_{\xi_\alpha}\xi_\gamma - \nabla_{[\xi_\alpha, \xi_\beta]}\xi_\gamma = 0, \tag{7.1}$$

Therefore G^3 is flat. □

When $n = 0$ of course we have no splitting, and M is necessarily a 3-torus in the compact case, as it is shown in the following proposition.

Proposition 7.2. *Suppose M^3 is a compact three dimensional 3-cosymplectic manifold. Then M^3 is a three dimensional torus.*

Proof. First of all M^3 is clearly flat. Indeed, in this case the three Reeb vector fields span all the vector fields over the ring of smooth functions. Furthermore, they commute with each other and are parallel. Thus, similarly to (7.1) we get $R(\xi_\alpha, \xi_\beta)\xi_\gamma = 0$ for any triple of indices $1 \leq \alpha, \beta, \gamma \leq 3$. The manifold M^3 is orientable, since $\eta_1 \wedge \eta_2 \wedge \eta_3 \neq 0$ is a volume form on M^3. Moreover η_1, η_2, η_3 are three linear independent harmonic forms of degree 1, so that $b_1\left(M^3\right) \geq 3$. The complete list of compact orientable Euclidean three-dimensional manifolds was obtained in Sections 2-3 of [14]. The unique manifold with $b_1 \geq 3$ in this list is the three dimensional torus. □

Due to Proposition 7.1, it is natural to ask whether there are examples of 3-cosymplectic manifolds which are not the global product of a hyper-Kähler manifold with an abelian Lie group. We will give an example of a compact 3-cosymplectic manifold in dimension seven that is not a product of a hyper-Kähler manifold and a three-dimensional torus. Before describing the construction, we remind the following well-known result.

Theorem 7.3. *If M^4 is a compact four-dimensional hyper-Kähler manifold, then M^4 is either a K3 surface or a four dimensional torus.*

Proof. From [29, Theorem 8.1] it follows that $b_1(M^4)$ is even. Moreover, since every hyper-Kähler manifold is Calabi-Yau, M^4 has a trivial canonical bundle. Therefore, by the Kodaira classification (cf. [16, Section 6A]) M^4 is either a K3 surface or a 4-torus. □

Let (M^{4n}, J_α, G) be a compact hyper-Kähler manifold, where (J_1, J_2, J_3) is the hyper-complex structure of M^{4n} and G is the compatible Riemannian metric. Let $f : M^{4n} \longrightarrow M^{4n}$ be a hyper-Kählerian isometry, that is f is an isometry such that

$$f_* \circ J_\alpha = J_\alpha \circ f_* \tag{7.2}$$

for each $\alpha \in \{1, 2, 3\}$. Let us define the action φ of $\mathbb{Z}^3$ on the product manifold $M^{4n} \times \mathbb{R}^3$ by

$$\varphi\left((k_1, k_2, k_3), (x, t_1, t_2, t_3)\right) = \left(f^{k_1+k_2+k_3}(x), t_1 + k_1, t_2 + k_2, t_3 + k_3\right).$$

Note that the action φ is free and properly discontinuous, hence the orbit space $M_f^{4n+3} := (M^{4n} \times \mathbb{R}^3)/\mathbb{Z}^3$ is a smooth manifold. We define a 3-cosymplectic structure on M_f^{4n+3} in the following way. Let $\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3$ be the vector fields on $M^{4n} \times \mathbb{R}^3$ given by $\hat{\xi}_\alpha := \frac{\partial}{\partial t_\alpha}$, and let $\hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_3$ be the 1-forms defined by $\hat{\eta}_\alpha := \hat{g}(\cdot, \hat{\xi}_\alpha)$, where

$$\hat{g} = G + dt_1 \otimes dt_1 + dt_2 \otimes dt_2 + dt_3 \otimes dt_3.$$

Let $\hat{\phi}_\alpha$ be the tensor field of type $(1, 1)$ on $M^{4n} \times \mathbb{R}^3$ defined as follows. Let E be a vector field on M. We can uniquely decompose E into the sum of a vector field X tangent to M^{4n} and its vertical part $\sum_{\beta=1}^{3} \hat{\eta}_\beta(E)\hat{\xi}_\beta$. Then we set

$$\hat{\phi}_\alpha E := J_\alpha X + \sum_{\beta,\gamma=1}^{3} \epsilon_{\alpha\beta\gamma}\hat{\eta}_\beta(E)\hat{\xi}_\gamma.$$

Since f is an isometry, $\hat{g}$ descends to a Riemannian metric on the quotient manifold M_f^{4n+3}. Furthermore, the vector fields $\hat{\xi}_1$, $\hat{\xi}_2$, $\hat{\xi}_3$, together with their dual 1-forms $\hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_3$, are clearly invariant under the action φ. Finally, because of (7.2), also the endomorphisms $\hat{\phi}_\alpha$ induce three endomorphisms on the tangent spaces of M_f^{4n+3}. We denote the induced structure by $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$, $\alpha \in \{1,2,3\}$. By a straightforward computation one can check that $(M_f^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is a 3-cosymplectic manifold. Moreover, M_f^{4n+3} is not in general a global product of a hyper-Kähler manifold by the torus $\mathbb{T}^3$. To see this we will consider the following more specific seven-dimensional example.

Let $\mathbb{H}$ be the algebra of quaternions. We consider $\mathbb{H}$ as a hyper-Kähler four-dimensional manifold with a hyper-complex structure given by left multiplication by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. Define the action of $\mathbb{Z}^4$ on $\mathbb{H}$ by

$$\begin{aligned} \mathbb{Z}^4 \times \mathbb{H} &\to \mathbb{H} \\ ((a,b,c,d), q) &\mapsto q + a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}. \end{aligned}$$

By distributivity of multiplication in $\mathbb{H}$ this action commutes with the left multiplication by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. Furthermore, the Euclidean metric on $\mathbb{H}$ is translation invariant. Thus the quotient space $\mathbb{H}/\mathbb{Z}^4$ is diffeomorphic to $\mathbb{T}^4$ and inherits a hyper-Kähler structure from $\mathbb{H}$.

Let $\bar{f}\colon \mathbb{H} \to \mathbb{H}$ be the map given by the right multiplication by $\mathbf{i}$. Then from associativity of multiplication on $\mathbb{H}$ it follows that $\bar{f}$ commutes with the hyper-complex structure maps on $\mathbb{H}$. Moreover, from the distributivity of multiplication in $\mathbb{H}$ it follows that $\bar{f}$ induces a hyper-Kählerian isometry f on $\mathbb{H}/\mathbb{Z}^4$.

Theorem 7.4. *Let $M^4 = \mathbb{T}^4$ and f be as above. Then M_f^7 is not the total space of a toric bundle over a hyper-Kähler four dimensional manifold. In particular, M_f^7 is not the global product of a compact hyper-Kähler four-manifold and the torus $\mathbb{T}^3$.*

Proof. Suppose M_f^7 has a regular Reeb foliation, in other words M_f^7 is the total space of a toric bundle over a hyper-Kähler manifold $K^4 = M_f^7/\mathcal{F}_3$, where $\mathcal{F}_3$ is the Reeb foliation on M_f^7. Since the harmonic forms η_α generate the cohomology groups of fibres in this bundle, by the Leray-Hirsch theorem

$$H^*\left(M_f^7\right) \cong H^*\left(\mathbb{T}^3\right) \otimes H^*\left(K^4\right).$$

By Theorem 7.3 there are only two possibilities for K^4: either $K^4 \cong \mathbb{H}/\mathbb{Z}^4$ or K^4 is a complex K3 surface. In the first case the Hilbert-Poincaré series of $H^*\left(\mathbb{T}^3\right) \otimes H^*\left(K^4\right)$ is $(1+t)^7 = 1 + 7t + 21t^2 + \dots$, in the second case it equals

$$(1 + 22t^2 + t^4)(1+t)^3 = 1 + 3t + 25t^2 + \dots$$

We will show in Proposition 7.7 that $b_2(M_f^7) < 21$. This will imply a contradiction with our initial assumption. □

To get an estimate on $b_2(M_f^7)$ we will define a structure of CW-complex on M_f^7.

Recall the definition of CW-complex (cf. [21, Definition 7.3.1]). We will modify it by replacing the balls in $\mathbb{R}^n$ by cubes $Q^k = \left\{ x \in \mathbb{R}^k \,\middle|\, 0 \le x_i \le 1,\ i = 1, \dots, k \right\}$.

Definition 7.5. A CW-*complex* is a Hausdorff space X, together with an indexing set I_k for each integer $k \ge 0$ and maps $\phi_\alpha^k\colon Q^k \to X$, $k \ge 0$, $\alpha \in I_k$ such that the following conditions are satisfied:

(1) $X=\bigcup_{k\geq 0}\bigcup_{\alpha\in I_k}\phi^k_\alpha(\mathring{Q}^k)$;
(2) $\phi^k_\alpha(\mathring{Q}^k)\cap\phi^l_\beta(\mathring{Q}^l)=\emptyset$ unless $k=l$ and $\alpha=\beta$;
(3) $\phi^k_\alpha|_{\mathring{Q}^k}$ is one-to-one;
(4) Let $X^k=\bigcup_{j\leq k}\bigcup_{\alpha\in I_j}\phi^j_\alpha(\mathring{Q}^j)$. Then $\phi^k_\alpha(\partial Q^k)\subset X^{k-1}$ for each $k\geq 1$ and $\alpha\in I_k$.
(5) A subset Z of X is closed if and only if $\left(\phi^k_\alpha\right)^{-1}(Z)$ is closed in Q^k for each $k\geq 0$ and $\alpha\in I_k$.
(6) For each $k\geq 0$ and $\alpha\in I_k$ the set $\phi^k_\alpha(Q^k)$ is contained in the union of a finite number of sets of the form $\phi^l_\beta(\mathring{Q}^l)$.

Let X be a CW-complex. Then we have the induced maps

$$\overline{\phi^k_\alpha}\colon S^k\cong\ Q^k\Big/\partial Q^k\ \to\ X^k\Big/X^{k-1}\ .$$

We will denote the image of this map by S^k_α. By [21, Example 7.3.15] we get a homeomorphism of topological spaces

$$\overline{\phi^k}=\bigvee_{\alpha\in I_k}\overline{\phi^k_\alpha}\colon\bigvee_{\alpha\in I_k}Q^k\Big/\partial Q^k\to\bigvee_{\alpha\in I_k}S^k_\alpha=\ X^k\Big/X^{k-1}\ ,$$

where $\bigvee_{\alpha\in I_k}S^k_\alpha$ denote the one point union (see e.g. [21], page 205). We denote by q_β the map from $\bigvee_{\alpha\in I_k}S^k_\alpha$ to S^k_β that acts as the identity on S^k_β and collapses all the other spheres to the basic point.

Now we explain how the homology groups of a CW-complex can be computed. We define $C_k(X)$ to be the free abelian group generated by I_k. For every pair $\alpha\in I_k$ and $\beta\in I_{k-1}$ we define the map $d_{\alpha,\beta}$ to be the composition

$$S^{k-1}\cong\partial Q^k\xrightarrow{\phi^k_\alpha}X^{k-1}\xrightarrow{\pi}\ X^{k-1}\Big/X^{k-2}=\bigvee_{\gamma\in I_k}S^{k-1}_\gamma\xrightarrow{q_\beta}S^{k-1}_\beta.$$

We denote by $[d_{\alpha,\beta}]$ the degree of the map $d_{\alpha,\beta}$. Now define the differential $\partial\colon C_k(X)\to C_{k-1}(X)$ by $\partial(\alpha)=\sum_{\beta\in I_{k-1}}[d_{\alpha,\beta}]\,\beta$. It is proved in Chapter 8 of [21] that the homology groups of the complex $(C_*(X),\partial)$ are isomorphic to the integral homology groups of the space X.

Since $\mathbb{R}$ is torsion free and $C_k(X)$ are free $\mathbb{Z}$-modules, it follows from the universal coefficient theorem that

$$H_k(C_*(X)\otimes_{\mathbb{Z}}\mathbb{R})\cong H_k(C_*(X))\otimes_{\mathbb{Z}}\mathbb{R}=H^{\mathbb{R}}_k(X),\qquad k\geq 0.$$

If X is an m-dimensional compact Riemannian manifold then we have by the Poincaré duality

$$H^{\mathbb{R}}_k(X)\cong H^{m-k}_{dR}(X)\cong\Omega^{m-k}_H(X),\qquad 0\leq k\leq m.$$

Define $\pi\colon\mathbb{H}\times\mathbb{R}^3\to M^7_f$ to be the composition

$$\mathbb{H}\times\mathbb{R}^3\xrightarrow{\pi_1}\left(\ \mathbb{H}\big/\mathbb{Z}^4\right)\times\mathbb{R}^3\xrightarrow{\pi_2}M^7_f,$$

where π_1 and π_2 are the natural projections.

Now we describe the cellular structure on M^7_f. For every $k\in\{0,\dots,7\}$ we denote by I_k the set of k-subsets in $\{1,\dots,7\}$. For every $S\in I_k$ and $x\in Q^k$ define $\theta_S(x)$ to be the element of $\mathbb{R}^7\equiv\mathbb{H}\times\mathbb{R}^3$, obtained from x by order preserving placing of coordinates of x into the places $s\in S$ and putting at all other places 0. Now we define $\phi^k_S\colon Q^k\to M^7_f$ to be the composition $\pi\circ\theta_S$.

Proposition 7.6. *The maps* $\left\{ \phi_S^k \,\middle|\, S \in I_k,\ k = 0, \dots, 7 \right\}$ *give a CW-complex structure on* M_f^7.

Proof. The topological space M_f^7 is Hausdorff since it is a manifold. Now we show that the restriction of π to $[0,1)^7$ is a bijection. For any $x \in \mathbb{R}$ we denote by $\lfloor x \rfloor$ the integral part of x and by $\{x\}$ the fractional part $x - \lfloor x \rfloor$ of x.

Let $[[q], \vec{x}] \in M_f^7$. Then $[[q], \vec{x}] = \left[\left[q\mathbf{i}^{-(\lfloor x_1 \rfloor + \lfloor x_2 \rfloor + \lfloor x_3 \rfloor)}\right], \{x_1\}, \{x_2\}, \{x_3\}\right]$ in M_f^7 by definition of the action of $\mathbb{Z}^3$ on $\mathbb{H}/\mathbb{Z}^4 \times \mathbb{R}^3$. Thus the restriction of π_2 to $\mathbb{H}/\mathbb{Z}^4 \times [0,1)^3$ is a surjection. To see that π_2 is a bijection we note that the map

$$\begin{aligned} f\colon \quad \mathbb{H}/\mathbb{Z}^4 \times \mathbb{R}^3 &\to \mathbb{H}/\mathbb{Z}^4 \times \mathbb{R}^3 \\ ([q], \vec{x}) &\mapsto \left(\left[q\mathbf{i}^{-(\lfloor x_1 \rfloor + \lfloor x_2 \rfloor + \lfloor x_3 \rfloor)}\right], \{x_1\}, \{x_2\}, \{x_3\}\right) \end{aligned}$$

is $\mathbb{Z}^3$-invariant. Since for different points of $\mathbb{H}/\mathbb{Z}^4 \times \mathbb{R}^3$ the values of f are obviously different we see that the restriction of π_2 to $\mathbb{H}/\mathbb{Z}^4 \times [0,1)^3$ is injective. Similarly we can show that the restriction of π_1 to $[0,1)^7$ gives a bijection between $[0,1)^7$ and $\mathbb{H}/\mathbb{Z}^4 \times [0,1)^3$. Thus we get that the restriction of π to $[0,1)^7$ gives a bijection between $[0,1)^7$ and M_f^7.

Now we check that the maps ϕ_S^k satisfy the properties of CW-structure.

(1) We have

$$\bigcup_{k=0}^{7} \bigcup_{S \in I_k} \theta_S^k \left(\mathring{Q}^k\right) = [0,1)^7,$$

which implies that the similar union with ϕ_S^k in place of θ_S^k gives M_f^7.

(2) Let $S \in I_k$ and $T \in I_l$. Then the points of $\theta_S^k\left(\mathring{Q}^k\right)$ have non-integer coordinates at places $s \in S$ and integer coordinates in all other places. Similarly for the points of $\theta_T^l\left(\mathring{Q}^l\right)$. This implies that if $S \neq T$ then there are no common points in the sets $\theta_S^k\left(\mathring{Q}^k\right)$ and $\theta_T^l\left(\mathring{Q}^l\right)$. As the restriction of π to $[0,1)^7$ is a bijection the same property holds for $\phi_S^k\left(\mathring{Q}^k\right)$ and $\phi_T^l\left(\mathring{Q}^l\right)$.

(3) As $\theta_S^k\left(\mathring{Q}^k\right) \subset [0,1)^7$ we see that the restriction of ϕ_S^k to $\mathring{Q}^k$ is one-to-one, for any $0 \le k \le 7$, $S \in I_k$.

(4) From the considerations at the beginning of the proof we can see that if two points (q, x), $(q', x') \in \mathbb{H} \times \mathbb{R}^3$ are representatives of the same point in M_f^7 then the number of integer coordinates in (q, x) and (q', x') is the same. Now $X^k \subset M_f^7$ can be identified with those points $[[q], x] \in M_f^7$ such that (q, x) has at most k fractional coordinates. Now every point ∂Q^k contains at least one integral coordinate. Therefore for $S \in I_k$, $\theta_S^k\left(\partial Q^k\right)$ contains at least $7 - k + 1 = 8 - k$ integral coordinates, or, in other words, at most $k - 1$ non-integral coordinates. Thus $\phi_S^k\left(\partial Q^k\right) = \pi \circ \theta_S^k\left(\partial Q^k\right)$ is a subset of X^{k-1}.

(5) If $Z \in M_f^7$ is closed then for any $0 \le k \le 7$ and $S \in I_k$ the sets $\left(\phi_S^k\right)^{-1}(Z)$ are obviously closed, as the maps ϕ_α^k are continuous. Suppose now that for every $0 \le k \le 7$ and $S \in I_k$ the sets $\left(\phi_S^k\right)^{-1}(Z)$ are closed. As M_f^7 has the quotient topology under the projection π, we have to show that $\pi^{-1}(Z)$ is a closed subset in $\mathbb{H} \times \mathbb{R}^3$. Let $(q_n, \vec{x}_n)$

be a sequence in $\pi^{-1}(Z)$ that converges to $(q,\vec{x}) \in \mathbb{H} \times \mathbb{R}^3$. We have to show that $(q,\vec{x}) \in \pi^{-1}(Z)$. Let $i \in \{1,2,3\}$. If x^i is fractional, then starting from some n we have $\lfloor x_n^i \rfloor = \lfloor x^i \rfloor$. If x^i is integer then for infinitely many n we have $x_n^i < \lfloor x^i \rfloor$ or $\lfloor x^i \rfloor \leq x_n^i$. By passing to an appropriate subsequence we can assume that for all n either $\lfloor x_n^i \rfloor = \lfloor x^i \rfloor - 1$ or $\lfloor x_n^i \rfloor = \lfloor x^i \rfloor$. We denote the common integer part of x_n^i by $\widetilde{x}^i$. Define

$$q_n' = q_n i^{-\widetilde{x}^1-\widetilde{x}^2-\widetilde{x}^3} \qquad (x_n')^i = x_n^i - \widetilde{x}^i$$
$$q' = q i^{-\widetilde{x}^1-\widetilde{x}^2-\widetilde{x}^3} \qquad (x')^i = x^i - \widetilde{x}^i.$$

Then (q_n', x_n') is a sequence of points in $\pi^{-1}(Z)$ that converges to (q', x'). Moreover $(q', x') \in \pi^{-1}(Z)$ if and only if $(q, x) \in \pi^{-1}(Z)$. We also have $0 \leq (x_n')^i < 1$ and $x^i \in [0,1]$.

Now, similarly to the considerations above, by passing to an appropriate subsequence we can assume that the integer parts of the coefficients of q_n' does not depend on n. Denote by $\widetilde{q}$ the quaternion with coefficients equal to the integer parts of q_n'. Define $q_n'' = q_n' - \widetilde{q}$ and $q'' = q' - \widetilde{q}$. Then $(q_n'', x_n') \in \pi^{-1}(Z)$ converges to (q'', x'). Moreover, $(q'', x') \in \pi^{-1}(Z)$ if and only if $(q', x') \in \pi^{-1}(Z)$ if and only if $(q, x) \in \pi^{-1}(Z)$.

Let $S = \{1, \ldots, 7\}$. Note that $\theta_S^7 \colon Q^7 \to \mathbb{R}^7$ is the identity map on Q^7. Therefore $\left(\phi_S^7\right)^{-1}(Z) = Q^7 \cap \pi^{-1}(Z)$. Thus the intersection $Q^7 \cap \pi^{-1}(Z)$ is closed in Q^7 and thus in $\mathbb{R}^7$. Since the sequence (q_n'', x_n'') lies in $Q^7 \cap \pi^{-1}(Z)$ we get that also its limit (q'', x'') is an element of $Q^7 \cap \pi^{-1}(Z) \subset \pi^{-1}(Z)$.

(6) Obvious, as we have only finitely many cells at every dimension.

□

With the cellular structure on M_f^7 given in Proposition 7.6 we get

Proposition 7.7. *The degree of the map* $d_{\{3,5\},\{3\}}$ *is* 1. *Therefore* $\partial_2(\{3,5\}) \neq 0$. *In particular,*

$$b_2\left(M_f^7\right) = \dim\left(H_2^{\mathbb{R}}\left(M_f^7\right)\right) \leq \dim\left(\ker\left(\partial_2\right)\right) < 21.$$

Proof. Below we identify $\mathbb{R}^7$ with $\mathbb{H} \times \mathbb{R}^3$. Note that X^0 consists of one point $[[\mathbf{0}], 0, 0, 0]$. Therefore $X^1 \big/ X^0 = X^1$. Now we describe the image of ∂Q^2 in X^1 under $\phi_{\{3,5\}}$. We have

$$\begin{aligned}\partial Q^2 = &\{\, (0,x) \,|\, 0 \leq x \leq 1\} \cup \{\, (x,1) \,|\, 0 \leq x \leq 1\} \\ &\cup \{\, (1,x) \,|\, 0 \leq x \leq 1\} \cup \{\, (x,0) \,|\, 0 \leq x \leq 1\}\end{aligned}$$

in $\mathbb{R}^2$. Now for all $0 \leq x \leq 1$

$$\phi^2_{\{3,5\}}(0,x) = [[\mathbf{0}], x, 0, 0] = \phi^1_{\{5\}}(x) \in S^1_{\{5\}}$$
$$\phi^2_{\{3,5\}}(x,1) = [[x\mathbf{j}], 1, 0, 0] = [[x\mathbf{j}(\mathbf{i})], 0, 0, 0] = [[x\mathbf{k}], 0, 0, 0] = \phi^1_{\{4\}}(x) \in S^1_{\{4\}}$$
$$\phi^2_{\{3,5\}}(1,x) = [[\mathbf{j}], x, 0, 0] = [[\mathbf{0}], x, 0, 0] = \phi^1_{\{5\}}(x) \in S^1_{\{5\}}$$
$$\phi^2_{\{3,5\}}(x,0) = [[x\mathbf{j}], 0, 0, 0] = \phi^1_{\{3\}}(x) \in S^1_{\{3\}}.$$

Therefore after composing $\phi_{\{3,5\}}$ with $q_{\{3\}}$ we get that for $0 \leq x \leq 1$

$$d_{\{3,5\}\{3\}}(0,x) = d_{\{3,5\}\{3\}}(x,1) = d_{\{3,5\}\{3\}}(1,x) = [[\mathbf{0}],0,0,0] \in S^1_{\{3\}}$$
$$d_{\{3,5\}\{3\}}(x,0) = [[x\mathbf{j}],0,0,0] \in S^1_{\{3\}}.$$

Now it is obvious that the degree of $d_{\{3,5\},\{3\}}$ is one. □

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