NORTH-HOLLAND

# The Inertia of Certain Hermitian Block Matrices

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ABSTRACT

We characterize sets of inertias of some partitioned Hermitian matrices by a system of inequalities involving the orders of the blocks, the inertias of the diagonal blocks, and the ranks of the nondiagonal blocks. The main result generalizes some well-known characterizations of Sá and Cain and others. © 1998 Elsevier Science Inc.

## 1. INTRODUCTION

Define the inertia of an $n \times n$ Hermitian matrix $H$ as the triple $\mathrm{In}(H) = (\pi, \nu, \delta)$, where $\pi$ is the number of positive eigenvalues, $\nu$ is the number of negative eigenvalues and $\delta = n - \pi - \nu$. We will simply write $(\pi, \nu, *)$ for the inertia of $H$, without any mention of the value of $\delta$.

We denote by $I_r$ the identity matrix of order $r$, and by $I$ the same matrix when we do not need to specify the order.

In [2] Cain and Sá characterized the inertia of a Hermitian skew-triangular $3 \times 3$ block matrix by a system of inequalities involving the orders of the blocks, the inertias of the diagonal blocks, and the ranks of the nondiagonal blocks.

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THEOREM 1.1 [2]. *Let us assume that $\pi_1, \nu_1, \pi_2, \nu_2, n_1, n_2, n_3$ are nonnegative and*

$$\pi_i + \nu_i \leqslant n_i \qquad \text{for} \quad i = 1, 2,$$

$$0 \leqslant r_{1i} \leqslant R_{1i} \leqslant \min\{n_1, n_i\} \qquad \text{for} \quad i = 2, 3.$$

*Then the following conditions are equivalent:*

(I) *For $i = 1, 2$, and $j = 2, 3$, there exist $n_i \times n_i$ Hermitian matrices $H_i$ and $n_1 \times n_j$ matrices $X_{1j}$ such that* $\operatorname{In}(H_i) = (\pi_i, \nu_i, *)$, $r_{1j} \leqslant \operatorname{rank} X_{1j} \leqslant R_{1j}$, *and*

$$H = \begin{bmatrix} H_1 & X_{12} & X_{13} \\ X_{12}^* & H_2 & 0 \\ X_{13}^* & 0 & 0 \end{bmatrix}$$

*has inertia* $(\pi, \nu, *)$.

(II) *Let $k \in \{1, 2\}$. Let $W_{kk}$ be any fixed $n_k \times n_k$ Hermitian matrix with inertia $(\pi_k, \nu_k, *)$.* (I) *holds with $H_k = W_{kk}$.*

(III) *Let $k \in \{2, 3\}$. Let $W_{1k}$ be any fixed $n_1 \times n_k$ matrix with $r_{1k} \leqslant$* $\operatorname{rank} W_{1k} \leqslant R_{1k}$. (I) *holds with $X_{1k} = W_{1k}$.*

(IV) *For $k = 1, 2$ let $W_{kk}$ be any fixed $n_k \times n_k$ Hermitian matrix with inertia $(\pi_k, \nu_k, *)$.* (I) *holds with $H_1 = W_{11}$ and $H_2 = W_{22}$.*

(V) *Let $W_{22}$ be any fixed $n_2 \times n_2$ Hermitian matrix with inertia $(\pi_2, \nu_2, *)$, and let $W_{13}$ be any fixed $n_1 \times n_3$ matrix with $r_{13} \leqslant \operatorname{rank} W_{13} \leqslant R_{13}$.* (I) *holds with $H_2 = W_{22}$ and $X_{13} = W_{13}$.*

(VI) *The following inequalities hold:*

$$\pi \geqslant \max\{\pi_1, \pi_2 + r_{13}, \pi_1 + \pi_2 - R_{12}, r_{12} - \nu_1, r_{12} - \nu_2\},$$

$$\nu \geqslant \max\{\nu_1, \nu_2 + r_{13}, \nu_1 + \nu_2 - R_{12}, r_{12} - \pi_1, r_{12} - \pi_2\},$$

$$\pi \leqslant \min\{n_1 + \pi_2, \pi_1 + n_2 + R_{13}, \pi_1 + \pi_2 + R_{12} + R_{13}\},$$

$$\nu \leqslant \min\{n_1 + \nu_2, \nu_1 + n_2 + R_{13}, \nu_1 + \nu_2 + R_{12} + R_{13}\},$$

$$\pi - \nu \leqslant \min\{\pi_1 + \pi_2, \pi_1 + \pi_2 + R_{12} - \nu_2\},$$

$$\nu - \pi \leqslant \min\{\nu_1 + \nu_2, \nu_1 + \nu_2 + R_{12} - \nu_2\},$$

$$\pi + \nu \geqslant \pi_1 + \pi_2 + \nu_1 + \nu_2 - R_{12},$$

$$\pi + \nu \leqslant \min\{n_1 + n_2 + R_{13}, n_1 + \pi_2 + \nu_2 + R_{12} + R_{13}, \\ \pi_1 + \nu_1 + n_2 + R_{12} + 2R_{13}\}.$$

In this work we will generalize this result by allowing a nonzero block in the (3, 3) entry. We will combine the tools used in [1], [2], and [3], with the Schur complement technique.

## 2. THE INERTIA OF SUMS OF SEVERAL HERMITIAN MATRICES

In this section we extend some results achieved by Sá in [6].

Let us assume the $n$, $p$, $\pi_i$, $\nu_i$, $n_i$, $r_i$, and $R_i$ are nonnegative integers such that $r_i \leqslant R_i \leqslant n_i \leqslant n$ for $i = 1, \dots, p$. We denote also by $\overline{\pi}_i$, $\overline{\nu}_i$ and $\overline{\rho}_i$ the nonnegative integers

$$\overline{\pi}_i = \min\{\pi_i, R_i\},$$

$$\overline{\nu}_i = \min\{\nu_i, R_i\},$$

$$\overline{\rho}_i = \min\{\pi_i + \nu_i, R_i\}$$

for $i = 1, \dots, p$.

THEOREM 2.1. *For $i = 1, \dots, p$, let $H_i$ be an $n_i \times n_i$ Hermitian matrix with inertia $(\pi_i, \nu_i, *)$. Then there exist matrices $S_i$ of dimensions $n \times n_i$ and $r_i \leqslant \operatorname{rank} S_i \leqslant R_i$, $i = 1, \dots, p$, such that*

$$\operatorname{In}(S_1 H_1 S_1^* + \cdots + S_p H_p S_p^*) = (\pi, \nu, *)$$

*if and only if (maximizing over $i \in \{1, \dots, p\}$) the following inequalities hold:*

$$\max_i \{\pi_i + \overline{\nu}_i + r_i - n_i\} - \sum_{t=1}^{p} \overline{\nu}_t \leqslant \pi \leqslant \sum_{t=1}^{p} \overline{\pi}_t,$$

$$\max_i \{\nu_i + \overline{\pi}_i + r_i - n_i\} - \sum_{t=1}^{p} \overline{\pi}_t \leqslant \nu \leqslant \sum_{t=1}^{p} \overline{\nu}_t,$$

$$\max_i \{\overline{\rho}_i + 2r_i - 2n_i + \pi_i + \nu_i\} - \sum_{t=1}^{p} \overline{\rho}_t \leqslant \pi + \nu \leqslant \sum_{t=1}^{p} \overline{\rho}_t,$$

$$\pi + \nu \leqslant n.$$

COROLLARY 2.2. *Let $H_1$ and $H_2$ be $n_i \times n_i$ Hermitian matrices with inertias $(\pi_i, \nu_i, *)$ for $i = 1, 2$. Then there exists a matrix $S$ of dimension $n_1 \times n_2$ and $r \leqslant \operatorname{rank} S \leqslant R$ such that*

$$\operatorname{In}(H_1 + SH_2S^*) = (\pi, \nu, *)$$

*if and only if the following inequalities hold:*

$$\pi \leqslant \min\{\pi_1 + \pi_2, \pi_1 + R\},$$

$$\nu \leqslant \min\{\nu_1 + \nu_2, \nu_1 + R\},$$

$$\pi \geqslant \max\{0, \pi_1 - \nu_2, \pi_1 - R, \pi_2 - \nu_1 + r - n_2\},$$

$$\nu \geqslant \max\{0, \nu_1 - \pi_2, \nu_1 - R, \nu_2 - \pi_1 + r - n_2\},$$

$$\pi + \nu \leqslant \min\{n_1, \pi_1 + \nu_1 + R\},$$

$$\pi + \nu \geqslant \pi_1 + \nu_1 - R.$$

## 3. THE MAIN RESULT

We present now the main result of this work.

THEOREM 3.1. *Let us assume that for $i = 1, 2, 3$, the quantities $\pi_i$, $\nu_i$, $n_i$ are nonnegative and*

$$\pi_i \geqslant 0, \quad \pi_i + \nu_i \leqslant n_i, \qquad i = 1, 2, 3,$$

$$0 \leqslant r_{1j} \leqslant R_{1j} \leqslant \min\{n_1, n_j\}, \qquad j = 2, 3.$$

*Then the following conditions are equivalent:*

(I) *For $i = 1, 2, 3$, and $j = 2, 3$, there exist $n_i \times n_i$ Hermitian matrices $H_i$ and $n_1 \times n_j$ matrices $X_{1j}$ such that* $\operatorname{In}(H_i) = (\pi_i, \nu_i, *)$, $r_{1j} \leqslant \operatorname{rank} X_{1j}$

$\leqslant R_{1j}$, *and*

$$H = \begin{bmatrix} H_1 & X_{12} & X_{13} \\ X_{12}^* & H_2 & 0 \\ X_{13}^* & 0 & H_3 \end{bmatrix}$$

*has inertia* $(\pi, \nu, *)$.

(II) *Let* $k \in \{1, 2, 3\}$. *Let* $W_{kk}$ *be any fixed* $n_k \times n_k$ *Hermitian matrix with inertia* $(\pi_k, \nu_k, *)$. (I) *holds with* $H_k = W_{kk}$.

(III) *Let* $k \in \{2, 3\}$. *Let* $W_{1k}$ *be any fixed* $n_1 \times n_k$ *matrix with* $r_{1k} \leqslant \operatorname{rank} W_{1k} \leqslant R_{1k}$. (I) *holds with* $X_{1k} = W_{1k}$.

(IV) *For* $k = 1, 2, 3$ *let* $W_{kk}$ *be any fixed* $n_k \times n_k$ *Hermitian matrix with inertia* $(\pi_k, \nu_k, *)$. (I) *holds with* $H_1 = W_{11}$, $H_2 = W_{22}$, *and* $H_3 = W_{33}$.

(V) *Let* $j, k \in \{2, 3\}$ *and* $j \neq k$. *Let* $W_{kk}$ *be any fixed* $n_k \times n_k$ *Hermitian matrix with inertia* $(\pi_k, \nu_k, *)$, *and let* $W_{1j}$ *be any fixed* $n_1 \times n_j$ *matrix with* $r_{1j} \leqslant \operatorname{rank} W_{1j} \leqslant R_{1j}$. (I) *holds with* $H_k = W_{kk}$ *and* $X_{1j} = W_{1j}$.

(VI) *The following inequalities hold*:

$$\begin{aligned}
\pi \geqslant \max\{&\pi_1, r_{13} - \nu_1, r_{12} - \nu_1, \\
&\pi_2 - \nu_1 + r_{13} - R_{12}, \pi_2 - \nu_3 + r_{13}, \\
&\pi_3 - \nu_1 + r_{12} - R_{13}, \pi_3 - \nu_2 + r_{12}, \\
&\pi_1 + \pi_2 - R_{12}, \pi_1 + \pi_3 - R_{13}, \pi_2 + \pi_3, \\
&\pi_1 + \pi_2 + \pi_3 - R_{12} - R_{13}\},
\end{aligned}$$

$$\begin{aligned}
\nu \geqslant \max\{&\nu_1, r_{13} - \pi_1, r_{12} - \pi_1, \\
&\nu_2 - \pi_1 + r_{13} - R_{12}, \nu_2 - \pi_3 + r_{13}, \\
&\nu_3 - \pi_1 + r_{12} - R_{13}, \nu_3 - \pi_2 + r_{12}, \\
&\nu_1 + \nu_2 - R_{12}, \nu_1 + \nu_3 - R_{13}, \nu_2 + \nu_3, \\
&\nu_1 + \nu_2 + \nu_3 - R_{12} - R_{13}\},
\end{aligned}$$

$$\begin{aligned}
\pi \leqslant \min\{&\pi_1 + n_2 + n_3, \\
&n_1 + \pi_2 + \pi_3, \pi_1 + n_2 + \pi_3 + R_{13}, \pi_1 + \pi_2 + n_3 + R_{12}, \\
&\pi_1 + \pi_2 + \pi_3 + R_{12} + R_{13}\},
\end{aligned}$$

$$\begin{aligned}
\nu \leqslant \min\{&\nu_1 + n_2 + n_3, \\
&n_1 + \nu_2 + \nu_3, \nu_1 + n_2 + \nu_3 + R_{13}, \nu_1 + \nu_2 + n_3 + R_{12}, \\
&\nu_1 + \nu_2 + \nu_3 + R_{12} + R_{13}\},
\end{aligned}$$

$$\begin{aligned}
\pi - \nu \leqslant \min\{&\pi_1 + \pi_2 + \pi_3, \\
&\pi_1 + \pi_2 + \pi_3 - \nu_2 + R_{12}, \pi_1 + \pi_2 + \pi_3 - \nu_3 + R_{13}\}
\end{aligned}$$

$$\nu - \pi \leqslant \min\{\nu_1 + \nu_2 + \nu_3,$$

$$\nu_1 + \nu_2 + \nu_3 - \pi_2 + R_{12}, \nu_1 + \nu_2 + \nu_3 - \pi_3 + R_{13}\},$$

$$\pi + \nu \geqslant \max\{\pi_1 + \nu_1 + \pi_2 + \nu_2 - R_{12}, \pi_1 + \nu_1 + \pi_3 + \nu_3 - R_{13},$$

$$\pi_1 + \nu_1 + \pi_2 + \nu_2 + \pi_3 + \nu_3 - R_{12} - R_{13},$$

$$\pi_2 + \nu_2 - \pi_1 - \nu_1 + 2r_{13} - R_{12},$$

$$\pi_3 + \nu_3 - \pi_1 - \nu_1 + 2r_{12} - R_{13}\},$$

$$\pi + \nu \leqslant \min\{n_1 + n_2 + n_3,$$

$$n_1 + n_2 + \pi_3 + \nu_3 + R_{13},$$

$$n_1 + \pi_2 + \nu_2 + n_3 + R_{12},$$

$$\pi_1 + \nu_1 + n_2 + n_3 + R_{12} + R_{13},$$

$$n_1 + \pi_2 + \nu_2 + \pi_3 + \nu_3 + R_{12} + R_{13},$$

$$\pi_1 + \nu_1 + n_2 + \pi_3 + \nu_3 + R_{12} + 2R_{13},$$

$$\pi_1 + \nu_1 + \pi_2 + \nu_2 + n_3 + 2R_{12} + R_{13}\}.$$

Of course this theorem can easily be adapted to the two other different prescribed $3 \times 3$ block decomposition of a Hermitian matrix $H$, when two of the nondiagonal blocks are zero, i.e., in the case

$$H = \begin{bmatrix} H_1 & 0 & X_{13} \\ 0 & H_2 & X_{23} \\ X_{13}^* & X_{23}^* & H_3 \end{bmatrix},$$

and when the decomposition is tridiagonal

$$H = \begin{bmatrix} H_1 & X_{12} & 0 \\ X_{12}^* & H_2 & X_{23} \\ 0 & X_{23}^* & H_3 \end{bmatrix}.$$

*Proof.* The proof that (I) is equivalent to each of (II)–(V) is the same as one can find in the proof of the Theorem 2.1 in [2]. We include this part of the proof for completeness.

It is obvious that each of (II)–(V) implies (I). Suppose now that $H$ satisfies (I). Let $M$ be a block diagonal matrix $M_1 \oplus M_2 \oplus M_3$, where $M_i$ denotes an $n_i \times n_i$ invertible matrix. For $i = 1, 2, 3$ and $j = 2, 3$ set $Y_{ii} = M_i^* H_i M_i$, $Y_{1j} = M_1^* X_{1j} M_j$, and $Y_{23} = 0$. We have $Y = (Y_{ij})_{i,j} = M^*HM$. Then rank $Y_{1j}$ = rank $X_{1j}$, and by Sylvester's theorem $\mathrm{In}(Y) = \mathrm{In}(H)$ and $\mathrm{In}(Y_{ii}) = \mathrm{In}(H_i)$. Thus $Y$ has all the rank and inertia properties required in (II)–(V). In each of these cases the only additional requirement is that, for certain $i, j$, $M_1^* X_{1j} M_j = W_{ij}$ and $M_i^* H_i M_i = W_{ii}$. Such $M_i$'s can always be found [5].

Let us prove that (II) is equivalent to (VI). Assume that $r_{13} = R_{13} = r$. We set

$$H_3 = \begin{bmatrix} \tilde{H}_3 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \text{where} \quad \tilde{H}_3 = \begin{bmatrix} I_{\pi_3} & 0 \\ 0 & -I_{\nu_3} \end{bmatrix}.$$

Our choice of $H_3$ allows us to partition $H$ as

$$H = \begin{bmatrix} H_1 & X_{12} & Y & Z \\ X_{12}^* & H_2 & 0 & 0 \\ Y^* & 0 & \tilde{H}_3 & 0 \\ Z^* & 0 & 0 & 0 \end{bmatrix},$$

where $[Y \;\; Z] = X_{13}$. Let $s$ be the rank of Z. There exist nonsingular matrices, say $U$ and $V$, such that

$$UZV = \begin{bmatrix} 0 & I_s \\ 0 & 0 \end{bmatrix}.$$

Let us define the matrix

$$H' = (U \oplus I \oplus I \oplus V^*)H(U^* \oplus I \oplus I \oplus V)$$

$$= \begin{bmatrix} UH_1U^* & UX_{12} & UY & \begin{matrix} 0 & I_s \\ 0 & 0 \end{matrix} \\ (UX_{12})^* & H_2 & 0 & 0 \\ (UY)^* & 0 & \tilde{H}_3 & 0 \\ \begin{matrix} 0 & 0 \\ I_s & 0 \end{matrix} & 0 & 0 & 0 \end{bmatrix},$$

which is conjunctive to $H$, so $\text{In}(H) = \text{In}(H')$. Note that for the same reason $\text{In}(H_1) = \text{In}(UH_1U^*)$.

Let us make a new partition of $H'$ in the following way:

$$H' = \begin{bmatrix} * & * & * & * & 0 I_s \\ * & \tilde{H}_1 & \tilde{X}_{12} & \tilde{X}_{13} & 0 0 \\ * & \tilde{X}_{12}^* & H_2 & 0 & 0 \\ * & \tilde{X}_{13}^* & 0 & \tilde{H}_3 & 0 \\ \begin{matrix} 0 \\ I_s \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & 0 & 0 & 0 \end{bmatrix}.$$

Then applying the Schur complement technique, $H'$ is conjunctive to $H''$ defined by

$$H'' = \begin{bmatrix} * & * & * & * & 0 \quad I_s \\ * & \tilde{H}_1 - \tilde{X}_{13}\tilde{H}_3^{-1}\tilde{X}_{13}^* & \tilde{X}_{12} & 0 & 0 \\ * & \tilde{X}_{12}^* & H_2 & 0 & 0 \\ * & 0 & 0 & \tilde{H}_3 & 0 \\ \begin{matrix} 0 \\ I_s \end{matrix} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Applying a corollary and a lemma of [4], we get

$$\operatorname{In}(H) = \operatorname{In}(H'') = (s, s, 0) + (\pi_3, \nu_3, 0) + \operatorname{In}(\overline{H}),$$

where

$$\overline{H} = \begin{bmatrix} \tilde{H}_1 - \tilde{X}_{13}\tilde{H}_3^{-1}\tilde{X}_{13}^* & \tilde{X}_{12} \\ \tilde{X}_{12}^* & H_2 \end{bmatrix}.$$

Now, describing the inertias $(\pi, \nu, 0)$ of $H$ is equivalent to describing the inertias $(\overline{\pi}, \overline{\nu}, *) = (\pi' - s, \nu' - s, *)$ of $\overline{H}$, where $\pi' = \pi - \pi_3$ and $\nu' = \nu - \nu_3$.

Applying the Claim of [2], as $X_{13}$ varies over the set of $n_1 \times n_3$ rank $r$ matrices with rank $Z = s$, the matrices $\tilde{X}_{13}$ form the set of all $(n_1 - s) \times (\pi_3 + \nu_3)$ matrices of rank $r - s$. On other hand, we easily prove that as $X_{12}$ varies over the set of $n_1 \times n_2$ matrices $X$ such that $r_{12} \leqslant \operatorname{rank} X \leqslant R_{12}$, $\tilde{X}_{12}$ ranges over all $(n_1 - s) \times n_2$ matrices $X$ such that $r_{12} - s \leqslant \operatorname{rank} X \leqslant R_{12}$. Hence we may apply Theorem 2.1 of [2]. According to it, $(\pi' - s, \nu' - s, *)$ will be the inertia of $\overline{H}$ for some (1, 2) and (2, 2) blocks, with the (1, 1) block fixed, if and only if

$$\pi' + \nu' \leqslant n_1 + n_2 + s,$$

$$\max\{x, \pi_2\} \leqslant \pi' - s \leqslant \min\{n_1 - s + \pi_2, x + n_2\},$$

$$\max\{y, \nu_2\} \leqslant \nu' - s \leqslant \min\{n_1 - s + \nu_2, y + n_2\},$$

$$\pi' - \nu' \leqslant x + \pi_2,$$

$$\nu' - \pi' \leqslant y + \nu_2,$$

$$r_{12} \leqslant \min\{\pi' + y, \pi' + \nu_2, \nu' + x, \nu' + \pi_2\},$$

$$R_{12} \geqslant \max\{|x + \pi_2 - \pi' + s|, |y + \nu_2 - \nu' + s|\},$$

$$R_{12} + 2s \geqslant \pi' + \nu' - \min\{x + y + n_2, \pi_2 + \nu_2 + n_1 - s\},$$

$$R_{12} \geqslant x + y + \pi_2 + \nu_2 - \pi' - \nu' + 2s, \tag{3.1}$$

where $(x, y, *)$ is the inertia of $\tilde{H}_1 - \tilde{X}_{13}\tilde{H}_3^{-1}\tilde{X}_{13}^*$.

We note that $s = \operatorname{rank} Z$ varies as

$$\max\{0, r - \pi_3 - \nu_3\} \leqslant s \leqslant \min\{n_3 - \pi_3 - \nu_3, r\},$$

and, since $r_{13} \leqslant r \leqslant R_{13}$ and $r - s = \operatorname{rank} \tilde{X}_{13}$, eliminating $r$, we conclude

$$\max\{0, r_{13} - \pi_3 - \nu_3\} \leqslant s \leqslant \min\{n_3 - \pi_3 - \nu_3, R_{13}\}$$

and

$$r_{13} - s \leqslant \operatorname{rank} \tilde{X}_{13} \leqslant R_{13} - s.$$

According to the Corollary 2.2, the set of inertias $(x, y, *)$ when $\operatorname{In}(\tilde{H}_1) = (\tilde{\pi}_1, \tilde{\nu}_1, *)$, $\operatorname{In}(-\tilde{H}_3^{-1}) = (\nu_3, \pi_3, *)$, and $r_{13} - s \leqslant \operatorname{rank} \tilde{X}_{13} \leqslant R_{13} - s$ is characterized by

$$
\begin{aligned}
x &\leqslant \min\{\tilde{\pi}_1 + \nu_3, \tilde{\pi}_1 + R_{13} - s\},\\
y &\leqslant \min\{\tilde{\nu}_1 + \pi_3, \tilde{\nu}_1 + R_{13} - s\},\\
x &\geqslant \max\{0, \tilde{\pi}_1 - \pi_3, \tilde{\pi}_1 - R_{13} + s, -\pi_3 - \tilde{\nu}_1 + r_{13} - s\},\\
y &\geqslant \max\{0, \tilde{\nu}_1 - \nu_3, \tilde{\nu}_1 - R_{13} + s, -\nu_3 - \tilde{\pi}_1 + r_{13} - s\},\\
x + y &\leqslant \min\{n_1 - s, \tilde{\pi}_1 + \tilde{\nu}_1 + R_{13} - s\},\\
x + y &\geqslant \tilde{\pi}_1 + \tilde{\nu}_1 - R_{13} + s, \qquad (3.2)
\end{aligned}
$$

while, by Theorem 1 of [6], the set of inertias $(\tilde{\pi}_1, \tilde{\nu}_1, *)$ which arise as $H_1$ varies is characterized by

$$
\begin{aligned}
\max\{0, \pi_1 - s\} &\leqslant \tilde{\pi}_1 \leqslant \pi_1,\\
\max\{0, \nu_1 - s\} &\leqslant \tilde{\nu}_1 \leqslant \nu_1\\
\tilde{\pi}_1 + \tilde{\nu}_1 &\leqslant n_1 - s. \qquad (3.3)
\end{aligned}
$$

We know now that $(x, y, *)$ is the inertia of $\tilde{H}_1 - \tilde{X}_{13}\tilde{H}_3^{-1}\tilde{X}_{13}^*$ if and only if there exist integers $\tilde{\pi}_1$ and $\tilde{\nu}_1$ satisfying (3.2) and (3.3). We combine these two sets of inequalities to get

$$a \leqslant \tilde{\pi}_1 \leqslant A,$$

$$b \leqslant \tilde{\nu}_1 \leqslant B,$$

$$c \leqslant \tilde{\pi}_1 + \tilde{\nu}_1 \leqslant C, \tag{3.4}$$

Where

$$a = \max\{0, \pi_1 - s, x - \nu_3, x - R_{13} + s, -\nu_3 - y + r_{13} - s\},$$

$$b = \max\{0, \nu_1 - s, y - \pi_3, y - R_{13} + s, -\pi_3 - x + r_{13} - s\},$$

$$c = x + y - R_{13} + s,$$

$$A = \min\{\pi_1, x + \pi_3, x + R_{13} - s\},$$

$$B = \min\{\nu_1, y + \nu_3, y + R_{13} - s\},$$

$$C = \min\{n_1 - s, x + y + R_{13} - s\}.$$

Then there is an integral solution $\tilde{\pi}_1$ and $\tilde{\nu}_1$ to (3.4) if and only if

$$a \leqslant A, \qquad b \leqslant B, \qquad c \leqslant C, \qquad a + b \leqslant C, \qquad c \leqslant A + B. \tag{3.5}$$

Eliminating redundant inequalities from (3.5) (e.g., some inequalities are redundant by the $\pi\nu$ duality) gives rise to $7 + 7 + 1 + 21 + 3 = 39$ inequalities, which can be reduced to

$$x \leqslant \min\{\pi_1 + \nu_3, n_1 + \nu_3 - s, \pi_1 + R_{13} - s\},$$

$$y \leqslant \min\{\nu_1 + \pi_3, n_1 + \pi_3 - s, \nu_1 + R_{13} - s\},$$

$$x \geqslant \max\{0, \pi_1 - \pi_3 - s, \pi_1 - R_{13}, -\pi_3 - \nu_1 + r_{13} - s\},$$

$$y \geqslant \max\{0, \nu_1 - \nu_3 - s, \nu_1 - R_{13}, -\nu_3 - \pi_1 + r_{13} - s\},$$

$$x + y \leqslant \min\{n_1 - s, \pi_1 + \nu_1 + R_{13} - s\},$$

$$x + y \geqslant \pi_1 + \nu_1 - R_{13} - s. \tag{3.6}$$

Using the same idea we have used before, we will eliminate $x$ and $y$. We know that $(\pi' + \pi_3, \nu' + \nu_3, *)$ is the inertia of $H$ if and only if there exist integers $x$ and $y$ satisfying (3.1) and (3.6). Again, we combine these two sets of inequalities to get

$$a \leqslant x \leqslant A,$$

$$b \leqslant y \leqslant B,$$

$$c \leqslant x + y \leqslant C,$$

and some inequalities not involving $x$ or $y$, where

$$a = \max\{0, \pi_1 - \pi_3 - s, \pi_1 - R_{13}, -\pi_3 - \nu_1 + r_{13} - s,$$
$$\pi' - s - n_2, \pi' - \nu' - \pi_2, r_{12} - \nu', \pi' - R_{12} - \pi_2 - s\},$$

$$b = \max\{0, \nu_1 - \nu_3 - s, \nu_1 - R_{13}, -\nu_3 - \pi_1 + r_{13} - s,$$
$$\nu' - s - n_2, \nu' - \pi' - \nu_2, r_{12} - \pi', \nu' - R_{12} - \nu_2 - s\},$$

$$c = \max\{\pi_1 + \nu_1 - R_{13} - s, \pi' + \nu' - n_2 - R_{12} - 2s\},$$

$$A = \min\{\pi_1 + \nu_3, \pi_1 + R_{13} - s, n_1 + \nu_3 - s, \pi' - s, \pi' + R_{12} - \pi_2 - s\},$$

$$B = \min\{\nu_1 + \pi_3, \nu_1 + R_{13} - s, n_1 + \pi_3 - s, \nu' - s, \nu' + R_{12} - \nu_2 - s\}$$

$$C = \min\{n_1 - s, \pi_1 + \nu_1 + R_{13} - s, \pi' + \nu' - \pi_2 - \nu_2 + R_{12} - 2s\}.$$

When the redundancies have been eliminated we have

$$\pi_i \geqslant 0, \quad \pi_i + \nu_i \leqslant n_i, \qquad i = 1, 2, 3,$$

$$0 \leqslant r_{1j} \leqslant R_{1j} \leqslant \min\{n_1, n_j\}, \qquad j = 2, 3,$$

$$\pi \geqslant \max\{\pi_1, r_{13} - \nu_1, r_{12} - \nu_1,$$
$$\pi_2 - \nu_1 + r_{13} - R_{12}, \pi_3 - \nu_1 + r_{12} - R_{13} + s,$$
$$\pi_3 - \nu_2 + r_{12}, \pi_2 + \pi_3 + s,$$
$$\pi_1 + \pi_2 - R_{12}, \pi_1 + \pi_3 - R_{13} + s, \pi_1 + \pi_2 + \pi_3$$
$$-R_{12} - R_{13} + s\},$$

$$\nu \geqslant \max\{\nu_1, r_{13} - \pi_1, r_{12} - \pi_1,$$
$$\nu_2 - \pi_1 + r_{13} - R_{12}, \nu_3 - \pi_1 + r_{12} - R_{13} + s,$$
$$\nu_3 - \pi_2 + r_{12}, \nu_2 + \nu_3 + s,$$
$$\nu_1 + \nu_2 - R_{12}, \nu_1 + \nu_3 - R_{13} + s, \nu_1 + \nu_2 + \nu_3 - R_{12} - R_{13} + s\},$$

$$\pi \leqslant \min\{n_1 + n_2 + \pi_3, \pi_1 + n_2 + \pi_3 + \nu_3 + s,$$
$$\pi_1 + n_2 + \pi_3 + R_{13}, \pi_1 + \pi_2 + \pi_3 + \nu_3 + R_{12} + s,$$
$$\pi_1 + \pi_2 + \pi_3 + R_{12} + R_{13}\},$$

$$\nu \leqslant \min\{n_1 + n_2 + \nu_3, \pi_1 + n_2 + \pi_3 + \nu_3 + s,$$
$$\pi_1 + n_2 + \nu_3 + R_{13}, \nu_1 + \nu_2 + \pi_3 + \nu_3 + R_{12} + s,$$
$$\nu_1 + \nu_2 + \nu_3 + R_{12} + R_{13}\},$$

$$\pi - \nu \leqslant \min\{\pi_1 + \pi_2 + \pi_3,$$
$$\pi_1 + \pi_2 + \pi_3 - \nu_3 + R_{13} - s\},$$

$$\nu - \pi \leqslant \min\{\nu_1 + \nu_2 + \nu_3,$$
$$\nu_1 + \nu_2 + \nu_3 - \pi_3 + R_{13} - s\},$$

$$\pi + \nu \geqslant \max\{\pi_1 + \nu_1 + \pi_2 + \nu_2 - R_{12},$$
$$\pi_1 + \nu_1 + \pi_3 + \nu_3 - R_{13} + s,$$
$$\pi_1 + \nu_1 + \pi_2 + \nu_2 + \pi_3 + \nu_3 - R_{12} - R_{13} + s,$$
$$\pi_2 + \nu_2 - \pi_1 - \nu_1 + 2r_{13} - R_{12},$$
$$\pi_3 + \nu_3 - \pi_1 - \nu_1 + 2r_{12} - R_{13} + s\},$$

$$\pi + \nu \leqslant \min\{n_1 + n_2 + \pi_3 + \nu_3 + s,$$
$$n_1 + \pi_2 + \nu_2 + \pi_3 + \nu_3 + R_{12} + s,$$
$$\pi_1 + \nu_1 + n_2 + \pi_3 + \nu_3 + R_{12} + R_{13} + s,$$
$$\pi_1 + \nu_1 + \pi_2 + \nu_2 + \pi_3 + \nu_3 + 2R_{12} + R_{13} + s,$$
$$\pi_1 + \nu_1 + n_2 + \pi_3 + \nu_3 + R_{12} + 2R_{13}\}.$$

Now we get a system of inequalities of the type

$$\mathcal{S}, \qquad d \leqslant s \leqslant D,$$

where $\mathcal{S}$ is a subsystem of inequalities not involving $s$, and $d$ and $D$ are defined below:

$$\begin{aligned} d = \max\{0,\, & r_{13} - \pi_3 - \nu_3, \\ & \pi + \nu - \pi_3 - \nu_3 - n_1 - n_2, \\ & \pi + \nu - \pi_3 - \nu_3 - n_1 - \pi_2 - \nu_2 - R_{12}, \\ & \pi - \pi_3 - \nu_3 - \pi_1 - n_2, \nu - \pi_3 - \nu_3 - \nu_1 - n_2, \\ & \pi - \pi_3 - \nu_3 - \pi_1 - \pi_2 - R_{12}, \nu - \pi_3 - \nu_3 - \nu_1 - \nu_2 - R_{12}, \\ & \pi + \nu - \pi_3 - \nu_3 - \pi_1 - \nu_1 - n_2 - R_{12} - R_{13}, \\ & \pi + \nu - \pi_3 - \nu_3 - \pi_1 - \nu_1 - \pi_2 - \nu_2 - 2R_{12} - R_{13}\}, \end{aligned}$$

$$\begin{aligned} D = \min\{ & n_3 - \pi_3 - \nu_3, R_{13}, \\ & \pi - \pi_3 - \pi_2, \nu - \nu_3 - \nu_2, \\ & \pi - \nu - \pi_3 + \nu_1 + \nu_2 + \nu_3 + R_{13}, \\ & \nu - \pi - \nu_3 + \pi_1 + \pi_2 + \pi_3 + R_{13}, \\ & \pi - \pi_3 + \nu_1 - r_{12} + R_{13}, \nu - \nu_3 + \pi_1 - r_{12} + R_{13}, \\ & \pi - \pi_1 - \pi_3 + R_{13}, \nu - \nu_1 - \nu_3 + R_{13}, \\ & \pi - \pi_1 - \pi_2 - \pi_3 + R_{12} + R_{13}, \nu - \nu_1 - \nu_2 - \nu_3 + R_{12} + R_{13}, \\ & \pi + \nu - \pi_1 - \nu_1 - \pi_2 - \nu_2 - \pi_3 - \nu_3 + R_{12} + R_{13}, \\ & \pi + \nu + \pi_1 + \nu_1 - \pi_3 - \nu_3 - 2r_{12} + R_{13}, \\ & \pi + \nu - \pi_1 - \nu_1 - \pi_3 - \nu_3 + R_{13}\}. \end{aligned}$$

Finally, eliminating $s$, we prove equivalence between (II) and the inequalities defined in (VI). ■

## 4. A GENERALIZATION

In this final section we generalize the Theorem 3.1 to the decompositions of $H$ of the type

$$H = \begin{bmatrix} H_1 & X_{12} & X_{13} & \cdots & X_{1p} \\ H_{12}^* & H_2 & 0 & \cdots & 0 \\ X_{13}^* & 0 & H_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{1p}^* & 0 & 0 & \cdots & H_p \end{bmatrix}. \tag{4.1}$$

Let us define

$$\Pi_p = \{0, \pi_2 - R_{12}, \ldots, \pi_p - R_{1p}\},$$

$$\Omega_p = \{0, \nu_2 - R_{12}, \ldots, \nu_p - R_{1p}\},$$

and, for $k = 1, \ldots, p - 1$,

$$\Sigma_k \Pi_p = \left\{ \sum_{a \in P} a \,\middle|\, P \subset \Pi_p \text{ and } \#P = k \right\},$$

$$\Sigma_k \Omega_p = \left\{ \sum_{a \in P} a \,\middle|\, P \subset \Omega_p \text{ and } \#P = k \right\}.$$

The symbol $r_p$ will denote the set $\{r_{12}, \ldots, r_{1p}\}$.
We define $\Gamma_p$ as the set

$$\left\{ \sum_{i=2}^{p} a_i \,\middle|\, a_i \in \{0, \pi_i + \nu_i - R_{1i}\} \right\} \setminus \{0\}.$$

The symbol $\Delta_p$ will be used to denote the set

$$\left\{ \sum_{i=2}^{p} a_i \,\middle|\, a_i \in \{n_i, \pi_i + \nu_i + R_{1i}\} \right\},$$

and $\Delta_p \setminus \{\sum_{i=2}^{p}(\pi_i + \nu_i + R_{1i})\}$ will be represented by $\Delta'_p$. The set

$$\left\{ r_{1j} - \nu_j + \sum_{i \in C} \pi_i \,\middle|\, C \subset \{2, \dots, p\},\ \#C = p - 2, \text{ and } j \in \{2, \dots, p\} \setminus C \right\}$$

is represented by $\Upsilon_p$, and $\Lambda_p$ represents the set

$$\left\{ r_{1j} - \pi_j + \sum_{i \in C} \nu_i \,\middle|\, C \subset \{2, \dots, p\},\ \#C = p - 2, \text{ and } j \in \{2, \dots, p\} \setminus C \right\}.$$

Finally, $\Phi_p$ denotes the set

$$\left\{ \sum_{i=2}^{p} a_i \,\middle|\, a_i \in \{n_i, \pi_i + R_{1i}\} \right\}$$

and, by $\pi\nu$ duality, $\Psi_p$ denotes the set

$$\left\{ \sum_{i=2}^{p} a_i \,\middle|\, a_i \in \{n_i, \nu_i + R_{1i}\} \right\}.$$

Now we are ready to state the result of this section.

THEOREM 4.1. *Let us assume that all symbols represent nonnegative integers and*

$$\pi_i \geqslant 0, \quad \pi_i + \nu_i \leqslant n_i, \qquad i = 1, \dots, p,$$

$$0 \leqslant r_{1j} \leqslant R_{1j} \leqslant \min\{n_1, n_j\}, \qquad j = 2, \dots, p.$$

*Then the following conditions are equivalent:*

(I) *For* $i = 1, \dots, p$ *and* $j = 2, \dots, p$, *there exist* $n_i \times n_i$ *Hermitian matrices* $H_i$ *and* $n_i \times n_j$ *matrices* $X_{1j}$ *such that* $\mathrm{In}(H_i) = (\pi_i, \nu_i, *)$, $r_{1j} \leqslant \operatorname{rank} X_{1j} \leqslant R_{1j}$, *and* $H$ *defined in* (4.1) *has inertia* $(\pi, \nu, *)$.

(II) *For* $k = 1, \dots, p$ *let* $W_{kk}$ *be any fixed* $n_k \times n_k$ *Hermitian matrix with inertia* $(\pi_k, \nu_k, *)$. (I) *holds with* $H_1 = W_{11}, \dots,$ *and* $H_p = W_{pp}$.

(III) *The following inequalities hold*:

$$\pi \geqslant \max\left\{\sum_{i=2}^{p}\pi_i, \Upsilon_p, r_p - \nu_1 + \Sigma_1\Pi_p, \dots, r_p - \nu_1 + \Sigma_{p-2}\Pi_p,\right.$$

$$\left.\pi_1 + \Sigma_1\Pi_p, \dots, \pi_1 + \Sigma_{p-1}\Pi_p\right\},$$

$$\nu \geqslant \max\left\{\sum_{i=2}^{p}\nu_i, \Lambda_p, r_p - \pi_1 + \Sigma_1\Omega_p, \dots, r_p - \pi_1 + \Sigma_{p-2}\Omega_p,\right.$$

$$\left.\nu_1 + \Sigma_1\Omega_p, \dots, \nu_1 + \Sigma_{p-1}\Omega_p\right\},$$

$$\pi \leqslant \min\left\{n_1 + \sum_{i=2}^{p}\pi_i, \pi_1 + \Phi_p\right\},$$

$$\nu \leqslant \min\left\{n_1 + \sum_{i=2}^{p}\nu_i, \nu_1 + \Psi_p\right\},$$

$$\pi - \nu \leqslant \min\left\{\sum_{i=1}^{p}\pi_i - \Sigma_1\Omega_p, \dots, \sum_{i=1}^{p}\pi_i - \Sigma_{p-2}\Omega_p\right\},$$

$$\nu - \pi \leqslant \min\left\{\sum_{i=1}^{p}\nu_i - \Sigma_1\Pi_p, \dots, \sum_{i=1}^{p}\nu_i - \Sigma_{p-2}\Pi_p\right\},$$

$$\pi + \nu \geqslant \max\{\pi_1 + \nu_1 + \Gamma_p, 2r_p + \Gamma_p - \pi_1 - \nu_1\},$$

$$\pi + \nu \leqslant \min\left\{n_1 + \Delta_p, \pi_1 + \nu_1 + \sum_{i=2}^{p}R_{1i} + \Delta'_p\right\}.$$

*Proof*. The proof is done by induction on $p$. ■

REMARK. Concerning (III), there occur some redundant inequalities. For instance, we have $\pi \geqslant r_{12} - \nu_1 + \pi_2 - R_{12} + \pi_3 - R_{13}$, which is clearly redundant, since $\pi \geqslant \Sigma_{i=2}^{p}\pi_i$ and $r_{12} \leqslant R_{12}$. Moreover, this phenomenon is even more general, since when $\pi_i - R_{1i}$ or $\nu_i - R_{1i}$ and $r_{1i}$ occur simultaneously in the same inequality, that inequality is redundant.

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REFERENCES

1 B. E. Cain and E. Marques de Sá, The inertia of a Hermitian matrix having prescribed complementary principal submatrices, *Linear Algebra Appl.* 37:161–171 (1981).
2 B. E. Cain and E. Marques de Sá, The inertia of Hermitian matrices with a prescribed 2 × 2 block decomposition, *Linear and Multilinear Algebra* 31:119–130 (1992).
3 B. E. Cain and E. Marques de Sá, The inertia of certain skew-triangular block matrices, *Linear Algebra Appl.* 160:75–85 (1992).
4 E. V. Haynsworth and A. M. Ostrowski, On the inertia of some classes of partitioned matrices, *Linear Algebra Appl.* 1:299–316 (1968).
5 S. Perlis, *Theory of Matrices*, Addison-Wesley, Reading, Mass., 1952.
6 E. Marques de Sá, On the inertia of sums of Hermitian matrices, *Linear Algebra Appl.* 37:143–159 (1981).

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