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LINEAR ALGEBRA AND ITS APPLICATIONS

On the number of invariant polynomials of the product of matrices with prescribed similarity classes

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Abstract

We study the possibilities for the number of nontrivial invariant polynomials of the product of two nonsingular matrices, with prescribed similarity classes, over an algebraically closed field.

1. Introduction

Let F be an algebraically closed field. For $A \in F^{n\times n}$, denote by $i(A)$ the number of nontrivial invariant polynomials of A.

In this paper, we study the range of $i(XAX^{-1}YBY^{-1})$, when A and B are given $n \times n$ nonsingular matrices over F and X, Y run over the set of nonsingular matrices over F.

Define

$$R(A) = \min_{\lambda\in F} \operatorname{rank}(A + \lambda I_n),$$

where I_n is the $n \times n$ identity matrix. In [1], it was proved that

$$i(A) = n - R(A).$$

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Thus the study of the range of $i(A'B')$ is equivalent to the study of the range of $R(A'B')$ with A' and B' similar to A and B, respectively. In this paper, instead of $i(A'B')$, we prefer to consider $R(A'B')$.

If X and Y are n-square invertible matrices over F, then $XAX^{-1}YBY^{-1}$ is similar to $(Y^{-1}X)A(X^{-1}Y)B$ and to $A(X^{-1}Y)B(Y^{-1}X)$, so our problem is equivalent to studying the range of $i(A'B)$ or $i(AB')$, with A' and B' similar to A and B, respectively.

Since a square matrix is similar to its transpose, the problem is also equivalent to studying the range of $i(B'A')$, with A' and B' similar to A and B, respectively.

For any polynomial $f(x)$ over F, we denote by $d(f)$ the degree of $f(x)$. Given two polynomials $f(x)$ and $g(x)$, we write $f(x)\,|\,g(x)$ whenever $f(x)$ divides $g(x)$.

Let $\alpha_1(x), \alpha_2(x), \ldots, \alpha_n(x)$ and $\beta_1(x), \beta_2(x), \ldots, \beta_n(x)$ be the invariant polynomials of A and B, respectively, and let $\gamma_1(x,\lambda), \gamma_2(x,\lambda), \ldots, \gamma_n(x,\lambda)$ be the invariant polynomials of λB^{-1}. We assume that the invariant polynomials are always monic and have been ordered so that each one divides the following of its group.

It is easy to see that if $\beta_i(x) = (x-b_1)^{\epsilon_1}(x-b_2)^{\epsilon_2}\cdots(x-b_p)^{\epsilon_p}$, then

$$\gamma_i(x,\lambda) = \left(x-\frac{\lambda}{b_1}\right)^{\epsilon_1}\left(x-\frac{\lambda}{b_2}\right)^{\epsilon_2}\cdots\left(x-\frac{\lambda}{b_p}\right)^{\epsilon_p}.$$

Let $r(=i(A))$ be the number of invariant polynomials of A which are different from 1. In the same manner, let $s := i(B)$. This means that $\alpha_1(x) = \cdots = \alpha_{n-r}(x) = 1$, and $\alpha_{n-r+1}(x)$ has degree at least one. Similarly, $\beta_1(x) = \cdots = \beta_{n-s}(x) = 1$ and $\beta_{n-s+1}(x)$ has degree at least one.

Given a monic polynomial $f(x) = x^k - a_k x^{k-1} - \cdots - a_2 x - a_1$ with degree $k \geqslant 1$, we denote by $C(f)$ and $C'(f)$ the *companion matrices* of $f(x)$, defined by

$$C(f) = [e_2, e_3, \ldots, e_k, a] \quad \text{and} \quad C'(f) = [a', e_1, \ldots, e_{k-1}],$$

where e_i is the ith column of the k-identity matrix, $i \in \{1,\ldots,k\}$ and

$$a = [a_1, a_2, \ldots, a_k]^{\mathrm{t}}, \quad a' = [a_k, a_{k-1}, \ldots, a_1]^{\mathrm{t}},$$

where the superscript "t" means transpose.

Let us define $f_i(x) := \alpha_{i+n-r}(x)$, and $g_j(x) := \beta_{i+n-s}(x)$ for $i = 1,\ldots,r$ and $j = 1,\ldots,s$, and let $h_i(x,\lambda)$, $i = 1,\ldots,s$ be the nontrivial invariant polynomials of λB^{-1}.

We take $K_i = C(f_i)$, $i = 1,\ldots,r$, $L_j = C'(g_j)$, $j = 1,\ldots,s$, and define $K = K_1 \oplus \cdots \oplus K_r$ and $L = L_1 \oplus \cdots \oplus L_s$. The matrices K and L are respectively similar to A and B.

We say that the pair (A,B) is *spectrally complete for the product* if for any n-tuple $(\lambda_1,\ldots,\lambda_n)$ of elements of F satisfying $\lambda_1 \ldots \lambda_n = \det(AB)$, there exist matrices A' and B' similar to A and B, respectively, such that $A'B'$ has eigenvalues $(\lambda_1,\ldots,\lambda_n)$.

In [2], Silva characterized all such pairs when F has at least four elements. The following is the corresponding result when F is algebraically closed.

Theorem 1 [2]. *Let A and B be $n \times n$ nonsingular matrices, over an algebraically closed field, with $n \geqslant 2$. Then (A, B) is spectrally complete for the product if and only if $i(A) + i(B) \leqslant n$ and at least one of the following conditions is satisfied:*
1. $n = 2$,
2. *at least one of the nontrivial invariant polynomials of A or B has degree different from two.*

The following theorem, proved in [3] will be used in the sequel.
Define

$$R(A, B) = \min_{c \in F} \{\operatorname{rank}(A - cI_n) + \operatorname{rank}(B - cI_n)\}.$$

Theorem 2 [3]. *Let A and B be $n \times n$ matrices over an algebraically closed field and $t \in \{0, 1, \ldots, n\}$. There exist matrices A' and B' similar to A and B, respectively, such that*

$$\operatorname{rank}(A' - B') = t$$

if and only if the following conditions are satisfied:

$$\alpha_i(x) | \beta_{i+t}(x), \quad \beta_i(x) | \alpha_{i+t}(x) \qquad \text{for} \quad i \in \{1, \ldots, n - t\} \text{ and } t \leqslant R(A, B). \tag{1}$$

If condition (1) is satisfied, we shall say (A, B) is a t-pair. It is easy to check that the set of integers $t \in \{0, 1, \ldots, n - 1\}$ for which there exists $\lambda \in F$, such that

$$\alpha_i(x) \,|\, \gamma_{i+t}(x, \lambda), \quad \gamma_i(x, \lambda) \,|\, \alpha_{i+t}(x), \qquad i = 1, \ldots, n - t \tag{2}$$

is not empty. So let t_0 be the minimum of this set.

Remark 1. Clearly, $t_0 \geqslant |R(A) - R(B)|$ and $R(A) + R(B) \leqslant R(A, \lambda B^{-1})$.

2. Main result

We are going to prove the following theorem, which is our main result.

Theorem 3. *For any $n \times n$ nonsingular matrices, A and B, over an algebraically closed field F, there exist A' and B' similar to A and B, respectively, such that $R(A'B') = t$ if and only if*

$$t_0 \leqslant t \leqslant \min\{n - 1, R(A) + R(B)\}. \tag{3}$$

Lemma 1 (Necessity). *For any $n \times n$ nonsingular matrices A and B, over an algebraically closed field F, we have*

$$t_0 \leqslant R(AB) \leqslant \min\{n - 1, R(A) + R(B)\}. \tag{4}$$

Proof. For any nonzero λ, $\beta \in F$, we have

$$\begin{aligned} R(AB) &\leqslant \operatorname{rank}(AB + \lambda I) \\ &= \operatorname{rank}(A + \lambda B^{-1}I + \beta I - \beta I) \\ &\leqslant \operatorname{rank}(A - \beta I) + \operatorname{rank}(\lambda B^{-1} + \beta I) \\ &= \operatorname{rank}(A - \beta I) + \operatorname{rank}\left(B^{-1} + \frac{\beta}{\lambda} I\right) \\ &= \operatorname{rank}(A - \beta I) + \operatorname{rank}\left(B + \frac{\lambda}{\beta} I\right). \end{aligned}$$

So $R(AB) \leqslant R(A) + R(B)$.

Denoting by "$\cong$" the similarity relation and bearing Theorem 2 in mind, we have

$$\begin{aligned} \min_{A'\cong A, B'\cong B} \min_{\lambda\in F} \operatorname{rank}(A'B' - \lambda I) &= \min_{A'\cong A, B'\cong B} \min_{\lambda\in F} \operatorname{rank}(A' - \lambda B'^{-1}) \\ &= \min_{\lambda\in F} \min_{A'\cong A, B'\cong B} \operatorname{rank}(A' - \lambda B'^{-1}) \\ &= \min_{\lambda\in F} \min_{A'\cong A, C\cong \lambda B^{-1}} \operatorname{rank}(A' - C) = t_0. \end{aligned}$$

So

$$R(AB) = \min_{\lambda\in F} \operatorname{rank}(A - \lambda B^{-1}) \geqslant t_0.$$

So (4) holds. □

To prove the sufficiency, we have to consider several cases.

Remark 2. If A and B are nonsingular matrices, it is easy to see that there exist A' and B' similar to A and B, respectively, such that $R(A'B') = 0$ if and only if A and B^{-1} are similar, up to a scalar factor (i.e., there exists $\alpha \in F$, so that $B^{-1} \cong \alpha A$).

Lemma 2. *If A and B are both $n \times n$ nonderogatory matrices, over an algebraically closed field, then for $t \in \{1, \ldots, n-1\}$, there exist A' and B' similar to A and B, respectively, such that $R(A'B') = t$.*

Proof. In this case, A and B are respectively similar to matrices of the forms

$$A' = \begin{bmatrix} 0 & & & a_1 \\ 1 & \ddots & & a_2 \\ & \ddots & 0 & \vdots \\ & & 1 & a_n \end{bmatrix}, \quad B' = \begin{bmatrix} b_n & 1 & & \\ \vdots & 0 & \ddots & \\ b_2 & & \ddots & 1 \\ b_1 & & & 0 \end{bmatrix}.$$

Let $X = \mathrm{diag}(c_1, \ldots, c_{t-1}, \lambda_0^{n-(t-1)}, \ldots, \lambda_0^2, \lambda_0)$, where c_i, $i = 1, \ldots, t-1$ and λ_0 are any nonzero elements of F.

Let $B'' = XB'X^{-1}$. Then we have

$$B'' = \begin{bmatrix} * & d_1 & & & & & \\ & 0 & \ddots & & & & \\ \vdots & & \ddots & d_{t-1} & & & \\ \vdots & & & \ddots & \lambda_0 & & \\ & & & & \ddots & \ddots & \\ * & & & & & 0 & \lambda_0 \\ * & & & & & & 0 \end{bmatrix},$$

where $d_i = c_i/c_{i+1}$, $i = 1, \ldots, t-2$, and $d_{t-1} = c_{t-1}/\lambda_0^{n-(t-1)}$. Then

$$A'B'' = \begin{bmatrix} a_1' & & & & & \\ & d_1 & & & & \\ * & & \ddots & & & \\ \vdots & & & d_{t-1} & & \\ & & & & \lambda_0 & \\ * & & & & & \ddots & \\ * & & & & & & \lambda_0 \end{bmatrix},$$

where $a_1' = a_1 b_1 \lambda_0 / c_1$.

Since F is an infinite field, we may choose the c_i's and λ_0, such that a_1', λ_0 and the d_i's are pairwise distinct. Then we have

$$R(A'B'') = \mathrm{rank}(A'B'' - \lambda_0 I) = t. \qquad \square$$

Lemma 3. *If A and B are $n \times n$ nonsingular matrices, over an algebraically closed field, and either A or B is nonderogatory, then for any t satisfying*

$$t_0 \leqslant t \leqslant n-1$$

there exist A' and B' similar to A and B, respectively, such that

$$R(A'B') = t.$$

Proof. Without loss of generality, we may assume that A is nonderogatory. Then $f_1(x)$ is the only invariant polynomial of A different from one. As before, let $g_1(x)|\cdots|g_s(x)$ be the nontrivial invariant polynomials of B, and $h_1(x,\lambda)|\cdots|h_s(x,\lambda)$ be the nontrivial invariant polynomials of λB^{-1}.

If $s = n$, then B is scalar. In this case, it is easy to see that $R(AB) = R(A) = n - 1$, and the lemma is trivial.

Now we assume that A is nonderogatory and B is nonscalar. We consider two cases.

Case 1: There does not exist $\lambda_0 \in F$ such that $h_1(x,\lambda_0)\,|\,f_1(x)$. Then we have $t_0 = R(A) - R(B) + 1 = s$. Factorize $f_1(x)$ in the following way: $f_1(x) = l_1(x)l_2(x)\ldots l_s(x)$, where the degree of $l_i(x)$ is the same as the degree of $g_i(x)$, $i = 1,\ldots,s$. For $R(A) - R(B) + 1 \leqslant t \leqslant n - 1$, we do the following. Let B' be the following normal form of B

$$B' = \begin{bmatrix} C(g_1) & & & \\ & C(g_2) & & \\ & & \ddots & \\ & & & C(g_s) \end{bmatrix} = \left[\begin{array}{c|c|c|c} \begin{matrix} a_{d(g_1)} & 1 & & \\ \vdots & 0 & \ddots & \\ a_2 & & \ddots & 1 \\ a_1 & & & 0 \end{matrix} & & & \\ \hline & \begin{matrix} b_{d(g_2)} & 1 & & \\ \vdots & 0 & \ddots & \\ b_2 & & \ddots & 1 \\ b_1 & & & 0 \end{matrix} & & \\ \hline & & \ddots & \\ \hline & & & \begin{matrix} c_{d(g_s)} & 1 & & \\ \vdots & 0 & \ddots & \\ c_2 & & \ddots & 1 \\ c_1 & & & 0 \end{matrix} \end{array}\right].$$

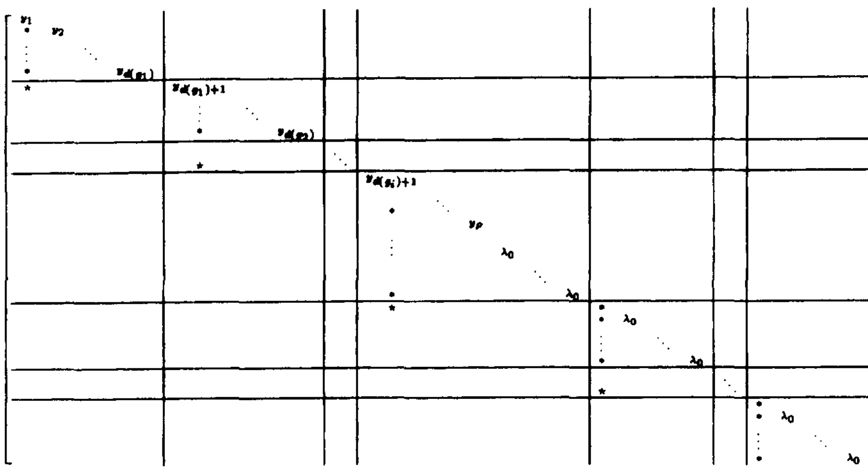
Since B is nonsingular, $a_1, b_1, \ldots, c_1$ are nonzero. A is similar to the following matrix

$$A' = \left[\begin{array}{cccc|cccc|c|cccc} 0 & & & p_1 & & & & & & & & & \\ 1 & \ddots & & p_2 & & & & & & & & & \\ & \ddots & 0 & \vdots & & & & & & & & & \\ & & 1 & p_{d(g_1)} & & & & & & & & & \\ \hline & & & 1 & 0 & & & m_1 & & & & & \\ & & & & 1 & \ddots & & m_2 & & & & & \\ & & & & & \ddots & 0 & \vdots & & & & & \\ & & & & & & 1 & m_{d(g_2)} & & & & & \\ \hline & & & & & & & 1 & \ddots & & & & \\ & & & & & & & & \ddots & & & & \\ \hline & & & & & & & & 1 & 0 & & & n_1 \\ & & & & & & & & & 1 & \ddots & & n_2 \\ & & & & & & & & & & \ddots & 0 & \vdots \\ & & & & & & & & & & & 1 & n_{d(g_s)} \end{array}\right],$$

where the diagonal blocks have $l_1(x), l_2(x), \ldots, l_s(x)$ as characteristic polynomials.

Let $X = \mathrm{diag}(x_1, \ldots, x_\rho, \lambda_0^{n-\rho}, \ldots, \lambda_0^2, \lambda_0)$, where x_i, $i = 1, \ldots, \rho$ and λ_0 are any nonzero elements of F, $\rho \in \{0, \ldots, n-1\}$.

Then $A'XB'X^{-1} =$



,

where the elements denoted with $\star$ are in the positions

$$\left(\sum_{i=1}^{k} d(g_i) + 1, \sum_{i=1}^{k-1} d(g_i) + 1\right), \quad k = 1, \ldots, s-1.$$

The blank places are zero. Because A and B are nonsingular matrices, the $\star$s and the diagonal elements are all nonzero. The number of the $\star$s equals $|R(A) - R(B)|$, so we have at least $|R(A) - R(B)|$ columns which are linearly independent. And we may choose the x_i's so that the y_i's are distinct and also different from λ_0. We can see that $\mathrm{rank}(A'XB'X^{-1} - \lambda I) \geqslant |R(A) - R(B)| + 1 + \rho - i$, for $\lambda \in F$.

So we have

$$R(A'XB'X^{-1}) = |R(A) - R(B)| + 1 + \rho - i,$$

where

$$d(g_0) + \cdots + d(g_i) \leqslant \rho < d(g_0) + \cdots + d(g_{i+1}) \quad i = 0, \ldots, s-1.$$

(Define $d(g_0) = 0$.)

Case 2: There exists $\lambda_0 \in F$ such that $h_1(x, \lambda_0) \,|\, f_1(x)$. Then we have $t_0 = |R(A) - R(B)| = s - 1$. Factorize $f_1(x)$ in the following way: $f_1(x) = l_1'(x) l_2'(x) \ldots l_s'(x)$, where

$$l'_1(x) = h_1(x, \lambda_0) = (x - a_1)(x - a_2)\cdots(x - a_{d(h_1)})$$

and the degree of $l'_i(x)$ is the same as the degree of h_i, $i = 1, \ldots, s$.

Clearly, A is similar to

$$A' = \left[\begin{array}{cccc|cccc|ccc|cccc}
a_1 & 1 & & & & & & & & & & & & & \\
 & a_2 & \ddots & & & & & & & & & & & & \\
 & & \ddots & 1 & & & & & & & & & & & \\
 & & & a_{d(h_1)} & 1 & & & & & & & & & & \\
\hline
 & & & & * & 1 & & & & & & & & & \\
 & & & & * & 0 & \ddots & & & & & & & & \\
 & & & & \vdots & & \ddots & 1 & & & & & & & \\
 & & & & * & & & 0 & 1 & & & & & & \\
\hline
 & & & & & & & & & \ddots & & & & & \\
 & & & & & & & & & & \ddots & 1 & & & \\
\hline
 & & & & & & & & & & & * & 1 & & \\
 & & & & & & & & & & & * & 0 & \ddots & \\
 & & & & & & & & & & & \vdots & & \ddots & 1 \\
 & & & & & & & & & & & * & & & 0
\end{array}\right],$$

where the diagonal blocks have $l'_1(x), l'_2(x), \ldots, l'_s(x)$ as characteristic polynomials. The matrix $\lambda_0 B^{-1}$ is similar to a matrix of the following form:

$$\lambda_0 B'^{-1} = \left[\begin{array}{cccc|cccc|c|cccc}
a_1 & 1 & & & & & & & & & & & \\
 & a_2 & \ddots & & & & & & & & & & \\
 & & \ddots & 1 & & & & & & & & & \\
 & & & a_{d(h_1)} & & & & & & & & & \\
\hline
 & & & & * & 1 & & & & & & & \\
 & & & & * & 0 & \ddots & & & & & & \\
 & & & & \vdots & & \ddots & 1 & & & & & \\
 & & & & * & & & 0 & & & & & \\
\hline
 & & & & & & & & \ddots & & & & \\
\hline
 & & & & & & & & & * & 1 & & \\
 & & & & & & & & & * & 0 & \ddots & \\
 & & & & & & & & & \vdots & & \ddots & 1 \\
 & & & & & & & & & * & & & 0
\end{array}\right],$$

where the diagonal blocks have $h_1(x, \lambda_0), h_2(x, \lambda_0), \ldots, h_s(x, \lambda_0)$ as characteristic polynomials. The inverse of B'^{-1} is in the form

$$B' = \lambda_0 \left[\begin{array}{cccc|cccc|c|cccc}
\frac{1}{a_1} & * & * & * & & & & & & & & & \\
 & \frac{1}{a_2} & * & * & & & & & & & & & \\
 & & \ddots & * & & & & & & & & & \\
 & & & \frac{1}{a_{d(h_1)}} & & & & & & & & & \\
\hline
 & & & & 0 & & & * & & & & & \\
 & & & & 1 & \ddots & & \vdots & & & & & \\
 & & & & & \ddots & 0 & * & & & & & \\
 & & & & & & 1 & * & & & & & \\
\hline
 & & & & & & & & \ddots & & & & \\
\hline
 & & & & & & & & & 0 & & & * \\
 & & & & & & & & & 1 & \ddots & & \vdots \\
 & & & & & & & & & & \ddots & & * \\
 & & & & & & & & & & & 1 & *
\end{array}\right],$$

where the diagonal blocks have $g_1(x), \ldots, g_s(x)$ as characteristic polynomials.
Then we have A′B′

$$\left[\begin{array}{ccc|cccc|cccc|c|cccc}
\lambda_0 & & & & & & & & & & & & & & & \\
 & \ddots & & & & & & & & & & & & & & \\
 & & \lambda_0 & & & & \star & & & & & & & & & \\
\hline
 & & & \lambda_0 & & & * & & & & & & & & & \\
 & & & & \ddots & & \vdots & & & & & & & & & \\
 & & & & & \lambda_0 & * & & & & & & & & & \\
 & & & & & & * & & & & \star & & & & & \\
\hline
 & & & & & & & \lambda_0 & & & * & & & & & \\
 & & & & & & & & \ddots & & \vdots & & & & & \\
 & & & & & & & & & \lambda_0 & * & & & & & \\
 & & & & & & & & & & * & & & & & \\
\hline
 & & & & & & & & & & & \ddots & & & & \star \\
\hline
 & & & & & & & & & & & & \lambda_0 & & & * \\
 & & & & & & & & & & & & & \ddots & & \vdots \\
 & & & & & & & & & & & & & & \lambda_0 & * \\
 & & & & & & & & & & & & & & & *
\end{array}\right],$$

where the $\star$s are in the positions $(\sum_{i=1}^{t} d(g_i), \sum_{i=1}^{t+1} d(g_i))$, $t = 1, \ldots, s-1$. Since A and B are nonsingular, the elements $\star$'s are nonzero. This way we get t_0 columns which are linearly independent. So

$$R(A'B') = \min_{\lambda \in F} \operatorname{rank}(A'B' - \lambda I) = \operatorname{rank}(A'B' - \lambda_0 I) = t_0 = |R(A) - R(B)|.$$

For $R(A) - R(B) + 1 \leqslant t \leqslant n - 1$, we do the same as we did for Case 1. □

Proof of Theorem 3. Here we assume that A and B are both derogatory. If one of them is scalar, the theorem is trivial. Now suppose neither of A and B is scalar or nonderogatory. The proof goes by induction on n. When $n = 1, 2, 3, 4$, it can be verified easily.

Suppose $n \geqslant 5$ and A and λB^{-1} satisfy (2). From Theorem 2, there are A' and B' similar to A and B, respectively, such that $\operatorname{rank}(A'B' - \lambda I) = \operatorname{rank}(A' - \lambda (B')^{-1}) = t$; then we have that

$$R(A' - \lambda (B')^{-1}) \leqslant t.$$

If we have equality, the proof is complete. Now suppose $R(A' - \lambda (B')^{-1}) \leqslant t - 1$, then $(A, \lambda B^{-1})$ is $(t-1)$-pair.

Note that, if $t \leqslant n/2$, equality always holds, otherwise there exists μ such that $R(A' - \mu (B')^{-1}) \leqslant t - 1$. That means μ is an eigenvalue of $A'B'$ of algebraic multiplicity at least $n - t + 1$. Since λ is an eigenvalue of $A'B'$ of algebraic multiplicity at least $n - t$, so $(n - t + 1) + (n - t) \leqslant n$, i.e., $t \geqslant (n + 1)/2$, a contradiction. Henceforth we consider only the case $t \geqslant (n + 1)/2$.

Without loss of generality, suppose that $d(f_1) \geqslant d(g_1) = p$ with $p < n$. Let $L' = L_2 \oplus \cdots \oplus L_s$. If $d(g_1) = d(f_1)$, take $K' = K_2 \oplus \cdots \oplus K_r$. (Bear in mind the definition of K_i and L_i.) If $d(f_1) > d(g_1) = p$, K_1 is similar to a matrix of the form

$$K_1' = \left[\begin{array}{cccc|c} & N & & & 0 \\ \hline 0 & \dots & 0 & 1 & \\ & & & 0 & \\ & 0 & & \vdots & M \\ & & & 0 & \end{array}\right],$$

where $N \in F^{p \times p}$. In this case, take $K' = M \oplus K_2 \oplus \cdots \oplus K_r$. Since $(A, \lambda B^{-1})$ is $(t-1)$-pair, it is not difficult to verify, in any case $(K', \lambda L'^{-1})$ is also a $(t-1)$-pair.

Case 1: Suppose that $d(f_1) \geqslant d(g_1) = p \geqslant 2$.

Without loss of generality, suppose that M and N are companion matrices, and suppose $N \neq \alpha L_1^{-1}$. Consider the polynomial $\phi(y) = -y^2 + (n+2)y - 2n$, with coefficients in the field of real numbers. Its roots are 2 and n. Since $2 \leqslant p \leqslant n$, we have $\phi(p) \geqslant 0$. We also have $r \leqslant n/p$ and $s \leqslant n/p$. Therefore

$$R(K') + R(L') \geqslant 2n - 2p - \frac{2n}{p} + 1 = \frac{1}{p}\phi(p) + n - p - 1 \geqslant n - p - 1.$$

So the maximum value t can be attained. On the other hand, from $t \geqslant (n+1)/2$ we can get $t-1 \geqslant |R(K') - R(L')|$.

Case 1.1: Suppose that $n-p \geqslant t$. By the induction assumption, there is $X \in F^{(n-p)\times(n-p)}$ such that $R(K'XL'X^{-1}) = \operatorname{rank}(K'XL'X^{-1} - \lambda I) = t-1$.

If $d(g_1) = d(f_1)$, then, by Lemma 2, there is a nonsingular matrix $Y \in F^{p\times p}$ such that

$$R(K_1YL_1Y^{-1}) = \operatorname{rank}(K_1YL_1Y^{-1} - \lambda I) = 1.$$

Then

$$R(K(Y \oplus X)L(Y \oplus X)^{-1}) = \operatorname{rank}(K(Y \oplus X)L(Y \oplus X)^{-1} - \lambda I) = t.$$

Now suppose that $d(f_1) > d(g_1)$. If the column $[*, 0, \ldots, 0]^{\mathrm{t}} \in F^{(n-p)\times 1}$ is a linear combination of the columns of $K'XL'X^{-1}$, then, by Lemma 2, there is a nonsingular matrix $Y \in F^{p\times p}$ such that $R(NYL_1Y^{-1}) = \operatorname{rank}(NYL_1Y^{-1} - \lambda I) = 1$; if not, by Lemma 2 again, there is $Y \in F^{p\times p}$ such that

$$NYL_1Y^{-1} = \left[\begin{array}{c|c} * & 0 \\ \hline * & \lambda I_{p-1} \end{array}\right].$$

In any case,

$$R(K(Y \oplus X)L(Y \oplus X)^{-1}) = \operatorname{rank}(K(Y \oplus X)L(Y \oplus X)^{-1} - \lambda I) = t.$$

Case 1.2: Suppose that $n-p < t < n-1$. Since $n-p-1 \leqslant R(K') + R(L')$, by the induction assumption, there exists a nonsingular matrix $X \in F^{(n-p)\times(n-p)}$ such that

$$R(K'XL'X^{-1}) = \operatorname{rank}(K'XL'X^{-1} - \lambda I) = n-p-1.$$

If $d(g_1) = d(f_1)$, then, by Lemma 2, there exists a nonsingular matrix $Y \in F^{p\times p}$ such that

$$R(K_1YL_1Y^{-1}) = \operatorname{rank}(K_1YL_1Y^{-1} - \lambda I) = t - (n-p-1).$$

Then

$$R(K(Y \oplus X)L(Y \oplus X)^{-1}) = \operatorname{rank}(K(Y \oplus X)L(Y \oplus X)^{-1} - \lambda I) = t.$$

Now suppose that $d(f_1) > d(g_1)$. If the column $[*, 0, \ldots, 0]^{\mathrm{t}} \in F^{(n-p)\times 1}$ is a linear combination of the columns of $K'XL'X^{-1}$, then, by lemma 2, there exists a nonsingular matrix $Y \in F^{p\times p}$ such that

$$R(NYL_1Y^{-1}) = \operatorname{rank}(NYL_1Y^{-1} - \lambda I) = t - (n-p-1).$$

Then

$$R(K(Y \oplus X)L(Y \oplus X)^{-1}) = \operatorname{rank}(K(Y \oplus X)L(Y \oplus X)^{-1} - \lambda I) = t.$$

If not, by Lemma 2 again, there is $Y \in F^{p\times p}$ such that

$$NYL_1Y^{-1} = \left[\begin{array}{c|c} * & 0 \\ \hline * & \lambda I_{n-t-1} \end{array}\right].$$

We have $R(NYL_1Y^{-1}) = t - (n-p-1)$. In any case,

$$R(K(Y\oplus X)L(Y\oplus X)^{-1}) = \operatorname{rank}(K(Y\oplus X)L(Y\oplus X)^{-1} - \lambda I) = t.$$

Case 1.3: Suppose that $t = n-1$. Because $p \geqslant 2$, we have $R(A) + R(B) \geqslant n$, that is $i(A) + i(B) \leqslant n$. Assume one of the nontrivial invariant polynomials of A and B is not of degree 2. By Theorem 1, (A, B) is spectrally complete for the product. We may choose λ_i's such that all of them are distinct. Then we have $R(A'B') = n-1$.

Assume all the nontrivial invariant polynomials of A and B are of degree 2. Without loss of generality, suppose

$$A = \left[\begin{array}{cc|c|cc} 0 & 1 & & & \\ a_1 & a_2 & & & \\ \hline & & \ddots & & \\ \hline & & & 0 & 1 \\ & & & a_1 & a_2 \end{array}\right] \text{ and } B = \left[\begin{array}{cc|c|cc} b_2 & b_1 & & & \\ 1 & 0 & & & \\ \hline & & \ddots & & \\ \hline & & & b_2 & b_1 \\ & & & 1 & 0 \end{array}\right].$$

Let $X = diag(c_1, 1, c_2, 1, \ldots, c_{(n/2)}, 1)$, where the c_i's are nonzero elements of F chosen so that the diagonal elements of the matrix

$$XAX^{-1}B = \left[\begin{array}{cc|c|cc} c_1 & 0 & & & \\ * & \frac{a_1b_1}{c_1} & & & \\ \hline & & \ddots & & \\ \hline & & & c_{\frac{n}{2}} & \\ & & & * & \frac{a_1b_1}{c_{\frac{n}{2}}} \end{array}\right]$$

are all distinct. Then we have $R(XAX^{-1}B) = n-1$.

Case 2: Suppose that $d(f_1) \geqslant d(g_1) = 1$.

Case 2.1: $d(f_1) = d(g_1) = 1$. In this case, $A' = a \oplus K'$ and $B' = b \oplus L'$. Since $R(A) = R(K')$ and $R(B) = R(L')$, we have $|R(K') - R(L')| \leqslant t \leqslant R(K') + R(L')$.

Now let $|R(K') - R(L')| \leqslant t \leqslant n-2$. By the induction assumption there exists X, such that

$$R(K'XL'X^{-1}) = \operatorname{rank}(K'XL'X^{-1} - abI) = t.$$

Consequently, $R(A'(1\oplus X)B'(1\oplus X)^{-1}) = t$.

Now assume $t = n - 1$.

1. $R(A) + R(B) \geqslant n$, that means $i(A) + i(B) \leqslant n$. According to Theorem 1, the pair (A, B) is spectrally complete for the product. Then we can get $R(A'B') = n - 1$.
2. $R(A) + R(B) = n - 1$. From $R(A) = R(K')$, $R(B) = R(L')$, we have $R(K') + R(L') = n - 1$. Again by Theorem 1, (K', L') is spectrally complete for the product. We may choose $\lambda_1, \ldots, \lambda_{n-1} \in F - \{ab\}$ to be distinct and also conclude that $R(A'B') = n - 1$.

Case 2.2: $d(f_1) = q > d(g_1) = 1$.

Let w be the number of linear invariant polynomials of B, and $u = d(g_{w+1})$. Note that $2 \leqslant q \leqslant n/2$.

(1). $w < q = u$.

In this case, A and B are similar to

$$A' = C(f_1) \oplus K' \quad \text{and} \quad B' = C(g_{w-1}) \oplus L''.$$

respectively. Since $R(K') \geqslant (n - q) - (n - q)/q$, $R(L'') \geqslant (n - q - w) -(n - q - w)/q$, we have $R(K') + R(L'') \geqslant 2n - 2q + 2 - 2n/q - (q - 1)w/q \geqslant 2n - 2q+2 - 2n/q - (q - 1)\ (q - 1)/q$. Notice the fact that the quadratic expression $2q^2 - (n + 5)q + 2n + 1$ is nonpositive for $2 \leqslant q \leqslant n/2$. We have that $R(K') + R(L'') \geqslant n - q - 1$, and we can verify that $(K', \lambda L''^{-1})$ is also a $(t - 1)$-pair.

We are going to use the same technique as we used in Case 1.

Suppose that $n - q \geqslant t$. By the induction assumption, there is $X \in F^{(n-q)\times(n-q)}$ such that $R(K'XL''X^{-1}) = \text{rank}(K'XL''X^{-1} - \lambda I) = t - 1$. Then, by Lemma 2, there is a nonsingular matrix $Y \in F^{q\times q}$ such that

$$R(C(f_1)YC(g_{w+1})Y^{-1}) = \text{rank}(C(f_1)YC(g_{w-1})Y^{-1} - \lambda I) = 1.$$

Then

$$\begin{aligned} &R((C(f_1) \oplus K')(Y \oplus X)(C(g_{w+1}) \oplus L'')(Y \oplus X)^{-1}) \\ &\quad = \text{rank}(A'(Y \oplus X)B'(Y \oplus X)^{-1} - \lambda I) = t. \end{aligned}$$

Suppose that $n - q < t < n - 1$. Since $R(K') + R(L'') \geqslant n - q - 1$, by the induction assumption there exists a nonsingular matrix $X \in F^{(n-q)\times(n-q)}$ such that

$$R(K'XL''X^{-1}) = \text{rank}(K'XL''X^{-1} - \lambda I) = n - q - 1.$$

Then, by Lemma 2, there exists a nonsingular matrix $Y \in F^{q\times q}$ such that

$$R(C(f_1)YC(g_{w+1})Y^{-1}) = \text{rank}(C(f_1)YC(g_{w+1})Y^{-1} - \lambda I) = t - (n - q - 1).$$

Then

$$\begin{aligned} &R((C(f_1) \oplus K')(Y \oplus X)(C(g_{w+1}) \oplus L'')(Y \oplus X)^{-1}) \\ &\quad = \text{rank}(A'(Y \oplus X)B'(Y \oplus X)^{-1} - \lambda I) = t. \end{aligned}$$

Suppose that $t = n - 1$.

1. $R(A) + R(B) \geqslant n$, that is the pair (A, B) is spectrally complete for the product.
2. $R(A) + R(B) = n - 1$. We prove this is impossible.

Note that $r \leqslant n/q$ and $s \leqslant (n-w)/2 + w$. Hence $n + 1 = r + s \leqslant w + (n-w)/2 + n/q$. As $w \leqslant q - 1$, we have $q^2 - (3+n)q + 2n \geqslant 0$, a contradiction because $q^2 - (3+n)q + 2n < 0$ for $2 \leqslant q \leqslant n/2$.

(2). $w < q < u$.

Note that A and B are similar to

$$A' = C(f_1) \oplus K' \text{ and } B' = \left[\begin{array}{cccc|c} & C(g) & & & 0 \\ \hline 0 & \cdots & 0 & 1 & \\ & & & & L'' \\ & 0 & & & \end{array}\right],$$

respectively, where $C(g) \in F^{q \times q}$. Assuming that $u > n/2$, then $R(K') \geqslant n - n/q - (q-1)$, $R(L'') = n - w - q - 1$. So $R(K') + R(L'') \geqslant 2n - w - 1 - q - n/q - (q-1) \geqslant 2n - (q-1) - 2q - n/q = 2n - 3q + 1 - n/q$. Notice the fact that the quadratic expression $2q^2 - (n+2)q + n \leqslant 0$ for $2 \leqslant q \leqslant n/2$ and $n \geqslant 4$. We have that $R(K') + R(L'') \geqslant n - q - 1$. It is also easy to check that $|R(K') - R(L'')| \leqslant |w + 1 - 1| \leqslant q - 1 \leqslant t - 1$. Assume that $u < n/2$. Note that $u \geqslant 3$ and u cannot be $n/2$ because, in this case, as B is nonderogatory, we will have $w = n/2$. So $R(L'') \geqslant (n-q) - (w + (n-w)/3)$ and $R(K') \geqslant (n-q) - (n/q - 1)$. We have that $R(K') + R(L'') \geqslant 2n - 2q + 1 - n/q \; -(2w+n)/3 \geqslant 2n - 2q + 1 - n/q - (2(q-1)+n)/3 = 2n - 2q - (2q^2 - 5q + nq + 3n)/3q$. Notice the fact that the quadratic expression $5q^2 - (8+2n)q + 3n \leqslant 0$ for $2 \leqslant q \leqslant n/3$ and $n \geqslant 4$. We have that $R(K') + R(L'') \geqslant n - q - 1$. When $n/3 < q \leqslant n/2$, we have $r = 2$, it is obvious $R(K') + R(L'') \geqslant n - q - 1$. So in any case we have $R(K') + R(L'') \geqslant n - q - 1$. On the other hand, $|R(K') - R(L'')| \leqslant |w + (n-w)/u - 1| \leqslant |u - 2 - (n-u-2)/2| \leqslant (n-1)/2$, the last inequality holds because $2 \leqslant u \leqslant n/2$. So the induction assumption holds for the pair (K', L'').

Bear in mind that a square matrix is similar to its transpose, so we may change the order of A' and B'. Now we reduced our problem to the same type as Case 1, and we may do the same analysis as we did before.

(3). $w < q$ and $u < q$.

Note that A and B are similar to

$$\left[\begin{array}{cccc|c} & C(f) & & & 0 \\ \hline 0 & \cdots & 0 & 1 & \\ & & & & K' \\ & 0 & & & \end{array}\right] \quad \text{and} \quad C(g_{w+1}) \oplus L'',$$

respectively, where $C(f) \in F^{u \times u}$. In this case we have that $u \geqslant 2$, $q \geqslant 3$, $w \leqslant q-1$. Since $R(K') \geqslant (n-u) - n/3$ and $R(L'') \geqslant (n-u) - (w + (n-u-w)/u) \geqslant (n-u) - (n/u - 1 + (q-1)(u-1)/u)$. We have $R(K') + R(L'') \geqslant 2n - 2u - n/3 - (n + (q-1)(u-1))/u + 1$. Notice the fact that the quadratic expression $3u^2 - (2n - 3q + 9)u + 3n - 3q + 3 \leqslant 0$ for $2 \leqslant u \leqslant q-1$ and $3 \leqslant q \leqslant n/3$. We have that $R(K') + R(L'') \geqslant n - u - 1$. When $n/3 < q \leqslant n/2$, we have $r = 2$, it is obvious $R(K') + R(L'') \geqslant n - u - 1$. So in any case we have $R(K') + R(L'') \geqslant n - u - 1$. And we can verify that $(K', \lambda L''^{-1})$ is also a $(t-1)$-pair.

Again we reduced our problem to the same form as Case 1, and we may do the same analysis as we did before.

(4). $q \leqslant w$.

In this case, A and B are similar to

$$A' = C(f_1) \oplus K' \quad \text{and} \quad B' = x_q \oplus L'',$$

respectively, where $x_q \in F^{q \times q}$ is a scalar matrix. First suppose that $R(K') \geqslant R(L'')$. As

$$|R(A') - R(B')| \leqslant t \leqslant R(A') + R(B'),$$

we have

$$(q-1) + R(K') - R(L'') \leqslant t \leqslant (q-1) + R(K') + R(L'').$$

That means $R(K') - R(L'') \leqslant t - (q-1) \leqslant R(K') + R(L'')$. Note that $t - (q-1) \leqslant n - q - 1$. By the induction assumption, there exists X such that

$$R(K'XL''X^{-1}) = \operatorname{rank}(K'XL''X^{-1} - \lambda I) = t - (q-1).$$

Then $R((C(f_1) \oplus K')(I_q \oplus X)(x_q \oplus L'')(I_q \oplus X)^{-1}) = t$.

Second, suppose that $R(B') > R(A')$. That means $R(L'') > (q-1) + R(K')$. If $t - (q-1) \geqslant R(L'') - R(K')$, then we may do the same as we did in the first case. Now consider $t - (q-1) < R(L'') - R(K')$. As $R(L'') > R(A')$, there must be one diagonal block in L'' whose order is greater or equal to the order of $C(f_1)$. Moving it to the first diagonal block we can get a matrix similar to B' of the form

$$\left[\begin{array}{cccc|c} & C(g) & & & 0 \\ \hline 0 & \dots & 0 & 1 & \\ & & & & L''' \\ & 0 & & & \end{array}\right],$$

where $C(g) \in F^{q \times q}$.

We have $R(L''') - R(K') = R(L'') - R(K') - q$. Because $t \geqslant |R(A') - R(B')| = R(L'') - R(K') - (q-1)$, by induction assumption there exists X such that

$$R(L'''XK'X^{-1}) = \operatorname{rank}(L'''XK'X^{-1} - \lambda I) = t_1 \geqslant R(L'') - R(K') - q.$$

Then by Lemma 2, there exists Y such that

$$R(C(g)YC(f_1)Y^{-1}) = \text{rank}(C(g)YC(f_1)Y^{-1} - \lambda I) = t - t_1.$$

Thus we have

$$R(A'B') = t.$$

Third, suppose that $R(A') \geqslant R(B') = R(L'') > R(K')$. If $t - (q-1) \geqslant R(L'') - R(K')$, then we may do the same as we did in the first case. Now consider $t - (q-1) < R(L'') - R(K')$. Let $z = R(L'') - R(K')$. Clearly, $z \leqslant q-1$, and $t \geqslant R(A') - R(B') = R(K') + (q-1) - R(L'') = (q-1) - z$. As $R(L'') > R(K')$, there must be one diagonal block in L'' whose order is greater or equal to the order of $C(f_1)$. Then we may get a matrix B'', similar to B', of the form

$$\left[\begin{array}{cccc|c} x_{q-z} & \oplus & \mathrm{C(g)} & & 0 \\ \hline 0 & \dots & 0 & 1 & \\ & & & & L''' \\ & 0 & & & \end{array}\right],$$

where $C(g) \in F^{z\times z}$ and $R(x_{q-z} \oplus C(g)) = z - 1$. (If there is a block in L'' with size $z+1$, then let $B'' = x_{q-(z+1)} \oplus C(g) \oplus L'''$.)

Since $C(f_1)$ is nonderogatory, by Lemma 3, there is $Y \in F^{q\times q}$ such that

$$\begin{aligned} R((x_{q-z} \oplus C(g))YC(f_1)Y^{-1}) &= \text{rank}((x_{q-z} \oplus C(g))YC(f_1)Y^{-1} - \lambda I) \\ &= (q-1) - (z-1) = q - z. \end{aligned}$$

On the other hand, by the induction assumption, there is $X \in F^{(n-q)\times(n-q)}$ such that

$$R(L'''XK'X^{-1})\text{rank}(L'''XK'X^{-1} - \lambda I) = t - (q - z).$$

So

$$R(A'B') = t.$$

Assume that $t = n - 1$.

1. $R(A) + R(B) \geqslant n$, the pair (A, B) is spectrally complete for the product.
2. $R(A) + R(B) = n - 1$, i.e., $(q-1) + R(K') + R(L'') = n - 1$. Then $R(K') + R(L'') = n - q$.

That means the pair (K', L'') is spectrally complete for the product. We may choose distinct nonzero elements, $\lambda_1, \dots, \lambda_{n-q}$, different from the eigenvalues of $C(f_1)x_q$ (we recall that x_q is a $q \times q$ scalar matrix) and satisfying $\lambda_1 \dots \lambda_{n-q} = \det K'L''$. Then it is easy to conclude that $R(A'B') = n - 1$. □

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