J. Math. Pures Appl.,
78, 1999, *p*. 819-840

DEGENERATE ELLIPTIC PROBLEMS IN A CLASS OF FREE DOMAINS

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Received 31 January 1999

ABSTRACT. – We study a mixed boundary value problem for an operator of p-Laplacian type. The main feature of the problem is the fact that the exact domain where it is considered is not known *a priori* and is to be determined so that a certain integral condition is satisfied. We establish the existence of a unique solution to the problem, by means of the analysis of the range of an appropriate real function, and we show the continuous dependence with respect to a family of operators. These results can be applied to the study of unidirectional non-Newtonian flows of power-law type, in particular to solve a simplified problem arising in theoretical glaciology and to show the existence of a Bingham flow in an open channel; the uniqueness in this case is an open problem. © Elsevier, Paris

Keywords: p-Laplacian operator, Free boundary problems, Continuous dependence, Unidirectional non-Newtonian flows

RÉSUMÉ. – On étudie un problème aux limites mixte pour un opérateur du type du p-Laplacian. L'intérêt principal du problème est dans le fait que le domaine exact où on le considère n'est pas *a priori* connu et il faut le determiner de façon telle qu'une certaine condition intégrale soit satisfaite. On démontre l'existence et l'unicité d'une solution du problème par l'analyse d'une fonction réele appropriée et on obtient aussi un résultat de dépendence continue pour une famille d'opérateurs. Ces résultats peuvent être appliqués à l'étude de l'écoulement unidirectionel de fluides non-newtoniens, en particulier, à la résolution d'un problème simplifié de la glaciologie théorique et à la démonstration de l'éxistence d'un écoulement de Bingham dans une conduite cylindrique ; l'únicité reste dans ce cas un problème ouvert. © Elsevier, Paris

1. Introduction and statement of the problem

We consider a free boundary value problem for a quasi-linear degenerate elliptic equation of the form

$$-\nabla \cdot \left[\alpha\left(|\nabla u|\right)\nabla u\right] = 1, \tag{1}$$

the model for the operator on the left hand side of (1) being the p-Laplacian, that corresponds to the choice $\alpha(|\nabla u|) = |\nabla u|^{p-2}$, with $1 < p < \infty$.

The new feature in this paper is the fact that the exact domain where the boundary value problem is to be considered is not known *a priori*. More precisely, it belongs to the class of open

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JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES. – 0021-7824/99/08

subsets of $\mathbb{R}^N$ of the form

$$\Omega_\xi = \{x = (x', x_N) \in \mathbb{R}^N:\ h(x') < x_N < \xi\}, \quad \xi \in \mathbb{R}, \tag{2}$$

where h is a given continuous function defined in $\mathbb{R}^{N-1}$. The unknown height ξ of the flat top surface of the domain is to be determined such that the solution u of the boundary value problem for (1) in Ω_ξ satisfies the integral condition

$$\int_{\Omega_\xi} u \,\mathrm{d}x = Q, \tag{3}$$

where $Q > 0$ is a given constant.

The boundary conditions considered on the domain Ω_ξ are of mixed type. Along the surface defined by h we take an homogeneous Dirichlet condition, and on the top surface an homogeneous Neumann condition.

This problem corresponds to the mathematical formulation of a family of interesting physical situations arising in classical fluid mechanics, for which recent interest has been raised by a problem in theoretical glaciology (see [5] and [12], for references). Although of simple formulation, it seems that few mathematical results exist, in spite of its study being possible in the framework of variational solutions. As we intend to show here, this allows a considerable degree of generality, in particular, in what concerns the low regularity of the domains, a possible consequence of the geometry of physical problems.

We denote

$$\Gamma_H^\xi = (H \cap \partial\Omega_\xi) \setminus \Pi_\xi = \{x \in \mathbb{R}^N:\ h(x') < \xi,\ x_N = h(x')\}, \tag{4}$$

$$\Gamma_\Pi^\xi = (\Pi_\xi \cap \partial\Omega_\xi) \setminus H = \{x \in \mathbb{R}^N:\ h(x') < \xi,\ x_N = \xi\}, \tag{5}$$

where $\Pi_\xi = \{x \in \mathbb{R}^N:\ x_N = \xi\}$ and $H = \{x \in \mathbb{R}^N:\ x_N = h(x')\}$ stands for the graph of h. The abridged classical formulation of the problem then reads:

PROBLEM (P). – *Given $Q > 0$, find a real number ξ and a function u_ξ such that*:

$$\left\{\begin{array}{ll} (1) & \left\{\begin{array}{ll} \nabla\cdot\big[\alpha\big(|\nabla u_\xi|\big)\nabla u_\xi\big] = -1 & \textit{in } \Omega_\xi, \\ u_\xi = 0 & \textit{on } \Gamma_H^\xi, \\ \partial u_\xi/\partial x_N = 0 & \textit{on } \Gamma_\Pi^\xi, \end{array}\right. \\ (2) & \displaystyle\int_{\Omega_\xi} u_\xi = Q. \end{array}\right.$$

Remark 1. – We are considering a constant second member in the equation. All the results in this paper can be easily extended to the case of a second member $f = f(x')$, i.e., independent of x_N, and such that $f(x') < 0$. We could also have considered a dependence in x' through α.

Since classical solutions are not to be expected in general, we next consider the problem in a weak form and state the assumptions under which we are able to establish an unique solvability result.

Let $C_H^1(\overline{\Omega_\xi})$ denote the subspace of $C^1(\overline{\Omega_\xi})$ consisting of those functions that vanish in a neighbourhood of H and define the space of test functions:

$$W_H^{1,p}(\Omega_\xi) = \text{closure of } C_H^1\big(\overline{\Omega_\xi}\big) \text{ in } W^{1,p}(\Omega_\xi), \quad 1 < p < \infty.$$

It is a reflexive Banach space where a Poincaré type inequality is valid; the usual $W^{1,p}$-norm of a function is then equivalent to the L^p norm of its gradient and this is the one that will be used.

Remark 2. – Note that with this definition we avoid some delicate questions concerning the regularity of the domain, namely the ones posed by the use of traces and those considered in the theory of elliptic problems in nonsmooth domains (see [6]).

A standard formal integration by parts, taking the boundary conditions into account, leads to the following:

DEFINITION. – *A weak solution to* (P) *is a pair* $(\xi, u_\xi) \in \mathbb{R}^+ \times W_H^{1,p}(\Omega_\xi)$ *such that*

$$\int_{\Omega_\xi} \alpha\big(|\nabla u_\xi|\big)\nabla u_\xi \cdot \nabla v = \int_{\Omega_\xi} v, \quad \forall v \in W_H^{1,p}(\Omega_\xi); \tag{6}$$

$$\int_{\Omega_\xi} u_\xi = Q. \tag{7}$$

This definition makes sense according to the following set of

ASSUMPTIONS. –
(A1) $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous function and* $\lim_{r\to 0^+} \alpha(r)r = 0$;
(A2) $\exists C_1, C_2 > 0, C_3 \geqslant 0$ *such that* $C_1 r^{p-2} \leqslant \alpha(r) \leqslant C_2 r^{p-2} + C_3 r^{-1}, \ \forall r > 0$;
(A3) $[\alpha(|\xi|)\xi - \alpha(|\eta|)\eta] \cdot [\xi - \eta] > 0, \ \forall \xi \neq \eta \in \mathbb{R}^N \setminus \{0\}$;
(A4) $h : \mathbb{R}^{N-1} \to \mathbb{R}$ *is a continuous function,* $\lim_{|x'|\to\infty} h(x') = +\infty$ *and* $\min_{x'\in\mathbb{R}^{N-1}} h(x') = 0$.

Remark 3. – In accordance with (A1), $\alpha(|\nabla u_\xi|)\nabla u_\xi$ is defined to be zero at each point where $\nabla u_\xi = 0$.

Remark 4. – Concerning (A4), we remark that the assumption $\min_{x'\in\mathbb{R}^{N-1}} h(x') = 0$ stands only to simplify the writing and allows us to search for a positive ξ. The fact that h is continuous and the coercivity condition $\lim_{|x'|\to\infty} h(x') = +\infty$ assure that, for each $\xi > 0$, Ω_ξ is an open and bounded subset of $\mathbb{R}^N$. Observe that no further regularity on h is required.

For each fixed $\zeta > 0$, we call (P_ζ) the problem consisting of solving (6). The assumptions on α assure that each (P_ζ) has a unique solution, by the general theory of (strictly) monotone operators (see [10], for instance). Moreover, we can formulate each (P_ζ) as a minimization problem. We introduce the real function

$$A(r) = \int_0^r \alpha\big(|s|\big)s \, \mathrm{d}s, \quad r \in \mathbb{R},$$

which, due to (A1) is differentiable in $\mathbb{R}$, with $A'(r) = \alpha(|r|)r$. From (A3) we conclude that A is strictly convex. Hence (6) is the Euler–Lagrange variational equation for

$$\min_{v\in W_H^{1,p}(\Omega_\zeta)} \left[\int_{\Omega_\zeta} A\big(|\nabla v|\big) - \int_{\Omega_\zeta} v \right], \tag{8}$$

so the solution u_ζ is the corresponding unique minimizer.

The regularity results for the solution, that we shall not use, are also well known. In fact, with our assumptions we can apply the results of [3] and conclude that we have:

$$u_\zeta \in C^{1,\alpha}_{\mathrm{loc}}(\Omega_\zeta).$$

As is pointed out there, this is the best regularity that is to be expected in general.

The mathematical analysis of degenerate elliptic equations has deserved considerable attention and is now well documented in the literature (see for example the recent books [8] and [11] and their references). To our knowledge, the corresponding problems in free domains have not been considered for mixed boundary conditions. In the nondegenerate case, a problem of this type was studied in [7] and an existence result was obtained without the assumption of a plane top boundary, i.e., taking the surface tension into account.

This paper is organized as follows. After establishing some auxiliary results in Section 2, we prove the existence and uniqueness of a weak solution to problem (P) in Section 3, by making a detailed analysis of the problem (P_ζ) when ζ varies in $\mathbb{R}^+$. In Section 4, we prove a continuous dependence with respect to variations of the operator, i.e., to the function α in the class (A1)–(A3). The main difficulty here is that the domain Ω_ζ also varies in order that (7) is kept. Finally, in the last section, we give two examples of applications of the preceding results in fluid mechanics, namely to the study of steady non-Newtonian flows in open channels or valleys. In particular, we solve a problem arising in theoretical glaciology and extend the existence theorem to the case of a Bingham flow, by adapting the continuous dependence result. The uniqueness of solution for this last problem seems to be an interesting open question.

2. Some auxiliary results

In this section we introduce some notation and gather a few lemmas that will be used later, simplifying the main proofs.

We start with the question of the extension of a Sobolev function defined in a set Ω_ζ, $\xi > 0$, of the form (2) to a larger set of this type. The fact that the top surface of Ω_ζ is flat allows us to use the results concerning the extension of $W^{1,p}$ functions by reflection. Introducing the reflected set of Ω_ζ, $\widehat{\Omega_\zeta} = \{x \in \mathbb{R}^N\colon \zeta < x_N < 2\zeta - h(x')\}$, we denote the open subset

$$\Omega^*_\zeta = \left\{x \in \mathbb{R}^N\colon h(x') < x_N < 2\zeta - h(x')\right\},$$

which is $\Omega_\zeta \cup \widehat{\Omega_\zeta}$ together with the interior of their common boundary. Given a function $w \in W^{1,p}_H(\Omega_\zeta)$, we define its extension w^* to Ω^*_ζ, through reflection with respect to Π_ζ, i.e.,

$$w^*(x', x_N) = \begin{cases} w(x', x_N) & \text{if } (x', x_N) \in \Omega_\zeta, \\ \widehat{w}(x', x_N) & \text{if } (x', x_N) \in \widehat{\Omega_\zeta}, \end{cases}$$

where $\widehat{w}(x', x_N) = w(x', 2\zeta - x_N)$. We have the following result:

LEMMA 1. – *Given $w \in W^{1,p}_H(\Omega_\zeta)$, its extension is such that $w^* \in W^{1,p}_0(\Omega^*_\zeta)$ and it satisfies the estimate*

$$\|w^*\|_{W^{1,p}_0(\Omega^*_\zeta)} \leqslant 2\|w\|_{W^{1,p}(\Omega_\zeta)}. \tag{9}$$

Moreover, we have

$$\nabla w^* = \left(\left(\frac{\partial w}{\partial x_1}\right)^*, \dots, \left(\frac{\partial w}{\partial x_{N-1}}\right)^*, \left(\frac{\partial w}{\partial x_N}\right)^\circ\right),$$

denoting, for w defined in Ω_ζ,

$$w^\circ(x', x_N) = \begin{cases} w(x', x_N) & \text{if } (x', x_N) \in \Omega_\zeta, \\ -\widehat{w}(x', x_N) & \text{if } (x', x_N) \in \widehat{\Omega_\zeta}. \end{cases}$$

Proof. – The fact that $w^* \in W^{1,p}(\Omega_\zeta^*)$, the expression for its gradient and the estimate (9) follow from standard arguments (see for instance [1, p. 158]). To show that $w^* \in W_0^{1,p}(\Omega_\zeta^*)$, observe that, since $w \in W_H^{1,p}(\Omega_\zeta)$, we have:

$$w = \lim w_n \text{ in } W^{1,p}(\Omega_\zeta), \quad w_n \in C_H^1\big(\overline{\Omega_\xi}\big).$$

We can extend each w_n by reflection, obtaining functions $w_n^* \in W^{1,p}(\Omega_\zeta^*)$ with compact support in Ω_ζ^*. Hence, they belong to $W_0^{1,p}(\Omega_\zeta^*)$ and since this space is closed we conclude from

$$\|w_n^* - w^*\|_{W_0^{1,p}(\Omega_\zeta^*)} = \big\|(w_n - w)^*\big\|_{W_0^{1,p}(\Omega_\zeta^*)} \leqslant 2\|w_n - w\|_{W^{1,p}(\Omega_\zeta)} \to 0. \qquad \square$$

If we want to extend a function $w \in W_H^{1,p}(\Omega_\zeta)$ to a set Ω_K for $K > 2\zeta$, we first obtain $w^* \in W_0^{1,p}(\Omega_\zeta^*)$ and then extend it canonically by zero to Ω_K, obtaing a function in $W_0^{1,p}(\Omega_K)$, still denoted with w^*. For an extension to Ω_K, with $K \leqslant 2\zeta$, we extend the function to some $\Omega_{K'}$, $K' > 2\zeta$, as before, and then restrict w^* to Ω_K. We also denote this restriction with w^* if it is clear from the context in which domain the function is to be considered.

As a consequence of the previous lemma, we obtain the following compactness result, that is valid for any Ω_ζ, $\zeta > 0$, irrespective of its regularity.

COROLLARY 1. – *The injection $W_H^{1,p}(\Omega_\zeta) \hookrightarrow L^p(\Omega_\zeta)$ is compact.*

Proof. – The injection can be seen as the following composition

$$\begin{array}{ccccccc} W_H^{1,p}(\Omega_\zeta) & \to & W_0^{1,p}\big(\Omega_\zeta^*\big) & \hookrightarrow & L^p\big(\Omega_\zeta^*\big) & \to & L^p(\Omega_\zeta), \\ w & \mapsto & w^* & \mapsto & w^* & \mapsto & w^*|_{\Omega_\zeta} = w, \end{array}$$

the first function being continuous due to Lemma 1, the second compact due to the theorem of Rellich–Kondrachov and the last one also continuous, since it is a restriction of an L^p function. $\square$

We proceed introducing some notation. Given a function w, the restriction of w to a subset S of its domain is as usually denoted by $w|_S$. If w is defined in a set like Ω_ζ, its restriction to $\Omega_{\zeta-\eta}$, for $0 < \eta < \zeta$, is simply denoted by $w|_{(\zeta-\eta)}$.

Another recurrent tool in this work is the translation of a function w defined in a subset D of $\mathbb{R}^N$ with respect to a vector parallel to $\vec{e}_N$. Given $\rho > 0$, we define the translated set

$$T_\rho D = \big\{x \in \mathbb{R}^N \colon (x', x_N - \rho) \in D\big\},$$

and the translated function $\tau_\rho w(x', x_N) = w(x', x_N - \rho)$, for all $(x', x_N) \in T_\rho D$.

The next lemma will be used in Section 3. Given $0 < \eta < \zeta$ and $w \in W_H^{1,p}(\Omega_{\zeta-\eta})$, define in Ω_ζ, the function:

$$\overline{\tau}_\eta w(x', x_N) = \begin{cases} 0 & \text{if } h(x') < x_N \leqslant h(x') + \eta, \\ \tau_\eta w(x', x_N) & \text{if } h(x') + \eta < x_N < \zeta \end{cases}$$

that clearly belongs to $W_H^{1,p}(\Omega_\zeta)$.

LEMMA 2. – *If $v \in W_H^{1,p}(\Omega_\zeta)$, then as $\eta \to 0$, $\overline{\tau}_\eta v|_{(\zeta-\eta)} \to v$ strongly in $W^{1,p}(\Omega_\zeta)$.*

Proof. – Adapt in the obvious way the well known result on translations of L^p functions (see [1, p. 75], for example), noting that $\nabla \overline{\tau}_\eta v|_{(\zeta-\eta)} = \overline{\tau}_\eta [\nabla v|_{(\zeta-\eta)}]$. □

The last lemma concerns a certain kind of approximation of extended functions, that will be essential in Section 4. Consider a sequence of positive real numbers $\zeta_\varepsilon \to \zeta$, with $\zeta_\varepsilon \leqslant M$, $\forall \varepsilon > 0$. Choose $K > 2M$ and for each $v \in W_H^{1,p}(\Omega_\zeta)$, define the sequence $(v_\varepsilon^\diamond)_{\varepsilon>0}$, with

$$v_\varepsilon^\diamond = \begin{cases} v^* & \text{if } \zeta_\varepsilon \geqslant \zeta, \\ (v|_{\Omega_{\zeta_\varepsilon}})^* & \text{if } \zeta_\varepsilon < \zeta, \end{cases}$$

both the reflections being followed by extensions by zero to Ω_K.

LEMMA 3. – *For each $\varepsilon > 0$, the function $v_\varepsilon^\diamond \in W_0^{1,p}(\Omega_K)$ and $v_\varepsilon^\diamond = 0$ in $\Omega_K \setminus \Omega_{\zeta_\varepsilon}^*$. Moreover, as $\zeta_\varepsilon \to \zeta$,*

$$v_\varepsilon^\diamond \to v^* \text{ strongly in } W^{1,p}(\Omega_K). \tag{10}$$

Proof. – Only the convergence result is non trivial. We just show that

$$\frac{\partial v_\varepsilon^\diamond}{\partial x_i} \to \frac{\partial v^*}{\partial x_i}, \text{ strongly in } L^p(\Omega_K), \quad 1 \leqslant i \leqslant N-1,$$

since the strong convergence $v_\varepsilon^\diamond \to v^*$ in $L^p(\Omega_K)$ is similar but simpler and the case for the derivative with respect to x_N only differs from this one in a few irrelevant minus signs. We have, for $1 \leqslant i \leqslant N-1$, using Lemma 1:

$$\left\| \frac{\partial v_\varepsilon^\diamond}{\partial x_i} - \frac{\partial v^*}{\partial x_i} \right\|_{L^p(\Omega_K)} = \begin{cases} 0 & \text{if } \zeta_\varepsilon \geqslant \zeta, \\ \left\| \left(\left(\frac{\partial v}{\partial x_i} \right) \Big|_{\Omega_{\zeta_\varepsilon}} \right)^* - \left(\frac{\partial v}{\partial x_i} \right)^* \right\|_{L^p(\Omega_K)} & \text{if } \zeta_\varepsilon < \zeta. \end{cases}$$

We focus on the relevant case $\zeta_\varepsilon < \zeta$:

$$\begin{aligned} &\left\| \left(\left(\frac{\partial v}{\partial x_i} \right) \Big|_{\Omega_{\zeta_\varepsilon}} \right)^* - \left(\frac{\partial v}{\partial x_i} \right)^* \right\|_{L^p(\Omega_K)}^p \\ &\quad = \int_{A_\varepsilon} \left| \frac{\partial v}{\partial x_i}(x', x_N) \right|^p + \int_{B_\varepsilon} \left| \frac{\partial v}{\partial x_i}(x', 2\zeta_\varepsilon - x_N) - \frac{\partial v}{\partial x_i}(x', x_N) \right|^p \\ &\qquad + \int_{C_\varepsilon} \left| \frac{\partial v}{\partial x_i}(x', 2\zeta_\varepsilon - x_N) - \frac{\partial v}{\partial x_i}(x', 2\zeta - x_N) \right|^p + \int_{D_\varepsilon} \left| \frac{\partial v}{\partial x_i}(x', 2\zeta - x_N) \right|^p, \end{aligned}$$

where we have:

$$A_\varepsilon = \{x \in \Omega_K\colon\ 2\zeta_\varepsilon - h(x') < x_N < \zeta\},$$
$$B_\varepsilon = \{x \in \Omega_K\colon\ \zeta_\varepsilon < x_N < \min(2\zeta_\varepsilon - h(x'), \zeta)\},$$
$$C_\varepsilon = \{x \in \Omega_K\colon\ \zeta < x_N < 2\zeta_\varepsilon - h(x')\},$$
$$D_\varepsilon = \{x \in \Omega_K\colon\ \max(2\zeta_\varepsilon - h(x'), \zeta) < x_N < 2\zeta - h(x')\}.$$

When we let $\zeta_\varepsilon \to \zeta^-$, $|A_\varepsilon| \to 0$ and $|D_\varepsilon| \to 0$ and so the corresponding integrals vanish. Concerning the integral over B_ε, we find, with $E_\varepsilon = \{x \in \Omega_K\colon\ \max(2\zeta_\varepsilon - \zeta, h(x')) < x_N < \zeta_\varepsilon\}$,

$$\int_{B_\varepsilon} \left|\frac{\partial v}{\partial x_i}(x', 2\zeta_\varepsilon - x_N) - \frac{\partial v}{\partial x_i}(x', x_N)\right|^p \leqslant 2^p\left(\int_{B_\varepsilon}\left|\frac{\partial v}{\partial x_i}(x', 2\zeta_\varepsilon - x_N)\right|^p + \int_{B_\varepsilon}\left|\frac{\partial v}{\partial x_i}(x', x_N)\right|^p\right)$$
$$= 2^p\left(\int_{E_\varepsilon}\left|\frac{\partial v}{\partial x_i}(x', x_N)\right|^p + \int_{B_\varepsilon}\left|\frac{\partial v}{\partial x_i}(x', x_N)\right|^p\right) \to 0,$$

because also $|B_\varepsilon| \to 0$ and $|E_\varepsilon| \to 0$. The remaining integral also vanishes since

$$\int_{C_\varepsilon} \left|\frac{\partial v}{\partial x_i}(x', 2\zeta_\varepsilon - x_N) - \frac{\partial v}{\partial x_i}(x', 2\zeta - x_N)\right|^p \leqslant \left\|\tau_{2(\zeta-\zeta_\varepsilon)}\frac{\partial v}{\partial x_i} - \frac{\partial v}{\partial x_i}\right\|^p_{L^p(\mathbb{R}^N)} \to 0,$$

since $\partial v/\partial x_i$ extended by zero to $\mathbb{R}^N$ belongs to $L^p(\mathbb{R}^N)$. □

3. Existence and uniqueness

In this section we establish our main result, namely:

THEOREM 1. – *Under assumptions* (A1)–(A4)*, given any* $Q > 0$*, there exists a unique weak solution to* (P)*.*

The proof will be reduced to the analysis of the function of one real parameter

$$\Phi : \mathbb{R}^+ \ni \zeta \mapsto (\Omega_\zeta, u_\zeta) \mapsto \int_{\Omega_\zeta} u_\zeta \in \mathbb{R},$$

where u_ζ is the solution of (P_ζ), and to the question: Does the range of Φ contain the given $Q > 0$?

We can consider the problem corresponding to the choice $\alpha(r) = r^{p-2}$, $r > 0$, in the non physical one dimensional case, to see what happens in this simpler situation. We have $\Omega_\zeta = (0, \zeta)$ and the problem (P_ζ) consists of solving the ordinary boundary value problem

$$\left(|u_\zeta'|^{p-2}u_\zeta'\right)' = -1 \quad \text{in } (0, \zeta),$$

with boundary conditions $u_\zeta(0) = 0$ and $u_\zeta'(\zeta) = 0$. We find

$$u_\zeta(z) = \frac{1-p}{p}\left[(-z+\zeta)^{\frac{p}{p-1}} - \zeta^{\frac{p}{p-1}}\right] \quad \text{and} \quad \Phi(\zeta) = \int_0^\zeta u_\zeta = \frac{p-1}{2p-1}\zeta^{\frac{2p-1}{p-1}},$$

so, Φ being bijective, given $Q>0$, we just have to choose

$$\xi=\left(\frac{2p-1}{p-1}Q\right)^{\frac{p-1}{2p-1}},$$

and the solution of the problem is

$$u(z)=\frac{1-p}{p}\left[\left\{-z+\left(\frac{2p-1}{p-1}Q\right)^{\frac{p-1}{2p-1}}\right\}^{\frac{p}{p-1}}-\left(\frac{2p-1}{p-1}Q\right)^{\frac{p}{2p-1}}\right]\quad \text{in } \left(0,\left(\frac{2p-1}{p-1}Q\right)^{\frac{p-1}{2p-1}}\right).$$

The analysis of the general case cannot be done explicitly in higher dimensions for general α and h. Therefore, we fix ζ and choose $v=-u_\zeta^-$ in (6), to find, from (A2):

$$\begin{aligned}C_1\int_{\Omega_\zeta\cap\{u_\zeta<0\}}|\nabla u_\zeta|^p &\leqslant \int_{\Omega_\zeta\cap\{u_\zeta<0\}}\alpha\big(|\nabla u_\zeta|\big)|\nabla u_\zeta|^2\\ &=-\int_{\Omega_\zeta}\alpha\big(|\nabla u_\zeta|\big)\nabla u_\zeta\cdot\nabla u_\zeta^-=-\int_{\Omega_\zeta}u_\zeta^-\leqslant 0.\end{aligned}$$

This implies $\int_{\Omega_\zeta}|\nabla u_\zeta^-|^p=0$, hence $u_\zeta^-=0$ and $u_\zeta\geqslant 0$. So $\Phi(\zeta)\geqslant 0,\ \forall\zeta\in\mathbb{R}^+$ and it would be zero only if $u_\zeta=0$ a.e. in Ω_ζ, which is not possible. So the range of Φ is contained in $\mathbb{R}^+$, thus choosing $Q>0$ is necessary. It turns out that it is also sufficient by the following result, that clearly implies Theorem 1.

PROPOSITION 1. – *The function $\Phi:\mathbb{R}^+\to\mathbb{R}$ is a continuous, strictly increasing function, whose range is the set* $(0,+\infty)$. *Hence, it is a bijection in* $\mathbb{R}^+$.

Proof. – For future reference, note that, using (A2), the unique solution of (P_ζ), $\zeta\in(0,\infty)$, obviously satisfies

$$C_1\int_{\Omega_\zeta}|\nabla u_\zeta|^p\leqslant\int_{\Omega_\zeta}\alpha\big(|\nabla u_\zeta|\big)|\nabla u_\zeta|^2=\int_{\Omega_\zeta}u_\zeta=\Phi(\zeta). \tag{11}$$

(i) *Monotonicity*: For each $\hat\zeta>\zeta$, put $\hat u\equiv u_{\hat\zeta}$ and $u\equiv u_\zeta$. Let $\varrho=\hat\zeta-\zeta>0$ and define the translation of the set Ω_ζ

$$\omega_\varrho=T_\varrho(\Omega_\zeta)=\big\{x\in\mathbb{R}^N:\ h(x')+\varrho<x_N<\zeta+\varrho\big\}.$$

It is clear that the function $\tau_\varrho u$, defined in ω_ϱ, satisfies

$$(\mathrm{P}_\varrho)\qquad \tau_\varrho u\in W^{1,p}_{H+\varrho}(\omega_\varrho):\int_{\omega_\varrho}\alpha\big(|\nabla\tau_\varrho u|\big)\nabla\tau_\varrho u\cdot\nabla v=\int_{\omega_\varrho}v,\quad \forall v\in W^{1,p}_{H+\varrho}(\omega_\varrho),$$

where $H+\varrho$ denotes the graph of the function $h+\varrho$.
We now show that:

$$\Phi(\hat\zeta)=\int_{\Omega_{\hat\zeta}}\hat u>\int_{\omega_\varrho}\hat u\geqslant\int_{\omega_\varrho}\tau_\varrho u=\int_{\Omega_\zeta}u=\Phi(\zeta). \tag{12}$$

The equality is an obvious consequence of translation and the strict inequality follows from the fact that $\hat{u}$ can not vanish in the open set $\mathcal{O} = \Omega_{\hat{\zeta}} \setminus \omega_\varrho$. In fact, if $\hat{u} = 0$ a.e. in $\mathcal{O}$, then also $|\nabla \hat{u}| = 0$ a.e. in $\mathcal{O}$ and by Remark 3 that would contradict the fact that $\hat{u}$ solves there the equation $\nabla \cdot (\alpha(|\nabla \hat{u}|)\nabla \hat{u}) = -1$, in the sense of distributions. To conclude it suffices to show that $\hat{u} \geqslant \tau_\varrho u$ a.e. in ω_ϱ. Take the function:

$$v = \begin{cases} (\hat{u} - \tau_\varrho u)^- & \text{in } \omega_\varrho, \\ 0 & \text{in } \Omega_{\hat{\zeta}} \setminus \omega_\varrho, \end{cases}$$

which can be used as a test function both in (P_ϱ) and $(\mathrm{P}_{\hat{\zeta}})$.

We obtain

$$\int_{\omega_\varrho} \alpha\big(|\nabla \tau_\varrho u|\big)\nabla \tau_\varrho u \cdot \nabla(\hat{u} - \tau_\varrho u)^- = \int_{\omega_\varrho} (\hat{u} - \tau_\varrho u)^- = \int_{\omega_\varrho} \alpha\big(|\nabla \hat{u}|\big)\nabla \hat{u} \cdot \nabla(\hat{u} - \tau_\varrho u)^-.$$

Subtracting, we get

$$\int_{\omega_\varrho \cap \{\hat{u} < \tau_\varrho u\}} \big[\alpha\big(|\nabla \tau_\varrho u|\big)\nabla \tau_\varrho u - \alpha\big(|\nabla \hat{u}|\big)\nabla \hat{u}\big] \cdot \nabla(\tau_\varrho u - \hat{u}) = 0,$$

and using (A3) we conclude that:

$$\big\|\nabla(\hat{u} - \tau_\varrho u)^-\big\|_{L^p(\omega_\varrho)} = 0, \quad \text{i.e.,} \quad \hat{u} \geqslant \tau_\varrho u \quad \text{a.e. in } \omega_\varrho.$$

(ii) *Continuity*: We start with right continuity. We want to show that, for fixed ζ, $\Phi(\zeta + \eta) \to \Phi(\zeta)$ when $\eta \to 0^+$. We start by showing that

$$\int_{\Omega_{\zeta+\eta} \setminus \Omega_\zeta} u_{\zeta+\eta} \to 0. \tag{13}$$

With $0 < \eta < 1$, consider the translation

$$\omega_{(1-\eta)} = T_{(1-\eta)}(\Omega_{\zeta+\eta} \setminus \Omega_\zeta) = \big\{x \in \mathbb{R}^N \colon \zeta + 1 - \eta \leqslant x_N < \zeta + 1,\ x_N > h(x') + 1 - \eta\big\}.$$

The same reasoning used to obtain (12), yields in this context

$$0 \leqslant \int_{\Omega_{\zeta+\eta} \setminus \Omega_\zeta} u_{\zeta+\eta} \leqslant \int_{\omega_{(1-\eta)}} u_{\zeta+1} \to 0,$$

because $|\omega_{(1-\eta)}| \leqslant C\{(\zeta + 1) - [\max[h(x'), \zeta] + (1 - \eta)]\} \leqslant C\eta \to 0$.

Then from

$$\Phi(\zeta + \eta) = \int_{\Omega_{\zeta+\eta} \setminus \Omega_\zeta} u_{\zeta+\eta} + \int_{\Omega_\zeta} u_{\zeta+\eta},$$

and (13), we need only show the second term converges to $\Phi(\zeta)$. First observe that, due to (11) and the monotonicity of Φ,

$$\|\nabla u_{\zeta+\eta}\|^p_{L^p(\Omega_\zeta)} \leqslant \int\limits_{\Omega_{\zeta+\eta}} |\nabla u_{\zeta+\eta}|^p \leqslant \frac{1}{C_1}\Phi(\zeta+\eta) \leqslant \frac{1}{C_1}\Phi(\zeta+1), \quad \forall 0<\eta<1,$$

and so $(u_{\zeta+\eta})$ is bounded in $W^{1,p}_H(\Omega_\zeta)$ and we can extract a subsequence such that $u_{\zeta+\eta} \rightharpoonup u$ in $W^{1,p}_H(\Omega_\zeta)$-weak, for some $u \in W^{1,p}_H(\Omega_\zeta)$. If we show that u is a solution to the limit problem (P_ζ) we get as a consequence, due to the uniqueness, that $u = u_\zeta$. Since the imbedding $W^{1,p}_H(\Omega_\zeta) \hookrightarrow L^1(\Omega_\zeta)$ is compact, due to Corollary 1, this gives

$$\int\limits_{\Omega_\zeta} u_{\zeta+\eta} \to \int\limits_{\Omega_\zeta} u_\zeta = \Phi(\zeta).$$

So take an arbitrary $v \in W^{1,p}_H(\Omega_\zeta)$ and consider its extension v^* (see Section 2), that can be used as a test function in $(\mathrm{P}_{\zeta+\eta})$. We obtain:

$$\begin{aligned}
0 &\leqslant \int\limits_{\Omega_{\zeta+\eta}} \big(\alpha\big(|\nabla u_{\zeta+\eta}|\big)\nabla u_{\zeta+\eta} - \alpha\big(|\nabla v^*|\big)\nabla v^*\big)\cdot(\nabla u_{\zeta+\eta} - \nabla v^*) \\
&= \int\limits_{\Omega_{\zeta+\eta}\setminus\Omega_\zeta} (u_{\zeta+\eta} - v^*) - \int\limits_{\Omega_{\zeta+\eta}\setminus\Omega_\zeta} \alpha\big(|\nabla v^*|\big)\nabla v^*\cdot(\nabla u_{\zeta+\eta} - \nabla v^*) \\
&\quad + \int\limits_{\Omega_\zeta} (u_{\zeta+\eta} - v) - \int\limits_{\Omega_\zeta} \alpha\big(|\nabla v|\big)\nabla v\cdot(\nabla u_{\zeta+\eta} - \nabla v).
\end{aligned}$$

Now we take the limit as $\eta \to 0^+$, getting

$$\int\limits_{\Omega_\zeta} (u - v) - \int\limits_{\Omega_\zeta} \alpha\big(|\nabla v|\big)\nabla v\cdot(\nabla u - \nabla v) \geqslant 0,$$

because all the integrals over $\Omega_{\zeta+\eta}\setminus\Omega_\zeta$ vanish. In fact, for two of them this is obvious, for another we can use (13) and for the remaining note that we have:

$$\left|\int\limits_{\Omega_{\zeta+\eta}\setminus\Omega_\zeta} \alpha\big(|\nabla v^*|\big)\nabla v^*\cdot\nabla u_{\zeta+\eta}\right| \leqslant \left(\int\limits_{\Omega_{\zeta+\eta}\setminus\Omega_\zeta} \big|\alpha\big(|\nabla v^*|\big)\nabla v^*\big|^{p'}\right)^{1/p'} \left(\int\limits_{\Omega_{\zeta+\eta}\setminus\Omega_\zeta} |\nabla u_{\zeta+\eta}|^p\right)^{1/p},$$

and here the first factor goes to zero while the second one is bounded by $[\frac{1}{C_1}\Phi(\zeta+1)]^{1/p}$ for $0<\eta<1$, thus yielding the result.

We can now use the technique of Minty for monotone operators. Since v is arbitrary, choose $v = u - \delta w$, with $\delta \in \mathbb{R}$ and $w \in W^{1,p}_H(\Omega_\zeta)$ and then take δ to zero, using the hemicontinuity, and conclude that

$$\int\limits_{\Omega_\zeta} \alpha\big(|\nabla u|\big)\nabla u\cdot\nabla w = \int\limits_{\Omega_\zeta} w, \quad \forall w \in W^{1,p}_H(\Omega_\zeta),$$

thus proving that u is a solution to (P_ζ).

Concerning left continuity, we show that $\Phi(\zeta-\eta)\to\Phi(\zeta)$ when $\eta\to 0^+$. The sequence $(\overline{\tau}_\eta u_{\zeta-\eta})_\eta$ is bounded in $W^{1,p}(\Omega_\zeta)$, independently of η. In fact, recalling the definition from Section 2 and using (11) and the monotonicity, we have

$$\|\overline{\tau}_\eta u_{\zeta-\eta}\|^p_{W^{1,p}(\Omega_\zeta)}=\|u_{\zeta-\eta}\|^p_{W^{1,p}(\Omega_{\zeta-\eta})}\leqslant\frac{1}{C_1}\Phi(\zeta-\eta)\leqslant\frac{1}{C_1}\Phi(\zeta);$$

we can then extract a subsequence such that

$$\overline{\tau}_\eta u_{\zeta-\eta}\rightharpoonup u\quad\text{in } W^{1,p}_H(\Omega_\zeta)\text{-weak.}\tag{14}$$

Once we show that this weak limit solves the problem in Ω_ζ, we obtain, from the uniqueness, $u=u_\zeta$ and we conclude from

$$0\leqslant\Phi(\zeta)-\Phi(\zeta-\eta)=\int\limits_{\Omega_\zeta}u_\zeta-\int\limits_{\Omega_{\zeta-\eta}}u_{\zeta-\eta}=\int\limits_{\Omega_\zeta}\alpha\big(|\nabla u_\zeta|\big)\nabla u_\zeta\cdot\nabla(u_\zeta-\overline{\tau}_\eta u_{\zeta-\eta})\to 0.$$

To this end, it is enough to show that

$$\int\limits_{\Omega_\zeta}\alpha\big(|\nabla u|\big)\nabla u\cdot\nabla v=\int\limits_{\Omega_\zeta}v,\quad\forall v\in W^{1,p}_H(\Omega_\zeta).$$

Given such a v and starting from

$$0\leqslant\int\limits_{\Omega_\zeta}\big[\alpha\big(|\nabla\overline{\tau}_\eta u_{\zeta-\eta}|\big)\nabla\overline{\tau}_\eta u_{\zeta-\eta}-\alpha\big(|\nabla\overline{\tau}_\eta v|_{(\zeta-\eta)}|\big)\nabla\overline{\tau}_\eta v|_{(\zeta-\eta)}\big]\cdot\nabla(\overline{\tau}_\eta u_{\zeta-\eta}-\overline{\tau}_\eta v|_{(\zeta-\eta)}),$$

we can conclude as before, through Minty's lemma, if we are able to pass to the limit. This can be done, having in mind the equation, the convergence (14) and Lemma 2.

(iii) *Range*: First we show that $\Phi(\zeta)\to 0$ when $\zeta\to 0$. In fact, using the inequalities of Hölder and Poincaré, we get:

$$\begin{aligned}\Phi(\zeta)=\int\limits_{\Omega_\zeta}u_\zeta&\leqslant|\Omega_\zeta|^{\frac{1}{p'}}\|u_\zeta\|_{L^p(\Omega_\zeta)}\leqslant C|\Omega_\zeta|^{\frac{1}{p'}+\frac{1}{N}}\|\nabla u_\zeta\|_{L^p(\Omega_\zeta)}\\&\leqslant C|\Omega_\zeta|^{\frac{1}{p'}+\frac{1}{N}}\left(\frac{1}{C_1}\right)^{\frac{1}{p}}\Phi(\zeta)^{\frac{1}{p}},\end{aligned}$$

using (A2) and with C depending only on p and N. We ultimately obtain

$$0\leqslant\Phi(\zeta)\leqslant C'|\Omega_\zeta|^{1+\frac{p}{N(p-1)}},$$

and let ζ converge to zero to conclude.

Next, we prove that $\Phi(\zeta)\to\infty$ when $\zeta\to\infty$. In fact, due to (8), we have, for any $v\in W^{1,p}_H(\Omega_\zeta)$,

$$-\Phi(\zeta)\leqslant\int\limits_{\Omega_\zeta}A\big(|\nabla u_\zeta|\big)-\int\limits_{\Omega_\zeta}u_\zeta\leqslant\int\limits_{\Omega_\zeta}A\big(|\nabla v|\big)-\int\limits_{\Omega_\zeta}v$$

and consequently, using (A2),

$$\Phi(\zeta) \geqslant \int_{\Omega_\zeta} v - \int_{\Omega_\zeta} A\big(|\nabla v|\big) \geqslant \int_{\Omega_\zeta} v - \frac{C_2}{p}\int_{\Omega_\zeta} |\nabla v|^p - C_3 \int_{\Omega_\zeta} |\nabla v|, \quad \forall v \in W_H^{1,p}(\Omega_\zeta). \tag{15}$$

We now make an appropriate choice of a sequence v_ζ, such that the second member of (15) will tend to infinity with ζ thus concluding that the same happens with Φ. Without loss of generality, we may choose $\rho > 0$ and $\sigma > 0$ such that, for all $\zeta > \sigma + 1$, the cylinder $\mathcal{C}_\zeta = B'(\rho) \times (\sigma, \zeta)$ is contained in Ω_ζ, where $B'(\rho)$ denotes the ball in $\mathbb{R}^{N-1}$ with center at the origin and radius ρ. Define $v_\zeta \in W_H^{1,p}(\Omega_\zeta)$, as

$$v_\zeta(x', x_N) = \begin{cases} f(x')g_\zeta(x_N) & \text{if } (x', x_n) \in \mathcal{C}_\zeta, \\ 0 & \text{if } (x', x_N) \in \Omega_\zeta \setminus \mathcal{C}_\zeta, \end{cases}$$

where $g_\zeta : (\sigma, \zeta) \to \mathbb{R}$ is the function

$$g_\zeta(x_N) = \begin{cases} x_N - \sigma & \text{if } x_N \in (\sigma, \sigma + 1], \\ 1 & \text{if } x_N \in (\sigma + 1, \zeta) \end{cases}$$

and $f : B'(\rho) \to \mathbb{R}$ is a radial function defined, for $r = |x'|$, by

$$f(r) = \begin{cases} 1 & \text{if } 0 \leqslant r \leqslant \frac{\rho}{2}, \\ 2 - \frac{2}{\rho} r & \text{if } \frac{\rho}{2} < r < \rho. \end{cases}$$

We have $\nabla v_\zeta = (g_\zeta \nabla' f, f g_\zeta')$ and $|\nabla v_\zeta|^p \leqslant 2^{(\frac{p-2}{2})^+}[g_\zeta^p |\nabla' f|^p + f^p |g_\zeta'|^p]$. Using the theorem of Fubini, we have:

$$\begin{aligned} &\int_{\Omega_\zeta} v_\zeta - \frac{C_2}{p}\int_{\Omega_\zeta} |\nabla v_\zeta|^p - C_3 \int_{\Omega_\zeta} |\nabla v_\zeta| \\ &\quad \geqslant \int_\sigma^\zeta g_\zeta \int_{B'(\rho)} f - \frac{C_2}{p} 2^{(\frac{p-2}{2})^+} \left[\int_\sigma^\zeta g_\zeta^p \int_{B'(\rho)} |\nabla' f|^p + \int_{B'(\rho)} f^p \int_\sigma^\zeta |g_\zeta'|^p \right] \\ &\quad\quad - C_3 \left[\int_\sigma^\zeta g_\zeta \int_{B'(\rho)} |\nabla' f| + \int_{B'(\rho)} f \int_\sigma^\zeta |g_\zeta'| \right] \\ &\quad = (\zeta - \sigma) \left[\int_{B'(\rho)} f - C_3 \int_{B'(\rho)} |\nabla' f| - \frac{C_2}{p} 2^{(\frac{p-2}{2})^+} \int_{B'(\rho)} |\nabla' f|^p \right] + C, \end{aligned} \tag{16}$$

with C a constant independent of ζ. This last identity holds since

$$\int_\sigma^\zeta g_\zeta = \zeta - \sigma - \frac{1}{2}, \quad \int_\sigma^\zeta g_\zeta^p = \zeta - \sigma - \frac{p}{p-1} \quad \text{and} \quad \int_\sigma^\zeta |g_\zeta'|^p = \int_\sigma^\zeta |g_\zeta'| = 1.$$

We now compute

$$\int_{B'(\rho)} f = C(N)\rho^{N-1} \quad \text{and} \quad \int_{B'(\rho)} |\nabla' f|^p = C(N,p)\rho^{N-p-1}, \quad p \geqslant 1,$$

and, from (15), we obtain, through (16),

$$\Phi(\zeta) \geqslant (\zeta - \sigma)\rho^{N-2}\big[k_1\rho - k_2 - k_3\rho^{1-p}\big] + C,$$

for positive constants k_1, k_2 and k_3 depending only on N, p and the constants in (A2). Now the second member in this inequality goes to infinity with ζ provided $k_1\rho - k_2 - k_3\rho^{1-p} > 0$. Since this expression goes to infinity when ρ goes to infinity this holds true if we choose ρ large enough, which is always possible due to our assumption (A4) on h and choosing σ also large enough. □

Remark 5. – In the case of the p-Laplacian, the proof of the left continuity can be done in a very easy way, that we briefly describe. Since we can use u_ζ as a test function in ($P_{\zeta-\eta}$), i.e., in (6) for $u_{\zeta-\eta}$, we so obtain:

$$\int_{\Omega_{\zeta-\eta}} u_\zeta = \int_{\Omega_{\zeta-\eta}} |\nabla u_{\zeta-\eta}|^{p-2}\nabla u_{\zeta-\eta}\cdot\nabla u_\zeta \leqslant \bigg(\int_{\Omega_{\zeta-\eta}} |\nabla u_{\zeta-\eta}|^p\bigg)^{\frac{1}{p'}}\bigg(\int_{\Omega_{\zeta-\eta}} |\nabla u_\zeta|^p\bigg)^{\frac{1}{p}},$$

using Hölder's inequality. Using (11), this can be written in the form

$$\Phi(\zeta-\eta) \geqslant \frac{\big(\int_{\Omega_{\zeta-\eta}} u_\zeta\big)^{p'}}{\big(\int_{\Omega_{\zeta-\eta}} |\nabla u_\zeta|^p\big)^{\frac{p'}{p}}}.$$

Taking the limit when $\eta \to 0^+$, this implies $\lim_{\eta\to 0^+}\Phi(\zeta-\eta) \geqslant \Phi(\zeta)$.

4. Continuous dependence on the operator

In this section we obtain some continuous dependence results with respect to perturbations of the operator. Consider a family of functions $\alpha_\varepsilon, \alpha_0 : \mathbb{R}^+ \to \mathbb{R}$, satisfying assumptions (A1)–(A3) uniformly in ε and let $(\xi_\varepsilon, u_{\xi_\varepsilon})$ be the solution of the problem corresponding to $\varepsilon \geqslant 0$:

$$(\mathrm{P}_\varepsilon) \quad \begin{cases} (\xi_\varepsilon, u_{\xi_\varepsilon}) \in \mathbb{R}^+ \times W_H^{1,p}(\Omega_{\xi_\varepsilon}) \\ \displaystyle\int_{\Omega_{\xi_\varepsilon}} \alpha_\varepsilon\big(|\nabla u_{\xi_\varepsilon}|\big)\nabla u_{\xi_\varepsilon}\cdot\nabla v = \int_{\Omega_{\xi_\varepsilon}} v, \quad \forall v \in W_H^{1,p}(\Omega_{\xi_\varepsilon}), \\ \displaystyle\int_{\Omega_{\xi_\varepsilon}} u_{\xi_\varepsilon} = Q. \end{cases}$$

We want to show that, as $\varepsilon \to 0$, $(\xi_\varepsilon, u_{\xi_\varepsilon})$ converge to (ξ_0, u_{ξ_0}) in some sense to be made clear.

We show that the following theorem holds:

THEOREM 2. – *If $\alpha_\varepsilon \to \alpha_0$ uniformly on compacts of $\mathbb{R}^+$ as $\varepsilon \to 0$, then the solutions of the corresponding problems satisfy*

$$\xi_\varepsilon \to \xi_0 \quad \textit{in } \mathbb{R},$$

$$u^*_{\xi_\varepsilon} \rightharpoonup u \quad \textit{in } W^{1,p}(\Omega_K),\ \textit{for some } K > 0,$$

where u is such that $u|_{\Omega_{\xi_0}} = u_{\xi_0}$.

Proof. – We start with an *a priori* estimate, showing that there exists a constant $C > 0$, independent of ε, such that

$$0 < \xi_\varepsilon < C. \tag{17}$$

Suppose not. This means that we can extract a subsequence (denoted with the same index) such that $\xi_\varepsilon \to +\infty$. Now observe that, like in (15), we can write

$$\Phi_\varepsilon(\xi_\varepsilon) = \int\limits_{\Omega_{\xi_\varepsilon}} u_{\xi_\varepsilon} \geqslant \int\limits_{\Omega_{\xi_\varepsilon}} v - \frac{C_2}{p} \int\limits_{\Omega_{\xi_\varepsilon}} |\nabla v|^p - C_3 \int\limits_{\Omega_{\xi_\varepsilon}} |\nabla v|, \quad \forall v \in W^{1,p}_H(\Omega_{\xi_\varepsilon})$$

and so a sequence (v_{ξ_ε}) can be chosen, using the same reasoning of the previous section, such that the right hand side of this inequality goes to infinity. Note that this is possible since $\xi_\varepsilon \to +\infty$. We reach a contradition since the left hand side is constant and equal to Q. We can then extract a subsequence (denoted with the same index) such that, as $\varepsilon \to 0$,

$$\xi_\varepsilon \to \xi, \quad \xi \geqslant 0.$$

We now observe that still due to the *a priori* estimate (17), $\Omega := \Omega_{2C}$ contains all the sets $\Omega^*_{\xi_\varepsilon}$. For the u_{ξ_ε} corresponding to this subsequence, we obtain, from (9) and (11), the independence of ε estimate for its extensions

$$\|u^*_{\xi_\varepsilon}\|^p_{W^{1,p}_0(\Omega)} \leqslant 2^p \|u_{\xi_\varepsilon}\|^p_{W^{1,p}(\Omega_{\xi_\varepsilon})} \leqslant \frac{2^p}{C_1} \int\limits_{\Omega_{\xi_\varepsilon}} \alpha_\varepsilon\big(|\nabla u_{\xi_\varepsilon}|\big)|\nabla u_{\xi_\varepsilon}|^2 = \frac{2^p Q}{C_1}. \tag{18}$$

We can then extract another subsequence such that

$$u^*_{\xi_\varepsilon} \rightharpoonup u \quad \text{in } W^{1,p}_0(\Omega)\text{-weak, } L^p(\Omega)\text{-strong and pointwise a.e.} \tag{19}$$

for a certain element $u \in W^{1,p}_0(\Omega)$.

We first observe that for $(x', x_N) \in \Omega \setminus \Omega^*_\xi$ and ε sufficiently small, $u^*_{\xi_\varepsilon}(x', x_N) = 0$; the convergence (19) of $u^*_{\xi_\varepsilon}$ to u gives

$$u = 0 \quad \text{a.e. in } \Omega \setminus \Omega^*_\xi. \tag{20}$$

Next we prove that u is symmetric with respect to Π_ξ. For any function w defined in Ω, consider its reflection with respect to Π_ξ:

$$(w \circ \mathcal{R})(x', x_N) = \begin{cases} w(x', 2\xi - x_N) & \text{if } (x', x_N) \in \Omega^*_\xi, \\ 0 & \text{if } (x', x_N) \in \Omega \setminus \Omega^*_\xi. \end{cases}$$

What we want to show is that $u = u \circ \mathcal{R}$, say in $L^p(\Omega)$. We start from:

$$\|u - u \circ \mathcal{R}\|_{L^p(\Omega)} \leqslant \|u - u^*_{\xi_\varepsilon}\|_{L^p(\Omega)} + \|u^*_{\xi_\varepsilon} - u^*_{\xi_\varepsilon} \circ \mathcal{R}\|_{L^p(\Omega)} + \|u^*_{\xi_\varepsilon} \circ \mathcal{R} - u \circ \mathcal{R}\|_{L^p(\Omega)} \tag{21}$$

and show that each term of the second member goes to zero thus obtaining the result. For the first it is obvious, due to (19). The same applies to the third one since

$$\|u^*_{\xi_\varepsilon} \circ \mathcal{R} - u \circ \mathcal{R}\|_{L^p(\Omega)} \leqslant \|u^*_{\xi_\varepsilon} - u\|_{L^p(\Omega)}.$$

The second term is more delicate. To begin with, we write

$$\|u^*_{\xi_\varepsilon} - u^*_{\xi_\varepsilon} \circ \mathcal{R}\|^p_{L^p(\Omega)} = \int\limits_{\Omega \setminus \Omega^*_\xi} |u^*_{\xi_\varepsilon} - u|^p + \|u^*_{\xi_\varepsilon} - u^*_{\xi_\varepsilon} \circ \mathcal{R}\|^p_{L^p(\Omega^*_\xi)}$$

and since the first term goes to zero, we are left with the second one. Now we need to consider two separate cases:

(i) $\xi_\varepsilon > \xi$: In this case, we have $\Omega^*_\xi \subset \Omega^*_{\xi_\varepsilon}$, so

$$\begin{aligned}\|u^*_{\xi_\varepsilon} - u^*_{\xi_\varepsilon} \circ \mathcal{R}\|^p_{L^p(\Omega^*_\xi)} &= \int\limits_{\Omega^*_\xi} \left|u^*_{\xi_\varepsilon}(x', 2\xi_\varepsilon - x_N) - u^*_{\xi_\varepsilon}(x', 2\xi - x_N)\right|^p \mathrm{d}x \\ &= \int\limits_{\Omega^*_\xi} \left| \int\limits_{2\xi - x_N}^{2\xi_\varepsilon - x_N} \frac{\partial}{\partial x_N} u^*_{\xi_\varepsilon}(x', \eta)\,\mathrm{d}\eta \right|^p \mathrm{d}x \leqslant \left[2(\xi_\varepsilon - \xi)\right]^{p-1} \int\limits_{2\xi - x_N}^{2\xi_\varepsilon - x_N} \int\limits_{\Omega^*_\xi} \left| \frac{\partial}{\partial x_N} u^*_{\xi_\varepsilon}(x', \eta)\right|^p \mathrm{d}x\,\mathrm{d}\eta,\end{aligned}$$

and this goes to zero, since we have an estimate for $u^*_{\xi_\varepsilon}$ in $W^{1,p}(\Omega)$ and $|2\xi_\varepsilon - x_N - 2\xi + x_N| \to 0$.

(ii) $\xi_\varepsilon < \xi$: Here we have

$$\begin{aligned}\|u^*_{\xi_\varepsilon} - u^*_{\xi_\varepsilon} \circ \mathcal{R}\|^p_{L^p(\Omega^*_\xi)} &= \int\limits_{\Omega^*_\xi} \left|u^*_{\xi_\varepsilon}(x', x_N) - u^*_{\xi_\varepsilon}(x', 2\xi - x_N)\right|^p \mathrm{d}x \\ &= \int\limits_{A_\varepsilon} \left|u^*_{\xi_\varepsilon}(x', x_N)\right|^p \mathrm{d}x + \int\limits_{B_\varepsilon} \left|u^*_{\xi_\varepsilon}(x', 2\xi_\varepsilon - x_N) - u^*_{\xi_\varepsilon}(x', 2\xi - x_N)\right|^p \mathrm{d}x \\ &\quad + \int\limits_{C_\varepsilon} \left|u^*_{\xi_\varepsilon}(x', 2\xi - x_N)\right|^p \mathrm{d}x,\end{aligned}$$

where we have:

$$\begin{aligned}A_\varepsilon &= \left\{x \in \Omega^*_\xi\colon\ h(x') < x_N < h(x') + 2(\xi - \xi_\varepsilon)\right\}, \\ B_\varepsilon &= \left\{x \in \Omega^*_\xi\colon\ h(x') + 2(\xi - \xi_\varepsilon) < x_N < 2\xi_\varepsilon - h(x')\right\}, \\ C_\varepsilon &= \left\{x \in \Omega^*_\xi\colon\ 2\xi_\varepsilon - h(x') < x_N < 2\xi - h(x')\right\}.\end{aligned}$$

Now we see that $|A_\varepsilon| \to 0$, $|C_\varepsilon| \to 0$ and so the estimate on $u^*_{\xi_\varepsilon}$ cancels both those integrals. For the integral over B_ε, which can be empty, we use the reasoning of case (i). We conclude that (21) converges to zero as we wanted.

Now, observe that for any $v \in W_0^{1,p}(\Omega^*_{\xi_\varepsilon})$,

$$\int\limits_\Omega \alpha_\varepsilon\big(|\nabla u^*_{\xi_\varepsilon}|\big)\nabla u^*_{\xi_\varepsilon} \cdot \nabla v = \int\limits_\Omega \chi_{\Omega^*_{\xi_\varepsilon}}\, v. \tag{22}$$

In fact, with

$$\nabla^- = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{N-1}}, -\frac{\partial}{\partial x_N}\right)$$

and, for a vector $\mathbf{v} = (v_1, \ldots, v_n)$, $\widehat{\mathbf{v}} = (\widehat{v_1}, \ldots, \widehat{v_n})$, we obtain, using Lemma 1,

$$\begin{aligned}
\int\limits_{\widehat{\Omega_{\xi_\varepsilon}}} \alpha_\varepsilon\big(|\nabla u^*_{\xi_\varepsilon}|\big)\nabla u^*_{\xi_\varepsilon} \cdot \nabla v &= \int\limits_{\widehat{\Omega_{\xi_\varepsilon}}} \alpha_\varepsilon\big(|\widehat{\nabla^- u_{\xi_\varepsilon}}|\big)\widehat{\nabla^- u_{\xi_\varepsilon}} \cdot \nabla v = \int\limits_{\Omega_{\xi_\varepsilon}} \alpha_\varepsilon\big(|\nabla^- u_{\xi_\varepsilon}|\big)\nabla^- u_{\xi_\varepsilon} \cdot \widehat{\nabla v} \\
&= \int\limits_{\Omega_{\xi_\varepsilon}} \alpha_\varepsilon\big(|\nabla^- u_{\xi_\varepsilon}|\big)\nabla^- u_{\xi_\varepsilon} \cdot \nabla^- \widehat{v} = \int\limits_{\Omega_{\xi_\varepsilon}} \alpha_\varepsilon\big(|\nabla u_{\xi_\varepsilon}|\big)\nabla u_{\xi_\varepsilon} \cdot \nabla \widehat{v} \\
&= \int\limits_{\Omega_{\xi_\varepsilon}} \widehat{v} = \int\limits_{\widehat{\Omega_{\xi_\varepsilon}}} v,
\end{aligned}$$

because $\widehat{v}$ also belongs to $W_H^{1,p}(\Omega_{\xi_\varepsilon})$. Now, given any $v \in W_H^{1,p}(\Omega_\xi)$, we have, using (22):

$$\begin{aligned}
0 &\leqslant \int\limits_\Omega \big[\alpha_\varepsilon\big(|\nabla u^*_{\xi_\varepsilon}|\big)\nabla u^*_{\xi_\varepsilon} - \alpha_\varepsilon\big(|\nabla v^\diamond_\varepsilon|\big)\nabla v^\diamond_\varepsilon\big] \cdot \nabla\big(u^*_{\xi_\varepsilon} - v^\diamond_\varepsilon\big) \\
&= \int\limits_\Omega \chi_{\Omega^*_{\xi_\varepsilon}}\big(u^*_{\xi_\varepsilon} - v^\diamond_\varepsilon\big) - \int\limits_\Omega \alpha_\varepsilon\big(|\nabla v^\diamond_\varepsilon|\big)\nabla v^\diamond_\varepsilon \cdot \nabla\big(u^*_{\xi_\varepsilon} - v^\diamond_\varepsilon\big),
\end{aligned}$$

and passing to the limit, using Lemma 3 and having in mind that $\chi_{\Omega^*_{\xi_\varepsilon}} \to \chi_{\Omega^*_\xi}$ in $L^p(\Omega)$, for all $1 \leqslant p < \infty$, we get

$$\int\limits_\Omega \chi_{\Omega^*_\xi}(u - v^*) - \int\limits_\Omega \alpha_0\big(|\nabla v^*|\big)\nabla v^* \cdot \nabla(u - v^*) \geqslant 0.$$

This is equivalent to

$$\int\limits_{\Omega_\xi} (u - v) - \int\limits_{\Omega_\xi} \alpha_0\big(|\nabla v|\big)\nabla v \cdot \nabla(u - v) \geqslant 0,$$

which can be shown using the same arguments that led to (22), having the symmetry in mind. We finally obtain, using the usual reasoning,

$$\int\limits_{\Omega_\xi} \alpha_0\big(|\nabla u|\big)\nabla u \cdot \nabla w = \int\limits_{\Omega_\xi} w, \quad \forall w \in W_H^{1,p}(\Omega_\xi).$$

We also have

$$2Q = \int\limits_\Omega u^*_{\xi_\varepsilon} \to \int\limits_\Omega u$$

and so, using (20), we obtain:

$$\int_{\Omega_\xi} u = \frac{1}{2}\int_{\Omega_\xi^*} u = Q.$$

Since $u \in W_0^{1,p}(\Omega)$ implies that $u|_{\Omega_\xi} \in W_H^{1,p}(\Omega_\xi)$, by the uniqueness, we have shown that $(\xi, u|_{\Omega_\xi}) = (\xi_0, u_{\xi_0})$ is the solution of the limit problem. □

Remark 6. – In the case of a regular domain, for example if $\partial\Omega_\xi \in C^{0,1}$ (for which it is sufficient to consider h convex), we could further conclude that $u \in W_0^{1,p}(\Omega_\xi^*)$ and so in that case, we would have $u = u_{\xi_0}^*$.

Remark 7. – We could have obtained the symmetry of u using regularity results. Take any $(x', x_N) \in \Omega_\xi$; since $\xi_\varepsilon \to \xi$, we can choose ε_0 such that

$$\varepsilon < \varepsilon_0 \Rightarrow |\xi_\varepsilon - \xi| < \frac{x_N - h(x')}{4}.$$

We then have that, for $\varepsilon < \varepsilon_0$,

$$(x', 2\xi_\varepsilon - x_N) \in B\left((x', 2\xi - x_N), \frac{x_N - h(x')}{2}\right) \subset \Omega_{\xi_\varepsilon}^*,$$

where $B(x, \delta)$ stands for the ball in $\mathbb{R}^N$ with centre in x and radius δ. Now, in this ball we can obtain an estimate for the Holder norm of $u_{\xi_\varepsilon}^*$. Observe that, due to (22), $u_{\xi_\varepsilon}^*$ solves the following Dirichlet problem:

$$w \in W_0^{1,p}(\Omega_{\xi_\varepsilon}^*): \int_{\Omega_{\xi_\varepsilon}^*} \alpha_\varepsilon\big(|\nabla w|\big)\nabla w \cdot \nabla v = \int_{\Omega_{\xi_\varepsilon}^*} v, \quad \forall v \in W_0^{1,p}(\Omega_{\xi_\varepsilon}^*).$$

So we can use the results in [9], namely Theorem 7.2 (p. 290), to obtain the estimate

$$|u_{\xi_\varepsilon}^*|_{\gamma,B} \leqslant C,$$

which can be seen to be uniform in ε, due to the fact the α_ε satisfy (A2) uniformly. This means, in particular, that $u_{\xi_\varepsilon}^* \to u$ uniformly in the ball B, and we have:

$$u(x', x_N) \leftarrow u_{\xi_\varepsilon}^*(x', x_N) = u_{\xi_\varepsilon}^*(x', 2\xi_\varepsilon - x_N) \to u(x', 2\xi - x_N).$$

We conclude that u is symmetric in (x', x_N), for a.e. $(x', x_N) \in \Omega_\xi$.

5. Applications in fluid mechanics

5.1. Fluids of power-law type

The results of the previous sections can be applied to the study of the motion of a general fluid in an infinite channel or valley, in the particular case in which the flow is unidirectional and parallel to its axis.

We consider an incompressible and viscous fluid, with a constant density $\rho \equiv 1$. In the steady-state case, the basic equations from continuum mechanics that describe the motion are the equations of conservation of mass and momentum (see [4], for instance)

$$\nabla \cdot \mathbf{u} = 0, \tag{23}$$

$$\mathbf{u} \cdot \nabla \mathbf{u} = \nabla \cdot \sigma + \mathbf{f}. \tag{24}$$

Here $\boldsymbol{\sigma} = [\sigma_{ij}]$ is the stress tensor, $\mathbf{f}$ represents the external forces and $\mathbf{u} = (u_i)$ is the velocity field, which for a motion of the type we are considering, is given by $\mathbf{u} = (u, 0, 0)$, if the geometric setting (in $\mathbb{R}^3$) considers the x_1 axis, $O\vec{e}_1$, in the direction of the flow, the x_2 axis across stream and the x_3 axis upwards. We can then simplify Eq. (24), observing that, from (23), $\partial u/\partial x_1 = 0$ and so u is a function of x_2 and x_3 alone. In particular, $\mathbf{u} \cdot \nabla \mathbf{u} = 0$ and (24) reduces to $\nabla \cdot \sigma = -\mathbf{f}$.

We next introduce a (possibly nonlinear) constitutive relation for the incompressible non-Newtonian fluid

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij},$$

p being the pressure and $\tau = [\tau_{ij}]$ the viscous stress tensor. We then need to relate τ and the deformation tensor $\mathbf{D}$, that, in this context, takes the form:

$$\mathbf{D} = \left(\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right)_{ij} = \frac{1}{2}\begin{pmatrix} 0 & \frac{\partial u}{\partial x_2} & \frac{\partial u}{\partial x_3} \\ \frac{\partial u}{\partial x_2} & 0 & 0 \\ \frac{\partial u}{\partial x_3} & 0 & 0 \end{pmatrix},$$

by considering a viscosity coefficient dependent on the second scalar invariant $D_{II}(u) = \frac{1}{2}D_{ij}D_{ij} = \frac{1}{4}|\nabla u|^2$, with $\nabla = (\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$:

$$\tau_{ij} = \nu\big(D_{II}(u)\big)D_{ij} = \alpha\big(|\nabla u|\big)D_{ij}. \tag{25}$$

We can recognize some classical examples of fluids of the differential type, when α is given, for $r = |\nabla u|$, for some viscosity parameter $\mu > 0$, by:

$$\alpha(r) = 2\mu, \quad \text{Newtonian fluid,}$$

$$\alpha(r) = \mu\left[\frac{r}{2}\right]^{q-2}, \quad \text{non-Newtonian fluid,}$$

which is called a pseudo-plastic fluid if $1 < q < 2$ and a dilatant fluid if $q > 2$. An expression for generalized non-Newtonian fluids, like

$$\alpha(r) = 2\mu\left(\frac{\sqrt{2}}{2}r\right),$$

where μ satisfies appropriate assumptions, includes some important asymptotically Newtonian models in which $\mu(r) \to \mu_\infty$ when $r \to \infty$, (see [2], for example).

Going back to (25), we have:

$$\tau_{11} = \tau_{22} = \tau_{33} = \tau_{23} = 0,$$

$$\tau_{12} = \alpha\big(|\nabla u|\big)\frac{\partial u}{\partial y} \quad \text{and} \quad \tau_{13} = \alpha\big(|\nabla u|\big)\frac{\partial u}{\partial z}$$

and so, assuming constant external forces $\mathbf{f} = (k_1, k_2, k_3)$, with $k_1 \geqslant 0$, the equations of motion (24) read

$$\frac{\partial p}{\partial x_1} = \nabla \cdot \big(\alpha\big(|\nabla u|\big)\nabla u\big) + k_1, \qquad \frac{\partial p}{\partial x_2} = k_2, \qquad \frac{\partial p}{\partial x_3} = k_3.$$

From the second and third equations we see that $\partial p/\partial x_1$ is a function of x_1 alone and so the left hand side of the first equation depends only on x_1 while the right hand side depends only on x_2 and x_3; there is then a constant, called drop in pressure per unit length, such that $\partial p/\partial x_1 = -c$. We can then determine the pressure up to this constant and, dividing by $c + k_1 > 0$, we are left with the degenerate elliptic equation

$$\nabla \cdot \big(\alpha\big(|\nabla u|\big)\nabla u\big) = -1$$

of the previous sections, now with $N = 2$ and $\nabla = (\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$.

The domain where the problem was considered corresponds to a cross section, which is in accordance with the fact that u is independent of x_1; in this setting we have the unknown open subset of $\mathbb{R}^2$

$$\Omega_\xi = \big\{(x_2, x_3) \in \mathbb{R}^2\colon\ h(x_2) < x_3 < \xi\big\},$$

where the function h describes the bed of the valley; ξ is unknown and is to be determined so that the flux of fluid is exactly equal to a given constant $Q > 0$:

$$\int\limits_{\Omega_\xi} \mathbf{u} \cdot \vec{e}_1 = \int\limits_{\Omega_\xi} u \,\mathrm{d}x_2 \,\mathrm{d}x_3 = Q.$$

The boundary conditions correspond to no slip at the base and absence of mass transfer through the top surface, which is then supposed to be flat but having unknown height.

As a particular case, we can solve a problem arising in [5] concerning the motion of a glacier in a valley. We briefly describe the model under consideration and show that in fact it falls into the general setting described previously. The material under consideration here is the ice. As is usual in theoretical glaciology, we assume that the flow of the glacier is so slow that we can restrict ourselves to the steady case. The glacier is taken to be isothermal, which means that we neglect heat transfer. The external forces correspond to the action of the gravity vector $\mathbf{g}$. Denoting with θ the mean valley slope, we can write $\mathbf{g} = (g \sin\theta, 0, -g\cos\theta)$.

The constitutive law that we consider is the usual one in glaciology, namely the nonlinear flow law of Glen

$$\tau_{ij} = 2\tau^{1-n} D_{ij}, \tag{26}$$

where τ is the stress second invariant, given by $2\tau^2 = \tau_{ij}\tau_{ij}$. The physical constant n is obtained from experimental data and the value $n = 3$ is now often accepted. The limit cases correspond to a viscous Newtonian fluid ($n = 1$) and an ideal plastic material ($n = \infty$).

A major simplification occurs with the shallow ice approximation. The idea is that the depth of the glacier is very small compared to its length. After an appropriate rescaling of the variables, in [5] this corresponds to considering a laminar unidirectional flow, in which the velocity is parallel to the axis of the valley and we can go through the simplifications made in the introduction. We get

$$\tau = |\nabla u|^{\frac{1}{n}},$$

and from the equations of conservation of mass and momentum we can obtain in the same way, the dimensionless equation

$$\Delta_p u_\xi \equiv \nabla \cdot \left(|\nabla u|^{p-2}\nabla u\right) = -1 \quad \text{in } \Omega_\xi,$$

where $p = 1 + \frac{1}{n}$, $1 < p < 2$, that corresponds to the choice $\alpha(r) = r^{p-2}$, that clearly satisfies assumptions (A1) to (A3).

The boundary conditions considered are appropriate to deal with a cold glacier but are inadequate for temperate glaciers. This problem was considered in [12], where part of the present results were announced.

5.2. Bingham flows

Here we adapt the results of Section 4 to obtain a solution to the problem analogous to (P) when a fluid of Bingham is considered. In this case, we have (see [4])

$$\alpha(r) = \mu + \gamma r^{-1},$$

the constants μ and γ standing for the viscosity and the threshold of plasticity, respectively. We can not apply Theorem 1 directly since this α does not satisfy (A1) and the corresponding $A(r) = \mu r^2/2 + \gamma|r|$ is not differentiable. But, for each ζ, we can still consider the minimization problem (8) for this A, and since it also has a unique solution, define the function Φ in the same way and ask whether its range exhausts $\mathbb{R}^+$ or not. The problem is easily seen to be equivalent to

$$(\mathrm{P}_B) \quad \text{Find } (\xi, u_\xi) \in \mathbb{R}^+ \times W_H^{1,2}(\Omega_\xi) \text{ such that } \int_{\Omega_\xi} u_\xi = Q \text{ and}$$

$$\mu \int_{\Omega_\xi} \nabla u_\xi \cdot \nabla(v - u_\xi) + \gamma \int_{\Omega_\xi} |\nabla v| - \gamma \int_{\Omega_\xi} |\nabla u_\xi| \geqslant \int_{\Omega_\xi} (v - u_\xi), \quad \forall v \in W_H^{1,2}(\Omega_\xi).$$

The fact that A is not differentiable leads us to a variational inequality instead of an equation. We shall prove the following existence result:

PROPOSITION 2. – *The problem* (P_B) *has at least one solution.*

Proof. – The idea is to consider

$$\alpha_\varepsilon(r) = \mu + \gamma r^{\varepsilon - 1}, \quad 0 < \varepsilon < 1,$$

that clearly satisfy the assumptions (A1)–(A3) with $p = 2$ and solve the corresponding problems using Theorem 1. Then let $\varepsilon \to 0$, and using the results of Section 4, obtain the solution to the Bingham case in the limit.

We then have the approximating problems:

$$(\mathrm{P}_\varepsilon) \quad \begin{cases} (\xi_\varepsilon, u_{\xi_\varepsilon}) \in \mathbb{R}^+ \times W_H^{1,2}(\Omega_{\xi_\varepsilon}) \\ \displaystyle\int_{\Omega_{\xi_\varepsilon}} \mu \nabla u_{\xi_\varepsilon} \cdot \nabla v + \int_{\Omega_{\xi_\varepsilon}} \gamma |\nabla u_{\xi_\varepsilon}|^{\varepsilon - 1} \nabla u_{\xi_\varepsilon} \cdot \nabla v = \int_{\Omega_{\xi_\varepsilon}} v, \quad \forall v \in W_H^{1,2}(\Omega_{\xi_\varepsilon}), \\ \displaystyle\int_{\Omega_{\xi_\varepsilon}} u_{\xi_\varepsilon} = Q \end{cases}$$

and, introducing the convex function

$$\varphi_\zeta^\varepsilon(v) = \frac{1}{1+\varepsilon} \int_{\Omega_\zeta} |\nabla v|^{1+\varepsilon}, \quad 0 \leqslant \varepsilon < 1,$$

we know that every solution of the first equation in (P_ε) also satisfies the variational inequality (see [4]):

$$\mu \int_{\Omega_{\xi_\varepsilon}} \nabla u_{\xi_\varepsilon} \cdot \nabla(v - u_{\xi_\varepsilon}) + \gamma \varphi_{\xi_\varepsilon}^\varepsilon(v) - \gamma \varphi_{\xi_\varepsilon}^\varepsilon(u_{\xi_\varepsilon}) \geqslant \int_{\Omega_{\xi_\varepsilon}} (v - u_{\xi_\varepsilon}), \quad \forall v \in W_H^{1,2}(\Omega_{\xi_\varepsilon}). \tag{27}$$

We now proceed as in the proof of Theorem 2, obtaining the estimates (17) and (18) and the convergences

$$\xi_\varepsilon \to \xi \quad \text{in } \mathbb{R},$$
$$u_{\xi_\varepsilon}^* \rightharpoonup u \quad \text{in } W_0^{1,2}(\Omega).$$

Using the same type of arguments as before, we can show that $u = 0$ outside of Ω_ξ^* and that u is symmetric with respect to Π_ξ. Moreover, for any $v \in W_H^{1,2}(\Omega_\xi)$, we have

$$\mu \int_\Omega \nabla u_{\xi_\varepsilon}^* \cdot \nabla\big(v_\varepsilon^\diamond - u_{\xi_\varepsilon}^*\big) + \gamma \varphi^\varepsilon\big(v_\varepsilon^\diamond\big) - \gamma \varphi^\varepsilon(u_{\xi_\varepsilon}^*) \geqslant \int_\Omega \chi_{\Omega_{\xi_\varepsilon}^*}\big(v_\varepsilon^\diamond - u_{\xi_\varepsilon}^*\big),$$

with

$$\varphi^\varepsilon(w) = \frac{1}{1+\varepsilon} \int_\Omega |\nabla w|^{1+\varepsilon}, \quad 0 \leqslant \varepsilon < 1.$$

We can pass to the limit, using the lower semi-continuity, and obtain:

$$\mu \int_\Omega \nabla u \cdot \nabla(v^* - u) + \gamma \varphi^0(v^*) - \gamma \varphi^0(u) \geqslant \int_\Omega \chi_{\Omega_\xi^*}(v^* - u),$$

which, due to the symmetry, reduces to

$$\mu \int_{\Omega_\xi} \nabla u \cdot \nabla(v - u) + \gamma \int_{\Omega_\xi} |\nabla v| - \gamma \int_{\Omega_\xi} |\nabla u| \geqslant \int_{\Omega_\xi} (v - u).$$

We find also, as before, that

$$\int_{\Omega_\xi} u = Q,$$

showing that the problem has the solution $u|_{\Omega_\xi} \in W_H^{1,2}(\Omega_\xi)$. $\square$

Remark 8. – The uniqueness in this case was not obtained. We recall from [4] that, in the absence of the condition $\int u = Q$, the solution u of the Bingham problem may be zero for γ sufficiently large, more precisely for $\gamma \geqslant \gamma_c$, this critical value being dependent on the domain.

This fact shows that, in general, the solution of the Bingham problem may vanish in connected regions that correspond to zones of rigid motion. Therefore, obtaining the strict monotonicity of the function Φ becomes a delicate matter and an open problem in this case.

Acknowledgements

We would like to thank the referee for his valuable suggestions that helped improving the manuscript.

J.M. Urbano is indebted to CMAF-University of Lisbon, for the warm hospitality.

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