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The number of Kronecker indices of square pencils of a special kind

E. Marques de Sá [a,*,1], Yu-Lin Zhang [b,2]

[a] Departamento de Matemática, Universidade de Coimbra, 3000 Coimbra, Portugal
[b] Departamento de Matemática, Universidade do Minho, 4710 Braga, Portugal

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Abstract

In this paper we describe the possible numbers of Kronecker indices of the pencils $xA + B$, where A and B run over two prescribed similarity classes, $\mathscr{A}$ and $\mathscr{B}$, respectively. © 2000 Published by Elsevier Science Inc. All rights reserved.

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1. Introduction

This paper is about matrices over an arbitrary field **F**. The script letters $\mathscr{A}$ and $\mathscr{B}$ denote $n \times n$ similarity classes. So if, say, $A \in \mathscr{A}$, then A is $n \times n$ matrix over **F** and $\mathscr{A}$ is the set of all matrices over **F** similar to A. The *invariant factors*, *eigenvalues*, *rank*, etc, of $\mathscr{A}$ are defined as the corresponding concepts of any $A \in \mathscr{A}$. There exists a matrix in $\mathscr{A}$ of the form $A_1 \oplus N$, where A_1 is nonsingular and N is nilpotent; the similarity classes of A_1 and N are well defined and called the *non-singular* and *nilpotent parts* of $\mathscr{A}$. A Jordan block of $\mathscr{A}$ with eigenvalue λ is called a *Jordan λ-block*. Througout the paper, the symbol $r_{\mathscr{A}}$ denotes the rank (of any element) of $\mathscr{A}$.

An interesting though extremely difficult problem is the following:

* Corresponding author.
E-mail address: emsa@mat.uc.pt (E. Marques de Sá).

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[2] Centro de Matemática da Universidade de Coimbra/FCT.

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Problem 1. Given two n-square similarity classes, say $\mathscr{A}$ and $\mathscr{B}$, describe the possible Kronecker invariants of the pencils $xA + B$, where A and B run over $\mathscr{A}$ and $\mathscr{B}$, respectively.

Here x denotes a variable over $\mathbf{F}$, and by '*Kronecker invariants*' we mean the invariant factors of the polynomial matrix $xA + B$ together with its *Kronecker minimal row* [*column*] *indices* (see, e.g., [1, chapter XII]).

In case $\mathscr{A}$ [or $\mathscr{B}$] is non-singular the above problem is equivalent to what we may call the *product problem*, namely, the determination of the similarity invariant factors of $A^{-1}B$ [or AB^{-1}, respectively] with $A \in \mathscr{A}$ and $B \in \mathscr{B}$, respectively, in this special case Kronecker indices do not occur, but the problem is still hopelessly difficult (see, e.g., [3,6]). If $\mathscr{A}$ and $\mathscr{B}$ are both singular, then things get much worse, for Kronecker row and column indices occur, for certain choices of $A \in \mathscr{A}$ and $B \in \mathscr{B}$ – precisely the choices for which $\det(xA + B)$ is the zero polynomial. This suggests an easier problem than the first one:

Problem 2. Describe the possible Kronecker row and/or column indices of $xA + B$, where A and B run over $\mathscr{A}$ and $\mathscr{B}$, respectively.

In this paper, the two problems above are left open, but a related, simpler one is completely solved, namely, we describe the possible numbers of Kronecker row [column] indices of $xA + B$, where A and B run over $\mathscr{A}$ and $\mathscr{B}$, respectively.

Recall that the number of Kronecker row [column] indices of a pencil $xA + B$ (even in the non-square case) is the dimension of the left [right] kernel of $xA + B$ as a matrix over the rational field $\mathbf{F}(x)$. So, as our pencils are square, there are as many Kronecker row indices as Kronecker column indices. Moreover, the problem we are adressing below is equivalent to the description of all possible ranks of $xA + B$, where A and B run over $\mathscr{A}$ and $\mathscr{B}$, respectively.

It is essential for the understanding of the paper the fact that the rank of a matrix over an integral domain D, defined, say, as the maximum number of D-linearly independent rows of the matrix, equals the maximum of the orders of the non-zero minors of the given matrix. So, as ranks and nullities are concerned, it does not matter whether we consider $xA + B$ as a polynomial matrix, or as a matrix over $\mathbf{F}(x)$, or over any other field extending $\mathbf{F}[x]$.

2. Results

Theorem 2.1. *For any $A \in \mathscr{A}$ and $B \in \mathscr{B}$ we have*

$$\max\{r_{\mathscr{A}}, r_{\mathscr{B}}\} \leqslant \operatorname{rank}(xA + B) \leqslant \min\{r_{\mathscr{A}} + r_{\mathscr{B}}, n\}. \tag{1}$$

In our main result we show that inequalities (1) characterize all possible ranks of our matrices $xA + B$.

Theorem 2.2. *If t is an integer satisfying*

$$\max\{r_{\mathscr{A}}, r_{\mathscr{B}}\} \leqslant t \leqslant \min\{r_{\mathscr{A}} + r_{\mathscr{B}}, n\}, \tag{2}$$

then there exist $A \in \mathscr{A}$ and $B \in \mathscr{B}$ such that $xA + B$ has rank t.

According to our previous comments, we may summarize the previous theorems as follows.

Theorem 2.3. *When A and B run over $\mathscr{A}$ and $\mathscr{B}$, respectively, the number of Kronecker column* [*row*] *indices of $xA + B$ describes the set of all integers in the interval*

$$[\max\{0, \kappa_{\mathscr{A}} + \kappa_{\mathscr{B}} - n\},\ \min\{\kappa_{\mathscr{A}}, \kappa_{\mathscr{B}}\}],$$

where $\kappa_{\mathscr{A}}$ [$\kappa_{\mathscr{B}}$] denotes the dimension of the kernel of any $A \in \mathscr{A}[B \in \mathscr{B}]$.

3. Proofs

Clearly we only have to prove Theorems 2.1 and 2.2.

Proof of Theorem 2.1. The right-hand side inequality in (1) follows from rank $xA =$ rank A and from a well-known elementary inequality.

The determinantal characterization of the rank implies $xA + B$, $(x/y)A + B$ and $xA + yB$ all have the same rank (where y is a new variable commuting with x) as well as

$$\text{rank}(\alpha A + \beta B) \leqslant \text{rank}(xA + yB)$$

for any α and β in $\mathbf{F}$ [or in any field containing $\mathbf{F}$]. The left inequality in (1) follows as a particular case. □

Proof of Theorem 2.2. By induction we assume the theorem holds for matrices of sizes smaller than n. In case $t = \max\{r_{\mathscr{A}}, r_{\mathscr{B}}\}$ the theorem follows easily: just take $A_0 \in \mathscr{A}$ and $B_0 \in \mathscr{B}$ of the form

$$A_0 = [A'0] \quad \text{and} \quad B_0 = [B'0],$$

where A' is $n \times r_{\mathscr{A}}$ and B' is $n \times r_{\mathscr{B}}$, obviously, $\text{rank}(xA_0 + B_0) \leqslant \max\{r_{\mathscr{A}}, r_{\mathscr{B}}\}$, and equality must hold because of (1).

In view of this we assume, from now on, that our integer t satisfies

$$\max\{r_{\mathscr{A}}, r_{\mathscr{B}}\} < t \leqslant \min\{r_{\mathscr{A}} + r_{\mathscr{B}}, n\}.$$

In particular, $\mathscr{A}$ and $\mathscr{B}$ are non-zero. We choose $A \in \mathscr{A}$ and $B \in \mathscr{B}$ exhibiting the Jordan form of the nilpotent parts, say

$$A = J_{p_1} \oplus \cdots \oplus J_{p_u} \oplus A_1 \quad \text{and} \quad B = J_{q_1} \oplus \cdots \oplus J_{q_v} \oplus B_1,$$

where the p's and q's are in non-increasing order, and A_1 and B_1 are non-singular.

Case 1: *When one of the Jordan* 0*-blocks of* $\mathscr{A}$ has order $\geqslant 3$, *and one of the Jordan* 0-blocks of $\mathscr{B}$ has order $\geqslant 2$, that is, $p_1 \geqslant 3$ and $q_1 \geqslant 2$. Let ν_A and ρ_A [ν_B and ρ_B] be, respectively, the order and the rank of the nilpotent part of A [B], thus $n - \nu_A = r_A - \rho_A =$ the order of A_1. Let $\nu := \min\{\nu_A, \nu_B\}$. For $1 \leqslant s \leqslant \nu$ we partition A and B as

$$A = \begin{bmatrix} A_s & X_s \\ 0 & A'_s \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_s & Y_s \\ 0 & B'_s \end{bmatrix},$$

where A_s and B_s are $s \times s$. We associate with A its *superdiagonal sequence*, $(a_1, a_2, \ldots, a_{\nu_{\mathscr{A}}})$, which is defined as follows: $a_1 := 0$ and $a_2, \ldots, a_{\nu_{\mathscr{A}}}$ are the 1's and 0's that occur, from top down, along the superdiagonal of A. We let

$$\bar{a}_k := a_1 + \cdots + a_k.$$

With the convention $a_{\nu_{\mathscr{A}}+1} := 0$, we clearly have $\bar{a}_{\nu_A} = \bar{a}_{\nu_A+1} = \rho_A$. Moreover

$$\operatorname{rank} A_s = \bar{a}_s \quad \text{and} \quad \operatorname{rank} A'_s = r_A - \bar{a}_{s+1}.$$

For $1 \leqslant k \leqslant \nu_B$, define b_k and $\bar{b}_k$ in the same manner, relative to matrix B. We are going to consider only those values of $s \in \{1, \ldots, \nu\}$ such that rank $A_s +$ rank $B_s \geqslant s$. With the notations so far considered we state and prove:

Lemma 3.1. *Assume the conditions of Case* 1 *hold. Let* m *be the maximum* $s \leqslant \nu$ *such that* $\bar{a}_s + \bar{b}_s \geqslant s$. *Assume that* $m < \nu$. *Then*

$$\bar{a}_s + \bar{b}_s \geqslant s, \quad \text{if } 2 \leqslant s \leqslant m, \tag{3}$$

$$\bar{a}_m + \bar{b}_m = m, \tag{4}$$

$$\bar{a}_s + \bar{b}_s < s, \quad \text{if } m < s \leqslant \nu, \tag{5}$$

$$\bar{a}_m = \bar{a}_{m+1}, \tag{6}$$

$$\bar{b}_m = \bar{b}_{m+1}, \tag{7}$$

$$\bar{a}_m = \rho_A \text{ or } \bar{b}_m = \rho_B. \tag{8}$$

Moreover, if $\bar{a}_m = \rho_A$ [$\bar{b}_m = \rho_B$], *then* m *is the sum of the orders of the greatest* ρ_A [ρ_B] *nilpotent Jordan blocks of* B [*of* A].

Proof of Lemma 3.1. First we point out that if there are two consecutive terms, a_i, a_{i+1} [b_i, b_{i+1}] equal to zero, then all other terms after these are zero as well, and so $\bar{a}_{i-1} = \rho_A$ [$\bar{b}_{i-1} = \rho_B$].

In Case 1 we have $\bar{a}_2 + \bar{b}_2 = 2$, and $\bar{a}_3 + \bar{b}_3 \geqslant 3$. As $m < \nu$, there is s such that $1 < s \leqslant \nu$ and $\bar{a}_s + \bar{b}_s < s$. Let σ be the smallest such s. Clearly $\sigma \geqslant 4$, $\bar{a}_{\sigma-1} + \bar{b}_{\sigma-1} = \sigma - 1$, and $a_\sigma = b_\sigma = 0$. We now get a contradiction from the assumption $a_{\sigma-1} = b_{\sigma-1} = 1$; in fact, if this holds, $a_{w-1} = 0$ [$b_{w-1} = 0$] implies $a_w = 1$ [$b_w = 1$], for $1 < w < \sigma$, because in the interval $[0, \sigma - 1]$ there are no two consecutive a_w [b_w] equal to zero; but the number of 0's in the sequence $a_1, \ldots, a_{\sigma-1}, b_1, \ldots, b_{\sigma-1}$

is equal to the number of 1's; therefore, this sequence takes alternate values 0, 1, 0, 1, ...; this is impossible, because $a_2 = a_3 = 1$. This proves that $a_{\sigma-1} = a_\sigma = 0$ or $b_{\sigma-1} = b_\sigma = 0$. Therefore

$$\bar{a}_{\sigma-1} = \rho_A \quad \text{or} \quad \bar{b}_{\sigma-1} = \rho_B. \tag{9}$$

Assume the former alternative holds (the latter has a similar treatment). Then $a_w = 0$ for $w \geqslant \sigma$, and therefore

$$\bar{a}_w + \bar{b}_w = \sigma - 1 + \bar{b}_w - \bar{b}_\sigma \leqslant \sigma - 1 + w - \sigma < w$$

for $w \geqslant \sigma$. This shows that $m = \sigma - 1$. Therefore (9) is nothing but (8), and the other properties (3)–(7) are as obvious.

For the last part of the lemma note that $s - \bar{b}_s$ $[s - \bar{a}_s]$ is the number of nilpotent Jordan blocks of the submatrix $B_s[A_s]$. If $\bar{a}_m = \rho_A$, then B_m has $m - \bar{b}_m$, that is, ρ_A nilpotent Jordan blocks; by (7) these are precisely the greatest ρ_A nilpotent Jordan blocks of B. The case $\bar{b}_m = \rho_B$ has a similar treatment. The lemma is proved. □

Continuing the proof of the theorem, still in Case 1, for a fixed $s \in \{2, \ldots, m\}$ we consider matrices $\tilde{B}$ similar to B of the form

$$\tilde{B} = \begin{bmatrix} \tilde{B}_s & \tilde{Y}_s \\ 0 & \tilde{B}'_s \end{bmatrix},$$

where $\tilde{B}_s$ $[\tilde{B}'_s]$ is similar to $B_s[B'_s]$. By [2, Theorem 3.1], we may fix $\tilde{B}_s$ such that $xA_s + \tilde{B}_s$ is non-singular. By induction, as $\tilde{B}'_s$ takes all values similar to B'_s, the rank of $xA'_s + \tilde{B}'_s$ covers the interval

$$T'_s := \left[\max\{r_A - \bar{a}_{s+1}, r_B - \bar{b}_{s+1}\}\ \min\{r_A + r_B - \bar{a}_{s+1} - \bar{b}_{s+1}, n - s\}\right].$$

Therefore, for any fixed $s \in \{2, \ldots, m\}$, the rank of $xA + B$ covers the interval $T_s = s + T'_s$ whose lower and upper bounds are

$$l_s := s + \max\{r_A - \bar{a}_{s+1}, r_B - \bar{b}_{s+1}\},$$
$$u_s = \min\{r_A + r_B + s - \bar{a}_{s+1} - \bar{b}_{s+1}, n\}.$$

It is obvious that $l_{s-1} \leqslant l_s \leqslant l_{s-1} + 1$ for $2 < s \leqslant m$; therefore all ranks in the interval $[l_2, u_m]$ are attained. To determine u_m we consider two subcases: (i) when $m = \nu$, and (ii) when $m < \nu$. In subcase (i), we have either $m = \nu_A$, or $m = \nu_B$; assume $m = \nu_A$ (for $\nu = \nu_B$ the argument is analogous); as $r_A + \nu_A - \rho_A = n$, we have

$$u_m = \min\{r_A + r_B + \nu_A - \rho_A - \bar{b}_{m+1}, n\} = \min\{n + r_B - \bar{b}_{m+1}, n\} = n.$$

In subcase (ii), Lemma 3.1 applies: we have $\bar{a}_{m+1} + \bar{b}_{m+1} = m$, and therefore $u_m = \min\{r_A + r_B, n\}$. So, in both subcases, the upper bound is

$$u_m = \min\{r_A + r_B, n\}.$$

As $\bar{a}_3$ and $\bar{b}_3$ are positive, our lower bound l_2 satisfies

$$l_2 \leqslant \max\{r_A, r_B\} + 1.$$

So we are done with Case 1.

Case 2: *When the nilpotent parts of* $\mathscr{A}$ *and* $\mathscr{B}$ *are zero. That means that* $p_1 = q_1 = 1$. Without loss of generality we may assume $r_{\mathscr{A}} \geqslant r_{\mathscr{B}}$. Let $s := t - r_{\mathscr{B}}$. Choose $A \in \mathscr{A}$ and $B \in \mathscr{B}$ of the form

$$A = A_1 \oplus 0 \quad \text{and} \quad B = 0_s \oplus B_1 \oplus 0_{n-t},$$

where A_1 and B_1 are non-singular, 0_k denotes a square zero block of order k, and the principal submatrix of B_1 of the last $t - r_{\mathscr{A}}$ rows and columns is non-singular. It is easy to see that $xA + B$ has rank t.

Case 3: *When the nilpotent part of* $\mathscr{A}$ *is non-zero and the nilpotent part of* $\mathscr{B}$ *is zero, that is to say,* $p_1 > q_1 = 1$. We use an argument close to [4, p. 57]. Choose $A \in \mathscr{A}$ and $B \in \mathscr{B}$ of the form $A = J \oplus A''$ and $B = 0 \oplus B'$, with J a nilpotent, upper-triangular, Jordan block of order $\geqslant 2$, and B' of order $n - 1$. Let A' be the principal submatrix of the last $n - 1$ rows and columns of A. As

$$\min\{r_{A'}, r_{B'}\} \leqslant t - 1 \leqslant \max\{r_{A'} + r_{B'}, n - 1\},$$

the induction hypothesis allows us to choose B' in such a way that $xA' + B'$ has rank $t - 1$. Now let $C(x) := xP^{-1}AP + B$, where P denotes the matrix

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \oplus I_{n-2}.$$

If we add the first row of $C(x)$ to the second row, and then subtract the first column from the second, we obtain $[x] \oplus (xA' + B')$. So $C(x)$ has rank t.

Case 4: *When both* $\mathscr{A}$ *and* $\mathscr{B}$ *have a* 2-*by*-2 *nilpotent Jordan block.* There exist $A \in \mathscr{A}$ and $B \in \mathscr{B}$ of the form

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \oplus A' \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus B'.$$

By induction, we may choose A' and B', such that $xA' + B'$ has any prescribed rank t' in the interval

$$T' := [\max\{r_{A'}, r_{B'}\},\ \min\{r_{A'} + r_{B'}, n - 2\}]. \tag{10}$$

Therefore, $t' + 2$ may be given any value in the interval

$$T' + 2 = [\max\{r_{\mathscr{A}}, r_{\mathscr{B}}\} + 1\ \min\{r_{\mathscr{A}} + r_{\mathscr{B}}, n\}].$$

So we are done with this case, and the proof of the theorem is complete. □

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