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TODA SYSTEMS AND EXPONENTS OF SIMPLE LIE GROUPS

BY

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ABSTRACT. – Results on the finite nonperiodic A_n Toda lattice are extended to the Bogoyavlesky Toda systems of type B_n and C_n. The areas investigated, include master symmetries, recursion operators, higher Poisson brackets and invariants. The results are presented both in Flaschka coordinates (a, b) as well as in the natural (q, p) coordinates. A conjecture which relates the degrees of higher Poisson brackets and the exponents of the corresponding Lie group is verified for these systems. © 2001 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – On étend des résultats sur le système Toda A_n, fini et non-périodique, aux systèmes Toda de Bogoyavlesky de types B_n et C_n. Les sujets traités comprennent les master symmetries, les opérateurs de récursion, les crochets de Poisson et les invariants. Les résultats sont présentés en coordonnées (a, b) de Flaschka et aussi en coordonnées naturelles (q, p). Une conjecture qui met en rapport les degrés des crochets de Poisson avec les exponents des groupes de Lie qui leur correspondent est vérifiée pour ces systèmes. © 2001 Éditions scientifiques et médicales Elsevier SAS

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1. Introduction

In this paper we study some aspects of the Bogoyavlesky Toda systems of type A_n, B_n and C_n. We mainly compute the hierarchy of Poisson brackets and master symmetries in the natural (q, p) coordinates instead of the more common Flaschka coordinates (a, b). There are some advantages for doing this. Some interesting connections with fundamental invariants of the corresponding Lie group, namely the exponents of the Lie group, become more transparent in these coordinates. To be more specific, the exponents of the Lie group appear through the action of the master symmetries on the Hamiltonian vector fields. The exponents are also realized as the degrees of the polynomial Poisson brackets generated by the recursion operator. This observation was predicted a long time ago (1986) by Hermann Flaschka, but at that time not too many people believed it would be true. The Toda lattice is a system of particles on the line where each particle interacts with its neighbour with an exponential force. The original Toda system with an infinite number of particles was considered by Toda [26] in 1967. The integrability of the system is due to Flaschka [8], Henon [12] and Manakov [17] all in 1974. The explicit solution of the finite lattice is due to Moser [19] in 1975. We restrict our attention to the finite, non-periodic version of the Toda lattice. The Poisson tensors, master symmetries and recursion operators for Toda systems in the Flaschka coordinates (a, b) were computed in [3–5]. In the case of A_n these results were duplicated in (q, p) coordinates by Das and Okubo [6], and Fernandes [7]. In principle their method is general and may work for other finite dimensional systems as well. The procedure is the following: One defines a second Poisson bracket in the space of canonical variables $(q_1, \ldots, q_n, p_1, \ldots, p_n)$. This gives rise to a recursion operator. The presence of a conformal symmetry as defined in Oevel [23] allows one, by using the recursion operator, to generate an infinite sequence of master symmetries. These, in turn, project to the space of the new variables (a, b) to produce a sequence of master symmetries in the reduced space. This approach was also used in [21] in the case of the Relativistic Toda lattice.

The equations for the Toda systems in consideration will be written in the form

$$\dot{L}(t) = [B(t), L(t)]. \tag{1}$$

The pair of matrices L, B is known as a Lax pair. In the case of the finite nonperiodic Toda lattice L is a symmetric tridiagonal matrix and B is the projection onto the skew-symmetric part in the decomposition of L into skew-symmetric plus lower triangular. In the case of B_n and C_n Toda systems the matrix L will lie in the corresponding Lie algebra and B will again be obtained from L by some projection associated with a decomposition of the Lie algebra. The decomposition plays an important role in the solution of the equations by factorization.

In the case of Toda lattice the Lax equation is obtained by the use of a transformation due to H. Flaschka [8] which changes the original (q, p) variables to new reduced variables (a, b). The symplectic bracket in the variables (q, p) transforms to a degenerate Poisson bracket in the variables (a, b). This linear bracket is an example of a Lie-Poisson bracket. The functions $H_n = \frac{1}{n}\operatorname{tr} L^n$ are in involution. A Lie algebraic interpretation of this bracket can be found in [14]. We denote this bracket by π_1. A quadratic Toda bracket, which we call π_2 appeared in a paper of Adler [1]. It is a Poisson bracket in which the Hamiltonian vector field generated by H_1 is the same as the Hamiltonian vector field generated by H_2 with respect to the π_1 bracket. This is an example of a bi-Hamiltonian system, an idea introduced by Magri [16]. A cubic bracket was found by Kupershmidt [15] via the infinite Toda lattice. We found the explicit formulas for both the quadratic and cubic brackets in some lecture notes by H. Flaschka. The Lenard relations are also in these notes. The Lenard relations show that the system is bi-Hamiltonian. In a situation like this, if one of the tensors is invertible one can find a recursion by inverting the symplectic tensor. The recursion operator is then applied to the initial symplectic bracket to produce an infinite sequence. However, in the case of Toda lattice (in Flaschka variables (a, b)) both operators are non-invertible and therefore this method fails. The absence of a recursion operator for the finite Toda lattice is also mentioned in Morosi and Tondo [18] where a Nijenhuis tensor for the infinite Toda lattice is calculated.

In [3], master symmetries were used to generate nonlinear Poisson brackets for the Toda lattice. For the definition and examples of master symmetries see [10,22,11].

The Toda lattice has been generalized in several directions. In this paper we consider the generalized Toda systems defined by Bogoyavlensky [2] and studied in Kostant [14] and Olshanetsky and Perelomov [25].

The Toda system is generalized to the tridiagonal coadjoint orbit of the Borel subgroup of an arbitrary simple Lie group. Therefore, for each simple Lie group there is a corresponding mechanical system of Toda type.

In Section 2 we present the necessary background on bi-Hamiltonian systems and master symmetries. We also define the exponents of a simple Lie group.

Section 3 is a review of the classical finite nonperiodic Toda lattice. This system was investigated in [8,9,12,13,19,20,26]. We define the quadratic and higher Toda brackets and show that they satisfy certain Lenard-type relations. We briefly describe the construction of master symmetries and the new Poisson brackets as in [3,4]. We also describe the method of Fernandes [7].

In Section 4 we define the integrable Toda systems associated with simple Lie groups. We present in detail the systems of type B_n and C_n. We make the computations both in Flaschka coordinates (a, b) and also in (q, p) variables. In each case we compute invariants, Poisson tensors, recursion operators and master symmetries. The invariants $H_2, H_4, \dots,$ are of even degree. Let χ_i denote the Hamiltonian vector field generated by H_i and let Z_i denote a master symmetry. Then we have

$$[Z_i, \chi_j] = f(j)\chi_{i+j}.$$

The values of $f(j)$ corresponding to independent χ_j generate the exponents. The exponents also appear as the degrees of the Poisson tensors, i.e., for the Toda systems of type A_n, B_n and C_n, the degrees of the higher Poisson brackets coincide with the exponents of the corresponding Lie group.

2. Background

Let M be a C^∞ manifold equipped with two Poisson tensors π_0 and π_1. The two tensors are called compatible if $\pi_0 + \pi_1$ is Poisson. If π_0 is symplectic, we call the Poisson pair (π_0, π_1) non-degenerate. In this case, the $(1, 1)$-tensor $\mathcal{R}$ defined by

$$\mathcal{R} = \pi_1 \pi_0^{-1} \tag{2}$$

is called the *recursion operator* associated with the non-degenerate pair. Recursion operators were introduced by Olver [24].

A *bi-Hamiltonian system* is defined by specifying two Hamiltonian functions H_0, H_1 satisfying:

$$X = \pi_0 \nabla H_1 = \pi_1 \nabla H_0, \tag{3}$$

where $\pi_i,\ i = 0, 1$, denotes the Poisson matrix of the tensor π_i. The theory of bi-Hamiltonian systems was developed by Magri [16]. He established the existence a hierarchy of mutually commuting functions $H_0, H_1, \ldots$, all in involution with respect to both brackets. They generate mutually commuting bi-Hamiltonian flows χ_i satisfying the Lenard recursion relations.

We recall the definition and basic properties of master symmetries following Fuchssteiner [11]. Consider a differential equation on a manifold M, defined by a vector field χ. We are mostly interested in the case where χ is a Hamiltonian vector field. A vector field Z is a *symmetry* of the equation if

$$[Z, \chi] = 0.$$

A vector field Z will be called a *master symmetry* if

$$\big[[Z, \chi], \chi\big] = 0,$$

but

$$[Z, \chi] \neq 0.$$

Suppose that we have a bi-Hamiltonian system defined by the Poisson tensors π_0, π_1 and the Hamiltonians H_0, H_1. Assume that π_0 is symplectic. We define the recursion operator $\mathcal{R} = \pi_1 \pi_0^{-1}$, the higher flows

$$\chi_i = \mathcal{R}^{i-1} \chi_1, \tag{4}$$

and the higher order Poisson tensors

$$\pi_i = \mathcal{R}^i \pi_0. \tag{5}$$

For a non-degenerate bi-Hamiltonian system, master symmetries can be generated using a method due to Oevel [23].

THEOREM 1. – *Suppose that Z_0 is a conformal symmetry for both π_0, π_1 and H_0. i.e., for some scalars λ, μ, and ν, we have*

$$\mathcal{L}_{Z_0}\pi_0 = \lambda\pi_0, \qquad \mathcal{L}_{Z_0}\pi_1 = \mu\pi_1, \qquad \mathcal{L}_{Z_0}H_0 = \nu H_0. \tag{6}$$

Then the vector fields

$$Z_i = \mathcal{R}^i Z_0 \tag{7}$$

are master symmetries and we have

$$[Z_i, \chi_j] = \big(\mu + \nu + (j-1)(\mu - \lambda)\big)\chi_{i+j}, \tag{8}$$

$$[Z_i, Z_j] = (\mu - \lambda)(j - i)Z_{i+j}, \tag{9}$$

$$\mathcal{L}_{Z_i}\pi_j = \big(\mu + (j - i - 1)(\mu - \lambda)\big)\pi_{i+j}, \tag{10}$$

$$\mathcal{L}_{Z_i}H_j = \big(\nu + (j + i)(\mu - \lambda)\big)H_{i+j}. \tag{11}$$

Finally, let us recall the definition of exponents for a semi-simple group G. Let G be a connected complex simple Lie Group G. We form the de Rham cohomology groups $H^i(G, \mathbf{C})$ and the corresponding Poincaré polynomial of G:

$$p_G(t) = \sum d_i t^i,$$

where $d_i = \dim H^i(G, \mathbf{C})$. A theorem of Hopf shows that the cohomology algebra is a finite product of $l =$ rank of $\mathcal{G}$ spheres of odd dimension. This theorem implies that

$$p_G(t) = \prod_i \big(1 + t^{2e_i+1}\big). \tag{12}$$

The positive integers $\{e_1, e_2, \ldots, e_l\}$ are called the *exponents* of G. One can also extract the exponents from the root space decomposition of $\mathcal{G}$. In this paper we propose another method of obtaining the exponents. The connection with the invariant polynomials is the following: Let $H_1, H_2, \ldots, H_l$ be algebraically independent homogeneous polynomials of degrees $m_1, m_2, \ldots, m_l$. Then $m_i = e_i + 1$. The exponents of a simple

Lie algebra of type A_{n-1} are given by

$$1, 2, 3, \ldots, n-1,$$

while for type B_n or C_n

$$1, 3, 5, \ldots, 2n-1.$$

3. Finite, non-periodic A_n Toda lattice

The Toda lattice is a completely integrable classical mechanical system consisting of n particles on the line and subject to a system of springs which behave exponentially. The Hamiltonian function of the system is

$$H(q_1, \ldots, q_n,\ p_1, \ldots, p_n) = \sum_{j=1}^{n} \frac{1}{2} p_j^2 + \sum_{j=1}^{n-1} e^{q_j - q_{j+1}}, \tag{13}$$

where $q_j(t)$ is the position of the jth particle and $p_j(t)$ is the corresponding momentum. Hamilton's equations are

$$\begin{aligned} \dot{q}_j &= p_j, \\ \dot{p}_j &= e^{q_{j-1} - q_j} - e^{q_j - q_{j+1}}. \end{aligned} \tag{14}$$

To determine the set of independent functions $\{H_1, \ldots, H_n\}$ which are constants of motion for Hamilton's equations, one uses Flaschka's transformation:

$$a_i = \frac{1}{2} e^{\frac{1}{2}(q_i - q_{i+1})}, \qquad b_i = -\frac{1}{2} p_i. \tag{15}$$

Then,

$$\begin{aligned} \dot{a}_i &= a_i(b_{i+1} - b_i), \\ \dot{b}_i &= 2(a_i^2 - a_{i-1}^2). \end{aligned} \tag{16}$$

These equations can be written as a Lax pair $\dot{L} = [B, L]$, where L is the symmetric Jacobi matrix

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & & \vdots \\ 0 & a_2 & b_3 & \ddots & & \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & a_{n-1} \\ 0 & \cdots & & \cdots & a_{n-1} & b_n \end{pmatrix} \tag{17}$$

and

$$B = \begin{pmatrix} 0 & a_1 & 0 & \cdots & \cdots & 0 \\ -a_1 & 0 & a_2 & \cdots & & \vdots \\ 0 & -a_2 & 0 & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & a_{n-1} \\ 0 & \cdots & \cdots & & -a_{n-1} & 0 \end{pmatrix}. \tag{18}$$

Consider $\mathbf{R}^{2n}$ with coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$, the standard symplectic bracket and the mapping $F : \mathbf{R}^{2n} \to \mathbf{R}^{2n-1}$ defined by

$$F : (q_1, \ldots, q_n, p_1, \ldots, p_n) \to (a_1, \ldots, a_{n-1}, b_1, \ldots, b_n). \tag{19}$$

The standard symplectic bracket on $\mathbf{R}^{2n}$ reduces, under the mapping F, to a linear bracket on $\mathbf{R}^{2n-1}$ determined by

$$\begin{aligned} \{a_i, b_i\} &= -a_i, \\ \{a_i, b_{i+1}\} &= a_i; \end{aligned} \tag{20}$$

all other brackets are zero. We denote this Poisson tensor by π_1. The only Casimir is $H_1 = b_1 + b_2 + \cdots + b_n$. The Hamiltonian turns out to be $H_2 = \frac{1}{2} \operatorname{tr} L^2$ and the functions H_j are in involution.

There is also a quadratic bracket π_2 which appeared in a paper of Adler [1] in 1979. The defining relations for the new bracket π_2 are:

$$\{a_i, a_{i+1}\} = \frac{1}{2} a_i a_{i+1},$$

$$\begin{aligned} \{a_i, b_i\} &= -a_i b_i, \\ \{a_i, b_{i+1}\} &= a_i b_{i+1}, \\ \{b_i, b_{i+1}\} &= 2a_i^2; \end{aligned} \tag{21}$$

all other brackets are zero. This bracket has $\det L$ as Casimir and $H_1 = \operatorname{tr} L$ is the Hamiltonian. The eigenvalues of L are still in involution and

$$\pi_2 \nabla \lambda_j = \lambda_j \pi_1 \nabla \lambda_j \quad \forall j. \tag{22}$$

It follows easily that

$$\pi_2 \nabla H_l = \pi_1 \nabla H_{l+1}. \tag{23}$$

These relations are similar to the Lenard relations for the KdV equation. They show that the Toda lattice is a bi-Hamiltonian system.

Since it is impossible to find a recursion operator for the non-periodic Toda lattice we use a different method to generate invariants. The idea is to define master symmetries, and use Lie derivatives to generate higher invariants.

We describe the construction following Refs. [3,4]. We denote the master symmetries by X_n. These vector fields generate an infinite sequence of contravariant 2-tensors π_n, for $n \geqslant 1$. We summarize the properties of X_n and π_n:

THEOREM 2. –
(i) *π_n are all Poisson.*
(ii) *The functions $H_n = \frac{1}{n} \operatorname{tr} L^n$ are in involution with respect to all of the π_n.*
(iii) $X_n(H_m) = (n+m) H_{n+m}$.
(iv) $\mathcal{L}_{X_n} \pi_m = (m-n-2)\pi_{n+m}$.
(v) $\pi_n \nabla H_l = \pi_{n-1} \nabla H_{l+1}$, *where π_n denotes the Poisson matrix of the tensor π_n.*

To define the vector fields X_n we consider expressions of the form

$$\dot{L} = [B, L] + L^n. \tag{24}$$

This equation is similar to a Lax equation, but in this case the eigenvalues satisfy $\dot{\lambda} = \lambda^n$ instead of $\dot{\lambda} = 0$.

There is another method of finding the master symmetries due to Fernandes [7] which we describe briefly:

The first step is to define a second Poisson bracket on the space of canonical variables $(q_1, \ldots, q_n, p_1, \ldots, p_n)$. This bracket appears in Das and Okubo [6] and Fernandes [7]. We follow the notation from [7]. Let J_0 be the symplectic bracket on $\mathbf{R}^{2n}$ and define J_1 as follows:

$$\begin{aligned} &\{q_i, q_j\} = 1, \\ &\{p_i, q_i\} = p_i, \\ &\{p_i, p_{i+1}\} = e^{q_i - q_{i+1}}, \end{aligned} \tag{25}$$

all other brackets are zero. Also define

$$h_0 = \sum_{i=1}^{n} p_i, \qquad h_1 = \sum_{i=1}^{n} \frac{p_i^2}{2} + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}}. \tag{26}$$

Since we have a non-degenerate pair (J_0, J_1), there exists a recursion operator defined by $\mathcal{R} = J_1 J_0^{-1}$. It follows easily that the vector field

$$Z_0 = \sum_{i=1}^{n} \frac{n+1-2i}{2} \frac{\partial}{\partial q_i} + \sum_{i=1}^{n} p_i \frac{\partial}{\partial p_i} \tag{27}$$

is a conformal symmetry for J_0, J_1 and h_0 and therefore, Oevel's theorem applies. The constants in Theorem 1 turn out to be $\lambda = -1$, $\mu = 0$ and $\nu = 1$. We end up with the following deformation relations:

$$[Z_i, \chi_j] = j\chi_{i+j}, \tag{28}$$

$$\mathcal{L}_{Z_i} J_j = (j - i - 1) J_{i+j}, \tag{29}$$

$$[Z_i, Z_j] = (j - i) Z_{i+j}. \tag{30}$$

Taking into account the way we defined the linear bracket π_1 on $\mathbf{R}^{2n-1}$, the mapping F is a Poisson mapping between J_0 and π_1. But it is also a Poisson mapping between J_1 and π_2. In fact, the Poisson tensor J_1 reduces, under the mapping F, to π_2. The Hamiltonians h_0 and h_1 correspond to the reduced Hamiltonians H_1 and H_2 respectively. The recursion operator $\mathcal{R}$ cannot be reduced. Actually, it is easy to see that there exists no recursion operator in the reduced space. The kernels of the two Poisson structures π_1 and π_2 are different and, therefore, it is impossible to find an operator that maps one to the other.

The deformation relations (28), (29), (30) also reduce and become precisely the deformation relations of Theorem 2. Of course, one has to replace j by $j-1$ in the formulas involving J_k because of the difference in notation between [3] and [7].

Note that (28) gives a procedure for generating the exponents of a simple Lie group of type A_n.

4. Orthogonal Toda systems

4.1. Definition of the systems

In this section we consider mechanical systems which generalize the finite, nonperiodic Toda lattice. These systems correspond to Dynkin diagrams. It is well known that irreducible root systems classify simple Lie groups. So, in this generalization for each simple Lie algebra there exists a mechanical system of Toda type.

The generalization is obtained from the following simple observation: In terms of the natural basis q_i of weights, the simple roots of A_{n-1} are

$$q_1 - q_2, q_2 - q_3, \dots, q_{n-1} - q_n.$$

On the other hand, the potential for the Toda lattice is of the form

$$e^{q_1-q_2} + e^{q_2-q_3} + \dots + e^{q_{n-1}-q_n}.$$

We note that the angle between $q_{i-1} - q_i$ and $q_i - q_{i+1}$ is $\frac{2\pi}{3}$ and the lengths of $q_i - q_{i+1}$ are all equal. The Toda lattice corresponds to a Dynkin diagram of type A_{n-1}.

Similarly, the potential

$$e^{q_1-q_2} + e^{q_2-q_3} + \dots + e^{q_{n-1}-q_n} + e^{jq_n}$$

corresponds to B_n for $j=1$ and C_n for $j=2$.

These systems are interesting not only because they are integrable, but also for their fundamental importance in the theory of semi-simple Lie groups. For example Kostant in [14] shows that the integration of these systems and the theory of the finite dimensional representations of semi-simple Lie groups are equivalent.

4.2. A recursion operator for B_n Toda systems in Flaschka coordinates

In this section, we show that higher polynomial brackets exist also in the case of B_n Toda systems. Using Flaschka coordinates, we will prove that these systems possess a recursion operator and we will construct an infinite sequence of compatible Poisson brackets in which the constants of motion are in involution.

The Hamiltonian for B_n is

$$H = \frac{1}{2}\sum_{1}^{n} p_j^2 + e^{q_1-q_2} + \cdots + e^{q_{n-1}-q_n} + e^{q_n}. \tag{31}$$

We make a Flaschka-type transformation, $F : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ defined by

$$F : (q_1, \ldots, q_n, p_1, \ldots, p_n) \to (a_1, \ldots, a_n, b_1, \ldots, b_n)$$

with

$$a_i = \frac{1}{2} e^{\frac{1}{2}(q_i-q_{i+1})}, \qquad a_n = \frac{1}{2} e^{\frac{1}{2}q_n}, \tag{32}$$

$$b_i = -\frac{1}{2} p_i.$$

Then

$$\begin{aligned} \dot{a}_i &= a_i(b_{i+1} - b_i) \quad i = 1, \ldots, n, \\ \dot{b}_i &= 2(a_i^2 - a_{i-1}^2) \quad i = 1, \ldots, n, \end{aligned} \tag{33}$$

with the convention that $a_0 = b_{n+1} = 0$.

These equations can be written as a Lax pair $\dot{L} = [B, L]$, where L is the symmetric matrix

$$\begin{pmatrix} b_1 & a_1 & & & & & & & \\ a_1 & \ddots & \ddots & & & & & & \\ & \ddots & \ddots & a_{n-1} & & & & & \\ & & a_{n-1} & b_n & a_n & & & & \\ & & & a_n & 0 & -a_n & & & \\ & & & & -a_n & -b_n & \ddots & & \\ & & & & & \ddots & \ddots & -a_1 & \\ & & & & & & -a_1 & -b_1 & \end{pmatrix}, \tag{34}$$

and B is the skew-symmetric part of L (in the decomposition, lower Borel plus skew-symmetric).

The mapping $F:\mathbf{R}^{2n} \to \mathbf{R}^{2n}$, $(q_i, p_i) \to (a_i, b_i)$, defined by (32), transforms the standard symplectic bracket J_0 into another symplectic bracket π_1 given by

$$\begin{aligned} \{a_i, b_i\} &= -a_i \quad i = 1, \dots, n, \\ \{a_i, b_{i+1}\} &= a_i \quad i = 1, \dots, n-1. \end{aligned} \tag{35}$$

The invariant polynomials for B_n, which we denote by

$$H_2, H_4, \dots H_{2n}$$

are defined by $H_{2i} = \frac{1}{2i} \operatorname{Tr} L^{2i}$. The degrees of the first n (independent) polynomials are $2, 4, \dots, 2n$ and the exponents of the corresponding Lie group are $1, 3, \dots, 2n-1$.

We look for a bracket π_3 which satisfies

$$\pi_3 \nabla H_2 = \pi_1 \nabla H_4. \tag{36}$$

Using trial and error, we end up with the following homogeneous cubic bracket π_3:

$$\begin{aligned} \{a_i, a_{i+1}\} &= a_i a_{i+1} b_{i+1}, \\ \{a_i, b_i\} &= -a_i b_i^2 - a_i^3 \quad i = 1, 2, \dots, n-1, \end{aligned}$$

$$
\begin{aligned}
&\{a_n, b_n\} = -a_n b_n^2 - 2a_n^3,\\
&\{a_i, b_{i+2}\} = a_i a_{i+1}^2,\\
&\{a_i, b_{i+1}\} = a_i b_{i+1}^2 + a_i^3,\\
&\{a_i, b_{i-1}\} = -a_{i-1}^2 a_i,\\
&\{b_i, b_{i+1}\} = 2a_i^2(b_i + b_{i+1}).
\end{aligned}
\tag{37}
$$

We summarize the properties of this new bracket in the following:

THEOREM 3. – *The bracket π_3 satisfies*:

(1) *π_3 is Poisson.*
(2) *π_3 is compatible with π_1.*
(3) *H_{2i} are in involution.*
Define $\mathcal{R} = \pi_3\pi_1^{-1}$. Then $\mathcal{R}$ is a recursion operator. We obtain a hierarchy

$$\pi_1, \pi_3, \pi_5, \ldots$$

consisting of compatible Poisson brackets of odd degree in which the constants of motion are in involution.
(4) $\pi_{j+2}\operatorname{grad} H_{2i} = \pi_j \operatorname{grad} H_{2i+2}\ \forall i, j$.

The proof of this result is in [4,5].

4.3. A bi-Hamiltonian formulation for B_n Toda systems in natural (q, p) coordinates

Now we will define a bi-Hamiltonian formulation for B_n Toda systems in natural (q_i, p_i) coordinates. Consider the following bracket in (q_i, p_i) coordinates:

$$
\begin{aligned}
&\{q_i, q_{i-1}\} = \{q_i, q_{i-2}\} = \cdots = \{q_i, q_1\} = 2p_i \quad i = 2, \ldots, n,\\
&\{p_i, q_{i-2}\} = \{p_i, q_{i-3}\} = \cdots = \{p_i, q_1\} = 2\bigl(e^{q_{i-1}-q_i} - e^{q_i-q_{i+1}}\bigr)\\
&\quad i = 3, \ldots, n-1,\\
&\{p_n, q_{n-2}\} = \{p_n, q_{n-3}\} = \cdots = \{p_n, q_1\} = 2\bigl(e^{q_{n-1}-q_n} - e^{q_n}\bigr),\\
&\{q_i, p_i\} = p_i^2 + 2e^{q_i-q_{i+1}} \quad i = 1, \ldots, n-1,\\
&\{q_n, p_n\} = p_n^2 + 2e^{q_n},\\
&\{q_{i+1}, p_i\} = e^{q_i-q_{i+1}},
\end{aligned}
\tag{38}
$$

$$\{q_i, p_{i+1}\} = 2e^{q_{i+1}-q_{i+2}} - e^{q_i - q_{i+1}} \quad i = 1, \ldots, n-2,$$
$$\{q_{n-1}, p_n\} = 2e^{q_n} - e^{q_{n-1}-q_n},$$
$$\{p_i, p_{i+1}\} = -e^{q_i - q_{i+1}}(p_i + p_{i+1}).$$

Denote this Poisson tensor by J_1. A simple computation leads to the following:

THEOREM 4. – *The bracket* J_1 *satisfies*:
(1) J_1 *is Poisson.*
(2) J_1 *is compatible with* J_0.
(3) *The mapping* F *given by* (32) *is a Poisson morphism between* J_1 *and the cubic bracket* π_3.

Thus, in (q, p) coordinates we also have a non-degenerate pair (J_0, J_1) for B_n Toda and so we may define a recursion operator $\mathcal{N} = J_1 J_0^{-1}$. We have then a hierarchy of mutually compatible Poisson tensors defined by $J_i = \mathcal{N}^i J_0$.

The vector field

$$Z_0 = \sum_{i=1}^{n} 2(n-i+1)\frac{\partial}{\partial q_i} + \sum_{i=1}^{n} p_i \frac{\partial}{\partial p_i} \tag{39}$$

is a conformal symmetry for the Poisson tensors J_0 and J_1 and for the Hamiltonian

$$h_0 = \frac{1}{2}\sum_{1}^{n} p_j^2 + e^{q_1-q_2} + \cdots + e^{q_{n-1}-q_n} + e^{q_n}. \tag{40}$$

We compute:

$$\mathcal{L}_{Z_0} J_0 = -J_0, \qquad \mathcal{L}_{Z_0} J_1 = J_1, \qquad \mathcal{L}_{Z_0} h_0 = 2h_0.$$

So Oevel's theorem applies. With $Z_i = \mathcal{N}^i Z_0$ one calculates easily that

$$[Z_i, \chi_j] = (2j+1)\chi_{i+j}, \tag{41}$$

$$[Z_i, Z_j] = 2(j-i)Z_{i+j}, \tag{42}$$

$$\mathcal{L}_{Z_i} J_j = (2(j-i)-1)J_{i+j}, \tag{43}$$

$$\mathcal{L}_{Z_i} h_j = 2(i + j + 1) h_{i+j}. \tag{44}$$

Note that (41) gives a procedure for generating the exponents of a simple Lie group of type B_n.

4.4. A recursion operator for C_n Toda systems in Flaschka coordinates

We now consider C_n Toda systems. Using Flaschka coordinates, we will prove that these systems also possess a recursion operator and we will construct an infinite sequence of compatible Poisson brackets as in the B_n case.

The Hamiltonian for C_n is

$$H = \frac{1}{2} \sum_{1}^{n} p_j^2 + e^{q_1 - q_2} + \cdots + e^{q_{n-1} - q_n} + e^{2q_n}. \tag{45}$$

We make a Flaschka-type transformation,

$$F : \mathbf{R}^{2n} \to \mathbf{R}^{2n}, \quad (q_1, \ldots, q_n, p_1, \ldots, p_n) \to (a_1, \ldots, a_n, b_1, \ldots, b_n)$$

with

$$a_i = \frac{1}{2} e^{\frac{1}{2}(q_i - q_{i+1})}, \quad a_n = \frac{1}{\sqrt{2}} e^{q_n}, \tag{46}$$

$$b_i = -\frac{1}{2} p_i.$$

The equations in (a, b) coordinates are:

$$\begin{aligned} \dot{a}_i &= a_i (b_{i+1} - b_i) \quad i = 1, \ldots, n-1, \\ \dot{a}_n &= -2 a_n b_n, \\ \dot{b}_i &= 2(a_i^2 - a_{i-1}^2) \quad i = 1, \ldots, n, \end{aligned} \tag{47}$$

with the convention that $a_0 = 0$.

These equations can be written as a Lax pair $\dot{L} = [B, L]$, where L is the matrix

$$\begin{pmatrix} b_1 & a_1 & & & & & & & \\ a_1 & \ddots & \ddots & & & & & & \\ & \ddots & \ddots & a_{n-1} & & & & & \\ & & a_{n-1} & b_n & a_n & & & & \\ & & & a_n & -b_n & -a_{n-1} & & & \\ & & & & -a_{n-1} & \ddots & \ddots & & \\ & & & & & \ddots & \ddots & -a_1 \\ & & & & & & -a_1 & -b_1 \end{pmatrix}, \tag{48}$$

and B is the skew-symmetric part of L.

The mapping $F : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ defined by (46) transforms the standard symplectic bracket J_0 into another symplectic bracket π_1 given by

$$\begin{aligned} \{a_i, b_i\} &= -a_i \quad i = 1, 2, \ldots, n-1, \\ \{a_i, b_{i+1}\} &= a_i \quad i = 1, 2, \ldots, n-1, \\ \{a_n, b_n\} &= -2a_n. \end{aligned} \tag{49}$$

The invariant polynomials for C_n, which we denote by

$$H_2, H_4, \ldots H_{2n}$$

are defined by $H_{2i} = \frac{1}{2i} \operatorname{Tr} L^{2i}$.

We look for a bracket π_3 which satisfies

$$\pi_3 \nabla H_2 = \pi_1 \nabla H_4. \tag{50}$$

We obtain the following homogeneous cubic bracket π_3:

$$\begin{aligned} \{a_i, a_{i+1}\} &= a_i a_{i+1} b_{i+1} \quad i = 1, 2, \ldots, n-2, \\ \{a_{n-1}, a_n\} &= 2a_{n-1} a_n b_n, \\ \{a_i, b_i\} &= -a_i b_i^2 - a_i^3 \quad i = 1, 2, \ldots, n-1, \\ \{a_n, b_n\} &= -2a_n b_n^2 - 2a_n^3, \\ \{a_i, b_{i+2}\} &= a_i a_{i+1}^2, \\ \{a_i, b_{i+1}\} &= a_i b_{i+1}^2 + a_i^3 \quad i = 1, 2, \ldots, n-2, \end{aligned} \tag{51}$$

$$\{a_{n-1},b_n\}=a_{n-1}^3+a_{n-1}b_n^2-a_{n-1}a_n^2,$$
$$\{a_i,b_{i-1}\}=-a_{i-1}^2a_i\quad i=2,3,\ldots,n-1,$$
$$\{a_n,b_{n-1}\}=-2a_{n-1}^2a_n,$$
$$\{b_i,b_{i+1}\}=2a_i^2(b_i+b_{i+1}).$$

We summarize the properties of this bracket in the following:

THEOREM 5. – *The bracket* π_3 *satisfies*:
(1) π_3 *is Poisson.*
(2) π_3 *is compatible with* π_1.
(3) H_{2i} *are in involution.*
Define $\mathcal{R}=\pi_3\pi_1^{-1}$. *Then* $\mathcal{R}$ *is a recursion operator. We obtain a hierarchy*

$$\pi_1,\pi_3,\pi_5,\ldots$$

consisting of compatible Poisson brackets of odd degree in which the constants of motion are in involution.
(4) $\pi_{j+2}\,\mathrm{grad}\,H_{2i}=\pi_j\,\mathrm{grad}\,H_{2i+2},\ \forall i,j.$

The proofs are precisely the same as in the case of B_n.

4.5. A bi-Hamiltonian formulation for C_n Toda systems in natural (q,p) coordinates

As in the case of B_n we will define a bi-Hamiltonian formulation for C_n Toda systems in (q_i,p_i) coordinates.

Consider the following bracket in (q,p) coordinates:

$$\{q_i,q_{i-1}\}=\{q_i,q_{i-2}\}=\cdots=\{q_i,q_1\}=2p_i\quad i=2,\ldots,n,$$
$$\{p_i,q_{i-2}\}=\{p_i,q_{i-3}\}=\cdots=\{p_i,q_1\}=2\bigl(e^{q_{i-1}-q_i}-e^{q_i-q_{i+1}}\bigr)$$
$$i=3,\ldots,n-1,$$
$$\{p_n,q_{n-2}\}=\{p_n,q_{n-3}\}=\cdots=\{p_n,q_1\}=2e^{q_{n-1}-q_n}-4e^{2q_n},$$

(52)

$$\{q_i,p_i\}=p_i^2+2e^{q_i-q_{i+1}}\quad i=1,\ldots,n-1,$$
$$\{q_n,p_n\}=p_n^2+2e^{2q_n},$$
$$\{q_{i+1},p_i\}=e^{q_i-q_{i+1}},$$
$$\{q_i,p_{i+1}\}=2e^{q_{i+1}-q_{i+2}}-e^{q_i-q_{i+1}}\quad i=1,\ldots,n-2,$$

$$\{q_{n-1}, p_n\} = 4e^{2q_n} - e^{q_{n-1}-q_n},$$
$$\{p_i, p_{i+1}\} = -e^{q_i - q_{i+1}}(p_i + p_{i+1}).$$

Denote this Poisson tensor by J_1. As in the case of the B_n Toda systems, we have the following:

THEOREM 6. – *The bracket J_1 satisfies*:
(1) *J_1 is Poisson.*
(2) *J_1 is compatible with J_0.*
(3) *The mapping F given by* (46) *is a Poisson morphism between J_1 and the cubic bracket π_3.*

As in the case of the B_n Toda, we also have a non-degenerate pair (J_0, J_1) and we may define a recursion operator in (q, p) coordinates, $\mathcal{N} = J_1 J_0^{-1}$. So there exists a hierarchy of pairing compatible Poisson tensors, defined by $J_i = \mathcal{N}^i J_0$.

For C_n, the conformal symmetry is the vector field

$$Z_0 = \sum_{i=1}^{n} (2n - 2i + 1)\frac{\partial}{\partial q_i} + \sum_{i=1}^{n} p_i \frac{\partial}{\partial p_i}, \tag{53}$$

and we have the same constants as in the case of B_n:

$$\mathcal{L}_{Z_0} J_0 = -J_0, \quad \mathcal{L}_{Z_0} J_1 = J_1, \quad \mathcal{L}_{Z_0} H_0 = 2H_0.$$

The relations of Oevel's theorem are the same of the B_n Toda

$$[Z_i, \chi_j] = (2j + 1)\chi_{i+j}, \tag{54}$$

$$[Z_i, Z_j] = 2(j - i)Z_{i+j}, \tag{55}$$

$$\mathcal{L}_{Z_i} J_j = (2(j - i) - 1)J_{i+j}, \tag{56}$$

$$\mathcal{L}_{Z_i} h_j = 2(i + j + 1)h_{i+j}. \tag{57}$$

Note that (54) gives a procedure for generating the exponents of a simple Lie group of type C_n.

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