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Drazin invertibility for matrices over an arbitrary ring

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Abstract

Characterizations are given for existence of the Drazin inverse of a matrix over an arbitrary ring. Moreover, the Drazin inverse of a product PAQ for which there exist a P' and Q' such that $P'PA = A = AQQ'$ can be characterized and computed. This generalizes recent results obtained for the group inverse of such products.

The results also apply to morphisms in (additive) categories.

As an application we characterize Drazin invertibility of companion matrices over general rings.
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1. Introduction

Let R be an arbitrary ring with unity 1 and $\mathrm{Mat}_n(R)$ the ring of square matrices over R. An $n \times n$ matrix A over the ring R is called (von Neumann) regular if there exists a (von Neumann) regular inverse A^- i.e. $AA^-A = A$. $A\{1\}$ denotes the set of all regular inverses of A.

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An $n \times n$ matrix T over the ring R is said to have Drazin index k if k is the smallest natural number such that there exists a (unique) solution T^{D_k} of the system of equations

$$T^k X T = T^k, \tag{1.a}$$

$$XTX = X, \tag{1.b}$$

$$TX = XT. \tag{1.c}$$

If k equals 1 then T^{D_1} is denoted by $T^\#$ and is called the group inverse of T. Then, T^2 and T^3 are also regular, $T = MT^2 = T^2 N$ for some M, N and

$$\begin{aligned} T^\# &= (T^2)^- T (T^2)^- T \\ &= T(T^3)^- T \\ &= MTN \\ &= TN^2 \\ &= M^2 T. \end{aligned}$$

Recently (see [10]) the group inverse $T^\#$ of a regular matrix T over an arbitrary ring with unity 1 has been characterized by the invertibility of the matrix

$$T^2 T^- + 1_n - TT^-$$

or equivalently of the matrix

$$T^- T^2 + 1_n - T^- T.$$

Then,

$$\begin{aligned} T^\# &= (T^2 T^- + 1_n - TT^-)^{-2} T \\ &= T(T^- T^2 + 1_n - T^- T)^{-2} \\ &= (T^2 T^- + 1_n - TT^-)^{-1} T \ (T^- T^2 + 1_n - T^- T)^{-1}, \end{aligned}$$

which shows that $T^\#$ is always equivalent with T.

Moreover, if A is regular and P and Q are such that PA and A generate the same left ideal and AQ and A generate the same right ideal then $(PAQ)^\#$ exists iff

$$U = AQPAA^- + 1_n - AA^-$$

is invertible or iff

$$V = A^- AQPA + 1_n - A^- A$$

is invertible. Then,

$$(PAQ)^\# = PU^{-1} AV^{-1} Q = PU^{-2} AQ = PAV^{-2} Q.$$

In this paper we want to extend these Drazin index 1 results to an arbitrary Drazin index.

As mentioned in the recent book [11, p. 124], many statements on Drazin invertibility for matrices over fields and integral domains, scattered in the literature, are not true for matrices over commutative rings. However, we show here that even for matrices over general rings there is a nice generalization of the recent index 1 results (see [10]) to an arbitrary index. Indeed, it turns out that also the existence of the Drazin inverse of a matrix T of index k is completely characterized by the (classical) invertibility of $T^{2k}T^{k-} + 1_n - T^kT^{k-}$ and that T is equivalent with its Drazin inverse. Also the Drazin inverse of products PAQ, with P and Q such that there exists $P'$and Q' for which $P'PA = A = AQQ'$, can be handled as in the index 1 case. Moreover, it must be emphasized that the von Neumann regularity of A, which is of course needed in the index 1 case, is not postulated. However, if A is von Neumann regular we can present a slightly different result. Remark also that the results apply to operators and morphisms in (additive) categories.

Finally we apply the statements to companion matrices over general rings. Also here we obtain a nice generalization of the recently obtained index 1 case (see [10,11]). It turns out that one of the conditions for a companion matrix to have Drazin index k is the von Neumann regularity of a triangular Toeplitz matrix of order k. This application also shows that, for matrices over general rings, the Drazin index can be greater than the order of the matrix.

2. Results

We first collect and extend some known facts (see [1,3,5,8]) for the Drazin inverse of matrices over fields and other particular rings to arbitrary rings. We complete it with some new expressions, using recent results obtained in [10], in the following.

Lemma. *Let T be an $n \times n$ matrix over R. The following conditions are equivalent*:

(1) *T has Drazin index k.*
(2) *There exists a matrix L and k is the smallest natural number such that*

$$T^k = T^{k+1}L = LT^{k+1}.$$

(3) *There exist matrices M and N and k is the smallest natural number such that*

$$T^k = T^{k+1}N = MT^{k+1}.$$

(4) *k is the smallest natural number such that T^k has a group inverse $(T^k)^\#$ and, independent of the choice of T^{k-}, also equivalent with.*
(5) *k is the smallest natural number such that T^k is regular and $T^{2k}T^{k-} + 1_n - T^kT^{k-}$ is invertible.*
(6) *k is the smallest natural number such that T^k is regular and $T^{k-}T^{2k} + 1_n - T^{k-}T^k$ is invertible.*

In this case, T^{k+n} *is regular for all natural numbers* n *with* $M^{n(k+1)}T^{(n-1)k}$ *belonging to* $T^{k+n}\{1\}$ *and*

$$\begin{aligned}T^{D_k} &= T^k(T^{2k})^-T^{2k-1}(T^{2k})^-T^k\\ &= T^k(T^{2k+1})^-T^k\\ &= T^{k-1}(T^k)^{\#}\\ &= T^{k-1}(M^{k+1}T^{k+2}N^{k+1})\\ &= T^{k-1}[T^{2k}T^{k-}+1_n-T^kT^{k-}]^{-1}T^k[T^{k-}T^{2k}+1_n-T^{k-}T^k]^{-1}\\ &= T^{k-1}[T^{2k}T^{k-}+1_n-T^kT^{k-}]^{-2}T^k\\ &= T[T^{k-}T^{2k}+1_n-T^{k-}T^k]^{-2},\end{aligned}$$

which shows that T^{D_k} *is always equivalent with* T.
Moreover, $(T^{D_k})^{\#}$ *exists and equals* $T^2T^{D_k}$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3): clear.
(3) $\Rightarrow$ (1): suppose $T^k = MT^{k+1} = T^{k+1}N$ then

$$\begin{aligned}T^k &= T^{k+(k+1)}N^{k+1}\\ &= T^k(T^{k+1}N)N^k\\ &= T^k(MT^{k+1}N^k)\\ &= \cdots\\ &= T^k(M^{k+1}T)T^k\end{aligned}$$

and

$$T^{k+1} = T^{k+1}(M^{k+1})T^{k+1},$$

$$\begin{aligned}T^{k+2} &= T^{k+2(k+1)+2}N^{2(k+1)}\\ &= T^{k+2}(T^{2(k+1)}N^{2(k+1)})\\ &= T^{k+2}(M^{2(k+1)}T^{2(k+1)})\\ &= T^{k+2}(M^{2(k+1)}T^k)T^{k+2},\end{aligned}$$

more generally,

$$T^{k+n} = T^{k+n}(M^{n(k+1)}T^{(n-1)k})T^{k+n}, \quad n = 1, 2, \ldots$$

It follows that $(T^{2k})^-$ and $(T^{2k+1})^-$ exist.

Now, we prove by straightforward computation that

$$T^k(T^{2k})^-T^{2k-1}(T^{2k})^-T^k = T^k(T^{2k+1})^-T^k$$

is the Drazin inverse of T of index k.

(I)

$$T^{D_k} = T^k(T^{2k})^-T^{2k-1}(T^{2k})^-T^k.$$

It follows from (3) that:

$$\begin{aligned} T^k &= DT^{2k+1} \quad \text{with } D = M^{k+1} \\ &= T^{2k+1}C \quad \text{with } C = N^{k+1}. \end{aligned}$$

Then,

$$\begin{aligned} T^k(T^{2k})^-T^{2k-1}(T^{2k})^-T^k &= DT^{2k+1}(T^{2k})^-T^{2k+1}CT^{k-1}(T^{2k})^-T^k \\ &= DT^{k+1}T^{k-1}(T^{2k})^-T^k \\ &= DT^{2k}(T^{2k})^-T^{2k+1}C \\ &= DT^{2k+1}C \\ &= T^kC. \end{aligned}$$

We verify now the three definition equations:

(1.c)

$$\begin{aligned} T(T^kC) &= T^{k+1}N^{k+1} \\ &= M^{k+1}T^{k+1} \\ &= (M^{k+1}T^{2k+1})CT \\ &= (T^kC)T, \end{aligned}$$

(1.a)

$$\begin{aligned} T^k(T^kC)T &= T^{k+1}(T^kC) \\ &= T^k, \end{aligned}$$

(1.b)

$$\begin{aligned} (T^kC)T(T^kC) &= T^{k+1}CT^kC \\ &= T(T^kCT)T^{k-1}C \\ &= T(TT^kC)T^{k-1}C \\ &= T^{k+2}CT^{k-1}C \\ &= T^{2k+1}CC \\ &= T^kC. \end{aligned}$$

(II)

$$T^{D_k} = T^k(T^{2k+1})^-T^k.$$

We verify the three definition equations:

(1.c)

$$\begin{aligned} TT^k(T^{2k+1})^-T^k &= TDT^{2k+1}(T^{2k+1})^-T^{2k+1}C \\ &= TDT^{2k+1}C \\ &= T^{k+1}C, \end{aligned}$$

$$
\begin{aligned}
T^k(T^{2k+1})^-T^kT &= DT^{2k+1}(T^{2k+1})^-TT^{2k+1}C\\
&= DT^{2k+1}TC\\
&= T^{k+1}C,
\end{aligned}
$$

(1.a)

$$
\begin{aligned}
T^{k+1}T^k(T^{2k+1})^-T^k &= T^{2k+1}(T^{2k+1})^-T^{2k+1}C\\
&= T^k,
\end{aligned}
$$

(1.b)

$$
\begin{aligned}
[T^k(T^{2k+1})^-T^k]^2T &= T^k(T^{2k+1})^-T^k[T^k(T^{2k+1})^-T^{k+1}]\\
&= T^k(T^{2k+1})^-T^kT^{k+1}C\\
&= T^k(T^{2k+1})^-T^k.
\end{aligned}
$$

(3) $\Leftrightarrow$ (4): it follows from (3) that:

$$
\begin{aligned}
T^k &= (M^{k+1}T)T^{2k}\\
&= T^{2k}(TN^{k+1}),
\end{aligned}
$$

which implies the existence of $(T^k)^\#$. Conversely, if $(T^k)^\#$ exists, for smallest k, then $T^{k-1}(T^k)^\#$ is the Drazin inverse of T since there exists a matrix G such that $T^k = GT^{2k} = T^{2k}G$, from which follows that:

$$
\begin{aligned}
T^k[T^{k-1}(T^k)^\#]T &= GT^{2k-1}[T^k(T^k)^\#]T\\
&= GT^{k-1}[T^k(T^k)^\#T^k]T\\
&= GT^{2k-1}T\\
&= T^k,
\end{aligned}
$$

$$
[T^{k-1}(T^k)^\#]T[T^{k-1}(T^k)^\#] = T^{k-1}(T^k)^\#
$$

and

$$
\begin{aligned}
T^k[T^{k-1}(T^k)^\#] &= T^{k-1}[T^k(T^k)^\#]\\
&= [T^{k-1}(T^k)^\#]T^k.
\end{aligned}
$$

Moreover,

$$
T^{D_k} = T^{k-1}(T^k)^\#
$$

implies

$$
\begin{aligned}
T^{D_k} &= T^{k-1}(M^{k+1}TT^kTN^{k+1})\\
&= T^{k-1}(M^{k+1}T^{k+2}N^{k+2}).
\end{aligned}
$$

(4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6): follows from [10], Theorem 1. Moreover,

$$
T^{D_k} = T^{k-1}(T^k)^\# = T(T^{k-}T^{2k} + 1_n - T^{k-}T^k)^{-2},
$$

i.e., T^{D_k} is always equivalent with T.

Finally, it follows from (2) that:

$$T^k = T^{2k}L^k = L^kT^{2k}$$

and therefore that $(T^{D_k})^\#$ always exists, if T^{D_k} exists. Since

$$T^{D_k}TT^{D_k} = T^{D_k} = T(T^{D_k})^2 = (T^{D_k})^2T,$$

we have

$$(T^{D_k})^\# = T^2T^{D_k}. \qquad \square$$

Remark. Statement (4) in the lemma shows that for rings R for which its free module R^n has finite length, the Drazin inverse of a matrix T over such rings always exists. Indeed, by Fitting's lemma (see [2, p. 355]) there exists a natural number k such that $R^n = \operatorname{Im} T^k \bigoplus \ker T^k$, which is known to be equivalent with T^k has a group inverse.

Recent developments in the theory of generalized invertibility (see [4,7,9,10]) show that the generalized inverses of products $T = PAQ$ such that A is von Neumann regular and for which there exist P' and Q' such that $P'PA = A = AQQ'$ can be characterized. Such nontrivial factorizations of T always exist if T is von Neumann regular because, in that case, there exist $T^{(1,2)}$—inverses satisfying $T = TT^{(1,2)}T$ and $T^{(1,2)}TT^{(1,2)} = T^{(1,2)}$. This means that a von Neumann regular T, having a $T^{(1,2)}$ of a simple form, has an interesting factorization $T = TT^{(1,2)}T$ and the following corollary of the next theorem shows how to compute the Drazin inverse of such a von Neumann regular T. But, it must be clear that the following theorem also handles matrices which are not von Neumann regular.

Theorem. *Let A be an $n \times n$ matrix over an arbitrary ring R and P, Q matrices over R for which there exist matrices P', Q' such that $P'PA = A = AQQ'$. Let $A_1 := A$ and for all natural numbers $i > 1$, let*

$$A_i := AQ(PAQ)^{i-2}PA.$$

Then, the following are equivalent:

(1) *$T = PAQ$ has Drazin index k.*
(2) *k is the smallest natural number such that A_k is regular and $A_kQPA_kA_k^- + 1_n - A_kA_k^-$ is invertible.*
(3) *k is the smallest natural number such that A_k is regular and $A_k^-A_kQPA_k + 1_n - A_k^-A_k$ is invertible and independent of the choice of A_k^-.*

In that case, with $U_k := A_kQPA_kA_k^- + 1_n - A_kA_k^-$ and $V_k := A_k^-A_kQPA_k + 1_n - A_k^-A_k$, we have

$$\begin{aligned}T^{D_k} &= T^{k-1}(PU_k^{-1}A_kV_k^{-1}Q)\\ &= T^{k-1}(PU_k^{-2}A_kQ)\\ &= T^{k-1}(PA_kV_k^{-2}Q).\end{aligned}$$

Proof. T has Drazin index k iff k is the smallest natural number such that T^k has Drazin index 1.

But,

$$\begin{aligned}T^k &= P(AQT^{k-2}PA)Q\\ &= PA_kQ\end{aligned}$$

and $AQT^{k-2}PA$ is von Neumann regular iff T^k is von Neumann regular, because there exist P' and Q' such that $P'PA = A = AQQ'$. Indeed,

$$AQT^{k-2}PA[Q(T^k)^-P]AQT^{k-2}PA = AQT^{k-2}PA$$

is equivalent to

$$T^k(T^k)^-T^k = T^k.$$

Moreover, since $P'PA = A = AQQ'$,

$$\begin{aligned}P'[P(AQT^{k-2}PA)] &= AQT^{k-2}PA\\ &= [(AQT^{k-2}PA)Q]Q',\end{aligned}$$

which means that $P'PA_k = A_k = A_kQQ'$.

We therefore can apply Corollary 1 of Theorem 1 in [10], with $p = P, a = AQT^{k-2}PA$ and $q = Q$, which implies T^k has Drazin index 1 iff T^k is von Neumann regular and $U_k := A_kQPA_kA_k^- + 1_n - A_kA_k^-$ is invertible iff T^k is von Neumann regular and $V_k := A_k^-A_kQPA_k + 1_n - A_k^-A_k$ is invertible. $\square$

Corollary. *If T is $n \times n$ and von Neumann regular with $T = TT^{(1,2)}T$ then the following are equivalent*:

(1) *T has Drazin index k.*
(2) *k is the smallest natural number such that $T_k := T^{(1,2)}T^kT^{(1,2)}$ is von Neumann regular and*

$$T_kT^2T_kT_k^{(1,2)} + 1_n - T_kT_k^{(1,2)}$$

is invertible.
(3) *k is the smallest natural number such that $T_k := T^{(1,2)}T^kT^{(1,2)}$ is von Neumann regular and*

$$T_k^{(1,2)}T_kT^2T_k + 1_n - T_k^{(1,2)}T_k$$

is invertible and independent of the choice of the reflexive von Neumann inverse $T_k^{(1,2)}$.

In that case, and with

$$U_k := T_kT^2T_kT_k^{(1,2)} + 1_n - T_kT_k^{(1,2)},$$

$$V_k := T_k^{(1,2)}T_kT^2T_k + 1_n - T_k^{(1,2)}T_k,$$

we obtain

$$\begin{aligned}T^{D_k} &= T^kU_k^{-1}T_kV_k^{-1}T\\ &= T^kU_k^{-2}T^{(1,2)}T^k\\ &= [T^kT^{(1,2)}]^2V_k^{-2}T.\end{aligned}$$

Indeed, apply the theorem to the factorization $T = TT^{(1,2)}T$ with $A = T^{(1,2)}$, $P = Q = T$ and $P' = Q' = T^{(1,2)}$.

Remark. The Drazin index 1 case, i.e. the group inverse of a $n \times n$ von Neumann regular T follows now easily from the corollary with $k = 1$ (see also [10]):

T has a group inverse iff $T^{(1,2)}T^2 + 1_n - T^{(1,2)}T$ is invertible iff $T^2T^{(1,2)} + 1_n - TT^{(1,2)}$ is invertible, for any $T^{(1,2)}$.

In that case, we obtain

$$\begin{aligned}T^{\#} &= T(T^{(1,2)}T^2 + 1_n - T^{(1,2)}T)^{-2}\\ &= (T^2T^{(1,2)} + 1_n - TT^{(1,2)})^{-2}T\\ &= (T^2T^{(1,2)} + 1_n - TT^{(1,2)})^{-1}T(T^{(1,2)}T^2 + 1_n - T^{(1,2)}T)^{-1}.\end{aligned}$$

Indeed, use the facts that

$$T^{(1,2)}T \text{ commutes with } T^{(1,2)}T^2 + 1_n - T^{(1,2)}T,$$

$$TT^{(1,2)} \text{ commutes with } T^2T^{(1,2)} + 1_n - TT^{(1,2)}$$

and

$$T(T^{(1,2)}T^2 + 1_n - T^{(1,2)}T)^{-1} = (T^2T^{(1,2)} + 1_n - TT^{(1,2)})^{-1}T,$$

$$(T^2T^{(1,2)} + 1_n - TT^{(1,2)})^{-1}T = T(T^{(1,2)}T^2 + 1_n - T^{(1,2)}T)^{-1}.$$

3. Drazin inverses of companion matrices over general rings

In [10], the Drazin index 1 case of a companion matrix $L_n = \begin{pmatrix} 0 & a \\ 1_{n-1} & B \end{pmatrix}$ with $B^{\mathrm{T}} = [a_1a_2 \cdots a_{n-1}]$ over a general ring was characterized by the von Neumann regularity of the element a together with the invertibility of $a - (1 - aa^-)a_1$ (or equivalently, the invertibility of $a - a_1(1 - a^-a)$).

For Drazin indices greater than 1 of the companion matrices L_n we can apply the lemma or the theorem, because L_n has a diagonal reduction $1_{n-1} \bigoplus a$, by invertible matrices. Remark that we only can apply the corollary of the theorem in case a is von Neumann regular. It follows that one of the conditions for L_n to have Drazin index k is that L_n^k is von Neumann regular, which can depend on the exponent k. This can be illustrated by the following example.

Over the ring Z_{12} of integers modulo 12, $L_2^k = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}^k$ is not von Neumann regular for $k = 2$, and for all other exponents it is von Neumann regular. Therefore, the Drazin index of L_2 must be greater than 2 if it exists, which is the case since the module Z_{12}^2 has finite length. Moreover, it follows from the theorem that L_2 has Drazin index 3 since

- the index is not 1, because 2 is not invertible in Z_{12}
- the index is not 2, because L_2^2 is not von Neumann regular and,
- L_2^3 is von Neumann regular and

$$A_3 = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix},\ A_3^- = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } U_3 = V_3 = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \text{ is invertible.}$$

Clearly, $L_2^{D_3} = \begin{pmatrix} 0 & 0 \\ 8 & 4 \end{pmatrix}$.

However, for Drazin indices $k \leqslant n$ we have, besides the lemma and the theorem, also the following characterization.

Proposition. *If* $L_n = \begin{pmatrix} 0 & a \\ 1_{n-1} & B \end{pmatrix}$ *with* $B^{\mathrm{T}} = [a_1 a_2 \cdots a_{n-1}]$ *is an* $n \times n$ *companion matrix over a general ring then the following are equivalent*:

1. L_n *has Drazin index* $k \leqslant n$.
2. $k \leqslant n$ *is the smallest natural number such that the lower triangular Toeplitz matrix*
$$T_k = \begin{pmatrix} a & 0 & 0 & \cdots & 0 & 0 \\ a_1 & a & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{k-1} & a_{k-2} & a_{k-3} & \cdots & a_1 & a \end{pmatrix}$$
is von Neumann regular in $\mathrm{Mat}_k(R)$ *and* $U_k = L_n^{2k}(L_n^k)^- + 1_n - L_n^k(L_n^k)^-$ *is invertible.*
3. $k \leqslant n$ *is the smallest natural number such that the lower triangular Toeplitz matrix* T_k *is von Neumann regular in* $\mathrm{Mat}_k(R)$ *and* $V_k = (L_n^k)^- L_n^{2k} + 1_n - (L_n^k)^- L_n^k$ *is invertible.*

Proof. If $k \leqslant n$ then L_n^k is equivalent with the matrix $\begin{pmatrix} 0_{k\times(n-k)} & T_k \\ 1_{n-k} & O_{(n-k)\times k} \end{pmatrix}$ in which T_k is the lower triangular Toeplitz matrix

$$T_k = \begin{pmatrix} a & 0 & 0 & \cdots & 0 & 0 \\ a_1 & a & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{k-1} & a_{k-2} & a_{k-3} & \cdots & a_1 & a \end{pmatrix}.$$

Clearly, L_n^k is von Neumann regular iff T_k is von Neumann regular. The remaining part follows directly from the theorem. □

Corollary. *If* $L_n = \begin{pmatrix} 0 & a \\ 1_{n-1} & B \end{pmatrix}$ *is an* $n \times n$ *companion matrix over a general ring such that* a *and*

$$T_{k-1} = \begin{pmatrix} a & 0 & \cdots & 0 & 0 \\ a_1 & a & \vdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{k-2} & a_{k-3} & \cdots & a_1 & a \end{pmatrix}$$

are von Neumann regular then the following are equivalent:

1. L_n *has Drazin index* $k \leqslant n$.
2. $k \leqslant n$ *is the smallest natural number such that the* $1 \times (k-1)$ *matrix*
   $$(1 - aa^-)[a_{k-1} \cdots a_1](1_{k-1} - T_{k-1}^- T_{k-1})$$
   is von Neumann regular and
   $$U_k = L_n^{2k}(L_n^k)^- + 1_n - L_n^k(L_n^k)^-$$
   is invertible.
3. $k \leqslant n$ *is the smallest natural number such that the* $1 \times (k-1)$ *matrix*
   $$(1 - aa^-)[a_{k-1} \cdots a_1](1_{k-1} - T_{k-1}^- T_{k-1})$$
   is von Neumann regular and
   $$V_k = (L_n^k)^- L_n^{2k} + 1_n - (L_n^k)^- L_n^k$$
   is invertible.

Indeed, Theorem 1 in [6] proves that, if U and W are von Neumann regular matrices then $\begin{pmatrix} U & 0 \\ V & W \end{pmatrix}$ is von Neumann regular iff $(1 - WW^-)V(1 - U^-U)$ is von Neumann regular. Therefore, if T_{k-1} and a are von Neumann regular,

$$T_k = \begin{pmatrix} T_{k-1} & O_{(k-1)\times 1} \\ a_{k-1} \cdots a_1 & a \end{pmatrix}$$

is von Neumann regular iff $(1 - aa^-)[a_{k-1} \cdots a_1](1_{k-1} - T_{k-1}^- T_{k-1})$ is von Neumann regular.

We therefore obtain the following examples:

1. If a is von Neumann regular, then L_n has Drazin index 2.
   $\Leftrightarrow$ L_n does not have Drazin index 1, $(1-aa^-)a_1(1-a^-a)$ is von Neumann regular and $L_n^4(L_n^2)^- + 1_n - L_n^2(L_n^2)^-$ is invertible.
   $\Leftrightarrow$ L_n does not have Drazin index 1, $(1-aa^-)a_1(1-a^-a)$ is von Neumann regular and $(L_n^2)^-L_n^4 + 1_n - (L_n^2)^-\,L_n^2$ is invertible.
2. If a and $(1-aa^-)a_1(1-a^-a)$ are von Neumann regular and $T_2 := \begin{pmatrix} a & 0 \\ a_1 & a \end{pmatrix}$, then $L_n, n > 2$ has Drazin index 3.
   $\Leftrightarrow$ L_n does not have Drazin index 1 or 2, $(1-aa^-)[a_1a_2](1_2 - T_2^-T_2)$ is von Neumann regular and $L_n^6(L_n^3)^- + 1_n - L_n^3(L_n^3)^-$ is invertible.
   $\Leftrightarrow$ L_n does not have Drazin index 1 or 2, $(1-aa^-)[a_1a_2](1_2 - T_2^-T_2)$ is von Neumann regular and $(L_n^3)^-L_n^6 + 1_n - (L_n^3)^-L_n^3$ is invertible.

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