On the computation of solutions of systems of interval polynomial equations

J.A. Ferreira*, F. Patrício, F. Oliveira

Universidade de Coimbra, Faculdade de Ciências e Tecnologia, Departamento de Matemática, Apartado 3008, 3000 Coimbra, Portugal

Received 17 November 2003; received in revised form 9 February 2004

Abstract

Systems of algebraic equations with interval coefficients are very common in several areas of engineering sciences. The computation of the solution of such systems is a central problem when the characterization of the variables related by such systems is desired.

In this paper we characterize the solution of systems of algebraic equations with real interval coefficients. The characterization is obtained considering the approach introduced in J. Comput. Appl. Math. 136 (2001) 271.

MSC: 65M06; 65M20; 65M15

Keywords: Interval polynomials; Systems of interval equations

1. Introduction

In certain areas of engineering sciences several behaviour laws of variables with a relevant role are formulated by algebraic expressions with perturbed coefficients which can be seen as real intervals [3,6–10]. The characterization of such variables is established solving systems of algebraic equations with interval coefficients. The case of one equation arises, for instance, in automatic control theory in [3], in dynamic systems in [6] and in astrophysics [4]. In this case in order to characterize the variable of interest, an interval polynomial equation must be solved. We mention that the computation of the roots of interval polynomials using the particular form of the polynomials was considered in [3]. In [1] the computation was made using the interval arithmetic. Attending that for certain interval

* Corresponding author.

E-mail address: ferreira@mat.uc.pt (J.A. Ferreira).

0377-0427/$ - see front matter © 2004 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2004.03.012
equations, the last approach does not enable us to obtain the desired roots at least approximately [2], in [5] a new approach was introduced which can be considered for very general interval polynomial equations.

In this paper, we use the approach introduced in [5] in order to compute the solution of systems of interval polynomials equations. Using this approach we are able to characterize the set containing the solution of the mentioned system.

Attending that the characterization of the solution of a system of interval polynomial equations with interval coefficients is obtained characterizing the set of solutions of each equation we start in Section 2 by generalizing the approach considered by the authors before in [5] to interval polynomials in several variables. The arguments presented in Section 2 are used in Section 3 on the characterization of the solution of systems of equations involving interval polynomials in several variables.

2. Interval polynomials in several variables

Let \( Q_{m_1,\ldots,m_n}(z_1,\ldots,z_n) \) be the interval polynomial in the variables \( z_i, \ i = 1,\ldots,n, \) of degree \( m_i \in \mathbb{N}, \ i = 1,\ldots,n, \) respectively, defined by

\[
Q_{m_1,\ldots,m_n}(z_1,\ldots,z_n) = \sum_{i=1}^{n} \mathbb{P}_m(z_i) + A_0,
\]

where \( A_0 = [a_0, b_0] \) is a closed real interval, \( \mathbb{P}_m(z_i) \) is an interval polynomial in the variable \( z_i, \) without independent term and of degree \( m_i, \) that is, \( \mathbb{P}_m(z_i) \) is defined by

\[
\mathbb{P}_m(z_i) = \sum_{j=1}^{m_i} A_{i,j} z_i^j, \quad A_{i,j} = [a_{i,j}, b_{i,j}], \quad j = 1,\ldots,m_i
\]

for \( i = 1,\ldots,n. \) The interval polynomials \( Q_{m_1,\ldots,m_n}(z_1,\ldots,z_n) \) can be seen as a family of real polynomials in the variables \( z_1,\ldots,z_n. \)

**Definition 1.** Let \( Q_{m_1,\ldots,m_n}(z_1,\ldots,z_n) \) be a interval polynomial in the variables \( z_i, \ i = 1,\ldots,n, \) defined by (1). By \( G(Q_{m_1,\ldots,m_n}) \) we denote the graph of \( Q_{m_1,\ldots,m_n}(z_1,\ldots,z_n) \) which is given by

\[
G(Q_{m_1,\ldots,m_n}) = \{(\tilde{z}_1,\ldots,\tilde{z}_n,\tilde{z}) \in \mathbb{R}^{n+1} : \exists p_{m_1,\ldots,m_n}(z_1,\ldots,z_n) \in Q_{m_1,\ldots,m_n}(z_1,\ldots,z_n), \quad \tilde{z} = p_{m_1,\ldots,m_n}(\tilde{z}_1,\ldots,\tilde{z}_n)\}.
\]

**Example 1.** Let us consider the interval polynomial of degree two in the variables \( z_1, z_2 \) defined by

\[
Q_{2,2}(z_1, z_2) = z_1^2 + [-4, -2]z_1 + z_2^2 + [-4, -2]z_2 + [1, 4].
\]

The graph of \( Q_{2,2}(z_1, z_2) \), \( G(Q_{2,2}) \), is the set of all points of \( \mathbb{R}^3 \) between the two surfaces plotted in Fig. 1.

Using the approach introduced in [5], we can define certain functions such that their graphs are the boundary surfaces of \( G(Q_{m_1,\ldots,m_n}) \).
Following the mentioned approach we associate with $Q_{m_1,\ldots,m_n}(z_1,\ldots,z_n)$ the real polynomials $q_{m_1,\ldots,m_n}(z_1,\ldots,z_n)$ and $r_{m_1,\ldots,m_n}(z_1,\ldots,z_n)$ defined in what follows. We start by noting that $\mathbb{R}^n$ is the union of the sets of the type $\prod_{j=1}^n \mathbb{R}^{s_j}$ where $s_j \in \{+, -\}$ for $j = 1,\ldots,n$, and we assume that when all $s_j = +$ or all $s_j = -$, by $\mathbb{R}^{s_j}$ we represent $\mathbb{R}_{0}^{s_j}$.

The expressions of $q_{m_1,\ldots,m_n}(z_1,\ldots,z_n)$ and $r_{m_1,\ldots,m_n}(z_1,\ldots,z_n)$ are depending of $\prod_{j=1}^n \mathbb{R}^{s_j}$.

We consider

$$q_{m_1,\ldots,m_n}(z_1,\ldots,z_n) = q^{s_1,\ldots,s_n}_{m_1,\ldots,m_n}(z_1,\ldots,z_n) \quad \text{if } (z_1,\ldots,z_n) \in \prod_{j=1}^n \mathbb{R}^{s_j},$$

(2)

where

$$q^{s_1,\ldots,s_n}_{m_1,\ldots,m_n}(z_1,\ldots,z_n) = \sum_{i=1}^n \sum_{j=1}^{m_i} q_{i,j} z_i^j + q_0,$$

with $q_0 = b_0$, and

$$q_{i,j} = \begin{cases} b_{i,j} & \text{if } s_i = + \text{ or } s_i = - \text{ and } j \text{ even}, \\ a_{i,j} & \text{if } s_i = - \text{ and } j \text{ odd}. \end{cases}$$

We represent by $r_{m_1,\ldots,m_n}(z_1,\ldots,z_n)$ the following function:

$$r_{m_1,\ldots,m_n}(z_1,\ldots,z_n) = r^{s_1,\ldots,s_n}_{m_1,\ldots,m_n}(z_1,\ldots,z_n) \quad \text{if } (z_1,\ldots,z_n) \in \prod_{j=1}^n \mathbb{R}^{s_j},$$

(3)

with

$$r^{s_1,\ldots,s_n}_{m_1,\ldots,m_n}(z_1,\ldots,z_n) = \sum_{i=1}^n \sum_{j=1}^{m_i} r_{i,j} z_i^j + r_0,$$
\[ r_0 = a_0, \text{ and} \]

\[ r_{i,j} = \begin{cases} a_{i,j} & \text{if } s_i = + \text{ or } s_i = - \text{ and } j \text{ even}, \\ b_{i,j} & \text{if } s_i = - \text{ and } j \text{ odd}. \end{cases} \]

We call \( q_{m_1,\ldots,m_n}(z_1,\ldots,z_n) \) and \( r_{m_1,\ldots,m_n}(z_1,\ldots,z_n) \), respectively the upper and lower polynomials associated with \( Q_{m_1,\ldots,m_n}(z_1,\ldots,z_n) \), and together, these two polynomials will be called in the following, boundary polynomials associated with \( Q_{m_1,\ldots,m_n}(z_1,\ldots,z_n) \).

The following lemma is consequence of the above definitions.

**Lemma 1.** Let \( Q_{m_1,\ldots,m_n}(z_1,\ldots,z_n) \) be a interval polynomial in the variables \( z_i, i = 1,\ldots,n \), defined by (1) and \( G(Q_{m_1,\ldots,m_n}) \) its graph. We have

\[ G(Q_{m_1,\ldots,m_n}) = \{(\tilde{z}_1,\ldots,\tilde{z}_n,\tilde{z}) \in \mathbb{R}^{n+1} : r_{m_1,\ldots,m_n}(\tilde{z}_1,\ldots,\tilde{z}_n) \leq \tilde{z} \leq q_{m_1,\ldots,m_n}(\tilde{z}_1,\ldots,\tilde{z}_n)\}. \]  

(4)

Let us suppose that we intent to characterize the set

\[ \mathcal{N}(Q_{m_1,\ldots,m_n}) = \{(\tilde{z}_1,\ldots,\tilde{z}_n) \in \mathbb{C}^n : \exists p_{m_1,\ldots,m_n}(z_1,\ldots,z_n) \in Q_{m_1,\ldots,m_n}(z_1,\ldots,z_n), p_{m_1,\ldots,m_n}(\tilde{z}_1,\ldots,\tilde{z}_n) = 0\}, \]

where \( \mathbb{C}^n = \{(z_1,\ldots,z_n) : z_i \in \mathbb{C}, i = 1,\ldots,n\} \) and \( \mathbb{C} \) denotes the complex plan. This characterization can be easily made using the boundary polynomials \( q_{m_1,\ldots,m_n}(z_1,\ldots,z_n) \) and \( r_{m_1,\ldots,m_n}(z_1,\ldots,z_n) \) induced by \( Q_{m_1,\ldots,m_n}(z_1,\ldots,z_n) \) and is established in the following theorem.

**Theorem 1.** Let \( Q_{m_1,\ldots,m_n}(z_1,\ldots,z_n) \) be a interval polynomial in the variables \( z_i, i = 1,\ldots,n \), defined by (1) and let \( q_{m_1,\ldots,m_n}(z_1,\ldots,z_n) \) and \( r_{m_1,\ldots,m_n}(z_1,\ldots,z_n) \) be the boundary polynomials associated with \( Q_{m_1,\ldots,m_n}(z_1,\ldots,z_n) \). We have

1. if \( r_{m_1,\ldots,m_n}(z_1,\ldots,z_n) > 0 \) for all \( (z_1,\ldots,z_n) \in \mathbb{R}^n \), then \( \mathcal{N}(Q_{m_1,\ldots,m_n}) \cap \mathbb{R}^n = \emptyset \);
2. if \( q_{m_1,\ldots,m_n}(z_1,\ldots,z_n) = 0 \) for some \( (z_1,\ldots,z_n) \in \mathbb{R}^n \), then \( \mathcal{N}(Q_{m_1,\ldots,m_n}) \subset \mathbb{R}^n \);
3. if \( q_{m_1,\ldots,m_n}(z_1,\ldots,z_n) > 0 \) for all \( (z_1,\ldots,z_n) \in \mathbb{R}^n \), and \( r_{m_1,\ldots,m_n}(z_1,\ldots,z_n) = 0 \) for some \( (z_1,\ldots,z_n) \in \mathbb{R}^n \), then \( \mathcal{N}(Q_{m_1,\ldots,m_n}) = I_1 \cup I_2 \) with \( I_1 \subset \mathbb{R}^n \) and \( I_2 \subset \mathbb{C}^n \).

**Example 2.** Let us consider Example 1. For the interval polynomial \( Q_{2,2}(z_1,z_2) \) consider there we have the following boundary polynomials:

1. \[ q_{2,2}(z_1,z_2) = \begin{cases} z_1^2 - 2z_1 + z_2^2 - 2z_2 + 4, & z_1 \geq 0, \ z_2 \geq 0, \\ z_1^2 - 2z_1 + z_2^2 - 4z_2 + 4, & z_1 \geq 0, \ z_2 < 0, \\ z_1^2 - 4z_1 + z_2^2 - 2z_2 + 4, & z_1 < 0, \ z_2 > 0, \\ z_1^2 - 4z_1 + z_2^2 - 4z_2 + 4, & z_1 < 0, \ z_2 < 0, \end{cases} \]
2.

\[ r_{2,2}(z_1, z_2) = \begin{cases} 
  z_1^2 - 4z_1 + z_2^2 - 4z_2 + 1, & z_1 \geq 0, z_2 \geq 0, \\
  z_1^2 - 4z_1 + z_2^2 - 2z_2 + 1, & z_1 \geq 0, z_2 < 0, \\
  z_1^2 - 2z_1 + z_2^2 - 4z_2 + 1, & z_1 < 0, z_2 > 0, \\
  z_1^2 - 2z_1 + z_2^2 - 2z_2 + 1, & z_1 < 0, z_2 < 0. 
\]

The graphics of the boundary polynomials are plotted in Fig. 1. We observe that \( r_{2,2}(z_1, z_2) > 0 \) for all \((z_1, z_2) \in \mathbb{R}^2\) and \( \{(z_1, z_2) \in \mathbb{R}^2 : r_{2,2}(z_1, z_2) = 0\} \) is the set

\[
\{(z_1, z_2) \in \mathbb{R}^2 : z_1 > 0, z_2 > 0, (z_1 - 2)^2 + (z_2 - 2)^2 = 7\}
\]

\( \cup \{(z_1, z_2) \in \mathbb{R}^2 : z_1 > 0, z_2 < 0, (z_1 - 2)^2 + (z_2 - 1)^2 = 3\}\)

\( \cup \{(z_1, z_2) \in \mathbb{R}^2 : z_1 < 0, z_2 > 0, (z_1 - 1)^2 + (z_2 - 2)^2 = 4\}\).

and, by Theorem 1, \( \mathcal{N}(Q_{2,2}) = I_1 \cup I_2 \) with \( I_1 \subset \mathbb{R}^2 \) and \( I_2 \subset \mathbb{C}^2 \).

3. Systems of interval polynomial equations

In this section we consider systems of interval polynomial equations of the type (1), that is,

\[ Q_{m_1, \ldots, m_n}(z_1, \ldots, z_n) = 0, \quad \ell = 1, \ldots, n, \quad \text{(5)} \]

where, for each \( \ell \),

\[ Q_{m_1, \ldots, m_n}(z_1, \ldots, z_n) = \sum_{i=1}^{n} \sum_{j=1}^{m_i} A_{i,j}^{(\ell)} z_i^{j} + A_0^{(\ell)} \]

and

\[ A_{i,j}^{(\ell)} := [a_{i,j}^{(\ell)}, b_{i,j}^{(\ell)}], \quad A_0^{(\ell)} = [a_0^{(\ell)}, b_0^{(\ell)}]. \]

Example 3. For example the system

\[
\begin{align*}
[1, 2]z_1 + [3, 4]z_2 &= 0, \\
[2, 3]z_1 + [-2, -1]z_2 + [-2, -1] &= 0
\end{align*}
\]

is of type (5). Our aim is to compute \((\hat{z}_1, \hat{z}_2) \in \mathbb{R}^2\) verifying the two interval equations, that is, such that exist \( p_{1,1}^{(1)}(z_1, z_2) \in [1, 2]z_1 + [3, 4]z_2 \) and \( p_{1,1}^{(2)}(z_1, z_2) \in [2, 3]z_1 + [-2, -1]z_2 + [-2, -1] \) verifying

\[ p_{1,1}^{(1)}(\hat{z}_1, \hat{z}_2) = p_{1,1}^{(2)}(\hat{z}_1, \hat{z}_2) = 0. \]

We intent to compute \((\hat{z}_1, \ldots, \hat{z}_n) \in \mathbb{R}^n\) such that, for each \( \ell \in \{1, \ldots, n\} \), exists \( p_{m_1, \ldots, m_n}^{(\ell)}(z_1, \ldots, z_n) \in Q_{m_1, \ldots, m_n}(z_1, \ldots, z_n) \) such that

\[ p_{m_1, \ldots, m_n}^{(\ell)}(\hat{z}_1, \ldots, \hat{z}_n) = 0. \]

The set of all these points of \( \mathbb{R}^n \) is denoted by \( \mathcal{S}(Q_{m_1, \ldots, m_n}, \ell = 1, \ldots, n) \).
Theorem 2. Let us consider the system (5) and the sets $\mathcal{N}(Q_{m_1,\ldots,m_\ell}^{(\ell)})$ for $\ell = 1,\ldots,n$. Then

$$\mathcal{S}(Q_{m_1,\ldots,m_n,\ell}^{(\ell)}, \ell = 1,\ldots,n) = \bigcap_{\ell=1}^n \mathcal{N}(Q_{m_1,\ldots,m_\ell}^{(\ell)}).$$

Applying Theorem 1 we are able to characterize each $\mathcal{N}(Q_{m_1,\ldots,m_\ell}^{(\ell)})$ and using Theorem 2, we obtain \{(\tilde{z}_1,\ldots,\tilde{z}_n)\in\mathbb{R}^n : Q_{m_1,\ldots,m_\ell}(\tilde{z}_1,\ldots,\tilde{z}_n) = 0, \ \ell = 1,\ldots,n\}.

Example 4. Let us consider Example 3. The graphics of the two interval polynomials in the variables $z_1$ and $z_2$, $Q_{1,1}^{(1)}(z_1,z_2)$, $Q_{1,1}^{(2)}(z_1,z_2)$, defined by the two equations are the points of $\mathbb{R}^3$ between the two surfaces plotted in Figs. 2 and 3, respectively.
In Figs. 4 and 5 we plot $\mathcal{N}(Q^{(1)}_{1,1})$ and $\mathcal{N}(Q^{(2)}_{1,1})$, respectively.
In Fig. 6 we plot $\mathcal{S}(Q^{(1)}_{1,1}, Q^{(2)}_{1,1}) = \mathcal{N}(Q^{(1)}_{1,1}) \cap \mathcal{N}(Q^{(2)}_{1,1})$. 
References


