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# On the number of invariant factors of matrix products

E. Marques de Sá [a,*,1], Yu Lin Zhang [b,2]

[a] *Departamento de Matemática, Universidade de Coimbra, 3000 Coimbra, Portugal*
[b] *Departamento de Matemática, Universidade do Minho, 4710 Braga, Portugal*

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**Abstract**

We prove an inequality relating the number of nontrivial invariant factors of $n \times n$ matrices $A$ and $B$, with those of $AB$, and get some results on the cases of equality. In particular, we characterize the similarity classes, $\mathscr{A}$ and $\mathscr{B}$, with all eigenvalues in the base field, such that $AB$ is nilpotent for some $A \in \mathscr{A}$ and $B \in \mathscr{B}$.
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## 1. Introduction

This paper is about matrices over an arbitrary field $\mathbb{F}$. We denote by $\overline{\mathbb{F}}$ its algebraic closure, and by $\mathbb{F}^*$ the set of nonzero elements of $\mathbb{F}$. We shall consider polynomials

* Corresponding author.
*E-mail addresses:* emsa@mat.uc.pt (E.M. de Sá), zhang@math.uminho.pt (Y.L. Zhang).
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from the polynomial ring $\mathbb{F}[x]$. The script letters $\mathscr{A}$ and $\mathscr{B}$ represent $n \times n$ similarity classes. So, if $A \in \mathscr{A}$, then $A$ is an $n \times n$ matrix over $\mathbb{F}$ and $\mathscr{A}$ is the set of all matrices over $\mathbb{F}$ similar to $A$. The *invariant factors*, *eigenvalues*, *rank*, etc, of $\mathscr{A}$ are defined as the corresponding concepts of any $A \in \mathscr{A}$. There exists a matrix in $\mathscr{A}$ of the form $A_1 \oplus N$, where $A_1$ is nonsingular and $N$ is nilpotent; the similarity classes of $A_1$ and $N$ are well defined and called the *nonsingular* and *nilpotent parts* of $\mathscr{A}$. Define

$$R(A) := \min\{\operatorname{rank}[\lambda I - A] : \lambda \in \overline{\mathbb{F}}\},$$

$$R^*(A) := \min\{\operatorname{rank}[\lambda I - A] : \lambda \in \overline{\mathbb{F}}^*\}.$$

As $R^*$ is invariant under similarity, we define $R^*(\mathscr{A})$, for any class $\mathscr{A}$, in the obvious way.

In recent literature, the problem of relating the similarity classes of two matrices with the similarity class of their product has received some attention. Those problems are, in general, of a very high degree of difficulty. In our references, we indicate some papers on that subject [1,3–6,8] for more information and related problems we send the reader to the references in [3].

In [9], the following theorem has been proved in the case $\mathscr{A}$ and $\mathscr{B}$ are nonsingular.

**Theorem 1.1.** *For any $A \in \mathscr{A}$ and $B \in \mathscr{B}$, we have*:

$$R^*(AB) \leqslant \min\{n, R^*(\mathscr{A}) + R^*(\mathscr{B})\}. \tag{1}$$

**Proof.** We give the following sketch of proof with no further ado:

$$\begin{aligned} R^*(AB) &= \min_{\lambda \neq 0} \operatorname{rank}(\lambda I - AB) \\ &= \min_{\mu \neq 0} \min_{\lambda \neq 0} \operatorname{rank}(\lambda I - AB + \mu B - \mu B) \\ &\leqslant \min_{\mu \neq 0} \min_{\lambda \neq 0} [\operatorname{rank}[(\mu I - A)B] + \operatorname{rank}(\lambda I - \mu B)] \\ &\leqslant \min_{\mu \neq 0} \min_{\lambda \neq 0} [\operatorname{rank}(\mu I - A) + \operatorname{rank}(\lambda I - \mu B)] \\ &= R^*(\mathscr{A}) + R^*(\mathscr{B}). \qquad \square \end{aligned}$$

The theorem may be obtained as a corollary of Theorem 4 of [3], but the argument given above is easier.

We shall denote by $\alpha_1, \ldots, \alpha_n$ the invariant factors of $\mathscr{A}$; the $\alpha$'s are monic polynomials ordered so that $\alpha_1 | \cdots | \alpha_n$. If we eliminate those $\alpha$'s equal to 1, we obtain the chain $f_1 | \cdots | f_r$, of the *nontrivial* invariant factors of $\mathscr{A}$; in the sequel, the number $r$ will be denoted by $i(\mathscr{A})$, and $i^*(\mathscr{A})$ denotes the number of invariant factors of $\mathscr{A}$ with at least one nonzero root in $\overline{\mathbb{F}}$. It is not difficult to prove that $i^*(\mathscr{A}) = n - R^*(\mathscr{A})$, and $i^*(\mathscr{A}) = i(\mathscr{A}_1)$, where $\mathscr{A}_1$ is the nonsingular part of $\mathscr{A}$. With this notation, (1) reads

$$i^*(AB) \geqslant \max\{0, i^*(\mathscr{A}) + i^*(\mathscr{B}) - n\}. \tag{2}$$

An interesting problem is the characterization of the similarity classes for which we have equality in (1)–(2) for some $A \in \mathscr{A}$ and $B \in \mathscr{B}$. This problem naturally splits into two subproblems:

**Problem I.** Characterize the classes $\mathscr{A}$ and $\mathscr{B}$, for which there exist $A \in \mathscr{A}$ and $B \in \mathscr{B}$ such that $i^*(AB) = 0$.

**Problem II.** Characterize the classes $\mathscr{A}$ and $\mathscr{B}$, for which there exist $A \in \mathscr{A}$ and $B \in \mathscr{B}$ such that $i^*(AB) = i^*(\mathscr{A}) + i^*(\mathscr{B}) - n$.

We obtain partial results on these problems. In particular, we solve Problem I when all eigenvalues of $\mathscr{A}$ and $\mathscr{B}$ lie in $\mathbb{F}$, and solve Problem II over algebraically closed fields.

## 2. Results on Problem I

As $i^*(X) = 0$ iff $X$ is nilpotent, Problem II consists in the characterization of $\mathscr{A}$ and $\mathscr{B}$, for which there exist $A \in \mathscr{A}$ and $B \in \mathscr{B}$ such that $AB$ is nilpotent. Clearly, if $AB$ is nilpotent, then either $\mathscr{A}$ or $\mathscr{B}$ is singular, and $i^*(\mathscr{A}) + i^*(\mathscr{B}) \leqslant n$, by inequality (2). We conjecture that the converse is true with a tiny exception. More precisely, we state

**Conjecture 2.1.** *Given two $n \times n$ similarity classes $\mathscr{A}$ and $\mathscr{B}$ over $\mathbb{F}$, there exist $A \in \mathscr{A}$ and $B \in \mathscr{B}$ such that $AB$ is nilpotent, if and only if one of $\mathscr{A}$, $\mathscr{B}$ is singular, $i^*(\mathscr{A}) + i^*(\mathscr{B}) \leqslant n$, and $\mathscr{A}$, $\mathscr{B}$ do not fall in the following*

> EXCEPTIONAL CASE. $n = 2$, the classes $\mathscr{A}$ and $\mathscr{B}$ are both nonzero, one of them is nilpotent and the characteristic polynomial of the other is irreducible over $\mathbb{F}$.

We shall prove the conjecture in several cases, namely:

**Theorem 2.2.** *Conjecture 2.1 holds if one of the following conditions holds*:

(a) *One of the classes, $\mathscr{A}$ or $\mathscr{B}$, has a zero Jordan block*;
(b) $i^*(\mathscr{A}) + i^*(\mathscr{B}) = n$;
(c) *All eigenvalues of $\mathscr{A}$ and $\mathscr{B}$ lie in $\mathbb{F}$.*

## 3. Results on Problem II

We consider two cases: (i) when $\mathscr{A}$ and $\mathscr{B}$ are both nonsingular; (ii) either $\mathscr{A}$ or $\mathscr{B}$ is singular. The first theorem of this section is a simple consequence of the main result of [9], so it goes with no proof.

**Theorem 3.1.** *Assume $\mathscr{A}$ and $\mathscr{B}$ are nonsingular, $i^*(\mathscr{A}) + i^*(\mathscr{B}) > n$, and $\mathbb{F}$ is algebraically closed. Then there exist $A \in \mathscr{A}$ and $B \in \mathscr{B}$ such that $i^*(AB) = i^*(\mathscr{A}) + i^*(\mathscr{B}) - n$.*

**Theorem 3.2.** *Assume either $\mathscr{A}$ or $\mathscr{B}$ is singular, and $i^*(\mathscr{A}) + i^*(\mathscr{B}) > n$. Then there exist $A \in \mathscr{A}$ and $B \in \mathscr{B}$ such that $i^*(AB) = i^*(\mathscr{A}) + i^*(\mathscr{B}) - n$.*

## 4. Proofs

First we check the 'exceptional case' of Conjecture 2.1. If, say, $\mathscr{B}$ is the nilpotent class, all products $AB$, with $A \in \mathscr{A}$ and $B \in \mathscr{B}$, are similar to

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

where the first factor has an irreducible characteristic polynomial. As $c$ is nonzero, $AB$ is not nilpotent.

**Proof of Theorem 2.2.** The result clearly holds for $n \leqslant 2$, or if one of the classes is scalar. So we may assume $n \geqslant 3$, and $\mathscr{A}$ and $\mathscr{B}$ are non-scalar.

**Proof of (a).** We argue by induction on $n$. We assume $\mathscr{B}$ has a zero Jordan block; this means we may pick $B \in \mathscr{B}$ of the form

$$B = \begin{bmatrix} B' & 0 \\ 0 & 0 \end{bmatrix},$$

where $B'$ is square of order $n-1$. Partition all matrices $A \in \mathscr{A}$ as

$$A = \begin{bmatrix} C_A & * \\ * & * \end{bmatrix},$$

with $C_A$ a square block of order $n-1$. Let $\alpha_1 | \cdots | \alpha_n$ and $\gamma_1 | \cdots | \gamma_{n-1}$ be the invariant factors of $A$ and $C_A$. According to the interlacing inequalities for similarity invariant factors [2,7], for a given class $\mathscr{A}$ the possible $\gamma$'s are characterized by

$$\deg(\gamma_1 \cdots \gamma_{n-1}) = n-1 \quad \text{and} \quad \alpha_i | \gamma_i | \alpha_{i+2}, \tag{3}$$

for $1 \leqslant i \leqslant n-1$ (with the convention $\alpha_{n+1} := 0$). Note that $\alpha_1 = 1$ and the degree of $\alpha_n$, call it $a$, satisfies $a \geqslant 2$, because $\mathscr{A}$ is nonscalar. Now let $z$ be the largest $i < n$ such that $\alpha_i$ is not a multiple of $x$ (recall: our polynomials are taken from $\mathbb{F}[x]$).

If $z < n-1$, define $\gamma_z := x\alpha_z$, $\gamma_{n-1} := x^{a-2}\alpha_{n-1}$, and $\gamma_i := \alpha_i$ for all $i \in \{1, \dots, n-2\}$, $i \neq z$. If $z = n-1$, define $\gamma_{n-1} := x(x-1)^{a-2}\alpha_{n-1}$, and $\gamma_i := \alpha_i$ for all $i \in \{1, \dots, n-2\}$.

In either case, the $\gamma$'s satisfy (3), and any $C_A$ having them as invariant factors satisfies the properties: $C_A$ is nilpotent if $\mathscr{A}$ is nilpotent; $i^*(C_A) = i^*(\mathscr{A}) - 1$, if $\mathscr{A}$

is not nilpotent; $C_A$ has a zero Jordan block (because 0 is a simple root of one of the $\gamma$'s). So we have $i^*(C_A) + i^*(B') \leqslant n - 1$; by induction on $n$, we may choose $C_A$ such that $C_A B'$ is nilpotent. Therefore $AB$ is nilpotent as well.

**Proof of (b).** We may assume $\mathscr{A}$ and $\mathscr{B}$ have no zero Jordan block, and $\mathscr{A}$ is singular. Note that $i^*(\mathscr{A}) + i^*(\mathscr{B}) = n$ implies $\mathscr{A}$, $\mathscr{B}$ do not fall in the exceptional case. The proof is by induction on $n$.

*Case* 1: *when $\mathscr{B}$ has an invariant factor of degree one*.

Assume $\mathscr{B}$ has $x - b$ as invariant factor. We consider matrices $A \in \mathscr{A}$ and $B \in \mathscr{B}$ of the form

$$A = \begin{bmatrix} A_1 & * \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & b \end{bmatrix},$$

where $A_1$ and $B_1$ are square matrices of order $n - 1$. As $\mathscr{A}$ and $\mathscr{B}$ have no zero Jordan block, $b$ is nonzero and $A_1$ is singular. Clearly, $i^*(A_1) + i^*(B_1) = n - 1$. By induction we may select $A_1$ and $B_1$ such that $A_1 B_1$ is nilpotent, and therefore get a nilpotent $AB$.

*Case* 2: *$\mathscr{B}$ has no invariant factor of degree one*.

As $\mathscr{A}$ is non-scalar, $i^*(\mathscr{A}) < n$; therefore $\mathscr{B}$ is non-nilpotent. So $\mathscr{B}$ has an invariant factor with a nonzero root over $\overline{\mathbb{F}}$; let $\beta$ be such an invariant factor of lowest positive degree, and let $d$ be the degree of $\beta$. The companion matrix $C_\beta$ is of order $d$, and $i^*(C_\beta) = 1$. Note that $i^*(\mathscr{A}) \leqslant n - 2$, because $\mathscr{A}$ has eigenvalue 0 with multiplicity $\geqslant 2$; so we have $i^*(\mathscr{B}) \geqslant 2$, and $n \geqslant 4$.

Now, from $i^*(\mathscr{B})d \leqslant n$, we obtain $d \leqslant n/2$, and $i^*(\mathscr{A}) \geqslant n - n/d$. Let $u$ be the number of invariant factors of the nonsingular part of $\mathscr{A}$ with degree one; these $u$ invariant factors are all equal to, say, $x - a$. We have $u + 2[i^*(\mathscr{A}) - u] \leqslant n - 2$ (the '$n - 2$' comes from the fact that $\mathscr{A}$ has at least two zero eigenvalues). Therefore $u \geqslant n + 2 - 2n/d$. For $n \geqslant 3$, the function $f(x) := n + 2 - 2n/x$ is strictly concave for $x > 0$, and satifies $f(2) = 2$, $f(n) = n$; we thus have $f(x) > x$ in the interval $]2, n[$. Therefore

$$n + 2 - 2n/d \geqslant d, \quad \textit{with equality iff } d \in \{2, n\}. \tag{4}$$

From this we get $u \geqslant d$. So the invariant factor $x - a$ occurs in $\mathscr{A}$ at least $d$ times. Accordingly, we choose $A \in \mathscr{A}$ and $B \in \mathscr{B}$ of the form

$$A = \left[\begin{array}{c|c} A_1 & \\ & 1 \\ \hline & \\ & D_a \end{array}\right] \quad \text{and} \quad B = \left[\begin{array}{c|c} B_1 & \\ \hline & \\ & C_\beta \end{array}\right], \tag{5}$$

where $A_1$ and $B_1$ are square matrices of order $n - d$, and $D_a$ is a $d \times d$ diagonal matrix with diagonal entries $0, a \ldots, a$ (with $a$ repeated $d - 1$ times).

Now we apply induction to the two pairs of diagonal blocks. As $D_a$ is singular and $i^*(D_a) + i^*(C_\beta) = d$, there exists $C$ similar to $C_\beta$ such that $D_a C$ is nilpotent. On the other hand, $A_1$ is singular and $i^*(A_1) + i^*(B_1) = n - d$; so there exists $B_1'$ similar to $B_1$ such that $A_1 B_1'$ is nilpotent. Therefore, $A(B_1' \oplus C)$ is nilpotent, and we are done.

**Proof of (c).** We go by induction on $n$. The previously proved items leave us with the case when $i^*(\mathscr{A}) + i^*(\mathscr{B}) < n$, and the classes have no zero Jordan block. Without loss of generality we assume $\mathscr{B}$ is singular.

There exist $A \in \mathscr{A}$ and $B \in \mathscr{B}$ of the form

$$A = \begin{bmatrix} A_1 & * \\ 0 & \tau \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_1 & * \\ 0 & 0 \end{bmatrix},$$

where $A_1$ and $B_1$ are square matrices of order $n - 1$. Clearly, $B_1$ is singular, and $i^*(A_1) + i^*(B_1) \leqslant n - 1$; by induction we may choose $A_1$ and $B_1$ such that $A_1 B_1$ is nilpotent, and so $AB$ is nilpotent as well. □

To prove Theorem 3.2 we need a lemma where the following notation is used. Let $f_1 | f_2 | \cdots | f_r$ and $g_1 | g_2 | \cdots | g_s$ be the nontrivial invariant factors of $\mathscr{A}$ and $\mathscr{B}$, respectively. We consider $A \in \mathscr{A}$ and $B \in \mathscr{B}$ in companion normal form:

$$A = C(f_1) \oplus \cdots \oplus C(f_r), \qquad B = C(g_1) \oplus \cdots \oplus C(g_s).$$

Here, $C(\varphi)$ is any companion matrix of polynomial $\varphi$ (in fact, we only need $C(\varphi)$ to be nonderogatory, with characteristic polynomial $\varphi$).

**Lemma 4.1.** *Assume $i^*(\mathscr{A}) + i^*(\mathscr{B}) > n$, $i^*(\mathscr{A}) \geqslant i^*(\mathscr{B})$, and $\mathscr{A}$ is not scalar. There exists $m$ such that the direct sum decompositions of the above matrices $A$ and $B$, as $A = A_1 \oplus A_2$, and $B = B_1 \oplus B_2$, where $A_2$ and $B_2$ are $m \times m$, satisfy $i^*(A_2) + i^*(B_2) = m$, and the block $A_1$ is scalar.*

**Proof.** We go by induction on $n$. As $i^*(\mathscr{A}) > n/2$, $f_1$ must have degree 1, and $f_1 = x - a$, for a *nonzero* $a$. Let $d$ be the degree of $g_1$, the first nontrivial invariant factor of $\mathscr{B}$. We have $i^*(\mathscr{B}) \leqslant n/d$, and $i^*(\mathscr{A}) \geqslant n - n/d + 1$. We may argue as in the proof of (4), to prove that the number of invariant factors of $\mathscr{A}$ of degree 1 is at least $d$. So $f_d = f_1$.

We now partition $A = (aI_d) \oplus A'$, and $B = C(g_1) \oplus B'$, where $A'$ and $B'$ are square of order $n' := n - d$. Clearly, $i^*(A') = i^*(A) - d$, and $i^*(B') \in \{i^*(B), i^*(B) - 1\}$; we thus have $i^*(A') + i^*(B') \geqslant n'$. If we have equality, the proof is done. Now assume that $i^*(A') + i^*(B') > n'$. To apply induction to $A'$, $B'$, we need to show

$$i^*(A') \geqslant i^*(B'). \tag{6}$$

If $d = 1$ and $i^*(A) > i^*(B)$, then (6) trivially holds; if $d = 1$ and $i^*(A) = i^*(B)$, then $i^*(B') = i^*(B) - 1$, and (6) holds as well. In case $d \geqslant 2$, we may take (4) into account, and get

$$\begin{aligned} i^*(A') &\geqslant n + 1 - i^*(B) - d \geqslant n + 1 - n/d - d \\ &\geqslant n/d - 1 \geqslant i^*(B) - 1. \end{aligned} \tag{7}$$

If one of the inequalities is strict, we have (6). If $i^*(A') = i^*(\mathscr{B}) - 1$ then all 4 inequalities in (7) are equalities; this implies $d = 2$, and $i^*(\mathscr{B}) = n/2$, *i.e.*, all invariant factors of $\mathscr{B}$ are equal, of degree 2; as $\mathscr{B}$ is not nilpotent, we have $i^*(B') = i^*(\mathscr{B}) - 1$, and we again get (6).

So, arguing by induction, we may apply the lemma to the submatrices $A'$, $B'$, and thus obtain the result for the initial matrices $A$, $B$. □

**Proof of Theorem 3.2.** The result is trivial if one of the classes is scalar; so we assume that both are nonscalar. Without loss of generality, we assume $i^*(\mathscr{A}) \geqslant i^*(\mathscr{B})$.

We apply Lemma 4.1 to get $A_1 \oplus A_2 \in \mathscr{A}$, and $B_1 \oplus B_2 \in \mathscr{B}$. One of the blocks $A_2$, $B_2$ is singular. So, by Theorem 2.2(b), there exist $A_2'$ and $B_2'$ similar to $A_2$ and $B_2$, respectively, such that $A_2'B_2'$ is nilpotent, *i.e.*, $i^*(A_2'B_2') = 0$. In this way, we get $A' := A_1' \oplus A_2$ and $B' := B_1' \oplus B_2$, satisfying

$$\begin{aligned} i^*(A'B') &= i^*(A_1B_1) = i^*(B_1) = i^*(\mathscr{B}) - i^*(B_2) \\ &= i^*(\mathscr{B}) + i^*(A_2) - m = i^*(\mathscr{B}) + i^*(\mathscr{A}) - n. \qquad \square \end{aligned}$$

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