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A superconvergent linear FE approximation for the solution of an elliptic system of PDEs

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Abstract

The aim of this work is to study a nonstandard piecewise linear finite element method for elliptic systems of partial differential equations. This nonstandard method was considered by the authors for scalar elliptic equations and for a planar elasticity problem. The method enables us to compute a superconvergent numerical approximation to the solution of the system of partial differential equations.

Keywords: Elliptic system of PDEs,; Finite element approximation; Finite difference method; Superconvergence; Supraconvergence

1. Introduction

Most physical applications quantities are governed by systems of partial differential equations. An example is given by the deformations and stresses of elastic and inelastic bodies subject to load, studied in solid mechanics.

For the computation of a numerical approximation of the solution of a system of partial differential equations, finite element methods—FEMs—and finite difference methods—FDMs—are the numerical methods usually used.

In this paper, we study the numerical approximation for the solution of a system of partial differential equations obtained using a nonstandard piecewise linear FEM. The method was studied in [8–10] for

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scalar elliptic equations and considered by the authors in [1] for a planar elasticity problem. In [1] the estimates for the $\mathbf{H}^1$-norm of the error were obtained using the results proved in [8]. As a consequence of the approach followed in [8], the estimates were proved under very restrictive assumptions for the solution of the continuous problem (it was assumed that the $\|.\|_\infty$-norms of the fourth-order partial derivatives of the solution are bounded). In [9,10], an alternative approach was introduced which allowed the authors to prove the same estimates under weaker assumptions. In the present paper, our aim is to generalize the results obtained in [1] to elliptic systems of partial differential equations defined on polygonal domains of $\mathbb{R}^2$.

We observe that our method allows a family of triangulations of the domain, which does not need to be quasi-uniform and regular and enables us to compute the numerical approximation of the displacement with an improved accuracy when compared with standard linear FEMs described in the literature, e.g. [2–5,16,27,28].

About more than 25 years ago, Zlámal [30] found superconvergence of the gradient for certain quadrature FE solutions on nearly rectangular grids. Furthermore, the superconvergence of the gradient of piecewise linear FE approximations was studied for instance in [6,14,17,20,22,25] but assuming that the triangulations are regular and quasi-uniform.

Assuming that the nonstandard FEM studied in this work is equivalent to a carefully defined FDM, we conclude that this last method is supraconvergent. Supraconvergent finite difference schemes have been largely studied in the literature and without being exhaustive, we mention [7–15,18,19,21,23,24,29–31].

The paper is organized as follows. In Section 2, we present the problem that we intend to solve. The nonstandard piecewise linear FEM is described in Section 3. In Section 4, we study the stability of the bilinear form that defines the nonstandard method. The study of the $\mathbf{H}^1$-norm of the error is dealt with in Section 5. In Section 6, we present a FDM equivalent to the piecewise linear FEM described in Section 3, which enables an easy computation of the finite element solution. Examples illustrating the performance of the method are considered in Section 7.

2. A second-order elliptic system of partial differential equations

Let us consider the system of partial differential equations

$$\sum_{j=1}^{n} \ell_{ij} u_j = g_i \text{ in } \Omega, \quad i = 1, \ldots, n, \tag{1}$$

subject to Dirichlet boundary conditions

$$u_i = 0 \text{ on } \partial\Omega, \quad i = 1, \ldots, n, \tag{2}$$

where Ω is a bounded polygonal domain of $\mathbb{R}^2$ with boundary $\partial\Omega$. In (1), ℓ_{ij} denotes the second-order differential operator defined by

$$\begin{aligned} \ell_{ij} u_j = & -\frac{\partial}{\partial x}\left(a_{ij}\frac{\partial u_j}{\partial x}\right) - \frac{\partial}{\partial y}\left(b_{ij}\frac{\partial u_j}{\partial x}\right) - \frac{\partial}{\partial x}\left(b_{ij}\frac{\partial u_j}{\partial y}\right) - \frac{\partial}{\partial y}\left(c_{ij}\frac{\partial u_j}{\partial y}\right) \\ & + d_{ij}\frac{\partial u_j}{\partial x} + e_{ij}\frac{\partial u_j}{\partial y} + f_{ij} u_j. \end{aligned} \tag{3}$$

In this paper, we deal with uniformly strongly elliptic systems of partial differential equations. Following [26], the system (1) is uniformly strongly elliptic in Ω if there exists a positive constant C_0 such that, for each $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$, the inequality

$$\sum_{i,j=1}^{n} (a_{ij}(x,y)\xi_1^2 + 2b_{ij}(x,y)\xi_1\xi_2 + c_{ij}(x,y)\xi_2^2)\eta_i\eta_j \geqslant C_0 \left(\sum_{i=1}^{n} \eta_i^2\right)(\xi_1^2 + \xi_2^2) \tag{4}$$

holds for all $(x, y) \in \Omega$.

We assume in the sequel that the coefficient functions are smooth enough ($a_{ij}, b_{ij}, c_{ij} \in W^{3,\infty}(\Omega)$, $d_{ij}, e_{ij}, f_{ij} \in W^{2,\infty}(\Omega)$ is sufficient) and $g_i \in H^1(\Omega)$.

For $s \in \mathbb{N}$, we define $\mathbf{H}^s(\Omega)$ by $\mathbf{H}^s(\Omega) = H^s(\Omega) \times \cdots \times H^s(\Omega)$ (n-times) endowed with the inner product

$$(v, w)_{\mathbf{H}^s(\Omega)} = \sum_{i=1}^{n} (v_i, w_i)_{H^s(\Omega)}$$

for $v, w \in \mathbf{H}^s(\Omega)$. The norm induced by the inner product $(.,.)_{\mathbf{H}^s(\Omega)}$ is represented by $\|.\|_s$.

The spaces $\mathbf{H}_0^s(\Omega)$ and $\mathbf{L}^2(\Omega)$ are defined analogously replacing in the definition of $\mathbf{H}^s(\Omega) H^s(\Omega)$ by $H_0^s(\Omega)$ and $L^2(\Omega)$, respectively, and their inner products are defined using the same modifications as before.

For the bilinear form

$$\begin{aligned} a(v, w) = \sum_{i,j=1}^{n} [&a_{ij}(v_j)_x(w_i)_x + b_{ij}((v_j)_x(w_i)_y + (v_j)_y(w_i)_x) + c_{ij}(v_j)_y(w_i)_y \\ &+ d_{ij}(v_j)_x w_i + e_{ij}(v_j)_y w_i + f_{ij} v_j w_i] \end{aligned} \tag{5}$$

defined in $\mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$, we introduce the variational problem:

$$\text{find } u \in \mathbf{H}_0^1(\Omega) \text{ such that } a(u, v) = (g, v)_{\mathbf{L}^2(\Omega)}, \text{ for all } v \in \mathbf{H}_0^1(\Omega). \tag{6}$$

The bilinear form $a(.,.)$ satisfies the following result:

Theorem 1. *Suppose that the system of partial differential equations* (1) *is uniformly strongly elliptic, i.e.,* Ω *verifies* (4). *Then the bilinear form* $a(.,.)$ *defined in* (5) *is* $\mathbf{H}_0^1(\Omega)$*-coercive, i.e., there exists* $C_K \in \mathbb{R}$ *and* $C_E > 0$ *such that*

$$a(u, u) \geqslant C_E \|u\|_1^2 - C_K \|u\|_0^2, \ \forall u \in \mathbf{H}_0^1(\Omega). \tag{7}$$

Details of the proof can be seen in [26].

3. The finite element method

Let us introduce the full discretization of the variational problem (6). Firstly, we define two special triangulations of the domain Ω associated with a nonuniform rectangular grid.

Let $h=(h_j)_{\mathbb{Z}}$ and $k=(k_\ell)_{\mathbb{Z}}$ be two sequences of positive numbers. We define the nonuniform rectangular grid $\mathbb{R}_H$ by $\mathbb{R}_H = \mathbb{R}_1 \times \mathbb{R}_2 \subset \mathbb{R}^2$, where $\mathbb{R}_1 = \{x_\kappa \in \mathbb{R} : x_{\kappa+1} = x_\kappa + h_\kappa,\ \kappa \in \mathbb{Z}\}$ and $\mathbb{R}_2 = \{y_\ell \in \mathbb{R} : y_{\ell+1} = y_\ell + k_\ell,\ \ell \in \mathbb{Z}\}$ with x_0 and y_0 given, The grid $\mathbb{R}_H$ induces the following sets of grid points: $\Omega_H = \Omega \cap \mathbb{R}_H$, $\partial\Omega_H = \partial\Omega \cap \mathbb{R}_H$ and $\bar{\Omega}_H = \bar{\Omega} \cap \mathbb{R}_H$.

We assume that the grid $\bar{\Omega}_H$ satisfies the following regularity condition with respect to the domain Ω:

(Reg) The intersection of any sub-rectangle $(x_k, x_{k+1}) \times (y_\ell, y_{\ell+1})$ with $\partial\Omega$ is either empty or is the diagonal of the rectangle.

As mentioned before, we consider two special triangulations related to the set $\bar{\Omega}_H$, which we call $\mathscr{T}_H^{(1)}$ and $\mathscr{T}_H^{(2)}$. They are obtained from the disjoint decomposition $\mathbb{R}_H = \mathbb{R}_H^{(1)} \cup \mathbb{R}_H^{(2)}$, where $\mathbb{R}_H^{(1)} = \{(x_\kappa, y_\ell) \in \mathbb{R}_H : \kappa + \ell \text{ is odd}\}$ and $\mathbb{R}_H^{(2)} = \{(x_\kappa, y_\ell) \in \mathbb{R}_H : \kappa + \ell \text{ is even}\}$. To simplify the following definition, we set $\mathbb{R}_H^{(3)} = \mathbb{R}_H^{(1)}$. For each point $(x_\kappa, y_\ell) \in \mathbb{R}_H$, we associate the triangles $\Delta_{\kappa\ell}^{(i)},\ i \in \{1, 2, 3, 4\}$, which have a right angle at (x_κ, y_ℓ) and two of the closest neighbour grid points of (x_κ, y_ℓ) as further vertices. Then, for $s \in \{1, 2\}$, we define the triangulations of $\bar{\Omega}$, $\mathscr{T}_H^{(s)} = \mathscr{T}_{H,1}^{(s)} \cup \mathscr{T}_{H,2}^{(s)}$, where

$$\mathscr{T}_{H,1}^{(s)} = \{\Delta_{\kappa,\ell}^{(i)} \subset \bar{\Omega}, (x_\kappa, y_\ell) \in \mathbb{R}_H^{(s)},\ i \in \{1, 2, 3, 4\}\},$$

$$\mathscr{T}_{H,2}^{(s)} = \left\{ \Delta_{\kappa,\ell}^{(i)} \subset \left(\bar{\Omega} \Big\backslash \bigcup_{\Delta \in \mathscr{T}_{H,1}^{(s)}} \overset{\circ}{\Delta} \right), (x_\kappa, y_\ell) \in \mathbb{R}_H^{(s+1)}, i \in \{1, 2, 3, 4\} \right\}$$

and Δ° denotes the interior of Δ. Fig. 1 shows an example of a triangulation $\mathscr{T}_H^{(s)}$.

We denote by $\overset{\circ}{W}_H$ the set of grid scalar functions zero at the boundary points $\partial\Omega_H$ and $\overset{\circ}{\mathbf{W}}_H$ stands for the space $\overset{\circ}{W}_H \times \cdots \times \overset{\circ}{W}_H$ (n-times).

The continuous piecewise linear interpolations of $v_H \in \overset{\circ}{\mathbf{W}}_H$, $P_H^{(s)} v_H = (P_H^{(s)} v_{H,1}, \ldots, P_H^{(s)} v_{H,n})$, $s = 1, 2$, are well defined for the triangulations $\mathscr{T}_H^{(s)}$.

The final part of this section deals with the discrete version of (6):

$$\text{find } u_H \in \overset{\circ}{\mathbf{W}}_H \text{ such that } a_H(u_H, v_H) = (g_H, v_H)_H, \text{ for all } v_H \in \overset{\circ}{\mathbf{W}}_H. \tag{8}$$

Let $\square_{\kappa,\ell}$ be the rectangle $[x_{\kappa-1/2}, x_{\kappa+1/2}] \times [y_{\ell-1/2}, y_{\ell+1/2}]$, where $x_{\kappa-1/2} = x_\kappa - h_{\kappa-1}/2$, $x_{\kappa+1/2} = x_\kappa + h_\kappa/2$; $y_{\ell-1/2}$, $y_{\ell+1/2}$ are defined analogously. Let g_H be the following grid function:

$$g_H(x_\kappa, y_\ell) = \frac{1}{|\square_{\kappa,\ell}|} \int_{\square_{\kappa,\ell}} g(x, y)\,\mathrm{d}x\,\mathrm{d}y, \tag{9}$$

where $|\square_{\kappa,\ell}|$ stands for the area of $\square_{\kappa,\ell}$.

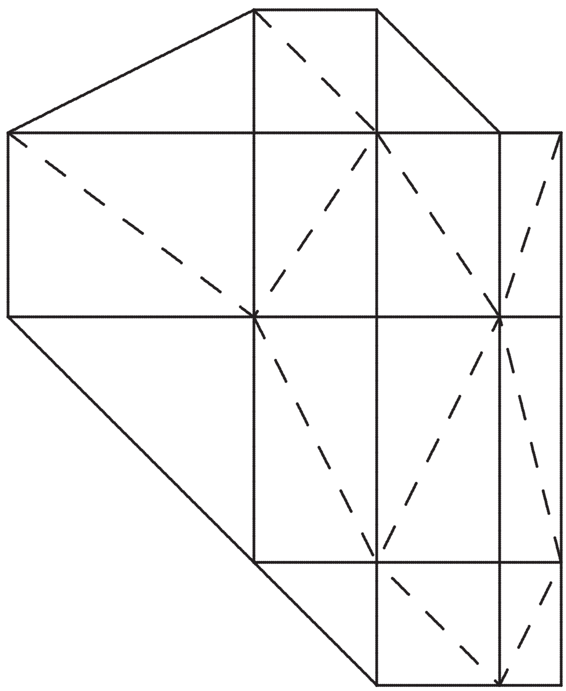

Fig. 1. Triangulation $\mathscr{T}_H^{(s)}$.

In (8), $(.,.)_H$ denotes the inner product in $\overset{\circ}{\mathbf{W}}_H$ defined by

$$(v_H, w_H)_H = \sum_{i=1}^{n} \sum_{(x_\kappa, y_\ell)\in\Omega_H} |\Box_{\kappa,\ell}| v_{H,i}(x_\kappa, y_\ell) w_{H,i}(x_\kappa, y_\ell) \tag{10}$$

for each $v_H, w_H \in \overset{\circ}{\mathbf{W}}_H$.

Let us now define the bilinear form $a_H(.,.)$. For each triangulation $\mathscr{T}_H^{(s)},\ s=1,2$, let $a_H^{(s)}(.,.)$ be the bilinear form

$$a_H^{(s)}(v_H, w_H) = \sum_{i,j=1}^{n} [a_{ij}^{(s)}(v_H, w_H) + b_{ij}^{(s)}(v_H, w_H) + c_{ij}^{(s)}(v_H, w_H) + d_{ij}^{(s)}(v_H, w_H) + e_{ij}^{(s)}(v_H, w_H) + f_{ij}^{(s)}(v_H, w_H)]. \tag{11}$$

for each $v_H, w_H \in \overset{\circ}{\mathbf{W}}_H$. The bilinear forms on the right-hand side of (11) are all constructed in a similar way by summing particular approximations of the "energy" related to each corresponding differential term over the triangles of $\mathscr{T}_H^{(s)}$. For each $\Delta \in \mathscr{T}_H^{(s)}$, let a_{ij,Δ_x} be the value of the coefficient function a_{ij}

at the midpoint of the side of Δ parallel to the x-axis and let $a_{ij}^{(s)}(.,.)$ be defined by

$$a_{ij}^{(s)}(v_H, w_H) = \sum_{\Delta \in \mathcal{T}_H^{(s)}} a_{ij,\Delta} \int_\Delta (P_H^{(s)} v_{H,j})_x (P_H^{(s)} \omega_{H,i})_x \, \mathrm{d}x \, \mathrm{d}y \tag{12}$$

for each $v_H, w_H \in \overset{\circ}{\mathbf{W}}_H$. Similarly, let c_{ij,Δ_y} be the value of the coefficient function c_{ij} at the midpoint of the side of Δ parallel to the y-axis and let $c_{ij}^{(s)}(.,.)$ be the bilinear form

$$c_{ij}^{(s)}(v_H, w_H) = \sum_{\Delta \in \mathcal{T}_H^{(s)}} c_{ij,\Delta} \int_\Delta (P_H^{(s)} w_{H,i})_y (P_H^{(s)} w_{H,i})_y \, \mathrm{d}x \, \mathrm{d}y \tag{13}$$

for each $v_H, w_H \in \overset{\circ}{\mathbf{W}}_H$.

We are now going to present the bilinear form associated with mixed derivative terms. Let $b_{ij,\Delta}$ be the value of the coefficient function b_{ij} at the vertex of Δ associated with the right angle of Δ. The bilinear form $b_{ij}^{(s)}(.,.)$ is defined by

$$\begin{aligned} b_{ij}^{(s)}(v_H, w_H) = \sum_{\Delta \in \mathcal{T}_H^{(s)}} b_{ij,\Delta} \int_\Delta &[(P_H^{(s)} v_{H,j})_x (P_H^{(s)} w_{H,i})_y \\ &+ (P_H^{(s)} v_{H,j})_y (P_H^{(s)} w_{H,i})_x] \, \mathrm{d}x \, \mathrm{d}y \end{aligned} \tag{14}$$

for each $v_H, w_H \in \overset{\circ}{\mathbf{W}}_H$.

In order to approximate the first-order terms, let $[P_H^{(s)}(d_{ij} w_{H,i})]_{\Delta_x}$ be the value of $P_H^{(s)}(d_{ij} w_{H,i})$ at the midpoint of the side of Δ parallel to the x-axis. Analogously, $[P_H^{(s)}(e_{ij} w_{H,i})]_{\Delta_y}$ denotes the value $[P_H^{(s)}(e_{ij} w_{H,i})]$ at the midpoint of Δ parallel to the y-axis. Then

$$d_{ij}^{(s)}(v_H, w_H) = \sum_{\Delta \in \mathcal{T}_H^{(s)}} [P_H^{(s)}(d_{ij} w_{H,i})]_{\Delta_x} \int_\Delta (P_H^{(s)} v_{H,j})_x \, \mathrm{d}x \, \mathrm{d}y, \tag{15}$$

$$e_{ij}^{(s)}(v_H, w_H) = \sum_{\Delta \in \mathcal{T}_H^{(s)}} [P_H^{(s)}(e_{ij} w_{H,i})]_{\Delta_y} \int_\Delta (P_H^{(s)} v_{H,j})_y \, \mathrm{d}x \, \mathrm{d}y \tag{16}$$

for each $v_H, w_H \in \overset{\circ}{\mathbf{W}}_H$.

The bilinear form $f_{ij}^{(s)}(.,.)$ is defined by

$$f_{ij}^{(s)}(v_H, w_H) = \sum_{(x_\kappa, y_\ell) \in \Omega_H} |\square_{\kappa,\ell}| f_{ij}(x_\kappa, y_\ell) v_{H,j}(x_\kappa, y_\ell) w_{H,i}(x_\kappa, y_\ell) \tag{17}$$

for each $v_H, w_H \in \overset{\circ}{\mathbf{W}}_H$.

Finally, in (8), the bilinear form $a_H(.,.)$ is defined by the arithmetical mean

$$a_H(v_H, w_H) = \frac{1}{2}(a_H^{(1)}(v_H, w_H) + a_H^{(2)}(v_H, w_H)) \tag{18}$$

for each $v_H, w_H \in \overset{\circ}{\mathbf{W}}_H$.

Remark. If, in (1), there are no mixed derivatives, then it is not necessary to define a_H as the arithmetic mean (18).

4. Stability

We consider a sequence of grids $\mathbb{R}_H$, $H = (h, k) \in \Lambda$, such that the maximal mesh-size $H_{\max}$ tends to zero.

The next theorem, which can be found for instance in [16], has a central rule in the proof of the stability of the bilinear form $a_H(.,.)$.

Theorem 2. *Assume that the homogeneous variational problem* (6) *has only the solution* $u = 0$. *Let the grids* $\bar{\Omega}_H$, *for each* $H \in \Lambda$, *satisfy condition* (*Reg*). *For each* $H \in \Lambda$, *let* $\mathscr{T}_H$ *be a triangulation of* $\bar{\Omega}$ *such that the nodes of* $\mathscr{T}_H$ *coincide with* $\bar{\Omega}_H$ *and let* P_H *be the corresponding piecewise linear interpolation operator*. *Then there exists a constant* C *for all* $H \in \Lambda$ *with* $H_{\max}$ *small enough, such that the inequality*

$$\|P_H v_H\|_1 \leqslant C \sup_{0 \neq w_H \in \overset{\circ}{\mathbf{W}}_H} \frac{|a(P_H v_H, P_H w_H)|}{\|P_H v_H\|_1} \tag{19}$$

holds for all $v_H \in \overset{\circ}{\mathbf{W}}_H$.

The behaviour of the difference between the bilinear forms $a_H^{(s)}(.,.)$ and $a(.,.)$ is established in the next result.

Theorem 3. *Let* $s \in \{1, 2\}$ *and* $v_H, w_H \in \overset{\circ}{\mathbf{W}}_H$ *with* $H \in \Lambda$ *be two sequences satisfying*

$$\|P_H^{(s)} v_H\|_1 \leqslant 1, \ \|P_H^{(s)} w_H\|_1 \leqslant 1, \quad H \in \Lambda.$$

Then

$$|a_H^{(s)}(v_H, w_H) - a(P_H^{(s)} v_H, P_H^{(s)} w_H)| \to 0 \ (H \in \Lambda). \tag{20}$$

Proof. It has been proved in [8] that

$$a_{ij}^{(s)}(v_{H,j}, w_{H,i}) - (a_{ij}(P_H^{(s)} v_{H,j})_x, (P_H^{(s)} w_{H,i})_x)_0 \to 0 \ (H \in \Lambda)$$

for $v_{H,j}, w_{H,i} \in \mathring{W}_H$, $i, j = 1, \dots, n$, and also that the corresponding relations for the bilinear forms $b_{ij}^{(s)}, c_{ij}^{(s)}, d_{ij}^{(s)}, e_{ij}^{(s)}, f_{ij}^{(s)}$ hold. Summing over i and j yields (20). □

The main result of this section is now established.

Theorem 4. *Suppose that the assumptions of Theorem* 2 *hold. Then exists a constant C for all $H \in \Lambda$ with $H_{\max}$ small enough, such that the inequality*

$$\|P_H v_H\|_1 \leqslant C \sup_{0 \neq w_H \in \mathring{\mathbf{W}}_H} \frac{|a_H(v_H, w_H)|}{\|P_H w_H\|_1} \tag{21}$$

holds for all $v_H \in \mathring{\mathbf{W}}_H$.

Proof. Following the proof of Theorem 2 of [8], from (19) and (20) we conclude (21). □

5. Convergence

Let u be the solution of (6) and let u_H be the solution of (8). In order to estimate the error $\|P_H R_H u - P_H u_H\|_1$, where $R_H u$ is the pointwise restriction of u to the grid $\bar{\Omega}_H$, we replace $P_H v_H$ in (21) by $P_H R_H u - P_H u_H$ and estimate the difference $|a_H(R_H u, v_H) - (g_H, v_H)_H|$. Following the proof of Theorem 3 of [1] and the procedure followed in [10] for the scalar case, the next result can be proved.

Theorem 5. *Let the grids $\bar{\Omega}_H$, for $H \in \Lambda$, satisfy condition* (*Reg*). *If the variational problem* (6) *is uniquely solvable, then for $H_{\max}$ small enough, the nonstandard finite element method* (8) *has a unique solution satisfying the error estimate*

$$\|P_H u_H - P_H R_H u\|_1 \leqslant C \left(\sum_{\Delta \in \mathcal{T}_H} (\operatorname{diam} \Delta)^4 \|u\|^2_{\mathbf{H}^3(\Delta)} + \sum_{\Delta \in \mathcal{T}_{H,2}} |\Delta| (\operatorname{diam} \Delta)^2 \|u\|^2_{\mathbf{W}^{2,\infty}(\Delta)} \right)^{1/2},$$

where $\mathcal{T}_{H,2} = \mathcal{T}^{(1)}_{H,2} \cup \mathcal{T}^{(2)}_{H,2}$ and $|\Delta|$ is the area of the triangle Δ.

Remarks. (1) If $\mathcal{T}_{H,2} = \emptyset$, i.e., Ω is a rectangle or a union of rectangles, then the convergence order is $\mathrm{O}(H^2_{\max})$, provided that $u \in \mathbf{H}^3_0(\Omega)$;

(2) If $\sum_{\Delta \in \mathcal{T}_{H,2}} |\Delta| = \mathrm{O}(H_{\max})$, then the convergence order is $\mathrm{O}(H^{3/2}_{\max})$, provided that $\|u\|_{\mathbf{W}^{2,\infty}(\Delta)}$ is bounded for $\Delta \in \mathcal{T}_{H,2}$.

6. On the computation of the finite element solution

In order to compute easily the solution of the discrete variational problem (8), we introduce in what follows an equivalent finite difference scheme.

For each grid point $(x_\kappa, y_\ell) \in \mathbb{R}_H$ we define the central finite difference quotients with respect to the variable x,

$$\delta_x^{(1/2)} w_H(x_\kappa, y_\ell) = \frac{w_H(x_{\kappa+1/2}, y_\ell) - w_H(x_{\kappa-1/2}, y_\ell)}{x_{\kappa+1/2} - x_{\kappa-1/2}},$$

$$\delta_x^{(1/2)} w_H(x_{\kappa+1/2}, y_\ell) = \frac{w_H(x_{\kappa+1}, y_\ell) - w_H(x_\kappa, y_\ell)}{x_{\kappa+1} - x_\kappa},$$

$$\delta_x w_H(x_\kappa, y_\ell) = \frac{w_H(x_{\kappa+1}, y_\ell) - w_H(x_{\kappa-1}, y_\ell)}{x_{\kappa+1} - x_{\kappa-1}}.$$

The central finite difference quotients with respect to the variable y are similarly defined by the natural change.

We now define the following finite difference problem:

$$\text{find } u_H \in \overset{\circ}{\mathbf{W}}_H \text{ such that } A_H u_H = g_H \text{ in } \Omega_H, \tag{22}$$

where

$$A_H u_H = [A_{H,i} u_H]_{i=1,\ldots,n}, \quad g_H = [g_{H,i}]_{i=1,\ldots,n}$$

and

$$A_{H,i} u_H = \sum_{j=1}^{n} A_{H,ij} u_{H,j}$$

with

$$\begin{aligned} A_{H,ij} u_{H,j} = & -\delta_x^{(1/2)}(a_{ij} \delta_x^{(1/2)} u_{H,j}) - \delta_y(b_{ij} \delta_x u_{H,j}) - \delta_x(b_{ij} \delta_y u_{H,j}) \\ & - \delta_y^{(1/2)}(c_{ij} \delta_y^{(1/2)} u_{H,j}) + d_{ij} \delta_x u_{H,j} + e_{ij} \delta_y u_{H,j} + f_{ij} u_{H,j}. \end{aligned}$$

We observe that if Ω has an oblique side, then for points $(x_\kappa, y_\ell) \in \Omega_H$ such that two of their neighbour grid points are on this side, $A_{H,ij} u_{H,j}$ involves grid points placed outside of Ω. In this case auxiliary boundary conditions must be considered. For instance, if $(x_\kappa, y_\ell) \in \Omega_H$ is such that $(x_{\kappa-1}, y_\ell)$, $(x_\kappa, y_{\ell+1}) \in \partial\Omega_H$, then on the definition of $A_{H,ij} u_{H,j}(x_\kappa, y_\ell)$ the grid point $(x_{\kappa-1}, y_{\ell+1})$ is used and so we introduce the auxiliary boundary condition

$$\begin{aligned} & \sum_{j=1}^{n} b_{ij}(x_\kappa, y_\ell)(-\delta_x^{(1/2)} u_{H,j}(x_{\kappa-1/2}, y_\ell) h_{\kappa-1} + \delta_y^{(1/2)} u_{H,j}(x_\kappa, y_{\ell+1/2}) k_\ell) \\ & \quad = \sum_{j=1}^{n} -b_{ij}(x_\kappa, y_{\ell+1}) \delta_x^{(1/2)} u_{H,j}(x_{\kappa-1/2}, y_{\ell+1}) h_{\kappa-1} \\ & \qquad + b_{ij}(x_{\kappa-1}, y_\ell) \delta_y^{(1/2)} u_{H,j}(x_{\kappa-1}, y_{\ell+1/2}) k_\ell. \end{aligned} \tag{23}$$

For other grid points the boundary conditions are defined in a similar way using in (23) the natural modifications.

By recalling the definition of $a_H(.,.)$ and A_H, it is easy to show the next result, which allows us to conclude the superconvergence of the finite difference operator A_H (see, for instance, [8,15,19,23]).

Theorem 6. *Let the bilinear form $a_H(.,.)$ be defined by* (18). *With A_H defined by* (22), *the equality*

$$a_H(v_H, w_H) = (A_H v_H, w_H)_H$$

holds for $v_H, w_H \in \overset{\circ}{\mathbf{W}}_H$.

7. Numerical results

The aim of this section is to give some examples that illustrate the performance of the method defined by (8) for planar elasticity problems.

We consider an isotropic material in the configuration space Ω and a body force g. The displacement u is the solution of the following system of partial differential equations:

$$-\mathrm{div}(\sigma(u)) = g \ \text{ in } \Omega, \tag{24}$$

with the displacement boundary condition

$$u = u_{\partial\Omega} \quad \text{on } \partial\Omega. \tag{25}$$

In (24), $\sigma(u)$ denotes the stress tensor defined by

$$\sigma(u) = 2\mu\varepsilon(u) + \lambda\,\mathrm{tr}(\varepsilon(u)) I_2,$$

where I_2 is the identity two-by-two matrix,

$$\varepsilon(u) = \frac{1}{2}(\mathrm{grad}(u) + \mathrm{grad}(u)^t) \quad \text{and} \quad \mathrm{grad}(u) = \begin{bmatrix} \dfrac{\partial u_1}{\partial x} & \dfrac{\partial u_1}{\partial y} \\ \dfrac{\partial u_2}{\partial x} & \dfrac{\partial u_2}{\partial y} \end{bmatrix}.$$

By μ, λ we represent the Lamé constants.

The system (24) is uniformly strongly elliptic in Ω (it satisfies the condition (4) with $C_0 = \mu$).

Let us consider the associated bilinear form

$$a(v, w) = \int_\Omega (2\mu\varepsilon(v) : \varepsilon(w) + \lambda\,\mathrm{div}(v)\,\mathrm{div}(w))\,\mathrm{d}x\,\mathrm{d}y$$

for $v, w \in \mathbf{H}^1(\Omega) = H^1(\Omega) \times H^1(\Omega)$. The first Korn inequality enables us to conclude that the bilinear form $a(.,.)$ is $\mathbf{H}_0^1$-elliptic and so the variational problem defined using $a(.,.)$ is uniquely solvable.

In the following, we consider two examples of (24) with different domains: a union of rectangles and a domain with an oblique side. The numerical approximation to the solution u is computed using the nonstandard piecewise linear finite element method (8). The rectangular case was considered in [1].

Example 1. Let us consider the boundary value problem (24) defined on the domain $\Omega = (0,1) \times (0,1) \backslash [1/2, 1) \times [1/2, 1)$.

Table 1

Grid	Number of points	H_{max}	$\|P_H u_H - P_H R_H u\|_1$
$\bar{\Omega}_{H,1}$	$N = 16$, $M = 20$	0.075	0.0155737
$\bar{\Omega}_{H,2}$	$N = 32$, $M = 32$	0.05	0.00765314
$\bar{\Omega}_{H,3}$	$N = 32$, $M = 40$	0.0375	0.00497282
$\bar{\Omega}_{H,4}$	$N = 64$, $M = 64$	0.025	0.00228567
$\bar{\Omega}_{H,5}$	$N = 64$, $M = 80$	0.01875	0.00149731
$\bar{\Omega}_{H,6}$	$N = 128$, $M = 128$	0.0125	0.000680543

We consider $\lambda = 1$, $\mu = 0.5$, $u_{\partial\Omega} = 0$ and g such that the problem (24) has the solution u with components $u_1(x, y) = 0.1\,\sin(2\pi x)\,\sin(2\pi y)$ and $u_2(x, y) = -\sin(2\pi x)y(y-1)(y-0.5)$.

In Table 1 we give the number of points that we consider in the x and y directions on the computation of the numerical approximation which we denote respectively by N and M, the maximum mesh-size H_{max} and the norm $\|.\|_1$ of the error. The grids $\bar{\Omega}_{H,1}$ and $\bar{\Omega}_{H,2}$ were defined taking $x_0 = y_0 = 0$ and, respectively, the following mesh-sizes:

$$h_j = 0.05,\ \ j = 1, 2, 7, \ldots, 10, 15, 16, \quad h_j = 0.075,\ \ j = 3, \ldots, 6, 11, \ldots, 14,$$
$$k_\ell = 0.05,\ \ \ell = 1, 2, 7, \ldots, 12, 17, \ldots, 20, \quad k_\ell = 0.075,\ \ \ell = 3, \ldots 6,$$
$$k_\ell = 0.025,\ \ \ell = 13, \ldots, 16,$$

and

$$h_j = 0.025,\ \ j = 1, \ldots, 4, 9, \ldots, 24, 29, \ldots, 32, \quad h_j = 0.05,\ \ j = 5, \ldots, 8, 25, \ldots, 28,$$
$$k_j = h_j,\ \ j = 1, \ldots, 32.$$

The grids $\bar{\Omega}_{H,3}$ and $\bar{\Omega}_{H,4}$ were generated introducing a new grid line between each grid line of $\bar{\Omega}_{H,1}$ and $\bar{\Omega}_{H,3}$, respectively. From $\bar{\Omega}_{H,3}$ and $\bar{\Omega}_{H,4}$ we construct the grids $\bar{\Omega}_{H,5}$ and $\bar{\Omega}_{H,6}$, respectively, using the procedure described below.

From the values presented in Table 1, we easily conclude that the average convergence rate is 2 which confirm the second-order convergence of the method stated in Theorem 5.

The $\mathbf{H}^1$-norm of the error against the square of the maximum mesh-size is plotted in Fig. 2. The values of Table 1 were used and the $\mathbf{H}^1$-norm of the errors of numerical approximations computed using other grids generated by the procedure described were also considered.

Example 2. Let Ω be the polygonal domain $\{(x, y) \in \mathbb{R}^2 : x, y \geqslant 0, x \leqslant 1, y \leqslant -\frac{1}{2}x + 1\}$.

We consider $\lambda = 1$, $\mu = 0.5$, $u_{\partial\Omega} = 0$ and g such that the problem (24) has the solution u with components $u_1(x, y) = 4xy(x-1)(y+0.5x-1)$ and $u_2(x, y) = 6x^2y(x-1)(y+0.5x-1)$.

We present the results obtained for this problem in Table 2. The grids $\bar{\Omega}_{H,1}$ and $\bar{\Omega}_{H,2}$ were defined taking $x_0 = y_0 = 0$ and, respectively, the following mesh-sizes:

$$h_j = 0.1,\ \ j = 1, \ldots, 10,$$
$$k_\ell = 0.1,\ \ \ell = 1, \ldots, 5, \quad k_\ell = 0.05,\ \ \ell = 6, \ldots, 15,$$

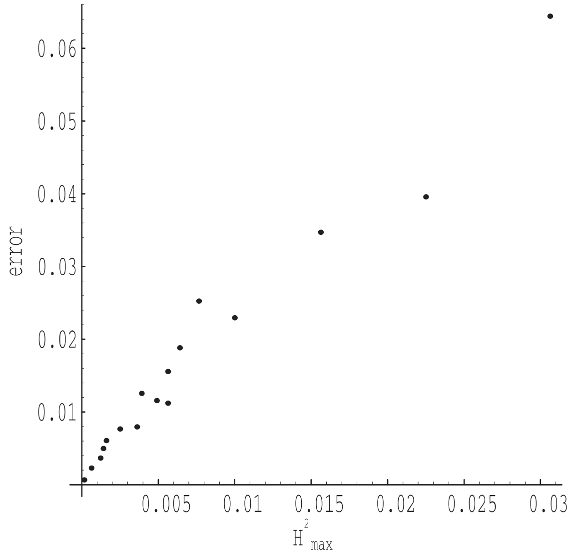

Fig. 2. The norm $\|.\|_1$ of the error.

Table 2

Grid	Number of points	H_{max}	$\|P_H u_H - P_H R_H u\|_1$
$\bar{\Omega}_{H,1}$	$N = 10$, $M = 15$	0.1	0.00868943
$\bar{\Omega}_{H,2}$	$N = 16$, $M = 23$	0.075	0.00506237
$\bar{\Omega}_{H,3}$	$N = 20$, $M = 30$	0.05	0.00280741
$\bar{\Omega}_{H,4}$	$N = 32$, $M = 46$	0.0375	0.00165363
$\bar{\Omega}_{H,5}$	$N = 40$, $M = 60$	0.025	0.000981383
$\bar{\Omega}_{H,6}$	$N = 64$, $M = 92$	0.01875	0.000590199

and

$$h_j = 0.075, \quad j = 1, \ldots, 4, 13, \ldots, 16, \qquad h_j = 0.05, \quad j = 5, \ldots, 12,$$
$$k_\ell = 0.075, \quad \ell = 1, \ldots, 6, \qquad k_7 = 0.05, \quad k_\ell = 0.0375, \qquad \ell = 8, \ldots, 11, 20, \ldots, 23,$$
$$k_\ell = 0.025, \quad \ell = 12, \ldots, 19.$$

The grids $\bar{\Omega}_{H,j}$, $j = 3, 4, 5, 6$, were generated using the procedure described in Example 1 such that condition (Reg) holds for all grids $\bar{\Omega}_{H,j}$, $j = 1, \ldots, 6$.

In Fig. 3, we plot the $\mathbf{H}^1$-norm of the error against $H_{max}^{1.5}$. We took the values of Table 2 and the $\mathbf{H}^1$-norm of the errors of several numerical approximations computed using grids satisfying condition (Reg) but generated by the procedure described in Example 1.

The average convergence rate computed using the results of Table 2 is 1.5, which confirm again the convergence order of the method stated in Theorem 5.

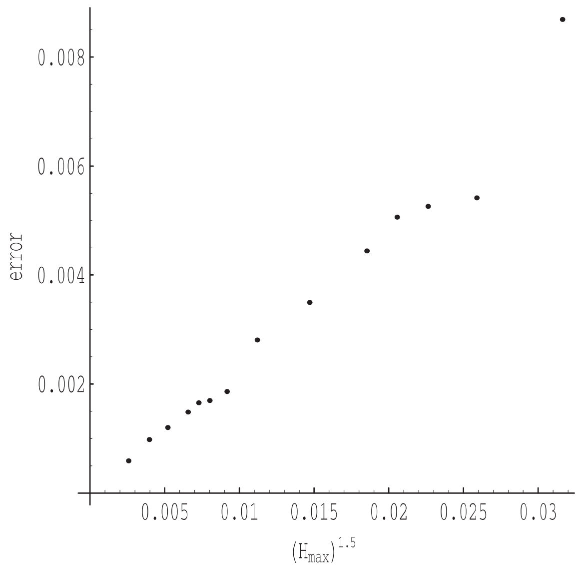

Fig. 3. The norm $\|.\|_1$ of the error.

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