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Reduction of Jacobi manifolds via Dirac structures theory

Fani Petalidou^a, Joana M. Nunes da Costa^{b,*}

^a Faculty of Sciences and Technology, University of Peloponnese, 22100 Tripoli, Greece

^b Department of Mathematics, University of Coimbra, Apartado 3008, 3001-454 Coimbra, Portugal

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Abstract

We first recall some basic definitions and facts about Jacobi manifolds, generalized Lie bialgebroids, generalized Courant algebroids and Dirac structures. We establish an one–one correspondence between reducible Dirac structures of the generalized Lie bialgebroid of a Jacobi manifold (M, Λ, E) for which 1 is an admissible function and Jacobi quotient manifolds of M . We study Jacobi reductions from the point of view of Dirac structures theory and we present some examples and applications.

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1. Introduction

The concept of a *Dirac structure* on a differentiable manifold M was introduced by T. Courant and A. Weinstein in [2] and developed by T. Courant in [3]. Its principal aim is to present a unified framework for the study of pre-symplectic forms, Poisson structures and foliations. More specifically, a *Dirac structure* on M is a subbundle $L \subset TM \oplus T^*M$ that is maximally isotropic with respect to the canonical symmetric bilinear form on $TM \oplus T^*M$ and satisfies a certain integrability condition. In order to for-

* Corresponding author.

E-mail addresses: petalido@uop.gr (F. Petalidou), jmcosta@mat.uc.pt (J.M. Nunes da Costa).

mulate this integrability condition, T. Courant defines a bilinear, skew-symmetric, bracket operation on the space $\Gamma(TM \oplus T^*M)$ of smooth sections of $TM \oplus T^*M$ which does not satisfy the Jacobi identity. The nature of this bracket was clarified by Z.-J. Liu, A. Weinstein and P. Xu in [18] by introducing the structure of a *Courant algebroid* on a vector bundle E over M and by extending the notion of a Dirac structure to the subbundles $L \subset E$. The most important example of Courant algebroid is the direct sum $A \oplus A^*$ of a Lie bialgebroid (A, A^*) over a smooth manifold M [22].

Alan Weinstein and his collaborators have studied several problems of Poisson geometry via Dirac structures theory. In [19], Z.-J. Liu et al. establish an one–one correspondence between Dirac subbundles of the double $TM \oplus T^*M$ of the triangular Lie bialgebroid (TM, T^*M, Λ) defined by a Poisson structure Λ on M and Poisson structures on quotient manifolds of M . Using this correspondence and the results concerning the pull-backs Dirac structures under Lie algebroid morphisms, Z.-J. Liu constructs in [20] the Poisson reduction in terms of Dirac structures.

On the other hand, it is well known that the notion of *Jacobi manifold*, i.e., a differentiable manifold M endowed with a bivector field Λ and a vector field E satisfying an integrability condition, introduced by A. Lichnerowicz in [17], is a rich geometrical notion that generalizes the Poisson, symplectic, contact and co-symplectic manifolds. Thus, it is natural to research a simple interpretation of Jacobi manifolds by means of Dirac structures. A first approach of this problem is presented in [32] by A. Wade. Taking into account that to any Jacobi structure (Λ, E) on M is canonically associated a generalized Lie bialgebroid structure on $(TM \times \mathbb{R}, T^*M \times \mathbb{R})$ [10], she considers the Whitney sum $\mathcal{E}^1(M) = (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$, introduces the notion of $\mathcal{E}^1(M)$ -Dirac structures by extending the Courant's bracket to the space $\Gamma(\mathcal{E}^1(M))$ of smooth sections of $\mathcal{E}^1(M)$ and shows that the graph of the vector bundle morphism $(\Lambda, E)^\#: T^*M \times \mathbb{R} \rightarrow TM \times \mathbb{R}$ is a Dirac subbundle of $\mathcal{E}^1(M)$. But the extended bracket does not endow $\mathcal{E}^1(M)$ with a Courant algebroid structure. A second approach of the problem is the one proposed by the second author and J. Clemente-Gallardo in the recent paper [30]. They introduce the notions of *generalized Courant algebroid* (which is equivalent to the notion of *Courant–Jacobi algebroid* independently defined by J. Grabowski and G. Marmo in [8]) and of *Dirac structure for a generalized Courant algebroid* and give several connections between Dirac structures for generalized Courant algebroids and Jacobi manifolds. We note that the construction of [30] includes as particular case the one of Wade and that the main example of generalized Courant algebroid over M is the direct sum of a generalized Lie bialgebroid over M .

In the present work, by using the results mentioned above, we establish a reduction theorem of Jacobi manifolds (Theorem 5.2). It is well known that there are already several geometric and algebraic treatments of the Jacobi reduction problem (see, for instance, [9,26–28]). But, it is an original goal of the Dirac structures theory to describe Jacobi reduction and to construct a more general framework for the study of the related problems concerning the projection of Jacobi structures and the existence of Jacobi structures on certain submanifolds of Jacobi manifolds. Precisely, on the way to our principal result, we construct an one to one correspondence between Dirac subbundles, satisfying a certain regularity condition, of the double $(TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$, where M is a Jacobi manifold, and quotient Jacobi manifolds of M (Theorem 4.14). Also, the reduction theorem (Theorem 5.2) allows us to state sufficient conditions under which a submanifold N of (M, Λ, E) inherits a Jacobi structure, that include as particular cases the results presented in [4,12].

The paper is organized as follows. In Sections 2 and 3 we recall some basic definitions and results concerning, respectively, Jacobi structures, generalized Lie bialgebroids and Dirac structures for generalized Courant algebroids. In Section 4 we establish a correspondence between Dirac structures and

quotient Jacobi manifolds (Theorem 4.14). Using this correspondence and the results for the pull-backs Dirac structures by Lie algebroid morphisms, we prove, in Section 5, a Jacobi reduction theorem (Theorem 5.2) which is essentially the Reduction Theorem proved in [27] and independently in [26]. Finally, in Section 6 we present some applications and examples.

Notation. In this paper, M is a C^∞ -differentiable manifold of finite dimension. We denote by TM and T^*M , respectively, the tangent and cotangent bundles over M , $C^\infty(M, \mathbb{R})$ the space of all real C^∞ -differentiable functions on M , $\Omega^k(M)$ the space of all differentiable k -forms on M and $\mathcal{X}(M)$ the space of all differentiable vector fields on M . Also, we denote by δ the usual differential operator on the graded space $\Omega(M) = \bigoplus_{k \in \mathbb{Z}} \Omega^k(M)$. For the Schouten bracket and the interior product of a form with a multi-vector field, we use the convention of sign indicated by Koszul [16] (see also [24]).

2. Jacobi structures and generalized Lie bialgebroids

A *Jacobi manifold* is a differentiable manifold M equipped with a bivector field Λ and a vector field E such that

$$[\Lambda, \Lambda] = -2E \wedge \Lambda \quad \text{and} \quad [E, \Lambda] = 0,$$

where $[\cdot, \cdot]$ denotes the Schouten bracket [17]. In this case, (Λ, E) defines on $C^\infty(M, \mathbb{R})$ the internal composition law $\{ \cdot, \cdot \}_{(\Lambda, E)} : C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R}) \rightarrow C^\infty(M, \mathbb{R})$ given, for all $f, g \in C^\infty(M, \mathbb{R})$, by

$$\{f, g\}_{(\Lambda, E)} = \Lambda(\delta f, \delta g) + \langle f \delta g - g \delta f, E \rangle, \tag{1}$$

which endows $C^\infty(M, \mathbb{R})$ with a local Lie algebra structure [14,17] (or with a Jacobi algebra structure in the terminology of J. Grabowski et al. [6,8]).

Let (M_1, Λ_1, E_1) and (M_2, Λ_2, E_2) be two Jacobi manifolds and $\Psi : M_1 \rightarrow M_2$ a differentiable map. If Λ_1 and E_1 are projectable by Ψ on M_2 and their projections are, respectively, Λ_2 and E_2 , i.e., $\Psi_* \Lambda_1 = \Lambda_2$ and $\Psi_* E_1 = E_2$, then $\Psi : M_1 \rightarrow M_2$ is said to be a *Jacobi morphism* or a *Jacobi map*. When $\Psi : M_1 \rightarrow M_2$ is a diffeomorphism, the Jacobi structures (Λ_1, E_1) and (Λ_2, E_2) are said to be *equivalent*.

A *Lie algebroid* over a smooth manifold M is a vector bundle $A \rightarrow M$ with a Lie algebra structure $[\cdot, \cdot]$ on the space $\Gamma(A)$ of the global cross sections of $A \rightarrow M$ and a bundle map $a : A \rightarrow TM$, called the *anchor map*, such that

1. the homomorphism $a : (\Gamma(A), [\cdot, \cdot]) \rightarrow (\mathcal{X}(M), [\cdot, \cdot])$, induced by the anchor map, is a Lie algebra homomorphism;
2. for all $f \in C^\infty(M, \mathbb{R})$ and for all $X, Y \in \Gamma(A)$,

$$[X, fY] = f[X, Y] + (a(X)f)Y.$$

We denote a Lie algebroid over M by the triple $(A, [\cdot, \cdot], a)$. For more details see, for example, [1,21,24].

A trivial example of a Lie algebroid over a differentiable manifold M is the triple $(TM \times \mathbb{R}, [\cdot, \cdot], \pi)$ for all $(X, f), (Y, g) \in \Gamma(TM \times \mathbb{R}) \cong \mathcal{X}(M) \times C^\infty(M, \mathbb{R})$,

$$[(X, f), (Y, g)] = ([X, Y], X \cdot g - Y \cdot f), \tag{2}$$

and $\pi : TM \times \mathbb{R} \rightarrow TM$ is the projection on the first factor.

The Lie algebroid of a Jacobi manifold (M, Λ, E) is defined in [13] as follows. We consider the vector bundle $T^*M \times \mathbb{R}$ over M and the vector bundle morphism $(\Lambda, E)^\# : T^*M \times \mathbb{R} \rightarrow TM \times \mathbb{R}$ given, for any $(\alpha, f) \in \Gamma(T^*M \times \mathbb{R})$, by

$$(\Lambda, E)^\#((\alpha, f)) = (\Lambda^\#(\alpha) + fE, -\langle \alpha, E \rangle).$$

On the space $\Gamma(T^*M \times \mathbb{R}) \cong \Omega^1(M) \times C^\infty(M, \mathbb{R})$ we define the Lie algebra bracket $[\cdot, \cdot]_{(\Lambda, E)}$ given, for all $(\alpha, f), (\beta, g) \in \Gamma(T^*M \times \mathbb{R})$, by

$$[(\alpha, f), (\beta, g)]_{(\Lambda, E)} := (\gamma, h), \tag{3}$$

where

$$\begin{aligned} \gamma &:= \mathcal{L}_{\Lambda^\#(\alpha)}\beta - \mathcal{L}_{\Lambda^\#(\beta)}\alpha - \delta(\Lambda(\alpha, \beta)) + f\mathcal{L}_E\beta - g\mathcal{L}_E\alpha - i_E(\alpha \wedge \beta), \\ h &:= -\Lambda(\alpha, \beta) + \Lambda(\alpha, \delta g) - \Lambda(\beta, \delta f) + \langle f\delta g - g\delta f, E \rangle. \end{aligned}$$

Then the triple $(T^*M \times \mathbb{R}, [\cdot, \cdot]_{(\Lambda, E)}, \pi \circ (\Lambda, E)^\#)$ is a Lie algebroid over M .

For a Lie algebroid $(A, [\cdot, \cdot], a)$ over M , we denote by A^* its dual vector bundle over M and by $\bigwedge A^* = \bigoplus_{k \in \mathbb{Z}} \bigwedge^k A^*$ the graded exterior algebra of A^* . Sections of $\bigwedge A^*$ are called *A-differential forms* (or *A-forms*) on M . There exists a graded endomorphism $d : \Gamma(\bigwedge A^*) \rightarrow \Gamma(\bigwedge A^*)$, of degree 1, of the exterior algebra of *A-forms*, called the *exterior derivative*, taking an *A-k-form* η to an *A-(k + 1)-form* $d\eta$ such that, for all $X_1, \dots, X_{k+1} \in \Gamma(A)$,

$$\begin{aligned} d\eta(X_1, \dots, X_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} a(X_i) \cdot \eta(X_1, \dots, \hat{X}_i, \dots, X_{k+1}) \\ &\quad + \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} \eta([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+1}). \end{aligned}$$

The Lie algebroid axioms of A imply that d is a $C^\infty(M, \mathbb{R})$ -multilinear superderivation of degree 1 such that $d^2 = 0$. Also, we denote by $\bigwedge A = \bigoplus_{k \in \mathbb{Z}} \bigwedge^k A$ the graded exterior algebra of A whose sections are called *A-multivector fields*. The Lie bracket on $\Gamma(A)$ can be extended to the exterior algebra of *A-multivector fields* and the result is a graded Lie bracket $[\cdot, \cdot]$, called the *Schouten bracket* of the Lie algebroid A . Details may be found, for instance, in [1,15,21].

Let $(A, [\cdot, \cdot], a)$ be a Lie algebroid over M and $\phi \in \Gamma(A^*)$ be an 1-cocycle in the Lie algebroid cohomology complex with trivial coefficients [10,21], i.e., for any $X, Y \in \Gamma(A)$,

$$\langle \phi, [X, Y] \rangle = a(X)\langle \phi, Y \rangle - a(Y)\langle \phi, X \rangle. \tag{4}$$

We modify the usual representation of the Lie algebra $(\Gamma(A), [\cdot, \cdot])$ on the space $C^\infty(M, \mathbb{R})$ by defining a new representation $a^\phi : \Gamma(A) \times C^\infty(M, \mathbb{R}) \rightarrow C^\infty(M, \mathbb{R})$ as

$$a^\phi(X, f) = a(X)f + \langle \phi, X \rangle f, \quad \forall (X, f) \in \Gamma(A) \times C^\infty(M, \mathbb{R}). \tag{5}$$

The resulting cohomology operator $d^\phi : \Gamma(\bigwedge A^*) \rightarrow \Gamma(\bigwedge A^*)$ of the new cohomology complex is called the ϕ -differential of A and its expression in terms of d is

$$d^\phi \eta = d\eta + \phi \wedge \eta, \quad \forall \eta \in \Gamma\left(\bigwedge^k A^*\right). \tag{6}$$

d^ϕ allows us to define, in a natural way, the ϕ -Lie derivative by $X \in \Gamma(A)$, $\mathcal{L}_X^\phi : \Gamma(\wedge^k A^*) \rightarrow \Gamma(\wedge^k A^*)$, as the commutator of d^ϕ and of the contraction by X , i.e., $\mathcal{L}_X^\phi = d^\phi \circ i_X + i_X \circ d^\phi$. Its expression in terms of the usual Lie derivative $\mathcal{L}_X = d \circ i_X + i_X \circ d$ is, for any $\eta \in \Gamma(\wedge^k A^*)$,

$$\mathcal{L}_X^\phi \eta = \mathcal{L}_X \eta + \langle \phi, X \rangle \eta. \tag{7}$$

Using ϕ we can also modify the Schouten bracket $[\cdot, \cdot]$ on $\Gamma(\wedge A)$ to the ϕ -Schouten bracket $[\cdot, \cdot]^\phi$ on $\Gamma(\wedge A)$. It is defined, for all $P \in \Gamma(\wedge^p A)$ and $Q \in \Gamma(\wedge^q A)$, by

$$[P, Q]^\phi = [P, Q] + (p - 1)P \wedge (i_\phi Q) + (-1)^p (q - 1)(i_\phi P) \wedge Q, \tag{8}$$

where $i_\phi Q$ can be interpreted as the usual contraction of a multivector field with an 1-form. We remark that, when $p = q = 1$, $[P, Q]^\phi = [P, Q]$, i.e., the brackets $[\cdot, \cdot]^\phi$ and $[\cdot, \cdot]$ coincide on $\Gamma(A)$. For a representation of the differential calculus using the ϕ -modified derivative, Lie derivative and Schouten bracket, see [10] and [7].

The notion of *generalized Lie bialgebroid* has been introduced by D. Iglesias and J.C. Marrero in [10] in such a way that a Jacobi manifold has a generalized Lie bialgebroid canonically associated and conversely. We consider a Lie algebroid $(A, [\cdot, \cdot], a)$ over M and an 1-cocycle $\phi \in \Gamma(A^*)$ and we assume that the dual vector bundle $A^* \rightarrow M$ admits a Lie algebroid structure $([\cdot, \cdot]_*, a_*)$ and that $W \in \Gamma(A)$ is an 1-cocycle in the Lie algebroid cohomology complex with trivial coefficients of $(A^*, [\cdot, \cdot]_*, a_*)$. Then, we say that:

Definition 2.1. The pair $((A, \phi), (A^*, W))$ is a generalized Lie bialgebroid over M if, for all $X, Y \in \Gamma(A)$ and $P \in \Gamma(\wedge^p A)$, the following conditions hold:

$$d_*^W [X, Y] = [d_*^W X, Y]^\phi + [X, d_*^W Y]^\phi \quad \text{and} \quad \mathcal{L}_{*\phi}^W P + \mathcal{L}_W^\phi P = 0, \tag{9}$$

where d_*^W and \mathcal{L}_*^W are, respectively, the W -differential and the W -Lie derivative of A^* .

An equivalent definition of this notion was presented in [7] by J. Grabowski and G. Marmo under the name of *Jacobi bialgebroid*. Precisely, they define that:

Definition 2.2. The pair $((A, \phi), (A^*, W))$ is a Jacobi bialgebroid over M , if for all $P \in \Gamma(\wedge^p A)$ and $Q \in \Gamma(\wedge^q A)$,

$$d_*^W [P, Q]^\phi = [d_*^W P, Q]^\phi + (-1)^{p+1} [P, d_*^W Q]^\phi.$$

In the particular case where $\phi = 0$ and $W = 0$, by the above two definitions we recover, respectively, the notion of *Lie bialgebroid* introduced by K. Mackenzie and P. Xu in [22] and its equivalent definition given by Yv. Kosmann-Schwarzbach in [15].

Remark 2.3. The property of duality of a Lie bialgebroid is also verified in the case of a generalized Lie bialgebroid: i.e., if $((A, \phi), (A^*, W))$ is a generalized Lie bialgebroid over M , so is $((A^*, W), (A, \phi))$ (see [7,10]). Consequently, conditions of Definition 2.1 as well as of Definition 2.2 can be replaced by their dual versions.

The fundamental results of [10], which will be used in the sequel, are the following theorems.

Theorem 2.4. Let (M, Λ, E) be a Jacobi manifold. Then $((TM \times \mathbb{R}, [,], \pi, (0, 1)), (T^*M \times \mathbb{R}, [,]_{(\Lambda, E)}, \pi \circ (\Lambda, E)^\#, (-E, 0))$ is a generalized Lie bialgebroid over M .

Theorem 2.5. Let $((A, \phi), (A^*, W))$ be a generalized Lie bialgebroid over M . Then the bracket $\{ , \}_J : C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R}) \rightarrow C^\infty(M, \mathbb{R})$ given, for all $f, g \in C^\infty(M, \mathbb{R})$, by

$$\{f, g\}_J := \langle d^\phi f, d_*^W g \rangle, \tag{10}$$

defines a Jacobi structure on M .

Corollary 2.6. Let $((TM \times \mathbb{R}, [,], \pi, (0, 1)), (T^*M \times \mathbb{R}, [,]_{(\Lambda, E)}, \pi \circ (\Lambda, E)^\#, (-E, 0))$ be the generalized Lie bialgebroid associated to a Jacobi manifold (M, Λ, E) . Then,

$$\{f, g\}_J = \{f, g\}_{(\Lambda, E)}, \quad \forall f, g \in C^\infty(M, \mathbb{R}). \tag{11}$$

Proof. Effectively, for all $f, g \in C^\infty(M, \mathbb{R})$,

$$\begin{aligned} \{f, g\}_J &\stackrel{(10)}{=} \langle d^{(0,1)} f, d_*^{(-E,0)} g \rangle = \langle (\delta f, f), (-\Lambda^\#(\delta g) - gE, \langle \delta g, E \rangle) \rangle \\ &= \Lambda(\delta f, \delta g) + \langle f\delta g - g\delta f, E \rangle \stackrel{(1)}{=} \{f, g\}_{(\Lambda, E)}. \quad \square \end{aligned}$$

An important class of generalized Lie bialgebroids is the one of *triangular generalized Lie bialgebroids* defined, also in [10,11], as follows:

Definition 2.7. A generalized Lie bialgebroid $((A, \phi), (A^*, W))$ is said to be a triangular generalized Lie bialgebroid if there exists $P \in \Gamma(\wedge^2 A)$ such that $[P, P]^\phi = 0$, $a_* = a \circ P^\#$, $W = -P^\#(\phi)$ and the Lie bracket $[,]_*$ on $\Gamma(A^*)$ is the bracket

$$[\alpha, \beta]_P = \mathcal{L}_{P^\#(\alpha)}^\phi \beta - \mathcal{L}_{P^\#(\beta)}^\phi \alpha - d^\phi(P(\alpha, \beta)), \quad \forall \alpha, \beta \in \Gamma(A^*). \tag{12}$$

A characteristic example of triangular generalized Lie bialgebroid is the generalized Lie bialgebroid of a Jacobi manifold (M, Λ, E) (Theorem 2.4), where $[(\Lambda, E), (\Lambda, E)]^{(0,1)} = 0$ holds.

3. Generalized Courant algebroids and Dirac structures

The notion of *generalized Courant algebroid* has been introduced by the second author and J. Clemente-Gallardo in [30] and independently, under the name of *Courant–Jacobi algebroid*, by J. Grabowski and G. Marmo in [8]. In this section, we recall some basic facts concerning this notion and its relation with Dirac and Jacobi structures.

Definition 3.1 [30]. A generalized Courant algebroid over a smooth manifold M is a vector bundle E over M equipped with: (i) a nondegenerate symmetric bilinear form $(,)$ on the bundle, (ii) a skew-symmetric bracket $[,]$ on $\Gamma(E)$, (iii) a bundle map $\rho : E \rightarrow TM$ and (iv) an E -1-form θ such that, for any $e_1, e_2 \in \Gamma(E)$, $\langle \theta, [e_1, e_2] \rangle = \rho(e_1)\langle \theta, e_2 \rangle - \rho(e_2)\langle \theta, e_1 \rangle$, which satisfy the following relations:

1. for any $e_1, e_2, e_3 \in \Gamma(E)$,

$$[[e_1, e_2], e_3] + \text{c.p.} = \mathcal{D}^\theta T(e_1, e_2, e_3);$$

2. for any $e_1, e_2 \in \Gamma(E)$,

$$\rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)]; \tag{13}$$

3. for any $e_1, e_2 \in \Gamma(E)$ and $f \in C^\infty(M, \mathbb{R})$,

$$[e_1, fe_2] = f[e_1, e_2] + (\rho(e_1)f)e_2 - (e_1, e_2)\mathcal{D}f; \tag{14}$$

4. for any $f, g \in C^\infty(M, \mathbb{R})$,

$$(\mathcal{D}^\theta f, \mathcal{D}^\theta g) = 0;$$

5. for any $e, e_1, e_2 \in \Gamma(E)$,

$$\rho(e)(e_1, e_2) + \langle \theta, e \rangle(e_1, e_2) = ([e, e_1] + \mathcal{D}^\theta(e, e_1), e_2) + (e_1, [e, e_2] + \mathcal{D}^\theta(e, e_2)).$$

For any $e_1, e_2, e_3 \in \Gamma(E)$, $T(e_1, e_2, e_3)$ is the function on the base M defined by

$$T(e_1, e_2, e_3) = \frac{1}{3}([e_1, e_2], e_3) + \text{c.p.}$$

$\mathcal{D}, \mathcal{D}^\theta : C^\infty(M, \mathbb{R}) \rightarrow \Gamma(E)$ are the maps defined, for any $f \in C^\infty(M, \mathbb{R})$, by $\mathcal{D}f = \frac{1}{2}\beta^{-1}\rho^*\delta f$ and $\mathcal{D}^\theta f = \mathcal{D}f + \frac{1}{2}f\beta^{-1}(\theta)$, β being the isomorphism between E and E^* defined by the nondegenerate bilinear form $(, \cdot)$. In other words, for any $e \in \Gamma(E)$,

$$(\mathcal{D}f, e) = \frac{1}{2}\rho(e)f \quad \text{and} \quad (\mathcal{D}^\theta f, e) = \frac{1}{2}(\rho(e)f + \langle \theta, e \rangle f).$$

The above definition is based on the original definition of Courant algebroid presented in [18] by Z.-J. Liu et al. while its equivalent definition proposed in [8] is based on the alternative definition of Courant algebroid given by D. Roytenberg in [31]. Their equivalence is established in [30].

By defining, for any $e \in \Gamma(E)$, the first order differential operator $\rho^\theta(e)$ by

$$\rho^\theta(e) = \rho(e) + \langle \theta, e \rangle, \tag{15}$$

we have that (13) is equivalent [30] to

$$\rho^\theta([e_1, e_2]) = [\rho^\theta(e_1), \rho^\theta(e_2)], \tag{16}$$

where the bracket on the right-hand side is the Lie bracket (2).

Definition 3.2. A Dirac structure for a generalized Courant algebroid (E, θ) over M is a subbundle $L \subset E$ that is maximal isotropic under $(, \cdot)$ and integrable, i.e., $\Gamma(L)$ is closed under $[,]$.

It is immediate from the above definition that a Dirac subbundle L of (E, θ) is a Lie algebroid under the restrictions of the bracket $[,]$ and of the anchor ρ to $\Gamma(L)$. If $\theta \in \Gamma(L^*)$, then it is an 1-cocycle for the Lie algebroid cohomology with trivial coefficients of $(L, [,], \rho|_L)$.

We consider now a generalized Lie bialgebroid $((A, \phi), (A^*, W))$ over M and we denote by E its vector bundle direct sum, i.e., $E = A \oplus A^*$. On E there exist two natural nondegenerate bilinear forms,

one symmetric $(,)_+$ and another skew-symmetric $(,)_-$ given, for any $e_1 = X_1 + \alpha_1, e_2 = X_2 + \alpha_2 \in E = A \oplus A^*$, by

$$(e_1, e_2)_\pm = (X_1 + \alpha_1, X_2 + \alpha_2)_\pm = \frac{1}{2}(\langle \alpha_1, X_2 \rangle \pm \langle \alpha_2, X_1 \rangle). \tag{17}$$

On $\Gamma(E)$, which is identified with $\Gamma(A) \oplus \Gamma(A^*)$, we introduce the bracket $[[,]]$ defined, for all $e_1 = X_1 + \alpha_1, e_2 = X_2 + \alpha_2 \in \Gamma(E)$, by

$$\begin{aligned} [[e_1, e_2]] &= [[X_1 + \alpha_1, X_2 + \alpha_2]] \\ &= ([X_1, X_2]^\phi + \mathcal{L}_{*\alpha_1}^W X_2 - \mathcal{L}_{*\alpha_2}^W X_1 - d_*^W(e_1, e_2)_-) \\ &\quad + ([\alpha_1, \alpha_2]_*^W + \mathcal{L}_{X_1}^\phi \alpha_2 - \mathcal{L}_{X_2}^\phi \alpha_1 + d^\phi(e_1, e_2)_-). \end{aligned} \tag{18}$$

Finally, let $\rho : E \rightarrow TM$ be the bundle map given by $\rho = a + a_*$, i.e., for any $X + \alpha \in E$,

$$\rho(X + \alpha) = a(X) + a_*(\alpha). \tag{19}$$

The following result, which is proved in [30], shows that the notion of generalized Courant algebroid permits us to generalize the double construction for Lie bialgebras (the *Drinfeld double*, [5]) and Lie bialgebroids [18] to generalized Lie bialgebroids.

Theorem 3.3 [30]. *If $((A, \phi), (A^*, W))$ is a generalized Lie bialgebroid over M , then $E = A \oplus A^*$ endowed with $([[,]], (,)_+, \rho)$ and $\theta = \phi + W \in \Gamma(E^*)$ is a generalized Courant algebroid over M . The operators \mathcal{D} and \mathcal{D}^θ are, respectively, $\mathcal{D} = (d_* + d)|_{C^\infty(M, \mathbb{R})}$ and $\mathcal{D}^\theta = (d_*^W + d^\phi)|_{C^\infty(M, \mathbb{R})}$.*

There are two important classes of Dirac structures for the generalized Courant algebroid $(E, \theta) = (A \oplus A^*, \phi + W)$ studied in [30].

The Dirac structure of the graph of an A-bivector field. Let Ω be an A -bivector field and $\Omega^\# : A^* \rightarrow A$ the associated vector bundle map. The graph of $\Omega^\#$ is the maximal isotropic vector subbundle

$$L = \{\Omega^\# \alpha + \alpha / \alpha \in A^*\}$$

of $(A \oplus A^*, (,)_+)$. L is a Dirac structure for $(A \oplus A^*, \phi + W)$ if and only if Ω satisfies the Maurer–Cartan type equation:

$$d_*^W \Omega + \frac{1}{2}[\Omega, \Omega]^\phi = 0.$$

Null Dirac structures. Let $D \subset A$ be a vector subbundle of A and $D^\perp \subset A^*$ its conormal bundle, i.e.,

$$D^\perp = \{\alpha \in A^* / \langle \alpha, X \rangle = 0, \forall X \in D\}. \tag{20}$$

Then, $L = D \oplus D^\perp$ is a Dirac structure for $(A \oplus A^*, \phi + W)$ if and only if D and D^\perp are Lie subalgebroids [21] of A and A^* , respectively. It is clear that in this context, as in the context of a Lie bialgebroid, $L = D \oplus D^\perp$ if and only if the skew-symmetric bilinear form $(,)_-$, defined on $E = A \oplus A^*$ by (17), vanishes on L . For this reason, L is said to be a *null Dirac structure*.

A third important category of Dirac structures for $(E, \theta) = (A \oplus A^*, \phi + W)$, also studied in [30], which generalizes both the above presented categories, is:

Dirac structures defined by a characteristic pair. We consider a pair (D, Ω) of a smooth subbundle $D \subset A$ and of an A -bivector field Ω . We construct, following [20], a subbundle $L \subset A \oplus A^*$ by setting:

$$L = \{X + \Omega^\# \alpha + \alpha / X \in D \text{ and } \alpha \in D^\perp\} = D \oplus \text{graph}(\Omega^\#|_{D^\perp}). \tag{21}$$

L is maximal isotropic with respect to $(,)_+$. The pair (D, Ω) is called the *characteristic pair of L* while the subbundle $D = L \cap (A \oplus \{0\})$, also denoted by $D = L \cap A$, is called the *characteristic subbundle of L* .

For simplicity, we will assume in the sequel that $D = L \cap A$ is of constant rank.

Moreover, since D^\perp may be considered as the dual bundle $(A/D)^*$ of the quotient bundle A/D , the restricted vector bundle map $\Omega^\#|_{D^\perp}$ can be seen as the bundle map associated to an A/D -bivector field. Hence, two pairs (D_1, Ω_1) and (D_2, Ω_2) of a smooth subbundle and of an A -bivector field determine the same subbundle $L \subset A \oplus A^*$ (given by (21)) if and only if

$$D_1 = D_2 =: D \quad \text{and} \quad \Omega_1^\#(\alpha) - \Omega_2^\#(\alpha) \in D, \quad \forall \alpha \in D^\perp. \tag{22}$$

Let $\text{pr}: \Gamma(\wedge A) \rightarrow \Gamma(\wedge(A/D))$ be the map on the spaces of sections, induced by the natural projection $A \rightarrow A/D$. In order to express that the projection under pr of an A -multivector field $\Omega \in \Gamma(\wedge A)$ vanishes in $\Gamma(\wedge(A/D))$, we write $\Omega \equiv 0 \pmod{D}$. Thus, the second condition of (22) can be written as $\Omega_1 - \Omega_2 \equiv 0 \pmod{D}$.

The conditions under which $L = D \oplus \text{graph}(\Omega^\#|_{D^\perp})$ is a Dirac subbundle of $(A \oplus A^*, \phi + W)$ are given by:

Theorem 3.4 [30]. *Let $L = D \oplus \text{graph}(\Omega^\#|_{D^\perp})$ be a maximal isotropic subbundle of $A \oplus A^*$. Then, L is a Dirac structure for the generalized Courant algebroid $(A \oplus A^*, \phi + W)$ if and only if*

- (i) D is a Lie subalgebroid of A ;
- (ii) $d_*^W \Omega + \frac{1}{2}[\Omega, \Omega]^\phi \equiv 0 \pmod{D}$;
- (iii) D^\perp is integrable for the sum bracket $[,]_* + [,]_\Omega$, i.e., for all $\alpha, \beta \in \Gamma(D^\perp)$, $[\alpha, \beta]_* + [\alpha, \beta]_\Omega \in \Gamma(D^\perp)$, where $[,]_\Omega$ is the bracket determined on $\Gamma(A^*)$ by (12).

In the particular case where $((A, \phi), (A^*, W))$ is a triangular generalized Lie bialgebroid (Definition 2.7), Theorem 3.4 takes the following form:

Corollary 3.5 [30]. *Let $((A, \phi), (A^*, W), P)$ be a triangular generalized Lie bialgebroid and $L \subset A \oplus A^*$, $L = D \oplus \text{graph}(\Omega^\#|_{D^\perp})$, a maximal isotropic subbundle of $A \oplus A^*$ with a fixed characteristic pair (D, Ω) . Then L is a Dirac structure for the generalized Courant algebroid $(A \oplus A^*, \phi + W)$ if and only if*

- (i) D is a Lie subalgebroid of A ;
- (ii) $[P + \Omega, P + \Omega]^\phi \equiv 0 \pmod{D}$;
- (iii) for any $X \in \Gamma(D)$, $\mathcal{L}_X^\phi(P + \Omega) \equiv 0 \pmod{D}$.

4. Jacobi structures and Dirac reducible subbundles

We consider a generalized Lie bialgebroid $((A, [\cdot, \cdot], a, \phi), (A^*, [\cdot, \cdot]_*, a_*, W))$ over M and we construct the associated generalized Courant algebroid $(A \oplus A^*, \llbracket \cdot, \cdot \rrbracket, (\cdot, \cdot)_+, \rho, \theta)$ over M , i.e., $\llbracket \cdot, \cdot \rrbracket$ is determined by (18), $\rho = a + a_*$ and $\theta = \phi + W$. We introduce the notions of *reducible Dirac structure* for $(A \oplus A^*, \llbracket \cdot, \cdot \rrbracket, (\cdot, \cdot)_+, \rho, \theta)$ and of *admissible function* of a Dirac structure for $(A \oplus A^*, \llbracket \cdot, \cdot \rrbracket, (\cdot, \cdot)_+, \rho, \theta)$ in an analog manner as in the case of a Dirac structure for a Lie bialgebroid [19].

Definition 4.1. We say that a Dirac subbundle L for $(A \oplus A^*, \llbracket \cdot, \cdot \rrbracket, (\cdot, \cdot)_+, \rho, \theta)$ is reducible if the image $a(D)$ of its characteristic subbundle $D = L \cap A$ by the anchor map a defines a simple foliation \mathcal{F} of M . By the term “simple foliation”, we mean that \mathcal{F} is a regular foliation such that the space M/\mathcal{F} is a nice manifold and the canonical projection $M \rightarrow M/\mathcal{F}$ is a submersion.

Definition 4.2. Let L be a Dirac subbundle for $(A \oplus A^*, \llbracket \cdot, \cdot \rrbracket, (\cdot, \cdot)_+, \rho, \theta)$. We say that a function $f \in C^\infty(M, \mathbb{R})$ is L -admissible if there exists $Y_f \in \Gamma(A)$ such that $Y_f + d^\phi f \in \Gamma(L)$.

Obviously, Y_f is unique up to a smooth section of $L \cap A$. We denote by $C_L^\infty(M, \mathbb{R})$ the set of all L -admissible functions of $C^\infty(M, \mathbb{R})$.

Let $L \subset A \oplus A^*$ be a Dirac structure for $(A \oplus A^*, \llbracket \cdot, \cdot \rrbracket, (\cdot, \cdot)_+, \rho, \theta)$. On $C_L^\infty(M, \mathbb{R})$ we define the bracket $\{ \cdot, \cdot \}_L$ by setting, for all $f, g \in C_L^\infty(M, \mathbb{R})$,

$$\{f, g\}_L := \rho^\theta(e_f)g, \tag{23}$$

where $e_f = Y_f + d^\phi f$. An equivalent expression of (23) is:

$$\{f, g\}_L = \langle Y_f, d^\phi g \rangle + \{f, g\}_J, \tag{24}$$

where $\{ \cdot, \cdot \}_J$ is the bracket (10) of the Jacobi structure on M defined by the generalized Lie bialgebroid structure $((A, \phi), (A^*, W))$ over M . Effectively,

$$\begin{aligned} \{f, g\}_L &\stackrel{(23)}{=} \rho^\theta(e_f)g = ((a^\phi + a_*^W)(Y_f + d^\phi f))g = a^\phi(Y_f)g + a_*^W(d^\phi f)g \\ &= \langle Y_f, d^\phi g \rangle + \langle d^\phi f, d_*^W g \rangle \stackrel{(10)}{=} \langle Y_f, d^\phi g \rangle + \{f, g\}_J. \end{aligned}$$

It is easy to check that (23) or equivalently (24) is well defined. In fact, if $Y'_f = Y_f + X$, with $X \in \Gamma(L \cap A)$, is an other section of A such that $e'_f = Y'_f + d^\phi f \in \Gamma(L)$, we have

$$\langle Y'_f, d^\phi g \rangle + \{f, g\}_J = \langle Y_f + X, d^\phi g \rangle + \{f, g\}_J = \{f, g\}_L + \langle X, d^\phi g \rangle = \{f, g\}_L,$$

since, L being isotropic, $(X + 0, Y_g + d^\phi g)_+ = 0 \Leftrightarrow \langle X, d^\phi g \rangle = 0$.

Proposition 4.3. The space $C_L^\infty(M, \mathbb{R})$ endowed with the bracket $\{ \cdot, \cdot \}_L$, given by (23), is a Lie algebra.

Proof. We must prove that $C_L^\infty(M, \mathbb{R})$ is closed under $\{ \cdot, \cdot \}_L$ and that $\{ \cdot, \cdot \}_L$ is a bilinear, skew-symmetric bracket which satisfies the Jacobi identity.

Closeness of $\{, \}_L$ in $C_L^\infty(M, \mathbb{R})$. Let $f, g \in C_L^\infty(M, \mathbb{R})$ be two L -admissible functions. Then, there exist $Y_f, Y_g \in \Gamma(A)$ such that $e_f = Y_f + d^\phi f, e_g = Y_g + d^\phi g \in \Gamma(L)$. We consider the bracket $[[e_f, e_g]]$; according to (18), its component in $\Gamma(A^*)$ is:

$$[d^\phi f, d^\phi g]_*^W + \mathcal{L}_{Y_f}^\phi d^\phi g - \mathcal{L}_{Y_g}^\phi d^\phi f + d^\phi(e_f, e_g)_-.$$

We have [10],

$$[d^\phi f, d^\phi g]_*^W = -\mathcal{L}_{d_*^W f}^\phi d^\phi g = -d^\phi \langle d_*^W f, d^\phi g \rangle = -d^\phi \{g, f\}_J = d^\phi \{f, g\}_J$$

and, on the other hand,

$$\mathcal{L}_{Y_f}^\phi d^\phi g - \mathcal{L}_{Y_g}^\phi d^\phi f + d^\phi(e_f, e_g)_- = -d^\phi(e_f, e_g)_- = -d^\phi(e_f, e_g)_- + \underbrace{d^\phi(e_f, e_g)_+}_{=0} = d^\phi \langle Y_f, d^\phi g \rangle.$$

Thus,

$$[d^\phi f, d^\phi g]_*^W + \mathcal{L}_{Y_f}^\phi d^\phi g - \mathcal{L}_{Y_g}^\phi d^\phi f + d^\phi(e_f, e_g)_- = d^\phi \{f, g\}_J + d^\phi \langle Y_f, d^\phi g \rangle \stackrel{(24)}{=} d^\phi \{f, g\}_L,$$

which means that $\{f, g\}_L$ is an L -admissible function, i.e., $\{f, g\}_L \in C_L^\infty(M, \mathbb{R})$, and that we can take $[[e_f, e_g]] = e_{\{f, g\}_L}$.

Bilinearity and skew-symmetry of $\{, \}_L$. It is obvious that $\{, \}_L$ is bilinear. Also, for any $f \in C_L^\infty(M, \mathbb{R})$, we have $(e_f, e_f)_+ = 0 \Leftrightarrow \langle Y_f, d^\phi f \rangle = 0$, so $\{f, f\}_L \stackrel{(24)}{=} \langle Y_f, d^\phi f \rangle + \{f, f\}_J = 0 + 0 = 0$, which implies the skew-symmetry of $\{, \}_L$.

Jacobi identity. By a straightforward, but long, calculation we get that, for any $f, g, h \in C_L^\infty(M, \mathbb{R})$, the Jacobi identity holds:

$$\{f, \{g, h\}_L\}_L + \{g, \{h, f\}_L\}_L + \{h, \{f, g\}_L\}_L = 0.$$

Hence, $(C_L^\infty(M, \mathbb{R}), \{, \}_L)$ is a Lie algebra. \square

In the particular case where the constant function 1 is an L -admissible function, $C_L^\infty(M, \mathbb{R})$ equipped with the usual product of functions “ \cdot ” is an associative commutative algebra with unit and $\{, \}_L$ is a first order differential operator on each of its arguments. In fact, $1 \in C_L^\infty(M, \mathbb{R})$ means that there exists $Y_1 \in \Gamma(A)$ such that $Y_1 + d^\phi 1 = Y_1 + \phi \in \Gamma(L)$. Then, for any $f, g \in C_L^\infty(M, \mathbb{R})$, $f \cdot g \in C_L^\infty(M, \mathbb{R})$ since, for $Y_{fg} = fY_g + gY_f - fgY_1 \in \Gamma(A)$, $Y_{fg} + d^\phi(fg) \in \Gamma(L)$. Moreover, for any $f, g, h \in C_L^\infty(M, \mathbb{R})$,

$$\begin{aligned} \{f, gh\}_L &\stackrel{(24)}{=} \langle Y_f, d^\phi(gh) \rangle + \{f, gh\}_J \\ &= \langle Y_f, gdh + hdg + gh\phi \rangle + g\{f, h\}_J + h\{f, g\}_J - gh\{f, 1\}_J \\ &= g(\langle Y_f, d^\phi h \rangle + \{f, h\}_J) + h(\langle Y_f, d^\phi g \rangle + \{f, g\}_J) - gh(\langle Y_f, \phi \rangle + \{f, 1\}_J) \\ &= g\{f, h\}_L + h\{f, g\}_L - gh\{f, 1\}_L \end{aligned}$$

and by the skew-symmetry of $\{, \}_L$ we obtain the desired result. Consequently,

Theorem 4.4. *If 1 is an L -admissible function, then $(C_L^\infty(M, \mathbb{R}), \{, \}_L)$ is a Jacobi algebra.*

The above result generalizes the one of A. Wade [32] for the $\mathcal{E}^1(M)$ -Dirac structures.

In the following we will establish the characteristic equations of a Dirac structure (see, also, [3]).

Lemma 4.5. *Let L be a Dirac structure for $(A \oplus A^*, \llbracket, \rrbracket, (\cdot, \cdot)_+, \rho, \theta)$, $\varpi : A \oplus A^* \rightarrow A$ and $\varpi_* : A \oplus A^* \rightarrow A^*$ the natural projections from $A \oplus A^*$ onto A and A^* , respectively. Then, $\ker \varpi|_L = L \cap A^*$ and $\ker \varpi_*|_L = L \cap A$. Also,*

$$\varpi(L)^\perp = L \cap A^* \quad \text{and} \quad \varpi_*(L) = (L \cap A)^\perp. \tag{25}$$

Proof. We denote by L_x, A_x and A_x^* the fibers over $x \in M$ of L, A and A^* , respectively. It is clear that, at each point $x \in M$, $\ker \varpi|_{L_x} = L_x \cap A_x^*$ and $\ker \varpi_*|_{L_x} = L_x \cap A_x$, thus $\dim(\ker \varpi|_{L_x}) = \dim(L_x \cap A_x^*)$ and $\dim(\ker \varpi_*|_{L_x}) = \dim(L_x \cap A_x)$. Also, at each point $x \in M$, $\varpi(L_x)^\perp = L_x \cap A_x^*$ and $\varpi_*(L_x) = (L_x \cap A_x)^\perp$. Effectively, if $\alpha(x) \in L_x \cap A_x^*$, then $0 + \alpha(x) \in L_x$, thus, for any $Y(x) + \beta(x) \in L_x$, $(0 + \alpha(x), Y(x) + \beta(x))_+ = 0 \Leftrightarrow \langle \alpha(x), Y(x) \rangle = 0$ that implies that $\alpha(x) \in \varpi(L_x)^\perp$, i.e., $L_x \cap A_x^* \subseteq \varpi(L_x)^\perp$, and by a dimension count we conclude the equality. Analogously, we prove the second equation of (25) for the fibers over x . Since the above results hold at each point $x \in M$, we get that the characteristic equations (25) of L hold. \square

Lemma 4.6. *The constant function 1 is an L -admissible function if and only if, for any $Y \in \Gamma(D)$,*

$$\langle \phi, Y \rangle = 0. \tag{26}$$

Proof. In fact, if $1 \in C_L^\infty(M, \mathbb{R})$, then there exists $Y_1 \in \Gamma(A)$ such that $Y_1 + d^\phi 1 = Y_1 + \phi \in \Gamma(L)$. Also, for every $Y \in \Gamma(D)$, $Y + 0 \in \Gamma(L)$ and $(L, (\cdot, \cdot)_+)$ is maximal isotropic. Thus, for any $Y \in \Gamma(D)$, $(Y + 0, Y_1 + \phi)_+ = 0 \Leftrightarrow \langle \phi, Y \rangle = 0$. Conversely, we suppose that, for any $Y \in \Gamma(D)$, $\langle \phi, Y \rangle = 0$; then we will prove that $1 \in C_L^\infty(M, \mathbb{R})$. Effectively, if 1 is not an L -admissible function, then, for any $Y_1 \in \Gamma(A)$, $Y_1 + d^\phi 1 = Y_1 + \phi$ is not a section of L , fact which implies that ϕ is not a section of $\varpi_*(L) \stackrel{(25)}{=} (L \cap A)^\perp = D^\perp$. Therefore, there exists $Y \in \Gamma(D)$ such that $\langle \phi, Y \rangle \neq 0$; contradiction. \square

Proposition 4.7. *Let $L \subset A \oplus A^*$ be a reducible Dirac structure for $(A \oplus A^*, \llbracket, \rrbracket, (\cdot, \cdot)_+, \rho, \theta)$ and \mathcal{F} the simple foliation of M defined by the distribution $a(D)$, $D = L \cap A$, on M . If 1 is an L -admissible function, then $f \in C_L^\infty(M, \mathbb{R})$ if and only if f is constant along the leaves of \mathcal{F} .*

Proof. Let f be an L -admissible function, i.e., there exists $Y_f \in \Gamma(A)$ such that $Y_f + d^\phi f \in \Gamma(L)$, and $X \in \Gamma(a(D))$ a section of the distribution $a(D)$. $X \in \Gamma(a(D))$ means that there exists $Y \in \Gamma(D)$ such that $X = a(Y)$ and $Y \in \Gamma(D)$ means that $Y + 0 \in \Gamma(L)$. Since L is maximally isotropic,

$$\begin{aligned} (Y + 0, Y_f + d^\phi f)_+ = 0 &\Leftrightarrow \langle d^\phi f, Y \rangle = 0 \Leftrightarrow \langle df, Y \rangle + f \langle \phi, Y \rangle = 0 \stackrel{(26)}{\Leftrightarrow} \langle \delta f, a(Y) \rangle = 0 \\ &\Leftrightarrow \langle \delta f, X \rangle = 0. \end{aligned}$$

By the last equation, which holds for any $X \in \Gamma(a(D))$, we get that f is constant along the leaves of \mathcal{F} . Conversely, let f be a function on M constant along the leaves of \mathcal{F} , i.e., for any $X \in \Gamma(a(D))$, $\langle \delta f, X \rangle = 0$. But, $X \in \Gamma(a(D))$ means that there exists $Y \in \Gamma(D)$ such that $X = a(Y)$. Thus, for any $X \in \Gamma(a(D))$, $X = a(Y)$ with $Y \in \Gamma(D)$,

$$\langle \delta f, X \rangle = 0 \Leftrightarrow \langle \delta f, a(Y) \rangle = 0 \Leftrightarrow \langle df, Y \rangle = 0 \stackrel{(26)}{\Leftrightarrow} \langle df + f\phi, Y \rangle = 0 \Leftrightarrow \langle d^\phi f, Y \rangle = 0, \tag{27}$$

for any $Y \in \Gamma(D)$. If f is not an L -admissible function, then, for any $Z \in \Gamma(A)$, $Z + d^\phi f$ is not a section of L . So, $d^\phi f$ is not a section of $\varpi_*(L) \stackrel{(25)}{=} (L \cap A)^\perp = D^\perp$. Therefore, there exists $Y \in \Gamma(D)$ such that $\langle d^\phi f, Y \rangle \neq 0$; contradiction. \square

By the above study we conclude:

Theorem 4.8. *Let L be a reducible Dirac subbundle for $(A \oplus A^*, \llbracket, \rrbracket, (\cdot, \cdot)_+, \rho, \theta)$. We suppose that 1 is an L -admissible function. Then L induces a Jacobi structure on M/\mathcal{F} defined by the Jacobi bracket $\{, \}_L$, which is given by (23) or (24).*

By applying Theorem 4.8 to the case of the generalized Lie bialgebroid defined by a Jacobi structure (Λ, E) on M (Theorem 2.4) we deduce:

Corollary 4.9. *Let (M, Λ, E) be a Jacobi manifold, $((TM \times \mathbb{R}, [,], \pi, (0, 1)), (T^*M \times \mathbb{R}, [,]_{(\Lambda, E)}, \pi \circ (\Lambda, E)^\#, (-E, 0))$ the associated generalized Lie bialgebroid and L a reducible Dirac structure for the generalized Courant algebroid $((TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R}), \llbracket, \rrbracket, (\cdot, \cdot)_+, \pi + \pi \circ (\Lambda, E)^\#, (0, 1) + (-E, 0))$. We suppose that 1 is an L -admissible function. Then L induces a Jacobi structure on M/\mathcal{F} , where \mathcal{F} is the foliation of M defined by the distribution $\pi(D)$, $D = L \cap (TM \times \mathbb{R})$, which is exactly the Jacobi structure defined by $\{, \}_L$.*

Remark 4.10. In the context of Corollary 4.9, the condition “ 1 is an L -admissible function” is equivalent to the one “ D has only sections of type $(X, 0)$ with $X \in \Gamma(TM)$ ”. In fact, according to Lemma 4.6, $1 \in C_L^\infty(M, \mathbb{R})$ if and only if, for any $(X, f) \in \Gamma(D)$, $\langle (0, 1), (X, f) \rangle \stackrel{(26)}{=} 0 \Leftrightarrow f = 0$.

Taking into account Corollary 2.6, Definition 4.2 and (24), we can easily establish:

Proposition 4.11. *Under the assumptions of Corollary 4.9,*

1. if $L = \text{graph}(\Lambda', E')^\#$ is the graph of a $(TM \times \mathbb{R})$ -bivector field (Λ', E') on M , then $C_L^\infty(M, \mathbb{R}) = C^\infty(M, \mathbb{R})$ and, for all $f, g \in C_L^\infty(M, \mathbb{R})$,

$$\{f, g\}_L = \{f, g\}_{(\Lambda', E')} + \{f, g\}_{(\Lambda, E)}; \tag{28}$$

2. if $L = D \oplus D^\perp$ is a null Dirac structure, then $C_L^\infty(M, \mathbb{R}) = \{f \in C^\infty(M, \mathbb{R}) / (\delta f, f) \in \Gamma(D^\perp)\}$ and, for all $f, g \in C_L^\infty(M, \mathbb{R})$,

$$\{f, g\}_L = \{f, g\}_{(\Lambda, E)}; \tag{29}$$

3. if $L = D \oplus \text{graph}(\Lambda', E')^\#|_{D^\perp}$ is defined by a characteristic pair $(D, (\Lambda', E'))$, then $C_L^\infty(M, \mathbb{R}) = \{f \in C^\infty(M, \mathbb{R}) / (\delta f, f) \in \Gamma(D^\perp)\}$ and, for all $f, g \in C_L^\infty(M, \mathbb{R})$,

$$\{f, g\}_L = \{f, g\}_{(\Lambda', E')} + \{f, g\}_{(\Lambda, E)}. \tag{30}$$

In what follows, we will prove that in the context of “generalized Lie bialgebroids—Jacobi structures”, as in the context of “Lie bialgebroids—Poisson structures” [19], the converse result of Corollary 4.9 also holds.

Theorem 4.12. *Let (M, Λ, E) be a Jacobi manifold, \mathcal{F} a simple foliation of M defined by a Lie subalgebroid $D \subset TM \times \mathbb{R}$ that has only sections of type $(X, 0)$ and $(\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$ a Jacobi structure on the quotient manifold M/\mathcal{F} . Then $(M/\mathcal{F}, \Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$ defines a reducible Dirac structure L in $(TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ such that $L \cap (TM \times \mathbb{R}) = D$, $1 \in C_L^\infty(M, \mathbb{R})$ and the Jacobi structure induced by L on M/\mathcal{F} , in the sense of Corollary 4.9, is the initially given $(\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$.*

Proof. We make the proof in several steps.

First step. Let $D \subset TM \times \mathbb{R}$ be a Lie subalgebroid of $(TM \times \mathbb{R}, [,], \pi)$, which has only sections of type $(X, 0)$, such that $\pi(D)$ defines a simple foliation \mathcal{F} of M and let D^\perp be its conormal bundle:

$$\begin{aligned} D^\perp &= \{(\alpha, g) \in T^*M \times \mathbb{R} / \langle (\alpha, g), (X, 0) \rangle = \langle \alpha, X \rangle = 0, \forall (X, 0) \in D\} \\ &= \pi(D)^\perp \times \mathbb{R}. \end{aligned} \tag{31}$$

We suppose that the quotient manifold M/\mathcal{F} is endowed with a Jacobi structure $(\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$ and we denote by $p: M \rightarrow M/\mathcal{F}$ the canonical projection.

Second step. We keep under control the fact that $p: M \rightarrow M/\mathcal{F}$ is not a Jacobi map by defining a “difference” bracket $\{ , \}_1: C^\infty(M/\mathcal{F}, \mathbb{R}) \times C^\infty(M/\mathcal{F}, \mathbb{R}) \rightarrow C^\infty(M, \mathbb{R})$ as follows:

$$\{f, g\}_1 = p^*\{f, g\}_{M/\mathcal{F}} - \{p^*f, p^*g\}_{(\Lambda, E)}, \quad \forall f, g \in C^\infty(M/\mathcal{F}, \mathbb{R}). \tag{32}$$

Obviously, $\{ , \}_1$ is a bilinear, skew-symmetric, first order differential operator on each of its arguments. Thus, $\{ , \}_1$ induces a skew-symmetric bilinear form (Λ_1, E_1) on $T^*(M/\mathcal{F}) \times \mathbb{R}$ so that, for all $f, g \in C^\infty(M/\mathcal{F}, \mathbb{R})$,

$$\{f, g\}_1 = \Lambda_1(\delta f, \delta g) + \langle f\delta g - g\delta f, E_1 \rangle.$$

In turn, (Λ_1, E_1) induces a vector bundle map $(\Lambda_1, E_1)^\# : T^*(M/\mathcal{F}) \times \mathbb{R} \rightarrow T(M/\mathcal{F}) \times \mathbb{R}$. But, $T^*(M/\mathcal{F}) \times \mathbb{R} \cong \pi(D)^\perp \times \mathbb{R} \stackrel{(31)}{=} D^\perp$ and $T(M/\mathcal{F}) \times \mathbb{R} \cong (TM/\pi(D)) \times \mathbb{R} \cong (TM \times \mathbb{R})/D$. Consequently, we can consider that $(\Lambda_1, E_1)^\# : D^\perp \rightarrow (TM \times \mathbb{R})/D$.

Third step. We denote by $\text{pr}: TM \times \mathbb{R} \rightarrow (TM \times \mathbb{R})/D$ the natural projection and we define a subbundle $L \subset (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ by

$$L = \{(X, f) + (\alpha, g) \in (TM \times \mathbb{R}) \oplus D^\perp / \text{pr}(X, f) = (\Lambda_1, E_1)^\#(\alpha, g)\}. \tag{33}$$

By construction, L is maximally isotropic, $C_L^\infty(M, \mathbb{R}) \cong C^\infty(M/\mathcal{F}, \mathbb{R})$ and $1 \in C_L^\infty(M, \mathbb{R})$. Effectively, by a straightforward calculation we show that, for any $e_1 = (X_1, f_1) + (\alpha_1, g_1), e_2 = (X_2, f_2) + (\alpha_2, g_2) \in L$, $(e_1, e_2)_+ = 0$ and $f \in C_L^\infty(M, \mathbb{R}) \Leftrightarrow d^{(0,1)}f = (\delta f, f) \in \Gamma(D^\perp) \cong \Gamma(T^*(M/\mathcal{F}) \times \mathbb{R}) \Leftrightarrow f \in C^\infty(M/\mathcal{F}, \mathbb{R})$. Obviously, $1 \in C_L^\infty(M, \mathbb{R})$ since $(0, 1) \in \Gamma(D^\perp) \cong \Gamma(T^*(M/\mathcal{F}) \times \mathbb{R})$. Also, by Definition 4.2, $f \in C_L^\infty(M, \mathbb{R})$ if and only if there exists $(Y_f, \varphi_f) \in \Gamma(TM \times \mathbb{R})$ such that $e_f = (Y_f, \varphi_f) + (\delta f, f) \in \Gamma(L)$. Hence, we have that $\Gamma(L)$ is spanned by all the sections of the type he_f , where $h \in C^\infty(M, \mathbb{R})$ and $f \in C_L^\infty(M, \mathbb{R})$. To verify the integrability of L , it suffices to verify the closeness of the bracket \llbracket , \rrbracket for the sections of L of the form $e_f = (Y_f, \varphi_f) + (\delta f, f)$ with $f \in C_L^\infty(M, \mathbb{R})$, since, according to (14) and because L is isotropic,

$$\begin{aligned} \llbracket e_f, he_g \rrbracket &= h \llbracket e_f, e_g \rrbracket + (\rho(e_f)h)e_g - (e_f, e_g)_+ \mathcal{D}h \\ &= h \llbracket e_f, e_g \rrbracket + (\rho(e_f)h)e_g, \end{aligned}$$

for all $e_f, e_g \in \Gamma(L)$, with $f, g \in C_L^\infty(M, \mathbb{R})$, and $h \in C^\infty(M, \mathbb{R})$.

Let $f, g \in C_L^\infty(M, \mathbb{R})$ be two L -admissible functions. Since $C_L^\infty(M, \mathbb{R}) \cong C^\infty(M/\mathcal{F}, \mathbb{R})$, $\{f, g\}_{M/\mathcal{F}} \in C_L^\infty(M, \mathbb{R})$, i.e., there is $(Y_{\{f,g\}_{M/\mathcal{F}}}, \varphi_{\{f,g\}_{M/\mathcal{F}}}) \in \Gamma(TM \times \mathbb{R})$ such that $e_{\{f,g\}_{M/\mathcal{F}}} = (Y_{\{f,g\}_{M/\mathcal{F}}}, \varphi_{\{f,g\}_{M/\mathcal{F}}}) + (\delta\{f, g\}_{M/\mathcal{F}}, \{f, g\}_{M/\mathcal{F}}) \in \Gamma(L)$. We show that

$$\{f, g\}_{M/\mathcal{F}} = \rho^\theta(e_f)g \stackrel{(23)}{=} \{f, g\}_L. \tag{34}$$

Effectively,

$$\begin{aligned} \{f, g\}_L = \rho^\theta(e_f)g &= [(\pi^{(0,1)} + (\pi \circ (\Lambda, E)^\#)^{-E,0})(Y_f, \varphi_f) + (\delta f, f)]g \\ &= (Y_f + \varphi_f + \Lambda^\#(\delta f) + fE - \langle \delta f, E \rangle)g \\ &= (\text{pr}(Y_f, \varphi_f) + \text{the component of } (Y_f, \varphi_f) \text{ on } D)g + \{f, g\}_{(\Lambda, E)} \\ &= \langle (\delta g, g), (\Lambda_1, E_1)^\#(\delta f, f) \rangle + \{f, g\}_{(\Lambda, E)} \\ &= \{f, g\}_1 + \{f, g\}_{(\Lambda, E)} \\ &\stackrel{(32)}{=} \{f, g\}_{M/\mathcal{F}}. \end{aligned}$$

On the other hand, since $\{, \}_{M/\mathcal{F}}$ is a Jacobi bracket, thus it verifies the Jacobi identity, for any $f, g, h \in C_L^\infty(M, \mathbb{R}) \cong C^\infty(M/\mathcal{F}, \mathbb{R})$,

$$\begin{aligned} \rho^\theta(\llbracket e_f, e_g \rrbracket - e_{\{f,g\}_{M/\mathcal{F}}})h &= \rho^\theta(\llbracket e_f, e_g \rrbracket)h - \rho^\theta(e_{\{f,g\}_{M/\mathcal{F}}})h \\ &\stackrel{(16)}{=} [\rho^\theta(e_f), \rho^\theta(e_g)]h - \rho^\theta(e_{\{f,g\}_{M/\mathcal{F}}})h \\ &= \rho^\theta(e_f)(\rho^\theta(e_g)h) - \rho^\theta(e_g)(\rho^\theta(e_f)h) - \rho^\theta(e_{\{f,g\}_{M/\mathcal{F}}})h \\ &\stackrel{(34)}{=} \{f, \{g, h\}_{M/\mathcal{F}}\}_{M/\mathcal{F}} - \{g, \{f, h\}_{M/\mathcal{F}}\}_{M/\mathcal{F}} - \{\{f, g\}_{M/\mathcal{F}}, h\}_{M/\mathcal{F}} \\ &= 0. \end{aligned} \tag{35}$$

From the proof of Proposition 4.3, we have that the component of $\llbracket e_f, e_g \rrbracket$ in $\Gamma(T^*M \times \mathbb{R})$ is $d^{(0,1)}\{f, g\}_{M/\mathcal{F}}$, therefore $\llbracket e_f, e_g \rrbracket - e_{\{f,g\}_{M/\mathcal{F}}} \in \Gamma(TM \times \mathbb{R})$. So, (35) means that $\rho^\theta(\llbracket e_f, e_g \rrbracket - e_{\{f,g\}_{M/\mathcal{F}}}) \in \Gamma(D)$. But $\rho^\theta(\llbracket e_f, e_g \rrbracket - e_{\{f,g\}_{M/\mathcal{F}}}) = \pi^{(0,1)}(\llbracket e_f, e_g \rrbracket - e_{\{f,g\}_{M/\mathcal{F}}}) = \llbracket e_f, e_g \rrbracket - e_{\{f,g\}_{M/\mathcal{F}}}$ and $\Gamma(D) \subset \Gamma(L)$. Consequently, $\llbracket e_f, e_g \rrbracket - e_{\{f,g\}_{M/\mathcal{F}}} \in \Gamma(L)$ which implies $\llbracket e_f, e_g \rrbracket \in \Gamma(L)$, whence the integrability of L .

For the constructed L , $L \cap (TM \times \mathbb{R}) = \{(X, f) + (0, 0) \in (TM \times \mathbb{R}) \oplus \{(0, 0)\} / \text{pr}(X, f) = (\Lambda_1, E_1)^\#(0, 0)\} = \{(X, f) \in TM \times \mathbb{R} / \text{pr}(X, f) = (0, 0)\} = D$ and the induced Jacobi structure on M/\mathcal{F} , in the sense of Corollary 4.9, is the initially given $(\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$ (see (34)). \square

Remark 4.13. The condition “ D has only sections of type $(X, 0)$ ” is indispensable in order to ensure that the constant function 1 is an L -admissible function for the constructed L . In the opposite case, i.e., if D has at least one section of type (X, f) with $f \neq 0$, we will have that there exists at least one section of D , the section (X, f) , such that $\langle (0, 1), (X, f) \rangle = f \neq 0$ and, according to Lemma 4.6, this implies that 1 is not an L -admissible function. Hence, $(C_L^\infty(M, \mathbb{R}), \{, \}_L)$ cannot be a Jacobi algebra and $C_L^\infty(M, \mathbb{R})$ does not coincide with $C^\infty(M/\mathcal{F}, \mathbb{R})$. Thus, for a Lie subalgebroid D of $TM \times \mathbb{R}$ that has at least one section (X, f) with $f \neq 0$ we cannot construct a reducible Dirac subbundle $L \subset (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ which induces, in the sense of Corollary 4.9, a Jacobi structure on M/\mathcal{F} .

In conclusion, we have proved:

Theorem 4.14. *Let (M, Λ, E) be a Jacobi manifold. There is a one–one correspondence between reducible Dirac subbundles of the generalized Courant algebroid $((TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R}), [\cdot, \cdot], (\cdot, \cdot)_+, \pi + \pi \circ (\Lambda, E)^\#, (0, 1) + (-E, 0))$ for which 1 is an admissible function and quotient Jacobi manifolds M/\mathcal{F} of M , where \mathcal{F} is a simple foliation of M defined by a Lie subalgebroid $D \subset TM \times \mathbb{R}$ that has sections only of type $(X, 0)$.*

Remark 4.15. If, in the proof of Theorem 4.12, $p: (M, \Lambda, E) \rightarrow (M/\mathcal{F}, \Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$ is a Jacobi map, then $(\Lambda_1, E_1) = (0, 0)$. Hence, in this case, $L = D \oplus D^\perp$ is a null Dirac structure. Thus:

Corollary 4.16. *A Lie subalgebroid $D \subset TM \times \mathbb{R}$ which has only sections of type $(X, 0)$ defines a simple foliation \mathcal{F} of (M, Λ, E) such that $p: (M, \Lambda, E) \rightarrow (M/\mathcal{F}, \Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$ is a Jacobi map if and only if $L = D \oplus D^\perp$.*

Remark 4.17. In the case where $D = \{(0, 0)\}$, a Jacobi structure on $M/\mathcal{F} \cong M$ is a new Jacobi structure (Λ', E') on M and the constructed L is the graph of $(\Lambda' - \Lambda, E' - E)^\#$. Since, by construction, L is a Dirac subbundle of $(TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$, $(\Lambda' - \Lambda, E' - E)$ is a Jacobi structure on M [30], fact which implies that (Λ, E) and (Λ', E') are compatible Jacobi structures in the sense of [29].

A geometric interpretation of Corollary 3.5. In the context of this paragraph, Corollary 3.5 can be formulated as: *Let (M, Λ, E) be a Jacobi manifold, $((TM \times \mathbb{R}, (0, 1)), (T^*M \times \mathbb{R}, (-E, 0)), (\Lambda, E))$ the associated triangular generalized Lie bialgebroid over M and (Λ', E') a $(TM \times \mathbb{R})$ -bivector field such that $L = D \oplus \text{graph}((\Lambda', E')^\#|_{D^\perp})$ is a maximal isotropic subbundle of $(TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ with fixed characteristic pair $(D, (\Lambda', E'))$. Then L is a Dirac structure for $((TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R}), (0, 1) + (-E, 0))$ if and only if*

- (i) D is a Lie subalgebroid of $TM \times \mathbb{R}$;
- (ii) $[(\Lambda + \Lambda', E + E'), (\Lambda + \Lambda', E + E')]^{(0,1)} \equiv 0 \pmod{D}$;
- (iii) for any $(X, f) \in \Gamma(D)$, $\mathcal{L}_{(X,f)}^{(0,1)}(\Lambda + \Lambda', E + E') \equiv 0 \pmod{D}$.

If $L = D \oplus \text{graph}((\Lambda', E')^\#|_{D^\perp})$ is a reducible Dirac structure and 1 is an L -admissible function, after the proofs of Theorems 4.8 and 4.12, we get that condition (iii) is equivalent to that $(\Lambda + \Lambda', E + E')$ can be reduced to a $(TM \times \mathbb{R})/D \cong (T(M/\mathcal{F}) \times \mathbb{R})$ -bivector field on M/\mathcal{F} and the condition (ii) is equivalent to the fact that the reduced bivector field is a Jacobi structure on M/\mathcal{F} . Furthermore, by Proposition 4.11 (case 3) we get that the induced Jacobi structure on M/\mathcal{F} is exactly the one defined by the bracket of L -admissible functions. Consequently, it is the Jacobi structure induced on M/\mathcal{F} by L in the sense of Corollary 4.9.

5. Dirac structures and Jacobi reduction

In this paragraph, we will establish a Jacobi reduction theorem in terms of Dirac structures. For its proof, we need to adapt the results concerning the pull-back Dirac structures of a Lie bialgebroid [20] to the pull-back Dirac structures for a generalized Lie bialgebroid.

Proposition 5.1. *Let (A_1, ϕ_1) be a Lie algebroid over a differentiable manifold M_1 with an 1-cocycle, $((A_2, \phi_2), (A_2^*, W_2), P_2)$ a triangular generalized Lie bialgebroid over a differentiable manifold M_2 and $\Phi : A_1 \rightarrow A_2$ a Lie algebroid morphism of constant rank, which covers a surjective map between the bases, such that $\Phi^*(\phi_2) = \phi_1$. Then the following two statements are equivalent.*

1. *There exists a Dirac structure for the triangular generalized Lie bialgebroid $((A_1, \phi_1), (A_1^*, 0), 0)$ whose characteristic pair is $(\ker \Phi, P_1)$ and $\bar{\Phi}(P_1) = P_2$.*
2. $\text{Im } P_2^\# \subseteq \text{Im } \Phi$.

We note that, since $\Phi : A_1 \rightarrow A_2$ is a Lie algebroid morphism such that $\Phi^*(\phi_2) = \phi_1$ then, for any $P \in \Gamma(\wedge^p A_1)$ and $Q \in \Gamma(\wedge^q A_1)$, $\Phi([P, Q]^{\phi_1}) = [\Phi(P), \Phi(Q)]^{\phi_2}$.

Proof. According to Corollary 3.5, it suffices to show that the following two statements are equivalent.

1. There exists $P_1 \in \Gamma(\wedge^2 A_1)$ such that $\Phi(P_1) = P_2$ and
 - (a) $\ker \Phi$ is a Lie subalgebroid of A_1 ;
 - (b) $[0 + P_1, 0 + P_1]^{\phi_1} \equiv 0 \pmod{\ker \Phi} \Leftrightarrow [P_1, P_1]^{\phi_1} \equiv 0 \pmod{\ker \Phi}$;
 - (c) for any $X \in \Gamma(\ker \Phi)$, $\mathcal{L}_X^{\phi_1}(0 + P_1) \equiv 0 \pmod{\ker \Phi} \Leftrightarrow \mathcal{L}_X^{\phi_1}(P_1) \equiv 0 \pmod{\ker \Phi}$.
2. $\text{Im } P_2^\# \subseteq \text{Im } \Phi$.

Obviously, $\ker \Phi$ is a Lie subalgebroid of A_1 since, for all $X, Y \in \Gamma(\ker \Phi)$,

$$\Phi([X, Y]) = [\Phi(X), \Phi(Y)] = [0, 0] = 0,$$

which means that $[X, Y] \in \Gamma(\ker \Phi)$. On the other hand, the subbundle $\ker \Phi^\perp = \{\alpha \in A_1^* / \langle \alpha, X \rangle = 0, \forall X \in \ker \Phi\}$ of A_1^* can be identified with the dual bundle $(A_1 / \ker \Phi)^*$ of $A_1 / \ker \Phi$. Also, $\ker \Phi^\perp = \text{Im } \Phi^*$, where $\Phi^* : A_2^* \rightarrow A_1^*$ is the dual map of Φ . Effectively, it is clear that, $\text{Im } \Phi^* \subseteq \ker \Phi^\perp$ and, since Φ is of constant rank, $\dim \text{Im } \Phi^* = \dim \ker \Phi^\perp$, thus $\text{Im } \Phi^* = \ker \Phi^\perp \cong (A_1 / \ker \Phi)^*$. Hence, $\Phi^* : A_2^* \rightarrow (A_1 / \ker \Phi)^*$ is a surjective map, i.e., for any $\bar{\alpha}_1, \bar{\beta}_1 \in \Gamma((A_1 / \ker \Phi)^*)$, there exist $\alpha_2, \beta_2 \in \Gamma(A_2^*)$ such that $\bar{\alpha}_1 = \Phi^*(\alpha_2)$ and $\bar{\beta}_1 = \Phi^*(\beta_2)$. If there is some $\bar{P}_1 \in \Gamma(\wedge^2(A_1 / \ker \Phi))$ which is Φ -related to P_2 , i.e., $\Phi(\bar{P}_1) = P_2$, then it should be defined, for all $\bar{\alpha}_1, \bar{\beta}_1 \in \Gamma((A_1 / \ker \Phi)^*)$, by

$$\bar{P}_1(\bar{\alpha}_1, \bar{\beta}_1) = P_2(\alpha_2, \beta_2).$$

It is clear that \bar{P}_1 is well-defined if and only if $\ker \Phi^* \subseteq \ker P_2^\#$, or equivalently, if and only if $\text{Im } P_2^\# \subseteq \text{Im } \Phi$. Let P_1 be an arbitrary representative of \bar{P}_1 in $\Gamma(\wedge^2 A_1)$. Since $\Phi : A_1 \rightarrow A_2$ is a Lie algebroid morphism such that $\Phi^*(\phi_2) = \phi_1$ and $((A_2, \phi_2), (A_2^*, W_2), P_2)$ is a triangular generalized Lie bialgebroid, we have that

$$\Phi([P_1, P_1]^{\phi_1}) = [\Phi(P_1), \Phi(P_1)]^{\phi_2} = [P_2, P_2]^{\phi_2} = 0 \Leftrightarrow [P_1, P_1]^{\phi_1} \equiv 0 \pmod{\ker \Phi}.$$

Moreover, for any $X \in \Gamma(\ker \Phi)$,

$$\Phi(\mathcal{L}_X^{\phi_1} P_1) = \Phi([X, P_1]^{\phi_1}) = [\Phi(X), \Phi(P_1)]^{\phi_2} = [0, \Phi(P_1)]^{\phi_2} = 0 \Leftrightarrow \mathcal{L}_X^{\phi_1} P_1 \equiv 0 \pmod{\ker \Phi}.$$

Consequently, there exists $P_1 \in \Gamma(\wedge^2 A_1)$ such that $\Phi(P_1) = P_2$ and $(\ker \Phi, P_1)$ defines a Dirac structure for the triangular generalized Lie bialgebroid $((A_1, \phi_1), (A_1^*, 0), 0)$ if and only if $\text{Im } P_2^\# \subseteq \text{Im } \Phi$. \square

Reduction of Jacobi manifolds. Let (M, Λ, E) be a Jacobi manifold, $N \subseteq M$ a submanifold of M and $i : N \hookrightarrow M$ the canonical inclusion, $D \subset TM \times \mathbb{R}$ a Lie subalgebroid of $(TM \times \mathbb{R}, [\cdot, \cdot], \pi)$ that has only sections of type $(X, 0)$ and $D_0 = D \cap (TN \times \mathbb{R})$. We suppose that D and D_0 define, respectively, a simple foliation \mathcal{F} of M and a simple foliation \mathcal{F}_0 of N and we denote by $p : M \rightarrow M/\mathcal{F}$ and $p_0 : N \rightarrow N/\mathcal{F}_0$ the canonical projections. Thus, we have the following commutative diagram:

$$\begin{array}{ccc}
 N & \xrightarrow{i} & M \\
 p_0 \downarrow & & \downarrow p \\
 N/\mathcal{F}_0 & \xrightarrow{\varphi} & M/\mathcal{F}
 \end{array} \tag{36}$$

Since any leaf of \mathcal{F}_0 is a connected component of the intersection between N and some leaf of \mathcal{F} , we can always suppose, under some clean intersection condition, that $\varphi : N/\mathcal{F}_0 \rightarrow M/\mathcal{F}$ is an immersion, locally injective.

We consider $L = D \oplus D^\perp$ and we suppose that L is a null Dirac structure for the triangular generalized Lie bialgebroid $((TM \times \mathbb{R}, (0, 1)), (T^*M \times \mathbb{R}, (-E, 0)), (\Lambda, E))$. By the hypothesis on D , we have that L is also reducible and that 1 is an L -admissible function. Then, by Corollary 4.9, we get that L induces a Jacobi structure $(\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$ on M/\mathcal{F} and by Corollary 4.16 and Remark 4.15, we obtain that $p : (M, \Lambda, E) \rightarrow (M/\mathcal{F}, \Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$ is a Jacobi map. We consider the triangular generalized Lie bialgebroids $((T(M/\mathcal{F}) \times \mathbb{R}, (0, 1)), (T^*(M/\mathcal{F}) \times \mathbb{R}, (-E_{M/\mathcal{F}}, 0)), (\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}}))$ over M/\mathcal{F} and $((TN \times \mathbb{R}, (0, 1)), (T^*N \times \mathbb{R}, (0, 0)), (0, 0))$ over N . We note that any function $\bar{f} \in C^\infty(N, \mathbb{R})$ can be seen as the image by $(p \circ i)^*$ of a function $f \in C^\infty(M/\mathcal{F}, \mathbb{R})$, i.e., $\bar{f} = (p \circ i)^* f$. Since \mathcal{F} is a regular foliation, p has constant rank, thus the map $p \circ i : N \rightarrow M/\mathcal{F}$ has also constant rank. Hence, the map $\Phi : TN \times \mathbb{R} \rightarrow T(M/\mathcal{F}) \times \mathbb{R} \cong (TM \times \mathbb{R})/D$ defined, for any $(X, \bar{f}) \in \Gamma(TN \times \mathbb{R})$, $\bar{f} = (p \circ i)^* f$ with $f \in C^\infty(M/\mathcal{F}, \mathbb{R})$, by

$$\Phi(X, \bar{f}) = ((p \circ i)_* X, f), \tag{37}$$

can be considered as a Lie algebroid morphism of constant rank such that $\Phi^*(0, 1) = (0, 1)$ and $\ker \Phi = D \cap (TN \times \mathbb{R}) = D_0$. Therefore, by Proposition 5.1, there exists a pull-back Dirac structure L_0 for the triangular generalized Lie bialgebroid $((TN \times \mathbb{R}, (0, 1)), (T^*N \times \mathbb{R}, (0, 0)), (0, 0))$ with characteristic pair $(D_0, (\Lambda_N, E_N))$ satisfying $\Phi(\Lambda_N, E_N) = (\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$ if and only if $\text{Im}(\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})^\# \subseteq \text{Im } \Phi$ holds on $T(M/\mathcal{F}) \times \mathbb{R}$, i.e.,

$$\Gamma((\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})^\#(D^\perp)) \subseteq \{((p \circ i)_* X, f) / X \in \Gamma(TN) \text{ and } f \in C^\infty(M/\mathcal{F}, \mathbb{R})\}. \tag{38}$$

But, $p : (M, \Lambda, E) \rightarrow (M/\mathcal{F}, \Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$ being a Jacobi map, $(\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}}) = p_*(\Lambda, E)$. Thus, on the submanifold $N \subseteq M$, by identifying $i_*(TN)$ with TN , condition (38) is equivalent to

$$(\Lambda, E)^\#(D^\perp) \subseteq TN \times \mathbb{R} + D. \tag{39}$$

On the other hand, since $D = \pi(D) \times \{0\}$, $D^\perp = \pi(D)^\perp \times \mathbb{R}$, consequently, (39) is equivalent to

$$\Lambda^\#(\pi(D)^\perp) \subseteq TN + \pi(D) \quad \text{and} \quad E|_N \in \Gamma(TN + \pi(D)). \tag{40}$$

Also, since $L_0 = D_0 \oplus \text{graph}(\Lambda_N, E_N)^\#|_{D_0^\perp}$ is a reducible Dirac structure of $((TN \times \mathbb{R}) \oplus (T^*N \times \mathbb{R}), (0, 1) + (0, 0))$ and $1 \in C_{L_0}^\infty(N, \mathbb{R})$, it induces a Jacobi structure $(\Lambda_{N/\mathcal{F}_0}, E_{N/\mathcal{F}_0})$ on N/\mathcal{F}_0 (see Corollary 4.9) such that $(\Lambda_{N/\mathcal{F}_0}, E_{N/\mathcal{F}_0}) = p_{0*}(\Lambda_N, E_N)$ (see Corollary 3.5 and its geometric interpretation). By the above results and by the commutativity of the diagram (36), we obtain:

$$\begin{aligned} (\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}}) &= \Phi(\Lambda_N, E_N) \\ &= ((p \circ i)_* \Lambda_N, (p \circ i)_* E_N) \\ &= ((\varphi \circ p_0)_* \Lambda_N, (\varphi \circ p_0)_* E_N) \\ &= (\varphi_*(p_{0*} \Lambda_N), \varphi_*(p_{0*} E_N)) \\ &= (\varphi_* \Lambda_{N/\mathcal{F}_0}, \varphi_* E_{N/\mathcal{F}_0}) \\ &= \varphi_*(\Lambda_{N/\mathcal{F}_0}, E_{N/\mathcal{F}_0}), \end{aligned} \tag{41}$$

which means that $\varphi : (N/\mathcal{F}_0, \Lambda_{N/\mathcal{F}_0}, E_{N/\mathcal{F}_0}) \rightarrow (M/\mathcal{F}, \Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$ is a Jacobi map.

The above study led us to the following theorem:

Theorem 5.2 (Reduction Theorem of Jacobi manifolds). *Let (M, Λ, E) be a Jacobi manifold, $N \subseteq M$ a submanifold of M , $D \subset TM \times \mathbb{R}$ a Lie subalgebroid of $(TM \times \mathbb{R}, [,], \pi)$ that has only sections of type $(X, 0)$ and $D_0 = D \cap (TN \times \mathbb{R})$. We suppose that D and D_0 define, respectively, a simple foliation \mathcal{F} of M and a simple foliation \mathcal{F}_0 of N and that $L = D \oplus D^\perp$ is a reducible Dirac structure for the triangular generalized Lie bialgebroid $((TM \times \mathbb{R}, (0, 1)), (T^*M \times \mathbb{R}, (-E, 0)), (\Lambda, E))$. Then, the following two statements are equivalent.*

1. *There exists a Jacobi structure $(\Lambda_{N/\mathcal{F}_0}, E_{N/\mathcal{F}_0})$ on N/\mathcal{F}_0 such that*

$$p_*(\Lambda, E) = \varphi_*(\Lambda_{N/\mathcal{F}_0}, E_{N/\mathcal{F}_0}).$$

2. *$\Lambda^\#(\pi(D)^\perp) \subseteq TN + \pi(D)$ holds on N and $E|_N \in \Gamma(TN + \pi(D))$.*

Remarks 5.3.

1. We remark that, in the context of the Reduction Theorem 5.2, the initial Jacobi manifold (M, Λ, E) and the reduced Jacobi manifold $(N/\mathcal{F}_0, \Lambda_{N/\mathcal{F}_0}, E_{N/\mathcal{F}_0})$ are connected by means of the Jacobi manifold $(M/\mathcal{F}, \Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$ with two Jacobi maps.
2. Reduction Theorem 5.2 holds for any reducible Dirac structure $L \subset (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ having a characteristic pair $(D, (\Lambda', E'))$, i.e., $L = D \oplus \text{graph}((\Lambda', E')^\#|_{D^\perp})$, such that D has only sections of type $(X, 0)$, so $1 \in C_L^\infty(M, \mathbb{R})$. Effectively, by Corollary 4.9 we get that L induces a Jacobi structure $(\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$ on M/\mathcal{F} which is exactly the induced Jacobi structure by $(\Lambda + \Lambda', E + E')$ (see the geometric interpretation of Corollary 3.5). If $(\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$ verifies (38) or, equivalently, $(\Lambda + \Lambda', E + E')$ verifies (40), then, by Proposition 5.1, there exists a pull-back Dirac structure L_0 for $((TN \times \mathbb{R}, (0, 1)), (T^*N \times \mathbb{R}, (0, 0)), (0, 0))$ with characteristic pair $(D_0, (\Lambda_N, E_N))$ such that $\Phi(\Lambda_N, E_N) = (\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$. The reducible Dirac subbundle $L_0 \subset (TN \times \mathbb{R}) \oplus (T^*N \times \mathbb{R})$ induces a Jacobi structure $(\Lambda_{N/\mathcal{F}_0}, E_{N/\mathcal{F}_0})$ on N/\mathcal{F}_0 and

$$p_{0*}(\Lambda_N, E_N) = (\Lambda_{N/\mathcal{F}_0}, E_{N/\mathcal{F}_0}).$$

Applying the calculus of (41) to the relation $(\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}}) = \Phi(\Lambda_N, E_N)$, we conclude that $\varphi: N/\mathcal{F}_0 \rightarrow M/\mathcal{F}$ is always a Jacobi map. But, the projection $p: M \rightarrow M/\mathcal{F}$ is a Jacobi map if and only if L is a null Dirac structure, fact which is equivalent to $(\Lambda', E') \equiv 0 \pmod{D}$.

3. As we have mentioned in introduction, there are already several works treating the Jacobi reduction problem. These results are, grosso-modo, equivalent to the ones established by the second author in [27] and, independently, by K. Mikami in [26]. They establish a geometric reduction theorem for Jacobi manifolds by extending the previous one proved by Marsden and Ratiu for Poisson manifolds [25], without mentioning Dirac structures. Precisely, they prove:

Theorem 5.4. *Let (M, Λ, E) be a Jacobi manifold, N a submanifold of M and Δ a vector subbundle of $T_N M$ such that: (i) $\Delta \cap TN$ defines a simple foliation \mathcal{F}_0 of N ; (ii) for any $f, g \in C^\infty(M, \mathbb{R})$ with differentials δf and δg , restricted to N , vanishing on Δ , the differential $\delta\{f, g\}_{(\Lambda, E)}$, restricted to N , vanishes on N . Then, (Λ, E) induces a unique Jacobi structure $(\Lambda_{N/\mathcal{F}_0}, E_{N/\mathcal{F}_0})$ on N/\mathcal{F}_0 if and only if $\Lambda^\#(\Delta^\perp) \subseteq TN + \Delta$ holds on N and $E|_N \in \Gamma(TN + \Delta)$. The associated bracket of $(\Lambda_{N/\mathcal{F}_0}, E_{N/\mathcal{F}_0})$ on $C^\infty(N/\mathcal{F}_0, \mathbb{R})$ is given, for any $f_0, g_0 \in C^\infty(N/\mathcal{F}_0, \mathbb{R})$ and any differentiable extensions f of $f_0 \circ p_0$ and g of $g_0 \circ p_0$ with differentials δf and δg , restricted to N , vanish on Δ , by $\{f_0, g_0\}_{(\Lambda_{N/\mathcal{F}_0}, E_{N/\mathcal{F}_0})} \circ p_0 = \{f, g\}_{(\Lambda, E)} \circ i$, where $p_0: N \rightarrow N/\mathcal{F}_0$ is the canonical projection and $i: N \rightarrow M$ is the canonical inclusion of N into M .*

We remark that the above theorem is slightly different from Theorem 5.2. In Theorem 5.2 we suppose that we have two simple foliations, a foliation \mathcal{F} of the initial phase space M determined by $\pi(D)$ and a foliation \mathcal{F}_0 of the considered submanifold N of M determined by $\pi(D_0) = \pi(D) \cap TN$, while in Theorem 5.4 we only suppose that we have a subbundle Δ of $T_N M$ such that $\Delta \cap TN$ defines a simple foliation of N , also denoted by \mathcal{F}_0 . But, in both theorems, the reducibility condition

$$\Lambda^\#(\pi(D)^\perp) \subseteq TN + \pi(D) \text{ holds on } N \text{ and } E|_N \in \Gamma(TN + \pi(D))$$

is exactly the same. Thus, it is natural to ask: *What is the advantage of using Dirac structures in the study of Jacobi reduction problem?* The answer can be founded in Remarks 1 and 2 of this paragraph. By using reducible Dirac structures in this study, we establish the existence, not only, of a reduced Jacobi manifold $(N/\mathcal{F}_0, \Lambda_{N/\mathcal{F}_0}, E_{N/\mathcal{F}_0})$, but also of a quotient Jacobi manifold $(M/\mathcal{F}, \Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$ which is always related with $(N/\mathcal{F}_0, \Lambda_{N/\mathcal{F}_0}, E_{N/\mathcal{F}_0})$ by means of a Jacobi map; very important fact when we treat reduction problems. On the other hand, this study, in this framework, allows us to investigate, in a future paper, the *Dirac reduction problem* and its relation with the one of Jacobi, Poisson and symplectic structures.

6. Applications and examples

6.1. Jacobi submanifolds

From Theorem 5.2 we obtain sufficient conditions under which a Jacobi structure (Λ, E) on a differentiable manifold M induces a Jacobi structure on a submanifold N of M . Effectively, under the assumptions of the above mentioned theorem, if $D_0 = D \cap (TN \times \mathbb{R}) = \{(0, 0)\}$ and $(\Lambda, E)^\#(D^\perp) \subseteq TN \times \mathbb{R} + D$ holds on N , then there exists a $(TN \times \mathbb{R})$ -bivector field (Λ_N, E_N) on N such that

$L_0 = D_0 \oplus \text{graph}(\Lambda_N, E_N)^\#|_{D_0^\perp} = \text{graph}(\Lambda_N, E_N)^\#$ is a reducible Dirac structure for the triangular generalized Lie bialgebroid $((TN \times \mathbb{R}, (0, 1)), (T^*N \times \mathbb{R}, (0, 0)), 0)$ and $\Phi(\Lambda_N, E_N) = (\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}})$. But, the fact “ $L_0 = \text{graph}(\Lambda_N, E_N)^\#$ is Dirac for $((TN \times \mathbb{R}, (0, 1)), (T^*N \times \mathbb{R}, (0, 0)), 0)$ ” is equivalent to the fact “ (Λ_N, E_N) is a Jacobi structure on N ” (see Proposition 5.2 in [30]) and

$$\begin{aligned} (\Lambda_{M/\mathcal{F}}, E_{M/\mathcal{F}}) = \Phi(\Lambda_N, E_N) &\Leftrightarrow p_*(\Lambda, E) = (p \circ i)_*(\Lambda_N, E_N) \\ &\Leftrightarrow p_*((\Lambda, E) - i_*(\Lambda_N, E_N)) = (0, 0). \end{aligned}$$

By the last equality we conclude either that $(\Lambda, E) - i_*(\Lambda_N, E_N) = (0, 0) \Leftrightarrow (\Lambda, E) = i_*(\Lambda_N, E_N)$, i.e., $i : (N, \Lambda_N, E_N) \rightarrow (M, \Lambda, E)$ is a Jacobi map, or that $\Lambda = i_*\Lambda_N + \sum_{j=1}^k X_j \wedge Y_j$ and $E = i_*E_N + X$, where $X_j, X \in \Gamma(\pi(D)), Y_j \in \Gamma(TM), j = 1, \dots, k$, are convenient vector fields such that $[\Lambda, \Lambda] = -2E \wedge \Lambda$ and $[E, \Lambda] = 0$.

Particular cases

- (a) When $D = \{(0, 0)\}$, then $D^\perp = T^*M \times \mathbb{R}$, and they verify the assumptions of Theorem 5.2. Condition $D_0 = D \cap (TN \times \mathbb{R}) = \{(0, 0)\}$ is automatically satisfied and the reducibility condition (40) takes the form

$$\Lambda^\#(T^*M) \subseteq TN \text{ on } N \text{ and } E|_N \in \Gamma(TN),$$

which is exactly the condition given in [4] and [23] for the submanifolds N of (M, Λ, E) of the first kind.

- (b) When $D = (\Lambda, E)^\#((TN \times \mathbb{R})^\perp)$, we have that D has only sections of type $(X, 0)$ if and only if $E|_N \in \Gamma(TN)$ and $D_0 = D \cap (TN \times \mathbb{R}) = \{(0, 0)\}$ if and only if $TN \cap \Lambda^\#(TN^\perp) = \{0\}$. Thus, under the assumptions

$$TN \cap \Lambda^\#(TN^\perp) = \{0\} \text{ on } N \text{ and } E|_N \in \Gamma(TN), \tag{42}$$

by a simple calculation we show that $D = \Lambda^\#(TN^\perp) \times \{0\}$ is a Lie subalgebroid of $(TM \times \mathbb{R}, [,], \pi)$ if and only if Λ belongs to the ideal generated by the space of smooth sections of TN . Also, since $\Lambda^\#((\Lambda^\#(TN^\perp))^\perp) \subseteq TN$ and $E|_N \in \Gamma(TN)$, it is easy to prove that $D^\perp = (\Lambda^\#(TN^\perp))^\perp \times \mathbb{R}$ is a Lie subalgebroid of $(T^*M \times \mathbb{R}, [,],_{(\Lambda, E)}, \pi \circ (\Lambda, E)^\#)$.

Consequently, if (42) holds and Λ belongs to the ideal generated by the space of smooth sections of TN , then we have that the requirements of Theorem 5.2 as the reducibility condition (39) are verified, therefore (Λ, E) induces a Jacobi structure on N . We note that conditions (42) are exactly those given in [12].

6.2. Reduction of Jacobi manifolds with symmetry

Let (M, Λ, E) be a Jacobi manifold, G a connected Lie group acting on M by a Jacobi action, \mathcal{G} the Lie algebra of G , \mathcal{G}^* the dual space of \mathcal{G} and $J : M \rightarrow \mathcal{G}^*$ an Ad^* -equivariant moment map for the considering action. Let D be the vector subbundle of $TM \times \mathbb{R}$ formed by the pairs $(X_M, 0)$, where X_M is the fundamental vector field on M associated to an element $X \in \mathcal{G}$, and D^\perp its conormal bundle which is $D^\perp = \{X_M \in TM/X \in \mathcal{G}\}^\perp \times \mathbb{R}$. It is easy to check that D and D^\perp are Lie subalgebroids of $(TM \times \mathbb{R}, [,], \pi)$ and $(T^*M \times \mathbb{R}, [,],_{(\Lambda, E)}, \pi \circ (\Lambda, E)^\#)$, respectively. (For D^\perp , we take into account that the action of G on M is a Jacobi action, thus, for any fundamental vector field X_M on M , $\mathcal{L}_{X_M}\Lambda = 0$ and $\mathcal{L}_{X_M}E = 0$.) Consequently, $L = D \oplus D^\perp$ is a Dirac subbundle of $((TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R}), (0, 1) +$

$(-E, 0)$). We suppose that 0 is a weakly regular value of the moment map J . Hence, $N = J^{-1}(0)$ is a submanifold of M and $D_0 = D \cap (TN \times \mathbb{R}) = \{(X_M, 0)/X \in \mathcal{G}_0\}$, where \mathcal{G}_0 is the Lie algebra of the isotropy subgroup G_0 of 0. Also, we suppose that $\pi(D)$ and $\pi(D_0)$ define, respectively, a simple foliation \mathcal{F} of M and a simple foliation \mathcal{F}_0 of N . Since, $(\Lambda, E)^\#(D^\perp) \subseteq TN \times \mathbb{R} + D$ holds on N , from the Reduction [Theorem 5.2](#) we get that (Λ, E) induces a Jacobi structure on N/\mathcal{F}_0 . For more details, see [\[9,26,28\]](#).

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References

- [1] A. Cannas da Silva, A. Weinstein, *Geometric Models for Noncommutative Algebras*, University of California Berkeley Mathematics Lecture Notes, vol. 10, Amer. Math. Soc., Providence, RI, 1999.
- [2] T. Courant, A. Weinstein, Beyond Poisson structures, in: *Séminaire Sud-Rhodanien de Géométrie, Travaux en cours* 27, Hermann, Paris, 1988, pp. 39–49.
- [3] T. Courant, Dirac manifolds, *Trans. Amer. Math. Soc.* 319 (1990) 631–661.
- [4] P. Dazord, A. Lichnerowicz, C.-M. Marle, Structure locale des variétés de Jacobi, *J. Math. Pures Appl.* 70 (1991) 101–152.
- [5] V. Drinfeld, Quantum groups, in: *Proceedings of the International Congress of Mathematicians*, Berkeley, Amer. Math. Soc., Providence, RI, 1986, pp. 798–820.
- [6] J. Grabowski, Abstract Jacobi and Poisson structures, *J. Geom. Phys.* 9 (1992) 45–73.
- [7] J. Grabowski, G. Marmo, Jacobi structures revisited, *J. Phys. A Math. Gen.* 34 (2001) 10975–10990.
- [8] J. Grabowski, G. Marmo, The graded Jacobi algebras and (co)homology, *J. Phys. A Math. Gen.* 36 (2003) 161–181.
- [9] A. Ibort, M. de Leon, G. Marmo, Reduction of Jacobi manifolds, *J. Phys. A Math. Gen.* 30 (1997) 2783–2798.
- [10] D. Iglesias, J.C. Marrero, Generalized Lie bialgebroids and Jacobi structures, *J. Geom. Phys.* 40 (2001) 176–200.
- [11] D. Iglesias, B. Lopez, J.C. Marrero, E. Padrón, Triangular generalized Lie bialgebroids: Homology and cohomology theories, in: *Lie Algebroids*, in: *Banach Center Publications*, vol. 54, Polish Acad. Sci., Warszawa, 2001, pp. 111–133.
- [12] D. Iglesias Ponte, $\mathcal{E}^1(M)$ -Dirac structures and Jacobi structures, in: *Differential Geometry and its Applications, Proc. Conf. Opava 2001*, Silesian Univ. Opava, Opava, 2001, pp. 275–283.
- [13] Y. Kerbrat, Z. Souici-Benhammedi, Variétés de Jacobi et groupoïdes de contact, *C. R. Acad. Sci. Paris, Sér. I* 317 (1993) 81–86.
- [14] A. Kirillov, Local Lie algebras, *Russian Math. Surv.* 31 (1976) 55–75.
- [15] Y. Kosmann-Schwarzbach, Exact Gerstenhaber algebras and Lie bialgebroids, *Acta Appl. Math.* 41 (1995) 153–165.
- [16] J.-L. Koszul, Crochet de Schouten–Nijenhuis et cohomologie, in: *Élie Cartan et les Mathématiques d’aujourd’hui*, *Astérisque*, Numéro Hors Série, 1985, pp. 257–271.
- [17] A. Lichnerowicz, Les variétés de Jacobi et leurs algèbres de Lie associées, *J. Math. Pures Appl.* 57 (1978) 453–488.
- [18] Z.-J. Liu, A. Weinstein, P. Xu, Manin triples for Lie bialgebroids, *J. Differential Geom.* 45 (1997) 547–574.
- [19] Z.-J. Liu, A. Weinstein, P. Xu, Dirac structures and poisson homogeneous spaces, *Comm. Math. Phys.* 192 (1998) 121–144.
- [20] Z.-J. Liu, Some remarks on Dirac structures and Poisson reductions, in: *Poisson Geometry*, in: *Banach Center Publications*, vol. 51, Polish Acad. Sci., Warszawa, 2000, pp. 165–173.
- [21] K. Mackenzie, *Lie Groupoids and Lie Algebroids in Differential Geometry*, *London Math. Soc. Lecture Notes Ser.*, vol. 124, Cambridge University Press, Cambridge, 1987.
- [22] K. Mackenzie, P. Xu, Lie bialgebroids and Poisson groupoids, *Duke Math. J.* 73 (1994) 415–452.
- [23] Ch.-M. Marle, On submanifolds and quotients of Poisson and Jacobi manifolds, in: *Poisson Geometry*, in: *Banach Center Publications*, vol. 51, Polish Acad. Sci., Warszawa, 2000, pp. 197–209.

- [24] Ch.-M. Marle, Differential calculus on a Lie algebroid and Poisson manifolds, The J.A. Pereira da Silva Birthday Schrift, Textos de Matemática 32, Departamento de Matemática da Universidade de Coimbra, Portugal, 2002, pp. 83–149, <http://www.math.jussieu.fr/~marle/>.
- [25] J. Marsden, T. Ratiu, Reduction of Poisson manifolds, *Lett. Math. Phys.* 11 (1986) 161–169.
- [26] K. Mikami, Reduction of local Lie algebra structures, *Proc. Amer. Math. Soc.* 105 (1989) 686–691.
- [27] J.M. Nunes da Costa, Réduction des variétés de Jacobi, *C. R. Acad. Sci. Paris Sér. I* 308 (1989) 101–103.
- [28] J.M. Nunes da Costa, Une généralisation, pour les variétés de Jacobi, du théorème de Marsden–Weinstein, *C. R. Acad. Sci. Paris Sér. I* 310 (1990) 411–414.
- [29] J.M. Nunes da Costa, Compatible Jacobi manifolds: geometry and reduction, *J. Phys. A Math. Gen.* 31 (1998) 1025–1033.
- [30] J.M. Nunes da Costa, J. Clemente-Gallardo, Dirac structures for generalized Lie bialgebroids, *J. Phys. A Math. Gen.* 37 (2004) 2671–2692.
- [31] D. Roytenberg, Courant algebroids, derived brackets and even symplectic supermanifolds, PhD Thesis, University of California, Berkeley 1999, Math.DG/9910078.
- [32] A. Wade, Conformal Dirac structures, *Lett. Math. Phys.* 53 (2000) 331–348.