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On the linear functionals associated to linearly related sequences of orthogonal polynomials

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Abstract

An inverse problem is solved, by stating that the regular linear functionals $\mathbf{u}$ and $\mathbf{v}$ associated to linearly related sequences of monic orthogonal polynomials $(P_n)_n$ and $(Q_n)_n$, respectively, in the sense

$$P_n(x) + \sum_{i=1}^{N} r_{i,n} P_{n-i}(x) = Q_n(x) + \sum_{i=1}^{M} s_{i,n} Q_{n-i}(x)$$

for all $n = 0, 1, 2, \ldots$ (where $r_{i,n}$ and $s_{i,n}$ are complex numbers satisfying some natural conditions), are connected by a rational modification, i.e., there exist polynomials ϕ and ψ, with degrees M and N, respectively, such that $\phi\mathbf{u} = \psi\mathbf{v}$. We also make some remarks concerning the corresponding direct problem, stating a characterization theorem in the case $N = 1$ and $M = 2$. As an example, we give a linear relation of the above type involving Jacobi polynomials with distinct parameters.

Keywords: Orthogonal polynomials; Moment linear functionals; Inverse problems; Locally convex spaces; Sobolev orthogonal polynomials

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1. Introduction and main result

In the last decade the study of the so-called Sobolev-type orthogonal polynomials have received much attention (see [10], where F. Marcellán and A. Ronveaux gave a updated list of 240 references on this subject). These are sequences of polynomials which are orthogonal with respect to nonstandard scalar inner products

$$(f,g) := \sum_{\nu=0}^{N} \int_{\mathbb{R}} f^{(\nu)} g^{(\nu)} \, \mathrm{d}\mu_\nu,$$

where $\mu_0, \ldots, \mu_N$ are positive Borel measures supported on the real line, their supports are infinite sets, and with finite moments, i.e., $\int_{\mathbb{R}} |x|^s \, \mathrm{d}\mu_\nu < \infty$ for all $s = 0, 1, 2, \ldots$ and $\nu = 0, \ldots, N$. Often the study of Sobolev-type orthogonal polynomials leads to a linear algebraic relation between the orthogonal polynomials associated to the measures $\mu_0, \ldots, \mu_N$. This suggests the study of the corresponding inverse problem—in the more general setting of moment linear functionals defined in the space $\mathcal{P}$ of all polynomials, with $\mathcal{P}$ and its dual $\mathcal{P}'$ carried with appropriate topologies—i.e., to state the relation between two regular linear functionals such that the corresponding sequences of orthogonal polynomials (OPS) are linearly related. Our main result is the following

Theorem 1.1. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be two regular functionals in* $\mathcal{P}'$*, and let* $\{P_n\}_{n \geqslant 0}$ *and* $\{Q_n\}_{n \geqslant 0}$ *be the corresponding monic OPS's, respectively. Assume that there exist nonnegative integer numbers* N *and* M*, and complex numbers* $r_{i,n}$ *and* $s_{k,n}$ $(i = 1, \ldots, N;\ k = 1, \ldots, M;\ n = 0, 1, \ldots)$*, such that the structure relation*

$$P_n(x) + \sum_{i=1}^{N} r_{i,n} P_{n-i}(x) = Q_n(x) + \sum_{i=1}^{M} s_{i,n} Q_{n-i}(x) \tag{1}$$

holds for all $n = 0, 1, 2, \ldots$ *. Further, assume that*

$$r_{N,M+N} \neq 0, \qquad s_{M,M+N} \neq 0, \qquad \det\bigl\{[\alpha_{i,j}]_{i,j=1}^{N+M}\bigr\} \neq 0, \tag{2}$$

where

$$\alpha_{ij} := \begin{cases} r_{j-i,j-1}, & \text{if } 1 \leqslant i \leqslant M \wedge i \leqslant j \leqslant N+i, \\ s_{j-i+M,j-1}, & \text{if } M+1 \leqslant i \leqslant M+N \wedge i-M \leqslant j \leqslant i, \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

with the convention $r_{0,k} = s_{0,\nu} = 1$ *for all* $k = 0, \ldots, M$ *and* $\nu = 0, \ldots, N$*. Then there exist two polynomials* ϕ *and* ψ*, with* $\deg \phi = M$ *and* $\deg \psi = N$*, such that*

$$\phi \mathbf{u} = \psi \mathbf{v}. \tag{4}$$

These polynomials ϕ *and* ψ *can be constructed explicitly.*

The left multiplication of a functional by a polynomial in (4) is defined in the usual distributional sense, i.e.,

$$\langle \phi \mathbf{u}, f \rangle := \langle \mathbf{u}, \phi f \rangle, \qquad f \in \mathcal{P},$$

where $\langle , \rangle$ means the duality bracket.

Theorem 1.1 generalizes and is motivated by some results contained in a recent paper by M. Alfaro, F. Marcellán, A. Peña, and M.L. Rezola [1], where the authors characterized linearly related sequences of orthogonal polynomials involving linear combinations with exactly two elements—corresponding to the case $M = N = 1$ in (1)—in terms of their functionals. Further, in [2] the above authors gave a complete discussion of the regularity conditions involving a rational modification as $(x - a)\mathbf{u} = \lambda(x - b)\mathbf{v}$ between two moment linear functionals $\mathbf{u}$ and $\mathbf{v}$ (thus, corresponding again to the case $M = N = 1$). Special cases of these type of relations were treated in [7,9,14]. Theorem 1.1 gives a solution for the inverse problem associated to (1). The corresponding direct problem, i.e., to find the relation between the polynomials starting from a relation such as (4) between the functionals (rational modification) is not completely solved here. In fact, we only state that, in general, a relation such as (4) between regular functionals $\mathbf{u}$ and $\mathbf{v}$, associated to monic OPS's (P_n) and (Q_n), always implies two structure relations (which, in principle, are independent) close to (1), i.e., (P_n) and (Q_n) are linearly related by (16) and (17) in bellow. However, the number of summands in both sides of (16) and (17) is not optimal, when compared with the number of summands in both sides of (1), namely $\deg\phi + 1$ in the sum involving the Q_ν's, and $\deg\psi + 1$ in the sum involving the P_ν's. We conjecture that an optimal structure relation can be obtained (under certain suitable conditions). In fact, it holds when $\deg\phi = \deg\psi = 1$, as follows from the results in [1]. Further, in the present paper we prove (again under certain natural conditions) that it also holds when $\deg\phi = 2$ and $\deg\psi = 1$ (an so, by symmetry, also when $\deg\phi = 1$ and $\deg\psi = 2$).

The paper is organized as follows. In Section 2 we review some general results needed for the proof of Theorem 1.1. This proof will be made in Section 3, and in Section 4 we will discuss the converse of the statement in Theorem 1.1. In Section 5 we state a characterization theorem in the case $N = 1$ and $M = 2$ which we use to establish a linear relation involving Jacobi polynomials (which we did no found in the literature). Finally, in Section 6 we see that the same technique applies to give an alternative proof to (a slightly modification of) Theorem 2.4 in [1] and we discuss integral representations for the involved moment linear functionals.

2. Background

All the facts presented in this section concerning general aspects in the theory of orthogonal polynomials can be found in the standard textbooks by Szegö [18], Freud [6], or Chihara [5], and its connection with the theory of locally convex spaces in the papers [11,13,15] by Maroni. The needed general facts about the theory of locally convex spaces are contained, e.g., in the books by Treves [19] or Reed and Simon [17]. See also [16], for a review on these topics.

With the usual operations of addition and scalar multiplication, $\mathbb{C}[x]$ is a linear space which will be denoted by $\mathcal{P}$. $\mathcal{P}_n$ will denote the linear subspace of $\mathcal{P}$ of polynomials with degree less than or equal to n. Since in a finite dimensional vector space all the norms are equivalent, we may adopt (without loss of generality)

$$\|f\|_n := \sum_{\nu=0}^{n} |a_\nu|$$

for an arbitrary polynomial $f \in \mathcal{P}_n$ such that $f(x) \equiv \sum_{\nu=0}^{n} a_\nu x^\nu$. Then $(\mathcal{P}_n, \|\cdot\|_n)$ is a Banach space (remember that a finite dimensional normed space is always complete). It is clear that $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for all $n = 0, 1, 2, \ldots$ (strict inclusion) and the topology of each $\mathcal{P}_n$ is identical to the one induced by $\mathcal{P}_{n+1}$. Further, each $\mathcal{P}_n$ is closed in $\mathcal{P}_{n+1}$. Therefore, since

$$\mathcal{P} = \bigcup_{n=0}^{+\infty} \mathcal{P}_n,$$

the theory of locally convex spaces leads us to consider in $\mathcal{P}$ its natural topology of the hyper-strict inductive limit (the term "hyper-strict" is used for the strict inductive limit $X = \operatorname{ind}\lim_n X_n$ of a family of l.c.s.'s $(X_n)_n$ when each X_n is closed in X_{n+1}) defined by the sequence $\{(\mathcal{P}_n, \|\cdot\|_n)\}_{n\in\mathbb{N}_0}$ (cf. Maroni [13,15]; Treves [19, pp. 126–131]).

Let $\mathcal{P}^*$ be the algebraic dual of $\mathcal{P}$, i.e., the set of all linear functionals $\mathbf{u} : \mathcal{P} \to \mathbb{C}$. Given an $\mathbf{u} \in \mathcal{P}^*$, the action of $\mathbf{u}$ over a polynomial f will be denoted by $\langle \mathbf{u}, f\rangle$. The topological dual of $\mathcal{P}$ will be represented by $\mathcal{P}'$ and the topology to be considered in this space is the dual weak topology, which, by definition, is characterized by the family of seminorms $\wp := \{p_f : f \in \mathcal{P}\}$, where

$$p_f(\mathbf{u}) := \left|\langle \mathbf{u}, f\rangle\right|, \quad \mathbf{u} \in \mathcal{P}',\ f \in \mathcal{P}$$

(see Treves [19, p. 197]). Alternatively, setting $\Im := \{|\cdot|_n : n \in \mathbb{N}_0\}$, where

$$|\mathbf{u}|_n := \sup_{0\leqslant\nu\leqslant n} \left|\langle \mathbf{u}, x^\nu\rangle\right|, \quad \mathbf{u} \in \mathcal{P}',\ n = 0, 1, 2, \ldots,$$

it can be proved that the families of seminorms $\wp$ and $\Im$ on $\mathcal{P}'$ are equivalent (cf. Maroni [15]). Hence $\mathcal{P}'$ is a Fréchet space. Further, the set equality

$$\mathcal{P}' = \mathcal{P}^* \tag{5}$$

holds, which follows essentially from

$$\left|\langle \mathbf{u}, f\rangle\right| \leqslant |\mathbf{u}|_n \|f\|_n \quad (\mathbf{u} \in \mathcal{P}^*,\ f \in \mathcal{P}_n),\ n = 0, 1, 2, \ldots,$$

and taking into account general facts concerning continuity in inductive limit topologies.

Any sequence of polynomials $\{f_n\}_{n\geqslant 0}$ such that $\deg f_n = n$ for all n will be called a simple set. Let $\{R_n\}_{n\geqslant 0}$ be a simple set of polynomials. Since it is an (algebraic) basis for $\mathcal{P}$, we can consider the corresponding dual basis in $\mathcal{P}^*$, say $\{\mathbf{a}_n\}_{n\geqslant 0}$, where, by definition, $\mathbf{a}_n : \mathcal{P} \to \mathbb{C}$ is the linear functional characterized by

$$\langle \mathbf{a}_n, R_\nu\rangle := \delta_{n,\nu} \quad (n, \nu = 0, 1, 2, \ldots),$$

$\delta_{n,\nu}$ being the usual Kronecker symbol. Under these conditions, making use of (5) it can be shown that any $\mathbf{v} \in \mathcal{P}^*$ admits a Fourier-type representation such as

$$\mathbf{v} = \sum_{n=0}^{\infty} \lambda_n \mathbf{a}_n, \quad \lambda_n := \langle \mathbf{v}, R_n\rangle\ (n = 0, 1, 2, \ldots), \tag{6}$$

in the sense of the weak dual topology in $\mathcal{P}'$ (see Maroni [14]).

Next we recall the concept of orthogonality. Let $\mathbf{u} \in \mathcal{P}^*$. A sequence of polynomials $(P_n)_n$ is said to be an orthogonal polynomial sequence (OPS) with respect to $\mathbf{u}$ if it is a simple set and

$$\langle \mathbf{u}, P_n P_m \rangle = k_n \delta_{n,m}, \quad n, m = 0, 1, 2, \ldots,$$

where k_n is a nonzero complex number for each n. When there exists an OPS with respect to $\mathbf{u}$, $\mathbf{u}$ is said to be regular or quasi-definite. Without loss of generality, usually one works with monic orthogonal polynomial sequences (MOPS). Any MOPS $\{P_n\}_{n\geqslant 0}$ is characterized by a three-term recurrence relation

$$x P_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n = 0, 1, 2, \ldots,$$

with initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$, where $\{\beta_n\}_{n\geqslant 0}$ and $\{\gamma_n\}_{n\geqslant 0}$ are sequences of complex numbers such that $\gamma_n \neq 0$ for all $n = 1, 2, \ldots$. This fact is known as Favard's theorem (see, e.g., [5, p. 21]) or spectral theorem for orthogonal polynomials.

Every MOPS $(P_n)_n$ is a simple set of polynomials, hence it has an associated dual basis in $\mathcal{P}'$. A very important fact asserts that if $\mathbf{u}$ is the regular moment linear functional in $\mathcal{P}$ corresponding to $(P_n)_n$, and $(\mathbf{a}_n)_n$ is the associated dual basis, then the functionals in the dual basis are explicitly given by (cf. [15])

$$\mathbf{a}_n = \frac{P_n}{\langle \mathbf{u}, P_n^2 \rangle} \mathbf{u}, \quad n = 0, 1, 2, \ldots. \tag{7}$$

This fact will of major importance in the proof of Theorem 1.1.

Finally, we would like to point out some remarks concerning the so-called distributional approach to the theory of orthogonal polynomials. In fact, orthogonal polynomials can be studied from many different points of view, according to the motivations of the authors dealing with the subject. However, from an algebraic point of view, it is very useful to consider the moment linear functional with respect to which a given sequence of polynomials is orthogonal as an element of the topological dual space $\mathcal{P}'$ of the space $\mathcal{P}$, with the above topologies. Notice that, according to (5) every linear functional defined in $\mathcal{P}$ is continuous. (This property never holds in an infinite dimensional normed space, since in such a space one can always ensure the existence of a linear functional which is not continuous—a well-known fact which can be proved by using Zorn's lemma—hence $\mathcal{P}$ is not normable; in fact, it is not metrizable.) Therefore we have the possibility of giving a meaning to the convergence of any sequence of functionals in $\mathcal{P}^*$, in the sense of the weak topology in $\mathcal{P}'$. In particular, a Fourier-type expansion as in (6) of any given linear functional as a linear combination (finite or not) of the elements of any dual basis corresponding to any simple set in $\mathbb{C}[x]$ is always possible. This fact enables us to deal directly in the dual space $\mathcal{P}'$—by choosing some appropriate dual basis when the corresponding simple sets are sequences of orthogonal polynomials or not—instead of work in the space $\mathcal{P}$ of the polynomials. Thus, the use of continued fractions and recurrence relations—which are the main classical tools when working with algebraic properties of orthogonal sequences in $\mathcal{P}$—is replaced by the use of dual basis in $\mathcal{P}'$ of appropriate chosen simple sets in $\mathcal{P}$. The application of these ideas produces a natural way for the study of the algebraic properties of sequences of orthogonal polynomials. Further, in the so-called positive definite case, often it also leads to the necessary understanding of the problem in consideration in order to get the analytic properties (in particular, the orthogonality measure) of the polynomials.

3. Proof of Theorem 1.1

Denote by $\{\mathbf{a}_n\}_{n\geqslant 0}$ and $\{\mathbf{b}_n\}_{n\geqslant 0}$ the dual basis in $\mathcal{P}'$ corresponding to $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$, respectively. Then, according to (7), the relations

$$\mathbf{a}_n = \frac{P_n}{\langle \mathbf{u}, P_n^2\rangle}\mathbf{u}, \qquad \mathbf{b}_n = \frac{Q_n}{\langle \mathbf{v}, Q_n^2\rangle}\mathbf{v}, \quad n = 0, 1, 2, \ldots, \tag{8}$$

hold. In view of the structure relation, we can define a simple set of polynomials $\{R_n\}_{n\geqslant 0}$ by

$$R_n(x) := \sum_{i=0}^{N} r_{i,n} P_{n-i}(x) = \sum_{i=0}^{M} s_{i,n} Q_{n-i}(x), \quad n = 0, 1, 2, \ldots, \tag{9}$$

with the convention $r_{0,n} = s_{0,n} = 1$ for all $n = 0, 1, 2, \ldots$ (we always assume $P_k = Q_k \equiv 0$ if $k < 0$). Let $\{\mathbf{c}_n\}_{n\geqslant 0}$ be the dual basis corresponding to $\{R_n\}_{n\geqslant 0}$. Expanding $\mathbf{a}_n$ in terms of $\{\mathbf{c}_n\}_{n\geqslant 0}$, by (6) we can write

$$\mathbf{a}_n = \sum_{i\geqslant 0} \lambda_{n,i}\mathbf{c}_i, \quad n = 0, 1, 2, \ldots, \tag{10}$$

where, using (9),

$$\lambda_{n,i} = \langle \mathbf{a}_n, R_i\rangle = \sum_{j=0}^{N} r_{j,i}\langle \mathbf{a}_n, P_{i-j}\rangle = \begin{cases} r_{i-n,i}, & \text{if } n \leqslant i \leqslant n+N, \\ 0, & \text{otherwise.}\end{cases}$$

Therefore (10) reduces to

$$\mathbf{a}_n = \sum_{i=n}^{n+N} r_{i-n,i}\mathbf{c}_i, \quad n = 0, 1, 2, \ldots. \tag{11}$$

Similarly, if we start by expanding $\mathbf{b}_n$ in the basis $\{\mathbf{c}_n\}_{n\geqslant 0}$, we find

$$\mathbf{b}_n = \sum_{i=n}^{n+M} s_{i-n,i}\mathbf{c}_i, \quad n = 0, 1, 2, \ldots. \tag{12}$$

We now consider the equations (11) for $n = 0, 1, \ldots, M-1$ and also (12) for $n = 0, 1, \ldots, N-1$, to get the following system:

$$A \begin{bmatrix} \mathbf{c}_0 \\ \vdots \\ \mathbf{c}_{M-1} \\ \mathbf{c}_M \\ \vdots \\ \mathbf{c}_{M+N-1} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_0 \\ \vdots \\ \mathbf{a}_{M-1} \\ \mathbf{b}_0 \\ \vdots \\ \mathbf{b}_{N-1} \end{bmatrix}, \tag{13}$$

where $A := [\alpha_{ij}]_{i,j=1}^{N+M}$ and the α_{ij}'s are defined by (3). Since, by hypothesis, $\det A \neq 0$, solving (13) for $\mathbf{c}_i$ we get

$$\mathbf{c}_i = \ell_{i,0}\mathbf{a}_0 + \cdots + \ell_{i,M-1}\mathbf{a}_{M-1} + \ell_{i,M}\mathbf{b}_0 + \cdots + \ell_{i,M+N-1}\mathbf{b}_{N-1} \tag{14}$$

for every $i = 0, 1, \ldots, N + M - 1$, where (for every i and j) the complex numbers $\ell_{i,j}$ depend only on the coefficients $r_{k,\nu}$ ($k = 0, \ldots, N$; $\nu = 0, \ldots, N + M - 1$) and $s_{k,\nu}$ ($k = 0, \ldots, M$; $\nu = 0, \ldots, M + N - 1$). Now consider the system with two equations, one of which is (11) for $n = M$ and the other one is (12) for $n = N$, then multiply the first of these equations by $s_{M,M+N}$ and the second one by $r_{N,M+N}$, and then subtract the resulting equations (this will eliminate $\mathbf{c}_{N+M}$) to get

$$s_{M,M+N}\mathbf{a}_M - r_{N,M+N}\mathbf{b}_N = \ell_1\mathbf{c}_K + \cdots + \ell_{M+N-K}\mathbf{c}_{N+M-1}, \tag{15}$$

where $K := \min\{N, M\}$ and the numbers $\ell_1, \ldots, \ell_{M+N-K}$ are explicitly determined only in terms of $r_{\nu,M+\nu}$ ($\nu = 0, \ldots, N$) and $s_{\nu,N+\nu}$ ($\nu = 0, \ldots, M$). Finally, substituting $\mathbf{c}_K, \ldots, \mathbf{c}_{N+M-1}$ given by (14) in the right-hand side of (15) and taking into account (8), after some straightforward computations we get (4), ϕ and ψ being defined by

$$\phi(x) := \frac{s_{M,M+N}}{\langle \mathbf{u}, P_M^2\rangle} x^M + \pi_{M-1}(x), \qquad \psi(x) := \frac{r_{N,M+N}}{\langle \mathbf{v}, Q_N^2\rangle} x^N + \varrho_{N-1}(x),$$

with $\pi_{M-1} \in \mathcal{P}_{M-1}$ and $\varrho_{N-1} \in \mathcal{P}_{N-1}$. This concludes the proof. □

Remark 3.1. From an algebraic viewpoint (i.e., using duality arguments) the method of the proof of the previous theorem can be applied to get the relation between linear functionals associated with sequences of orthogonal polynomials such that their derivatives are linearly related. This is connected with some problems solved by S. Bonan, D. Lubinsky, and P. Nevai [3,4], among others. We point out that when derivatives of some of the involved families of orthogonal polynomials appear in the structure relation, the framework of this class of (inverse) problems is the theory of semiclassical orthogonal polynomials (see, e.g., [8,12,13,15]).

4. Some remarks on the converse of Theorem 1.1

The natural question at this point is to know when the converse of Theorem 1.1 holds, i.e., if a relation such as (4) between two regular moment linear functionals implies a structure relation of the type (1) between the corresponding MOPS's. In order to analyze this question, let $\mathbf{u}$ and $\mathbf{v}$ be two regular moment linear functionals, $(P_n)_n$ and $(Q_n)_n$ their associated MOPS's, respectively, and let us assume that there exist polynomials ϕ and ψ, with $\deg\phi = M$ and $\deg\psi = N$, such that (4) holds. Then, by computing the Fourier coefficients of ϕQ_n with respect to $(P_n)_n$, one gets

$$\phi Q_n = \sum_{\nu=n-N}^{n+M} r_{n,\nu} P_\nu \quad (n = 0, 1, 2, \ldots),$$

where $r_{n,\nu} = \langle \mathbf{u}, \phi Q_n P_\nu\rangle / \langle \mathbf{u}, P_\nu^2\rangle$ for all $n = 0, 1, 2, \ldots$ and $\nu = n - N, \ldots, n + M$. Similarly, we also have

$$\psi P_n = \sum_{\nu=n-M}^{n+N} s_{n,\nu} Q_\nu \quad (n = 0, 1, 2, \ldots),$$

where $s_{n,\nu} = \langle \mathbf{v}, \psi P_n Q_\nu\rangle / \langle \mathbf{v}, Q_\nu^2\rangle$ for all $n = 0, 1, 2, \ldots$ and $\nu = n - M, \ldots, n + N$.

Now, by using successively the three-term recurrence relation for the sequence $(Q_n)_n$, we can expand ϕQ_n as a finite linear combination (with at most $2M+1$ terms, independent of n) to get a relation of the form

$$\sum_{\nu=n-M}^{n+M} \tilde{r}_{n,\nu} Q_\nu = \sum_{\nu=n-N}^{n+M} r_{n,\nu} P_\nu \quad (n = 0, 1, 2, \ldots), \tag{16}$$

where the coefficients $\tilde{r}_{n,\nu}$ only depend on the polynomial ϕ and the coefficients of the three-term recurrence relation for the sequence $(Q_n)_n$. In the same way, expanding ψP_n using the three-term recurrence relation for $(P_n)_n$, we find

$$\sum_{\nu=n-N}^{n+N} \tilde{s}_{n,\nu} P_\nu = \sum_{\nu=n-M}^{n+N} s_{n,\nu} Q_\nu \quad (n = 0, 1, 2, \ldots), \tag{17}$$

where the numbers $\tilde{s}_{n,\nu}$ only depend on the polynomial ψ and the coefficients of the three-term recurrence relation for the sequence $(P_n)_n$.

Notice that both (16) and (17) show that the sequences $(P_n)_n$ and $(Q_n)_n$ are linearly related in the required sense. However, comparing with (1) in Theorem 1.1 we conjecture that (up to some natural conditions) relations with a smaller number of summands in both sides of (16) and (17) can be obtained, namely $N+1$ terms in the sums involving the P_ν's, and $M+1$ terms in the sums involving the Q_ν's. In fact this holds, for instance, when $N = M = 1$, as follows from Theorem 2.4 in [1] (cf. also Theorem 6.1 below), and when $N = 1$ and $M = 2$, as follows from the next proposition (and so, of course, also when $N = 2$ and $M = 1$).

5. The case $M = 2$, $N = 1$

Theorem 5.1. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be two regular functionals in* $\mathcal{P}'$, *and let* $(P_n)_n$ *and* $(Q_n)_n$ *be the corresponding MOPS's, respectively. Then the following two conditions are equivalent*:

(i) *There exist complex numbers* a, b, c, λ *such that*

$$(x-a)(x-b)\mathbf{u} = \lambda(x-c)\mathbf{v}, \tag{18}$$

with $\langle \mathbf{v}, P_2\rangle \neq 0$ *and* $\langle \mathbf{u}, P_n^2\rangle \neq \lambda\langle \mathbf{v}, P_n Q_{n-1}\rangle$ *for all* $n = 2, 3, \ldots$.

(ii) *There exist complex numbers* r_n, s_n, *and* t_n, *with* $r_3 t_3 \neq 0$ *and* $t_n \neq r_n(s_{n-1} - r_{n-1})$ *for all* $n = 2, 3, \ldots$, *such that*

$$P_n(x) + r_n P_{n-1}(x) = Q_n(x) + s_n Q_{n-1}(x) + t_n Q_{n-2}(x) \tag{19}$$

for all $n = 1, 2, \ldots$.

Proof. (ii) $\Rightarrow$ (i) follows from Theorem 1.1. In fact, if (ii) holds then the hypotheses of Theorem 1.1 are fulfilled since in this case ($N = 1$, $M = 2$) we have

$$A := [\alpha_{ij}]_{i,j=1}^3 = \begin{bmatrix} 1 & r_1 & 0 \\ 0 & 1 & r_2 \\ 1 & s_1 & t_2 \end{bmatrix},$$

hence $\det A = t_2 - r_2(s_1 - r_1) \neq 0$ and so (18) holds for some $a, b, c, \lambda \in \mathbb{C}$, with $\lambda \neq 0$. Further, multiplying both sides of (19) by $(x-a)(x-b)P_{n-3}$ and then applying $\mathbf{u}$ and taking into account (18), we find

$$r_n\langle \mathbf{u}, P_{n-1}^2\rangle = t_n\lambda\langle \mathbf{v}, Q_{n-2}^2\rangle, \quad n = 3, 4, \ldots. \tag{20}$$

This, together with the hypothesis $t_n \neq r_n(s_{n-1} - r_{n-1})$ for all $n = 2, 3, \ldots,$ implies $r_n t_n \neq 0$ for $n = 3, 4, \ldots$. Now, multiplying both sides of (19) by Q_{n-1} and then applying $\mathbf{v}$, we get

$$\langle \mathbf{v}, P_n Q_{n-1}\rangle = (s_n - r_n)\langle \mathbf{v}, Q_{n-1}^2\rangle, \quad n = 1, 2, \ldots. \tag{21}$$

In the same way, multiplying both sides of (19) by Q_{n-2} and applying $\mathbf{v}$, and then taking into account (21), we deduce

$$\langle \mathbf{v}, P_n Q_{n-2}\rangle = \big[t_n - r_n(s_{n-1} - r_{n-1})\big]\langle \mathbf{v}, Q_{n-2}^2\rangle, \quad n = 2, 3, \ldots. \tag{22}$$

For $n = 2$ this gives $\langle \mathbf{v}, P_2\rangle = [t_2 - r_2(s_1 - r_1)]\langle \mathbf{v}, 1\rangle \neq 0$. Now, changing n into $n+1$ in (22) and then substituting in the right-hand side of the resulting equation the expressions for r_{n+1} and $s_n - r_n$ given by (20) and (21), respectively, we find

$$\langle \mathbf{v}, P_{n+1} Q_{n-1}\rangle = t_{n+1}\big[\langle \mathbf{u}, P_n^2\rangle - \lambda\langle \mathbf{v}, P_n Q_{n-1}\rangle\big]\frac{\langle \mathbf{v}, Q_{n-1}^2\rangle}{\langle \mathbf{u}, P_n^2\rangle} \tag{23}$$

for all $n = 2, 3, \ldots$. Finally, from (20) and (23) we deduce

$$\lambda\langle \mathbf{v}, P_{n+1} Q_{n-1}\rangle = r_{n+1}\big[\langle \mathbf{u}, P_n^2\rangle - \lambda\langle \mathbf{v}, P_n Q_{n-1}\rangle\big], \quad n = 2, 3, \ldots. \tag{24}$$

Therefore, comparing (22) and (24), and taking into account that $r_{n+1} \neq 0$ and $t_n \neq r_n(s_{n-1} - r_{n-1})$ for all $n = 2, 3, \ldots,$ we see that $\langle \mathbf{u}, P_n^2\rangle \neq \lambda\langle \mathbf{v}, P_n Q_{n-1}\rangle$ for all $n = 2, 3, \ldots$. Thus (ii) $\Rightarrow$ (i).

In order to prove that (i) $\Rightarrow$ (ii), denote by $\{\beta_n, \gamma_{n+1}\}_{n\geqslant 0}$ and $\{\tilde\beta_n, \tilde\gamma_{n+1}\}_{n\geqslant 0}$ the sets of parameters which appear in the three-term recurrence relations for the MOPS's $(P_n)_n$ and $(Q_n)_n$, respectively. Then, according to the considerations we have made just before the statement of this theorem, the functional equation (18), with $\phi(x) := (x-a)(x-b)$ and $\psi(x) := \lambda(x-c)$, implies two relations corresponding to (16) and (17), namely

$$\begin{aligned} &P_{n+2} + r_{n,n+1}P_{n+1} + r_{n,n}P_n + r_{n,n-1}P_{n-1} \\ &\quad = Q_{n+2} + \tilde r_{n,n+1}Q_{n+1} + \tilde r_{n,n}Q_n + \tilde r_{n,n-1}Q_{n-1} + \tilde r_{n,n-2}Q_{n-2}, \end{aligned} \tag{25}$$

and

$$\begin{aligned} &P_{n+1} + (\beta_n - c)P_n + \gamma_n P_{n-1} \\ &\quad = Q_{n+1} + \frac{s_{n,n}}{\lambda}Q_n + \frac{s_{n,n-1}}{\lambda}Q_{n-1} + \frac{s_{n,n-2}}{\lambda}Q_{n-2}, \end{aligned} \tag{26}$$

where $r_{n,\nu} = \langle \mathbf{u}, \phi Q_n P_\nu\rangle/\langle \mathbf{u}, P_\nu^2\rangle$ $(\nu = n-1, n, n+1)$, $s_{n,\nu} = \langle \mathbf{v}, \psi P_n Q_\nu\rangle/\langle \mathbf{v}, Q_\nu^2\rangle$ $(\nu = n-2, n-1, n, n+1)$, and

$$\begin{aligned} &\tilde r_{n,n+1} = \tilde\beta_{n+1} + \tilde\beta_n - a - b, \qquad \tilde r_{n,n} = \tilde\gamma_{n+1} + \tilde\gamma_n + \phi(\tilde\beta_n), \\ &\tilde r_{n,n-1} = \tilde\gamma_n(\tilde\beta_n + \tilde\beta_{n-1} - a - b), \quad \tilde r_{n,n-2} = \tilde\gamma_n\tilde\gamma_{n-1} \end{aligned}$$

for all $n = 2, 3, \ldots$. Notice that, according to (18) and the operational rules of the distributional calculus, for all $n = 2, 3, \ldots$ one has

$$s_{n,n-2} = \frac{\langle \mathbf{v}, \psi P_n Q_{n-2}\rangle}{\langle \mathbf{v}, Q_{n-2}^2\rangle} = \frac{\langle \mathbf{u}, \phi Q_{n-2} P_n\rangle}{\langle \mathbf{v}, Q_{n-2}^2\rangle} = \frac{\langle \mathbf{u}, P_n^2\rangle}{\langle \mathbf{v}, Q_{n-2}^2\rangle} \neq 0$$

(the last equality holds since $\deg \phi = 2$), hence from (26) we can write Q_{n-2} as a linear combination of $P_{n+1}, P_n, P_{n-1}, Q_{n+1}, Q_n$ and Q_{n-1}. Further, by changing n into $n+1$ in (26), we find an expression for Q_{n+2} as a linear combination of $P_{n+2}, P_{n+1}, P_n, Q_{n+1}, Q_n$ and Q_{n-1}. Now, substituting these two expressions for Q_{n-2} and Q_{n+2} in the right-hand side of (25), after some straightforward computations, we find

$$a_n P_{n+1} + b_n P_n + c_n P_{n-1} = d_n Q_{n+1} + e_n Q_n + f_n Q_{n-1} \tag{27}$$

for all $n = 2, 3, 4, \ldots$, where

$$\begin{aligned} a_n &:= r_{n,n+1} - \beta_{n+1} + c - \lambda \tilde{r}_{n,n-2}/s_{n,n-2}, \\ b_n &:= r_{n,n} - \gamma_{n+1} - \lambda(\beta_n - c)\tilde{r}_{n,n-2}/s_{n,n-2}, \\ c_n &:= r_{n,n-1} - \lambda\gamma_n \tilde{r}_{n,n-2}/s_{n,n-2}, \end{aligned}$$

and

$$\begin{aligned} d_n &:= \tilde{r}_{n,n+1} - s_{n+1,n+1}/\lambda - \lambda \tilde{r}_{n,n-2}/s_{n,n-2}, \\ e_n &:= \tilde{r}_{n,n} - s_{n+1,n}/\lambda - s_{n,n}\tilde{r}_{n,n-2}/s_{n,n-2}, \\ f_n &:= \tilde{r}_{n,n-1} - s_{n+1,n-1}/\lambda - s_{n,n-1}\tilde{r}_{n,n-2}/s_{n,n-2}. \end{aligned}$$

Since P_{n+1} and Q_{n+1} are monic polynomials, (27) implies $a_n = d_n$ for every $n = 2, 3, \ldots$. Further, $\langle \mathbf{u}, P_{n-1}^2\rangle r_{n,n-1} = \langle \mathbf{u}, \phi Q_n P_{n-1}\rangle = \langle \mathbf{v}, \psi P_{n-1} Q_n\rangle = \lambda\langle \mathbf{v}, Q_n^2\rangle$, so that

$$c_n = \lambda \frac{\langle \mathbf{v}, Q_n^2\rangle}{\langle \mathbf{u}, P_{n-1}^2\rangle} - \lambda\gamma_n \frac{\tilde{\gamma}_n \tilde{\gamma}_{n-1}}{\langle \mathbf{u}, P_n^2\rangle}\langle \mathbf{v}, Q_{n-2}^2\rangle = 0, \quad n = 2, 3, \ldots,$$

the zero equality being justified since the relations $\gamma_n = \langle \mathbf{u}, P_n^2\rangle/\langle \mathbf{u}, P_{n-1}^2\rangle$ and $\tilde{\gamma}_n = \langle \mathbf{v}, Q_n^2\rangle/\langle \mathbf{v}, Q_{n-1}^2\rangle$ hold for all $n = 1, 2, \ldots$. Notice that (27) also holds for $n = 0$ and $n = 1$, as follows trivially from (26), with

$$\begin{aligned} &a_0 = a_1 = d_0 = d_1 = 1, \qquad b_0 = \beta_0 - c, \qquad c_0 = c_1 = 0, \qquad e_0 = \tilde{\beta}_0 - c, \\ &b_1 = \beta_1 - c, \qquad e_1 = s_{1,1}/\lambda, \qquad f_1 = s_{1,0}/\lambda - \gamma_1 \end{aligned} \tag{28}$$

(remark that this choice is not unique). Therefore, one see that (27) reduces to (19), after changing n into $n-1$, provided we can show that $a_n \neq 0$ for all $n = 2, 3, \ldots$, being

$$r_n = \frac{b_{n-1}}{a_{n-1}}, \qquad s_n = \frac{e_{n-1}}{a_{n-1}}, \qquad t_n = \frac{f_{n-1}}{a_{n-1}}, \quad n = 1, 2, 3, \ldots. \tag{29}$$

In order to prove that, in fact, $a_n \neq 0$ for all $n = 2, 3, \ldots$, notice first that

$$b_n = \frac{\lambda\langle \mathbf{v}, Q_n P_{n+1}\rangle - \langle \mathbf{u}, P_{n+1}^2\rangle}{\langle \mathbf{u}, P_n^2\rangle}, \quad n = 2, 3, \ldots. \tag{30}$$

This follows from the definition of b_n and taking into account that

$$\begin{aligned}\langle \mathbf{u}, \phi Q_n P_n\rangle &= \langle \mathbf{v}, \psi Q_n P_n\rangle = \lambda\big(\langle \mathbf{v}, (x-\beta_n)P_n Q_n + (\beta_n - c)P_n Q_n\rangle\big)\\ &= \lambda\langle \mathbf{v}, Q_n P_{n+1}\rangle + \lambda(\beta_n - c)\langle \mathbf{v}, Q_n^2\rangle.\end{aligned}$$

Next, apply $\mathbf{v}$ to both sides of (27) to find

$$a_n\langle \mathbf{v}, P_{n+1}\rangle = -b_n\langle \mathbf{v}, P_n\rangle, \quad n = 2, 3, \ldots. \tag{31}$$

Since $b_n \neq 0$ for all $n = 2, 3, \ldots$ and $\langle \mathbf{v}, P_2\rangle \neq 0$, as follows from (30) and the hypothesis, we deduce recursively from (31) that $\langle \mathbf{v}, P_n\rangle \neq 0$ for all $n = 2, 3, \ldots$, and so, again by (31), we may conclude that $a_n \neq 0$ for all $n = 2, 3, \ldots$. So, (19) holds with r_n, s_n and t_n defined by (29), and it remains to prove that the conditions $r_3 t_3 \neq 0$ and $t_n \neq r_n(s_{n-1} - r_{n-1})$ hold for all $n = 2, 3, \ldots$. In fact, since (19) holds, as well as (18), as in the proof of (ii) $\Rightarrow$ (i) one immediately see that relations (20)–(24) hold. Therefore, since $r_n = b_{n-1}/a_{n-1} \neq 0$ for all $n = 3, 4, \ldots$ we conclude from (20) that also $t_n \neq 0$ for all $n = 3, 4, \ldots$, hence by the hypothesis we see that the right-hand side of (24) does not vanish, so that $\langle \mathbf{v}, P_{n+1}Q_{n-1}\rangle \neq 0$ for all $n = 2, 3, \ldots$, and since by hypothesis we also have $\langle \mathbf{v}, P_2\rangle \neq 0$, then $\langle \mathbf{v}, P_n Q_{n-2}\rangle \neq 0$ for all $n = 2, 3, \ldots$. Thus, it follows from (22) that $t_n \neq r_n(s_{n-1} - r_{n-1})$ hold for all $n = 2, 3, \ldots$. □

Remark 5.1. As a consequence of the proof, if (ii) holds in Theorem 5.1 then $r_n t_n \neq 0$ for all $n = 3, 4, \ldots$.

Remark 5.2. If (i) holds in Theorem 5.1 then the coefficients r_n, s_n and t_n in (ii) can be computed successively from (24), (21) and (20), for all $n = 3, 4, \ldots$. Further, setting $P_n(x) = x^n + p_{n,1}x^{n-1} + p_{n,2}x^{n-2} + \cdots$ and $Q_n(x) = x^n + q_{n,1}x^{n-1} + q_{n,2}x^{n-2} + \cdots$, then it is easy to see that

$$\begin{aligned}&\langle \mathbf{v}, P_n Q_{n-1}\rangle = (p_{n,1} - q_{n,1})\langle \mathbf{v}, Q_{n-1}^2\rangle,\\ &\langle \mathbf{v}, P_n Q_{n-2}\rangle = -\big[q_{n,2} - p_{n,2} + q_{n-1,1}(p_{n,1} - q_{n,1})\big]\langle \mathbf{v}, Q_{n-2}^2\rangle\end{aligned} \tag{32}$$

for all $n = 2, 3, \ldots$. (In fact, this is true for any MOPS's $(P_n)_n$ and $(Q_n)_n$.) As a consequence, the parameters r_n, s_n and t_n in the structure relation (19) can be computed only in terms of the coefficients of x^{n-1} and x^{n-2} in the polynomials P_n and Q_n and the quantities $\langle \mathbf{u}, P_n^2\rangle$ and $\langle \mathbf{v}, Q_n^2\rangle$ (in the positive-case, these are the squares of the norms of P_n and Q_n, respectively). In fact,

$$\begin{aligned}&r_n = \frac{p_{n,2} - q_{n,2} - (p_{n,1} - q_{n,1})q_{n-1,1}}{q_{n-1,1} - p_{n-1,1} + \langle \mathbf{u}, P_{n-1}^2\rangle/(\lambda\langle \mathbf{v}, Q_{n-2}^2\rangle)},\\ &s_n = r_n + p_{n,1} - q_{n,1}, \qquad t_n = r_n\langle \mathbf{u}, P_{n-1}^2\rangle/\big(\lambda\langle \mathbf{v}, Q_{n-2}^2\rangle\big)\end{aligned} \tag{33}$$

for all $n = 3, 4, \ldots$.

Example 5.1. Let $P_n^{(\alpha,\beta)}$, with $\alpha > -1$ and $\beta > -1$, be the Jacobi polynomial of degree n with the usual normalization $P_n^{(\alpha,\beta)}(1) = (1+\alpha)_n/n!$, $(\nu)_n$ being the Pochhammer symbol. Then

$$P_n^{(\alpha,\beta)}(x) = k_n^{(\alpha,\beta)}\hat{P}_n^{(\alpha,\beta)}(x), \qquad k_n^{(\alpha,\beta)} := \frac{(1+\alpha+\beta)_{2n}}{2^n n!(1+\alpha+\beta)_n},$$

where $\hat{P}_n^{(\alpha,\beta)}$ denotes the monic Jacobi polynomial of degree n, so that

$$\hat{P}_n^{(\alpha,\beta)}(x) = x^n + p_{n,1}^{(\alpha,\beta)} x^{n-1} + p_{n,2}^{(\alpha,\beta)} x^{n-2} + \cdots,$$
$$p_{n,1}^{(\alpha,\beta)} := \frac{(\alpha-\beta)n}{\alpha+\beta+2n}, \qquad p_{n,2}^{(\alpha,\beta)} := -\frac{n(n-1)[2n+\alpha+\beta-(\alpha-\beta)^2]}{2(\alpha+\beta+2n)(\alpha+\beta+2n-1)} \tag{34}$$

for all possible n (see [5,18], e.g.). The Jacobi polynomials are orthogonal with respect to the functional (beta distribution) $\mathbf{u}^{\alpha,\beta} : \mathcal{P} \to \mathbb{C}$ defined by

$$\left\langle \mathbf{u}^{\alpha,\beta}, f \right\rangle := c^{(\alpha,\beta)} \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta f(x)\,\mathrm{d}x, \quad f \in \mathcal{P},$$

where $c^{(\alpha,\beta)}$ is a constant, chosen such that $\langle \mathbf{u}^{\alpha,\beta}, 1\rangle = 1$. Further,

$$\left\langle \mathbf{u}^{\alpha,\beta}, \left[\hat{P}_n^{(\alpha,\beta)}\right]^2 \right\rangle = \frac{2^{2n} n!(1+\alpha)_n(1+\beta)_n(1+\alpha+\beta)_n}{(1+\alpha+\beta)_{2n}(2+\alpha+\beta)_{2n}}, \quad n = 0, 1, 2, \ldots. \tag{35}$$

Fix $\alpha > 1$ and $\beta > 0$. Then it is easy to check that

$$(x-1)^2 \mathbf{u}^{\alpha-2,\beta} = \lambda(x+1)\mathbf{u}^{\alpha,\beta-1} \quad \text{in } \mathcal{P}',$$

where $\lambda := \frac{2\alpha(\alpha-1)}{\beta(\alpha+\beta)}$, so that (18) holds with $a = b = -c = 1$ and, of course, taking $P_n \equiv \hat{P}_n^{(\alpha-2,\beta)}$ and $Q_n \equiv \hat{P}_n^{(\alpha,\beta-1)}$. Now, using (32), (35) and (34), we find

$$\begin{aligned}\langle \mathbf{v}, P_n Q_{n-1}\rangle &= \left(p_{n,1}^{(\alpha-2,\beta)} - p_{n,1}^{(\alpha,\beta-1)}\right)\left\langle \mathbf{u}^{\alpha,\beta-1}, \left[\hat{P}_{n-1}^{(\alpha,\beta-1)}\right]^2\right\rangle \\ &= -\frac{2^{2n-1} n!(\alpha+2\beta-2+3n)(1+\alpha)_{n-1}(\beta)_{n-1}(\alpha+\beta)_{n-1}}{(\alpha+\beta)_{2n-1}(1+\alpha+\beta)_{2n-1}}\end{aligned}$$

for all $n = 1, 2, \ldots$, hence

$$\left\langle \mathbf{u}, P_n^2\right\rangle - \lambda\langle \mathbf{v}, P_n Q_{n-1}\rangle = \frac{2^{2n} n!(\alpha-1)_{n+1}(1+\beta)_{n-2}(\alpha+\beta+1)_{n-2}}{(\alpha+n-1)(\alpha+\beta)_{2n-2}(\alpha+\beta+1)_{2n-2}}$$

holds for all $n = 2, 3, \ldots$, and we see that the hypotheses in (i), Theorem 5.1, are fulfilled. Henceforth, since

$$\frac{\langle \mathbf{u}, P_n^2\rangle}{\lambda\langle \mathbf{v}, Q_{n-1}^2\rangle} = \frac{2n(n+\beta-1)(n+\beta)}{(n+\alpha-1)(2n+\alpha+\beta-2)(2n+\alpha+\beta-1)}$$

holds for all $n = 1, 2, \ldots$, and using (34) and (35), from (33) we get

$$\begin{aligned} r_n &= \frac{2n(n+\alpha-2)}{(2n+\alpha+\beta-3)(2n+\alpha+\beta-2)}, \\ s_n &= -\frac{4n(n+\beta-1)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta-3)}, \\ t_n &= \frac{4n(n-1)(n+\beta-2)(n+\beta-1)}{(2n+\alpha+\beta-4)(2n+\alpha+\beta-3)^2(2n+\alpha+\beta-2)} \end{aligned}$$

for all $n = 3, 4, \ldots$. This gives us the following linear relation for Jacobi polynomials (with $\alpha > 1$ and $\beta > 0$)

$$
\begin{aligned}
&P_n^{(\alpha-2,\beta)} + \frac{n+\alpha-2}{n+\alpha+\beta-2} P_{n-1}^{(\alpha-2,\beta)} \\
&\quad = \frac{n+\alpha+\beta-1}{2n+\alpha+\beta-1} P_n^{(\alpha,\beta-1)} - \frac{2(n+\beta-1)(2n+\alpha+\beta-2)}{(2n+\alpha+\beta-3)(2n+\alpha+\beta-1)} P_{n-1}^{(\alpha,\beta-1)} \\
&\qquad + \frac{(n+\beta-2)(n+\beta-1)}{(n+\alpha+\beta-2)(2n+\alpha+\beta-3)} P_{n-2}^{(\alpha,\beta-1)}
\end{aligned}
$$

for all $n = 1, 2, 3, \ldots$. (Actually, we have proved this relation only for $n \geqslant 3$, but when $n = 1$ and $n = 2$ it can be checked directly.)

6. The case $M = N = 1$

This case was completely solved in [1,2]. Following the proof of Theorem 5.1, we can state the next proposition, which is a slightly modification of Theorem 2.4 in [1] (see also [16]).

Theorem 6.1. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be two regular functionals in* $\mathcal{P}'$, *and let* $(P_n)_n$ *and* $(Q_n)_n$ *be the corresponding MOPS's, respectively. Then the following two conditions are equivalent*:

(i) *There exist complex numbers* a, b, λ *such that*

$$(x-a)\mathbf{u} = \lambda(x-b)\mathbf{v} \tag{36}$$

and $\langle \mathbf{u}, P_n^2 \rangle \neq \lambda \langle \mathbf{v}, Q_n^2 \rangle$ *for all* $n = 1, 2, \ldots$.

(ii) *There exist complex numbers* r_n *and* s_n, *with* $r_n s_n \neq 0$ *and* $r_n \neq s_n$ *for all* $n = 1, 2, \ldots$, *such that*

$$P_n(x) + r_n P_{n-1}(x) = Q_n(x) + s_n Q_{n-1}(x) \tag{37}$$

for all $n = 1, 2, \ldots$.

Remark 6.1. In order to have the condition $r_n s_n \neq 0$ for all $n \geqslant 2$ in (ii), it suffices to prove that $r_2 s_2 \neq 0$ (see [1]).

Remark 6.2. Under the conditions of Theorem 6.1 we can show that the formal Stieltjes series $S_{\mathbf{u}}(z) := -\sum_{\nu \geqslant 0} u_\nu z^{-\nu-1}$ and $S_{\mathbf{v}}(z) := -\sum_{\nu \geqslant 0} v_\nu z^{-\nu-1}$ associated to $\mathbf{u}$ and $\mathbf{v}$, respectively, satisfy

$$-\lambda(z-b)S_{\mathbf{v}}(z) + (z-a)S_{\mathbf{u}}(z) = \lambda - 1 \tag{38}$$

(with the normalization condition $\langle \mathbf{u}, 1 \rangle = \langle \mathbf{v}, 1 \rangle = 1$). This gives us a simple relation between the moments of the functionals $\mathbf{u}$ and $\mathbf{v}$. Further, if one of the functionals $\mathbf{u}$ or $\mathbf{v}$ is semiclassical, so that $D(\varphi\mathbf{u}) = \psi\mathbf{u}$ for some polynomials φ and ψ, then, being a rational modification of $\mathbf{u}$, $\mathbf{v}$ is also semiclassical. The class of $\mathbf{v}$ can then be easily obtained (by standard methods) from (38).

To conclude this presentation we will consider the analytical problem concerning integral representations of the regular functionals involved in Theorem 6.1. As usual, the principal Cauchy value will be denoted by

$$P\int_{-\infty}^{+\infty}\frac{V(x)}{x-b}\,\mathrm{d}x := \lim_{\epsilon\to 0^+}\left(\int_{-\infty}^{b-\epsilon}+\int_{b+\epsilon}^{+\infty}\right)\frac{V(x)}{x-b}\,\mathrm{d}x,$$

assuming the limit exists.

Corollary 6.2. *Under the conditions of Theorem* 6.1*, if* **u** *has the integral representation*

$$\langle \mathbf{u}, f\rangle = \int_{-\infty}^{+\infty} f(x)W(x)\,\mathrm{d}x, \qquad f\in\mathcal{P},$$

where W is a locally integrable function with rapid decay and continuous at the point $x=b$, then **v** *has the integral representation*

$$\langle \mathbf{v}, f\rangle = -\frac{f(b)}{\lambda}\left\{1-\lambda+(b-a)P\int_{-\infty}^{+\infty}\frac{W(x)}{x-b}\,\mathrm{d}x\right\} + \frac{1}{\lambda}P\int_{-\infty}^{+\infty} f(x)\frac{x-a}{x-b}W(x)\,\mathrm{d}x, \qquad f\in\mathcal{P}, \tag{39}$$

where we have assumed the normalization condition $\langle \mathbf{u},1\rangle = \langle \mathbf{v},1\rangle = 1$.

Proof. Set $u_0 := \langle \mathbf{u},1\rangle$ and $v_0 := \langle \mathbf{v},1\rangle$ (only in the conclusion of the proof we will assume these quantities equal to one). Recall that, by definition, the division of **u** by $x-c$ is the functional in $\mathcal{P}'$, denoted by $(x-c)^{-1}\mathbf{u}$, such that

$$\left\langle (x-c)^{-1}\mathbf{u}, f\right\rangle := \left\langle \mathbf{u}, \big(f(x)-f(c)\big)/(x-c)\right\rangle, \qquad f\in\mathcal{P}.$$

Therefore the relation (36) between the functionals **u** and **v** can be written as

$$\mathbf{v} = v_0\delta_b + \frac{1}{\lambda}(x-b)^{-1}(x-a)\mathbf{u},$$

where, as usual, for any $c\in\mathbb{C}$, the Dirac functional at the point c, δ_c, is defined by $\langle \delta_c, f\rangle := f(c)$ for all $f\in\mathcal{P}$. From the above relation it is easy to get

$$\mathbf{v} = -\frac{u_0-\lambda v_0}{\lambda}\delta_b + \frac{1}{\lambda}\mathbf{u} + \frac{b-a}{\lambda}(x-b)^{-1}\mathbf{u}.$$

From this and after some straightforward computations (39) follows. □

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