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A new look at localic interpolation theorems

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Abstract

This paper presents a new treatment of the localic Katětov–Tong interpolation theorem, based on an analysis of special properties of normal frames, which shows that it does not hold in full generality. Besides giving us the conditions under which the localic Katětov–Tong interpolation theorem holds, this approach leads to a especially transparent and succinct proof of it. It is also shown that this pointfree extension of Katětov–Tong theorem still covers the localic versions of Urysohn's Lemma and Tietze's Extension Theorem.

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Urysohn's Lemma shows that normal spaces are precisely the spaces that admit "plenty of real-valued continuous functions" [3], that is, the spaces where all sets that could possibly be separated by continuous functions actually are. Indeed, it states that a space X is normal if and only if for any closed subset F, contained in any open subset G, there is a real-valued continuous function h on X such that $\chi_F \leqslant h \leqslant \chi_G$. The well-known Katětov–Tong interpolation theorem ([10,13]; see also [11]) strengthens this characterization by replacing χ_F and χ_G by arbitrary upper and lower semicontinuous real-valued functions f and g, respectively.

In Theorem 2.2 of [12], the authors extend this characterization to normal frames:

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A frame L is normal if and only if for every upper semicontinuous real function $f:\overline{\mathcal{L}}_u(\mathbb{R})\to L$ *and every lower semicontinuous real function* $g:\overline{\mathcal{L}}_l(\mathbb{R})\to L$ *with* $f\leqslant g$*, there exists a continuous real function* $h:\overline{\mathcal{L}}(\mathbb{R})\to L$ *such that* $f\leqslant h\leqslant g$.

However, a detailed analysis of their proof reveals that it does not work for all f and g; in fact, the statement above is not valid in its full generality, as Example 4.2 in the sequel shows.

The purpose of this paper is twofold: firstly, to correct that by establishing the conditions under which the localic version of Katětov–Tong interpolation theorem holds; secondly, to see as directly as possible how the normality of the frame provides the construction of the continuous map h to be inserted between the given upper f and lower g.

Apart from giving us the right formulation for the localic Katětov–Tong theorem, we believe that our approach leads to a specially transparent proof of it, avoiding the use of the sublocale lattice in [12], where the mistake occurred. In particular, it gives us a new proof of Theorem 2.1 of [12] (a weak form of the localic Katětov–Tong theorem).

Our investigation was, originally, motivated by the aim to study the basic properties of the localic upper and lower semicontinuous functions that appeared naturally in our investigation, with Ferreira [5,6], on the construction of compatible frame quasi-uniformities. The recognition that the notions of upper and lower semicontinuous functions in frames provide a pointfree axiomatization of semicontinuity in spaces naturally raised the question of pointfree interpolation theorems.

It is felt that the present approach should help to make the subject of interpolation theorems somewhat more transparent. As Ball and Walters-Wayland wrote in [2] "*what the pointfree formulation adds to the classical theory is a remarkable combination of elegance of statement, simplicity of proof, and increase of extent*".

1. Preliminaries

Pointfree topology is motivated by the goal of building topology on the intuition of "places of non-trivial extent" rather than on points; so it regards the points of a space as subsidiary to its open sets and accordingly deals with "lattices of open sets" abstractly defined as follows:

A *frame* (also *locale*) is a complete lattice L satisfying the infinite distributive law

$$x\wedge\bigvee S=\bigvee\{x\wedge s\mid s\in S\}$$

for every $x\in L$ and every $S\subseteq L$, and a *frame homomorphism* is a map $h:L\to M$ between frames which preserves the respective operations $\bigwedge$ (including the top element 1) and $\bigvee$ (including the bottom element 0). Frm is then the corresponding category of frames and their homomorphisms.

The most familiar examples of frames are the finite distributive lattices, the complete chains, the complete Boolean algebras and, for any topological space X, the lattice $\mathcal{O}X$ of open subsets of X.

We recall the basic notions involving frames that will be of particular importance here. As a general reference to frames, the reader can consult Johnstone [8] or Vickers [15].

For $x \in L$ we write x^* for its *pseudocomplement* $\bigvee\{a \in L \mid a \wedge x = 0\}$. Notice that, in any frame, $x \leqslant x^{**}$, $x^* = x^{***}$ and the first De Morgan law

$$\Bigl(\bigvee_{i\in I} x_i\Bigr)^* = \bigwedge_{i\in I} x_i^* \tag{1.1}$$

holds; moreover

$$x \vee y = 1 \implies y^* \leqslant x. \tag{1.2}$$

A frame L is called *normal* if $x \vee y = 1$ implies that there exist $a, b \in L$ such that $x \vee a = 1 = y \vee b$ and $a \wedge b = 0$. Clearly, this is equivalent to saying that $x \vee y = 1$ implies the existence of $a \in L$ such that $x \vee a = 1 = y \vee a^*$.

By the algebraic nature of frames, there is the notion of a *congruence* on a frame L, as an equivalence relation θ on L which is a subframe of $L \times L$ in the obvious sense, and the corresponding quotient frame L/θ is then defined just as quotients are always defined for algebraic systems, making the map $L \to L/\theta$ taking each $x \in L$ to its θ-block a frame homomorphism. The lattice of frame congruences on L under set inclusion is a frame, denoted by $\mathfrak{C}L$. Here, we shall need the following properties:

(1) For any $x \in L$, $\nabla_x := \{(a, b) \in L \times L \mid a \vee x = b \vee x\}$ is the least congruence containing $(0, x)$; $\Delta_x := \{(a, b) \in L \times L \mid a \wedge x = b \wedge x\}$ is the least congruence containing $(1, x)$. The ∇_x are called *closed* and the Δ_x *open*.
(2) For each closed congruence ∇_x, L/∇_x is isomorphic to the frame $\uparrow x := \{y \in L \mid y \geqslant x\}$ and the *closed quotient* $L \twoheadrightarrow \uparrow x$ is given by $y \mapsto x \vee y$.
(3) $\nabla L := \{\nabla_x \mid x \in L\}$ is a subframe of $\mathfrak{C}L$. Further, let ΔL denote the subframe of $\mathfrak{C}L$ generated by $\{\Delta_x \mid x \in L\}$. The map $x \mapsto \nabla_x$ is a frame isomorphism $L \to \nabla L$, whereas the map $x \mapsto \Delta_x$ is a dual poset embedding $L \to \Delta L$ taking finitary meets to finitary joins and arbitrary joins to arbitrary meets.

The fact that Frm is an algebraic category (in particular, one has free frames and quotient frames) also permits a procedure familiar from traditional algebra, namely, the definition of a frame by *generators and relations*: take the quotient of the free frame on the given set of generators modulo the congruence generated by the pairs (u, v) for the given relations $u = v$. So, it is natural and very useful to introduce the reals in the following pointfree way, independent of any notion of real number (in the sequel we denote by $\mathbb{Q}$ the usual totally ordered set of rational numbers):

The *frame of reals* [9,1] is the frame $\bar{\mathfrak{L}}(\mathbb{R})$ generated by all ordered pairs (α, β) where $\alpha, \beta \in \mathbb{Q}$, subject to the relations

(R_1) $(\alpha, \beta) \wedge (\gamma, \delta) = (\alpha \vee \gamma, \beta \wedge \delta)$,
(R_2) $(\alpha, \beta) \vee (\gamma, \delta) = (\alpha, \delta)$ whenever $\alpha \leqslant \gamma < \beta \leqslant \delta$,
(R_3) $(\alpha, \beta) = \bigvee\{(\gamma, \delta) \mid \alpha < \gamma < \delta < \beta\}$,
(R_4) $1 = \bigvee\{(\alpha, \beta) \mid \alpha, \beta \in \mathbb{Q}\}$.

Classically, this is just the interval topology of the real line, but under the point of view of constructiveness, these two notions are not the same (cf. [1,7]; see also [14] for more elucidation on the relation between this notion of real number and Dedekind cuts on the level of their propositional theories).

Continuous real functions play a central role in general topology, and the corresponding localic continuous real functions, are no less important to pointfree topology (cf. [1,2,12]): for any space X, there is a one–one onto map

$$\mathsf{Frm}\big(\overline{\mathfrak{L}}(\mathbb{R}), \mathcal{O}X\big) \to \mathsf{Top}(X, \mathbb{R}) \tag{1.3}$$

given by the correspondence $\tilde{h} \mapsto h$ such that $\alpha < h(x) < \beta$ iff $x \in \tilde{h}(\alpha, \beta)$ whenever $\alpha < \beta$ in $\mathbb{Q}$; this led Banaschewski [1] to define a *continuous real function* on L as a frame homomorphism $h : \overline{\mathfrak{L}}(\mathbb{R}) \to L$.

The definition of $\overline{\mathfrak{L}}(\mathbb{R})$ implies immediately that, for any frame L, a map from the set of all pairs (α, β), $\alpha, \beta \in \mathbb{Q}$, into L determines a (unique) continuous real function $h : \overline{\mathfrak{L}}(\mathbb{R}) \to L$ if and only if it transforms the above relations (R_1)–(R_4) into identities in L.

Throughout the paper, we write $\overline{\mathcal{R}}(L)$ for the set of all continuous real functions on L. Further, we denote by $\mathfrak{L}(\mathbb{R})$ the frame generated by all ordered pairs (α, β), subject to the relations (R_1), (R_2) and (R_3), and we write $\mathcal{R}(L)$ for the corresponding set of all frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to L$.

An obvious equivalent representation of the frame of reals is the following [12]: $\overline{\mathfrak{L}}(\mathbb{R})$ (respectively, $\mathfrak{L}(\mathbb{R})$) is the frame generated by elements $(-, \alpha)$ and $(\alpha, -)$, $\alpha \in \mathbb{Q}$, subject to the relations

(R'_1) $\beta < \alpha \Longrightarrow (-, \alpha) \vee (\beta, -) = 1$,
(R'_2) $\alpha \leqslant \beta \Longrightarrow (-, \alpha) \wedge (\beta, -) = 0$,
(R'_3) $\bigvee_{\beta < \alpha} (-, \beta) = (-, \alpha)$,
(R'_4) $\bigvee_{\beta > \alpha} (\beta, -) = (\alpha, -)$,
(R'_5) $\bigvee_{\alpha \in \mathbb{Q}} (-, \alpha) = 1$,
(R'_6) $\bigvee_{\alpha \in \mathbb{Q}} (\alpha, -) = 1$

(respectively (R'_1), (R'_2), (R'_3) and (R'_4)).

2. Localic semicontinuous real functions

In a previous paper [5], motivated by our investigation of compatible frame quasi-uniformities and by the work of Banaschewski [1] on localic continuous real functions, the concept of localic semicontinuous real function appeared naturally. We point out that equivalent concepts appear in [12], in terms of upper and lower continuous chains. Here we present the basic facts about them.

Let $\overline{\mathfrak{L}}_u(\mathbb{R})$ be the frame generated by elements $(-, \alpha)$, $\alpha \in \mathbb{Q}$, subject to the relations

(U_1) $\alpha \leqslant \beta \Longrightarrow (-, \alpha) \leqslant (-, \beta)$,
(U_2) $\bigvee_{\beta < \alpha} (-, \beta) = (-, \alpha)$,
(U_3) $\bigvee_{\alpha \in \mathbb{Q}} (-, \alpha) = 1$,

and, dually, let $\overline{\mathfrak{L}}_l(\mathbb{R})$ be the frame generated by elements $(\alpha, -)$, $\alpha \in \mathbb{Q}$, subject to the relations

(L$_1$) $\alpha \leqslant \beta \Longrightarrow (\alpha, -) \geqslant (\beta, -)$,
(L$_2$) $\bigvee_{\beta > \alpha} (\beta, -) = (\alpha, -)$,
(L$_3$) $\bigvee_{\alpha \in \mathbb{Q}} (\alpha, -) = 1$.

Classically, $\overline{\mathfrak{L}}_u(\mathbb{R})$ and $\overline{\mathfrak{L}}_l(\mathbb{R})$ are, respectively, the upper and lower topologies of the real line, but not in the constructive view.

We will also need the frame $\mathfrak{L}_u(\mathbb{R})$, defined by omitting the relation (U$_3$) in the definition of $\overline{\mathfrak{L}}_u(\mathbb{R})$, and the frame $\mathfrak{L}_l(\mathbb{R})$, defined dually, by omitting the relation (L$_3$) in the definition of $\overline{\mathfrak{L}}_l(\mathbb{R})$.

Recall that, for a space X, a real-valued map $f : X \to \mathbb{R}$ is *upper semi-continuous* if $f : X \to \mathbb{R}_u$ is continuous, where $\mathbb{R}_u$ denotes the space of reals with the upper topology $\{(-\infty, a) \mid a \in \mathbb{R}\}$. Correspondingly, we say that an *upper semicontinuous real function* on a frame L is a frame homomorphism $\overline{\mathfrak{L}}_u(\mathbb{R}) \to L$. Dually, a *lower semicontinuous real function* is a frame homomorphism $\overline{\mathfrak{L}}_l(\mathbb{R}) \to L$.

Obviously, a map f from the generators of $\overline{\mathfrak{L}}_u(\mathbb{R})$ into L defines an upper semicontinuous real function if and only if it transforms the relations (U$_1$)–(U$_3$) into identities in L; that is, if and only if $\{f(-, \alpha)\}_{\alpha \in \mathbb{Q}}$ is a *proper continuous ascending chain* [12] in L, that is, an ascending chain satisfying

$$\bigvee_{\beta < \alpha} f(-, \beta) = f(-, \alpha) \quad (\textit{continuity})$$

and

$$\bigvee_{\alpha \in \mathbb{Q}} f(-, \alpha) = 1 \quad (\textit{properness}).$$

Similarly, a map $g : \overline{\mathfrak{L}}_l(\mathbb{R}) \to L$ is a lower semicontinuous real function if and only if $\{g(\alpha, -)\}_{\alpha \in \mathbb{Q}}$ is a *proper continuous descending chain* in L, that is,

$$\alpha \leqslant \beta \implies g(\alpha, -) \geqslant g(\beta, -) \quad (\textit{descending})$$
$$\bigvee_{\beta > \alpha} g(\beta, -) = g(\alpha, -) \quad (\textit{continuity})$$
$$\bigvee_{\alpha \in \mathbb{Q}} g(\alpha, -) = 1 \quad (\textit{properness}).$$

Of course, frame homomorphisms $f : \mathfrak{L}_u(\mathbb{R}) \to L$ (respectively frame homomorphisms $g : \mathfrak{L}_l(\mathbb{R}) \to L$) correspond precisely to continuous ascending chains (respectively continuous descending chains) in L (cf. [12, Lemma 1.1]).

We write $\overline{\mathcal{R}}_u(L)$ for the set of all upper semicontinuous functions on L, $\mathcal{R}_u(L)$ for the set of all frame homomorphisms $f : \mathfrak{L}_u(\mathbb{R}) \to L$, $\overline{\mathcal{R}}_l(L)$ for the set of all lower semicontinuous functions on L and $\mathcal{R}_l(L)$ for the set of all frame homomorphisms $f : \mathfrak{L}_l(\mathbb{R}) \to L$.

$\mathcal{R}_u(L)$ (and, in particular, $\overline{\mathcal{R}}_u(L)$) is partially ordered by

$$f_1 \leqslant f_2 \equiv f_1(-, \alpha) \geqslant f_2(-, \alpha) \quad \text{for every } \alpha \in \mathbb{Q}$$

and $\mathcal{R}_l(L)$ (in particular, $\overline{\mathcal{R}}_l(L)$) is partially ordered by

$$g_1 \leqslant g_2 \equiv g_1(\alpha, -) \leqslant g_2(\alpha, -) \quad \text{for every } \alpha \in \mathbb{Q}.$$

Notice that $(\overline{\mathcal{R}}_l(L), \leqslant)$ is a poset with unit $\mathbb{1}$, given by $\mathbb{1}(\alpha, -) = 1$ for every $\alpha \in \mathbb{Q}$, and $(\mathcal{R}_l(L), \leqslant)$ is a frame, with zero $\mathbb{O}$ given by $\mathbb{O}(\alpha, -) = 0$ for every $\alpha \in \mathbb{Q}$. This is an interesting feature of the theory of localic semicontinuous functions, that shows that this theory is more general and interesting than the classical one even when the frame is spatial (cf. [12]).

Dually, $(\overline{\mathcal{R}}_u(L), \leqslant)$ is a poset with zero $\mathbb{O}$, given by $\mathbb{O}(-, \alpha) = 1$ for every $\alpha \in \mathbb{Q}$, and $(\mathcal{R}_l(L), \leqslant)$ is a co-frame.

For $f \in \mathcal{R}_u(L)$ and $g \in \mathcal{R}_l(L)$ we define

$$f \leqslant g \equiv f(-, \alpha) \vee g(\beta, -) = 1 \quad \text{whenever } \beta < \alpha$$

and

$$g \leqslant f \equiv f(-, \alpha) \wedge g(\alpha, -) = 0 \quad \text{for every } \alpha \in \mathbb{Q}.$$

Let $x \in L$. It is easy to see that the correspondence

$$(-, \alpha) \mapsto \begin{cases} 1 & \text{if } \alpha > 1, \\ x & \text{if } 0 < \alpha \leqslant 1, \\ 0 & \text{if } \alpha \leqslant 0 \end{cases}$$

defines an upper semicontinuous function $\chi_x^u : \bar{\mathfrak{L}}_u(\mathbb{R}) \to L$; similarly,

$$(\alpha, -) \mapsto \begin{cases} 1 & \text{if } \alpha < 0, \\ x & \text{if } 0 \leqslant \alpha < 1, \\ 0 & \text{if } \alpha \geqslant 1 \end{cases}$$

defines a lower semicontinuous function $\chi_x^l : \bar{\mathfrak{L}}_l(\mathbb{R}) \to L$.

The following proposition has a straightforward proof.

Proposition 2.1. *For any $x, y \in L$,*

(a) *if $x \leqslant y$ then $\chi_x^u \geqslant \chi_y^u$ and $\chi_x^l \leqslant \chi_y^l$,*
(b) *$\chi_x^u \leqslant \chi_y^l$ if and only if $x \vee y = 1$.*

It is also easy to see that continuous real functions $h \in \overline{\mathcal{R}}(L)$ are completely described by *proper continuous chains* on L, that is, pairs (f, g), with $f \in \overline{\mathcal{R}}_u(L)$ and $g \in \overline{\mathcal{R}}_l(L)$, such that $f \leqslant g$ and $g \leqslant f$: for each $h \in \overline{\mathcal{R}}(L)$ the corresponding $f \in \overline{\mathcal{R}}_u(L)$ is given by $f(-, p) = \bigvee_{q \in \mathbb{Q}} h(q, p)$, for every $p \in \mathbb{Q}$, and the corresponding $g \in \overline{\mathcal{R}}_l(L)$ is given by $g(p, -) = \bigvee_{q \in \mathbb{Q}} h(p, q)$, for every $p \in \mathbb{Q}$; conversely, for each proper continuous chain (f, g), the corresponding $h \in \overline{\mathcal{R}}(L)$ is given by $h(p, q) = f(-, q) \wedge g(p, -)$, for every $p, q \in \mathbb{Q}$ [12, Lemma 1.1].

Similarly, maps $h \in \mathcal{R}(L)$ are completely described by *continuous chains* on L, that is, pairs (f, g), with $f \in \mathcal{R}_u(L)$ and $g \in \mathcal{R}_l(L)$, such that $f \leqslant g$ and $g \leqslant f$ [12].

Proposition 2.2. *Let (f, g) be a continuous chain in L. Then*:

(a) $\bigvee_{\alpha\in\mathbb{Q}} \nabla_{f(-,\alpha)} = \bigvee_{\alpha\in\mathbb{Q}} \Delta_{g(\alpha,-)}$;
(b) $\bigvee_{\alpha\in\mathbb{Q}} \Delta_{f(-,\alpha)} = \bigvee_{\alpha\in\mathbb{Q}} \nabla_{g(\alpha,-)}$.

Proof. We only prove (a) since (b) may be proved similarly.

For each $\alpha \in \mathbb{Q}$, $f(-,\alpha) \wedge g(\alpha,-) = 0$, so $\nabla_{f(-,\alpha)} \leqslant \Delta_{g(\alpha,-)}$. Therefore

$$\bigvee_{\alpha\in\mathbb{Q}} \nabla_{f(-,\alpha)} \leqslant \bigvee_{\alpha\in\mathbb{Q}} \Delta_{g(\alpha,-)}.$$

On the other hand, for each $\alpha \in \mathbb{Q}$, $f(-,\beta) \vee g(\alpha,-) = 1$ whenever $\beta > \alpha$. Thus $\nabla_{g(\alpha,-)} \vee (\bigvee_{\beta\in\mathbb{Q}} \nabla_{f(-,\beta)}) = 1$ and, consequently,

$$\begin{aligned}\Delta_{g(\alpha,-)} &= \Delta_{g(\alpha,-)} \wedge \left(\nabla_{g(\alpha,-)} \vee \left(\bigvee_{\beta\in\mathbb{Q}} \nabla_{f(-,\beta)}\right)\right)\\ &= \Delta_{g(\alpha,-)} \wedge \left(\bigvee_{\beta\in\mathbb{Q}} \nabla_{f(-,\beta)}\right).\end{aligned}$$

Hence $\bigvee_{\alpha\in\mathbb{Q}} \Delta_{g(\alpha,-)} \leqslant \bigvee_{\alpha\in\mathbb{Q}} \nabla_{f(-,\alpha)}$. $\square$

$\overline{\mathcal{R}}(L)$ may be partially ordered by

$$(f_1, g_1) \leqslant (f_2, g_2) \equiv f_1 \leqslant f_2$$

(which is easily seen to be equivalent to $g_1 \leqslant g_2$, also $f_1 \leqslant g_2$, also $g_1 \leqslant f_2$).

3. The basic lemmas

In order to simplify the notation, given $f \in \mathcal{R}_u(L)$ and $g \in \mathcal{R}_l(L)$, we denote throughout the element $f(-,\alpha)$ by f_α and the element $g(\alpha,-)$ by g_α.

Lemma 3.1. *Let $f \in \mathcal{R}_u(L)$ and $g \in \mathcal{R}_l(L)$. If $f \leqslant g$ then*:

(a) *For every $\alpha \in \mathbb{Q}$ and every $\gamma > \alpha$, $f_\gamma \vee (\bigwedge_{\beta<\alpha} g_\beta) = 1$.*
(b) *For every $\alpha \in \mathbb{Q}$ and every $\gamma < \alpha$, $(\bigwedge_{\beta>\alpha} f_\beta) \vee g_\gamma = 1$.*

Proof. We only prove (a) (assertion (b) may be proved in a similar way):

Since $g_\beta \geqslant g_\alpha$ for every $\beta < \alpha$, we have $\bigwedge_{\beta<\alpha} g_\beta \geqslant g_\alpha$ and therefore

$$f_\gamma \vee \left(\bigwedge_{\beta<\alpha} g_\beta\right) \geqslant f_\gamma \vee g_\alpha = 1. \quad \square$$

Notice that Lemma 3.1 holds, more generally, for any ascending chain f and any descending chain g.

The following lemma gives us the characterization of normal frames which will be fundamental in our approach.

Lemma 3.2. *A frame L is normal if and only if, for every countable subsets $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ of L, satisfying $x_i \vee (\bigwedge_{j\in\mathbb{N}} y_j) = 1$ and $y_i \vee (\bigwedge_{j\in\mathbb{N}} x_j) = 1$ for every $i \in \mathbb{N}$, there exists $u \in L$ such that $x_i \vee u = 1$ and $y_i \vee u^* = 1$ for every $i \in \mathbb{N}$.*

Proof. Let L be a normal frame. Then, for each $i \in \mathbb{N}$, $x_i \vee (\bigwedge_{j\in\mathbb{N}} y_j) = 1$ implies, by normality, the existence of $u_i \in L$ satisfying $x_i \vee u_i = 1$ and $(\bigwedge_{j\in\mathbb{N}} y_j) \vee u_i^* = 1$. Similarly, $y_i \vee (\bigwedge_{j\in\mathbb{N}} x_j) = 1$ implies that there exists $v_i \in L$ such that $y_i \vee v_i = 1$ and $(\bigwedge_{j\in\mathbb{N}} x_j) \vee v_i^* = 1$. Then, for each $i \in \mathbb{N}$, we have

$$x_i \vee \left(\bigwedge_{k=1}^{i} v_k^*\right) \geqslant \left(\bigwedge_{j\in\mathbb{N}} x_j\right) \vee \left(\bigwedge_{k=1}^{i} v_k^*\right) = \bigwedge_{k=1}^{i}\left(\left(\bigwedge_{j\in\mathbb{N}} x_j\right) \vee v_k^*\right) = 1$$

and, similarly,

$$y_i \vee \left(\bigwedge_{k=1}^{i} u_k^*\right) \geqslant \left(\bigwedge_{j\in\mathbb{N}} y_j\right) \vee \left(\bigwedge_{k=1}^{i} u_k^*\right) = \bigwedge_{k=1}^{i}\left(\left(\bigwedge_{j\in\mathbb{N}} y_j\right) \vee u_k^*\right) = 1.$$

Now define, for each $i \in \mathbb{N}$, $u_i' := u_i \wedge \bigwedge_{k=1}^{i} v_k^*$ and $v_i' := v_i \wedge \bigwedge_{k=1}^{i} u_k^*$ and consider $u := \bigvee_{i\in\mathbb{N}} u_i'$ and $v := \bigvee_{i\in\mathbb{N}} v_i'$. Evidently,

$$x_i \vee u \geqslant x_i \vee u_i' = (x_i \vee u_i) \wedge \left(x_i \vee \left(\bigwedge_{k=1}^{i} v_k^*\right)\right) = 1$$

and

$$y_i \vee v \geqslant y_i \vee v_i' = (y_i \vee v_i) \wedge \left(y_i \vee \left(\bigwedge_{k=1}^{i} u_k^*\right)\right) = 1.$$

Furthermore,

$$u \wedge v = \bigvee_{i\in\mathbb{N}} \bigvee_{j\in\mathbb{N}} (u_i' \wedge v_j') = \bigvee_{i\in\mathbb{N}} \bigvee_{j\in\mathbb{N}} \left(u_i \wedge v_j \wedge \bigwedge_{k=1}^{i} v_k^* \wedge \bigwedge_{l=1}^{j} u_l^*\right) = 0.$$

Thus $v \leqslant u^*$ and, consequently, $y_i \vee u^* \geqslant y_i \vee v = 1$.

The converse is trivial. □

In the sequel let $\{\alpha_i \mid i \in \mathbb{N}\}$ be an indexation of $\mathbb{Q}$ by natural numbers.

Lemma 3.3. *Let L be a normal frame. Given $f \in \mathcal{R}_u(L)$ and $g \in \mathcal{R}_l(L)$ such that $f \leqslant g$, there exists $\{u_{\alpha_i}\}_{i\in\mathbb{N}} \subseteq L$ satisfying*

$$(\gamma > \alpha_i) \implies (f_\gamma \vee u_{\alpha_i} = 1),$$
$$(\delta < \alpha_i) \implies (g_\delta \vee u_{\alpha_i}^* = 1),$$
$$(\alpha_{j_1} < \alpha_{j_2}) \implies (u_{\alpha_{j_1}} \vee u_{\alpha_{j_2}}^* = 1).$$

Proof. We shall prove this by showing, applying induction over $\mathbb{N}$, that, for every $i \in \mathbb{N}$, there exists $u_{\alpha_i} \in L$ such that

$$(\gamma > \alpha_i) \implies (f_\gamma \vee u_{\alpha_i} = 1),$$
$$(\delta < \alpha_i) \implies (g_\delta \vee u^*_{\alpha_i} = 1),$$
$$(\alpha_{j_1} < \alpha_{j_2}) \implies (u_{\alpha_{j_1}} \vee u^*_{\alpha_{j_2}} = 1), \quad \text{for all } j_1, j_2 \leqslant i.$$

Since $f \leqslant g$, we may assume, by virtue of Lemma 3.1, that $f_\gamma \vee (\bigwedge_{\beta<\alpha_1} g_\beta) = 1$, for every $\gamma > \alpha_1$, and $(\bigwedge_{\beta>\alpha_1} f_\beta) \vee g_\delta = 1$, for every $\delta < \alpha_1$. Then, by Lemma 3.2, there exists $u_{\alpha_1} \in L$ satisfying $f_\gamma \vee u_{\alpha_1} = 1$, for every $\gamma > \alpha_1$, and $g_\delta \vee u^*_{\alpha_1} = 1$, for every $\delta < \alpha_1$, which shows the first step of the induction.

Now, consider $k \in \mathbb{N}$, and assume, by inductive hypothesis, that for any $i < k$ there is $u_{\alpha_i} \in L$ satisfying $f_\gamma \vee u_{\alpha_i} = 1$, for every $\gamma > \alpha_i$, $g_\delta \vee u^*_{\alpha_i} = 1$, for every $\delta < \alpha_i$, and

$$(\alpha_{j_1} < \alpha_{j_2}) \implies (u_{\alpha_{j_1}} \vee u^*_{\alpha_{j_2}} = 1), \quad \text{for all } j_1, j_2 \leqslant k-1.$$

Further, define

$$\{f'_n\}_{n\in\mathbb{N}} := \{f_\gamma \mid \gamma > \alpha_k\} \cup \{u^*_{\alpha_i} \mid i < k,\ \alpha_k < \alpha_i\},$$
$$\{g'_n\}_{n\in\mathbb{N}} := \{g_\delta \mid \delta < \alpha_k\} \cup \{u_{\alpha_i} \mid i < k,\ \alpha_i < \alpha_k\}.$$

Then $\{f'_n\}_{n\in\mathbb{N}}$ and $\{g'_n\}_{n\in\mathbb{N}}$ satisfy the conditions of Lemma 3.2:

(1) For each $\gamma > \alpha_k$,

$$f_\gamma \vee \left(\bigwedge_{\delta<\alpha_k} g_\delta \wedge \bigwedge_{i<k,\, \alpha_i<\alpha_k} u_{\alpha_i} \right)$$
$$= \left(f_\gamma \vee \left(\bigwedge_{\delta<\alpha_k} g_\delta \right)\right) \wedge \left(f_\gamma \vee \left(\bigwedge_{i<k,\ \alpha_i<\alpha_k} u_{\alpha_i} \right)\right) = 1$$

since $f_\gamma \vee (\bigwedge_{\delta<\alpha_k} g_\delta) = 1$, by Lemma 3.1, and, by inductive hypothesis,

$$f_\gamma \vee \left(\bigwedge_{i<k,\ \alpha_i<\alpha_k} u_{\alpha_i} \right) = \bigwedge_{i<k,\, \alpha_i<\alpha_k} (f_\gamma \vee u_{\alpha_i}) = 1.$$

(2) For each $i < k$ such that $\alpha_k < \alpha_i$,

$$u^*_{\alpha_i} \vee \left(\bigwedge_{\delta<\alpha_k} g_\delta \wedge \bigwedge_{j<k,\, \alpha_j<\alpha_k} u_{\alpha_j} \right)$$
$$= \left(u^*_{\alpha_i} \vee \left(\bigwedge_{\delta<\alpha_k} g_\delta \right)\right) \wedge \left(u^*_{\alpha_i} \vee \left(\bigwedge_{j<k,\, \alpha_j<\alpha_k} u_{\alpha_j} \right)\right) = 1,$$

since $u^*_{\alpha_i} \vee (\bigwedge_{\delta<\alpha_k} g_\delta) \geqslant u^*_{\alpha_i} \vee g_{\alpha_k} = 1$ and, by inductive hypothesis,

$$u^*_{\alpha_i} \vee \left(\bigwedge_{j<k,\ \alpha_j<\alpha_k} u_{\alpha_j} \right) = \bigwedge_{j<k,\, \alpha_j<\alpha_k} (u^*_{\alpha_i} \vee u_{\alpha_j}) = 1.$$

(3) Similarly to (1) and (2), respectively, one can prove that, for each $\delta < \alpha_k$,

$$\Big(\bigwedge_{\gamma>\alpha_k} f_\gamma \wedge \bigwedge_{i<k,\alpha_k<\alpha_i} u^*_{\alpha_i}\Big) \vee g_\delta = 1,$$

and that, for each $i < k$ satisfying $\alpha_i < \alpha_k$,

$$\Big(\bigwedge_{\gamma>\alpha_k} f_\gamma \wedge \bigwedge_{i<k,\alpha_k<\alpha_i} u^*_{\alpha_i}\Big) \vee u_{\alpha_i} = 1.$$

So, it follows from Lemma 3.2, that there exists $u_{\alpha_k} \in L$ such that

$$\begin{aligned}
&\forall \gamma > \alpha_k, \quad f_\gamma \vee u_{\alpha_k} = 1,\\
&(\alpha_k < \alpha_i) \implies (u^*_{\alpha_i} \vee u_{\alpha_k} = 1) \quad \text{for all } i < k,\\
&\forall \delta < \alpha_k, \quad g_\delta \vee u^*_{\alpha_k} = 1,\\
&(\alpha_i < \alpha_k) \implies (u_{\alpha_i} \vee u^*_{\alpha_k} = 1) \quad \text{for all } i < k.
\end{aligned}$$

This, together with the inductive hypothesis, gives us the required $u_{\alpha_k} \in L$ satisfying

$$\begin{aligned}
&\forall \gamma > \alpha_k, \quad f_\gamma \vee u_{\alpha_k} = 1,\\
&\forall \delta < \alpha_k, \quad g_\delta \vee u^*_{\alpha_k} = 1,\\
&(\alpha_{j_1} < \alpha_{j_2}) \implies (u_{\alpha_{j_1}} \vee u^*_{\alpha_{j_2}} = 1) \quad \text{for all } j_1, j_2 \leqslant k. \quad \square
\end{aligned}$$

4. Interpolation theorems

Given $f \in \mathcal{R}_u(L)$, $g \in \mathcal{R}_l(L)$ and $h = (h_1, h_2) \in \mathcal{R}(L)$, we write $f \leqslant h$ whenever $f \leqslant h_1$ (or, equivalently, $f \leqslant h_2$) and we write $h \leqslant g$ whenever $h_1 \leqslant g$ (or, equivalently, $h_2 \leqslant g$).

Next, we present a new proof of Theorem 2.1 of [12].

Theorem 4.1. *A frame L is normal if and only if for every $f \in \mathcal{R}_u(L)$ and $g \in \mathcal{R}_l(L)$ with $f \leqslant g$ there exists $h \in \mathcal{R}(L)$ such that $f \leqslant h \leqslant g$.*

Proof. Suppose that L is normal and consider $f \in \mathcal{R}_u(L)$ and $g \in \mathcal{R}_l(L)$ with $f \leqslant g$. By virtue of Lemma 3.3, we may construct $\{u_{\alpha_i}\}_{i\in\mathbb{N}} \subseteq L$ such that, for every $i, j_1, j_2 \in \mathbb{N}$,

(a) $\forall \gamma > \alpha_i\, f_\gamma \vee u_{\alpha_i} = 1$,
(b) $\forall \delta < \alpha_i\, g_\delta \vee u^*_{\alpha_i} = 1$,
(c) $(\alpha_{j_1} < \alpha_{j_2}) \implies (u_{\alpha_{j_1}} \vee u^*_{\alpha_{j_2}} = 1)$.

Then $h_1(-,\alpha_k) := \bigvee_{\alpha_j<\alpha_k} u^*_{\alpha_j}$ defines a homomorphism $h_1 : \mathfrak{L}_u(\mathbb{R}) \to L$; indeed, $\{h_1(-,\alpha_k)\}_{k\in\mathbb{N}}$ is clearly an ascending chain in L; further, it is continuous since, by the density of $\mathbb{Q}$ in itself,

$$\bigvee_{\alpha_k<\alpha_i} h_1(-,\alpha_k) = \bigvee_{\alpha_k<\alpha_i} \bigvee_{\alpha_j<\alpha_k} u^*_{\alpha_j} = \bigvee_{\alpha_j<\alpha_i} u^*_{\alpha_j} = h_1(-,\alpha_i).$$

Analogously, $h_2(\alpha_k,-) := \bigvee_{\alpha_j>\alpha_k} u_{\alpha_j}$ defines a homomorphism $h_2 : \mathfrak{L}_l(\mathbb{R}) \to L$.

In order to show that the pair (h_1, h_2) gives the required continuous real function h on L, it remains to show that this is a continuous chain in L (that is, (1) $h_1 \leqslant h_2$ and (2) $h_2 \leqslant h_1$) interpolating f and g (that is, (3) $f \leqslant h_1$ and (4) $h_2 \leqslant g$).

(1) Let $\alpha_i < \alpha_j$ and consider $\alpha_k, \alpha_l \in \mathbb{Q}$ such that $\alpha_i < \alpha_l < \alpha_k < \alpha_j$. Then

$$h_1(-, \alpha_j) \vee h_2(\alpha_i, -) = \bigvee_{\alpha_k < \alpha_j} u^*_{\alpha_k} \vee \bigvee_{\alpha_l > \alpha_i} u_{\alpha_l} \geqslant u^*_{\alpha_k} \vee u_{\alpha_l} = 1.$$

(2) For every $k \in \mathbb{N}$ we have

$$h_1(-, \alpha_k) \wedge h_2(\alpha_k, -) = \Big(\bigvee_{\alpha_i < \alpha_k} u^*_{\alpha_i}\Big) \wedge \Big(\bigvee_{\alpha_j > \alpha_k} u_{\alpha_j}\Big) = \bigvee_{\alpha_i < \alpha_k,\, \alpha_j > \alpha_k} (u^*_{\alpha_i} \wedge u_{\alpha_j}).$$

By (c), $u_{\alpha_i} \vee u^*_{\alpha_j} = 1$ and, consequently, using (1.1), $u^*_{\alpha_i} \wedge u^{**}_{\alpha_j} = 0$, that is, $u^*_{\alpha_i} \leqslant u^{***}_{\alpha_j} = u^*_{\alpha_j}$. Thus $u^*_{\alpha_i} \wedge u_{\alpha_j} \leqslant u^*_{\alpha_j} \wedge u_{\alpha_j} = 0$ and $\bigvee_{\alpha_i < \alpha_k,\, \alpha_j > \alpha_k} (u^*_{\alpha_i} \wedge u_{\alpha_j}) = 0$.

(3) We need to show that $f_{\alpha_k} \geqslant \bigvee_{\alpha_j < \alpha_k} u^*_{\alpha_j}$ for every $k \in \mathbb{N}$, that is, $u^*_{\alpha_j} \leqslant f_{\alpha_k}$ for every $k \in \mathbb{N}$ and $\alpha_j < \alpha_k$. This is obvious, since $f_{\alpha_k} \vee u_{\alpha_j} = 1$ by (a).

(4) By (b), $g_{\alpha_k} \vee u^*_{\alpha_j} = 1$ whenever $\alpha_k < \alpha_j$. This implies, using property (1.2), that $u_{\alpha_j} \leqslant u^{**}_{\alpha_j} \leqslant g_{\alpha_k}$. Hence $h_2(\alpha_k, -) = \bigvee_{\alpha_j > \alpha_k} u_{\alpha_j} \leqslant g_{\alpha_k}$.

For the converse we apply the interpolation hypothesis with f, g characteristic functions of appropriate elements:

Let $x \vee y = 1$. By Proposition 2.1, $\chi^u_x \leqslant \chi^l_y$, so there is $h \in \mathcal{R}(L)$ such that $\chi^u_x \leqslant h \leqslant \chi^l_y$. Let $a := h_2(1/2, -)$. Then

$$1 = \chi^u_x(-, 3/4) \vee h_2(1/2, -) = x \vee a,$$

since $\chi^u_x \leqslant h_2$, and, on the other hand,

$$y \vee a^* = \chi^l_y(1/4, -) \vee h_2(1/2, -)^* \geqslant \chi^l_y(1/4, -) \vee h_1(-, 1/2) = 1,$$

since $h_2 \leqslant h_1 \leqslant \chi^l_y$. This proves that L is normal. □

The following example shows that Theorem 4.1 is no longer true if we replace $\mathcal{R}_u(L)$, $\mathcal{R}_l(L)$ and $\mathcal{R}(L)$ by, respectively, $\overline{\mathcal{R}}_u(L)$, $\overline{\mathcal{R}}_l(L)$ and $\overline{\mathcal{R}}(L)$ (and, therefore, it shows that the statement of Theorem 2.2 of [12] is not valid for all functions f, g).

Example 4.2. Let L be the complete chain $\mathbb{Z} \cup \{-\infty, \infty\}$ obtained by adding a top element ∞ and a bottom element $-\infty$ to $\mathbb{Z}$. This is, clearly, a normal frame. Consider $f : \overline{\mathfrak{L}_u}(\mathbb{R}) \to L$ given by $f(-, \alpha) := \min\{n \in \mathbb{Z} \mid \alpha \leqslant n\}$ (that we denote by $\bar{\alpha}$) and $g : \overline{\mathfrak{L}_l}(\mathbb{R}) \to L$ given by $g(\alpha, -) := \infty$. Of course, $f \leqslant g$. Now consider $h \in \mathcal{R}(L)$ given by Theorem 4.1. Then $h(-, \alpha) \leqslant \bar{\alpha}$ for all $\alpha \in \mathbb{Q}$, since $f \leqslant h$; therefore, for every $\alpha \in \mathbb{Q}$ and $\beta < \alpha$, $\bar{\alpha} \vee h(\beta, -) \geqslant h(-, \alpha) \vee h(\beta, -) = \infty$. Consequently, $h(\beta, -) = \infty$ for all $\beta \in \mathbb{Q}$. But $h(-, \beta) \wedge h(\beta, -) = -\infty$, so $h(-, \beta) = -\infty$ for every $\beta \in \mathbb{Q}$. Thus $\bigvee_{\beta \in \mathbb{Q}} h(-, \beta) = -\infty$, that is, $h(\bigvee_{\beta \in \mathbb{Q}} (-, \beta)) = -\infty$, which shows that $h \notin \overline{\mathcal{R}}(L)$.

This raises the question of finding the conditions under which the h provided by Theorem 4.1 belongs to $\overline{\mathcal{R}}(L)$ whenever $f \in \overline{\mathcal{R}}_u(L)$ and $g \in \overline{\mathcal{R}}_l(L)$.

Proposition 4.3. *Let $f \in \overline{\mathcal{R}}_u(L)$ and $g \in \overline{\mathcal{R}}_l(L)$ with $f \leqslant g$ and consider $h := (h_1, h_2) \in \mathcal{R}(L)$ given by Theorem* 4.1. *Then*:

(a) *$h_1 \in \overline{\mathcal{R}}_u(L)$ if and only if for each $\alpha \in \mathbb{Q}$ there exists $\beta \in \mathbb{Q}$ such that*

$$f(-,\alpha) \wedge g(\beta,-) \leqslant \bigvee_{\gamma\in\mathbb{Q}} h_1(-,\gamma);$$

(b) *$h_2 \in \overline{\mathcal{R}}_l(L)$ if and only if for each $\beta \in \mathbb{Q}$ there exists $\alpha \in \mathbb{Q}$ such that*

$$f(-,\alpha) \wedge g(\beta,-) \leqslant \bigvee_{\gamma\in\mathbb{Q}} h_2(\gamma,-).$$

Proof. We only prove (a) since assertion (b) may be proved dually.

The implication "⇒" is trivial. Conversely, for each $\alpha \in \mathbb{Q}$ consider $\beta \in \mathbb{Q}$ for which $f(-,\alpha) \wedge g(\beta,-) \leqslant \bigvee_{\gamma\in\mathbb{Q}} h_1(-,\gamma)$. Since $h_1 \leqslant g$, we have

$$g(\beta,-) \vee \Big(\bigvee_{\gamma\in\mathbb{Q}} h_1(-,\gamma)\Big) = 1.$$

Thus

$$\begin{aligned} f(-,\alpha) &= f(-,\alpha) \wedge \Big(g(\beta,-) \vee \Big(\bigvee_{\gamma\in\mathbb{Q}} h_1(-,\gamma)\Big)\Big) \\ &\leqslant \bigvee_{\gamma\in\mathbb{Q}} h_1(-,\gamma) \end{aligned}$$

for every $\alpha \in \mathbb{Q}$, from which it follows that $\bigvee_{\alpha\in\mathbb{Q}} f(-,\alpha) \leqslant \bigvee_{\gamma\in\mathbb{Q}} h_1(-,\gamma)$. Hence $\bigvee_{\gamma\in\mathbb{Q}} h_1(-,\gamma) = 1$ and $h_1 \in \overline{\mathcal{R}}_u(L)$. □

Corollary 4.4. *Let $f \in \overline{\mathcal{R}}_u(L)$ and $g \in \overline{\mathcal{R}}_l(L)$ with $f \leqslant g$ and consider $h = (h_1, h_2) \in \mathcal{R}(L)$ given by Theorem* 4.1. *Then*:

(a) *Each one of the following conditions implies that $h_1 \in \overline{\mathcal{R}}_u(L)$.*
  (a_1) $\forall\alpha \in \mathbb{Q}\ \exists\beta \in \mathbb{Q}$: $f(-,\alpha) \wedge g(\beta,-) = 0$.
  (a_2) $\bigvee_{\alpha\in\mathbb{Q}} \Delta_{g(\alpha,-)} = 1$.
(b) *Each one of the following conditions implies that $h_2 \in \overline{\mathcal{R}}_l(L)$.*
  (b_1) $\forall\beta \in \mathbb{Q}\ \exists\alpha \in \mathbb{Q}$: $f(-,\alpha) \wedge g(\beta,-) = 0$.
  (b_2) $\bigvee_{\alpha\in\mathbb{Q}} \Delta_{f(-,\alpha)} = 1$.

Proof. (a_1) It follows immediately from Proposition 4.3(a).

(a_2) Since $h_1 \in \overline{\mathcal{R}}_u(L)$ if and only if $\bigvee_{\alpha\in\mathbb{Q}} h_1(-,\alpha) = 1$ and, in terms of congruences, this condition means that $\bigvee_{\alpha\in\mathbb{Q}} \nabla_{h_1(-,\alpha)} = 1$, we may conclude, by Proposition 2.2, that $h_1 \in \overline{\mathcal{R}}_u(L)$ if and only if $\bigvee_{\alpha\in\mathbb{Q}} \Delta_{h_2(\alpha,-)} = 1$. Since $h_2 \leqslant g$, this is clearly implied by condition (a_2).

(b_1) It follows immediately from Proposition 4.3(b).

(b_2) Proved similarly to (a_2). □

Remarks 4.5.

(1) Notice that condition (a_1) (respectively (b_1)) trivially implies condition (a_2) (respectively (b_2)).
(2) A careful analysis of the proof of Theorem 2.2 in [12] reveals that conditions (a_2) and (b_2) are precisely the conditions on f and g under which the proof works without problems. Our proof above is more direct and transparent and avoids the use of the sublocale lattice.
(3) Condition (b_2) implies, in particular, that $\bigwedge_{\alpha\in\mathbb{Q}} f(-,\alpha)=0$. The converse is not true, and $\bigwedge_{\alpha\in\mathbb{Q}} f(-,\alpha)=0$ is not a sufficient condition for $h_2\in\overline{\mathcal{R}}_l(L)$, as Example 4.2 above shows. Condition (b_2) holds if $\bigwedge_{\alpha\in\mathbb{Q}} f(-,\alpha)=0$ and $\{f(-,\alpha)\}_{\alpha\in\mathbb{Q}}$ is, what we call in [5], an *interior-preserving cover* of L (i.e., $\bigvee_{\alpha\in\mathbb{Q}}\Delta_{f(-,\alpha)}=\Delta_{\bigwedge_{\alpha\in\mathbb{Q}} f(-,\alpha)}$).

We are finally in the position to establish the pointfree extension of Katětov–Tong interpolation theorem.

Theorem 4.6. *Let L be a frame. The following assertions are equivalent*:

(i) *L is normal.*
(ii) *For any $f\in\overline{\mathcal{R}}_u(L)$ and $g\in\overline{\mathcal{R}}_l(L)$, with $f\leqslant g$, satisfying conditions* (a_1) *and* (b_1), *there exists $h\in\overline{\mathcal{R}}(L)$ such that $f\leqslant h\leqslant g$.*
(iii) *For any $f\in\overline{\mathcal{R}}_u(L)$ and $g\in\overline{\mathcal{R}}_l(L)$, with $f\leqslant g$, satisfying conditions* (a_2) *and* (b_2), *there exists $h\in\overline{\mathcal{R}}(L)$ such that $f\leqslant h\leqslant g$.*

Proof. (i) $\Rightarrow$ (iii): It is clear by Theorem 4.1 and Corollary 4.4.

(iii) $\Rightarrow$ (ii): It is obvious since conditions (a_1) and (b_1) imply conditions (a_2) and (b_2), respectively.

(ii) $\Rightarrow$ (i): Since $\chi_x^u\in\overline{\mathcal{R}}_u(L)$ and $\chi_y^l\in\overline{\mathcal{R}}_l(L)$ satisfy conditions (a_1) and (b_1), for every $x,y\in L$, the proof of the normality of L goes on as in the proof of Theorem 4.1. □

5. Some noteworthy applications

The Katětov–Tong interpolation theorem has broad application in the theory of topological spaces, and so it should come as no surprise that its localic analogue (Theorem 4.6) also has broad application for frames. We outline a few of these applications in this section together with the remark that classical Katětov–Tong theorem follows immediately from its localic analogue.

5.1. The classical Katětov–Tong theorem

Applied to $\mathcal{O}X$ for a normal space X, the "(i) $\Rightarrow$ (ii)" part of 4.6 yields the non-trivial implication of the classical Katětov–Tong interpolation theorem [10,13]:

For any upper semicontinuous real-valued function $f:X\to\mathbb{R}$ and any lower semicontinuous real-valued function $g:X\to\mathbb{R}$ with $f\leqslant g$, let $\tilde{f}:\overline{\mathfrak{L}}_u(\mathbb{R})\to\mathcal{O}X$, defined by $\tilde{f}(-,\alpha) := f^{-1}((-\infty,\alpha))$, and let $\tilde{g} : \overline{\mathfrak{L}}_l(\mathbb{R}) \to \mathcal{O}X$, defined by $\tilde{g}(\alpha,-) := g^{-1}((\alpha,+\infty))$. The condition $f \leqslant g$ implies $f^{-1}((-\infty,\alpha)) \supseteq g^{-1}((\alpha,+\infty))$, thus, $\tilde{f} \leqslant \tilde{g}$. Furthermore, we may assume without loss of generality that $f(x) \geqslant 0$ and $g(x) \leqslant 1$ for every $x \in X$ (cf. [3, 15.13]). This implies that $\tilde{f}$ and $\tilde{g}$ satisfy conditions (a_1) and (b_1). Take $\tilde{h} \in \overline{\mathcal{R}}(\mathcal{O}X)$ as given by the Theorem, and consider $h : X \to \mathbb{R}$ given by (1.3). Then, immediately, $f \leqslant h \leqslant g$.

5.2. *The localic Urysohn's Lemma*

Let L be a normal frame and consider $x, y \in L$ with $x \vee y = 1$. Applied to $\chi_x^u \in \overline{\mathcal{R}}_u(L)$ and $\chi_y^l \in \overline{\mathcal{R}}_l(L)$, the "(i) $\Rightarrow$ (ii)" part of 4.6 yields the non-trivial implication of the localic Urysohn's Lemma [4,1]:

Since χ_x^u and χ_y^l satisfy conditions (a_1) and (b_1), there exists $h \in \overline{\mathcal{R}}(L)$ such that $\chi_x^u \leqslant h \leqslant \chi_y^l$. This is clearly equivalent to $h(0,-) \leqslant y$, $h(-,1) \leqslant x$ and $h((-,0) \vee (1,-)) = 0$.

5.3. *The localic Tietze's Extension Theorem*

Let $\overline{\mathfrak{L}}[a,b] := \uparrow((-,a) \vee (b,-))$, for $a < b$, be a closed interval frame of reals and consider a closed quotient $F := \uparrow x$ of a normal frame L, given by $c_F : L \twoheadrightarrow F$ $(y \mapsto y \vee x)$, and a bounded continuous real function $h : \overline{\mathfrak{L}}[a,b] \to F$. Applied to

$$f : \overline{\mathfrak{L}}_u(\mathbb{R}) \to L$$
$$(-,\alpha) \mapsto \begin{cases} 1 & \text{if } \alpha > b, \\ h((-,\alpha) \vee (b,-)) & \text{if } a < \alpha \leqslant b, \\ 0 & \text{if } \alpha \leqslant a \end{cases}$$

and

$$g : \overline{\mathfrak{L}}_l(\mathbb{R}) \to L$$
$$(\alpha,-) \mapsto \begin{cases} 1 & \text{if } \alpha < a, \\ h((-,a) \vee (\alpha,-)) & \text{if } a \leqslant \alpha < b, \\ 0 & \text{if } \alpha \geqslant b \end{cases}$$

the "(i) $\Rightarrow$ (ii)" part of Theorem 4.6 yields the non-trivial implication of the localic Tietze's extension theorem ([16]; see also Section 8.3 of [2]).

Indeed, it is obvious that f and g satisfy conditions (a_1) and (b_1) and $f \leqslant g$. So, there is $\tilde{h} : \overline{\mathfrak{L}}(\mathbb{R}) \to L$ such that $f \leqslant \tilde{h} \leqslant g$. Since $\tilde{h}((-,a) \vee (b,-)) \leqslant f(-,a) \vee g(b,-) = 0$, the restriction of $\tilde{h}$ to $\overline{\mathfrak{L}}[a,b]$ is a frame homomorphism $\tilde{h} : \overline{\mathfrak{L}}[a,b] \to L$. Finally, let us check that this is the required extension of h to L

$$\begin{tikzcd} \overline{\mathfrak{L}}[a,b] \arrow[r, "h"] \arrow[dr, "\tilde{h}"'] & F \\ & L \arrow[u, two heads, "c_F"'] \end{tikzcd}$$

that is, $\tilde{h}(\alpha,\beta) \vee x = h(\alpha,\beta)$, for every $\alpha, \beta \in \mathbb{Q}$.

The inequality $\tilde{h}(\alpha, \beta) \vee x \leqslant h(\alpha, \beta)$ follows immediately from the fact that $f \leqslant \tilde{h} \leqslant g$. Conversely, we have

$$h(\alpha, \beta) = h(-, \beta) \wedge h(\alpha, -) = \Big(\bigvee_{\gamma<\beta} h(-, \gamma)\Big) \wedge \Big(\bigvee_{\delta>\alpha} h(\delta, -)\Big).$$

But, for each $\gamma < \beta$,

$$\begin{aligned} h(-, \gamma) &= h(-, \gamma) \wedge \big(\tilde{h}(-, \beta) \vee \tilde{h}(\gamma, -)\big) \\ &= \big(h(-, \gamma) \wedge \tilde{h}(-, \beta)\big) \vee \big(h(-, \gamma) \wedge \tilde{h}(\gamma, -)\big) \\ &\leqslant \tilde{h}(-, \beta) \vee \big(h(-, \gamma) \wedge h(\gamma, -)\big) \\ &= \tilde{h}(-, \beta) \vee h(0) \\ &= \tilde{h}(-, \beta) \vee x. \end{aligned}$$

Similarly, $h(\delta, -) \leqslant \tilde{h}(\alpha, -) \vee x$ whenever $\delta > \alpha$. Hence,

$$h(\alpha, \beta) \leqslant \big(\tilde{h}(-, \beta) \vee x\big) \wedge \big(\tilde{h}(\alpha, -) \vee x\big) = \tilde{h}(\alpha, \beta) \vee x.$$

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