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On the location of the eigenvalues of Jacobi matrices

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Abstract

Using some well known concepts on orthogonal polynomials, some recent results on the location of eigenvalues of tridiagonal matrices of very large order are extended. A significant number of important papers are unified.

Keywords: Jacobi matrix; Eigenvalues; Orthogonal polynomials

1. Introduction

There has been increasing interest in tridiagonal matrices in the last few decades in various areas, such as numerical analysis, special functions, partial differential equations, and, naturally, linear algebra. The evaluation of the spectra of the matrices involved seems to be essential.

Our aim is unifying a significant number of important papers which deal with the spectra of tridiagonal matrices, bringing the results together in one place, unifying them in the context of the theory of orthogonal polynomials. We have generalized some recent works on the behavior of the eigenvalues of some tridiagonal matrices which contain a Toeplitz matrix in the upper left block to the case of a general Jacobi matrix. Finally, we give some examples.

2. Orthogonal polynomials

One of the most important tools in the study of orthogonal polynomials is the so-called Favard theorem, which states that any orthogonal polynomial sequence (OPS) $\{P_n\}_{n\geq 0}$ is characterized by a three-term recurrence relation

$$x P_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n = 0, 1, 2, \ldots \tag{2.1}$$

with initial conditions $P_{-1}(x) = 0$ and $P_0(x) = \text{const.} \neq 0$, where $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$ are sequences of complex numbers such that $\alpha_n \gamma_{n+1} \neq 0$ for all $n = 0, 1, 2, \ldots$.

The next proposition is known as the Separation Theorem for the zeros and tells us that the (distinct) zeros of P_n and P_{n+1} are mutually separate.

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Theorem 2.1 *([3, p. 28]). In (2.1) let $\beta_n \in \mathbb{R}$ and $\gamma_{n+1} > 0$ for all $n = 0, 1, 2, \ldots$. Then, for each n, P_n has n real and distinct zeros, denoted in increasing order by $x_{n1} < x_{n2} < \cdots < x_{nn}$. Furthermore, the interlacing inequalities $x_{n+1,i} < x_{ni} < x_{n+1,i+1}$ $(i = 1, \ldots, n)$ hold for every $n = 1, 2, \ldots$.*

Notice that the three-term recurrence relation (2.1) can be written in matrix form as

$$x \begin{pmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_n(x) \end{pmatrix} = J_{n+1} \begin{pmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_n(x) \end{pmatrix} + \alpha_n P_{n+1}(x) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

where J_{n+1} is a Jacobi matrix of order $n + 1$, defined by

$$J_{n+1} := \begin{pmatrix} \beta_0 & \alpha_0 & & & \\ \gamma_1 & \beta_1 & \alpha_1 & & \\ & \gamma_2 & \ddots & \ddots & \\ & & \ddots & \beta_{n-1} & \alpha_{n-1} \\ & & & \gamma_n & \beta_n \end{pmatrix} \quad (n = 0, 1, 2, \ldots).$$

It follows that if $\{x_{nj}\}_{j=1}^{n}$ is the set of zeros of the polynomial P_n, then each x_{nj} is an eigenvalue of the corresponding Jacobi matrix J_n of order n, and an associated eigenvector is $[P_0(x_{nj}), P_1(x_{nj}), \ldots, P_{n-1}(x_{nj})]^t$. From Theorem 2.1 the eigenvalues of J_{n+1} are distinct and interlace strictly with the eigenvalues of J_n.

Given a family of orthogonal polynomials $\{P_n\}_{n\geq 0}$ defined by (2.1) with $\alpha_n = 1$ for all n (so that $\{P_n\}_{n\geq 0}$ is a monic OPS) with $\gamma_n > 0$ for all $n = 1, 2, \ldots$, we may define the *associated polynomials* of order r (r a positive integer) $\{P_n^{(r)}\}_{n\geq 0}$, $n = 0, 1, 2, \ldots$, via the shifted recurrence

$$P_{n+1}^{(r)}(x) = (x - \beta_{n+r}) P_n^{(r)}(x) - \gamma_{n+r} P_{n-1}^{(r)}(x), \quad n = 0, 1, 2, \ldots \tag{2.2}$$

with $P_{-1}^{(r)} = 0$ and $P_0^{(r)} = 1$ [2].

Led by the above definition, Ronveaux and Van Assche constructed in [9] a new family of orthogonal polynomials, the *anti-associated polynomials* for the family $\{P_n\}_{n\geq 0}$, denoted by $\{P_{n+r}^{(-r)}\}_{n\geq 0}$, obtained by pushing down a given Jacobi matrix and by introducing in the empty upper left corner new coefficients β_{-i} $(i = r, r-1, \ldots, 1)$, with 1's on the upper subdiagonal and new coefficients $\gamma_{-i} > 0$, $i = r-1, r-2, \ldots, 0$ on the lower subdiagonal. The new Jacobi matrix is then of the form

$$\left(\begin{array}{cccc|ccc} \beta_{-r} & 1 & & & & & \\ \gamma_{-r+1} & \beta_{-r+1} & 1 & & & & \\ & \gamma_{-r+2} & \ddots & \ddots & & & \\ & & \ddots & \beta_{-1} & 1 & & \\ \hline & & & \gamma_0 & \beta_0 & 1 & \\ & & & & \gamma_1 & \ddots & \ddots \\ & & & & & \ddots & \ddots \end{array}\right).$$

If $\{Q_n\}_{n\geq 0}$ satisfies $Q_{-1} = 0$, $Q_0 = 1$ and

$$Q_{n+1}(x) = (x - \beta_{-r+n}) Q_n(x) - \gamma_{-r+n} Q_{n-1}(x), \quad n = 0, 1, \ldots, r-1,$$

then, clearly,

$$P_n^{(-r)}(x) = Q_n(x), \quad n = 0, 1, \ldots, r.$$

For $n > r$ the anti-associated polynomials satisfy the three-term recurrence relation

$$P_{n+r+1}^{(-r)}(x) = (x - \beta_n) P_{n+r}^{(-r)}(x) - \gamma_n P_{n+r-1}^{(-r)}(x), \quad n = 0, 1, 2, \ldots.$$

The anti-associated polynomial $P_{n+r}^{(-r)}$ can be represented as a linear combination of the original family P_n and the associated polynomials $P_{n-1}^{(1)}$ in the following way:

$$P_{n+r}^{(-r)}(x) = Q_r(x)P_n(x) - \gamma_0 Q_{r-1}(x)P_{n-1}^{(1)}(x), \quad n = 0, 1, 2, \ldots. \tag{2.3}$$

3. Eigenvalues of tridiagonal Toeplitz matrices

A most important OPS is the Chebyshev polynomial of the second kind, $\{U_n\}_{n\geq 0}$, which satisfies the three-term recurrence relations

$$xU_n(x) = U_{n+1}(x) + U_{n-1}(x),$$

for all $n = 1, 2, \ldots$, with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$. It is well known (cf. [3], e.g.) that each U_n also satisfies

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta \quad (0 \leq \theta < \pi)$$

for all $n = 0, 1, 2 \ldots$, from which one easily deduce the orthogonality relations

$$\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}\mathrm{d}x = \frac{\pi}{2}\delta_{n,m}.$$

Let T_n be an n-by-n real tridiagonal Toeplitz matrix defined by

$$T_n = \begin{pmatrix} \beta & 1 & & \\ \gamma & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & \gamma & \beta \end{pmatrix} \in \mathbb{R}^{n\times n}, \tag{3.1}$$

with $\gamma > 0$. The characteristic polynomial of T_n is $p_n(\lambda) = (\sqrt{\gamma})^n U_n\left(\frac{\lambda-\beta}{2\sqrt{\gamma}}\right)$ and the eigenvalues of T_n are the zeros of $p_n(\lambda)$ given by:

Theorem 3.1. *The eigenvalues of T_n are*

$$\lambda_\ell = \beta - 2\sqrt{\gamma}\cos\left(\frac{\ell\pi}{n+1}\right),$$

for $\ell = 1, 2 \ldots, n$.

If $\gamma < 0$, then $p_n(\lambda) = (i\sqrt{-\gamma})^n U_n\left(\frac{\lambda-\beta}{2i\sqrt{-\gamma}}\right)$.

4. Eigenvalues of Jacobi matrices

Let us consider the following block decomposition of a tridiagonal matrix:

$$T_r^k = \left(\begin{array}{ccccc|cccc} \beta_{-r} & 1 & & & & & & & \\ \gamma_{-r+1} & \beta_{-r+1} & 1 & & & & & & \\ & \gamma_{-r+2} & \ddots & \ddots & & & & & \\ & & \ddots & \ddots & 1 & & & & \\ & & & \gamma_{-1} & \beta_{-1} & 1 & & & \\ \hline & & & & \gamma_0 & \beta_0 & 1 & & \\ & & & & & \gamma_1 & \ddots & \ddots & \\ & & & & & & \ddots & \ddots & 1 \\ & & & & & & & \gamma_{k-1} & \beta_{k-1} \end{array}\right).$$

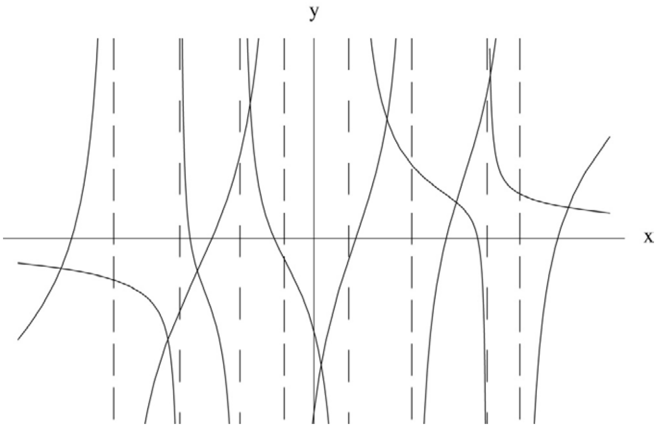

Fig. 1. A possible intersection of the graphs $y = Q_r(x)/Q_{r-1}(x)$ and $y = \gamma_0 P^{(1)}_{k-1}(x)/P_k(x)$.

Kulkarni et al. [8] referred to this matrix as a pseudo-Toeplitz matrix whenever the left upper block is a Toeplitz matrix.

The characteristic polynomial of T^k_r is given by $P^{(-r)}_{k+r}(x)$ defined in (2.3), i.e.,

$$P^{(-r)}_{k+r}(x) = Q_r(x)P_k(x) - \gamma_0 Q_{r-1}(x)P^{(1)}_{k-1}(x).$$

This means that the eigenvalues of T^k_n are the intersections of the graphics of $\frac{Q_r(x)}{Q_{r-1}(x)}$ and $\frac{\gamma_0 P^{(1)}_{k-1}(x)}{P_k(x)}$, i.e., the solutions of the equation

$$\frac{Q_r(x)}{Q_{r-1}(x)} = \frac{\gamma_0 P^{(1)}_{k-1}(x)}{P_k(x)} \tag{4.1}$$

are eigenvalues of T^k_r.

Under these conditions, let

$$p_r(x) := \frac{Q_r(x)}{Q_{r-1}(x)}.$$

According to [3, p. 24],

$$Q'_r(x)Q_{r-1}(x) - Q_r(x)Q'_{r-1}(x) > 0.$$

Therefore p_r is strictly increasing. Using a similar argument we may prove that the function defined by $\frac{\gamma_0 P^{(1)}_{k-1}(x)}{P_k(x)}$ is strictly decreasing (cf. [3, p. 88]), with $\gamma_0 > 0$. (See Fig. 1.)

The next proposition is a generalization to any Jacobi matrix of Theorem 3.3 of [8] and the proof uses essentially the same arguments.

Theorem 4.1. *Let $\xi_1, \ldots, \xi_r$ be the zeros of Q_r. Then, for $j = 1, \ldots, r+1$, each interval (ξ_{j-1}, ξ_j) contains one more root of $P^{(-r)}_{k+r}(x)$ than there are zeros of $P^{(1)}_{k-1}(x)/P_k(x)$.*

In [6] da Fonseca generalized this result to any acyclic matrix in a graph theoretical context.

As an immediate consequence of the Theorem 4.1, if we join the sets of zeros of Q_r and of $P^{(1)}_{k-1}$ and order them non-decreasingly, then in each interval with as extremes two consecutive elements, there exists one zero of T^k_r. This is essentially the main result of [7] and of [1], which relates the spread of the eigenvalues of the Jacobi matrix to the spread of the eigenvalues of its upper left and lower right blocks.

Corollary 4.2. *Let*

$$\{-\infty = \gamma_0 < \gamma_1 \leq \cdots \leq \gamma_{n-1} < \gamma_n = \infty\}$$

the union of the zeros of Q_r *and those of* $P_{k-1}^{(1)}$*, be ordered in a non-decreasingly way, with* $n = r + k$*. Then in each interval* (γ_{i-1}, γ_i)*, for* $i = 1, \ldots, n$*, there is exactly one different eigenvalue of* T_r^k*.*

Bar-On [1] considered a tridiagonal Toeplitz matrix of order 1024 with the main diagonal equal to 2 and unit subdiagonals, and gave sharper bounds for the exact eigenvalues of the matrix. Namely, from Theorem 3.1 the eigenvalues of such a matrix are given analytically by

$$\lambda_k = 4\sin^2 \frac{k\pi}{2050}, \quad k = 1, \ldots, 1024.$$

Suppose that we are looking for the 307th eigenvalue which is

$$\lambda_{307} = 0.821951265246423.$$

According to Corollary 4.2, with $r = 511$, λ_{307} lies in the interval

$$(\gamma_{306} = 0.818480596282252, \gamma_{307} = 0.825420593702442).$$

Since γ_{306} is the 153th zero of Q_{511} and

$$\frac{P_{522}^{(1)}(\gamma_{306})}{P_{523}(\gamma_{306})} < 0,$$

λ_{307} is in the interval

$$(0.819593618127087, 0.825420593702442),$$

where the new limit is the 157th zero of P_{523}, a pole of $\frac{P_{522}^{(1)}}{P_{523}}$. Due to monotonicity we can easily generalize this result, improving the bounds of the original interval, for example in the following way.

Theorem 4.3. *Suppose that* T_r^k *has an eigenvalue* λ *in the interval* $(\gamma_\ell, \gamma_{\ell+1})$*. If* γ_ℓ *is a zero of* Q_r *and* $\frac{P_r^{(1)}(\gamma_\ell)}{P_r(\gamma_\ell)} < 0$*, then* γ_k *is in the interval* $(\theta, \gamma_{\ell+1})$*, where* θ *is the minimum of all the zeros of* P_r *and of* Q_{r-1} *greater than* γ_ℓ*.*

5. Examples

The results obtained by Kulkarni et al. in [8] are in fact consequences of the previews sections. For example, they analyzed some particular cases of pseudo-Toeplitz matrices like

$$T_1^r(a,b,c) = \left(\begin{array}{cccc|c} a & b & & & \\ c & \ddots & \ddots & & \\ & \ddots & \ddots & b & \\ & & c & a & b_1 \\ \hline & & & c_1 & a_1 \end{array}\right)$$

which is a particular case of

$$T_r^1 = \left(\begin{array}{cccccc|c} \beta_{-r} & 1 & & & & & \\ \gamma_{-r+1} & \beta_{-r+1} & 1 & & & & \\ & \gamma_{-r+2} & \ddots & \ddots & & & \\ & & \ddots & \ddots & 1 & & \\ & & & \gamma_{-1} & \beta_{-1} & 1 & \\ \hline & & & & & \gamma_0 & \beta_0 \end{array}\right). \quad (5.1)$$

From (4.1), there emerges $P_1(z) = z - \beta_0$ and, thus, $P_{k-1}^{(1)}(z) = 1$. Let $\eta_1, \ldots, \eta_{r-1}$ be the zeros of Q_{r-1}. We may state the following:

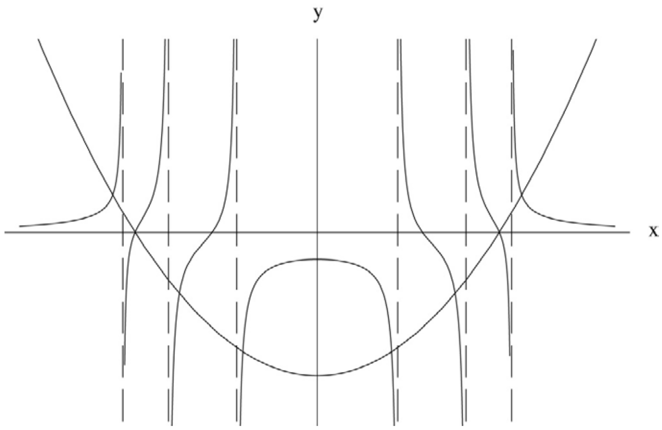

Fig. 2. A possible intersection of the graphs $y = \phi_{r-2}(x)/\phi_r(x)$ and $y = (x^2 - 2\alpha)/\alpha(2\epsilon - \alpha)$.

Theorem 5.1. *If $\eta_{j-1} < \beta_0 < \eta_j$ for some j, then there are precisely two additional roots of (4.1), exactly one lying in each of the intervals (η_{j-1}, β_0) and (β_0, η_j).*

Taking motivation from the study of antipodal tridiagonal patterns (cf. [4,5]), an interesting case to study is that when the Toeplitz matrix is affected by a perturbation in the upper left and in the lower right blocks. Let us consider the tridiagonal matrix with main diagonal equal to zero, where $\alpha, \beta > 0$:

$$T_r^{1,1} := \left(\begin{array}{c|ccccc|c} 0 & 1 & & & & & \\ \hline \alpha & 0 & 1 & & & & \\ & \epsilon & \ddots & \ddots & & & \\ & & \ddots & \ddots & 1 & & \\ & & & \epsilon & 0 & 1 & \\ \hline & & & & & \alpha & 0 \end{array}\right). \tag{5.2}$$

The characteristic polynomial of $T_r^{1,1}$ is given by

$$p_n(x) := (x^2 - 2\alpha)\phi_r(x) - \alpha(2\epsilon - \alpha)\phi_{r-2}(x),$$

where

$$\phi_r(x) = \left(\sqrt{\epsilon}\right)^r U_r\left(\frac{x}{2\sqrt{\epsilon}}\right).$$

If $\alpha = 2\epsilon$, the eigenvalues of $T_r^{1,1}$ are the zeros of ϕ_r and $2\sqrt{\epsilon}$ and $-2\sqrt{\epsilon}$. Otherwise, if x is an eigenvalue of $T_r^{1,1}$, then it is an intersection of the graphics of the functions defined by

$$\frac{\phi_{r-2}(x)}{\phi_r(x)} \quad \text{and} \quad \frac{x^2 - 2\alpha}{\alpha(2\epsilon - \alpha)}.$$

The function ϕ_{r-2}/ϕ_r is even and on the positive real axis is decreasing, and the second function is a parabola with vertex $-\frac{2}{2\epsilon-\alpha}$. (See Fig. 2.)

If $\lambda_\ell = 2\sqrt{\epsilon}\cos\left(\frac{\ell\pi}{r+1}\right)$, for $\ell = 1, 2 \ldots, r$, then there is always one eigenvalue of (5.2) in each interval $(\lambda_\ell, \lambda_{\ell+1})$, for $\ell = 1, 2 \ldots, \lfloor\frac{r}{2}\rfloor - 1$ and for $\ell = \lfloor\frac{r}{2}\rfloor + 1, \ldots, r - 1$. If $\alpha > 2\epsilon$, then there are two eigenvalues in the interval $(\lambda_r, +\infty)$ and two in the interval $(-\infty, \lambda_1)$. Otherwise, there is one eigenvalue in the interval $\left(\lambda_{\lfloor\frac{r}{2}\rfloor}, 0\right)$ and another in $\left(0, \lambda_{\lceil\frac{r}{2}\rceil+1}\right)$.

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