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What is a first countable space?

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Abstract

The definition of first countable space is standard and its meaning is very clear. But is that the case in the absence of the Axiom of Choice? The answer is negative because there are at least three choice-free versions of first countability. And, most likely, the usual definition does not correspond to what we want to be a first countable space. The three definitions as well as other characterizations of first countability are presented and it is discussed under which set-theoretic conditions they remain equivalent.

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1. Introduction

A topological space is *first countable* if there is a countable neighborhood base (or local base) at each of its points. In general, that is in the presence of the Axiom of Choice, this definition is clear and there is no room for two different interpretations. But what happens when the Axiom of Choice does not hold? The first consequence is that the definition does not say how to choose, simultaneously, a countable neighborhood base at each point of a first countable space. The existence of a solution for this kind of problem is one of the reasons for the use of the Axiom of Choice. Although, there are many first countable spaces where such a choice can be done without using the Axiom of Choice. Furthermore, in some cases one can built, at the same time, surjections from the natural numbers to a local base at each point. For instance, in a metric space such a construction is done using the open balls of radius $1/n$. These three different situations induce three possible definitions of first countability in ZF, *Zermelo–Fraenkel set theory without the Axiom of Choice*.

In the literature may be found discussions about the equivalence, in ZF, of alternative ways of defining same topological concepts such as compact space [14,4,10] or complete metric space [1,8]. Naturally, different definitions of the same concept arise to different versions of several theorems. The splitting of the concepts of compactness and completeness originated the study of the relations between the Axiom of Choice and "new" versions of the Tychonoff's Compactness Theorem [10,3,14] or the Baire Category Theorem [1,11]. Following these ideas, we introduce three def-

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initions of first countable space and their relations in ZF are investigated. After doing that, we search for ZF-alternatives to a well-known result (Theorem 3.1).

We introduce next some definitions of set-theoretic axioms which will be used throughout the paper. All results take place in the setting of ZF.

Definitions 1.1.

(a) The *Axiom of Choice* (**AC**) states that every family of non-empty sets has a choice function.
(b) The *Axiom of Multiple Choice* (**MC**) states that for every family $(X_i)_{i\in I}$ of non-empty sets, there is a family $(A_i)_{i\in I}$ of non-empty finite sets with $A_i \subseteq X_i$ for each $i \in I$.
(c) The *Axiom of Countable Choice* (**CC**) states that every countable family of non-empty sets has a choice function.
(d) $\mathbf{MC}_\omega$ states that for every family $(X_i)_{i\in I}$ of non-empty sets, there is a family $(A_i)_{i\in I}$ of non-empty countable sets with $A_i \subseteq X_i$ for each $i \in I$.
(e) The *Countable Union Condition* (**CUC**) states that the countable union of countable sets is countable.

We denote by $\mathbf{AC}(\alpha)$ the Axiom of Choice restricted to families of sets each of which has cardinality at most equal to α and by $\mathbf{AC}(\mathbb{R})$ the axiom of choice for families of subsets of $\mathbb{R}$. Similar notation is used with **MC**, **CC** and $\mathbf{MC}_\omega$.

One should remark that in ZF the Axiom of Choice is equivalent to the Axiom of Multiple Choice [15]. Although their restrictions are not equivalent in general.

Proposition 1.2. ([6, p. 76], [12]) *The following conditions are equivalent to* **CC** (*respectively* $\mathbf{CC}(\mathbb{R})$):

(i) *every countable family of non-empty sets* (*respectively subsets of* $\mathbb{R}$) *has an infinite subfamily with a choice function*;
(ii) *for every countable family* $(X_n)_{n\in\mathbb{N}}$ *of non-empty sets* (*respectively subsets of* $\mathbb{R}$), *there is a sequence which take values in an infinite number of the sets* X_n.

Lemma 1.3. *Let* $(X,\mathcal{T})$ *be a topological space.*

(a) *If* $(X,\mathcal{T})$ *is second countable, then* $|\mathcal{T}| \leqslant |\mathbb{R}| = 2^{\aleph_0}$.
(b) *If* $(X,\mathcal{T})$ *is a* T_0*-space, then* $|X| \leqslant |\mathcal{T}|$.

2. Definitions

We start this section with three versions of the First Axiom of Countability, which are equivalent in the presence of the Axiom of Choice. They will be denoted by A, B and C, A being the usual definition. Later, other conditions *choice-equivalent* to these ones will be introduced and they will also be denoted in alphabetic order. To make the new definitions easier to understand and to compare, they are presented in symbolic language.

Definitions 2.1. Let X be a topological space. One says that X satisfies:

A if $(\forall x \in X)(\exists \mathcal{B}(x))|\mathcal{B}(x)| \leqslant \aleph_0$ and $\mathcal{B}(x)$ is a local base at x;
B if $(\exists (\mathcal{B}(x))_{x\in X})(\forall x \in X)|\mathcal{B}(x)| \leqslant \aleph_0$ and $\mathcal{B}(x)$ is a local base at x;
C if $(\exists (B(n,x))_{n\in\mathbb{N},x\in X})(\forall x \in X)\{B(n,x)\colon n \in \mathbb{N}\}$ is a local base at x.

In the definitions of A, B and C, one can take only the open neighborhoods without changing the logic value of them. This fact is pointed out because it will be seen later situations where that is not the case.

Lemma 2.2.

(a) *If a topological space satisfies* B*, then satisfies* A.
(b) *If a topological space satisfies* C*, then satisfies* B.

Proposition 2.3. *Every metric or second countable space satisfies* C*, and then also* B *and* A.

The first intuitive idea which one might have is that the Axiom of Choice is necessary to proof the equivalence between A and B, because the implication A $\Rightarrow$ B has some formal similarity to the Axiom of Choice. Although, it is possible to prove that A is equivalent to B from a choice principle weaker then **AC**.

Theorem 2.4. [9] *If* $\mathbf{MC}_\omega$ *holds, then a topological space satisfies* A *if and only if satisfies* B.

There are several models of ZF where **AC** does not hold, but $\mathbf{MC}_\omega$ does, for instance the Cohen/Pincus Model ($\mathcal{M}1(\langle\omega_1\rangle)$ in [13]).

Unfortunately, it is not known if the equivalence between A and B is provable in ZF.

Proposition 2.5. *If* $\mathbf{MC}(2^{\aleph_0})$ *holds, then a topological space satisfies* B *if and only if satisfies* C.

Proof. Since C always implies B, it is only necessary to prove the other direction.

Let X be a topological space which satisfies B. The definition of B says that there is $(\mathcal{B}(x))_{x\in X}$ such that $\mathcal{B}(x)$ is a countable local base at x. Define the sets

$$\mathsf{L}(x) := \big\{\big(f:\mathbb{N}\to\mathcal{B}(x)\big):\ f \text{ is a surjection}\big\}.$$

Each of the sets $\mathsf{L}(x)$ is non-empty and it has cardinality at most $2^{\aleph_0}$, since $|\mathsf{L}(x)| \leqslant |\mathcal{B}(x)^{\mathbb{N}}| \leqslant {\aleph_0}^{\aleph_0} = 2^{\aleph_0}$. By $\mathbf{MC}(2^{\aleph_0})$, there is a family $(\mathsf{M}(x))_{x\in X}$ such that, for every x, $\mathsf{M}(x)$ is finite, non-empty and it is contained in $\mathsf{L}(x)$.

It is clear that for $B(n,x) := \bigcap_{f\in\mathsf{M}(x)} f(n)$, $\{B(n,x):\ n\in\mathbb{N}\}$ is a local base at x, which finishes the proof. □

Lemma 2.6. [7] *The following conditions are equivalent*:

(i) $\mathbf{AC}(\aleph_0)$;
(ii) *for every family of countable sets* $(X_i)_{i\in I}$*, there is a family of functions* $(f_i)_{i\in I}$ *such that* f_i *is a bijection between an ordinal* α_i *and* X_i.

Corollary 2.7. *If* $\mathbf{AC}(\aleph_0)$ *and* $\mathbf{AC}(\mathbb{R})$ *hold, then for every family of countable sets* $(X_i)_{i\in I}$*, there is a family of functions* $(f_i)_{i\in I}$ *such that* f_i *is an injection from* X_i *to* $\mathbb{N}$.

A. Church [2] proved that the set of all well-ordered rearrangements of $\mathbb{N}$ has the cardinality of $\mathbb{R}$. This is sufficient to prove this corollary from the previous lemma.

Corollary 2.8. *If* $\mathbf{AC}(\aleph_0)$ *and* $\mathbf{AC}(\mathbb{R})$ *hold, then a topological space satisfies* B *if and only satisfies* C.

If for any family of countable sets there is a family of injections from each of them to $\mathbb{N}$, then B implies C. So, this corollary is straightforward after Corollary 2.7.

Remark 2.9. If it is possible to choose injective functions from $\mathcal{B}(x)$ to $\mathbb{N}$, for every family $(\mathcal{B}(x))_{x\in X}$ of local bases ($\star$), then B implies C. The reverse implication is not necessarily true, because one not needs to consider the same local base in both definitions. The condition ($\star$) is, in fact, equivalent to the thesis of Corollary 2.7.

Proposition 2.10. *If a topological space satisfies* B *if and only if satisfies* C*, then* $\mathbf{MC}(\aleph_0)$ *holds.*

Proof. Let $(X_i)_{i\in I}$ be a family of non-empty countable sets. Consider each X_i with the discrete topology, $X_i \dot\cup \{i\}$ its Alexandroff compactification and the disjoint union $X := \bigcup(X_i \cup \{i\})$ with the sum topology. We know that $|X_i| \leqslant \aleph_0$, and then $|\mathcal{P}_{\mathrm{fin}}(X_i)| \leqslant \aleph_0$. This implies that each of the points $i\in I$ has a countable local base. The space X satisfies B and then also C. This means that there is a local base $\{B(n,i):\ n\in\mathbb{N}\}$ for each i. Without lost of generality, one consider that $B(1,i)$ does not contain $X_i\cup\{i\}$ and consequently $(X_i \setminus B(1,i))_{i\in I}$ is a multiple choice on $(X_i)_{i\in I}$. □

Remark 2.11. With an iteration of this process it is possible to write the sets X_i as well ordered unions of finite sets, which can be regarded as a special case of Lévy's characterization of the Axiom of Multiple Choice [15].

3. Characterizations

The motivation for the work presented in this section is the attempt to find out the set theoretic status in ZF of Theorem 3.1. This work has already been started in [9] and originated the idea to find the "best" definition of first countable space.

We try to see in what conditions we can take a countable local base from any local base in a first countable space.

Following what was done in the previous section, there are three ways to do it: one local and two global, in accordance with each of the definitions A, B and C.

Theorem 3.1. (ZF + **AC**) *Every neighborhood base at a point of a first countable space contains a countable neighborhood base.*

A proof of this theorem can be seen, for instance, in [5, 2.4.12].

This is the usual version of the theorem. However, it is not necessary to consider a first countable space, it suffices to consider that a specific point has a countable neighborhood base. For that reason, perhaps it is more appropriate to consider a global version of the theorem.

We introduce now several characterization of first countability in ZFC, which are not equivalent in general. They are introduced in order to help a better understanding of the possible choice free versions of the previous theorem.

Definitions 3.2. Let X be a topological space. We say that X satisfies:

- D if $(\forall x)(\forall \mathcal{V}(x)$ open local base at $x)(\exists \mathcal{B}(x) \subseteq \mathcal{V}(x))|\mathcal{B}(x)| \leqslant \aleph_0$ and $\mathcal{B}(x)$ is a local base at x;
- E if $(\forall(\mathcal{V}(x))_{x\in X}$ with $\mathcal{V}(x)$ open local base at $x)(\exists(\mathcal{B}(x))_{x\in X})$ $(\forall x)\mathcal{B}(x) \subseteq \mathcal{V}(x)$, $|\mathcal{B}(x)| \leqslant \aleph_0$ and $\mathcal{B}(x)$ is a local base at x;
- F if $(\forall(\mathcal{V}(x))_{x\in X}$ with $\mathcal{V}(x)$ open local base at $x)(\exists(B(n,x))_{n\in\mathbb{N},x\in X})$ $(\forall x)[(\forall n)B(n,x) \in \mathcal{V}(x)$ and $\{B(n,x)\colon n \in \mathbb{N}\}$ is a local base at $x]$;
- G if $(\forall x)(\forall \mathcal{V}(x)$ local base at $x)(\exists \mathcal{B}(x) \subseteq \mathcal{V}(x))|\mathcal{B}(x)| \leqslant \aleph_0$ and $\mathcal{B}(x)$ is a local base at x;
- H if $(\forall(\mathcal{V}(x))_{x\in X}$ with $\mathcal{V}(x)$ local base at $x)(\exists(\mathcal{B}(x))_{x\in X})$ $(\forall x)\mathcal{B}(x) \subseteq \mathcal{V}(x)$, $|\mathcal{B}(x)| \leqslant \aleph_0$ and $\mathcal{B}(x)$ local base at x;
- I if $(\forall(\mathcal{V}(x))_{x\in X}$ with $\mathcal{V}(x)$ local base at $x)(\exists(B(n,x))_{n\in\mathbb{N},x\in X})$ $(\forall x)[(\forall n)B(n,x) \in \mathcal{V}(x)$ and $\{B(n,x)\colon n \in \mathbb{N}\}$ is a local base at $x]$.

Together with the definitions G–I which try to transfer to ZF the characterization of Theorem 3.1, one includes three other definitions where the given local bases are open.

Proposition 3.3. *For the classes* A–I*, the following inclusions hold*:

(a) C ⊆ B ⊆ A;
(b) F ⊆ E ⊆ D;
(c) I ⊆ H ⊆ G;
(d) G ⊆ D ⊆ A;
(e) H ⊆ E ⊆ B;
(f) I ⊆ F ⊆ C.

$$\begin{tikzcd}
G \arrow[d, Rightarrow] & H \arrow[l, Rightarrow] \arrow[d, Rightarrow] & I \arrow[l, Rightarrow] \arrow[d, Rightarrow] \\
D \arrow[d, Rightarrow] & E \arrow[l, Rightarrow] \arrow[d, Rightarrow] & F \arrow[l, Rightarrow] \arrow[d, Rightarrow] \\
A & B \arrow[l, Rightarrow] & C \arrow[l, Rightarrow]
\end{tikzcd}$$

Lemma 3.4. *Every topological space with a countable topology satisfies* F*, and then* E *and* D*.*

The next two theorems enlarge Theorems 3.1 and 3.5 of [9], respectively.

Theorem 3.5. *The following conditions are equivalent to* **CC**:

(i) *if, in a topological space, x has a countable local base, then every local base at x contains a countable local base*;
(ii) *a topological space satisfies* A *if and only if satisfies* G;
(iii) *a topological space satisfies* A *if and only if satisfies* D;
(iv) *a topological space satisfies* D *if and only if satisfies* G.

Note that condition (ii) is Theorem 3.1.

Proof. The proof that **CC** implies (i) is the usual one and it can be seen in [5, 2.4.12]. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv) are clear.

It remains to show only that (iii) $\Rightarrow$ **CC** and (iv) $\Rightarrow$ **CC**.

Assume that **CC** does not hold. From Proposition 1.2, there is a family $(X_n)_{n\in\mathbb{N}}$ of non-empty sets such that every sequence intersects only a finite number of the sets X_n. One can suppose that each two sets are disjoint.

Define now the sets $X := (\bigcup X_n) \dot{\cup} \{\infty\}$ and $Y_n := (\bigcup_{k\geqslant n+1} X_k) \dot{\cup} \{\infty\}$. One consider in X the topologies $\mathcal{T}_1$, $\mathcal{T}_2$, and a base $\mathcal{B}_1$ for $\mathcal{T}_1$:

$$\mathcal{B}_1 := \big\{\{x\}\colon x \in X \setminus \{\infty\}\big\} \cup \{Y_n\colon n \in \mathbb{N}\};$$
$$\mathcal{T}_2 := \{Y_n\colon n \in \mathbb{N}\}.$$

The topological space $(X, \mathcal{T}_1)$ satisfies A and $(X, \mathcal{T}_2)$ satisfies D.

One also has that $\mathcal{V} := \{Y_n \cup \{x\}\colon x \in X_n,\ n \in \mathbb{N}\}$ is a local base at ∞ which does not contain any countable base, because otherwise there would be a sequence converging to ∞. That is not possible from the hypothesis. In conclusion, $(X, \mathcal{T}_2)$ does not satisfies G and $(X, \mathcal{T}_1)$ does not satisfies D, since $\mathcal{V} \subseteq \mathcal{T}_1$. $\square$

It is easy to see that $|\mathcal{T}_2| = \aleph_0$ and that $(X, \mathcal{T}_1)$ is metrizable by the metric

$$d(x, y) := \begin{cases} 0 & \text{if } x = y, \\ \frac{1}{n} & \text{if } (x \in X_n \text{ and } y = \infty) \text{ or } (y \in X_n \text{ and } x = \infty), \\ \frac{1}{n} + \frac{1}{m} & \text{if } x \neq y,\ x \in X_n \text{ and } y \in X_m, \end{cases}$$

which gives us the next corollary.

Corollary 3.6. *The following conditions are equivalent to* **CC**:

(i) *every metric space satisfies* G (*respectively* D);
(ii) *every second countable space satisfies* G;
(iii) *every space with a countable topology satisfies* G.

In this point, we will see how Theorem 3.1 can be generalized in a more global way. As it was said before there are two options for that. One is in the next theorem and the other in Theorem 3.10.

Theorem 3.7. *The following conditions are equivalent*:

(i) $\mathbf{MC}_\omega$ *and* **CC**;
(ii) $\mathbf{MC}_\omega$ *and* **CUC**;
(iii) $\mathbf{MC}_\omega$ *and* $\mathbf{CC}(\aleph_0)$;
(iv) *a topological space satisfies* B *if and only if satisfies* H;
(v) *a topological space satisfies* B *if and only if satisfies* E;
(vi) *a topological space satisfies* E *if and only if satisfies* H.

Proof. We know that $\mathbf{CC} \Rightarrow \mathbf{CUC} \Rightarrow \mathbf{CC}(\aleph_0)$ (see [13]). If $\mathbf{MC}_\omega$ holds, then **CC** is equivalent to $\mathbf{CC}(\aleph_0)$, and finally (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

The implications (iv) $\Rightarrow$ (v) and (iv) $\Rightarrow$ (vi) are obvious. It is enough to prove that (ii) $\Rightarrow$ (iv), (v) $\Rightarrow$ (i) and (vi) $\Rightarrow$ (i).

(ii) $\Rightarrow$ (iv) See [9, 3.5].

(v) $\Rightarrow$ (i) and (vi) $\Rightarrow$ (i). From Theorem 3.5 it is (almost) straightforward that each of the conditions (v) and (vi) implies **CC**.

To prove that (v) or (vi) imply $\mathbf{MC}_\omega$, let $(X_i)_{i\in I}$ be a family of non-empty sets. Without lost of generality, one consider each two sets disjoint and its union disjoint from I. Define the sets $X := \bigcup_i [(X_i \times \mathbb{N}) \cup \{(i,\infty)\}]$ and $Y_n := \{(x,k){:}\ x \in X_i \text{ for some } i \in I,\ k > n\} \cup (I \times \{\infty\})$.

One consider the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ in X, and also a base $\mathcal{B}_1$ for the topology $\mathcal{T}_1$:

$$\mathcal{B}_1 := \Big\{\big\{(x,n)\big\}{:}\ x \notin I,\ n \in \mathbb{N}\Big\} \cup \{Y_n{:}\ n \in \mathbb{N}\};$$
$$\mathcal{T}_2 := \{Y_n{:}\ n \in \mathbb{N}\}.$$

From the definitions of the topologies, it is clear that $\mathcal{T}_1$ satisfies B and $\mathcal{T}_2$ satisfies E. In both cases, each of the elements $(x,n) \notin I \times \{\infty\}$ has a local base with only one element, and then we do not need to consider local bases for those points. The set $\mathcal{V}(i) := \{Y_n \cup \{(x,n)\}{:}\ x \in X_i,\ n \in \mathbb{N}\}$ is a local base at (i,∞) for both topologies, and $\mathcal{V}(i) \subseteq \mathcal{T}_1$. By hypothesis, one of the conditions (v) or (vi) hold. This means that there is $(\mathcal{B}(i))_{i\in I}$ such that $\mathcal{B}(i)$ is a countable base at (i,∞) and it is contained $\mathcal{V}(i)$.

The sets $A_i := \{x \in X_i{:}\ (\exists B \in \mathcal{B}(i)) B \setminus Y_n = \{(x,n)\} \text{ for some } n\}$ are countable, which finish the proof. □

Corollary 3.8. *Every first countable space* (*i.e. satisfies* A) *satisfies H if and only if* $\mathbf{MC}_\omega$ *and* **CC** *hold.*

This is a possible alternative, in ZF, to Theorem 3.1.

Corollary 3.9. *The following conditions are equivalent to* $\mathbf{MC}_\omega$ *and* **CC**:

(i) *every metric space satisfies* H (*respectively* E);
(ii) *every second countable space satisfies* H;
(iii) *every space with a countable topology satisfies* H.

Proof. It follows that $\mathbf{MC}_\omega + \mathbf{CC} \Leftrightarrow$ (ii) $\Leftrightarrow$ (iii), because the topology $\mathcal{T}_2$ defined in the proof of Theorem 3.7 is countable.

By Corollary 3.6, (i) $\Rightarrow$ **CC**.

To show that (i) $\Rightarrow \mathbf{MC}_\omega$, one takes a family $(X_i)_{i\in I}$ as in the proof of (v) $\Rightarrow$ (i) in Theorem 3.7. Define the metric space (X,d) with $X := \bigcup_i (X_i \times \mathbb{N} \times \{i\} \cup \{(i,\infty,i)\})$ and

$$d\big((x,n,i),(y,m,j)\big) := \begin{cases} 0 & \text{if } (x,n,i) = (y,m,j), \\ 2 & \text{if } i \neq j, \\ \frac{1}{n} + \frac{1}{m} & \text{if } i = j \text{ and } (x,n) \neq (y,m), \end{cases}$$

and the sets $Y(i,n) := \{(i,\infty,i)\} \cup \bigcup_{k\geqslant n+1} (X_i \times \{k\} \times \{i\})$.

For each $i \in I$, $\mathcal{V}(i) := \{Y(i,n) \cup \{(x,n)\}{:}\ x \in X_i,\ n \in \mathbb{N}\}$ is a local base at (i,∞,i). From here, the proof follows as the proof of Theorem 3.7. □

Theorem 3.10. *The following conditions are equivalent to the Axiom of Choice*:

(i) *every first countable space* ($\equiv$ A) *satisfies* I;
(ii) *a topological space satisfies* C *if and only if satisfies* I;
(iii) *a topological space satisfies* C *if and only if satisfies* F;
(iv) *a topological space satisfies* F *if and only if satisfies* I.

Proof. Since A, C, F and I are equivalent in $\mathsf{ZF} + \mathbf{AC}$, each of the conditions (i)–(iv) follow from **AC**.

It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and that (ii) $\Rightarrow$ (iv).

The proofs for (iii) $\Rightarrow$ **AC** and (iv) $\Rightarrow$ **AC** are similar to the correspondent proofs of Theorem 3.7. By hypothesis, there is a surjection between $\mathbb{N}$ and each $\mathcal{B}(i)$. This surjection allows us to choose an element in each of the sets A_i of that proof. □

The condition (i) of this theorem is another alternative to Theorem 3.1.

Corollary 3.11. *The following conditions are equivalent to* **AC**:

(i) *every metric space satisfies* I (*respectively* F);
(ii) *every second countable space satisfies* I;
(iii) *every space with a countable base satisfies* I.

The proof is done from Corollary 3.9 as the proof of Theorem 3.10 was done from Theorem 3.7.

It is somehow surprising that an apparently so weak condition, such as condition (iii), is equivalent to the Axiom of Choice itself.

Proposition 3.12. *If* $\mathbf{MC}_\omega$ *holds, then*:

(a) *a topological space satisfies* D *if and only satisfies* E;
(b) *a topological space satisfies* G *if and only satisfies* H.

The proof of this proposition is similar to the proof of Theorem 2.4.

Corollary 3.13. *If the Axiom of Countable Choice holds, then the following conditions are equivalent*:

(i) $\mathbf{MC}_\omega$;
(ii) *a topological space satisfies* D *if and only satisfies* E;
(iii) *a topological space satisfies* G *if and only satisfies* H.

Proof. (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) Proposition 3.12.

(ii) $\Rightarrow$ (i) If D is equivalent to E and **CC** holds, then A is equivalent to E by Theorem 3.5. Since B implies A, then B implies E and follows that $\mathbf{MC}_\omega$ holds by Theorem 3.7.

(iii) $\Rightarrow$ (i) Similar to (ii) $\Rightarrow$ (i). □

Proposition 3.14. *If* E *is equivalent to* F *or* H *is equivalent to* I, *then* $\mathbf{MC}(\aleph_0)$ *holds.*

If $\mathbf{MC}(\aleph_0)$ does not hold, then the topological space constructed in the proof of Proposition 2.10 satisfies E and H, but not F and I.

4. The real numbers

We have seen under which conditions some classes of topological spaces satisfy each of the characterizations of first countability. Now, we will study the special case of the topological space $\mathbb{R}$ with the Euclidean topology.

We recall that $\mathbb{R}$ satisfies each of the three definitions A–C.

Theorem 4.1. *The following conditions are equivalent to* $\mathbf{CC}(\mathbb{R})$:

(i) *every second countable space satisfies* D;
(ii) $\mathbb{R}$ *satisfies* D.

Proof. $\mathbf{CC}(\mathbb{R}) \Rightarrow$ (i) We follow the usual proof, taking into consideration that the cardinal of the topology of a second countable space has at most the cardinal of $\mathbb{R}$ (Lemma 1.3).

(ii) $\Rightarrow$ **CC**($\mathbb{R}$) Let $(X_n)_n$ be a countable family of non-empty subsets of $\mathbb{R}$. One can consider that $X_n \subseteq (\frac{1}{n+1}, \frac{1}{n})$, for all $n \in \mathbb{N}$. Consider also the local base $\mathcal{V} := \{(-\frac{1}{n}, x)\colon\ n \in \mathbb{N},\ x \in X_n\}$ at 0. By hypothesis, there is a local base $\mathcal{B} := \{B_k\colon\ k \in \mathbb{N}\}$ at 0 contained in $\mathcal{V}$. Define $s(k) := \sup B_k$, which belongs to X_n for some n, from the definition of $\mathcal{V}$. Since $\mathcal{B}$ is a local base, $\{s(k)\colon\ k \in \mathbb{N}\}$ has non-empty intersection with an infinite number of the sets X_n. Proposition 1.2 says that this fact suffices to prove **CC**($\mathbb{R}$). □

Proposition 4.2. *The following conditions are equivalent to the Axiom of Countable Choice for families of subsets of* $\mathcal{P}(\mathbb{R})$ (**CC**($\mathcal{P}\mathbb{R}$)):

(i) *every second countable* T_0*-space satisfies* G;
(ii) $\mathbb{R}$ *satisfies* G.

Proof. **CC**($\mathcal{P}\mathbb{R}$) $\Rightarrow$ (i) If X is a second countable T_0-space, then $|X| \leqslant |\mathbb{R}|$, and to prove (i) it is enough to use countable choice in a family of subsets of $\mathcal{P}(X)$.

(ii) $\Rightarrow$ **CC**($\mathcal{P}\mathbb{R}$) Let $(X_n)_n$ be a family of non-empty subsets of $\mathcal{P}(\mathbb{R})$. Consider $X_n \subseteq \mathcal{P}((\frac{1}{n+1}, \frac{1}{n}))$ and define a local base $\mathcal{V} := \{(-\frac{1}{n}, \frac{1}{n+1}) \cup A\colon\ n \in \mathbb{N},\ A \in X_n\}$ at 0. If $\mathbb{R}$ satisfies G, then there is $\mathcal{B} := \{B_k\colon\ k \in \mathbb{N}\}$ such that $\mathcal{B} \subseteq \mathcal{V}$ and $\mathcal{B}$ is a local base at 0. The existence of $\mathcal{B}$ implies the existence of a sequence which take values in X_n for an infinite number of the sets X_n. This fact completes the proof. □

Proposition 4.3. *Every second countable space satisfies* F *if and only if the Axiom of Choice holds in* $\mathbb{R}$ (**AC**($\mathbb{R}$)).

Proof. ($\Leftarrow$) Let $(X, \mathcal{T})$ be a second countable space and $\{B_n\colon\ n \in \mathbb{N}\}$ one of its bases. By Lemma 1.3, $|\mathcal{T}| \leqslant |\mathbb{R}|$ and, using (**AC**($\mathbb{R}$)), we know that $\mathbb{R}$ is well ordered and then also $\mathcal{T}$ is well ordered.

Consider a family $(\mathcal{V}(x))_{x \in X}$ such that $\mathcal{V}(x)$ is an open local base at x. For $(x, n) \in X \times \mathbb{N}$, one defines $\mathcal{C}(n, x) := \{V \in \mathcal{V}(x)\colon\ V \subseteq B_n\}$. If $x \in B_n$, then $\mathcal{C}(n, x) \neq \emptyset$. Since $\mathcal{T}$ has a well order, the sets $B(n, x) := \min \mathcal{C}(n, x)$ are well defined. The set $\mathcal{B}(x) = \{B(n, x)\colon\ n \in \mathbb{N},\ x \in B_n\}$ is a local base at x.

($\Rightarrow$) Let $(X_i)_{i \in I}$ be a family of non-empty subsets of $\mathbb{R}$. We define the topological space $Y := \mathbb{R} \setminus \{0\} \cup I$, with $I \cap \mathbb{R} = \emptyset$ and having the initial (or induced) topology with respect to the function

$$f : Y \to \mathbb{R},$$
$$y \mapsto \begin{cases} 0 & \text{if } y \in I, \\ y & \text{if } y \notin I. \end{cases}$$

The topological space $\mathbb{R}$ is second countable, then Y is also second countable.

There are constructive bijections $f_n : \mathbb{R} \to (\frac{1}{n+1}, \frac{1}{n})$. One defines $X_{in} := f_n(X_i)$. For each $i \in I$, $\mathcal{V}(i) := \{I \cup (-\frac{1}{n}, 0) \cup (0, x)\colon\ x \in X_{in},\ n \in \mathbb{N}\} \subseteq \mathcal{T}_Y$ is a local base at i. By hypothesis, there is a local base at i $\{B(k, i)\colon\ k \in \mathbb{N}\} \subseteq \mathcal{V}(i)$.

For each $i \in I$, $\sup B(1, i)$ is an element of X_{in} for some n and X_{in} is in bijection with X_i, which provides the desired choice. □

Corollary 4.4. *The topological space* $\mathbb{R}$ *satisfies* F *if and only if the Axiom of Choice holds for families* $(X_i)_{i \in \mathbb{R}}$ *of non-empty subsets of* $\mathbb{R}$.

Proof. ($\Leftarrow$) We follow the previous proof. For that, we only have to notice that $|\mathbb{R} \times \mathbb{N}| = 2^{\aleph_0}$.

($\Rightarrow$) Let $(X_i)_{i \in \mathbb{R}}$ be a family of non-empty subsets of $\mathbb{R}$. The proof is similar to the proof of Proposition 4.3, with $Y = \mathbb{R}$, $X_{in} \subseteq (i + \frac{1}{n+1}, i + \frac{1}{n})$ and $\mathcal{V}(i) := \{(i - \frac{1}{n}, x)\colon\ x \in X_{in},\ n \in \mathbb{N}\}$. □

Taking into consideration the proofs of Propositions 3.7, 4.2, 4.3 and of Corollary 4.4, and also that **CC**($\mathbb{R}$) (respectively **CC**($\mathcal{P}\mathbb{R}$)) implies that *the countable union of countable subsets of* $\mathbb{R}$ (*respectively* $\mathcal{P}(\mathbb{R})$) *is countable*, one can deduce the next results.

Corollary 4.5. *The following conditions are equivalent*:

(i) $\mathbf{MC}_\omega(\mathbb{R})$ *and* $\mathbf{CC}(\mathbb{R})$;
(ii) *every second countable space satisfies* E.

Corollary 4.6. *The following conditions are equivalent*:

(i) $\mathbf{MC}_\omega$ *holds for families* $(X_i)_{i\in\mathbb{R}}$ *of non-empty subsets of* $\mathbb{R}$ *and* $\mathbf{CC}(\mathbb{R})$ *also holds*;
(ii) $\mathbb{R}$ *satisfies* E.

Corollary 4.7. *The topological space* $\mathbb{R}$ *satisfies* I *if and only if the Axiom of Choice holds for families* $(X_i)_{i\in\mathbb{R}}$ *of non-empty subsets of* $\mathcal{P}(\mathbb{R})$.

To show that $\mathbb{R}$ satisfies I, one uses the usual prove and that for a given local base $\mathcal{V}(x)$ at $x \in \mathbb{R}$, $|\{(f:\mathbb{N}\to\mathcal{V}(x))\colon\ f(\mathbb{N}) \text{ is a local base at } x\}| \leqslant |\mathcal{P}(\mathbb{R})^{\mathbb{N}}| = |\mathcal{P}(\mathbb{R})|$.

To prove the other implication, one uses a construction of the type of the ones used in the proofs of Proposition 4.2 and Corollary 4.4.

Corollary 4.8. *The following conditions are equivalent*:

(i) $\mathbf{MC}_\omega$ *holds for families* $(X_i)_{i\in\mathbb{R}}$ *of non-empty subsets of* $\mathcal{P}(\mathbb{R})$ *and* $\mathbf{CC}(\mathcal{P}\mathbb{R})$ *also holds*;
(ii) $\mathbb{R}$ *satisfies* H.

In the third section, we have shown that the Axiom of Choice is a necessary condition to prove that *every space with a countable topology satisfies* I. As we did in other situations, we will look now to the situation of T_0-spaces with a countable topology. The results are surprising, mainly because they are identical for the classes G, H and I.

Proposition 4.9. *The following conditions are equivalent to* $\mathbf{CC}(\mathbb{R})$:

(i) *every* T_0*-space with a countable topology satisfies* I;
(ii) *every* T_0*-space with a countable topology satisfies* H;
(iii) *every* T_0*-space with a countable topology satisfies* G.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear (3.3).

$\mathbf{CC}(\mathbb{R}) \Rightarrow$ (i) Let $(X,\mathcal{T})$ be a topological T_0-space such that $|\mathcal{T}| \leqslant \aleph_0$. Since $(X,\mathcal{T})$ is T_0, $|X| \leqslant |\mathcal{T}| \leqslant \aleph_0$ and then $|\mathcal{P}(X)| \leqslant 2^{\aleph_0} = |\mathbb{R}|$. Let also $(\mathcal{V}(x))_{x\in X}$ be a family such that $\mathcal{V}(x)$ is a local base at x.

Define $\mathbb{M} := \{(x,A)\colon\ x \in A \in \mathcal{T}\}$ and, for each pair (x,A) in $\mathbb{M}$, $\mathcal{L}(x,A) := \{V \in \mathcal{V}(x)\colon\ V \subseteq A\}$. We have that $|\mathbb{M}| \leqslant |X\times\mathcal{T}| \leqslant \aleph_0$ and that $\mathcal{V}(x)$ is a local base at x, then $(\mathcal{L}(x,A))_{(x,A)\in\mathbb{M}}$ is a countable family of non-empty subsets of $\mathcal{P}(X)$. By hypothesis $\mathbf{CC}(\mathbb{R})$ holds and then we choose an element $V(x,A)$ from each $\mathcal{L}(x,A)$. The set $\mathcal{B}(x) := \{V(x,A)\colon\ x\in A\}$ is a local base at x. The set $\mathbb{M}$ is countable, which allows us to built simultaneously an injective function from each $\mathcal{B}(x)$ to $\mathbb{N}$.

(iii) $\Rightarrow \mathbf{CC}(\mathbb{R})$ Consider the topological space $(X,\mathcal{T})$, with $X := [0,\omega^2]$ and $\mathcal{T} := \{(\alpha,\omega^2]\colon\ \alpha\in\omega^2\}\cup\{\emptyset, X\}$. The ordinal number ω^2 is countable and then $\mathcal{T}$ is also countable. It is clear that $(X,\mathcal{T})$ is a T_0-space.

Let $(X_n)_{n\in\mathbb{N}}$ be a family of non-empty subsets of $\mathbb{R}$. Since $\mathbb{R}$ is in bijection with $\mathcal{P}(\mathbb{N})$, and there is a constructive bijection between $\mathbb{N}$ and $((n-1)\omega, n\omega)$, we can take $X_n \subseteq \mathcal{P}(((n-1)\omega, n\omega))$.

The set $\mathcal{V} := \{(n\omega,\omega^2]\cup A\colon\ A\in X_n,\ n\in\mathbb{N}\}$ is a local base at ω^2. By hypothesis, there is a local base $\{B_k\colon\ k\in\mathbb{N}\}$ at ω^2 contained in $\mathcal{V}$. We define now $\varphi(k) := \min\{n\in\mathbb{N}\colon\ (n\omega,\omega^2]\subseteq B_k\}$ and $A_k := B_k \setminus (\varphi(k)\omega,\omega^2] \subseteq X_{\varphi(k)}$.

The sequence $(\varphi(k))_{k\in\mathbb{N}}$ converges to ω because $\{B_k\colon\ k\in\mathbb{N}\}$ is a local base. It is now obvious that $(A_k)_{k\in\mathbb{N}}$ is a sequence which takes values in an infinite number of sets X_n, which together with Proposition 1.2 finishes the proof. $\square$

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