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Pseudovarieties defining classes of sofic subshifts closed under taking shift equivalent subshifts

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Abstract

For a pseudovariety V of ordered semigroups, let $\mathscr{S}(\mathsf{V})$ be the class of sofic subshifts whose syntactic semigroup lies in V. It is proved that if V contains Sl^- then $\mathscr{S}(\mathsf{V} * \mathsf{D})$ is closed under taking shift equivalent subshifts, and conversely, if $\mathscr{S}(\mathsf{V})$ is closed under taking conjugate subshifts then V contains LSl^- and $\mathscr{S}(\mathsf{V}) = \mathscr{S}(\mathsf{V} * \mathsf{D})$. Almost finite type subshifts are characterized as the irreducible elements of $\mathscr{S}(\mathsf{LInv})$, which gives a new proof that the class of almost finite type subshifts is closed under taking shift equivalent subshifts.

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1. Introduction

Given a finite alphabet A, a *subshift* of $A^{\mathbb{Z}}$ is a non-empty compact subset of $A^{\mathbb{Z}}$ that is closed under the shift operation and its inverse. There is a natural bijection between subshifts and non-empty factorial prolongable languages. The subshift is called *sofic* if the corresponding language is rational. Two subshifts are *conjugate* if there is a shift commuting homeomorphism between them. It is an open question whether there is an algorithm for deciding if two sofic subshifts are conjugate or not. The notion of *shift equivalence* is strictly weaker than conjugacy. For a long time it was an open problem whether the two notions coincided or not [22,23]. The shift equivalence between sofic subshifts is decidable [21].

Pseudovarieties of semigroups are useful for classifying varieties of rational languages, via Eilenberg's correspondence theorem [17]. A more refined classification of rational languages using pseudovarieties of ordered semigroups was successfully introduced by Pin [30]. It is natural to ask which pseudovarieties define classes of sofic subshifts closed under taking conjugate subshifts. To be more precise, for a pseudovariety V of ordered semigroups let $\mathscr{S}(\mathsf{V})$ be the class of sofic subshifts whose (ordered) syntactic semigroup lies in V, where the syntactic semigroup of a subshift is the syntactic semigroup of the corresponding factorial prolongable language. In this paper it is proved that if V contains the pseudovariety Sl^- of commutative idempotent monoids in which the neutral element is a global

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minimum, then $\mathscr{S}(\mathsf{V} * \mathsf{D})$ is closed under taking conjugate subshifts. After obtaining this result, the author has recently observed that its unordered version can be easily deduced from Theorem 2.7 in [14], which is a theorem about ζ-semigroups as recognition structures for sofic subshifts. Conversely, we prove that if $\mathscr{S}(\mathsf{V} * \mathsf{D})$ is closed under taking conjugate subshifts then V contains LSI^- and $\mathscr{S}(\mathsf{V}) = \mathscr{S}(\mathsf{V} * \mathsf{D})$.

One of the most successful approaches in the research on pseudovarieties of semigroups over the last two decades involves profinite methods, namely through the study of free and relatively free profinite semigroups. The elements of free profinite semigroups are sometimes called *profinite words* or *pseudowords*. They can be seen as a generalization of ordinary words. The equational description of pseudovarieties by means of formal identities between pseudowords established by Reiterman [36] is one of the seminal motivations for the profinite approach in the study of pseudovarieties. The author developed in [15] some tools for using pseudowords in the study of subshifts. With them he obtained some new conjugacy invariants. The present paper is a sequel to [15], namely through the exploration of one of its main instrumental results, which appears here in Theorem 2.10. The exploration of links between the theory of profinite semigroups and concepts from symbolic dynamics began with the papers [2,5]. Almeida also established in [3] a connection between the minimal subshifts over a given alphabet and the corresponding free profinite semigroup, which leads to a better understanding of the structure of such semigroups.

The search for conjugacy invariants in the syntactic semigroup of a sofic subshift is also performed in [10], where a shift equivalence invariant is introduced, which defines a hierarchy of irreducible sofic subshifts, and it is proved that the first level of the hierarchy is the class of almost finite type subshifts. This class has practical interest for coding theory, and for several reasons it is a meaningful class above the class of irreducible finite type subshifts, as stated in [10]; see [25, Chapter 13.1] and [8].

The paper is organized as follows. Section 2 is dedicated to preliminary definitions and results, some of which are recovered from [15]. Section 3 contains the results describing which classes defined by pseudovarieties of semigroups are closed under taking conjugate subshifts. Section 4 is dedicated to the characterization of some significant classes of sofic subshifts defined by pseudovarieties in the way described in Section 3. We deduce a new proof of the conjugacy invariance of the class of almost finite type subshifts by showing that they are the irreducible members of $\mathscr{S}(\mathsf{LInv})$. Finally, in Section 5 we prove that the conjugacy invariants that we have established are also shift equivalence invariants, with a proof depending on previous results about conjugacy invariance.

Our main reference for symbolic dynamics is the book of Lind and Marcus [25]. For background on classical semigroup theory, rational languages and finite automata see for example [29]. For the study of pseudovarieties from a profinite semigroup theory perspective, see the introductory text [4].

2. Preliminaries

2.1. Subshifts and sliding block codes

Let A be an alphabet. All alphabets in this paper are assumed to be finite. The semigroup of finite non-empty words (or blocks) on letters of A is denoted by A^+; the empty word is denoted by 1 and A^* is the monoid $A^+ \cup \{1\}$. The set of words over A with length n is A^n. Let $A^{\mathbb{Z}}$ be the set of sequences of letters of A indexed by $\mathbb{Z}$. The *shift* in $A^{\mathbb{Z}}$ is the bijective function σ_A (or just σ) from $A^{\mathbb{Z}}$ to $A^{\mathbb{Z}}$ defined by $\sigma_A((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$. We endow $A^{\mathbb{Z}}$ with the product topology with respect to the discrete topology of A. Note that $A^{\mathbb{Z}}$ is a compact Hausdorff space. From here on *compact* will mean both compact and Hausdorff. A *shift dynamical system* or *subshift* of $A^{\mathbb{Z}}$ is a non-empty closed subset $\mathcal{X}$ of $A^{\mathbb{Z}}$ such that $\sigma_A(\mathcal{X}) \subseteq \mathcal{X}$ and $\sigma_A^{-1}(\mathcal{X}) \subseteq \mathcal{X}$. A factor of $(x_i)_{i\in\mathbb{Z}}$ is a finite sequence $x_i x_{i+1} \ldots x_{i+n-1} x_{i+n}$, where $i \in \mathbb{Z}$ and $n \geq 0$. If $\mathcal{X}$ is a subset of $A^{\mathbb{Z}}$ then we denote by $L(\mathcal{X})$ the set of factors of elements of $\mathcal{X}$. A subset K of a semigroup S is *factorial* if it is closed under taking factors, and it is *prolongable* if for every element u of K there are $a, b \in S$ such that $aub \in K$. It is easy to prove that the correspondence $\mathcal{X} \mapsto L(\mathcal{X})$ is a bijection between the subshifts of $A^{\mathbb{Z}}$ and the non-empty factorial prolongable languages of A^+.

A *sliding block code* G between the subshifts $\mathcal{X}$ of $A^{\mathbb{Z}}$ and $\mathcal{Y}$ of $B^{\mathbb{Z}}$ is a function $G : \mathcal{X} \to \mathcal{Y}$ for which there are integers $k, l \geq 0$ and a function $g : A^{k+l+1} \to B$ such that $G(x) = (g(x_{[i-k,i+l]}))_{i\in\mathbb{Z}}$. We say that g is a *block map* of G with *memory* k and *anticipation* l. The sliding block code G depends only on the restriction of g to $A^{k+l+1} \cap L(\mathcal{X})$. We use the notation $G = g^{[-k,l]} : \mathcal{X} \to \mathcal{Y}$. If $n \geq l$, $m \geq k$ and $h : A^{m+n+1} \to B$ is defined by $h(a_{-m}a_{-m+1} \ldots a_{n-1}a_n) = g(a_{-k}a_{-k+1} \ldots a_{l-1}a_l)$, with $a_i \in A$, then h is a block map of G with memory m and anticipation n. In particular, one can choose a block map with equal memory and anticipation.

It is well known [19] that a map $G : \mathcal{X} \subseteq A^{\mathbb{Z}} \to \mathcal{Y} \subseteq B^{\mathbb{Z}}$ between subshifts is a sliding block code if and only if it is a continuous function such that $G \circ \sigma_A = \sigma_B \circ G$. Note that the identity transformation of a subshift is a sliding block code, the composition of two sliding block codes is a sliding block code and the inverse of a bijective sliding block code is a sliding block code. A bijective sliding block code is called a *conjugacy*. Two subshifts are *conjugate* if there is a conjugacy between them. A *conjugacy invariant* is a property of subshifts that is preserved under taking conjugate subshifts. See [25] for the definition and computation of ordinary conjugacy invariants like the zeta function and entropy.

Given an alphabet A and $k \geq 1$, consider the alphabet A^k. To avoid ambiguities, we represent an element $w_1 \ldots w_n$ of $(A^k)^+$ (with $w_i \in A^k$) by $\langle w_1, \ldots, w_n \rangle$. For $k \geq 0$ let Φ_k be the function from A^+ to $(A^{k+1})^*$ defined by

$$\Phi_k(a_1 \ldots a_n) = \begin{cases} 1 & \text{if } n \leq k, \\ \langle a_{[1,k+1]}, a_{[2,k+2]}, \ldots a_{[n-k-1,n-1]}, a_{[n-k,n]} \rangle & \text{if } n > k, \end{cases}$$

where $a_i \in A$ and $a_{[i,j]} = a_i a_{i+1} \ldots a_{j-1} a_j$. It is easy to see that, if $\mathcal{X}$ is a subshift of $A^{\mathbb{Z}}$ and $i, j \geq 0$ are such that $i + j = k$, then the restriction of the sliding block code $\Phi_k^{[-i,j]}$ to $\mathcal{X}$ is a conjugacy between $\mathcal{X}$ and $\Phi_k^{[-i,j]}(\mathcal{X})$. A *one-block code* is a sliding block code having a block map with memory and anticipation zero.

Remark 2.1. For every sliding block code G there are one-block codes G_1 and G_2 such that G_1 is a conjugacy and $G = G_2 \circ G_1^{-1}$.

Proof. For a sliding block code $G = g^{[-k,k]} : \mathcal{X} \to \mathcal{Y}$ let G_1 be the inverse of the restriction $\Phi_{2k}^{[-k,k]} : \mathcal{X} \to \Phi_{2k}^{[-k,k]}(\mathcal{X})$ and let $G_2 = g^{[0,0]} : \Phi_{2k}^{[-k,k]}(\mathcal{X}) \to \mathcal{Y}$. $\square$

A subshift $\mathcal{X}$ is *sofic* if $L(\mathcal{X})$ is rational. We use the term *graph-automaton* for an automaton such that all states are initial and final. An automaton is *essential* if all states lie in a bi-infinite path on the automaton. One can see that $\mathcal{X}$ is sofic if and only if $L(\mathcal{X})$ is recognized by an essential finite graph-automaton. We say that a graph-automaton *presents* the subshift $\mathcal{X}$ if it recognizes $L(\mathcal{X})$.

A subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$ is *irreducible* if for all $u, v \in L(\mathcal{X})$ there is $w \in A^*$ such that $uwv \in L(\mathcal{X})$. Irreducibility is a conjugacy invariant. A sofic subshift is irreducible if and only if it is presented by a strongly connected finite graph-automaton [18].

A subshift of $A^{\mathbb{Z}}$ is of *finite type* if there is a finite subset F of A^+ such that $L(\mathcal{X}) = A^+ \setminus A^* F A^*$. Note that finite type subshifts are sofic. Moreover, the class of sofic subshifts is the smallest class of subshifts containing the finite type subshifts and closed under taking the image of a sliding block code. This was how sofic subshifts were first introduced by Weiss in [38], together with the first characterizations using finite automata and semigroups.

A subshift presented by a finite graph-automaton in which every letter acts on at most one state is called an *edge subshift*. The subshifts of finite type are precisely those that are conjugate with an edge subshift. The following result is well known (see [25, Theorem 2.1.8]).

Proposition 2.2. *A subshift $\mathcal{X}$ is of finite type if and only if there is an integer $n \geq 0$ such that whenever $uv, vw \in L(\mathcal{X})$ and v has length greater than n, then $uvw \in L(\mathcal{X})$.*

A state v of the minimal automaton of $L(\mathcal{X})$ is a *K-state* if there is $x \in \mathcal{X}$ such that the set of words labeling a path from the initial state to v contains infinitely many words of the form $x_{-n} x_{-(n-1)} \ldots x_{-1}$, with $n \geq 1$. The *Krieger cover* of a sofic subshift $\mathcal{X}$ is the essential graph-automaton obtained from the minimal automaton of $L(\mathcal{X})$ by deleting all the states that are not K-states [28, Section 5].

The edge subshift obtained from the Krieger cover of $\mathcal{X}$ by labeling in its graphical representation different arrows with different letters is called the *Krieger edge subshift* of $\mathcal{X}$. Krieger proved in [24] that if $\mathcal{X}$ and $\mathcal{Y}$ are conjugate sofic subshifts, then their Krieger edge subshifts are also conjugate. If the sofic subshift $\mathcal{X}$ is irreducible then its Krieger cover has a unique terminal strongly connected component which is a graph-automaton presenting $\mathcal{X}$ [11]. This graph-automaton is named the *Fischer cover* of $\mathcal{X}$.

2.2. Pseudowords

A *compact semigroup* is a semigroup endowed with a compact topology for which the semigroup operation is continuous; if moreover the topology is zero-dimensional (that is, generated by open sets that are closed) then we say that it is a *profinite semigroup*. In [4] we can find other equivalent definitions of profinite semigroup. Note that finite semigroups are profinite with respect to the discrete topology. Given an alphabet A, there is a profinite semigroup $\widehat{A^+}$, in which A^+ embeds as a dense subsemigroup, such that for every map φ from A into a profinite semigroup S, there is a unique continuous homomorphism $\hat{\varphi} : \widehat{A^+} \to S$ whose restriction to A is φ. The semigroup $\widehat{A^+}$ is, up to isomorphism of compact semigroups, the unique profinite semigroup with this property; for that reason it is called the *free A-generated profinite semigroup*. For constructions of $\widehat{A^+}$ see [4]. The definition of the *free A-generated profinite monoid* $\widehat{A^*}$ is similar to that of $\widehat{A^+}$. Considering the empty word as an isolated point of $\widehat{A^+} \cup \{1\}$, we can view $\widehat{A^+} \cup \{1\}$ as being $\widehat{A^*}$. Elements of $\widehat{A^*}$ are called *pseudowords*.

Let w be a pseudoword of $\widehat{A^+}$. For $a \in A$, we say that *a is a letter of w* if a is a factor of w. A *prefix* (respectively, *suffix*) of w is a pseudoword u of $\widehat{A^*}$ such that $w = u\pi$ (respectively, $w = \pi u$) for some π in $\widehat{A^*}$. For $n \geq 1$, let $A^{<n}$ be the set of words of A^+ with length less than n. If $w \in \widehat{A^+} \setminus A^{<n}$ then w has a unique prefix and a unique suffix of length n, denoted respectively by $\mathrm{i}_n(w)$ and $\mathrm{t}_n(w)$ [1]. If $w \in A^{<n}$ then we define $\mathrm{i}_n(w) = \mathrm{t}_n(w) = w$.

Let us consider within the alphabet $A = \{a_1, \ldots, a_n\}$ with n elements the order in which a_i is the i-th letter. Let $\pi \in \widehat{A^+}$. For a profinite semigroup S, denote by π_S the n-ary operation on S that maps $(s_1, \ldots, s_n) \in S^n$ to the image of π under the unique continuous homomorphism $\varphi : \widehat{A^+} \to S$ such that $\varphi(a_i) = s_i$. Note that if $\psi : S \to T$ is a continuous homomorphism between profinite semigroups then $\psi(\pi_S(s_1, \ldots, s_n)) = \pi_T(\psi(s_1), \ldots, \psi(s_n))$. In the absence of confusion we may drop the index S in $\pi_S(s_1, \ldots, s_n)$ and write $\pi(s_1, \ldots, s_n)$.

The next lemma generalizes to pseudowords the way in which a word appears as a factor of a finite product of finite words.

Lemma 2.3 (*[5, Lemma 8.2]*). *Let $X = \{x_1, \ldots, x_n\}$ be an alphabet with n elements. Let A be also an alphabet. Consider pseudowords $w \in \widehat{X^+}$ and $v_1, \ldots, v_n \in \widehat{A^+}$. Suppose that u is a finite factor of $w_{\widehat{A^+}}(v_1, \ldots, v_n)$. Then u is either a factor of some v_i or w has a factor $x_{i_0} x_{i_1} \ldots x_{i_k} x_{i_{k+1}}$ (with $x_{i_j} \in X$) such that u factors as $u = u_{i_0} v_{i_1} \ldots v_{i_k} u_{i_{k+1}}$ where u_{i_0} is a suffix of v_{i_0} and $u_{i_{k+1}}$ is a prefix of $v_{i_{k+1}}$.*

The following lemma is easily proved using the fact that the closure of a rational language is open [4, Theorem 3.6].

Lemma 2.4 (*[15]*). *If L is a factorial rational language of A^+ then the closure of L in $\widehat{A^+}$ is factorial.*

If s is an element of a profinite semigroup S, then $s^{n!}$ converges to the unique idempotent in the closure of the subsemigroup generated by s; this idempotent is denoted by s^ω. Let e and f be idempotents of S. We say that an element u of S is *bounded* by e and f (in this order) if $u = euf$. An element is *idempotent-bound* if it is bounded by some pair of idempotents.

In [1, Lemma 10.6.1] it is proved that $\Phi_k : A^+ \to (A^{k+1})^*$ has a unique continuous extension $\widehat{A^+} \to \widehat{(A^{k+1})^*}$, which we also denote by Φ_k. For a map $g : A^{2k+1} \to B$ let $\hat{g}$ be the unique continuous monoid homomorphism from $\widehat{(A^{2k+1})^*}$ into $\widehat{B^*}$ that extends g. Denote by $\bar{g}$ the map $\hat{g} \circ \Phi_{2k}$. The coding process described by g is extended to every pseudoword of $\widehat{A^+}$ by $\bar{g}$:

Lemma 2.5. *For all $u, v \in \widehat{A^+}$ we have:*

$$\bar{g}(uv) = \bar{g}(u)\bar{g}(\mathrm{t}_{2k}(u)v) = \bar{g}(u\,\mathrm{i}_{2k}(v))\bar{g}(v) = \bar{g}(u\,\mathrm{i}_k(v))\bar{g}(\mathrm{t}_k(u)v). \tag{2.1}$$

Proof. We first prove by induction on the length of v that for all $u, v \in A^+$ we have

$$\Phi_k(uv) = \Phi_k(u)\,\Phi_k(\mathrm{t}_k(u)v). \tag{2.2}$$

If $u \in A^+$ and $a \in A$, then

$$\Phi_k(ua) = \Phi_k(u) \cdot \mathrm{t}_{k+1}(ua) = \Phi_k(u) \cdot \mathrm{t}_k(u)a = \Phi_k(u) \cdot \Phi_k(\mathrm{t}_k(u)a),$$

which proves the initial step of the induction. Suppose that $u, v \in A^+$ verify (2.2). Then, by the already proved initial step,

$$\Phi_k(uva) = \Phi_k(uv) \cdot \Phi_k(\mathrm{t}_k(uv)a)$$

$$= \Phi_k(u) \cdot \Phi_k(\mathrm{t}_k(u)v) \cdot \Phi_k(\mathrm{t}_k(uv)a)$$
$$= \Phi_k(u) \cdot \Phi_k(\mathrm{t}_k(u)v) \cdot \Phi_k(\mathrm{t}_k(\mathrm{t}_k(u)v)a)$$
$$= \Phi_k(u) \cdot \Phi_k(\mathrm{t}_k(u)va).$$

This finishes the induction proof of the equality (2.2) for all $u, v \in A^+$. Since A^+ is dense in $\widehat{A^+}$ and Φ_k, t_k are continuous, equality (2.2) also holds for all $u, v \in \widehat{A^+}$. The proof that for all $u, v \in \widehat{A^+}$ we have $\Phi_k(uv) = \Phi_k(u\,\mathrm{i}_k(v))\,\Phi_k(v)$ is similar. Since $\bar{g}$ is the composition of the homomorphism $\hat{g}$ with Φ_{2k}, we have therefore proved the two first equalities of (2.1).

It remains to prove that $\bar{g}(uv) = \bar{g}(u\,\mathrm{i}_k(v))\,\bar{g}(\mathrm{t}_k(u)v)$. We divide the proof into three cases. First, if $u = u_1u_2$ and $v = v_1v_2$, with $u_i, v_i \in \widehat{A^*}$ and $u_2, v_1 \in A^k$, then

$$\begin{aligned}\bar{g}(uv) &= \bar{g}(u_1 \cdot u_2v_1v_2)\\ &= \bar{g}(u_1\mathrm{i}_{2k}(u_2v_1v_2)) \cdot \bar{g}(u_2v_1v_2)\\ &= \bar{g}(u_1u_2v_1) \cdot \bar{g}(\mathrm{t}_k(u)v)\\ &= \bar{g}(u\mathrm{i}_k(v)) \cdot \bar{g}(\mathrm{t}_k(u)v).\end{aligned}$$

If u is a word of A^+ with length less than k, then $u\mathrm{i}_k(v)$ has length less than $2k$, thus $\bar{g}(u\mathrm{i}_k(v)) = \hat{g}(1) = 1$ and $\bar{g}(uv) = \bar{g}(\mathrm{t}_k(u)v) = \bar{g}(u\mathrm{i}_k(v))\,\bar{g}(\mathrm{t}_k(u)v)$. The case where v is a word of A^+ with length less than k is similar. $\square$

Given a subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$, let $\mathrm{Mir}(\mathcal{X})$ be the set of pseudowords whose finite factors belong to $L(\mathcal{X})$. We call $\mathrm{Mir}(\mathcal{X})$ the *mirage* of $\mathcal{X}$ in $\widehat{A^+}$. Note that $\mathrm{Mir}(\mathcal{X})$ is a union of $\mathcal{J}$-classes. We have $\overline{L(\mathcal{X})} \subseteq \mathrm{Mir}(\mathcal{X})$. In general $\mathrm{Mir}(\mathcal{X})$ and $\overline{L(\mathcal{X})}$ are different, when $\mathcal{X}$ is sofic [15].

Lemma 2.6 ([15]). *Let $G = g^{[-k,k]} : \mathcal{X} \to \mathcal{Y}$ be a sliding block code. Then $\bar{g}\left(\overline{L(\mathcal{X})}\right) \subseteq \overline{L(\mathcal{Y})} \cup \{1\}$ and $\bar{g}(\mathrm{Mir}(\mathcal{X})) \subseteq \mathrm{Mir}(\mathcal{Y}) \cup \{1\}$.*

Lemma 2.7 ([15]). *Let $G = g^{[-k,k]} : \mathcal{X} \subseteq A^{\mathbb{Z}} \to \mathcal{Y} \subseteq B^{\mathbb{Z}}$ be a conjugacy and let $G^{-1} = h^{[-l,l]} : \mathcal{Y} \to \mathcal{X}$ be its inverse. Consider an element v of $\widehat{A^+}$. If r and s are words of length $k+l$ such that $rvs \in \mathrm{Mir}(\mathcal{X})$ then $v = \bar{h}\bar{g}(rvs)$.*

2.3. The syntactic semigroup of a sofic subshift

A binary relation $\mathcal{K}$ on a semigroup S is *stable* if $r\,\mathcal{K}\,s$ implies $tr\,\mathcal{K}\,ts$ and $rt\,\mathcal{K}\,st$ for all $r, s, t \in S$. The semigroup congruences are the stable equivalence relations. Let L be a language of A^+. The following quasi-order, called *syntactic order*, is stable:

$$v \leq_L u \Leftrightarrow [\forall x, y \in A^*, xuy \in L \Rightarrow xvy \in L].$$

The equivalence relation generated by $\leq_L$ is a semigroup congruence, the *syntactic congruence* of L. The quotient of A^+ by the syntactic congruence of L is called the *syntactic semigroup* of L. We denote it by $\mathrm{Syn}(L)$. Let δ_L be the canonical homomorphism from A^+ into $\mathrm{Syn}(L)$. Consider in $\mathrm{Syn}(L)$ the relation also denoted $\leq_L$ (or simply $\leq$) such that $\delta_L(v) \leq_L \delta_L(u)$ if and only if $v \leq_L u$. It is a well-defined partial order. An *ordered semigroup* is a semigroup equipped with a stable partial order for multiplication. The syntactic semigroup of L equipped with the partial order $\leq_L$ is an ordered semigroup, which in the absence of confusion is also denoted $\mathrm{Syn}(L)$ and named syntactic semigroup of L. The language L is rational if and only if $\mathrm{Syn}(L)$ is finite, in which case δ_L has a unique extension to a continuous homomorphism $\hat{\delta}_L : \widehat{A^+} \to \mathrm{Syn}(L)$.

Lemma 2.8 ([15]). *Let u and v be elements of $\widehat{A^+}$. If L is a rational language of A^+ then*

$$\hat{\delta}_L(v) \leq_L \hat{\delta}_L(u) \Leftrightarrow \left[\forall x, y \in \widehat{A^*}, xuy \in \overline{L} \Rightarrow xvy \in \overline{L}\right].$$

Let $\mathcal{X}$ be a subshift of $A^{\mathbb{Z}}$ and let $\mathrm{Syn}(\mathcal{X})$ be the syntactic semigroup of $L(\mathcal{X})$. We denote respectively by $\delta_{\mathcal{X}}$ and $\hat{\delta}_{\mathcal{X}}$ the homomorphisms $\delta_{L(\mathcal{X})}$ and $\hat{\delta}_{L(\mathcal{X})}$. The subshift $A^{\mathbb{Z}}$ is usually named the *full shift* of $A^{\mathbb{Z}}$; its syntactic semigroup is trivial. Suppose that $\mathcal{X}$ is not the full shift. Then $\mathrm{Syn}(\mathcal{X})$ is a non-trivial semigroup with a zero denoted by 0. One can easily prove that, for all $u \in A^+$, $\delta_{\mathcal{X}}(u) = 0$ if and only if $u \notin L(\mathcal{X})$ [12]. This implies that if $\mathcal{X}$ is sofic then, for all $u \in A^+$, $\hat{\delta}_{\mathcal{X}}(u) = 0$ if and only if $u \notin \overline{L(\mathcal{X})}$. The zero is the maximal element of $\mathrm{Syn}(\mathcal{X})$ for $\leq_{L(\mathcal{X})}$, because if $u \in A^+ \setminus L(\mathcal{X})$ then $xuy \notin L(\mathcal{X})$ for all $x, y \in A^*$.

Lemma 2.9. *Let $G = g^{[-k,k]} : \mathcal{X} \subseteq A^{\mathbb{Z}} \to \mathcal{Y} \subseteq B^{\mathbb{Z}}$ be a conjugacy between sofic subshifts. Let u be an idempotent-bound element of* $\mathrm{Mir}(\mathcal{X})$. *If $\bar{g}(u) \in \overline{L(\mathcal{Y})}$ then $u \in \overline{L(\mathcal{X})}$.*

Proof. Let h be a block map of G^{-1} with memory and anticipation l. Let e and f be idempotents of $\widehat{A^+}$ such that $u = euf$, and let $r = \mathrm{i}_{k+l}(e)$ and $s = \mathrm{t}_{k+l}(f)$. Then there are e_0, f_0 such that $u = re_0uf_0s$. By Lemma 2.7 we have $e_0uf_0 = \bar{h}\bar{g}(u)$, thus u is a factor of $\bar{h}\bar{g}(u)$. Since $\bar{g}(u) \in \overline{L(\mathcal{Y})}$, by Lemma 2.6 we have $\bar{h}\bar{g}(u) \in \overline{L(\mathcal{X})}$. Hence $u \in \overline{L(\mathcal{X})}$ by Lemma 2.4. □

Theorem 2.10 *([15]). Let $G = g^{[-k,k]} : \mathcal{X} \subseteq A^{\mathbb{Z}} \to \mathcal{Y} \subseteq B^{\mathbb{Z}}$ be a conjugacy between sofic subshifts. Let e and f be idempotents of $\widehat{A^+}$. Let u and v be elements of $\widehat{A^+}$ such that $u = euf$, $v = evf$, $u \in \overline{L(\mathcal{X})}$ and $v \in$* $\mathrm{Mir}(\mathcal{X})$. *Then $\hat{\delta}_{\mathcal{X}}(v) \le \hat{\delta}_{\mathcal{X}}(u)$ if and only if $\hat{\delta}_{\mathcal{Y}}(\bar{g}(v)) \le \hat{\delta}_{\mathcal{Y}}(\bar{g}(u))$.*

2.4. Pseudovarieties of ordered semigroups

A *pseudovariety of ordered semigroups* is a class of finite ordered semigroups closed under taking ordered subsemigroups, finite direct products and images under order-preserving homomorphisms of semigroups. A *pseudovariety of semigroups* is a pseudovariety of ordered semigroups closed under taking images under homomorphisms of semigroups; since the identity map is a homomorphism, the order takes no role in this notion, which therefore corresponds to the usual notion of pseudovariety of (unordered) semigroups. The class Com of finite commutative semigroups is a pseudovariety of semigroups. Pseudovarieties of ordered monoids and pseudovarieties of monoids are defined similarly, using the notions of submonoid and homomorphism of monoid. The class Sl^- of commutative ordered monoids such that every element is idempotent and greater than or equal to the neutral element is a pseudovariety of ordered monoids. It is not a pseudovariety of monoids. The smallest pseudovariety of monoids containing Sl^- is the class Sl of commutative monoids whose elements are idempotents. If V is a pseudovariety of ordered semigroups or monoids then the class LV of ordered semigroups whose ordered submonoids are in V is a pseudovariety of ordered semigroups.

For an alphabet A with n letters, let π and ρ be elements of $\widehat{A^+}$. We say that the formal inequality $\pi \le \rho$ is a *pseudoidentity over A*. The formal equality $\pi = \rho$ is seen as the set of pseudoidentities $\{\pi \le \rho, \rho \le \pi\}$. If S is a profinite ordered semigroup with order $\le_S$, then we say that S satisfies the pseudoidentity $\pi \le \rho$ if for all n-tuples $(s_1, \ldots, s_n)$ in S^n we have $\pi_S(s_1, \ldots, s_n) \le_S \rho_S(s_1, \ldots, s_n)$. A class V is a pseudovariety of ordered semigroups if and only if there is a set Σ of pseudoidentities (possibly over distinct alphabets) such that V is the class of finite ordered semigroups satisfying all pseudoidentities in Σ [32,27]. We denote by $[\![\Sigma]\!]$ the pseudovariety V defined by Σ, and we then say that Σ is a *basis of pseudoidentities* for V. Furthermore, V is a pseudovariety of semigroups if and only if it has a basis of formal equalities between pseudowords [36]. Similar definitions and results hold for pseudovarieties of ordered monoids, with the obvious changes. For example,

$$\begin{aligned} \mathsf{Sl}^- &= [\![xy = yx, x^2 = x, 1 \le x]\!], \\ \mathsf{LSl}^- &= [\![z^\omega x z^\omega y z^\omega = z^\omega y z^\omega x z^\omega, z^\omega x z^\omega x z^\omega = z^\omega x z^\omega, z^\omega \le z^\omega x^\omega z^\omega]\!]. \end{aligned}$$

A *variety of languages* is a family $\mathcal{W}$ that associates to each finite alphabet A a set $\mathcal{W}A^+$ of rational languages of A^+ with the following properties:

1. for every alphabet A, the set $\mathcal{W}A^+$ is closed under taking a finite number of unions and intersections;
2. for every alphabet A, if $L \in \mathcal{W}A^+$ then for every $a \in A$ the languages $\{w \in A^+ : aw \in L\}$ and $\{w \in A^+ : wa \in L\}$ belong to $\mathcal{W}A^+$;
3. if $\varphi : A^+ \to B^+$ is a homomorphism and $L \in \mathcal{W}B^+$ then $\varphi^{-1}(L) \in \mathcal{W}A^+$.

For a pseudovariety V of ordered semigroups let $\mathcal{V}$ be the class of languages whose syntactic semigroup belongs to V. The correspondence $\mathsf{V} \to \mathcal{V}$ is a bijection between pseudovarieties of ordered semigroups and varieties of languages [30], and $\mathcal{V}A^+$ is closed under taking complements in A^+, for an arbitrary alphabet A, if and only if V is a pseudovariety of semigroups [17].

The *locally testable languages* of A^+ are the languages that can be obtained from the languages of the form A^*wA^*, wA^* and A^*w, where $w \in A^+$, applying a finite number of unions, intersections and complements in A^+. The following characterization is a fundamental result in finite semigroup theory.

Theorem 2.11 (*[13,26,41,40]*). *The class of locally testable languages is the variety of languages corresponding to* LSl.

J.-E. Pin and P. Weil proved in [35] an ordered version of Theorem 2.11. The *negatively locally* testable languages of A^+ are the languages that can be expressed with a finite number of unions and intersections of languages of the form $A^+ \setminus A^*wA^*$, $A^+ \setminus wA^*$, $A^+ \setminus A^*w$ and $A^+ \setminus \{w\}$, with $w \in A^+$.

Theorem 2.12 (*[35]*). *The class of negatively locally testable languages is the variety of languages corresponding to* LSl^-.

3. Invariant pseudovarieties

For a class C of ordered semigroups, let $\mathscr{S}(\mathsf{C})$ be the class of subshifts whose syntactic semigroup is in C. We say that a class of subshifts is a conjugacy invariant if it is closed under taking conjugate subshifts. In this section we identify all conjugacy invariants $\mathscr{S}(\mathsf{V})$ such that V is a pseudovariety of ordered semigroups.

Proposition 3.1. *Let $G = g^{[0,0]} : \mathcal{X} \subseteq A^{\mathbb{Z}} \to \mathcal{Y} \subseteq B^{\mathbb{Z}}$ be a one-block conjugacy between sofic subshifts. Let ρ and π be pseudowords over an alphabet X with n elements such that the finite factors of π are factors of ρ, and such that $\rho = e\rho f$ and $\pi = e\pi f$ for some idempotents e and f of $\widehat{X^+}$. If* $\mathrm{Syn}(\mathcal{Y})$ *satisfies $\pi \leq \rho$, then so does* $\mathrm{Syn}(\mathcal{X})$.

Proof. Suppose that $\mathrm{Syn}(\mathcal{X})$ does not satisfy $\pi \leq \rho$. Then there is a n-tuple $(s_1, \ldots, s_n)$ of elements of $\mathrm{Syn}(\mathcal{X})$ such that $\pi_{\mathrm{Syn}(\mathcal{X})}(s_1, \ldots, s_n) \not\leq \rho_{\mathrm{Syn}(\mathcal{X})}(s_1, \ldots, s_n)$. For each i let w_i be a word of A^+ such that $\delta_{\mathcal{X}}(w_i) = s_i$. Because $\hat{\delta}_{\mathcal{X}}$ is a continuous homomorphism, we have $\hat{\delta}_{\mathcal{X}}(\pi(w_1, \ldots, w_n)) \not\leq \hat{\delta}_{\mathcal{X}}(\rho(w_1, \ldots, w_n))$. Then $\hat{\delta}_{\mathcal{X}}(\rho(w_1, \ldots, w_n)) \neq 0$, because 0 is the maximal element of $\mathrm{Syn}(\mathcal{X})$. Hence $\rho(w_1, \ldots, w_n) \in \overline{L(\mathcal{X})}$. By Lemma 2.4, every finite factor of $\rho(w_1, \ldots, w_n)$ belongs to $L(\mathcal{X})$. By Lemma 2.3 this implies $\pi(w_1, \ldots, w_n) \in \mathrm{Mir}(\mathcal{X})$, because the finite factors of π are factors of ρ. Then, since $\rho(w_1, \ldots, w_n)$ and $\pi(w_1, \ldots, w_n)$ are bounded by the idempotents $e(w_1, \ldots, w_n)$ and $f(w_1, \ldots, w_n)$, from Theorem 2.10 we deduce $\hat{\delta}_{\mathcal{Y}}\bar{g}(\pi(w_1, \ldots, w_n)) \not\leq \hat{\delta}_{\mathcal{Y}}\bar{g}(\rho(w_1, \ldots, w_n))$. Hence, since $\hat{\delta}_{\mathcal{Y}}\bar{g}$ is a continuous homomorphism, we have

$$\pi_{\mathrm{Syn}(\mathcal{Y})}(\hat{\delta}_{\mathcal{Y}}\bar{g}(w_1), \ldots \hat{\delta}_{\mathcal{Y}}\bar{g}(w_n)) \not\leq \rho_{\mathrm{Syn}(\mathcal{Y})}(\hat{\delta}_{\mathcal{Y}}\bar{g}(w_1), \ldots \hat{\delta}_{\mathcal{Y}}\bar{g}(w_n)). \quad \square$$

Let us recall that a *graph* is a 4-tuple $\Gamma = (V(\Gamma), E(\Gamma), \alpha, \beta)$ such that $V(\Gamma)$ and $E(\Gamma)$ are disjoint sets, and α, β are maps from $E(\Gamma)$ to $V(\Gamma)$. The elements of $V(\Gamma)$ and $E(\Gamma)$ are the *vertices* and the *edges* of Γ, respectively. We say that an edge x goes from u to v if $\alpha(x) = u$ and $\beta(x) = v$. If $\beta(x) = \alpha(y)$ then x and y are said to be *consecutive*. Denote by $A(\Gamma)$ the alphabet $E(\Gamma) \cup V(\Gamma)$. Let ζ_Γ be the unique continuous homomorphism from $\widehat{E(\Gamma)^+}$ to $\widehat{A(\Gamma)^+}$ that sends an element x from $E(\Gamma)$ to $\alpha(x)^\omega x\, \beta(x)^\omega$. We say that an element of $\widehat{E(\Gamma)^+}$ is a *Γ-profinite-path* if every factor of π with length two is a product of consecutive edges of Γ. Two Γ-profinite-paths π and ρ are *coterminal* if $\alpha(\mathrm{i}_1(\pi)) = \alpha(\mathrm{i}_1(\rho))$ and $\beta(\mathrm{t}_1(\pi)) = \beta(\mathrm{t}_1(\rho))$.

Proposition 3.2. *Let Γ be a finite graph. Let π and ρ be coterminal Γ-profinite-paths. Suppose that every letter of π is a letter of ρ. Then $\mathscr{S}([\![\zeta_\Gamma(\pi) \leq \zeta_\Gamma(\rho)]\!])$ is a conjugacy invariant.*

Proof. Let n and m be the number of edges and vertices of Γ, respectively. Let x_i be the i-th edge of Γ, and let y_j be the j-th vertex, with $1 \leq i \leq n$ and $1 \leq j \leq m$. Denote by αi and βi the integers such that $\alpha(x_i) = y_{\alpha i}$ and $\beta(x_i) = y_{\beta i}$.

By the Remark 2.1, we are reduced to the case where there is a one-block conjugacy $G = g^{[0,0]} : \mathcal{X} \to \mathcal{Y}$. Let u be a finite factor of $\zeta_\Gamma(\pi)$. By Lemma 2.3 there is i such that x_i is a factor of π and u is a factor of $y_{\alpha i}^\omega x_i y_{\beta i}^\omega$, or there are i, j such that $x_i x_j$ is a factor of π and u is a factor of $(y_{\alpha i}^\omega x_i y_{\beta i}^\omega)(y_{\alpha j}^\omega x_j y_{\beta j}^\omega)$. The arguments for the first case are included in the second case, so we only consider the latter. Since π is a Γ-profinite-path, the edges x_i and x_j are consecutive. Hence $\beta i = \alpha j$ and u is a finite factor of $y_{\alpha i}^\omega x_i y_{\beta i}^\omega x_j y_{\beta j}^\omega$. Again by Lemma 2.3, u is a finite factor of $y_{\alpha i}^\omega x_i y_{\beta i}^\omega$ or of $y_{\alpha j}^\omega x_j y_{\beta j}^\omega$. These pseudowords are factors of $\zeta_\Gamma(\rho)$, because every letter of π is a letter of ρ. Hence every finite factor of $\zeta_\Gamma(\pi)$ is a factor of $\zeta_\Gamma(\rho)$. Since π and ρ are coterminal, the pseudowords $\zeta_\Gamma(\pi)$ and $\zeta_\Gamma(\rho)$ are bounded by some idempotents $y_{i_0}^\omega$ and $y_{j_0}^\omega$. Therefore, by Proposition 3.1, if $\mathrm{Syn}(\mathcal{Y})$ satisfies $\zeta_\Gamma(\pi) \leq \zeta_\Gamma(\rho)$ then so does $\mathrm{Syn}(\mathcal{X})$.

Conversely, suppose that $\mathrm{Syn}(\mathcal{X})$ satisfies $\zeta_\Gamma(\pi) \le \zeta_\Gamma(\rho)$. Let A and B be the alphabets of $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let h be a block map of G^{-1} with memory and anticipation k. Let $t_1, \ldots, t_n, c_1, \ldots, c_m \in \mathrm{Syn}(\mathcal{Y})$. The remainder of the proof amounts to showing that

$$\pi(c_{\alpha 1}^{\omega} t_1 c_{\beta 1}^{\omega}, \ldots, c_{\alpha n}^{\omega} t_n c_{\beta n}^{\omega}) \le \rho(c_{\alpha 1}^{\omega} t_1 c_{\beta 1}^{\omega}, \ldots, c_{\alpha n}^{\omega} t_n c_{\beta n}^{\omega}). \tag{3.1}$$

Since 0 is the maximal element of $\mathrm{Syn}(\mathcal{Y})$, we only consider the case where the right side is different from 0. Since the letters of π are letters of ρ, we can assume that every edge of Γ is a letter of ρ. Then for all $i \in \{1, \ldots, n\}$ we have $c_{\alpha i}^{\omega} t_i c_{\beta i}^{\omega} \neq 0$. For every $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$, there are $\tau_i, \gamma_j \in \widehat{B^+}$ such that $\hat{\delta}_{\mathcal{Y}}(\tau_i) = t_i$ and $\hat{\delta}_{\mathcal{Y}}(\gamma_j) = c_j$. Then $\hat{\delta}_{\mathcal{Y}}(\gamma_{\alpha i}^{\omega} \tau_i \gamma_{\beta i}^{\omega}) = c_{\alpha i}^{\omega} t_i c_{\beta i}^{\omega} \neq 0$, thus $\gamma_{\alpha i}^{\omega} \tau_i \gamma_{\beta i}^{\omega} \in \overline{L(\mathcal{Y})}$. Consider the pseudowords

$$e_j = \bar{h}(\mathrm{t}_k(\gamma_j^{\omega})\, \gamma_j^{\omega} \mathrm{i}_k(\gamma_j^{\omega})), \qquad w_i = \bar{h}(\mathrm{t}_k(\gamma_{\alpha i}^{\omega}) \cdot \gamma_{\alpha i}^{\omega} \tau_i \gamma_{\beta i}^{\omega} \cdot \mathrm{i}_k(\gamma_{\beta i}^{\omega})).$$

Observe that $w_i = e_{\alpha i} w_i e_{\beta i}$. The pseudoword $\mathrm{t}_k(\gamma_{\alpha i}^{\omega}) \cdot \gamma_{\alpha i}^{\omega} \tau_i \gamma_{\beta i}^{\omega} \cdot \mathrm{i}_k(\gamma_{\beta i}^{\omega})$ is a factor of $\gamma_{\alpha i}^{\omega} \cdot \gamma_{\alpha i}^{\omega} \tau_i \gamma_{\beta i}^{\omega} \cdot \gamma_{\beta i}^{\omega} = \gamma_{\alpha i}^{\omega} \tau_i \gamma_{\beta i}^{\omega}$, thus it belongs to $\overline{L(\mathcal{Y})}$ by Lemma 2.4. Hence $w_i \in \overline{L(\mathcal{X})}$ by Lemma 2.6, and $\bar{g}(w_i) = \gamma_{\alpha i}^{\omega} \tau_i \gamma_{\beta i}^{\omega}$ by Lemma 2.7. Then, since $\hat{\delta}_{\mathcal{Y}} \bar{g}$ is a continuous homomorphism,

$$\begin{aligned}\text{for } \theta \in \{\pi, \rho\}, \quad \theta(c_{\alpha 1}^{\omega} t_1 c_{\beta 1}^{\omega}, \ldots, c_{\alpha n}^{\omega} t_n c_{\beta n}^{\omega}) &= \theta(\hat{\delta}_{\mathcal{Y}} \bar{g}(w_1), \ldots, \hat{\delta}_{\mathcal{Y}} \bar{g}(w_n)) \\ &= \hat{\delta}_{\mathcal{Y}} \bar{g}(\theta(w_1, \ldots, w_n)).\end{aligned} \tag{3.2}$$

Because $w_i = e_{\alpha i} w_i e_{\beta i}$ and e_j is idempotent, for $\theta \in \{\pi, \rho\}$ we have

$$\hat{\delta}_{\mathcal{X}}(\theta(w_1, \ldots, w_n)) = \theta(\hat{\delta}_{\mathcal{X}}(e_{\alpha 1})^{\omega} \hat{\delta}_{\mathcal{X}}(w_1) \hat{\delta}_{\mathcal{X}}(e_{\beta 1})^{\omega}, \ldots, \hat{\delta}_{\mathcal{X}}(e_{\alpha n})^{\omega} \hat{\delta}_{\mathcal{X}}(w_n) \hat{\delta}_{\mathcal{X}}(e_{\beta n})^{\omega}).$$

Therefore, since $\mathrm{Syn}(\mathcal{X})$ satisfies $\zeta_\Gamma(\pi) \le \zeta_\Gamma(\rho)$, we have

$$\hat{\delta}_{\mathcal{X}}(\pi(w_1, \ldots, w_n)) \le \hat{\delta}_{\mathcal{X}}(\rho(w_1, \ldots, w_n)). \tag{3.3}$$

Let u be a finite factor of $\rho(w_1, \ldots, w_n)$. By Lemma 2.3 there is i such that u is a factor of w_i, or there are i, j such that $x_i x_j$ is a factor of ρ and u is a factor of $w_i w_j$. In the first case we have $u \in L(\mathcal{X})$ because $w_i \in \overline{L(\mathcal{X})}$. Consider the second case. Since $w_i w_j = w_i e_{\beta i} w_j$, we conclude that u is a factor of $w_i e_{\beta i} = w_i$ or a factor of $e_{\beta i} w_j$, by Lemma 2.3. Since $x_i x_j$ is a factor of ρ, we have $\beta i = \alpha j$, thus $e_{\beta i} w_j = w_j$. Hence u is a factor of w_i or of w_j, which are both elements of $\overline{L(\mathcal{X})}$, thus $u \in L(\mathcal{X})$. Hence $\rho(w_1, \ldots, w_n) \in \mathrm{Mir}(\mathcal{X})$. Since $\rho(c_{\alpha 1}^{\omega} t_1 c_{\beta 1}^{\omega}, \ldots, c_{\alpha n}^{\omega} t_n c_{\beta n}^{\omega}) \neq 0$, by (3.2) we have $\bar{g}(\rho(w_1, \ldots, w_n)) \in \overline{L(\mathcal{Y})}$. For $\theta \in \{\pi, \rho\}$ the pseudowords $w_{i_0} = e_{i_0} w_{i_0}$ and $w_{j_0} = w_{j_0} f_{j_0}$ are respectively a prefix and a suffix of $\theta(w_1, \ldots, w_n)$, thus $\theta(w_1, \ldots, w_n)$ is bounded by the idempotents e_{i_0} and f_{j_0}. Then by Lemma 2.9 the pseudoword $\rho(w_1, \ldots, w_n)$ belongs to $\overline{L(\mathcal{X})}$. Hence we also have $\pi(w_1, \ldots, w_n) \in \overline{L(\mathcal{X})}$ by (3.3). From (3.3) and Theorem 2.10 we conclude that $\hat{\delta}_{\mathcal{Y}} \bar{g}(\pi(w_1, \ldots, w_n)) \le \hat{\delta}_{\mathcal{Y}} \bar{g}(\rho(w_1, \ldots, w_n))$. By (3.2) this is the same as (3.1). $\square$

A *semigroupoid* is a graph endowed with an associative rule of composition between consecutive edges. A morphism of semigroupoids is a morphism of graphs that respects the rule of composition. Sets and semigroups can be viewed as one-vertex graphs and semigroupoids, respectively. Just like a finite set A defines a unique free profinite A-generated semigroup, a finite graph Γ defines a unique free profinite Γ-generated semigroupoid, denoted by $\widehat{\Gamma^+}$ [6,20]. The two concepts coincide when Γ is a set. Then there is a unique continuous semigroupoid morphism $\varepsilon_\Gamma : \widehat{\Gamma^+} \to \widehat{E(\Gamma)^+}$ whose restriction to $E(\Gamma)$ is the identity. The image of the edges of $\widehat{\Gamma^+}$ under ε_Γ is the set of Γ-profinite-paths.

We refer the reader to [31] for a straightforward introduction to the notions of ordered semigroupoid and pseudovariety of ordered semigroupoids. Since an intersection of pseudovarieties of ordered semigroupoids is also a pseudovariety of ordered semigroupoids, if V is a pseudovariety of ordered semigroups then we can consider the smallest pseudovariety of ordered semigroupoids containing V, called the *global of* V and denoted by gV. Given a finite graph Γ, let π and ρ be coterminal edges of $\widehat{\Gamma^+}$; the formal triple $(\pi \le \rho; \Gamma)$ is called a pseudoidentity over Γ; we say that a semigroupoid S satisfies $(\pi \le \rho; \Gamma)$ if $\varphi(\pi) \le \varphi(\rho)$ for all continuous morphisms of semigroupoids $\varphi : \widehat{\Gamma^+} \to S$. In the same way as with semigroups, every pseudovariety of ordered semigroupoids is defined by a set of pseudoidentities over finite graphs. This is explicitly proved in [6,20] for the unordered case, and in [32,27] for pseudovarieties of ordered semigroups; the proof for the general case is a routine based on those cases.

For an ordered semigroup S, let S_E be the ordered semigroupoid defined as follows: the vertices are the idempotents of S, the edges from e to f are the triples (e, s, f) such that $s = esf$, the composition of edges is given by $(e, s, f)(f, t, g) = (e, st, g)$, and $(e, s, f) \leq (e, t, f)$ if and only if $s \leq t$.

In [34,31] the reader can find information about the *semidirect product* of two pseudovarieties of ordered semigroups. For this paper it is only necessary to know that such semidirect product is itself a pseudovariety of ordered semigroups, together with some more facts that we shall provide. We are interested in semidirect products in which the second factor is one of the pseudovarieties $\mathsf{D}_k = [\![yx_1 \dots x_k = x_1 \dots x_k]\!]$ with $k \geq 1$, or $\mathsf{D} = \bigcup_{k\geq 1} \mathsf{D}_k = [\![yx^\omega = x^\omega]\!]$.

Theorem 3.3 (*Delay Theorem*). *Let V be a pseudovariety of ordered semigroups containing some non-trivial monoid. Let S be a finite semigroup. Then $S \in \mathsf{V} * \mathsf{D}$ if and only if $S_E \in \mathsf{gV}$.*

The Delay Theorem for pseudovarieties of ordered semigroupoids was proved in [31] in the monoidal context, with a proof that also holds for the version presented here.

Theorem 3.4. *If V is a pseudovariety of ordered semigroups containing Sl^- then $\mathscr{S}(\mathsf{V} * \mathsf{D})$ is a conjugacy invariant.*

Proof. Let Σ be a basis of pseudoidentities for gV. Let S be a finite semigroup. By the Delay Theorem, we have $S \in \mathsf{V} * \mathsf{D}$ if and only if $S_E \in \mathsf{gV}$. On the other hand, S_E satisfies $(\pi \leq \rho; \Gamma)$ if and only if S satisfies $\zeta_\Gamma(\varepsilon_\Gamma(\pi)) \leq \zeta_\Gamma(\varepsilon_\Gamma(\rho))$. Therefore,

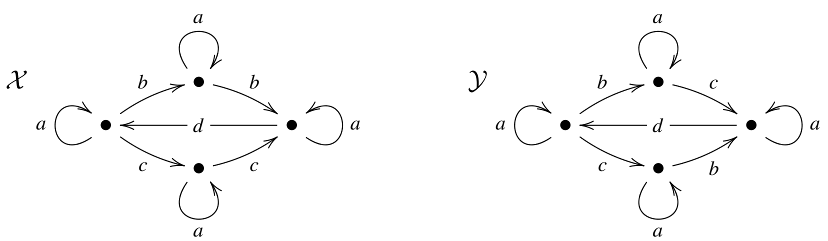
$$\mathsf{V} * \mathsf{D} = \bigcap_{(\pi \leq \rho; \Gamma) \in \Sigma} [\![\zeta_\Gamma(\varepsilon_\Gamma(\pi)) \leq \zeta_\Gamma(\varepsilon_\Gamma(\rho))]\!].$$

By Proposition 3.2 we only have to show that all letters of $\varepsilon_\Gamma(\pi)$ are letters of $\varepsilon_\Gamma(\rho)$. Suppose that there is a letter z that is a factor of $\varepsilon_\Gamma(\pi)$ but not of $\varepsilon_\Gamma(\rho)$. Since gV contains Sl^-, it contains the two-element monoid $M = \{0, 1\}$ such that 0 is a zero and $1 \leq 0$ (in fact Sl^- is generated by M). Hence M satisfies $(\pi \leq \rho; \Gamma)$. Since M is a one-vertex semigroupoid, that means that M satisfies $\varepsilon_\Gamma(\pi) \leq \varepsilon_\Gamma(\rho)$. Let φ be the unique continuous homomorphism from $\widehat{E(\Gamma)^+}$ to M such that $\varphi(z) = 0$ and $\varphi(x) = 1$ if x is a letter distinct from z. Then $0 = \varphi(\varepsilon_\Gamma(\pi)) \leq \varphi(\varepsilon_\Gamma(\rho)) = 1$, which is absurd. $\square$

Corollary 3.5. *If V is a pseudovariety of ordered semigroups or monoids containing Sl^- then $\mathscr{S}(\mathsf{LV})$ is a conjugacy invariant.*

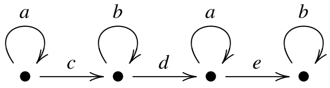
Proof. We have $\mathsf{LV} = \mathsf{LV} * \mathsf{D}$, for any pseudovariety V (in [1, Proposition 10.6.13] we find a proof for the unordered case easily adaptable for the ordered case). $\square$

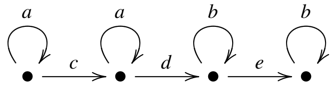
Example 3.6. Let $\mathcal{X}$ and $\mathcal{Y}$ be the irreducible sofic subshifts with the following presentations:



The pseudovariety $\mathsf{V} = [\![x^3 = x^2]\!]$ contains Sl, thus $\mathscr{S}(\mathsf{LV})$ is a conjugacy invariant. We have $\mathcal{X} \notin \mathscr{S}(\mathsf{LV})$, since $\delta_\mathcal{X}(aba)^3 = 0 \neq \delta_\mathcal{X}(aba)^2$ and $\delta_\mathcal{X}(a)\mathrm{Syn}(\mathcal{X})\,\delta_\mathcal{X}(a)$ is a submonoid of $\mathrm{Syn}(\mathcal{X})$. On the other hand, with some calculations one verifies that $\mathcal{Y} \in \mathscr{S}(\mathsf{LV})$. Hence $\mathcal{X}$ and $\mathcal{Y}$ are not conjugate. The subshifts $\mathcal{X}$ and $\mathcal{Y}$ have equal entropy, zeta function and Krieger edge shift. Moreover, the invariant for sofic subshifts obtained in [15, Theorem 4.12] is the same in $\mathcal{X}$ and $\mathcal{Y}$. This invariant is also related with the syntactic semigroup.

Example 3.7. The classes $\mathscr{S}(\mathsf{LSl})$, $\mathscr{S}(\mathsf{Com} * \mathsf{D})$ and $\mathscr{S}(\mathsf{LCom})$ are all distinct. Consider the following sofic subshifts:

$\mathcal{X}$: 

$\mathcal{Y}$: 

We can decide if a subshift belongs to $\mathsf{Com}*\mathsf{D}$, since Thérien and Weiss proved that $\mathsf{Com}*\mathsf{D} = [\![y^\omega x_1 z^\omega x_2 y^\omega x_3 z^\omega = y^\omega x_3 z^\omega x_2 y^\omega x_1 z^\omega]\!]$ [37]. Performing some computations, we conclude that $\mathcal{X} \in \mathscr{S}(\mathsf{LCom}) \setminus \mathscr{S}(\mathsf{Com} * \mathsf{D})$ and $\mathcal{Y} \in \mathscr{S}(\mathsf{Com} * \mathsf{D}) \setminus \mathscr{S}(\mathsf{LSl})$. In particular $\mathcal{X}$ and $\mathcal{Y}$ are not conjugate.

We proceed with the determination of all conjugacy invariants of the form $\mathscr{S}(\mathsf{V})$, with V a pseudovariety of ordered semigroups.

Proposition 3.8. *Let $\mathcal{V}$ be a variety of languages. If $\mathcal{V}$ contains all the languages of the form A^*wA^* with $w \in A^+$ and A an alphabet, then $\mathcal{V}$ also contains the languages of the form wA^*, A^*w or $\{w\}$.*

Proof. Let V be the pseudovariety of ordered semigroups corresponding to $\mathcal{V}$. If Σ is a basis of pseudoidentities for V, then $\mathsf{V} = \bigcap_{(\pi \le \rho) \in \Sigma} [\![\pi \le \rho]\!]$. It follows that it suffices to assume that $\mathsf{V} = [\![\pi \le \rho]\!]$ for some pseudowords π, ρ over an alphabet $X = \{x_1, \ldots, x_n\}$. Let b a letter which is not in X, and let $B = X \cup \{b\}$. Let $L = B^* b\, \mathrm{i}_k(\rho) B^*$. Then $L \in \mathcal{V}B^+$, and so

$$\hat{\delta}_L(\pi) = \pi(\hat{\delta}_L(x_1), \ldots, \hat{\delta}_L(x_n)) \le \rho(\hat{\delta}_L(x_1), \ldots, \hat{\delta}_L(x_n)) = \hat{\delta}_L(\rho).$$

Therefore $\hat{\delta}_L(b\pi) \le \hat{\delta}_L(b\rho)$. Since $b\rho \in \overline{L}$, it follows from Lemma 2.8 that $b\pi \in \overline{L}$. Then there are $z, t \in \widehat{B^*}$ such that $b\pi = zb\, \mathrm{i}_k(\rho)t$. Suppose that $z \ne 1$. Then there is $z' \in \widehat{B^*}$ such that $b\pi = bz'\, b\mathrm{i}_k(\rho)t$. In an equality between pseudowords, equal prefixes (and suffixes) can be canceled [1, Exercise 10.2.10]. Therefore $\pi = z'\, b\mathrm{i}_k(\rho)t$, which is impossible since b is not a factor of π. Hence $z = 1$ and $b\pi = b\mathrm{i}_k(\rho)t$, and so $\mathrm{i}_k(\pi) = \mathrm{i}_k(\rho)$. Similarly, $\mathrm{t}_k(\pi) = \mathrm{t}_k(\rho)$. Since k is arbitrary, it follows that $\pi = \rho$, or π and ρ are both infinite pseudowords.

For an alphabet A and an element w of A^+, let K be one of the sets $\{w\}$, wA^* or A^*w. Its closure $\overline{K}$ in $\widehat{A^+}$ equals, respectively, $\{w\}$, $w\widehat{A^*}$ or $\widehat{A^*}w$. Let $z_1, \ldots, z_n \in A^+$ and $x, y \in \widehat{A^*}$. Let $u = x\pi(z_1, \ldots, z_n)y$ and $v = x\rho(z_1, \ldots, z_n)y$. Then $u = v$ or u and v are both infinite pseudowords such that $\mathrm{i}_k(u) = \mathrm{i}_k(v)$ and $\mathrm{t}_k(u) = \mathrm{t}_k(v)$ for all $k \ge 1$. Therefore $u \in \overline{K}$ if and only if $v \in \overline{K}$. Hence $\pi(\hat{\delta}_K(z_1), \ldots, \hat{\delta}_K(z_n)) = \rho(\hat{\delta}_K(z_1), \ldots, \hat{\delta}_K(z_n))$ by Lemma 2.8. Since the words z_i are arbitrary, this means that the syntactic semigroup of K satisfies $\pi = \rho$, and so $K \in \mathcal{V}$. $\square$

The version of Proposition 3.8 for varieties corresponding to pseudovarieties of (unordered) semigroups was proved in [16], with arguments depending on the fact that such varieties are closed under complementation.

Proposition 3.9. *Let V be a pseudovariety of ordered semigroups. If $\mathscr{S}(\mathsf{V})$ is a conjugacy invariant then $\mathsf{V} \supseteq \mathsf{LSl}^-$. Moreover, if V is a pseudovariety of semigroups then $\mathsf{V} \supseteq \mathsf{LSl}$.*

Proof. Let $\mathcal{V}$ be the variety of languages corresponding to V. By Theorem 2.12 and the dual of Proposition 3.8, to prove $\mathsf{V} \supseteq \mathsf{LSl}^-$ it suffices to show that the languages of the form $A^+ \setminus A^*wA^*$ are in $\mathcal{V}A^+$.

For $n \ge 2$, denote by B_n^- the unique finite aperiodic ordered semigroup (up to isomorphism) with a zero and with a unique non-null $\mathcal{J}$-class having n idempotents and just one idempotent in each $\mathcal{R}$-class and in each $\mathcal{L}$-class, and where the order relation is given by $s \le t$ if and only if $s = t$ or $t = 0$. Let B_1^- be the trivial semigroup. Let C be a two-letter alphabet. The syntactic semigroup of $C^{\mathbb{Z}}$ is trivial, thus $C^{\mathbb{Z}}$ belongs to $\mathscr{S}(\mathsf{V})$. Therefore the conjugate subshift $\Phi_1^{[0,1]}(C^{\mathbb{Z}})$ also belongs to $\mathscr{S}(\mathsf{V})$. The syntactic semigroup of $\Phi_1^{[0,1]}(C^{\mathbb{Z}})$ is isomorphic to B_2^-, thus $B_2^- \in \mathsf{V}$. As is stated in [31], it is not difficult to verify that B_n^- is an ordered subsemigroup of an image under an order-preserving homomorphism of a direct product of copies of B_2^-. Hence $B_n^- \in \mathsf{V}$. It is easy to see that the syntactic semigroup of an irreducible edge subshift whose corresponding presentation has n vertices is isomorphic to B_n^- (see the argument in the proof of Theorem 12 of [9]). Hence $\mathscr{S}(\mathsf{V})$ contains all irreducible finite type subshifts, since they are conjugate with irreducible edge subshifts.

Consider an alphabet A and an element w of A^+. Let b be a letter not in A, and consider the alphabet $B = A \cup \{b\}$. Denote by φ the inclusion homomorphism $A^+ \to B^+$. The language $L = B^+ \setminus B^*wB^*$ is clearly factorial, and it is prolongable because if $u \in L$ then $bub \in L$. Moreover, if u and v are elements of L then $ubv \in L$. Thus L defines an irreducible finite type subshift. Hence $L \in \mathcal{V}$. Since $A^+ \setminus A^*wA^* = \varphi^{-1}(L)$, we have $A^+ \setminus A^*wA^* \in \mathcal{V}$.

The varieties of languages corresponding to pseudovarieties of semigroups are closed under complementation. Then it follows from Theorems 2.11 and 2.12 that every pseudovariety of semigroups containing LSl^- must contain LSl. $\square$

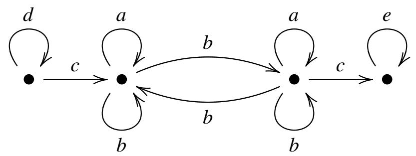
The languages of finite type subshifts are negatively locally testable. Therefore, from Proposition 3.9 we deduce that it is not possible to use an invariant of the form $\mathscr{S}(\mathsf{V})$ to detect non-conjugate subshifts of finite type, where V is a pseudovariety of ordered semigroups.

Before we go to the next proposition, we note that $\mathsf{LSl}^- = \mathsf{LSl} \cap \mathsf{L}[\![1 \leq x]\!]$, thus $\mathscr{S}(\mathsf{LSl}^-) = \mathscr{S}(\mathsf{LSl}) \cap \mathscr{S}(\mathsf{L}[\![1 \leq x]\!])$.

Proposition 3.10. *The classes $\mathscr{S}(\mathsf{LSl}^-)$, $\mathscr{S}(\mathsf{LSl})$ and $\mathscr{S}(\mathsf{L}[\![1 \leq x]\!])$ are distinct.*

Proof. It is proved in [33] that the syntactic semigroup of a language L of A^+ belongs to $\mathsf{L}[\![1 \leq x]\!]$ if and only if L is a finite intersection of languages of the form $A^+ \setminus u_0A^*u_1A^* \cdots u_{k-1}A^*u_k$, with $k \geq 0$ and $u_i \in A^*$. Therefore, if A is the two-letter alphabet $\{a, b\}$, the subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$ defined by the factorial prolongable language $A^+ \setminus A^*abA^*a^2bA^*$ belongs to $\mathscr{S}(\mathsf{L}[\![1 \leq x]\!])$. We have $\delta_{\mathcal{X}}(b) = \delta_{\mathcal{X}}(b)^2$. Since $ba^2ba^2b \notin L(\mathcal{X})$ and $ba^2b \in L(\mathcal{X})$, we have $\delta_{\mathcal{X}}(b)^{\omega}\delta_{\mathcal{X}}(a^2)\delta_{\mathcal{X}}(b)^{\omega}\delta_{\mathcal{X}}(a^2)\delta_{\mathcal{X}}(b)^{\omega} \neq \delta_{\mathcal{X}}(b)^{\omega}\delta_{\mathcal{X}}(a^2)\delta_{\mathcal{X}}(b)^{\omega}$, thus $\mathcal{X} \notin \mathscr{S}(\mathsf{LSl})$.

On the other hand, let $\mathcal{Y}$ be the subshift with the following presentation:



We have $cabac \in L(\mathcal{Y})$ and $cac \notin L(\mathcal{Y})$, thus $\delta_{\mathcal{Y}}(a) \not\leq \delta_{\mathcal{Y}}(aba)$. Since $\delta_{\mathcal{Y}}(a) = \delta_{\mathcal{Y}}(a)^2$, we deduce that $\mathcal{Y} \notin \mathscr{S}(\mathsf{L}[\![1 \leq x]\!])$. One can verify that $\mathcal{Y} \in \mathscr{S}(\mathsf{LSl})$. $\square$

A consequence of Propositions 3.9 and 3.10 is that there is not a pseudovariety of semigroups V such that $\mathscr{S}(\mathsf{LSl}^-) = \mathscr{S}(\mathsf{V})$. More generally, we do not know if there are distinct pseudovarieties of ordered semigroups V and W such that $\mathscr{S}(\mathsf{V})$ is a conjugacy invariant and $\mathscr{S}(\mathsf{V}) = \mathscr{S}(\mathsf{W})$. On the other for all $k, l \geq 1$, if $k \neq l$ then $\mathsf{D}_k \neq \mathsf{D}_l$, and one can prove that $\mathscr{S}(\mathsf{D}_k)$ is the class of the full shifts, which is not closed under taking conjugate subshifts.

Lemma 3.11. *Let V be a pseudovariety of ordered semigroups containing LSl^- and let k be a positive integer. If L belongs to the variety of languages defined by $\mathsf{V} * \mathsf{D}_k$ then $\Phi_k(L) \setminus \{1\}$ belongs to the variety of languages defined by V.*

Proof. The variety of languages corresponding to $\mathsf{W} * \mathsf{D}_k$ is described in [35, Theorem 4.22] when W is a pseudovariety of ordered monoids, but the corresponding statement and proof also holds when W is a pseudovariety of ordered semigroups, with obvious modifications. Let $A^{\leq k}$ be the set of words over A with length less or equal than k. Let $\mathcal{V}$ be the variety of languages defined by V. By the referred version of [35, Theorem 4.22], the language $L \setminus A^{\leq k}$ is the union of a finite family $(R_i)_{i \in I}$ of sets of the form $R_i = p_iA^* \cap A^*s_i \cap \Phi_k^{-1}(K_i)$, with $p_i, s_i \in A^{k+1}$ and $K_i \in \mathcal{V}(A^{k+1})^+$. One can easily verify that

$$\Phi_k(L) \setminus \{1\} = \bigcup_{i \in I} \Big[(\Phi_k(A^+) \setminus \{1\}) \cap p_i(A^{k+1})^* \cap (A^{k+1})^*s_i \cap K_i \Big].$$

The languages $\Phi_k(A^+) \setminus \{1\}$, $p_i(A^{k+1})^*$ and $(A^{k+1})^*s_i$ are negatively locally testable, hence they are in $\mathcal{V}(A^{k+1})^+$. Therefore $\Phi_k(L) \setminus \{1\} \in \mathcal{V}(A^{k+1})^+$, since $K_i \in \mathcal{V}(A^{k+1})^+$ and $\mathcal{V}(A^{k+1})^+$ is closed under finite intersections and unions. $\square$

Theorem 3.12. *Let V be a pseudovariety of ordered semigroups. Then $\mathscr{S}(\mathsf{V})$ is a conjugacy invariant if and only if V contains LSl^- and $\mathscr{S}(\mathsf{V}) = \mathscr{S}(\mathsf{V} * \mathsf{D})$.*

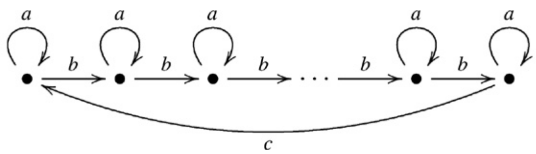
Proof. Suppose that $\mathscr{S}(\mathsf{V})$ is a conjugacy invariant. Then V contains LSl^- by Proposition 3.9. Let $\mathcal{X}$ be a subshift of $A^{\mathbb{Z}}$ belonging to $\mathscr{S}(\mathsf{V} * \mathsf{D})$. Since $\mathsf{V} * \mathsf{D} = \bigcup_{k \geq 1} \mathsf{V} * \mathsf{D}_k$, there is $k \geq 1$ such that $\mathcal{X} \in \mathscr{S}(\mathsf{V} * \mathsf{D}_k)$. The set $\Phi_k(L(\mathcal{X})) \setminus \{1\}$ is the language of a subshift $\mathcal{Y}$ of $(A^{k+1})^{\mathbb{Z}}$ which is conjugate with $\mathcal{X}$. By Lemma 3.11 we have $\mathcal{Y} \in \mathscr{S}(\mathsf{V})$, thus $\mathcal{X} \in \mathscr{S}(\mathsf{V})$. Hence $\mathscr{S}(\mathsf{V} * \mathsf{D}) \subseteq \mathscr{S}(\mathsf{V})$. The reverse inclusion follows from the fact that $\mathsf{V} \subseteq \mathsf{V} * \mathsf{W}$ for every pseudovariety W. The converse is an immediate consequence of Theorem 3.4. $\square$

4. Syntactic characterizations of some invariant classes of irreducible sofic subshifts

For a pseudovariety V of ordered semigroups, let $\mathscr{S}_I(\mathsf{V})$ be the class of irreducible subshifts in $\mathscr{S}(\mathsf{V})$. Theorem 3.12 also holds for the operator $\mathscr{S}_I$. If $\mathsf{Sl}^- \subseteq \mathsf{V}$ then $\mathscr{S}_I(\mathsf{LV})$ is a conjugacy invariant by Corollary 3.5. There is an infinity of such invariant classes:

Example 4.1. Consider the sequence $(\mathcal{X}_n)_{n \geq 1}$ of irreducible sofic subshifts with the following presentations:

$\mathcal{X}_n$:



(n b's)

Then $\mathcal{X}_n \in \mathscr{S}_I(\mathsf{L}[\![x^{n+2} = x^{n+1}]\!]) \setminus \mathscr{S}_I(\mathsf{L}[\![x^{n+1} = x^{n}]\!])$, thus

$$\mathscr{S}_I(\mathsf{L}[\![x^2 = x]\!]) \subsetneq \mathscr{S}_I(\mathsf{L}[\![x^3 = x^2]\!]) \subsetneq \mathscr{S}_I(\mathsf{L}[\![x^4 = x^3]\!]) \subsetneq \cdots.$$

There are some relevant classes of irreducible sofic subshifts of the form $\mathscr{S}_I(\mathsf{V})$. We proceed with the description of some of them.

Proposition 4.2. *Let $\mathcal{X}$ be a subshift of $A^{\mathbb{Z}}$. Then $\mathcal{X}$ is an irreducible subshift of finite type if and only if $\mathcal{X} \in \mathscr{S}_I(\mathsf{LCom})$.*

Proof. Every subshift of finite type is in $\mathscr{S}(\mathsf{LSl}^-)$, therefore it is also in $\mathscr{S}(\mathsf{LCom})$. Conversely, suppose that $\mathcal{X} \in \mathscr{S}_I(\mathsf{LCom})$. Consider elements u, v, w of A^+ such that $uv, vw \in L(\mathcal{X})$ and v has length greater than the cardinality of $\mathrm{Syn}(\mathcal{X})$. We can see with a simple combinatorial argument [1, Proposition 3.7.1] that there are $v_1, e, v_2 \in A^+$ such that $v = v_1 e v_2$ and $\delta_{\mathcal{X}}(e)$ is an idempotent. Since $e, uv_1e, ev_2w \in L(\mathcal{X})$ and $\mathcal{X}$ is irreducible, there are $x, y \in A^+$ such that $ev_2w \cdot x \cdot e \cdot y \cdot uv_1e \in L(\mathcal{X})$. This means that $\delta_{\mathcal{X}}(ev_2wxeyuv_1e) \neq 0$. Since the submonoid $\delta_{\mathcal{X}}(e)\mathrm{Syn}(\mathcal{X})\,\delta_{\mathcal{X}}(e)$ of $\mathrm{Syn}(\mathcal{X})$ is commutative, we have

$$\delta_{\mathcal{X}}(eyuvwxe) = \delta_{\mathcal{X}}(eyuv_1e)\delta_{\mathcal{X}}(ev_2wxe) = \delta_{\mathcal{X}}(ev_2wxe)\delta_{\mathcal{X}}(eyuv_1e) \neq 0.$$

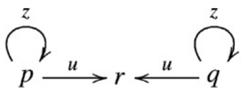
Hence $eyuvwxe \in L(\mathcal{X})$ and so $uvw \in L(\mathcal{X})$. From Proposition 2.2 we conclude that $\mathcal{X}$ is a subshift of finite type. $\square$

Let A be the class of aperiodic semigroups. We have $\mathsf{Sl} \subseteq \mathsf{A}$ and $\mathsf{A} = \mathsf{LA}$. A sliding block code $G : \mathcal{X} \to \mathcal{Y}$ is *aperiodic* if, for all $x \in \mathcal{X}$ such that $\{n \in \mathbb{Z}^+ : \sigma^n(x) = x\} \neq \emptyset$, the integer $\min\{n \in \mathbb{Z}^+ : \sigma^n(x) = x\}$ is equal to $\min\{n \in \mathbb{Z}^+ : \sigma^n(G(x)) = G(x)\}$. The class $\mathscr{S}_I(\mathsf{A})$ was characterized in [9] as being the class of irreducible sofic subshifts that are the image of a subshift of finite type under an aperiodic sliding block code. It was also proved in [9] that $\mathscr{S}_I(\mathsf{A})$ is a conjugacy invariant, using a weak version of the invariant obtained in [15, Theorem 4.12].

Let Inv be the pseudovariety generated by semigroups of partial one-to-one transformations. Ash [7] proved that $\mathsf{Inv} = [\![x^{\omega}y^{\omega} = y^{\omega}x^{\omega}]\!]$. An *almost finite type subshift* is an irreducible sofic subshift whose Fischer cover does not admit a labeled subgraph as in Fig. 1 [8]. It was proved in [10] that the almost finite type subshifts are in $\mathscr{S}_I(\mathsf{LInv})$. We next prove the converse. Note that $\mathscr{S}_I(\mathsf{LInv})$ is a conjugacy invariant since $\mathsf{Sl} \subseteq \mathsf{Inv}$.

Theorem 4.3. *The class $\mathscr{S}_I(\mathsf{LInv})$ is the class of almost finite type subshifts.*

Proof. We prove the missing part. Suppose that $\mathcal{X} \in \mathscr{S}_I(\mathsf{LInv})$ and that $\mathcal{X}$ is not of almost finite type. Let $\mathfrak{F}$ be the Fischer cover of $\mathcal{X}$. Then there is in $\mathfrak{F}$ a pattern as in Fig. 1. Since $\mathfrak{F}$ is strongly connected, it has paths $r \to p$ and $r \to q$ labeled v and w. Then $p \cdot (z^{\omega}uvz^{\omega})^{\omega}(z^{\omega}uwz^{\omega})^{\omega} = q$ and $p \cdot (z^{\omega}uwz^{\omega})^{\omega}(z^{\omega}uvz^{\omega})^{\omega} = p$. The monoid



p, q, r states, $p \neq q$, $u, z \in A^+$.

Fig. 1. Forbidden pattern in the Fischer cover of almost finite type subshifts.

$\hat{\delta}_{\mathcal{X}}(z)^{\omega} \cdot \mathrm{Syn}(\mathcal{X}) \cdot \hat{\delta}_{\mathcal{X}}(z)^{\omega}$ is in $\mathsf{Inv} = [\![x^{\omega}y^{\omega} = y^{\omega}x^{\omega}]\!]$, thus $(z^{\omega}uvz^{\omega})^{\omega}(z^{\omega}uwz^{\omega})^{\omega}$ and $(z^{\omega}uwz^{\omega})^{\omega}(z^{\omega}uvz^{\omega})^{\omega}$ have the same action on the states of $\mathfrak{F}$. Hence $p = q$. This is a contradiction. $\square$

All examples of irreducible sofic subshifts that we have so far presented are of almost finite type.

5. Shift equivalence

Let $\mathcal{X}$ be a subshift of $A^{\mathbb{Z}}$ and let $l \geq 1$ be an integer. Consider the alphabet A^l of the elements in A^+ with length l. We can naturally embed $(A^l)^+$ in A^+. The set $L(\mathcal{X}) \cap (A^l)^+$ is a non-empty factorial prolongable language of $(A^l)^+$, thus it defines a subshift $\mathcal{X}^l$ of $(A^l)^{\mathbb{Z}}$. Recall that $\delta_{\mathcal{X}}(u)$ is the equivalence class of u in A^+ for the syntactic congruence of $L(\mathcal{X})$. It is easy to see that if $u \in (A^l)^+$ then $\delta_{\mathcal{X}^l}(u) = \delta_{\mathcal{X}}(u) \cap (A^l)^+$, and so the map that sends $\delta_{\mathcal{X}^l}(u)$ into $\delta_{\mathcal{X}}(u)$ is a well-defined one-to-one homomorphism from $\mathrm{Syn}(\mathcal{X}^l)$ into $\mathrm{Syn}(\mathcal{X})$. Hence we can consider $\mathrm{Syn}(\mathcal{X}^l)$ as a subsemigroup of $\mathrm{Syn}(\mathcal{X})$. The following lemma was proved in [15]. It isolates and generalizes an argument in the proof of the last theorem of [9].

Lemma 5.1. *Let $\mathcal{X}$ be a sofic subshift of $A^{\mathbb{Z}}$. For every $l \geq 1$ there is $l' > l$ such that the set of idempotent-bound elements of* $\mathrm{Syn}(\mathcal{X})$ *is contained in* $\mathrm{Syn}(\mathcal{X}^{l'})$.

Two subshifts $\mathcal{X}$ and $\mathcal{Y}$ are *shift equivalent* if there is $l \geq 1$ such that $\mathcal{X}^l$ and $\mathcal{Y}^l$ are conjugate. If $\mathcal{X}^l$ and $\mathcal{Y}^l$ are conjugate then for all $k \geq l$ the subshifts $\mathcal{X}^k$ and $\mathcal{Y}^k$ are also conjugate. Conjugate subshifts are shift equivalent, but the validity of the converse in the finite type case was a major open problem for a long time, until Kim and Roush found examples showing that it was false [22,23]. There is an algorithm for deciding if two sofic subshifts are shift equivalent or not, but it is very complicated [21].

Theorem 5.2. *Let* V *be a pseudovariety of ordered semigroups. If* $\mathscr{S}(\mathsf{V})$ *is a conjugacy invariant then it is also a shift equivalence invariant.*

Proof. By Theorem 3.12, we have $\mathscr{S}(\mathsf{V}) = \mathscr{S}(\mathsf{V} * \mathsf{D})$, and V contains some non-trivial monoid. By the Delay Theorem we have

$$\mathscr{S}(\mathsf{V} * \mathsf{D}) = \{\mathcal{Z} : \mathcal{Z} \text{ is a sofic subsift and } \mathrm{Syn}(\mathcal{Z})_E \in \mathsf{gV}\}.$$

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are shift equivalent sofic subshifts. Let l be an integer such that $\mathcal{X}^l$ and $\mathcal{Y}^l$ are conjugate. Let l' be as in Lemma 5.1. Since $l' > l$, the subshifts $\mathcal{X}^{l'}$ and $\mathcal{Y}^{l'}$ are conjugate. Therefore

$$\mathrm{Syn}(\mathcal{X}^{l'})_E \in \mathsf{gV} \Leftrightarrow \mathrm{Syn}(\mathcal{Y}^{l'})_E \in \mathsf{gV}.$$

But $\mathrm{Syn}(\mathcal{X})_E = \mathrm{Syn}(\mathcal{X}^{l'})_E$ and $\mathrm{Syn}(\mathcal{Y})_E = \mathrm{Syn}(\mathcal{Y}^{l'})_E$. $\square$

As a corollary of Theorems 4.3 and 5.2, we recover the following theorem:

Theorem 5.3. *The class of almost finite type subshifts is closed under shift equivalence.*

See [10] for a different proof of Theorem 5.3 using shift equivalence invariant properties of syntactic semigroups. In [8, Proposition 4.1] there is a direct proof of the closure of the class of almost finite type subshifts under conjugacy, from which one can deduce Theorem 5.3 after some additional reasoning.

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