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Supraconvergent cell-centered scheme for two dimensional elliptic problems

S. Barbeiro

CMUC, Department of Mathematics, University of Coimbra, Apartado 3008, 3001-454 Coimbra, Portugal

Abstract

In this paper we study the convergence properties of a cell-centered finite difference scheme for second order elliptic equations with variable coefficients subject to Dirichlet boundary conditions. We prove that the finite difference scheme on nonuniform meshes although not even being consistent are nevertheless second order convergent. More precisely, second order convergence with respect to a discrete version of $L^2(\Omega)$-norm is shown provided that the exact solution is in $H^4(\Omega)$. Estimates for the difference between the pointwise restriction of the exact solution on the discretization nodes and the finite difference solution are proved. The convergence is studied with the aid of an appropriate negative norm. A numerical example support the convergence result.

Keywords: Cell-centered scheme; Nonuniform mesh; Supraconvergence

1. Introduction

In the last decades there has been a strong mathematical interest in numerical discretization methods that have higher convergence order than expected by analyzing the truncation error in a standard way. In the context of finite difference schemes on nonequidistant grids this behavior is called supraconvergence. Different methods of proving supraconvergence of finite difference schemes for ordinary differential equations have been used by the various authors (see e.g. [3,6,12,15,18,19,24] and [27]). The phenomenon of supraconvergence in more than one space dimension has also been studied in the literature (see e.g. [7,10,11] and [21]). The topic in the context of finite element methods has been treated in the papers [3,4,10,13,16,17,20,22,23,25,33].

We are interested in studying this phenomenon in a variant of finite differences, the so called cell-centered schemes, which are used in many codes. In fact, these schemes are not even consistent but nevertheless second order convergent. This fact is noticed by Tikhonov and Samarskii in [31]. Russell and Wheeler [26] use the equivalence of a cell-centered finite difference method and a mixed finite element method with a special quadrature formula for proving first order convergence of the solution and its gradient. Manteuffel and White [24] show second order convergence in both vertex-centered finite difference schemes and cell-centered finite difference schemes for scalar problems, on nonuniform meshes. Supraconvergence results for a two-dimensional cell-centered scheme are presented by Forsyth and Sammon [11] and also by Weiser and Wheeler [32], among others.

E-mail address: silvia@mat.uc.pt.
URL: http://www.mat.uc.pt/~silvia.

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Our main purpose is to analyze how supraconvergence come out for more general problems than considered so far. The analysis of the present paper is based on using negative norms. The analysis of supraconvergence with one additional order of convergence in [3] and [7] is more or less explicitly based on the concept of negative norms. In these two papers discrete analogues of the H^{-1}-norm were considered. The concept of negative norms in the analysis of supraconvergence was also used in [4,7–10,15] and [17]. The idea in the present paper is to work instead with a discrete version of the H^{-2}-norm. The convergence result relies on a stability inequality with respect to this norm. We consider the error as the difference between the pointwise restriction of the solution on the discretization nodes and the finite difference solution. Error estimates of order 2, in a discrete version of the $L^2(\Omega)$-norm, are proved if the exact solution is in the Sobolev space $H^4(\Omega)$.

The analysis of supraconvergence with two additional orders of convergence for the one-dimensional case is considered in [2], with the aid of so-called Spijker norms [28] which are defined using certain summation operators. These operators are applied twice corresponding to the two gained additional orders of convergence. The use of Spijker norms is restricted to one dimension but they give the idea for a generalization to higher dimensions because they are related to the negative norms (for more details see [2]). In the present paper we use this kind of analysis for two dimensional problems.

We consider the discretization of the following elliptic differential equation

$$Au := -(au_x)_x - (cu_y)_y + du_x + eu_y + fu = g \quad \text{on } \Omega, \tag{1}$$

subject to the Dirichlet boundary condition

$$u = \psi \quad \text{in } \partial\Omega. \tag{2}$$

The coefficients of A are assumed to satisfy $a, c \in W^{3,\infty}(\Omega)$, $d, e, f \in W^{2,\infty}(\Omega)$ and $a(x, y) \geqslant \underline{a} > 0$, $c(x, y) \geqslant \underline{c} > 0$, $\forall (x, y) \in \Omega$. We also assume $\psi \in H^{1/2}(\partial\Omega)$. The domain Ω is a union of rectangles.

In order to prepare the definition of the cell-centered finite difference approximation of (1)–(2) let us first introduce the nonuniform grid G_H. Let x_{-1}, x_{N+1}, y_{-1} and y_{M+1} be the vertices of a rectangle $R = (x_{-1}, x_{N+1}) \times (y_{-1}, y_{M+1})$ which contains Ω. We define the grid $G_H := R_1 \times R_2$, where

$$R_1 := \{x_{-1} < x_0 < \cdots < x_N < x_{N+1}\}, \qquad R_2 := \{y_{-1} < y_0 < \cdots < y_M < y_{M+1}\}.$$

The grid G_H is assumed to satisfy the following condition: the vertices of Ω are centers of the rectangles formed by G_H. Let

$$S_H := \big\{(x_{j-1/2}, y_{\ell-1/2}):\ j = 0, \ldots, N+1, \ell = 0, \ldots, M+1\big\},$$

where $x_{j-1/2} := (x_{j-1} + x_j)/2$, $y_{\ell-1/2} := (y_{\ell-1} + y_\ell)/2$. Our aim is to obtain numerical solutions in $\Omega_H := S_H \cap \Omega$. We define also $\partial\Omega_H := S_H \cap \partial\Omega$ and $\bar{\Omega}_H := \Omega_H \cup \partial\Omega_H$.

Fig. 1 illustrates the cell-centered grid in the domain.

In the case of a rectangular domain $\Omega = (x_0, x_N) \times (y_0, y_M)$, we allow both $R = \Omega$ and $R \supset \Omega$, i.e. we consider $x_{-1} \leqslant x_0$, $x_N \leqslant x_{N+1}$, $y_{-1} \leqslant y_0$ and $y_M \leqslant y_{M+1}$.

For the formulation of the difference problem we use the centered divided difference in x-direction

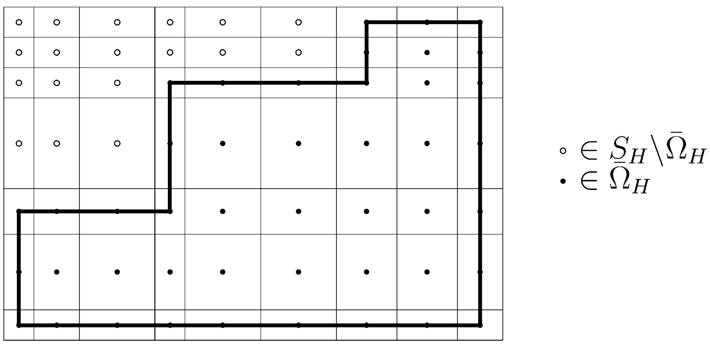

Fig. 1. Domain and grid points.

$$(\delta_x v_H)_{j,\ell+1/2} := \frac{v_{j+1/2,\ell+1/2} - v_{j-1/2,\ell+1/2}}{h_{j-1/2}},$$
$$(\delta_x w_H)_{j-1/2,\ell+1/2} := \frac{w_{j,\ell+1/2} - w_{j-1,\ell+1/2}}{h_{j-1}},$$

where $h_{j-1/2} := x_{j+1/2} - x_{j-1/2}$, $h_{j-1} := x_j - x_{j-1}$. Correspondingly, the divided difference with respect to the y variable are defined, with the mesh-size vector k in place of h. The difference problem is to find the solution u_H such that

$$A_H u_H = M_H R_{G_H} g \quad \text{in } \Omega_H, \tag{3}$$
$$u_H = R_H \psi \quad \text{on } \partial\Omega_H, \tag{4}$$

where the difference operator A_H is given by

$$A_H u_H := -\delta_x(a\delta_x u_H) - \delta_y(c\delta_y u_H) + M_x(d\delta_x u_H) + M_y(e\delta_y u_H) + f u_H, \tag{5}$$

and

$$(M_x w_H)_{j-1/2,\ell-1/2} := \frac{w_{j-1,\ell-1/2} + w_{j,\ell-1/2}}{2},$$
$$(M_y w_H)_{j-1/2,\ell-1/2} := \frac{w_{j-1/2,\ell-1} + w_{j-1/2,\ell}}{2},$$
$$(M_H w_H)_{j-1/2,\ell-1/2} := \frac{w_{j-1,\ell-1} + w_{j-1,\ell} + w_{j,\ell-1} + w_{j,\ell}}{4},$$

for $(x_{j-1/2}, y_{\ell-1/2}) \in \Omega_H$. These last three quantities are zero on $\partial\Omega_H$. R_H and R_{G_H} are the operators that define pointwise restrictions to $\bar{\Omega}_H$ and $G_H \cap \Omega$, respectively.

In the sequel we need norms for grid functions. To this end we introduce in the next section discrete versions of the Sobolev spaces $L^2(\Omega)$, $W_0^{1,2}(\Omega)$ and $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

2. Discrete $W^{m,2}(\Omega)$ spaces

For grid functions defined on S_H we define

$$|w_H|_{0,H}^2 := \sum_{j=1}^{N}\sum_{\ell=1}^{M} h_{j-1}k_{\ell-1}|w_{j-1/2,\ell-1/2}|^2,$$
$$|w_H|_{1,H}^2 := \sum_{j=0}^{N}\sum_{\ell=1}^{M} h_{j-1/2}k_{\ell-1}\big|(\delta_x w_H)_{j,\ell-1/2}\big|^2 + \sum_{j=1}^{N}\sum_{\ell=0}^{M} h_{j-1}k_{\ell-1/2}\big|(\delta_y w_H)_{j-1/2,\ell}\big|^2,$$
$$|w_H|_{2,H}^2 := \sum_{j=1}^{N}\sum_{\ell=1}^{M} h_{j-1}k_{\ell-1}\Big(\big|(\delta_x^2 w_H)_{j-1/2,\ell-1/2}\big|^2 + \big|(\delta_y^2 w_H)_{j-1/2,\ell-1/2}\big|^2\Big)$$
$$+ 2\sum_{j=0}^{N}\sum_{\ell=0}^{M} h_{j-1/2}k_{\ell-1/2}\big|(\delta_{xy} w_H)_{j,\ell}\big|^2,$$

with δ_{xy} given by $(\delta_{xy} w_H)_{j,\ell} := ((\delta_x w_H)_{j,\ell+1/2} - (\delta_x w_H)_{j,\ell-1/2})/k_{\ell-1/2}$. Let us now introduce discrete counterparts of $L^2(\Omega)$, $W_0^{1,2}(\Omega)$ and $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

We are going to consider the extension on $S_H \backslash \bar{\Omega}_H$ by zero for grid functions defined on $\bar{\Omega}_H$. When it is clear from the context that we use the extended function we use the same notation as for the function on $\bar{\Omega}_H$.

We denote by $\mathring{L}_H^2(\Omega)$, $\mathring{W}_H^{1,2}(\Omega)$ and $W_H^{2,2}(\Omega) \cap \mathring{W}_H^{1,2}(\Omega)$, respectively, the space of functions defined on $\bar{\Omega}_H$ which are zero on $\partial\Omega_H$ equipped with the norm $\|\cdot\|_{0,H}$, $\|\cdot\|_{1,H}$ and $\|\cdot\|_{2,H}$,

$$\|v_H\|_{m,H} := \left(\sum_{r=0}^{m} |v_H|_{r,H}^2\right)^{1/2}, \quad m = 0, 1, 2.$$

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The symbol $\|\cdot\|_{0,H}$ does not always represent a norm in spaces of grid functions. In order to overcome this fact we restrict the definition to spaces of grid functions which are zero in $\partial\Omega_H$. The space $\mathring{L}^2_H(\Omega)$ is endowed with the inner product

$$(v_H, w_H)_H := \sum_{j=1}^{N}\sum_{\ell=1}^{M} h_{j-1}k_{\ell-1}(v_H)_{j-1/2,\ell-1/2}(\bar{w}_H)_{j-1/2,\ell-1/2},$$

which is a discrete version of the usual $L^2(\Omega)$-inner product, $(\cdot,\cdot)_0$. The spaces $\mathring{L}^2_H(\Omega)$ and $\mathring{W}^{1,2}_H(\Omega)$ form discrete approximations of $L^2(\Omega)$ and $W^{1,2}_0(\Omega)$, respectively, in the sense explained in what follows [29,30].

Let Λ be a sequence of positive vectors of step-sizes, $H=(h,k)$, such that the maximum step-size, $H_{\max}$, converges to zero. A sequence $(v_H)_\Lambda \in \Pi\mathring{L}^2_H(\Omega)$ converges discretely to v, $v_H \to v$, in $(L^2(\Omega), \Pi\mathring{L}^2_H(\Omega))$ $(H\in\Lambda)$, if for each $\epsilon>0$ there exists $\varphi\in C^\infty(\Omega)$ such that

$$\|v-\varphi\|_{L^2(\Omega)} \leqslant \epsilon, \qquad \lim_{H_{\max}\to 0}\sup\big\{\|v_H - R_H\varphi\|_{0,H}\big\} \leqslant \epsilon.$$

A sequence $(v_H)_\Lambda \in \Pi\mathring{W}^{1,2}_H(\Omega)$ converges discretely to $v\in W^{1,2}_0(\Omega)$, $v_H\to v$, in $(W^{1,2}_0(\Omega), \Pi\mathring{W}^{1,2}_H(\Omega))$ $(H\in\Lambda)$, if for each $\epsilon>0$ there exists $\varphi\in C^\infty(\Omega)$ such that

$$\|v-\varphi\|_{W^{1,2}(\Omega)} \leqslant \epsilon, \qquad \lim_{H_{\max}\to 0}\sup\big\{\|v_H - R_H\varphi\|_{1,H}\big\} \leqslant \epsilon.$$

A sequence $(v_H)_\Lambda$ converges weakly to v, $v_H \rightharpoonup v$, in $(L^2(\Omega), \Pi\mathring{L}^2_H(\Omega))$ $(H\in\Lambda)$, if

$$(w_H, v_H)_H \to (w, v)_0 \quad (H\in\Lambda)$$

for all $w\in L^2(\Omega)$ and $(w_H)_\Lambda \in \Pi\mathring{L}^2_H(\Omega)$ such that $(w_H)_\Lambda$ converges weakly to w in $(L^2(\Omega), \Pi\mathring{L}^2_H(\Omega))$. The following lemma [1] is an important technical tool in the stability analysis which uses the concept of discrete compactness [30].

Lemma 1. *Let $(v_H)_\Lambda \in \Pi\mathring{W}^{1,2}_H(\Omega)$ be a bounded sequence. Then there exists a subsequence $\Lambda'\subset\Lambda$ such that $(v_H)_{\Lambda'}\in\Pi\mathring{L}^2_H(\Omega)$ converges discretely in $(L^2(\Omega), \Pi\mathring{L}^2_H(\Omega))$ $(H\in\Lambda')$. Moreover, if*

$$v_H \rightharpoonup v \quad in\ \big(L^2(\Omega), \Pi\mathring{L}^2_H(\Omega)\big) \quad (H\in\Lambda)$$

then $v\in W^{1,2}_0(\Omega)$.

The proof in [1] is specific for the normed spaces of cell-centered grid functions which we consider in this paper. For general results of discrete compactness in spaces of vertex-centered grid functions we cite [14]. The proof of Lemma 2 can be found in [29].

Lemma 2. *Let $(v_H)_\Lambda \in \Pi\mathring{L}^2_H(\Omega)$ be a bounded sequence. Then there exists a subsequence $\Lambda'\subset\Lambda$ and an element $v\in L^2(\Omega)$ such that*

$$v_H \rightharpoonup v \quad in\ \big(L^2(\Omega), \Pi\mathring{L}^2_H(\Omega)\big) \quad (H\in\Lambda').$$

3. Stability

Our aim in this section is to show an inverse stability condition for A_H.

We first define the operator $A_H^*: W^{2,2}_H(\Omega)\cap\mathring{W}^{1,2}_H(\Omega)\to\mathring{L}^2_H(\Omega)$, $A_H^* := A_H^{(2)*} + A_H^{(1)*}$, with $A_H^{(1)*}v_H := A_H^{(2)*}v_H := 0$ on $\partial\Omega_H$,

$$A_H^{(2)*}v_H := -\delta_x(a\delta_x v_H) - \delta_y(c\delta_y v_H) \quad \text{in } \Omega_H,$$
$$A_H^{(1)*}v_H := -\delta_x(\bar{d}M_x^* v_H) - \delta_y(\bar{e}M_y^* v_H) + \bar{f}v_H \quad \text{in } \Omega_H,$$

where

$$(M_x^* v_H)_{j,\ell-1/2} := \frac{v_{j-1/2,\ell-1/2} h_{j-1} + v_{j+1/2,\ell-1/2} h_j}{2h_{j-1/2}},$$
$$(M_y^* v_H)_{j-1/2,\ell} := \frac{v_{j-1/2,\ell-1/2} k_{\ell-1} + v_{j-1/2,\ell+1/2} k_\ell}{2k_{\ell-1/2}}. \tag{6}$$

Before we prove the desired stability inequality we present the proof of the following result: there exists $C > 0$ such that

$$\|v_H\|_{1,H} \leqslant C \|A_H^* v_H\|_{0,H} \quad \forall v_H \in W_H^{2,2}(\Omega) \cap \mathring{W}_H^{1,2}(\Omega).$$

This will be made with the aid of Lemmas 3–6. The first result is obvious using the definitions.

Lemma 3. *If $(v_H)_\Lambda \in \Pi \mathring{W}_H^{1,2}(\Omega)$ is a bounded sequence and $\alpha \in C(\bar{\Omega})$ then $(M_x(\alpha \delta_x v_H))_\Lambda$ and $(M_y(\alpha \delta_y v_H))_\Lambda$ are bounded on $\Pi \mathring{L}_H^2(\Omega)$.*

Lemma 4. *Let $H \in \Lambda$. Then there exists positive constants C_1, C_2 and C_3 not depending on H such that*

$$\big(-\delta_x(a\delta_x v_H) - \delta_y(c\delta_y v_H), v_H\big)_H \geqslant C_1 \|v_H\|_{1,H}^2 \quad \forall v_H \in \mathring{W}_H^{1,2}(\Omega), \tag{7}$$

and

$$(A_H v_H, v_H)_H \geqslant C_2 \|v_H\|_{1,H}^2 - C_3 \|v_H\|_{0,H}^2 \quad \forall v_H \in \mathring{W}_H^{1,2}(\Omega). \tag{8}$$

Proof. Since a has a lower bound $\underline{a}$,

$$\begin{aligned}
&\underline{a} \sum_{j=0}^{N} \sum_{\ell=1}^{M} h_{j-1/2} k_{\ell-1} \big|(\delta_x v_H)_{j,\ell-1/2}\big|^2 \\
&\quad \leqslant \sum_{j=0}^{N} \sum_{\ell=1}^{M} h_{j-1/2} k_{\ell-1} a(x_j, y_{\ell-1/2}) \big|(\delta_x v_H)_{j,\ell-1/2}\big|^2 \\
&\quad = \sum_{j=1}^{N-1} \sum_{\ell=1}^{M} k_{\ell-1} a(x_j, y_{\ell-1/2}) (\delta_x v_H)_{j,\ell-1/2} (\bar{v}_{j+1/2,\ell-1/2} - \bar{v}_{j-1/2,\ell-1/2}) \\
&\qquad + \sum_{\ell=1}^{M} k_{\ell-1} a(x_0, y_{\ell-1/2}) (\delta_x v_H)_{0,\ell-1/2} \bar{v}_{1/2,\ell-1/2} - \sum_{\ell=1}^{M} k_{\ell-1} a(x_N, y_{\ell-1/2}) (\delta_x v_H)_{N,\ell-1/2} \bar{v}_{N-1/2,\ell-1/2},
\end{aligned}$$

and then

$$\begin{aligned}
\big(-\delta_x(a\delta_x v_H), v_H\big)_H &= -\sum_{j=1}^{N} \sum_{\ell=1}^{M} k_{\ell-1} \big((a\delta_x v_H)_{j,\ell-1/2} - (a\delta_x v_H)_{j-1,\ell-1/2}\big) \bar{v}_{j-1/2,\ell-1/2} \\
&\geqslant \underline{a} \sum_{j=0}^{N} \sum_{\ell=1}^{M} h_{j-1/2} k_{\ell-1} \big|(\delta_x v_H)_{j,\ell-1/2}\big|^2.
\end{aligned}$$

In the same way we can prove a correspondent estimate for $(-\delta_y(c\delta_y v_H), v_H)_H$. Then (7) follows. From Lemma 3 and using a discrete version of the Poincaré–Friedrichs inequality which is simple to prove, there exists $C_4 > 0$ such that

$$\big|\big(M_x(d\delta_x v_H) + M_y(e\delta_y v_H) + f v_H, v_H\big)_H\big| \leqslant C_4 \|v_H\|_{1,H} \|v_H\|_{0,H}.$$

We conclude (8) using the fact that there exists $C_3 > 0$ such that

$$C_4 \|v_H\|_{1,H} \|v_H\|_{0,H} \leqslant \frac{C_1}{2} \|v_H\|_{1,H}^2 + C_3 \|v_H\|_{0,H}^2. \qquad \square$$

Lemma 5. *Let* $(v_H)_\Lambda \in \Pi \mathring{W}_H^{1,2}(\Omega)$ *and* $v \in W_0^{1,2}(\Omega)$ *such that*

$$v_H \to v \quad \textit{in } \big(W_0^{1,2}(\Omega), \Pi \mathring{W}_H^{1,2}(\Omega)\big) \quad (H \in \Lambda)$$

and let $\alpha \in C(\bar{\Omega})$. *Then*

$$M_x(\alpha \delta_x v_H) \to \alpha v_x \quad \textit{and} \quad M_y(\alpha \delta_y v_H) \to \alpha v_y \quad \textit{in } \big(L^2(\Omega), \Pi \mathring{L}_H^2(\Omega)\big)\ (H \in \Lambda).$$

Proof. Let C satisfy $\|\alpha\|_{L^\infty(\Omega)} \leqslant C$. For any positive real number ϵ there exists $\varphi \in C^\infty(\bar{\Omega})$ such that

$$\|v - \varphi\|_{W^{1,2}(\Omega)} \leqslant \epsilon, \qquad \lim_{H_{\max}\to 0} \sup\big\{\|v_H - R_H\varphi\|_{1,H}\big\} \leqslant \frac{1}{4C}\epsilon.$$

Since

$$\big\|M_x(\alpha\delta_x v_H) - M_x(\alpha\delta_x R_H\varphi)\big\|_{0,H} \leqslant 2\|\alpha\|_{L^\infty(\Omega)}\|v_H - R_H\varphi\|_{1,H}$$

and for $H_{\max}$ small enough

$$\big\|M_x(\alpha\delta_x R_H\varphi) - R_H(\alpha\varphi_x)\big\|_{0,H} \leqslant \frac{\epsilon}{2},$$

there exists a final section $\Lambda' \subset \Lambda$ such that, for $H \in \Lambda'$,

$$\big\|M_x(\alpha\delta_x v_H) - R_H(\alpha\varphi_x)\big\|_{0,H} \leqslant \epsilon. \qquad \square$$

Lemma 6. *Let* $(v_H)_\Lambda$ *be a bounded sequence in* $\Pi \mathring{W}_H^{1,2}(\Omega)$ *and* $\alpha \in C(\bar{\Omega})$. *Then there exists a subsequence* $\Lambda' \subseteq \Lambda$ *and an element* $v \in W_0^{1,2}(\Omega)$ *such that*

$$v_H \to v \quad \textit{in } \big(L^2(\Omega), \Pi \mathring{L}_H^2(\Omega)\big)\ (H \in \Lambda')$$

and the following weak convergence hold

$$M_x(\alpha\delta_x v_H) \rightharpoonup \alpha v_x \quad \textit{and} \quad M_y(\alpha\delta_y v_H) \rightharpoonup \alpha v_y \quad \textit{in } \big(L^2(\Omega), \Pi \mathring{L}_H^2(\Omega)\big)\ (H \in \Lambda').$$

Proof. It follows from Lemma 3 that $(M_x(\alpha\delta_x v_H))_\Lambda$ is bounded on $\mathring{L}_H^2(\Omega)$. Taking Lemma 2 into account we have

$$\big(M_x(\alpha\delta_x v_H)\big)_\Lambda \rightharpoonup w \quad \text{in } \big(L^2(\Omega), \Pi \mathring{L}_H^2(\Omega)\big)\ (H \in \Lambda''),$$

for a subsequence $\Lambda'' \subseteq \Lambda$ and $w \in L^2(\Omega)$. Then for any $\varphi \in C_0^\infty(\Omega)$

$$\big(R_H\varphi, M_x(\alpha\delta_x v_H)\big)_H \to (\varphi, w)_0 \quad (H \in \Lambda''). \tag{9}$$

From Lemma 1, there exists $v \in W_0^{1,2}(\Omega)$ and $\Lambda' \subseteq \Lambda''$, such that

$$v_H \to v \quad \text{in } \big(L^2(\Omega), \Pi \mathring{L}_H^2(\Omega)\big)\ (H \in \Lambda').$$

Let us prove that

$$\delta_x(\alpha M_x^* R_H\varphi) \rightharpoonup (\alpha\varphi)_x \quad \text{in } \big(L^2(\Omega), \Pi \mathring{L}_H^2(\Omega)\big) \quad (H \in \Lambda'), \tag{10}$$

with $(M_x^* R_H\varphi)_{j,\ell-1/2}$ given by (6). Let $\psi \in C_0^\infty(\Omega)$. From Lemma 5

$$\begin{aligned}\big(-\delta_x(\alpha M_x^* R_H\varphi), R_H\psi\big)_H &= \big(R_H\varphi, M_x(\alpha\delta_x R_H\psi)\big)_H \\ &\to (\varphi, \alpha\psi_x)_0,\end{aligned}$$

or equivalently,

$$\big(-\delta_x(\alpha M_x^* R_H\varphi), R_H\psi\big)_H \to \big(-(\alpha\varphi)_x, \psi\big)_0. \tag{11}$$

From Lemma 2, there exists $z \in L^2(\Omega)$ such that

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$$\delta_x(\alpha M_x^* R_H\varphi) \rightharpoonup z \quad \text{in } \big(L^2(\Omega), \Pi \mathring{L}_H^2(\Omega)\big)\ (H \in \Lambda')$$

and consequently

$$\big(-\delta_x(\alpha M_x^* R_H\varphi), R_H\psi\big)_H \to (-z, \psi)_0. \tag{12}$$

From (11) and (12) we obtain (10). Since

$$\begin{aligned}\big(R_H\varphi, M_x(\alpha\delta_x v_H)\big)_H &= \big(-\delta_x(\alpha M_x^* R_H\varphi), v_H\big)_H \\ &\to \big(-(\alpha\varphi)_x, v\big)_0 = (\varphi, \alpha v_x)_0,\end{aligned}$$

using (9) we conclude that

$$M_x(\alpha\delta_x v_H) \rightharpoonup \alpha v_x \quad \text{in } \big(L^2(\Omega), \Pi \mathring{L}_H^2(\Omega)\big)\ (H \in \Lambda'). \qquad \square$$

Theorem 1. *There exists a final sequence $\Lambda' \subset \Lambda$ and a constant $C > 0$ not depending on H such that*

$$\|v_H\|_{1,H} \leqslant C\|A_H^* v_H\|_{0,H} \quad \forall v_H \in W_H^{2,2}(\Omega) \cap \mathring{W}_H^{1,2}(\Omega),\ H \in \Lambda'. \tag{13}$$

Proof. Assuming (13) not to hold we can find a subsequence $\Lambda'' \subseteq \Lambda$ and elements v_H, $H \in \Lambda''$, such that

$$\|v_H\|_{1,H} = 1 \quad \text{and} \quad \|A_H^* v_H\|_{0,H} \to 0 \quad (H \in \Lambda''). \tag{14}$$

Lemma 1 allow the sequence Λ'' and $v \in W_0^{1,2}(\Omega)$ to be chosen such that $(v_H)_{\Lambda''}$ converges discretely to v in $(L^2(\Omega), \Pi \mathring{L}_H^2(\Omega))$. Let $w \in W_0^{1,2}(\Omega)$ be the solution of

$$(aw_x, z_x)_0 + (cw_y, z_y)_0 = \big((dv)_x + (ev)_y + fv, z\big)_0 \quad \forall z \in W_0^{1,2}(\Omega) \tag{15}$$

and $(w_H)_\Lambda \in \Pi \mathring{W}_H^{1,2}(\Omega)$ such that $w_H \to w$ in $(W_0^{1,2}(\Omega), \Pi \mathring{W}_H^{1,2}(\Omega))$ $(H \in \Lambda)$. Let us prove the convergence

$$|z_H|_{1,H} \to 0, \tag{16}$$

for $z_H = v_H - w_H$. Lemma 4 gives the existence of $C > 0$ such that

$$|z_H|_{1,H}^2 \leqslant C\big((A_H^* v_H, z_H)_H + a(w_H, z_H) + c(w_H, z_H) + (v_H, A_H^{(1)} z_H)_H\big), \tag{17}$$

where

$$a(w_H, z_H) := \sum_{j=0}^{N}\sum_{\ell=1}^{M} h_{j-1/2}k_{\ell-1}a(x_j, y_{\ell-1/2})(\delta_x w_H)_{j,\ell-1/2}(\delta_x \bar{z}_H)_{j,\ell-1/2},$$

$$c(w_H, z_H) := \sum_{j=1}^{N}\sum_{\ell=0}^{M} h_{j-1}k_{\ell-1/2}c(x_{j-1/2}, y_\ell)(\delta_y w_H)_{j-1/2,\ell}(\delta_y \bar{z}_H)_{j-1/2,\ell},$$

$$A_H^{(1)} z_H := M_x(d\delta_x z_H) + M_y(e\delta_y z_H) + f z_H.$$

Since $\|A_H^* v_H\|_{0,H} \to 0$, it follows that $(A_H^* v_H, z_H)_H \to 0$. Let $z = v - w$. Our aim is to prove that

$$a(w_H, z_H) \to (aw_x, z_x)_0, \quad c(w_H, z_H) \to (cw_y, z_y)_0 \quad (H \in \Lambda''). \tag{18}$$

Lemma 5 yields

$$M_x(\delta_x w_H) \to w_x \quad \text{in } \big(L^2(\Omega), \Pi \mathring{L}_H^2(\Omega)\big)\ (H \in \Lambda).$$

Since $(z_H)_\Lambda$, $H \in \Lambda''$, is bounded on $\Pi \mathring{W}_H^{1,2}(\Omega)$, Lemma 6 allows a subsequence $\Lambda' \subset \Lambda''$ to be chosen such that

$$\big(M_x(a\delta_x z_H)\big)_\Lambda \rightharpoonup az_x, \quad \big(M_y(c\delta_y z_H)\big)_\Lambda \rightharpoonup cz_y \quad \text{in } \big(L^2(\Omega), \Pi \mathring{L}_H^2(\Omega)\big)\ (H \in \Lambda'),$$

and consequently (18) holds. For the last term of (17) we have

$$(v_H, A_H^{(1)} z_H)_H \to (v, A^{(1)} z)_0 \quad (H \in \Lambda'').$$

Since w is the solution of (15), it follows that

$$a(w_H, z_H) + c(w_H, z_H) + (v_H, A_H^{(1)} z_H)_H \to 0 \quad (H \in \Lambda'')$$

and (16) follows. Then

$$v_H = z_H + w_H \to w \quad \text{in } \big(L^2(\Omega), \Pi \mathring{L}_H^2(\Omega)\big)\ (H \in \Lambda'),$$

and

$$(Aw, z)_0 = 0 \quad \forall z \in W_0^{1,2}(\Omega).$$

For A being injective $\|v_H\|_{1,H} = 1$ is not possible. □

Let us now prove a stability result for $A_H^{(2)*}$.

Lemma 7. *There exists C not depending on H such that, for $H \in \Lambda$,*

$$\|v_H\|_{2,H} \leqslant C\big(\|A_H^{(2)*} v_H\|_{0,H} + \|v_H\|_{1,H}\big) \quad \forall v_H \in W_H^{2,2}(\Omega) \cap \mathring{W}_H^{1,2}(\Omega). \tag{19}$$

Proof. Let $v_H \in W_H^{2,2}(\Omega) \cap \mathring{W}_H^{1,2}(\Omega)$. We define

$$\begin{aligned}
B_x^{(1)} v_H :=& \sum_{j=1}^{N}\sum_{\ell=1}^{M} h_{j-1}k_{\ell-1}(\delta_x^2 \bar{v}_H)_{j-1/2,\ell-1/2}\bigg[\frac{a(x_{j-1/2}, y_{\ell-1/2}) - a(x_{j-1}, y_{\ell-1/2})}{h_{j-1}}(\delta_x v_H)_{j-1,\ell-1/2} \\
&+ \frac{a(x_j, y_{\ell-1/2}) - a(x_{j-1/2}, y_{\ell-1/2})}{h_{j-1}}(\delta_x v_H)_{j,\ell-1/2}\bigg] \\
&+ \sum_{j=0}^{N}\sum_{\ell=0}^{M} h_{j-1}k_{\ell-1/2}\frac{c(x_j, y_\ell) - c(x_{j-1/2}, y_\ell)}{h_{j-1}}(\delta_y v_H)_{j-1/2,\ell}(\delta_{xy}\bar{v}_H)_{j,\ell} \\
&+ \sum_{j=0}^{N}\sum_{\ell=0}^{M} h_j k_{\ell-1/2}\frac{c(x_{j+1/2}, y_\ell) - c(x_j, y_\ell)}{h_j}(\delta_y v_H)_{j+1/2,\ell}(\delta_{xy}\bar{v}_H)_{j,\ell}, \\
B_x^{(2)} v_H :=& \sum_{j=1}^{N}\sum_{\ell=1}^{M} h_{j-1}k_{\ell-1} a(x_{j-1/2}, y_{\ell-1/2})\big|(\delta_x^2 v_H)_{j-1/2,\ell-1/2}\big|^2 \\
&+ \sum_{j=0}^{N}\sum_{\ell=0}^{M} h_{j-1/2}k_{\ell-1/2} c(x_j, y_\ell)\big|(\delta_{xy} v_H)_{j,\ell}\big|^2,
\end{aligned}$$

$B_y^{(1)}$ and $B_y^{(2)}$ similar to $B_x^{(1)}$ and $B_x^{(2)}$, respectively, replacing a with c, x with y and the indexes in a obvious way. We have

$$(A_H^{(2)*} v_H, \delta_x^2 v_H + \delta_y^2 v_H)_H = -B_H^{(1)} v_H - B_H^{(2)} v_H, \tag{20}$$

where $B_H^{(1)} := B_x^{(1)} + B_y^{(1)}$ and $B_H^{(2)} := B_x^{(2)} + B_y^{(2)}$. The conditions assumed for a and c give the existence of $C_E > 0$ and $C_L > 0$ such that

$$C_E |v_H|_{2,H}^2 \leqslant B_H^{(2)} v_H \quad \text{and} \quad B_H^{(1)} v_H \leqslant C_L |v_H|_{1,H} |v_H|_{2,H},$$

which together with (20) yield

$$\begin{aligned}
C_E |v_H|_{2,H}^2 &\leqslant \big|(A_H^{(2)*} v_H, \delta_x^2 v_H + \delta_y^2 v_H)_H\big| + |B_H^{(1)} v_H| \\
&\leqslant \|A_H^{(2)*} v_H\|_{0,H} |v_H|_{2,H} + C_L |v_H|_{1,H} |v_H|_{2,H}.
\end{aligned}$$

Then (19) follows with $C = \max\{1/C_E, C_L/C_E\}$. □

The main result of this section is the following stability theorem.

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Theorem 2. *There exists $C > 0$ and a final section $\Lambda' \subset \Lambda$ such that*

$$\|v_H\|_{0,H} \leqslant C \sup_{0 \neq w_H \in W_H^{2,2}(\Omega)\cap \mathring{W}_H^{1,2}(\Omega)} \frac{|(A_H v_H, w_H)_H|}{\|w_H\|_{2,H}} \tag{21}$$

$\forall v_H \in W_H^{2,2}(\Omega) \cap \mathring{W}_H^{1,2}(\Omega)$, $H \in \Lambda'$.

Proof. Let $v_H \in W_H^{2,2}(\Omega)\cap \mathring{W}_H^{1,2}(\Omega)$. Since $A_H^{(1)*}\colon \mathring{W}_H^{1,2}(\Omega) \to \mathring{L}_H^2(\Omega)$ is bounded, there exists $C_L > 0$ such that

$$\|A_H^{(2)*} v_H\|_{0,H} \leqslant \|A_H^* v_H\|_{0,H} + \|A_H^{(1)*} v_H\|_{0,H} \leqslant \|A_H^* v_H\|_{0,H} + C_L\|v_H\|_{1,H}.$$

Lemma 7 gives the existence of $C' > 0$ such that

$$\begin{aligned}\|v_H\|_{2,H} &\leqslant C'\big(\|A_H^{(2)*} v_H\|_{0,H} + \|v_H\|_{1,H}\big)\\ &\leqslant C'\|A_H^* v_H\|_{0,H} + (C' + C'C_L)\|v_H\|_{1,H}.\end{aligned}$$

The existence of $C > 0$ such that

$$\|v_H\|_{2,H} \leqslant C\|A_H^* v_H\|_{0,H} \quad \forall v_H \in W_H^{2,2}(\Omega)\cap \mathring{W}_H^{1,2}(\Omega) \tag{22}$$

follows from Theorem 1. Then $(A_H^*)^{-1} : \mathring{L}_H^2(\Omega) \to W_H^{2,2}(\Omega) \cap \mathring{W}_H^{1,2}(\Omega)$ exists and (22) is equivalent to

$$\big\|(A_H^*)^{-1}\varphi_H\big\|_{2,H} \leqslant C\|\varphi_H\|_{0,H} \quad \forall \varphi_H \in \mathring{L}_H^2(\Omega).$$

Combining this last inequality with

$$\begin{aligned}\sup_{0 \neq w_H \in W_H^{2,2}(\Omega)\cap \mathring{W}_H^{1,2}(\Omega)} \frac{|(A_H v_H, w_H)_H|}{\|w_H\|_{2,H}} &= \sup_{0\neq \varphi_H \in \mathring{L}_H^2(\Omega)} \frac{|(A_H v_H, (A_H^*)^{-1}\varphi_H)_H|}{\|(A_H^*)^{-1}\varphi_H\|_{2,H}}\\ &= \sup_{0\neq \varphi_H \in \mathring{L}_H^2(\Omega)} \frac{|(v_H, \varphi_H)_H|}{\|(A_H^*)^{-1}\varphi_H\|_{2,H}},\end{aligned}$$

results (21). □

The estimate (21) can be given in an alternative form which uses a negative norm. We introduce the discrete Laplace operator

$$\Delta_H v_H := \delta_x^2 v_H + \delta_y^2 v_H, \quad v_H \in W_H^{2,2}(\Omega) \cap \mathring{W}_H^{1,2}(\Omega)$$

and the norm

$$\|v_H\|_{-\Delta_H} := \sup_{0 \neq w_H \in W_H^{2,2}(\Omega)\cap \mathring{W}_H^{1,2}(\Omega)} \frac{|(v_H, w_H)_H|}{\|\Delta_H w_H\|_{0,H}}, \tag{23}$$

where the extension by zero on $S_H \backslash \bar{\Omega}_H$ of v_H and w_H is considered.

Some trivial algebraic manipulations lead to the next result [2]: the norms $\|\cdot\|_{2,H}$ and $\|\Delta_H \cdot\|_{0,H}$ are equivalent in $W_H^{2,2}(\Omega) \cap \mathring{W}_H^{1,2}(\Omega)$. With the definition (23) the estimate (21) is equivalent to

$$\|v_H\|_{0,H} \leqslant C\|A_H v_H\|_{-\Delta_H} \quad \forall v_H \in W_H^{2,2}(\Omega) \cap \mathring{W}_H^{1,2}(\Omega).$$

4. Convergence

The main result of this paper in Theorem 3 relies on the stability result of Theorem 2. An estimate for $\|R_H u - u_H\|_{0,H}$ will be obtained with the aid of (21) replacing v_H by $R_H u - u_H$ and bounding

$$\big(A_H(R_H u) - M_H(R_{G_H} g), v_H\big)_H.$$

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The bounds in Lemmas 8–10 are for that purpose. The Bramble–Hilbert Lemma (see e.g. [5]) and Schwarz's inequality

$$\|v\|_{L^1(\square)} \leqslant h^{1/2}k^{1/2}\|v\|_{L^2(\square)} \quad \forall v \in L^2(\square),$$

being $h \times k$ the dimension of the rectangle $\square$, for estimating the local contributions measured in the L^1-norm in terms of L^2-norm, are the main technical tools to obtain the desired convergence order.

In what follows we use the notation $\sum_{\Omega_H}$ for the sum over the set of indexes (j, ℓ) such that $(x_{j+1/2}, y_{\ell+1/2}) \in \Omega_H$.

Lemma 8. *Let $u \in H^4(\Omega)$. Then, for all $v_H \in W_H^{2,2}(\Omega) \cap \mathring{W}_H^{1,2}(\Omega)$ there holds*

$$\begin{aligned} &\big|\big(-\delta_x(a\delta_x u), v_H\big)_H - \big(M_H R_{G_H}(au_x)_x, v_H\big)_H\big| \\ &\quad \leqslant C\|a\|_{W^{3,\infty}(\Omega)}\Big(\sum_{\Omega_H}(h_j^2+k_\ell^2)^2\|u\|^2_{H^4((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))}\Big)^{1/2}\|v_H\|_{2,H}, \end{aligned} \tag{24}$$

$$\begin{aligned} &\big|\big(-\delta_y(c\delta_y u), v_H\big)_H - \big(M_H R_{G_H}(cu_y)_y, v_H\big)_H\big| \\ &\quad \leqslant C\|c\|_{W^{3,\infty}(\Omega)}\Big(\sum_{\Omega_H}(h_j^2+k_\ell^2)^2\|u\|^2_{H^4((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))}\Big)^{1/2}\|v_H\|_{2,H}. \end{aligned} \tag{25}$$

Proof. Let $v_H \in W_H^{2,2}(\Omega) \cap \mathring{W}_H^{1,2}(\Omega)$. We consider, in first place, only the terms in

$$(\delta_x a\delta_x u, v_H)_H \quad \text{and} \quad \big(M_H R_{G_H}(au_x)_x, v_H\big)_H$$

which have the factor $\bar{v}_{j+1/2,\ell+1/2}$, for some j, with ℓ given. Let us suppose, without loss of generality, that the set of the points in the form $(\cdot, y_{\ell+1/2})$ belonging to Ω_H is

$$\big\{(x_{p_\ell+1/2}, y_{\ell+1/2}), (x_{p_\ell+3/2}, y_{\ell+1/2}), \ldots, (x_{p_\ell+N_\ell-1/2}, y_{\ell+1/2})\big\}.$$

Let

$$S_1 := \sum_{j=p_\ell}^{p_\ell+N_\ell-1} h_j k_\ell (\delta_x a\delta_x u)_{j+1/2,\ell+1/2}\bar{v}_{j+1/2,\ell+1/2}$$

and

$$S_1^{(1)} := -\sum_{j=p_\ell}^{p_\ell+N_\ell}\Bigg(\int_{y_\ell}^{y_{\ell+1}}\int_{x_{j-1/2}}^{x_{j+1/2}} a(x_j, y)u_x(x, y)\,dx\,dy\Bigg)(\delta_x\bar{v}_H)_{j,\ell+1/2}.$$

We have

$$\begin{aligned} S_1 &= \sum_{j=p_\ell}^{p_\ell+N_\ell-1} k_\ell\big((a\delta_x u)_{j+1,\ell+1/2} - (a\delta_x u)_{j,\ell+1/2}\big)\bar{v}_{j+1/2,\ell+1/2} \\ &= -\sum_{j=p_\ell}^{p_\ell+N_\ell} k_\ell \int_{x_{j-1/2}}^{x_{j+1/2}} a(x_j, y_{\ell+1/2})u_x(x, y_{\ell+1/2})\,dx\,(\delta_x\bar{v}_H)_{j,\ell+1/2}. \end{aligned}$$

The functional $\lambda(g) := g(\frac{1}{2}) - \int_0^1 g(\xi)\,d\xi$ is bounded on $W^{2,1}(0,1)$ and vanishes for $g = 1$ and ξ. Thus the Bramble–Hilbert Lemma gives the existence of a positive constant C such that

$$\big|\lambda(g)\big| \leqslant C\|g''\|_{L^1(0,1)}.$$

From the last estimate applied to $g = w$, where

$$w(\xi) := a(x_j, y_\ell + \xi k_\ell)\int_{x_{j-1/2}}^{x_{j+1/2}} u_x(x, y_\ell + \xi k_\ell)\,dx, \quad \xi \in [0,1],$$

follows

$$S_1 = S_1^{(1)} - \sum_{j=p_\ell}^{p_\ell+N_\ell} \int_{y_\ell}^{y_{\ell+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} E_{j,\ell} (\delta_x \bar{v}_H)_{j,\ell+1/2},$$

with

$$|E_{j,\ell}| \leqslant C k_\ell^2 \left| a(x_j, \cdot) \int_{x_{j-1/2}}^{x_{j+1/2}} u_x(x, \cdot)\, dx \right|_{W^{2,1}((y_\ell, y_{\ell+1}))}.$$

Let

$$S_2 := \sum_{j=p_\ell}^{p_\ell+N_\ell-1} h_j k_\ell \big(M_H R_{G_H}(a u_x)_x\big)_{j+1/2,\ell+1/2} \bar{v}_{j+1/2,\ell+1/2},$$

which can be written in the form

$$S_2 = S_2^{(1)} + \sum_{j=p_\ell}^{p_\ell+N_\ell-1} F_{j,\ell} \bar{v}_{j+1/2,\ell+1/2},$$

where

$$S_2^{(1)} := \sum_{j=p_\ell}^{p_\ell+N_\ell-1} \int_{y_\ell}^{y_{\ell+1}} \int_{x_j}^{x_{j+1}} (a u_x)_x(x, y)\, dx\, dy\, \bar{v}_{j+1/2,\ell+1/2},$$

$$F_{j,\ell} := \big(M_H R_{G_H}(a u_x)_x\big)_{j+1/2,\ell+1/2} - \int_{y_\ell}^{y_{\ell+1}} \int_{x_j}^{x_{j+1}} (a u_x)_x(x, y)\, dx\, dy.$$

$F_{j,\ell}$ can be bounded with the aid of the Bramble–Hilbert Lemma. Let the function w be defined by

$$w(\xi, \eta) := (a u_x)_x(x_j + \xi h_j, y_\ell + \eta k_\ell), \quad (\xi, \eta) \in (0, 1) \times (0, 1).$$

Then

$$F_{j,\ell} = h_j k_\ell \left(\frac{w(0,0) + w(1,0) + w(0,1) + w(1,1)}{4} - \int_0^1 \int_0^1 w(\xi, \eta)\, d\xi\, d\eta \right).$$

The functional

$$\lambda(g) := \frac{g(0,0) + g(1,0) + g(0,1) + g(1,1)}{4} - \int_0^1 \int_0^1 g(\xi, \eta)\, d\xi\, d\eta,$$

$g \in W^{2,2}((0,1) \times (0,1))$, is bounded and vanishes for $g = 1, \xi$ and η. Again, by Bramble–Hilbert Lemma the estimate

$$\big|\lambda(g)\big| \leqslant C |g|_{W^{2,1}((0,1)\times(0,1))}$$

holds and we obtain the bound

$$|F_{j,\ell}| \leqslant C \big(h_j^2 \big\| (a u_x)_{xxx} \big\|_{L^1((x_j, x_{j+1})\times(y_\ell, y_{\ell+1}))} + k_\ell h_j \big\| (a u_x)_{xxy} \big\|_{L^1((x_j, x_{j+1})\times(y_\ell, y_{\ell+1}))} + k_\ell^2 \big\| (a u_x)_{xyy} \big\|_{L^1((x_j, x_{j+1})\times(y_\ell, y_{\ell+1}))} \big).$$

Let us finally consider the difference $S_1^{(1)} - S_2^{(1)}$. For $S_2^{(1)}$ we have

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$$S_2^{(1)} = \sum_{j=p_\ell}^{p_\ell+N_\ell-1} \int_{y_\ell}^{y_{\ell+1}} \big((au_x)(x_{j+1},y) - (au_x)(x_j,y)\big)\,dy\,\bar{v}_{j+1/2,\ell+1/2}$$
$$= -\sum_{j=p_\ell}^{p_\ell+N_\ell} \int_{y_\ell}^{y_{\ell+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} (au_x)(x_j,y)\,dx\,dy\,(\delta_x\bar{v}_H)_{j,\ell+1/2}$$

and then

$$S_1^{(1)} - S_2^{(1)} = (T_1+T_2)/2 + T_3 + T_4,$$

with

$$T_1 := -\sum_{j=p_\ell+1}^{p_\ell+N_\ell} \int_{y_\ell}^{y_{\ell+1}} \left[\frac{h_{j-1}}{2}\big(u_x(x_{j-1},y)+u_x(x_j,y)\big) - \int_{x_{j-1}}^{x_j} u_x(x,y)\,dx\right]$$
$$\times \big(a(x_{j-1},y)(\delta_x\bar{v}_H)_{j-1,\ell+1/2} + a(x_j,y)(\delta_x\bar{v}_H)_{j,\ell+1/2}\big)\,dy,$$
$$T_2 := -\sum_{j=p_\ell+1}^{p_\ell+N_\ell} \int_{y_\ell}^{y_{\ell+1}} \left[\frac{h_{j-1}}{2}\big(u_x(x_j,y)-u_x(x_{j-1},y)\big)\right.$$
$$\left. + \int_{x_{j-1}}^{x_{j-1/2}} u_x(x,y)\,dx - \int_{x_{j-1/2}}^{x_j} u_x(x,y)\,dx\right]\big(a(x_j,y)(\delta_x\bar{v}_H)_{j,\ell+1/2} - a(x_{j-1},y)(\delta_x\bar{v}_H)_{j-1,\ell+1/2}\big)\,dy,$$
$$T_3 := -\int_{y_\ell}^{y_{\ell+1}} \left[\frac{h_{p_\ell-1}}{2}u_x(x_{p_\ell},y) - \int_{x_{p_\ell-1/2}}^{x_{p_\ell}} u_x(x,y)\,dx\right]a(x_{p_\ell},y)\,dy\,(\delta_x\bar{v}_H)_{p_\ell,\ell+1/2},$$
$$T_4 := -\int_{y_\ell}^{y_{\ell+1}} \left[\frac{h_{p_\ell+N_\ell}}{2}u_x(x_{p_\ell+N_\ell},y) - \int_{x_{p_\ell+N_\ell}}^{x_{p_\ell+N_\ell+1/2}} u_x(x,y)\,dx\right]a(x_{p_\ell+N_\ell},y)\,dy\,(\delta_x\bar{v}_H)_{p_\ell+N_\ell,\ell+1/2}.$$

The sum in T_1 contains the errors of the trapezoidal rule that can be bounded with the aid of the Bramble–Hilbert Lemma by

$$|T_1| \leqslant C \sum_{j=p_\ell+1}^{p_\ell+N_\ell} h_{j-1}^2 \|u_{xxx}\|_{L^1((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))} \|a\|_{L^\infty((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))}$$
$$\times \big(\big|(\delta_x\bar{v}_H)_{j-1,\ell+1/2}\big| + \big|(\delta_x\bar{v}_H)_{j,\ell+1/2}\big|\big).$$

For T_2 we have only the first order bound but the factor

$$a(x_j,y)(\delta_x\bar{v}_H)_{j,\ell+1/2} - a(x_{j-1},y)(\delta_x\bar{v}_H)_{j-1,\ell+1/2}$$

allows to estimate T_2 with the same order as T_1. We have

$$a(x_j,y)(\delta_x\bar{v}_H)_{j,\ell+1/2} - a(x_{j-1},y)(\delta_x\bar{v}_H)_{j-1,\ell+1/2}$$
$$= a(x_{j-1/2},y)\big((\delta_x\bar{v}_H)_{j,\ell+1/2} - (\delta_x\bar{v}_H)_{j-1,\ell+1/2}\big) + \big(a(x_{j-1/2},y) - a(x_{j-1},y)\big)(\delta_x\bar{v}_H)_{j-1,\ell+1/2}$$
$$+ \big(a(x_j,y) - a(x_{j-1/2},y)\big)(\delta_x\bar{v}_H)_{j,\ell+1/2}$$
$$= h_{j-1}a(x_{j-1/2},y)(\delta_x^2\bar{v}_H)_{j-1/2,\ell+1/2} + \frac{h_{j-1}}{2}\big(a_x(\eta_1,y)(\delta_x\bar{v}_H)_{j-1,\ell+1/2} + a_x(\eta_2,y)(\delta_x\bar{v}_H)_{j,\ell+1/2}\big),$$

for some $\eta_1 \in [x_{j-1}, x_{j-1/2}]$, $\eta_2 \in [x_{j-1/2}, x_j]$, and then

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$$|T_2| \leqslant C \sum_{j=p_\ell+1}^{p_\ell+N_\ell} h_{j-1}^2 \|u_{xx}\|_{L^1((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))} \|a\|_{W^{1,\infty}((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))}$$
$$\times \left(\left|(\delta_x^2 \bar{v}_H)_{j-1/2,\ell+1/2}\right| + \left|(\delta_x \bar{v}_H)_{j-1,\ell+1/2}\right| + \left|(\delta_x \bar{v}_H)_{j,\ell+1/2}\right|\right).$$

For T_3 and T_4 we have

$$|T_3| \leqslant \int_{y_\ell}^{y_{\ell+1}} \frac{h_{p_\ell-1}}{8} \left\|u_{xx}(\cdot,y)\right\|_{L^1((x_{p_\ell-1/2},x_{p_\ell}))} \left|a(x_{p_\ell},y)\right| dy \left|(\delta_x \bar{v}_H)_{p_\ell,\ell+1/2}\right|,$$

$$|T_4| \leqslant \int_{y_\ell}^{y_{\ell+1}} \frac{h_{p_\ell+N_\ell}}{8} \left\|u_{xx}(\cdot,y)\right\|_{L^1((x_{p_\ell+N_\ell},x_{p_\ell+N_\ell+1/2}))} \left|a(x_{p_\ell+N_\ell},y)\right| dy \left|(\delta_x \bar{v}_H)_{p_\ell+N_\ell,\ell+1/2}\right|.$$

Considering the equality

$$(\delta_x \bar{v}_H)_{p_\ell,\ell+1/2} = -\sum_{i=p_\ell}^{j} h_i (\delta_x^2 \bar{v}_H)_{i+1/2,\ell+1/2} + (\delta_x \bar{v}_H)_{j+1,\ell+1/2},$$

$j = p_\ell, \ldots, p_\ell + N_\ell - 1$, follows

$$\left|(\delta_x \bar{v}_H)_{p_\ell,\ell+1/2}\right| \leqslant \sum_{j=p_\ell}^{p_\ell+N_\ell-1} h_j \left|(\delta_x^2 \bar{v}_H)_{j+1/2,\ell+1/2}\right| + \frac{1}{x_{p_\ell+N_\ell} - x_{p_\ell}} \sum_{j=p_\ell}^{p_\ell+N_\ell-1} h_j \left|(\delta_x \bar{v}_H)_{j+1,\ell+1/2}\right|.$$

For T_3 we have

$$|T_3| \leqslant \frac{h_{p_\ell-1}}{8} \|u_{xx}\|_{L^1((x_{p_\ell-1/2},x_{p_\ell})\times(y_\ell,y_{\ell+1}))} \left\|a(x_{p_\ell},\cdot)\right\|_{L^\infty((y_\ell,y_{\ell+1}))}$$
$$\times \left(\sum_{j=p_\ell}^{p_\ell+N_\ell-1} h_j \left|(\delta_x^2 \bar{v}_H)_{j+1/2,\ell+1/2}\right| + \frac{1}{x_{p_\ell+N_\ell} - x_{p_\ell}} \sum_{j=p_\ell}^{p_\ell+N_\ell-1} h_j \left|(\delta_x \bar{v}_H)_{j+1,\ell+1/2}\right|\right),$$

and in the same way for T_4 we obtain

$$|T_4| \leqslant \frac{h_{p_\ell+N_\ell}}{8} \|u_{xx}\|_{L^1((x_{p_\ell+N_\ell},x_{p_\ell+N_\ell+1/2})\times(y_\ell,y_{\ell+1}))} \left\|a(x_{p_\ell+N_\ell},\cdot)\right\|_{L^\infty((y_\ell,y_{\ell+1}))}$$
$$\times \left(\sum_{j=p_\ell}^{p_\ell+N_\ell-1} h_j \left|(\delta_x^2 \bar{v}_H)_{j+1/2,\ell+1/2}\right| + \frac{1}{x_{p_\ell+N_\ell} - x_{p_\ell}} \sum_{j=p_\ell}^{p_\ell+N_\ell-1} h_j \left|(\delta_x \bar{v}_H)_{j+1,\ell+1/2}\right|\right).$$

Using the Schwarz's inequality we obtain (24). □

Lemma 9. *Let $u \in H^3(\Omega)$. Then the following estimates hold*

$$\left|\left(M_x(d\delta_x u), v_H\right)_H - \left(M_H R_{G_H}(du_x), v_H\right)_H\right|$$
$$\leqslant C\|d\|_{W^{2,\infty}(\Omega)} \left(\sum_{\Omega_H} (h_j^2 + k_\ell^2)^2 \|u_{xxx}\|^2_{L^2((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))}\right)^{1/2} \|v_H\|_{1,H}, \tag{26}$$

$$\left|\left(M_y(e\delta_y u), v_H\right)_H - \left(M_H R_{G_H}(eu_y), v_H\right)_H\right|$$
$$\leqslant C\|e\|_{W^{2,\infty}(\Omega)} \left(\sum_{\Omega_H} (h_j^2 + k_\ell^2)^2 \|u_{yyy}\|^2_{L^2((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))}\right)^{1/2} \|v_H\|_{1,H}, \tag{27}$$

for all $v_H \in \mathring{W}_H^{1,2}(\Omega)$.

Proof. Let us consider the terms in $(M_x(d\delta_x u), v_H)_H$ and $(M_H R_{G_H}(du_x), v_H)_H$ which have the factor $\bar{v}_{j+1/2,\ell+1/2}$, for some j, with ℓ given. We obtain for $(M_x(d\delta_x u), v_H)_H$ and $(M_H R_{G_H}(du_x), v_H)_H$, respectively,

$$\sum_{j=p_\ell}^{p_\ell+N_\ell-1} k_\ell h_j \big(M_x(d\delta_x u)\big)_{j+1/2,\ell+1/2} \bar{v}_{j+1/2,\ell+1/2}$$
$$= -\sum_{j=p_\ell}^{p_\ell+N_\ell} k_\ell h_{j-1/2} \sum_{i=p_\ell}^{j-1} h_i \big(M_x(d\delta_x u)\big)_{i+1/2,\ell+1/2} (\delta_x \bar{v}_H)_{j,\ell+1/2}$$

and

$$\sum_{j=p_\ell}^{p_\ell+N_\ell-1} k_\ell h_j \big(M_H R_{G_H}(du_x)\big)_{j+1/2,\ell+1/2} \bar{v}_{j+1/2,\ell+1/2}$$
$$= -\sum_{j=p_\ell}^{p_\ell+N_\ell} k_\ell h_{j-1/2} \sum_{i=p_\ell}^{j-1} h_i \big(M_x(du_x)\big)_{i+1/2,\ell+1/2} (\delta_x \bar{v}_H)_{j,\ell+1/2}$$
$$+ \sum_{j=p_\ell}^{p_\ell+N_\ell-1} \frac{h_j}{2} k_\ell \big((E_y)_{j,\ell+1/2} + (E_y)_{j+1,\ell+1/2}\big) \bar{v}_{j+1/2,\ell+1/2},$$

where

$$(E_y)_{j,\ell+1/2} := \frac{(du_x)_{j,\ell} + (du_x)_{j,\ell+1}}{2} - (du_x)_{j,\ell+1/2}.$$

Let $w(\xi) := (du_x)(x_j, y_\ell + \xi k_\ell)$, $\xi \in [0,1]$. Then

$$(E_y)_{j,\ell+1/2} = \frac{w(0)+w(1)}{2} - w\left(\frac{1}{2}\right).$$

The functional $\lambda(g) := \frac{g(0)+g(1)}{2} - g(\frac{1}{2})$ is bounded on $W^{2,1}(0,1)$ and vanishes for $g = 1$ and ξ. Again by the Bramble–Hilbert Lemma the estimate

$$\big|\lambda(g)\big| \leqslant C\|g''\|_{L^1(0,1)}, \quad g \in W^{2,1}(0,1),$$

holds and we obtain the bound

$$\sum_{j=p_\ell}^{p_\ell+N_\ell-1} \frac{h_j}{2} k_\ell \big|(E_y)_{j,\ell+1/2} + (E_y)_{j+1,\ell+1/2}\big| |v_{j+1/2,\ell+1/2}|$$
$$\leqslant \sum_{j=p_\ell}^{p_\ell+N_\ell-1} \frac{h_j}{2} k_\ell^2 \big(\big\|\big((du_x)_{xx}\big)(x_j,\cdot)\big\|_{L^1(I_\ell)} + \big\|\big((du_x)_{xx}\big)(x_{j+1},\cdot)\big\|_{L^1(I_\ell)}\big) |v_{j+1/2,\ell+1/2}|. \tag{28}$$

We have

$$\sum_{i=p_\ell}^{j-1} h_i \big[\big(M_x(d\delta_x u)\big)_{i+1/2,\ell+1/2} - \big(M_x(du_x)\big)_{i+1/2,\ell+1/2}\big]$$
$$= \sum_{i=p_\ell+1}^{j-1} h_{i-1/2} d_{i,\ell+1/2} \big((\delta_x u)_{i,\ell+1/2} - u_x(x_i, y_{\ell+1/2})\big) + \frac{h_{j-1}}{2} d_{j,\ell+1/2} \big((\delta_x u)_{j,\ell+1/2} - u_x(x_j, y_{\ell+1/2})\big)$$
$$+ \frac{h_{p_\ell}}{2} d_{p_\ell,\ell+1/2} \big((\delta_x u)_{p_\ell,\ell+1/2} - u_x(x_{p_\ell}, y_{\ell+1/2})\big).$$

Using (28) we obtain the bound (26). □

Lemma 10. *Let $w \in H^2(\Omega)$. Then, for all $v_H \in \mathring{L}^2_H(\Omega)$,*

$$\left|(R_H w, v_H)_H - (M_H R_{G_H} w, v_H)_H\right| \leqslant C\left(\sum_{\Omega_H}(h_j^2 + k_\ell^2)^2 \|w\|^2_{H^2((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))}\right)^{1/2} \|v_H\|_{0,H}. \tag{29}$$

Proof. We can write

$$(M_H R_{G_H} w)_{j+1/2,\ell+1/2} = w_{j+1/2,\ell+1/2} + (E_x)_{j+1/2,\ell} + (E_x)_{j+1/2,\ell+1} + (E_y)_{j+1/2,\ell+1/2},$$

where

$$(E_x)_{j+1/2,\ell} := \frac{w_{j,\ell} + w_{j+1,\ell}}{4} - \frac{w_{j+1/2,\ell}}{2},$$
$$(E_y)_{j+1/2,\ell+1/2} := \frac{w_{j+1/2,\ell} + w_{j+1/2,\ell+1}}{2} - w_{j+1/2,\ell+1/2}.$$

Using the Bramble–Hilbert Lemma as before we obtain (29). □

Let us consider in (29) $w = fu$. For all $v_H \in \mathring{L}^2_H(\Omega)$ we obtain

$$\begin{aligned}&\left|\big(fu, v_H\big)_H - \big(M_H R_{G_H}(fu), v_H\big)_H\right| \\ &\quad\leqslant C\|f\|_{W^{2,\infty}(\Omega)} H^2_{\max}\left(\sum_{\Omega_H}\|u\|^2_{H^2((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))}\right)^{1/2} \|v_H\|_{0,H}.\end{aligned} \tag{30}$$

The next result follows from Theorem 2 and from the bounds (24)–(27) and (30).

Theorem 3. *Let Ω be a union of rectangles. Assume that the solution u of* (1)–(2) *lies in $H^4(\Omega)$. Then for $H \in \Lambda$, with $H_{\max}$ small enough, the discrete problem* (3)–(4) *has a unique solution u_H which satisfies*

$$\begin{aligned}\|R_H u - u_H\|_{0,H} &\leqslant C\left(\sum_{\Omega_H}(h_j^2 + k_\ell^2)^2\|u\|^2_{H^4((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))}\right)^{1/2} \\ &\leqslant C H^2_{\max}\|u\|_{H^4(\Omega)}.\end{aligned}$$

5. Numerical results

We present numerical results for the problem

$$-\Delta u = f \quad \text{on } \Omega = (0,1)\times(0,1),$$
$$u = 0 \quad \text{on } \partial\Omega,$$

with $f(x,y) = -((x(x-1)y(y-1))^2)_{xx} - ((x(x-1)y(y-1))^2)_{yy}$. Fig. 2 shows the numerical solution on 500 random meshes ($N-1 \times M-1$ points placed in Ω at random), where N and M ranges from 10 to 110. The logarithm of the norm of the error, $\log(\|R_H u - u_H\|_{0,H})$, is plotted versus the logarithm of the maximum step-size. The straight line is the least-squares fit to the points and has the slope 2.1721, which confirms the estimates given in Theorem 3.

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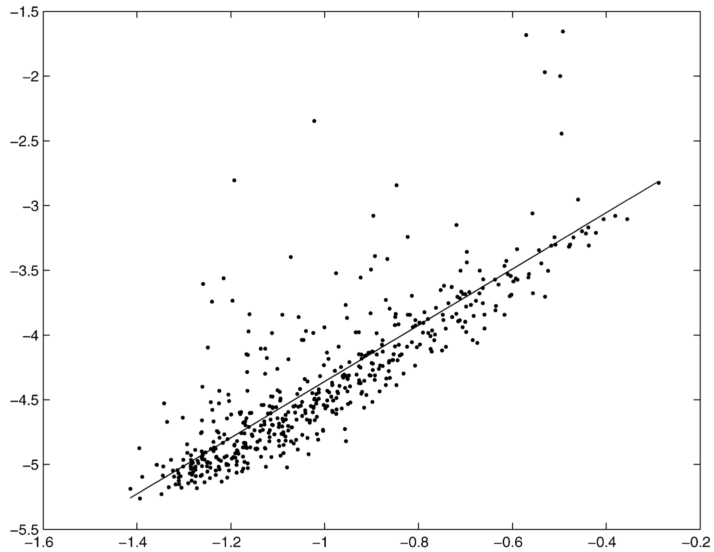

Fig. 2. $\log(\|R_H u - u_H\|_{0,H})$ versus $\log(h_{\max})$.

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