Fibonacci numbers, alternating parity sequences and faces of the tridiagonal Birkhoff polytope

C.M. da Fonseca, E. Marques de Sá

Departamento de Matemática, Universidade de Coimbra, 3001-454 Coimbra, Portugal

Received 3 May 2006; received in revised form 2 March 2007; accepted 28 March 2007
Available online 6 April 2007

Abstract

We determine the number of alternating parity sequences that are subsequences of an increasing \( m \)-tuple of integers. For this and other related counting problems we find formulas that are combinations of Fibonacci numbers. These results are applied to determine, among other things, the number of vertices of any face of the polytope of tridiagonal doubly stochastic matrices.

© 2007 Elsevier B.V. All rights reserved.

MSC: 05A15; 11B39; 52B11; 15A51

Keywords: Doubly stochastic matrix; Birkhoff polytope; Tridiagonal matrix; Number of vertices; Faces

1. Introduction

An alternating parity sequence (\( a.p. \) sequence, for short) is a (strictly) increasing sequence of integers, with a finite number of entries, such that any two adjacent entries have opposite parities. These are well-known objects in combinatorics under the name alternating subsets of integers. The first known reference on them goes back to the nineteenth century; since then, many counting results have been obtained involving sequences of this kind and of several of its generalizations. The precise references may be found in the papers [14,17,10]. Most generalizations go over, instead of the \( i \)th entry “parity”, its residue class modulo a fixed number \( m \), or modulo an \( m_j \) depending on the entry’s position; and, very often, the counting involves the sequences of a fixed length, and fixed lower and upper bounds. Our approach here is of a different kind: we fix an arbitrary increasing sequence of integers, \( \tau = (\tau_1, \tau_2, \ldots, \tau_w) \), and count the \( a.p. \) subsequences of \( \tau \) whose leftmost [rightmost] entry has a prescribed parity.

It is well known that the number \( n \) of \( a.p. \) subsequences of \( (1, 2, \ldots, k) \) is \( F_{k+3} - 2 \), where \( F_n \) is the \( n \)th Fibonacci number, determined by the usual recursion \( F_{n+2} = F_n + F_{n+1} \), with initial conditions \( F_0 = 0 \) and \( F_1 = 1 \). Our research on \( a.p. \) subsequences of \( \tau \) lead us to several counting formulas that are sums of products of Fibonacci numbers (cf. Theorems 4.3, 4.4 and 4.5). The methods range from \( \text{ad hoc} \) techniques to the use of the inclusion–exclusion principle.

The motivation for this research was the study of the facial structure of \( \Omega_n \), the polytope of all \( n \)-by-\( n \) nonnegative doubly stochastic matrices, known as transportation polytope, or the Birkhoff polytope. This has been extensively considered in the literature, lying at the crossroads of several branches of mathematics. For example, the Birkhoff polytope...
arises in the optimal assignment problem which can be seen as a special Hitchcock problem (cf. [9]). Nonnegative doubly stochastic matrices are also connected with probability theory since each row (column) can be identified as a discrete probability law (cf. [15]).

We are particularly concerned with the set $T_n$ whose elements are the $n \times n$ tridiagonal doubly stochastic matrices, which is a face of the Birkhoff polytope. The facial structure of $\Omega_n$ has been the object of a systematic study in the series of papers [3–6], and also in [2,8]. However, the tridiagonal case has interesting combinatorial peculiarities that deserve further analysis.

In [8] it is proven that $T_n$ has $F_n + 1$ vertices. In this paper, we find a closer connection of vertex counting in $T_n$ with Fibonacci numbers. In particular, our results on a.p. subsequences will be applied to determine the number of vertices of an arbitrarily given face of $T_n$. We also give an expression for the number of edges of $T_n$.

For the general theory of polytopes, and on the number, $f_d(K)$, of faces of dimension $d$ of a polytope $K$, we refer the reader to [11].

2. The faces of $T_n$

As $T_n$ is a face of $\Omega_n$, the faces of $T_n$ are the faces of $\Omega_n$ which are contained in $T_n$.

According to [3], the faces of $T_n$ are in one-to-one correspondence with the $n \times n$ tridiagonal matrices $A$, with entries 0 or 1 having total support (this means that $A$ is a Boolean sum of $n \times n$, tridiagonal, permutation matrices). In this paper, $A$ will always denote a matrix of this kind. The face of $T_n$ corresponding to $A$ is

$$F_A := \{X \in T_n : a_{ij} = 0 \Rightarrow x_{ij} = 0\}.$$  

A famous result of G. Birkhoff asserts that the vertices of $\Omega_n$ are the $n \times n$ permutation matrices (cf. [1,13]). So the vertices of $T_n$ are the tridiagonal permutation matrices; as these matrices are symmetric, all elements of $T_n$, and all 0–1 matrices $A$ to be considered in the sequel, are symmetric as well.

Taking a look at the super-diagonal entries of $A$ (i.e., the $a_{ij}$ with $j = i + 1$), we see that $A$ is a direct sum of square blocks, $A = A_1 \oplus \cdots \oplus A_p$, where each $A_t$ is of one of the following types:

* **Type 1:** $A_t = I$, a one-by-one matrix;
* **Type 2:** $A_t = K$, where $K$ is the $2 \times 2$ matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$;
* **Type 3:** $A_t$ is not of the previous two types, and all super-diagonal entries of $A_t$ are 1’s.

If $A_t$ is of type 3, then its first and last diagonal entries are 1, otherwise $A$ would not have total support. We display a $7 \times 7$ example of a type 3 matrix, where unspecified entries are 0:

$$\begin{bmatrix}
1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 0 & 1 \\
1 & 0 & 1 \\
1 & 1
\end{bmatrix}. \quad (1)$$

**Definition 2.1.** An $S$-matrix is a symmetric tridiagonal matrix of 0–1 entries, different from 1 and $K$, with all super-diagonal entries =1, and with first and last diagonal entries =1. The blocks $A_t$ of type 3 are called the $S$-blocks of $A$.

Let $B$ be an $S$-matrix of order $m$. We say that $b_{ii}$ is an inner entry of $B$, if $b_{ii} = 1$ and $1 < i < m$. (In example (1) the inner entries are $b_{33}$ and $b_{44}$.)

Note that $F_A = F_{A_1} \times \cdots \times F_{A_p}$, and $F_{A_t}$ is a singleton if $A_t$ is a block of type 1 or 2. Now, if we omit these singleton faces from the cartesian product, and reorder the $F_{A_t}$ corresponding to the $S$-blocks $A_t$, we get a polytope that is affinely isometric to $F_A$ (with respect to the usual inner product for square matrices: $\langle U | V \rangle = \text{tr}(UV^T)$). This implies that only the multi-set of $S$-blocks of $A$ matters in the study of a single face $F_A$. 


As a simple example of this we determine the dimension of $\mathcal{F}_A$. Recall that an $n \times n$ matrix, with $n > 1$, is fully indecomposable if it cannot be brought to the form
\[
\begin{bmatrix}
* & O \\
* & *
\end{bmatrix}
\]
by a permutation of its rows and a permutation of its columns, where $O$ is a $p \times q$, nonempty, zero matrix with $p + q = n$.

Lemma 2.2. Any S-matrix has total support and is fully indecomposable.

Proof. For $m > 1$, let $H$ be the $m \times m$ matrix defined by $h_{ij} = 1$ if and only if: $|i - j| = 1$, or $i = j \in \{1, m\}$ (the $m \times m$ S-matrix with minimum number of 1’s). Clearly, $H$ has total support, and if we transform a 0 diagonal entry into a 1 we also get a matrix of total support. As the Boolean sum of matrices of total support has total support, we may conclude that any S-matrix has total support.

The matrix $H$ gives rise to a simple bipartite graph, with $2m$ vertices, $u_1, \ldots, u_m, v_1, \ldots, v_m$, and the unordered pair $(u_i, v_j)$ is an edge of the graph iff $h_{ij} = 1$. This graph is obviously connected. So any S-matrix gives rise to a connected graph. Therefore, by [7, Theorem 4.2.7], any S-matrix is fully indecomposable. \qed

According to [3, Theorem 2.5], if $B$, of order $m$, is fully indecomposable, $\dim \mathcal{F}_B = \sigma_B - 2m + 1$, where $\sigma_B$ is the number of 1’s in $B$. So, for an S-matrix $B$, $\dim \mathcal{F}_B = 1 + w$, where $w$ is the number of inner entries of $B$. As $\dim \mathcal{F}_A$ is the sum of the dim $\mathcal{F}_{A_i}$ for the S-blocks of $A$, we get:

Lemma 2.3. $\dim \mathcal{F}_A = s_A + \iota_A$, where $s_A$ is the number of S-blocks of $A$, and $\iota_A$ is the sum of the numbers of inner entries in the S-blocks of $A$.

3. Vertices of $\mathcal{F}_n$ and alternating parity sequences

We shall give explicit formulas for $f_0(\mathcal{F}_A)$ (cf. [11]), the number of vertices of the face $\mathcal{F}_A$. A vertex of $\mathcal{F}_A$ is a permutation matrix which is entrywise $\leq A$, so $f_0(\mathcal{F}_A)$ is the permanent of $A$ (cf. [3]). Clearly, per($A$) is the product of the permanents of the S-blocks of $A$, so we first determine the permanent of an S-matrix.

For any increasing sequence of integers, $\tau = (\tau_1, \tau_2, \ldots, \tau_w)$, we denote by $N(\tau)$, or $N_{\tau}$, the number of nonempty a.p. subsequences of $\tau$.

Lemma 3.1. Let $B$ be an S-matrix. The number of vertices of $\mathcal{F}_B$ is $N_{\tau} + 2$, where $\tau$ is the (strictly) increasing sequence of the positions of the inner entries of $B$ (so $1 < \tau_1 < \cdots < \tau_w$, $n$, and $b_{\tau_i, \tau_i} = 1$).

Proof. There are exactly 2 permutation matrices $\leq B$ with no inner entries. So we have to prove that $\mathcal{M}$, the set of the permutation matrices $\leq B$ with at least one inner entry, has cardinality $N_{\tau}$.

Let $\mathcal{F}(P)$ be the increasing sequence of the positions of the inner entries of $P \in \mathcal{M}$. $P$ is a direct sum of blocks, each of which is either 1 or $K$. In between two consecutive blocks 1 of $P$, there are only blocks $K$ (possibly none); so the difference between the positions of these consecutive diagonal entries of $P$ is odd. Therefore, $\mathcal{F}(P)$ is an a.p. sequence.

Next, given an a.p. sequence $\gamma$, $1 \leq \gamma_1 < \cdots < \gamma_w \leq n$, the previous argument makes the converse clear: that there exists a unique permutation matrix $P_{\gamma}$ such that $\mathcal{F}(P_{\gamma}) = \gamma$. Moreover, $\gamma$ is a nonempty subsequence of $\tau$ if and only if $P_{\gamma} \in \mathcal{M}$. So $\mathcal{M}$ has cardinality $N_{\tau}$. \qed

Examples. For an S-matrix $B$:

(a) If $B$ has no inner entry, $N_{\tau} = 0$, and so $\mathcal{F}_B$ has 2 vertices.
(b) If $B$ has only one inner entry, $N_{\tau} = 1$, and so $\mathcal{F}_B$ is a triangle.
(c) Suppose $B$ has exactly two inner entries, $b_{\tau_1, \tau_1} = b_{\tau_2, \tau_2} = 1$, $\tau_1 < \tau_2$. We have two cases: (i) $\tau_2 - \tau_1$ is even; then the two singletons, ($\tau_1$), ($\tau_2$), are the only a.p. subsequences of $\tau$. So $\mathcal{F}_B$ has four vertices. (ii) $\tau_2 - \tau_1$ is odd; in this case, $N_{\tau} = 3$, so $\mathcal{F}_B$ has five vertices.
4. Alternating parity sequences and Fibonacci numbers

In this section we prove closed formulas for $N_\tau$ in terms of Fibonacci numbers, where $\tau = (\tau_1, \tau_2, \ldots, \tau_w)$ is an arbitrary increasing sequence of integers.

The variable $X$ [$Y$] may take one of three values, $A$, $E$, $O$, meaning “any”, “even”, “odd”, respectively. So, the expression “$X$ number” means “any number”, “even number”, or “odd number”, according to the current value of $X$. The symbol $\overline{X}$ denotes the opposite of $X$, i.e.,

$$\overline{E} = O, \quad \overline{O} = E, \quad \overline{A} = A.$$ 

For $X \neq Y$, the symbol $XY(\tau)$ (or just $XY_\tau$) denotes the number of all a.p. subsequences of $\tau$ started with an $X$ number and ended with a $Y$ number, including the empty sequence; if $X = Y$, $XY(\tau)$ is the number of nonempty a.p. subsequences of $\tau$ started and ended with an $X$ number. The curly notation $\mathcal{C} \mathcal{Y}(\tau)$, or $\mathcal{C} \mathcal{Y}_\tau$, denotes the set of the a.p. subsequences of $\tau$ started with an $X$ number and ended with an $Y$ number, including the empty sequence in case $X \neq Y$. So $XY_\tau$ is the cardinality of $\mathcal{C} \mathcal{Y}_\tau$.

For example, $\mathcal{C} \mathcal{Y}(\tau)$ is the set of a.p. subsequences of $\tau$ started with an even number and ended with an odd number, and $\emptyset \in \mathcal{C} \mathcal{Y}(\tau)$; $\mathcal{A} \mathcal{A}_\tau$ is the set of all nonempty a.p. subsequences of $\tau$, and $\mathcal{A} \mathcal{A}_\tau = N_\tau$. Note that, for $X \in \{O, E\}$:

$$AX_\tau = OX_\tau + EX_\tau, \quad XA_\tau = XO_\tau + XE_\tau, \quad AA_\tau = AO_\tau + AE_\tau - 2. \quad (2)$$

(The “$-2$” in the last equation comes from our conventions on the empty sequence.) In the sequel, we denote by $L_\tau$ [$R_\tau$] the parity of the leftmost [resp., rightmost] entry of $\tau$.

Given a subsequence of $\tau$, say $\kappa = (\kappa_1, \ldots, \kappa_r)$, and an integer $z$, the $z$-reverse of $\kappa$, denoted $\kappa^z$, is defined by $\kappa^z_i := z + 1 - \kappa_{r+1-i}$, for $i = 1, \ldots, r$. If $z$ is odd [even], then $z$-reversion preserves [resp., reverses] parity, in the sense that $\kappa^z_i$ and $\kappa^z_{r+1-i}$ have the same parity [resp., opposite parity], for all $i$. Clearly, $z$-reversion is an involution that maps the set of a.p. subsequences of $\tau$, onto the set of a.p. subsequences of $\tau^z$. In the sequel, we shall use the following identities with no further comment: if $z$ is odd, $[\mathcal{C} \mathcal{Y}(\tau)]^z = \mathcal{C} \mathcal{Y}^z(\tau^z)$; and, if $z$ is even, $[\mathcal{C} \mathcal{Y}(\tau)]^z = \mathcal{C} \mathcal{Y}(\tau^z)$.

To determine $XY(\tau)$ in case $\tau$ is the sequence $(1, 2, \ldots, w)$, we use the notation $XY_w := XY(1, 2, \ldots, w)$. The sequences $OO_w$, $OE_w$ satisfy the following recursions and initial conditions:

$$OO_1 = 1; \quad OE_1 = 1; \quad OE_2 = 2; \quad (3)$$

$$OO_w = OE_{w-1} + OO_{w-2}, \quad OE_w = OE_{w-1} \quad \text{for odd } w > 2; \quad (4)$$

$$OE_w = OO_{w-1} + OE_{w-2}, \quad OO_w = OO_{w-1} \quad \text{for even } w > 2. \quad (5)$$

The initial conditions (3) are trivial to check. To prove the first equation in (4), note that any $\gamma$ in the set $\mathcal{C} \mathcal{C}(1, 2, \ldots, w)$ is of one of the following mutually exclusive types: (i) $\gamma$ ends up with $w$; (ii) $\gamma$ ends with an odd number $\leq w - 2$. Clearly, there are $OE_{w-1}$ sequences of type (i); and $OO_{w-2}$ sequences of type (ii). The second identity in (4) is obvious. To prove (5) we argue in a similar manner.

The recursion (3)–(5) determines uniquely the $OO_w$ and $OE_w$; if we replace $OO_w$ and $OE_w$ by the following values

$$OO_w = F_{w+1}, \quad OE_w = F_w \quad \text{for odd } w \geq 1, \quad (6)$$

$$OO_w = F_w, \quad OE_w = F_{w+1} \quad \text{for even } w \geq 1, \quad (7)$$

then (3)–(5) are satisfied for all $w \geq 1$; therefore, $OO_w$ and $OE_w$ are given by (6)–(7). To determine $EE_w$, $EO_w$, $EA_w$, we $z$-reverse (6)–(7) to get: $EO_w = F_w$, for odd $w \geq 1$; and $EE_w = F_w$ for even $w \geq 1$. Then we use the identities $EO_w = EO_{w-1}$, for even $w$, and $EE_w = EE_{w-1}$ for odd $w$, to get the remaining values of $EE_w$ and $EO_w$. And we get

$$EE_w = F_{w-1}, \quad EO_w = F_w \quad \text{for odd } w \geq 1,$$

$$EE_w = F_w, \quad EO_w = F_{w-1} \quad \text{for even } w \geq 1.$$
Now the numbers $X_A w$, $X w$ and $A w$ are obtained at once from (2):

$$OA_w = F_{w+2} \quad EA_w = F_{w+1},$$
$$AO_w = F_{w+2} \quad AE_w = F_{w+1}, \quad \text{for odd } w \geq 1,$$
$$AO_w = F_{w+1} \quad AE_w = F_{w+2}, \quad \text{for even } w \geq 1,$$

and $AA_w = F_{w+3} - 2$. From this we get, with an easy proof:

**Theorem 4.1.** Let $\gamma = (\gamma_1, \ldots, \gamma_w)$ be an a.p. sequence. Recall $L_\gamma [R_\gamma]$ is the parity of the leftmost [resp., rightmost] entry of $\gamma$. For $X, Y \in \{E, O\}$, we have

$$XY_\gamma = \begin{cases} 
F_{w+1} & \text{if } X = L_\gamma \text{ and } Y = R_\gamma, \\
F_w & \text{if } X \neq L_\gamma \text{ and } Y = R_\gamma, \\
F_w & \text{if } X = L_\gamma \text{ and } Y \neq R_\gamma, \\
F_{w-1} & \text{if } X \neq L_\gamma \text{ and } Y \neq R_\gamma.
\end{cases} \quad (8)$$

$$XA_\gamma = \begin{cases} 
F_{w+2} & \text{if } X = L_\gamma, \\
F_{w+1} & \text{if } X \neq L_\gamma.
\end{cases} \quad (9)$$

$$AY_\gamma = \begin{cases} 
F_{w+2} & \text{if } Y = R_\gamma, \\
F_{w+1} & \text{if } Y \neq R_\gamma.
\end{cases}$$

4.1. Homogeneous formulas

We now fix two values, $L$ and $R$, in the set $\{E, O\}$, and seek a formula for $LR_\gamma$, the cardinality of $L R_\gamma$ (recall this is the set of the a.p. subsequences of $\gamma$ beginning [ending] with an $L$ [resp., $R$] number).

We may represent $\tau$ as a concatenation

$$\tau = \gamma_1 \gamma_2 \cdots \gamma_m,$$  \quad (10)

where $\gamma_i$ is a nonempty a.p. subsequence of $\gamma$, such that the concatenation $\gamma_i \gamma^{i+1}$ is not an a.p. sequence, for $1 \leq i < m$. So $\gamma'$ is made up of consecutive entries of $\gamma$ of alternating parities, and it is maximal under these conditions. The $\gamma'$'s are uniquely determined, and called a.p. components of $\gamma$. The length of $\gamma'$ will be denoted by $r_i$. The formulas given in Theorems 4.3 and 4.4 for $LR_\gamma$ are homogeneous in the sense that they are sums of $m$-fold products of Fibonacci numbers.

Any $\gamma \in L R_\gamma$ will be represented, according to the a.p. decomposition (10) of $\gamma$, as a concatenation

$$\gamma = \gamma_1 \gamma_2 \cdots \gamma_m,$$

where $\gamma_i$ is a, possibly empty, subsequence of $\gamma'$. To each such $\gamma$ we associate a sequence of $m + 1$ parities,

$$P_\gamma = (Z_1, Z_2, \ldots, Z_{m+1}),$$

satisfying the condition

$$\gamma_i \in Z_i \bar{Z}_{i+1}(\gamma') \quad \text{for } 1 \leq i \leq m. \quad (11)$$

In case $\gamma_i$ is nonempty, (11) implies that $Z_i [\bar{Z}_{i+1}]$ is the parity of the first [resp., last] entry of $\gamma_i$. But if $\gamma_i$ is empty, (11) only says that $Z_i = Z_{i+1}$. In any case the “boundary conditions”

$$Z_1 = L \quad \text{and} \quad Z_{m+1} = \bar{R}$$

together with (11), determine $P_\gamma$ in a unique way. To show this, suppose $p$ and $q$ are integers such that $\gamma_i$ is empty, for $p < i < q$, and $\gamma^p$ and $\gamma^q$ are nonempty. Then $Z_i = Z_{i+1}$ for $p < i < q$. On the other hand, $\bar{Z}_{p+1}$ is the parity, $R_{\gamma^p}$, of the rightmost entry of $\gamma^p$; so $\bar{Z}_i = R_{\gamma^p}$, for $p < i \leq q$. Note that the value $Z_q = \bar{R}_{\gamma^p}$ agrees with the fact that the last
entry of $\gamma^p$ and the first entry of $\gamma^q$, are consecutive entries of $\gamma$. The case when $\gamma^1 [\gamma^m]$ is empty is similarly treated, taking (12) into account. Note also that, if $\gamma$ is empty, then $R$ is opposite to $L$, and $Z_i = L$ for all $i$.

Let us denote by $\mathcal{S}$ the set of all sequences of parities
\[
S = (X_1, \ldots, X_{m+1}),
\]
and by $\mathcal{S}(L, R)$ the set of the elements of $\mathcal{S}$ satisfying $X_1 = L$ and $X_{m+1} = R$. The mapping defined above, $\gamma \mapsto P_{\gamma}$, maps $\mathcal{S}(L, R)$ onto $\mathcal{S}(L, R)$. Given (13), how many $\gamma \in \mathcal{S}(L, R)$ satisfy $P_{\gamma} = S$? The answer is, obviously, $\prod_{i=1}^{m} X_i X_{i+1}(\xi')$.

This implies the following formula
\[
LR_T = \sum_{S \in \mathcal{S}(L, R)} \prod_{i=1}^{m} X_i X_{i+1}(\xi'),
\]
where the sum is extended to all sequences (13) in $\mathcal{S}(L, R)$. In the next theorem, we express $LR_T$ as a combination of Fibonacci numbers.

**Theorem 4.3.** For any increasing integer sequence $\tau$, with $m$ a.p. components of lengths $r_1, \ldots, r_m$, we have
\[
LR_T = \sum_{\omega \in \Psi(u, v)} \prod_{i=1}^{m} F_{r_i + \omega_i},
\]
where $u = 1 [u = -1]$ if $L = L_T$ [resp., $L \neq L_T$], and $v = 1 [v = -1]$ if $R = R_T$ [resp., $R \neq R_T$].

**Proof.** For any $S = (X_1, \ldots, X_{m+1})$, Theorem 4.1 implies
\[
X_i X_{i+1}(\xi') = F_{r_i + \varepsilon_i(S)},
\]
where the coefficients $\varepsilon_i(S)$ are given according to the table (8):
\[
\varepsilon_i(S) = \begin{cases} 
1 & \text{if } X_i = L_{\alpha'} \text{ and } X_{i+1} \neq L_{\alpha' + 1}, \\
0 & \text{if } X_i \neq L_{\alpha'} \text{ and } X_{i+1} \neq L_{\alpha' + 1}, \\
0 & \text{if } X_i = L_{\alpha'} \text{ and } X_{i+1} = L_{\alpha' + 1}, \\
-1 & \text{if } X_i \neq L_{\alpha'} \text{ and } X_{i+1} = L_{\alpha' + 1}.
\end{cases}
\]

We may then write (14) as
\[
LR_T = \sum_{S \in \mathcal{S}(L, R)} \prod_{i=1}^{m} F_{r_i + \varepsilon_i(S)},
\]
and we are left with the proof that $\Psi(u, v)$ is precisely the set of $m$-tuples ($\varepsilon_1(S), \varepsilon_2(S), \ldots, \varepsilon_m(S)$) occurring on the right-hand side of (16), and that these $m$-tuples occur with no repetition. So we examine in detail the mapping
\[
\varepsilon(S) = (\varepsilon_1(S), \varepsilon_2(S), \ldots, \varepsilon_m(S)).
\]

Assume that $\varepsilon_i = 1$, that is, $X_i = L_{\alpha'}$ and $X_{i+1} \neq L_{\alpha' + 1}$. Therefore $\varepsilon_{i+1}$ is either 0 or $-1$; if it is 0, then $\varepsilon_{i+2}$ is either 0 or $-1$; and if $\varepsilon_{i+2}$ is also 0, then $\varepsilon_{i+3}$ is either 0 or $-1$; etc. So, by induction, we see that if $\varepsilon_i = 1$, the next nonzero $\varepsilon_j$ equals $-1$. Similarly we obtain: if $\varepsilon_i = -1$, the next nonzero $\varepsilon_j$ equals 1. This means that $\varepsilon$ maps $\{E, O\}^{m+1}$ into $\Psi$. 

To determine the kernel of $\varepsilon$, let $\varepsilon(S) = 0$. From (15) we have the alternative: (a) $X_1 \not= L_{2^1}$ and $X_2 \not= L_{2^2}$; or (b) $X_1 = L_{2^1}$ and $X_2 = L_{2^2}$. In case (a), $X_2 \not= L_{2^2}$ and $\varepsilon_2(S) = 0$ imply $X_3 \not= L_{2^3}$; in this way, we may prove by induction that $X_i \not= L_{2^{i+1}}$, for all $i$; therefore, $S = (L_{2^1}, \ldots, L_{2^{m+1}})$. In case (b) a similar argument proves $S = (L_{2^1}, \ldots, L_{2^{m+1}})$. By now, it is obvious that $\varepsilon$ transforms $(L_{2^1}, \ldots, L_{2^{m+1}})$ and $(L_{2^1}, \ldots, L_{2^{m+1}})$ into 0. So these two $m$-tuples form the kernel of $\varepsilon$.

Now let $\omega$ be a nonzero element of $\Psi$, and let $\{i_1, \ldots, i_t\}$ be the support of $\omega$, $i_1 < \cdots < i_t$. Partition the integer interval $[0, m + 1]$ into $t + 1$ subintervals, $J_k := [i_k, i_{k+1}]$, $k = 0, \ldots, t$ (with $i_0 := 0, i_{t+1} := m + 1$). Now suppose that $\omega_{i_j} = 1$ (the case $\omega_{i_j} = -1$ is analogous). Define $Z_t = L_{2^t} [Z_t = L_{2^t}]$ for $i$ inside the intervals $J_k$ with odd [resp., even] $k$. It is easy to check that $\varepsilon(Z_1, \ldots, Z_{m+1}) = \omega$. So $\varepsilon$ is onto $\Psi$.

Note that $\Psi$ has $2^{m+1} - 1$ elements, one less than $[E, O]^{m+1}$. As the kernel of $\varepsilon$ has two elements, $\varepsilon$ has to be injective outside its kernel.

Let $S = (X_1, \ldots, X_{m+1}) \in \mathcal{F}(L, R)$. By definition (13) have $X_1 = L, X_{m+1} = R$. The proof here splits into four cases, according to the value of the pair $(u, v)$. These cases are quite similar to each other, and so we chose to consider only one, namely: $(u, v) = (-1, 1)$, that is, the parity $L$ is opposed to the parity of the first entry of $\tau$, and $R$ is the parity of the last entry of $\tau$. So, in table (15), we enter $X_1 \not= L_{2^1}$ and $X_{m+1} \not= L_{2^{m+1}}$. We get: $\varepsilon_1(S)$ is either 0 or $-1$; if it is 0, then $\varepsilon_2(S)$ is either 0 or $-1$; etc. So, either $\varepsilon(S) = 0$ or the first nonzero entry of $\varepsilon(S)$ is $-1$. Now enter $X_{m+1} \not= L_{2^{m+1}}$, to obtain: $\varepsilon_m(S)$ is either 0 or 1; if it is 0, then $\varepsilon_{m-1}(S)$ is either 0 or 1; etc. So, either $\varepsilon(S) = 0$ or the last nonzero entry of $\varepsilon(S)$ is 1. So we proved $\varepsilon(S)$ belongs to $\Psi(-1, 1)$. And in general we have $\varepsilon(S) \in \Psi(u, v)$, for the prescribed $(u, v)$.

Finally, note that one of the kernel elements of $\varepsilon$, determined above, may lie in $\mathcal{F}(L, R)$, but not both (for fixed $L, R$). Therefore, $\varepsilon$ is one-to-one from $\mathcal{F}(L, R)$ into $\Psi(u, v)$; it is in fact onto $\Psi(u, v)$, because $\mathcal{F}(L, R)$ and $\Psi(u, v)$ both have $2^{m-1}$ elements. This ends the proof. $\Box$

**Theorem 4.4.** With the notation of Theorem 4.3, we have

$$LA_\tau = \sum_{\omega \in \Psi(u, 1) \cup \Psi(-1, 1)} \prod_{i=1}^{m} F_{r_i + c_{\omega_i}}, \quad AR_\tau = \sum_{\omega \in \Psi(1, v) \cup \Psi(-1, v)} \prod_{i=1}^{m} F_{r_i + c_{\omega_i}}$$

and

$$N_\tau = \prod_{i=1}^{m} F_{r_i} + \sum_{\omega \in \Psi} \prod_{i=1}^{m} F_{r_i + c_{\omega_i}} - 2.$$

**Proof.** The formulas are direct consequences of Theorem 4.3, combined with (2), and the fact that $\Psi(u, 1) \cup \Psi(u, -1)$ and $\Psi(1, v) \cup \Psi(-1, v)$ are disjoint unions. For the last formula, take into account that these two sets have union $\Psi$, and have exactly one element in common: the zero $m$-tuple. $\Box$

### 4.2. Inclusion–exclusion formulas

Other formulas for $LR_\tau$, $LA_\tau$, $AR_\tau$ and $N_\tau$ may be obtained based on the well-known inclusion–exclusion theorem. Let us go back to (10), the a.p. decomposition of $\tau$, where the a.p. component $a^j$ has length $r_j$. The number $r_1 + \cdots + r_m + m - 1$ is denoted by $M$; the numbers $c_0, \ldots, c_m$, given by

$$c_k := r_1 + r_2 + \cdots + r_k + k,$$

are called the gaps of $\tau$ (note that $c_0 = 0$, $c_m = M + 1$).

**Theorem 4.5.** With the notation just introduced, we have

$$LR_\tau = F_{M+m+u+v-1} + \sum_{i=1}^{m-1} (-1)^i \sum_{0 < k_1 < \cdots < k_i < m} F_{e_k} + u - 1 F_{M-e_{k_i}+v} \prod_{0 < c_i < t} F_{c_{k_i+1} - c_{k_i}},$$

$$LA_\tau = F_{M+m+u+1} + \sum_{i=1}^{m-1} (-1)^i \sum_{0 < k_1 < \cdots < k_i < m} F_{e_k} + u - 1 F_{M-e_{k_i}+2} \prod_{0 < c_i < t} F_{c_{k_i+1} - c_{k_i}},$$
We now prove the following formula:

\[ A R_\tau = F_{M+u+1} + \sum_{t=1}^{m-1} (-1)^t \sum_{0<k_1<\cdots<k_t<m} F_{c_{k_1}+1} F_{M-c_{k_t}+u} \prod_{0<i<t} F_{c_{k_i+1}-c_{k_i}}, \]

\[ N_\tau = F_{M+3} + \sum_{t=1}^{m-1} (-1)^t \sum_{0<k_1<\cdots<k_t<m} F_{c_{k_1}+1} F_{M-c_{k_t}+2} \prod_{0<i<t} F_{c_{k_i+1}-c_{k_i}} - 2, \]

where \( u = 1 \) if \( L = L_\tau \), \( u = 0 \) if \( L \neq L_\tau \), \( v = 1 \) if \( R = R_\tau \), \( v = 0 \) if \( R \neq R_\tau \).

**Proof.** To prove the theorem we assume, without loss of generality, that \( \tau_1 \) is odd. (If \( \tau_1 \) is even, then the proof goes the same way with appropriate reversion of parities.) And also assume, without loss of generality, that the entries of \( \tau \) are the smallest positive integers compatible with the conditions that \( \tau \) is an increasing sequence, and the a.p. components of \( \tau \) have lengths \( r_1, \ldots, r_m \) (from left to right), that is

\[ \gamma^i = (c_{i-1} + 1, c_{i-1} + 2, \ldots, c_{i-1} + r_i). \]

So, in this context, \( M \) is the maximum entry of \( \tau \), and the gaps of \( \tau \) are the elements of \( \{0, 1, \ldots, M + 1\} \) that are not entries of \( \tau \).

We only prove the formula for \( LR_\tau \); the others may be obtained from this one as in Corollary 4.4.

For \( 0 < k_1 < \cdots < k_t < m \), we let \( W_{k_1 \ldots k_t} \) be the number of elements of \( \mathcal{LR}(1, 2, \ldots, M) \) having all \( t \) gaps \( c_{k_1}, \ldots, c_{k_t} \) among its entries; and we denote by \( W(t) \) the sum of all \( W_{k_1 \ldots k_t} \) for a fixed \( t \). For example, \( W(0) \) is the number of elements of \( \mathcal{LR}(1, \ldots, M) \); table (8) gives its value

\[ W(0) = F_{M+u+v-1}, \]  

where \( u \) and \( v \) are as given in the theorem’s statement.

According to the inclusion–exclusion formula [16, Theorem 1.2, p. 19], the number of elements of \( \mathcal{LR}(1, 2, \ldots, M) \) is given by

\[ LR_\tau = W(0) - W(1) + W(2) - \cdots + (-1)^{m-1} W(m - 1). \]  

(18)

We now prove the following formula:

\[ W_{k_1 \ldots k_t} = F_{c_{k_t}+u-1} \left( \prod_{0<i<t} F_{c_{k_i+1}-c_{k_i}} \right) F_{M-c_{k_t}+v}, \]  

(19)

for positive \( t \). It is easy to describe how one can generate all elements of \( \mathcal{LR}(1, \ldots, M) \) that have \( c_{k_1}, \ldots, c_{k_t} \) among their entries. Each sequence \( \gamma \) has the following structure:

\[ \gamma = I^0(c_{k_1}) I^1(c_{k_2}) I^2 \cdots (c_{k_t}) I^t, \]  

(20)

i.e., \( \gamma \) is a concatenation of \( t + 1 \) sequences \( I^i \), and the \( t \) singleton \( s (c_{k_i}) \). To specify the parities of the extreme entries of each \( I^i \), let \( X_i \) be the parity of \( c_i + 1 \), for \( 0 < i < m \). For notational reasons, we define \( k_0 = 0 \), \( k_{t+1} = m \), \( X_0 = L \) and \( X_m = R \). Then \( I^i \) is an arbitrary a.p. sequence in the integer interval \( [c_{k_i} + 1, c_{k_{i+1}} - 1] \), beginning with an \( X_{k_i} \) number, and ending with an \( X_{k_{i+1}} \) number. Therefore, the number of possible \( I^i \)’s is

\[ X_{k_i} X_{k_{i+1}} (c_{k_i} + 1, c_{k_i} + 2, \ldots, c_{k_{i+1}} - 1). \]  

(21)

Let \( \ell_i \) denote the length of the a.p. sequence \( (c_{k_i} + 1, c_{k_i} + 2, \ldots, c_{k_{i+1}} - 1) \). Clearly, \( \ell_i = c_{k_{i+1}} - c_{k_i} - 1 \). To apply Theorem 4.1 we consider three cases:

*Case 1:* \( 0 < i < t \). We are in the first instance of (8), therefore (21) = \( F_{\ell_i+1} \).

*Case 2:* \( i = 0 \). We have to determine \( L X_{k_1} (1, 2, \ldots, \ell_0) \); table (8) yields (21) = \( F_{\ell_0+1} \) if \( L \) is odd, and (21) = \( F_{\ell_0} \) if \( L \) is even. As we are assuming \( L_\tau \) is odd, we get (21) = \( F_{\ell_0+u} \) with \( u \) as given in the statement of the theorem.

*Case 3:* \( i = t \). We determine \( X_{k_t} R (c_{k_t} + 1, c_{k_t} + 2, \ldots, c_{k_t} + \ell_t) \); as \( X_{k_t} \) is the parity of \( c_{k_t} + 1 \), (8) yields (21) = \( F_{\ell_t+1} \) if \( R = R_\tau \), and (21) = \( F_{\ell_t} \) if \( R \neq R_\tau \). So we get (21) = \( F_{\ell_t+v} \) with \( v \) as in the statement of the theorem.
As the $I^i$’s may vary independently of each other, the number of all a.p. sequences (20) is the product of these $t + 1$ Fibonacci numbers, namely

$$F_{\ell_0+u} \prod_{0 < i < t} F_{\ell_i+1} F_{\ell_i+u}.$$ 

As $\ell_0 = c_{k_1} - 1$ and $\ell_t = M - c_{k_t}$, this is precisely the right-hand side of (19) in a different notation. The first formula of the theorem results by entering (17) and (19) in (18). □

**Remark 4.6.** To have the flavor of the preceding results, we exhibit the formulas for $N_\tau$, for small values of $m$. For $m = 1$, we have

$$N_\tau = F_{r_1} + F_{r_1} + F_{r_1+1} + F_{r_1-1} - 2$$

$$= F_{r_1+3} - 2. \quad (22)$$

Formula (22) is what we get from Theorem 4.4, and (23) is from Theorem 4.5. For $m = 2$, we also give the two formulas for $N_\tau$, according to Theorems 4.4 and 4.5, respectively:

$$N_\tau = F_{r_1} F_{r_2} + F_{r_1} F_{r_2} + F_{r_1+1} F_{r_2} + F_{r_1} F_{r_2+1}$$

$$+ F_{r_1-1} F_{r_2} + F_{r_1} F_{r_2-1} + F_{r_1+1} F_{r_2-1} + F_{r_1-1} F_{r_2+1} - 2$$

$$= F_{r_1+r_2+4} - F_{r_1+2} F_{r_2+2} - 2. \quad (23)$$

For $m = 3$, Theorems 4.4 and 4.5 offer the following two formulas:

$$N_\tau = F_{r_1} F_{r_2} F_{r_3} + F_{r_1} F_{r_2} F_{r_3+1} + F_{r_1+1} F_{r_2} F_{r_3} + F_{r_1} F_{r_2+1} F_{r_3}$$

$$+ F_{r_1} F_{r_2} F_{r_3} + F_{r_1-1} F_{r_2} F_{r_3} + F_{r_1+1} F_{r_2} F_{r_3-1} + F_{r_1} F_{r_2-1} F_{r_3+1}$$

$$+ F_{r_1+1} F_{r_2} F_{r_3} + F_{r_1} F_{r_2-1} F_{r_3} + F_{r_1-1} F_{r_2+1} F_{r_3} + F_{r_1+1} F_{r_2-1} F_{r_3}$$

$$+ F_{r_1} F_{r_2+1} F_{r_3} + F_{r_1} F_{r_2} F_{r_3-1} + F_{r_1-1} F_{r_2} F_{r_3+1} + F_{r_1-1} F_{r_2+1} F_{r_3-1} - 2$$

$$= F_{r_1+r_2+r_3+5} - F_{r_1+2} F_{r_2+r_3} - F_{r_1+r_2+3} F_{r_3+2} + F_{r_1+2} F_{r_2+1} F_{r_3+2} - 2. \quad (24)$$

Later on we need the following instances of Theorem 4.5 when $m = 2$ and the first entry of $\tau$ is odd (note that $R_\tau$ is, therefore, the parity of $r_1 + r_2 + 1$. So $v = 0$ if $R$ has [resp., has not] the same parity as $r_1 + r_2$):

$$\text{AO}(x^1 x^2) = \begin{cases} F_{r_1+r_2+2} - F_{r_1+2} F_{r_2} & \text{if } r_1 + r_2 \text{ is odd,} \\ F_{r_1+r_2+3} - F_{r_1+2} F_{r_2+1} & \text{if } r_1 + r_2 \text{ is even,} \end{cases} \quad (25)$$

We may change slightly the indices of “$F$” in (24)–(25), by using the following well-known identities on Fibonacci numbers (see, e.g., [18]):

$$F_{r_1+r_2+2} - F_{r_1+2} F_{r_2} = F_{r_1+r_2+1} - F_{r_1} F_{r_2-2},$$

$$F_{r_1+r_2+3} - F_{r_1+2} F_{r_2+1} = F_{r_1+r_2+2} - F_{r_1} F_{r_2-1}. \quad (26)$$

### 4.3. An upper bound to $N_\tau$

Assume, for a moment, that $\tau$ is not an a.p. sequence. Then it may be represented as

$$\tau = x^1 x^2 \gamma.$$ \hfill (27)

We are assuming that $\tau$ is not an a.p. sequence; $x^1$ and $x^2$ are the first two a.p. components of $\tau$ [cf. (10)], and $\gamma$ is the (eventually empty) concatenation of all other a.p. components. Let $\tau^* \gamma$ be the sequence obtained from $\tau$ by adding 1 to all entries of $x$, i.e., $\tau^* = x^* x^2 \gamma$, where $x^* := (\tau_1 + 1, \ldots, \tau_{r_1} + 1)$. Note that $x^* x^2$ is an a.p. sequence of length $r_1 + r_2$, and $\tau^*$ has one less a.p. component than $\tau$. 
Lemma 4.7. Assume the first entry of $\tau$ is odd. Then

$$N_{\tau^*} - N_{\tau} = \begin{cases} F_{r_1} F_{r_2 - 2} E A_{\gamma} + F_{r_1} F_{r_2 - 1} O A_{\gamma} & \text{if } r_1 + r_2 \text{ is odd}, \\ F_{r_1} F_{r_2 - 1} E A_{\gamma} + F_{r_1} F_{r_2 - 2} O A_{\gamma} & \text{if } r_1 + r_2 \text{ is even}. \end{cases} \quad (28)$$

Proof. We obviously have $N_{\tau} = AO(x^1 x^2) E A_{\gamma} + AE(x^1 x^2) O A_{\gamma} - 2$, and $N_{\tau^*} = AO(x^* x^2) E A_{\gamma} + AE(x^* x^2) O A_{\gamma} - 2$ (note that, when $\gamma$ is empty, $E A_{\gamma} = O A_{\gamma} = 1$, and these expressions are samples of (2)). Therefore, the difference $N_{\tau^*} - N_{\tau}$ is given by

$$[AO(x^* x^2) - AO(x^1 x^2)] E A_{\gamma} + [AE(x^* x^2) - AE(x^1 x^2)] O A_{\gamma}. \quad (29)$$

As the rightmost entry of $x^* x^2$ has the parity of $r_1 + r_2 + 1$, (9) yields

$$AO(x^* x^2) = \begin{cases} F_{r_1 + r_2 + 1} & \text{if } r_1 + r_2 \text{ is odd}, \\ F_{r_1 + r_2 + 2} & \text{if } r_1 + r_2 \text{ is even}, \end{cases}$$

$$AE(x^* x^2) = \begin{cases} F_{r_1 + r_2 + 2} & \text{if } r_1 + r_2 \text{ is odd}, \\ F_{r_1 + r_2 + 1} & \text{if } r_1 + r_2 \text{ is even}. \end{cases}$$

The lemma follows by entering, in (29), the values just obtained for $AO(x^* x^2)$ and $AE(x^* x^2)$, and the values of $AO(x^1 x^2)$ and $AE(x^1 x^2)$ given in (24)–(25) and modified according to (26). $\square$

Theorem 4.8. Let $w$ be the length of $\tau$. Then $N_{\tau} \leq F_{w+3} - 2$, with equality if and only if $\tau$ is an a.p. sequence.

Proof. If $\tau$ is not an a.p. sequence, the right-hand side of (28) is positive, so $N_{\tau} < N_{\tau^*}$. We may apply the star operation, $\tau \mapsto \tau^*$, repeatedly, and obtain $\tau^*, \tau^{**}, \tau^{***}$, ... until we get (after $m - 1$ iterations) an a.p. sequence of length $w$. This proves the theorem. $\square$

5. Back to the tridiagonal Birkhoff polytope

To apply the preceding results to a general face $\mathcal{F}_A$ of $\mathcal{I}_n$, we let $B_1, \ldots, B_q$ be the S-blocks of $A$, and assume $B_k$ has $w_k$ inner entries. We denote by $\tau^{(k)}$ the sequence of the positions of the inner entries of $B_k$ (so $\tau^{(k)}$ is an increasing integer sequence of length $w_k$). The next theorem follows easily from Lemma 3.1, and Theorems 4.4, 4.5 and 4.8.

Theorem 5.1. The number of vertices of $\mathcal{F}_A$ is

$$f_0(\mathcal{F}_A) = [N(\tau^{(1)}) + 2][N(\tau^{(2)}) + 2] \cdots [N(\tau^{(w)}) + 2],$$

where the $N(\tau^{(k)})$ is given by Fibonacci formulas as those of Theorems 4.4 and 4.5. Moreover

$$f_0(\mathcal{F}_A) \leq F_{w_1+3} F_{w_2+3} \cdots F_{w_p+3},$$

with equality if and only if all $\tau^{(k)}$ are a.p. sequences.

It is easy to determine the number of edges of $\mathcal{I}_n$, denoted $f_1(\mathcal{I}_n)$. An edge is a face $\mathcal{F}_A$ of dimension 1. So, from the dimensional formula of [3, Corollary 2.6], or its specialized form given by Lemma 2.3, we get

Proposition 5.2. $\mathcal{F}_A$ is an edge of $\mathcal{I}_n$ if and only if $A$ has only one S-block, and no inner entry.

It is easy to check that this agrees with the characterization of the pairs of vertices that form an edge of $\mathcal{I}_n$ given in [8, Theorem 2(iii)]. The matrices $A$ as in Proposition 5.2 are those of the form $U \oplus M_k \oplus V$, where $U$ and $V$ are tridiagonal permutation matrices of orders $i$ and $j$, respectively, $M_k$ is the S-matrix of order $k$ with no inner entry, and $i + j + k = n$. Here, $k$ runs over $\{2, 3, \ldots, n\}$, and $i, j \geq 0$. For each such $i, j, k$, there exist $F_{i+1}$ possible matrices $U$, and $F_{j+1}$ possible matrices $V$ (cf. [8]). Therefore,

$$f_1(\mathcal{I}_n) = \sum_{0 \leq i+j \leq n-2} F_{i+1} F_{j+1}.$$
This may be simplified, or given many different forms, by means of well-known identities involving summations of order two products of Fibonacci numbers (see, e.g., [12, 18]). Finally, we briefly consider the determination of $f_2(\mathcal{I}_n)$, the number of polygons, i.e., 2-dimensional faces $\mathcal{F}_A$, of the tridiagonal Birkhoff polytope. By Lemma 2.3, $\dim \mathcal{F}_A = 2$ splits into two cases: (') $A$ has only one S-block, which has 1 inner entry; (''') $A$ has two S-blocks, and no inner entries. Accordingly, $f_2(\mathcal{I}_n) = f_2' + f_2''$, each term corresponding to the respective case. Clearly

$$f_2' = \sum_{0 \leq i + j \leq n - 3} (n - i - j - 1) F_{i+1} F_{j+1}$$

$$f_2'' = \sum_{0 \leq i + j + k \leq n - 4} (n - i - j - k + 1) F_{i+1} F_{j+1} F_{k+1}.$$

Note that in case ('''), per $A = 4$, and so $\mathcal{F}_A$ is a quadrilateral (as a matter of fact, it is a square for the standard inner product. Hint: it is enough to consider the 4-by-4 case). In case ('), $\mathcal{F}_A$ is a triangle.

References