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Hardy-type theorem for orthogonal functions with respect to their zeros. The Jacobi weight case

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Abstract

Motivated by the G.H. Hardy's 1939 results [G.H. Hardy, Notes on special systems of orthogonal functions II: On functions orthogonal with respect to their own zeros, J. London Math. Soc. 14 (1939) 37–44] on functions orthogonal with respect to their real zeros λ_n, $n = 1, 2, \ldots$, we will consider, under the same general conditions imposed by Hardy, functions satisfying an orthogonality with respect to their zeros with Jacobi weights on the interval $(0, 1)$, that is, the functions $f(z) = z^\nu F(z)$, $\nu \in \mathbb{R}$, where F is entire and

$$\int_0^1 f(\lambda_n t) f(\lambda_m t) t^\alpha (1-t)^\beta \, dt = 0, \quad \alpha > -1 - 2\nu,\ \beta > -1,$$

when $n \neq m$. Considering all possible functions on this class we obtain a new family of generalized Bessel functions including Bessel and hyperbessel functions as special cases.

Keywords: Zeros of special functions; Orthogonality; Jacobi weights; Mellin transform on distributions; Entire functions; Bessel functions; Hyperbessel functions

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1. Introduction

In his 1939 paper [4], G.H. Hardy proved that, under certain conditions, the only functions satisfying

$$\int_0^1 f(\lambda_n t) f(\lambda_m t)\, dt = 0$$

if $m \neq n$, are the Bessel functions.

In order to extend the Hardy's results to the q-case, it was observed in [1] that a substantial part of the Hardy's argument could be carried out without virtually any change, when the measure dt is replaced by an arbitrary positive measure $du(t)$ supported on the interval (0, 1) and functions satisfying

$$\int_0^1 f(\lambda_n t) f(\lambda_m t)\, du(t) = 0 \tag{1.1}$$

if $m \neq n$, are considered.

In particular, the completeness of the set $\{f(\lambda_n t)\}$ holds for such a general orthogonal system. Furthermore, there exists an associated Lagrange-type sampling theorem for functions defined by integral transforms whose kernel is defined by means of f [1].

It is however not possible to identify, under such a degree of generalization, the whole class of functions satisfying (1.1). Therefore, we raise the following question: Given a specific positive real measure on the interval (0, 1), what are the resulting orthogonal functions? In the case where this measure is the $d_q x$ arising from the Jackson's q-integral, it was shown in [1] that the corresponding functions are the Jackson q-Bessel functions of the third type. In this note we will answer this question in the case where $du(t) = t^\alpha (1-t)^\beta\, dt$ is the measure associated with the orthogonal Jacobi polynomials on the interval (0, 1). That is, we will characterize in some sense the functions $f(z) = z^\nu F(z)$, where $\nu \in \mathbb{R}$, F is entire, satisfying

$$\int_0^1 f(\lambda_n t) f(\lambda_m t) t^\alpha (1-t)^\beta\, dt = 0, \quad \alpha > -1 - 2\nu,\ \beta > -1, \tag{1.2}$$

and belonging to the classes $\mathcal{A}$ or $\mathcal{B}$ defined as follows (the same general restrictions are imposed in [4] and [1]).

Definition 1. Let $\nu, \alpha \in \mathbb{R}$ be such that $2\nu + \alpha > -1$. The class $\mathcal{A}$ is constituted by all entire functions f of order less than two or of order two and minimal type of the form

$$f(z) = z^\nu \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right).$$

The class $\mathcal{B}$ is constituted by all entire functions f of the form

$$f(z) = z^\nu F(z),$$

where $F(z)$ is an entire function with real but not necessarily positive zeros and order less than one or order one and minimal type, with $F(0) = 1$.

In this paper we will deal mainly with functions of the class $\mathcal{B}$. Namely, we will show that these functions satisfy the Abel-type integral equation of the second kind [7] and we will study some relevant properties as well as their representation in a series expansion. Furthermore, in the last section we will derive a similar integral equation for the class $\mathcal{A}$ and will also give its explicit solutions in terms of the series.

We will appeal in the sequel to the theory of the Mellin transform [6–8]. As it is known the Mellin direct and inverse transforms are defined by the formulas

$$f^*(s) = \int_0^\infty f(x) x^{s-1}\,dx, \tag{1.3}$$

$$f(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s) x^{-s}\,ds, \quad s = \gamma + i\tau,\ x > 0, \tag{1.4}$$

where integrals (1.3)–(1.4) exist as Lebesgue ones if we assume the conditions $f \in L_1(\mathbb{R}_+; x^{\gamma-1}\,dx)$, $f^* \in L_1((\gamma - i\infty, \gamma + i\infty); d\tau)$, respectively. However functions from the classes $\mathcal{A}$ or $\mathcal{B}$ can have a non-integrable singularity at infinity and the classical Mellin transform (1.3) generally does not exist. Therefore according to [8, Chapter 4] we introduce the Mellin transform for distributions from the dual space M'_{γ_1,γ_2} to Zemanian's testing-function space M_{γ_1,γ_2} into the space of analytic functions in the open vertical strip $\Omega_f := \{s \in \mathbb{C}\colon\ \gamma_1 < \operatorname{Re} s < \gamma_2\}$ by the formula

$$f^*(s) := \langle f(x), x^{s-1} \rangle, \quad s \in \Omega_f. \tag{1.5}$$

In this case the inversion integral (1.4) is convergent in $\mathcal{D}'(\mathbb{R}_+)$, i.e. for any smooth function $\theta \in \mathcal{D}(\mathbb{R}_+) \subset M_{\gamma_1,\gamma_2}$ with compact support on $\mathbb{R}_+$ it holds

$$\langle f(x), \theta(x) \rangle = \lim_{r\to\infty} \left\langle \frac{1}{2\pi i} \int_{\gamma-ir}^{\gamma+ir} f^*(s) x^{-s}\,ds, \theta(x) \right\rangle. \tag{1.6}$$

2. The Hardy-type integral equation

Suppose that $f \in \mathcal{B}$ and satisfies (1.2). Let

$$A_n = \int_0^1 \left[f(\lambda_n t)\right]^2 t^\alpha (1-t)^\beta\,dt, \quad \alpha > -1 - 2\nu,\ \beta > -1.$$

If $a_n(z)$ stands for the Fourier coefficients of the expansion of $f(zt)$ in terms of the orthonormal basis $\{A_n^{-1/2} f(\lambda_n t)\}$, then it is possible to show, following the arguments in [4], that

$$a_n(z) = \frac{1}{A_n^{1/2}} \int_0^1 f(zt) f(\lambda_n t) t^\alpha (1-t)^\beta\,dt = \frac{A_n^{1/2} f(z)}{f'(\lambda_n)(z - \lambda_n)}. \tag{2.1}$$

In the sequel we will essentially follow the lines of [4], but since considerable number of adaptations have to be made in order to deal with the presence of the Jacobi weight, we find it better to provide the proof.

Using the Parseval identity for inner products we get

$$\int_0^1 f(zt) f(\zeta t) t^\alpha (1-t)^\beta\,dt = \sum_{n=1}^\infty a_n(z) a_n(\zeta). \tag{2.2}$$

Theorem 1. *Let $f \in \mathcal{B}$ and satisfy* (1.2) *with $\alpha > -1 - 2\nu$, $\nu \in \mathbb{R}$, $\beta > -1$. Then it satisfies the integral equation*

$$a \int_0^z u^{\nu+\alpha+1} (z-u)^\beta f(u)\,du = (az+1) \int_0^z u^{\nu+\alpha} (z-u)^\beta f(u)\,du + z^{\nu+\alpha+\beta+1} f(z) A, \tag{2.3}$$

where $a = F'(0)$, $A = -B(2\nu + \alpha + 1, \beta + 1)$, and B denotes the Beta-function [3].

Proof. Substituting (2.1) into (2.2), we obtain, after some simplifications,

$$\int_0^1 f(zt)f(\zeta t)t^\alpha(1-t)^\beta\,dt = -f(z)f(\zeta)\frac{q(z)-q(\zeta)}{z-\zeta}, \tag{2.4}$$

where

$$q(z) = \sum_{n=1}^\infty \frac{A_n}{\{f'(\lambda_n)\}^2}\left[\frac{1}{z-\lambda_n}+\frac{1}{\lambda_n}\right].$$

Letting $\zeta \to 0$ in (2.4), the result is

$$\int_0^1 t^{\nu+\alpha} f(zt)(1-t)^\beta\,dt = -\frac{f(z)q(z)}{z}.$$

The change of variables $u = zt$ gives

$$\int_0^z u^{\nu+\alpha}(z-u)^\beta f(u)\,du = -z^{\nu+\alpha+\beta} f(z)q(z). \tag{2.5}$$

Now, when z is small enough, $f(z) \backsim z^\nu$ and $q(z) \sim zq'(0)$. Therefore, when $z \to 0$ we have

$$\int_0^z u^{2\nu+\alpha}(z-u)^\beta\,du \sim -z^{2\nu+\alpha+\beta+1} q'(0)$$

and, as a consequence,

$$q'(0) = -B(2\nu+\alpha+1, \beta+1).$$

Rewriting (2.4) in the form

$$\int_0^1 t^{2\nu+\alpha} F(zt)F(\zeta t)(1-t)^\beta\,dt = -F(z)F(\zeta)\frac{q(z)-q(\zeta)}{z-\zeta}$$

and differentiating in ζ we obtain

$$\int_0^1 t^{2\nu+\alpha+1} F'(\zeta t)F(zt)(1-t)^\beta\,dt = -F(z)F'(\zeta)\frac{q(z)-q(\zeta)}{z-\zeta} - F(z)F(\zeta)\frac{-q'(\zeta)(z-\zeta)+q(z)-q(\zeta)}{(z-\zeta)^2}.$$

Setting $\zeta = 0$, it gives

$$F'(0)\int_0^1 t^{\nu+\alpha+1}(1-t)^\beta f(zt)\,dt = -\frac{f(z)}{z^2}\big[q(z)\big(F'(0)z+1\big) - Az\big].$$

So, if $a = F'(0)$, the change of variable $zt = u$ yields

$$a\int_0^z u^{\nu+\alpha+1}(z-u)^\beta f(u)\,du = -z^{\nu+\alpha+\beta} f(z)\big[(za+1)q(z) - Az\big].$$

Combining with (2.5) we get the integral equation (2.3). □

3. The Hardy-type theorem

In this section we will prove the Hardy-type theorem [4], which will characterize all possible orthogonal functions with respect to their zeros from the class $\mathcal{B}$ under the Jacobi weight. Indeed, we have

Theorem 2. *Let f satisfy conditions of Theorem 1. Then*

$$f(z) = \text{const.}\, z^{\nu} \sum_{n=0}^{\infty} a_n z^n, \tag{3.1}$$

where

$$a_n = \frac{(a(\beta+1))^n}{\Gamma(\mu+n)} \prod_{j=1}^{n} \frac{(\mu)_j}{(\mu+\beta+1)_j - (\mu)_j}, \quad n = 0, 1, \ldots. \tag{3.2}$$

Here $\mu = 2\nu + \alpha + 1$, $\Gamma(z)$ is Euler's Gamma-function, $(b)_j$ is the Pochhammer symbol [3], and the empty product is equal to 1. Moreover, the series (3.1) represents an entire function of the order $\rho = \frac{1}{\beta+2} < 1$, when $\beta > -1$. In particular, the case $\beta = 0$ leads to the Hardy solutions in terms of the Bessel functions (cf. [4, p. 43]).

Proof. Making again an elementary substitution $u = zt$ and simplifying the factor $z^{\nu+\alpha+\beta+1}$ we write (2.3) in the form

$$az \int_0^1 t^{\nu+\alpha+1}(1-t)^{\beta} f(zt)\,dt = (az+1) \int_0^1 t^{\nu+\alpha}(1-t)^{\beta} f(zt)\,dt - B(2\nu+\alpha+1, \beta+1) f(z). \tag{3.3}$$

Hence we observe that (3.3) is a second kind integral equation containing two Erdélyi–Kober fractional integration operators with linear coefficients (see [7, Chapter 3]). In order to find a solution in terms of the series $z^{\nu} \sum_{n=0}^{\infty} a_n z^n$ where $a_n \neq 0$, $a_0 = 1$, and such that no two consecutive a_n vanish (see [4, p. 43]) we substitute it in (3.3). After changing the order of integration and summation because of the uniform convergence, from the calculation of the inner Beta-integrals and some elementary substitutions we get

$$\begin{aligned} &a \sum_{n=1}^{\infty} a_{n-1} z^{n+\nu} \big[B(\mu+n, \beta+1) - B(\mu+n-1, \beta+1) \big] \\ &\quad = \sum_{n=1}^{\infty} a_n z^{n+\nu} \big[B(\mu+n, \beta+1) - B(\mu, \beta+1) \big], \quad \mu = 2\nu + \alpha + 1. \end{aligned} \tag{3.4}$$

Hence equating coefficients of the series and taking into account that $B(\mu+n, \beta+1) - B(\mu, \beta+1) \neq 0$, $n \in \mathbb{N}$, $\mu > 0$, $\beta > -1$ we obtain the following recurrence relations:

$$\begin{aligned} a_n &= a a_{n-1} \frac{B(\mu+n, \beta+1) - B(\mu+n-1, \beta+1)}{B(\mu+n, \beta+1) - B(\mu, \beta+1)} \\ &= a^n \prod_{j=1}^{n} \frac{B(\mu+j, \beta+1) - B(\mu+j-1, \beta+1)}{B(\mu+j, \beta+1) - B(\mu, \beta+1)}, \end{aligned} \tag{3.5}$$

where $n \in \mathbb{N}_0$ and the empty product is equal to 1. Finally, we use the formulas

$$(b)_k = \frac{\Gamma(b+k)}{\Gamma(b)}, \qquad \Gamma(z+1) = z\Gamma(z), \qquad B(b,c) = \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)}$$

in order to write (3.5) in the form (3.2).

Further, from (3.5) and the asymptotic formula for the ratio of Gamma-functions [3] we get

$$\begin{aligned}\left|\frac{a_n}{a_{n-1}}\right| &= \left|a\frac{B(\mu+n,\beta+1)-B(\mu+n-1,\beta+1)}{B(\mu+n,\beta+1)-B(\mu,\beta+1)}\right| \\ &= \frac{|a|\Gamma(\beta+2)\Gamma(\mu+n-1)}{\Gamma(\mu+n+\beta+1)[B(\mu,\beta+1)-B(\mu+n,\beta+1)]} \\ &= O\big(n^{-(\beta+2)}\big)\to 0, \quad n\to\infty,\ \beta>-1.\end{aligned}$$

Thus $F(z)=\sum_{n=0}^{\infty}a_n z^n$ is an entire function. Let us compute its order ρ. This can be done directly from its series coefficients

$$\begin{aligned}\rho &= \limsup_{n\to\infty}\frac{n\log n}{\log|a_n|^{-1}} = \lim_{n\to\infty}\frac{(n+1)\log(n+1)-n\log n}{\log|a_{n+1}|^{-1}-\log|a_n|^{-1}} \\ &= \lim_{n\to\infty}\frac{\log n}{\log n^{\beta+2}} = \frac{1}{\beta+2}<1, \quad \beta>-1.\end{aligned}$$

The same arguments as in [4] guarantee that all our solutions belong to the class $\mathcal{B}$. In particular, for $\beta=0$ we easily get from (3.2) that

$$a_n = \frac{a^n}{\Gamma(\mu+n)}\prod_{j=1}^{n}\frac{(\mu)_j}{(\mu+1)_j-(\mu)_j} = \frac{(a\mu)^n}{\Gamma(\mu+n)n!}, \quad n=0,1,\ldots.$$

Then the series representation for the Bessel function [3] yields

$$f(z)=\text{const.}\, z^{-\alpha/2}J_{\mu-1}\big(cz^{1/2}\big), \quad c^2=-4a\mu,$$

which slightly generalizes Hardy's solutions [4, p. 43]. Thus Theorem 2 is proved. □

4. Properties of solutions and their particular cases

First in this section we will apply the Mellin transform (1.5) to the integral equation (2.3), reducing it to a certain functional equation. Then we will solve this equation by methods of the calculus of finite differences [5] to obtain the value of the Mellin transform for solutions (3.1).

In fact, returning to (3.3) we consider $z=x\in\mathbb{R}_+$ and we apply through the Mellin transform (1.5) taking into account the operational formula

$$\big\langle f(xt), x^{s-1}\big\rangle = t^{-s}f^*(s).$$

Then using values of the elementary Beta-integrals we deduce the following homogeneous functional equation:

$$\begin{aligned}aB(\nu+\alpha-s+1,\beta+1)f^*(s+1) &= aB(\nu+\alpha-s,\beta+1)f^*(s+1)+B(\nu+\alpha-s+1,\beta+1)f^*(s) \\ &\quad - B(2\nu+\alpha+1,\beta+1)f^*(s),\end{aligned} \tag{4.1}$$

where s is a parameter of the Mellin transform (1.5) such that $s+1\in\Omega_f$. Hence denoting by $h(s)=B(\nu+\alpha-s+1,\beta+1)$ and invoking the condition $a\neq 0$ (see [4, p. 43]) we rewrite (4.1) as

$$f^*(s+1)=H(s)f^*(s), \tag{4.2}$$

where the kernel $H(s)$ is given by

$$H(s)=\frac{1}{a}\frac{h(s)-h(-\nu)}{h(s)-h(s+1)}. \tag{4.3}$$

We can simplify $H(s)$ appealing to the basic properties for Beta-functions [3]. After straightforward calculations we get finally

$$H(s)=\frac{(s-\nu-\alpha)(h(s)-h(-\nu))}{a\Gamma(\beta+2)h(s)}. \tag{4.4}$$

Meanwhile,

$$h(s)-h(-\nu)=\int_0^1 (1-t)^{\beta}\bigl(t^{\nu+\alpha-s}-t^{2\nu+\alpha}\bigr)\,dt$$
$$=\sum_{n=0}^{\infty}\frac{(-\beta)_n}{n!}\left[\frac{1}{\nu+\alpha-s+n+1}-\frac{1}{2\nu+\alpha+n+1}\right]=(\nu+s)\chi(s),$$

where

$$\chi(s)=\sum_{n=0}^{\infty}\frac{(-\beta)_n}{n!(2\nu+\alpha+n+1)(\nu+\alpha+n+1-s)}.$$

Notice that $\chi(s)$ and $h(s)$ have the same simple poles at the points $s=\nu+\alpha+n+1$, $n=0,1,\dots$. As a consequence, taking into account the expression of $h(s)$ in terms of Gamma-functions, we find that $H(s)$ is meromorphic with simple poles $s=\nu+\alpha+\beta+n+2$, $n\in\mathbb{N}_0$. Furthermore, from the asymptotic behavior of the Beta-function [3], we get $H(s)=O(s^{\beta+2})$, $s\to\infty$.

The functional equation (4.2) can be formally solved using the methods proposed in [5, Chapter 11]. So it has a general solution

$$f^*(s)=\omega(s)\exp\left(\sum_c^s \log H(z)\Delta z\right), \tag{4.5}$$

where $\omega(s)$ is an arbitrary periodic function of s with period 1, $\sum_c^s$ means the operation of general summation [5] with a given constant c, and the function $\log H(s)$ is summable in this sense (cf. [5, Chapter 8]). Clearly that it may be necessary to make suitable cuts in the s-plane according to the possible multi-valued nature of the right-hand side of (4.5).

Let us consider some interesting particular cases of the solutions (3.1) when $\beta=k\in\mathbb{N}_0$. Indeed, putting $\beta=k$ in (3.2) and using the definition of the Pochhammer's symbol we obtain

$$\frac{(\mu+k+1)_j}{(\mu)_j}-1=\frac{\Gamma(\mu+k+j+1)}{\Gamma(\mu+k+1)(\mu)_j}-1=\frac{(\mu)_{k+j+1}}{(\mu)_{k+1}(\mu)_j}-1$$
$$=\frac{(\mu+j)(\mu+j+1)\dots(\mu+k+j)}{(\mu)_{k+1}}-1=\frac{jP_k(j)}{(\mu)_{k+1}},$$

where we denote by $P_k(j)=(j-\alpha_1)\dots(j-\alpha_k)$, $\alpha_i\in\mathbb{C}$, $i=1,\dots,k$, a polynomial of degree k with respect to j and, evidently, $P_k(j)>0$, $j\in\mathbb{N}$. Substituting the above expression in (3.2) it becomes

$$a_n=\frac{(a(k+1)(\mu)_{k+1})^n}{\Gamma(\mu+n)n!}\prod_{j=1}^{n}\prod_{i=1}^{k}\frac{1}{j-\alpha_i},\quad n,k\in\mathbb{N}_0, \tag{4.6}$$

and the empty products are equal to 1. As a consequence, the solutions (3.1) represent the so-called hyperbessel functions [7, Chapter 19] of the order $k+1$, namely

$$f(z)=\text{const.}\,z^{\nu}{}_0F_{k+1}\bigl(2\nu+\alpha+1,1-\alpha_1,\dots,1-\alpha_k;a(k+1)(2\nu+\alpha+1)_{k+1}z\bigr),\quad k\in\mathbb{N}_0. \tag{4.7}$$

As it is known [7] functions

$$F(z)={}_0F_{k+1}\bigl(2\nu+\alpha+1,1-\alpha_1,\dots,1-\alpha_k;\ a(k+1)(2\nu+\alpha+1)_{k+1}z\bigr)$$

in (4.7) satisfy the following $(k+2)$th-order linear differential equation:

$$\frac{d}{dz}\left(z\frac{d}{dz}+2\nu+\alpha\right)\prod_{i=1}^{k}\left(z\frac{d}{dz}-\alpha_i\right)F-a(k+1)(2\nu+\alpha+1)_{k+1}F=0. \tag{4.8}$$

Remark 1. We point out that in [2] the authors considered other generalizations of the Bessel functions which also satisfy higher order linear differential equations.

Remark 2. In particular, for $k = 0$ we easily get the Hardy's solutions in terms of the ${}_0F_1$-functions and strictly we prove their orthogonality (1.2) by using the corresponding second-order linear differential equation (4.8). However, even for $k = 1$ the direct proof of the orthogonality property (1.2) for solutions (4.7) is a difficult task since we deal in this case with the third-order linear differential equation. As we are aware, the orthogonality property for the hyperbessel functions with respect to their zeros is yet unknown. So, following the conclusions of Theorem 2 we conjecture here this fact as well as the orthogonality for all solutions (3.1) of the class $\mathcal{B}$.

5. Orthogonal functions of the class $\mathcal{A}$

In the case $f \in \mathcal{A}$ we recall again the arguments in [4] to write accordingly, the formula for the Fourier coefficients $a_n(z)$ of $f(zt)$ (see (2.1)) as

$$a_n(z) = \frac{2A_n^{1/2}\lambda_n}{f'(\lambda_n)}\frac{f(z)}{z^2 - \lambda_n^2}. \tag{5.1}$$

We have

Theorem 3. *If $f \in \mathcal{A}$ and satisfy* (1.2) *with $\alpha > -1 - 2\nu$, $\nu \in \mathbb{R}$, $\beta > -1$, then the integral equation holds*

$$a\int_0^z u^{\nu+\alpha+2}(z-u)^\beta f(u)\,du = \left(az^2+2\right)\int_0^z u^{\nu+\alpha}(z-u)^\beta f(u)\,du + z^{\nu+\alpha+\beta+1} f(z)A, \tag{5.2}$$

where $a = F''(0)$ and $A = -2B(2\nu+\alpha+1, \beta+1)$.

Proof. As in the proof of Theorem 1 we substitute (5.1) in (2.2) and we get the equality

$$\int_0^1 f(zt)f(\zeta t)t^\alpha(1-t)^\beta\,dt = -f(z)f(\zeta)\frac{q(z)-q(\zeta)}{z^2-\zeta^2}, \tag{5.3}$$

where now

$$q(z) = 4\sum_{n=1}^\infty \frac{A_n\lambda_n^2}{\{f'(\lambda_n)\}^2}\left[\frac{1}{z^2-\lambda_n^2} + \frac{1}{\lambda_n^2}\right].$$

Letting $\zeta \to 0$ in (5.3), we obtain

$$\int_0^1 t^\nu f(zt)t^\alpha(1-t)^\beta\,dt = -\frac{f(z)q(z)}{z^2},$$

which yields after the change $u = zt$,

$$\int_0^z u^{\nu+\alpha}(z-u)^\beta f(u)\,du = -z^{\nu+\alpha+\beta-1} f(z)q(z). \tag{5.4}$$

When z is small enough, $f(z) \backsim z^\nu$ and $q(z) \sim \frac{z^2}{2}q''(0)$. Therefore, as $z \to 0$ we get

$$2\int_0^z u^{2\nu+\alpha}(z-u)^\beta\,du \sim -z^{2\nu+\alpha+\beta+1}q''(0).$$

Thus $q''(0) = -2B(2\nu+\alpha+1, \beta+1) = A$. If we rewrite (5.3) in the form

$$\int_0^1 t^{2\nu+\alpha} F(zt)F(\zeta t)(1-t)^\beta\,dt = -F(z)F(\zeta)\frac{q(z)-q(\zeta)}{z^2-\zeta^2},$$

then after differentiation with respect to ζ we get

$$\int_0^1 t^{2\nu+\alpha+1} F'(\zeta t)F(zt)(1-t)^\beta\,dt = -F(z)F'(\zeta)\frac{q(z)-q(\zeta)}{z^2-\zeta^2} - F(z)F(\zeta)\frac{-q'(\zeta)(z^2-\zeta^2)+2\zeta(q(z)-q(\zeta))}{(z^2-\zeta^2)^2}.$$

Differentiating again with respect to ζ and setting $\zeta = 0$, we deduce after some simplifications

$$F''(0)\int_0^1 t^{\nu+\alpha+2}(1-t)^\beta f(zt)\,dt = -f(z)\left[\frac{F''(0)q(z)}{z^2} - \frac{A}{z^2} + \frac{2q(z)}{z^4}\right].$$

Letting $a = F''(0)$ and making the change $zt = u$ we obtain the equation

$$a\int_0^z u^{\nu+\alpha+2}(z-u)^\beta f(u)\,du = -z^{\nu+\alpha+\beta-1} f(z)\left[\left(z^2a+2\right)q(z) - Az^2\right].$$

Taking into account (5.4) we finally obtain the integral equation (5.2). $\square$

An analog of the Hardy theorem for the class $\mathcal{A}$ is

Theorem 4. *Let f satisfy the conditions of Theorem 3. Then*

$$f(z) = \text{const.}\, z^\nu \sum_{n=0}^\infty a_{2n} z^{2n}, \tag{5.5}$$

where

$$a_{2n} = \left(\frac{a(\beta+1)}{2}\right)^n \prod_{j=1}^n \frac{(\beta+2(\mu+2j-1))(\mu)_{2(j-1)}}{(\mu+\beta+1)_{2j}-(\mu)_{2j}}, \quad n = 0, 1, \ldots,$$

$\mu = 2\nu+\alpha+1$ *and the empty product is equal to* 1.

Proof. The substitution $u = zt$ and the value of A reduce Eq. (5.2) to

$$az^2\int_0^1 t^{\nu+\alpha}\left(t^2-1\right)(1-t)^\beta f(zt)\,dt = 2\left[\int_0^1 t^{\nu+\alpha}(1-t)^\beta f(zt)\,dt - f(z)B(2\nu+\alpha+1, \beta+1)\right]. \tag{5.6}$$

Since the case $f \in \mathcal{A}$ presumes the following series representation:

$$f(z) = \text{const.}\, z^\nu \sum_{n=0}^\infty a_{2n} z^{2n},$$

with $a_{2n} \neq 0$ for any $n \in \mathbb{N}_0$, we substitute it into (5.6), change the order of integration and summation, and calculate the inner Beta-integrals. Therefore denoting by $\mu = 2\nu+\alpha+1$ we obtain

$$a\sum_{n=1}^{\infty} a_{2(n-1)} z^{2n}\big[B(\mu+2n,\beta+1)-B\big(\mu+2(n-1),\beta+1\big)\big]$$
$$=2\sum_{n=1}^{\infty} a_{2n} z^{2n}\big[B(\mu+2n,\beta+1)-B(\mu,\beta+1)\big].$$

Hence equating coefficients of the series the following recurrence relations appear

$$a_{2n}=\frac{a}{2}a_{2(n-1)}\frac{B(\mu+2n,\beta+1)-B(\mu+2(n-1),\beta+1)}{B(\mu+2n,\beta+1)-B(\mu,\beta+1)}$$
$$=\left(\frac{a}{2}\right)^n\prod_{j=1}^{n}\frac{B(\mu+2j,\beta+1)-B(\mu+2(j-1),\beta+1)}{B(\mu+2j,\beta+1)-B(\mu,\beta+1)}, \tag{5.7}$$

where $n=0,1,2,\ldots,$ and the empty product is equal to 1. In the same way as (3.2) it can be written in the form

$$a_{2n}=\left(\frac{a(\beta+1)}{2}\right)^n\prod_{j=1}^{n}\frac{(\beta+2(\mu+2j-1))(\mu)_{2(j-1)}}{(\mu+\beta+1)_{2j}-(\mu)2j}, \quad n=0,1,\ldots. \qquad \square$$

Putting $\beta=0$ in (5.7) by straightforward calculations the functions (5.5) become

$$f(z)=\text{const.}\, z^{\frac{1-\alpha}{2}} J_{\frac{\mu}{2}-1}\big((-a\mu)^{1/2}z\big),$$

which slightly generalizes the Hardy's solutions [4, p. 42].

It is easily seen that the series in (5.5) is an entire function. Finally we compute the order of these solutions. Using the same arguments as in Theorem 2 we have

$$\rho=\limsup_{n\to\infty}\frac{2n\log 2n}{\log|a_{2n}|^{-1}}=\lim_{n\to\infty}\frac{2(n+1)\log 2(n+1)-2n\log 2n}{\log|a_{2(n+1)}|^{-1}-\log|a_{2n}|^{-1}}$$
$$=\lim_{n\to\infty}\frac{2\log n}{\log n^{\beta+2}}=\frac{2}{\beta+2}<2, \quad \beta>-1.$$

Therefore all solutions belong to the class $\mathcal{A}$.

Remark 3. Our final conjecture is that the solutions $f\in\mathcal{A}$ of the Hardy-type integral equation (5.2) are orthogonal with respect to their own zeros for the Jacobi weight.

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