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Protolocalisations of homological categories

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To the memory of Max Kelly

Abstract

A protolocalisation of a homological (resp. semi-abelian) category is a regular full reflective subcategory, whose reflection preserves short exact sequences. We study the closure operator and the torsion theory associated with such a situation. We pay special attention to the fibered, the regular epireflective and the monoreflective cases. We give examples in algebra, topos theory and functional analysis.

MSC: 18A05; 18A40; 18B40; 18E35; 18E40

0. Introduction

In an abelian category $\mathcal{C}$, hereditary torsion theories are in bijection with universal closure operators and, when $\mathcal{C}$ is a Grothendieck category, these are further in bijection with the localisations of $\mathcal{C}$ (see [21]). This last point is important since a localisation of an abelian category is again abelian.

For some years, the notion of semi-abelian category imposed itself as an elegant and powerful "non-commutative" substitute for the notion of abelian category (see [27]) and, more recently, it has been observed that the weaker notion of homological category is still sufficient to force the validity of all diagram lemmas of homological algebra (see [6]).

Torsion theories and closure operators in semi-abelian and homological categories have already been studied by various authors (see [13,24,17,28]), but, to the best of our knowledge, the possible link with an adequate notion of localisation remains to be investigated. This is one of the purposes of the present paper.

It is immediate to observe that a localisation of a semi-abelian (resp. homological) category is again semi-abelian (resp. homological). But, in the semi-abelian context, the notion of localisation may not be the most adequate one.

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Let us recall that a reflection of a category with finite limits is *a localisation* when it preserves finite limits. In the abelian context, this is equivalent to simply preserving monomorphisms, or to preserving short exact sequences, or to preserving left exact sequences, and so on. But, in the semi-abelian case, all these properties are no longer equivalent, so that deciding what a "semi-abelian localisation" is should be considered very seriously.

To give evidence of the pertinence of this question, we recall first a known result in the case of groups . . . the somehow basic "prototype" of a semi-abelian category. *The category* Gp *of groups does not have any non-trivial localisation!* But of course, the category Gp of groups admits plenty of interesting full reflective semi-abelian subcategories: for example, the category Ab of abelian groups and all its well-known localisations.

The first step of our study is to characterise those full reflective subcategories of a regular (resp. exact) category (see [3]) which are still regular (resp. exact). In both cases, this reduces to the preservation of some finite limits by the reflection: conditions which are of course valid in the case of a localisation. We call such a reflection *protoregular* (resp. *protoexact*).

Let us recall that a *homological category* is a regular category with a zero object which is Bourn protomodular (see [9]) or equivalently, which satisfies the split short five lemma. A semi-abelian category is an exact homological category with binary coproducts; this forces the existence of all finite colimits. A reflective subcategory of a homological (resp. semi-abelian) category is still homological (resp. semi-abelian) if and only if the reflection is protoregular (resp. protoexact).

We are then ready to handle the main notion of this paper: we call *protolocalisation*, of a homological category, a full reflective subcategory whose reflection is protoregular and preserves short exact sequences. A protolocalisation of a homological (resp. semi-abelian) category is still homological (resp. semi-abelian).

A protolocalisation of a homological category $\mathcal{C}$ – as every reflection – induces a prefactorisation system $(\mathcal{E}, \mathcal{M})$ on $\mathcal{C}$. We call *stable* a monomorphism admitting an $(\mathcal{E}, \mathcal{M})$-factorisation both of whose parts are still monomorphisms. We show that every protolocalisation of a homological category $\mathcal{C}$ induces a closure operator on stable subobjects in $\mathcal{C}$. This closure operator respects the normality of subobjects and induces further a torsion theory in $\mathcal{C}$. But, more importantly, when considered on stable subobjects, this closure operator is sufficient to characterise the original protolocalisation.

A special case of interest is given by the *fibered protolocalisations* of a homological category: the reflection functor of the protolocalisation is a fibration (see [8,13]). This additional property turns out to be equivalent to what is called a *semi-left-exact reflection* in [16]: another generalisation of the notion of localisation. We characterise the fibered protolocalisations in terms of stability properties of the class $\mathcal{E}$, generalising so the fact that having a localisation is equivalent to the stability of $\mathcal{E}$ under all pullbacks.

We devote special attention to the case of regular epireflections. A regular epireflection of a homological category is a fibered protolocalisation as soon as it preserves short exact sequences. We characterise the closure operators, the torsion theories and the radical functors corresponding to regular epireflective protolocalisations of semi-abelian categories. In this situation, it suffices to define the closure operator on normal subobjects to characterise the original protolocalisation, whose objects are exactly the closed ones.

We consider also the special case of monoreflections. We prove that, for a protolocalisation, being monoreflective is equivalent to each dense monomorphism being an epimorphism. We show also that the objects in the reflection coincide with the absolutely closed objects.

We finally provide various examples of protolocalisations. The category of Boolean rings is a protolocalisation of the category of commutative von Neumann regular rings. Every arithmetical semi-abelian category is a protolocalisation of its category of equivalence relations. Examples are also provided in the case of the dual of the category of pointed objects of a topos and in the context of $\mathbb{C}^*$-algebras. We observe that many of these examples involve arithmetical semi-abelian categories. And, of course, all well-known examples of localisations of abelian or semi-abelian categories fit into our context.

1. A quick review of known results

Every full reflective subcategory $\iota, \lambda \colon \mathcal{L} \underset{\longrightarrow}{\overset{\longleftarrow}{}} \mathcal{C}$, $\lambda \dashv \iota$, is entirely characterised by a prefactorisation system $(\mathcal{E}, \mathcal{M})$ on $\mathcal{C}$ (see [16]): $\mathcal{E}$ is the class of those morphisms inverted by λ while $m \in \mathcal{M}$ when $e \perp m$ for every morphism $e \in \mathcal{E}$ (let us recall that $e \perp m$ means that given a commutative square $m \circ f = g \circ e$, there exists a unique diagonal d yielding $m \circ d = g$, $d \circ e = f$). The prefactorisation system is a factorisation system when each morphism factors uniquely (up to isomorphism) as $f = m \circ e$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$. The class $\mathcal{M}$ is stable under limits and composition and contains all the morphisms of $\mathcal{L}$. The class $\mathcal{E}$ is stable under colimits and if two sides of a commutative triangle lie in $\mathcal{E}$, so does the third side. And so on.

When $\mathcal{C}$ has finite limits, λ preserves them precisely when the class $\mathcal{E}$ is stable under arbitrary pullbacks (see [22]). Such a situation is called a *localisation*. That notion is very important since being abelian, a topos, regular, exact, homological, semi-abelian, and so on, are notions preserved under localisation. In the abelian case, being a localisation is also equivalent to λ preserving monomorphisms, or kernels, or short exact sequences.

When the class $\mathcal{E}$ is only stable under pullbacks along morphisms in $\mathcal{M}$, the reflection is called *semi-left-exact* (see [16]); in that case, the prefactorisation system is at once a factorisation system and a morphism $f: A \longrightarrow B$ belongs to the class $\mathcal{M}$ precisely when it is the pullback of $\iota\lambda(f)$ along the unit η_B of the adjunction. And when each inverse image of a unit η_B still lies in $\mathcal{E}$, the reflection is called *unit-stable*: a property stronger than semi-left-exactness.

Let us now recall that a category $\mathcal{C}$ with a zero object is Bourn-protomodular (see [9]) when the split short five lemma holds, that is, given a diagram where all squares commute

$$\begin{array}{ccccccccc}
\mathbf{0} & \longrightarrow & K & \xrightarrow{k} & A & \underset{q}{\overset{s}{\leftrightarrows}} & Q & \longrightarrow & \mathbf{0} \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
\mathbf{0} & \longrightarrow & L & \xrightarrow{l} & B & \underset{p}{\overset{t}{\leftrightarrows}} & P & \longrightarrow & \mathbf{0}
\end{array}$$

and $q \circ s = \mathsf{id}$, $p \circ t = \mathsf{id}$, $k = \mathsf{Ker}\, q$, $l = \mathsf{Ker}\, p$, if α and γ are isomorphisms, β is an isomorphism as well.

A category $\mathcal{C}$ is *homological* (see [6]) when it has a zero object, is regular (see [3]) and protomodular. An exact homological category with binary coproducts is called *semi-abelian* (see [27]). In both cases a sequence of morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is called *exact* when the image of f coincides with the kernel of g. In a homological category, all the classical diagram lemmas of homological algebra hold true (see [10]); every normal monomorphism (= kernel of a morphism) has a cokernel; being a monomorphism is equivalent to having a zero kernel (see [9]). In the semi-abelian case, all finite colimits exist, as well as a notion of semi-direct product (see [14]); moreover, the image of a normal monomorphism along a regular epimorphism is still a normal monomorphism (see [27]).

And rather trivially:

Proposition 1. *Let $\iota, \lambda: \mathcal{L} \leftrightarrows \mathcal{C}$ be a full reflective subcategory, where $\mathcal{C}$ has a zero object and is protomodular. Then $\mathcal{L}$ has a zero object and is protomodular as well.* $\square$

2. Localisations of the category of groups

The following result, which can already be found in [4], seems to have been overlooked by many authors interested in localisation theory. We give here a direct proof.

Proposition 2. *The only localisations of the category* Gp *of groups are the trivial ones:* (0) *and* Gp.

Proof. Consider a localisation $\iota, \lambda: \mathcal{L} \leftrightarrows \mathsf{Gp}$ of the category Gp of groups. Our Theorem 34 proves that this localisation is entirely determined by those monomorphisms s such that $\lambda(s)$ is an isomorphism.

Given a group G, the family of all morphisms $f: \mathbb{Z} \longrightarrow G$ constitutes a strongly epimorphic family: that is, a subobject $s: S \rightarrowtail G$ is an isomorphism if and only if all the morphisms f factor through it. Strongly epimorphic families are preserved by every reflection, thus the family of all morphisms $\lambda(f)$ is strongly epimorphic in $\mathcal{L}$.

Notice now that $\lambda(s)$ is an isomorphism if and only if each $\lambda\left(f^{-1}(s)\right)$ is an isomorphism. The condition is indeed necessary since λ preserves pullbacks. It is also sufficient because, if each $\lambda\left(f^{-1}(s)\right)$ is an isomorphism, then each $\lambda(f)$ factors through $\lambda(s)$ and thus $\lambda(s)$ is an isomorphism.

So a localisation of the category Gp of groups is entirely determined by those subobjects $s\colon S \rightarrowtail \mathbb{Z}$ such that $\lambda(s)$ is an isomorphism. In particular, the identity on Gp is the localisation such that $\mathbb{Z} = \mathbb{Z}$ is the only subgroup of $\mathbb{Z}$ mapped by λ to an isomorphism, while the inclusion of the zero category in Gp is the localisation for which all subgroups $S \rightarrowtail \mathbb{Z}$ are inverted by λ. We must thus prove that if some proper inclusion $s\colon S \rightarrowtail \mathbb{Z}$ is inverted by λ, then all inclusions $s'\colon S' \rightarrowtail \mathbb{Z}$ are inverted by λ.

Now each subgroup of $\mathbb{Z}$ has the form $n\mathbb{Z} \rightarrowtail \mathbb{Z}$ for some integer n. But if $n\mathbb{Z} \rightarrowtail \mathbb{Z}$, with $n \neq 1$, is mapped by λ to an isomorphism, so is the coproduct of this monomorphism with itself, which is the subgroup

$$\langle\, x^n, y^n \,\rangle \rightarrowtail \langle\, x, y \,\rangle$$

where $\langle x, y\rangle$ indicates the free group on the two generators x, y, while $\langle x^n, y^n\rangle$ indicates the subgroup generated by x^n and y^n. Again, since λ is a localisation, the pullback of this subobject along the morphism

$$f\colon \mathbb{Z} \longrightarrow \langle\, x, y \,\rangle, \quad m \mapsto (xy)^m$$

is inverted by λ. But this pullback is the zero subgroup $(0) \rightarrowtail \mathbb{Z}$. Thus $(0) \rightarrowtail \mathbb{Z}$ is mapped by λ to $(0) = (0)$, which forces the same conclusion for every subgroup $S \rightarrowtail \mathbb{Z}$, simply because λ preserves monomorphisms. □

3. Protoregular and protoexact reflections

In this section we first investigate the very general question: *when is a full reflective subcategory of a regular (resp. exact) category again regular (resp. exact)?* (see [3]). It is well known that the reflection being a localisation is a sufficient condition, but this assumption is definitely too strong. For example, it is proved in [30] that a semi-left-exact reflection (see [16] or our Section 1) of a regular category is still regular. But this condition is not yet necessary.

Proposition 3. *Let $\iota, \lambda\colon \mathcal{L} \rightleftarrows \mathcal{C}$ be a full reflective subcategory of a regular category $\mathcal{C}$. For a morphism $f\colon L \longrightarrow M$ of $\mathcal{L}$, the following conditions are equivalent:*

1. *f is a regular epimorphism in $\mathcal{L}$;*
2. *if $f = s \circ p$ is the image factorisation of f in $\mathcal{C}$, then $\lambda(s)$ is an isomorphism.*

Proof. ($1 \Rightarrow 2$). Write (u, v) for the kernel pair of f in $\mathcal{L}$, thus also in $\mathcal{C}$. Since f is a regular epimorphism in $\mathcal{L}$, $f = \mathsf{Coeq}(u, v)$ in $\mathcal{L}$. The construction of the image of f in the regular category $\mathcal{C}$ yields $p = \mathsf{Coeq}(u, v)$ in $\mathcal{C}$. Thus in $\mathcal{L}$, $\lambda(p) = \mathsf{Coeq}(u, v)$. But $f = \lambda(s) \circ \lambda(p)$, proving that $\lambda(s)$ is an isomorphism, by uniqueness of the coequaliser.

($2 \Rightarrow 1$). Since p is a regular epimorphism in $\mathcal{C}$, $\lambda(p)$ is a regular epimorphism in $\mathcal{L}$. But $f = \lambda(s) \circ \lambda(p)$ and since $\lambda(s)$ is an isomorphism, $f \cong \lambda(p)$ is a regular epimorphism. □

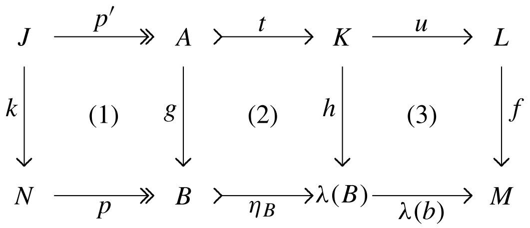
Theorem 4. *Let $\iota, \lambda\colon \mathcal{L} \rightleftarrows \mathcal{C}$ be a full reflective subcategory of a regular category $\mathcal{C}$. The following conditions are equivalent:*

1. *$\mathcal{L}$ is regular;*
2. *λ preserves the pullbacks of the form*

$$\begin{array}{ccc} A & \overset{a}{\rightarrowtail} & L \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ B & \underset{b}{\rightarrowtail} & M \end{array}$$

where $f \in \mathcal{L}$ and b is the image in $\mathcal{C}$ of a morphism of $\mathcal{L}$.

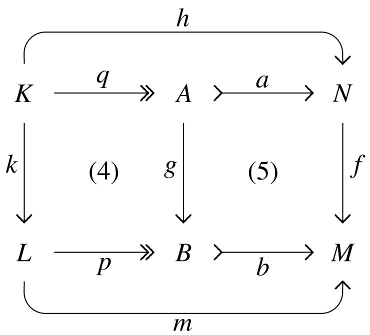
Proof. ($1 \Rightarrow 2$). Consider a pullback as in condition 2, where $h = b \circ p$ is the image factorisation of a morphism $h \in \mathcal{L}$. Write $\eta\colon \mathsf{id}_{\mathcal{C}} \Rightarrow \iota\lambda$ for the unit of the adjunction. We have $b = \lambda(b) \circ \eta_B$ and $\lambda(p) = \eta_B \circ p$; in particular, η_B is a monomorphism since so is b. Consider further the following pullbacks:



Since L, M, N and $\lambda(B)$ are in $\mathcal{L}$, K and J are in $\mathcal{L}$ as well.

The pullback (3) is preserved by λ, since it is a pullback in $\mathcal{L}$. On the other hand p is a regular epimorphism in $\mathcal{C}$, thus $\lambda(p) = \eta_B \circ p$ is a regular epimorphism in $\mathcal{L}$. Since $\mathcal{L}$ is regular by assumption, $t \circ p'$ is a regular epimorphism in $\mathcal{L}$. By Proposition 3, $\lambda(t)$ is an isomorphism; and of course $\lambda(\eta_B)$ is an isomorphism; so trivially, λ transforms the square (2) in a pullback. Thus λ preserves both pullbacks (2) and (3) and therefore also the pullback of the statement.

($2 \Rightarrow 1$). Consider a regular epimorphism $m: L \twoheadrightarrow M$ in $\mathcal{L}$ and its image factorisation $m = b \circ p$ in $\mathcal{C}$. Consider further the two pullbacks (4) and (5)



where $f \in \mathcal{L}$; in particular, $K \in \mathcal{L}$. By assumption, the pullback (5) is preserved by λ and by Proposition 3, $\lambda(b)$ is an isomorphism. Therefore $\lambda(a)$ is an isomorphism as well and, again by Proposition 3, the pullback $h = a \circ q$ of m along f is a regular epimorphism in $\mathcal{L}$. □

Definition 5. A reflection of a regular category satisfying the conditions of Theorem 4 is called *protoregular*.

By Theorem 4 and Proposition 1, we have thus:

Corollary 6. *Let $\iota, \lambda: \mathcal{L} \rightleftarrows \mathcal{C}$ be a full reflective subcategory of a homological category $\mathcal{C}$. The category $\mathcal{L}$ is homological if and only if the reflection is protoregular.* □

Let us recall (see [3]) that an *exact fork* in a regular category is a triple (u, v, q) where $q = \mathsf{Coeq}(u, v)$ and (u, v) is the kernel pair of q.

Theorem 7. *Let $\iota, \lambda: \mathcal{L} \rightleftarrows \mathcal{C}$ be a full reflective subcategory of an exact category $\mathcal{C}$. The following conditions are equivalent:*

1. *$\mathcal{L}$ is exact;*
2. *λ is protoregular and preserves the exact forks of the form*

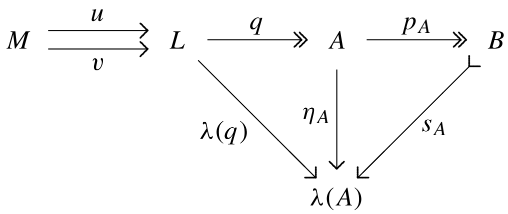
$$M \underset{v}{\overset{u}{\rightrightarrows}} L \overset{q}{\twoheadrightarrow} A$$

   where M and L are objects in $\mathcal{L}$;
3. *λ is protoregular and, given an exact fork as in condition 2, the unit $\eta_A: A \longrightarrow \iota\lambda(A)$ of the adjunction is a monomorphism.*

Proof. Notice that a reflection preserves coequalisers, thus condition 2 reduces to the preservation of the kernel pair of q.

(1 ⇒ 2). With the notation of condition 2, we have $\lambda(q) = \mathsf{Coeq}(u, v)$ since λ preserves colimits. But since ι preserves and reflects limits, $u, v : M \rightrightarrows L$ is an equivalence relation in $\mathcal{L}$ because it is so in $\mathcal{C}$. And since $\mathcal{L}$ is exact, $(u, v) = (\lambda(u), \lambda(v))$ is the kernel pair of $\lambda(q)$.

(2 ⇒ 3). Factoring η_A through its image $\eta_A = s_A \circ p_A$, we have now the following situation in $\mathcal{C}$



where (u, v) is the kernel pair of $\lambda(q) = s_A \circ p_A \circ q$. Since s_A is a monomorphism, (u, v) is also the kernel pair of the regular epimorphism $p_A \circ q$. Thus $p_A \circ q \cong \mathsf{Coeq}(u, v) = q$ so that p_A is an isomorphism. Thus $\eta_A \cong s_A$ is a monomorphism.

(3 ⇒ 1). Using the same diagram, when η_A is a monomorphism, the kernel pair of $\lambda(q)$ is the same as that of q, which is (u, v). □

Definition 8. A reflection of an exact category satisfying the conditions of Theorem 7 is called *protoexact*.

By Theorem 7 and Proposition 1, we conclude that

Corollary 9. *Let $\iota, \lambda : \mathcal{L} \rightleftarrows \mathcal{C}$ be a full reflective subcategory of a semi-abelian category $\mathcal{C}$. The category $\mathcal{L}$ is semi-abelian if and only if the reflection is protoexact.* □

Example 10. Every localisation of a regular (resp. exact) category is protoregular (resp. protoexact).

Proof. Protoregularity and protoexactness mean the preservation of some finite limits, while the localisation case assumes the preservation of all finite limits. □

Example 11. Every reflection $\iota, \lambda : \mathcal{L} \rightleftarrows \mathcal{C}$ of a regular category $\mathcal{C}$ whose units are regular epimorphisms is protoregular.

Proof. By regularity of $\mathcal{C}$, (ι, λ) the units being regular epimorphisms is equivalent to $\mathcal{L}$ being stable in $\mathcal{C}$ for subobjects (see [5], Vol. 1). The pullback of condition 2 in Theorem 4 is thus entirely in $\mathcal{L}$ and therefore is mapped to itself by λ. □

Let us recall some other piece of terminology borrowed from [26]:

Definition 12. By a *Birkhoff subcategory* of a regular category is meant a regular epireflective subcategory which is closed under regular quotients.

Example 13. A reflection $\iota, \lambda : \mathcal{L} \rightleftarrows \mathcal{C}$ with regular epimorphic units of an exact category $\mathcal{C}$ is protoexact if and only if $\mathcal{L}$ is a Birkhoff subcategory of $\mathcal{C}$.

Proof. When $\mathcal{L}$ is stable in $\mathcal{C}$ under regular quotients, the exact fork of condition 2 in Theorem 7 lies entirely in $\mathcal{L}$, thus is mapped to itself by λ.

Conversely assume that $\mathcal{L}$ is exact. Consider a regular epimorphism $q : L \twoheadrightarrow A$ in $\mathcal{C}$, with $L \in \mathcal{L}$. The product $L \times L$ is still in $\mathcal{L}$, thus also, by epireflectivity, the kernel pair M of q. By Theorem 7 we get a monomorphism $\eta_A : A \rightarrowtail \iota\lambda(A)$, thus $A \in \mathcal{L}$ by regular epireflectivity. □

Finally, let us recall that, in a homological category, being a right exact sequence

$$A \xrightarrow{\;f\;} B \xrightarrow{\;g\;} C \longrightarrow \mathbf{0}$$

is no longer a pure colimit condition – namely, $g = \mathsf{Coker}\, f$ as in the abelian case – but forces also f to be a *proper* morphism, that is, the image of f is a normal monomorphism.

Definition 14. A reflection $\iota, \lambda\colon \mathcal{L} \rightleftarrows \mathcal{C}$ of a homological category $\mathcal{C}$ is *sequentially right exact* when λ preserves right exact sequences.

And trivially, since a reflection preserves cokernels:

Proposition 15. *A protoregular reflection $\iota, \lambda\colon \mathcal{L} \rightleftarrows \mathcal{C}$ of a homological category $\mathcal{C}$ is sequentially right exact if and only if λ preserves proper morphisms.* □

4. The protolocalisations

Here we want to investigate – in the homological and semi-abelian cases – those reflections which preserve short exact sequences. Let us observe at once that:

Lemma 16. *Let $\iota, \lambda\colon \mathcal{L} \rightleftarrows \mathcal{C}$ be a protoregular reflection of a homological category $\mathcal{C}$. The following conditions are equivalent:*

1. *λ preserves short exact sequences;*
2. *λ preserves the kernels of regular epimorphisms;*
3. *λ preserves normal monomorphisms.*

Proof. $\mathcal{L}$ is homological by Corollary 6. The result holds because λ preserves cokernels and, in homological categories, every normal monomorphism is the kernel of its cokernel. □

Definition 17. A *protolocalisation* of a homological category $\mathcal{C}$ is a full reflective subcategory $\iota, \lambda\colon \mathcal{L} \rightleftarrows \mathcal{C}$ whose reflection λ is protoregular and preserves short exact sequences.

Proposition 18. *Let $\iota, \lambda\colon \mathcal{L} \rightleftarrows \mathcal{C}$ be a protolocalisation of a homological category $\mathcal{C}$. Then $\mathcal{L}$ is homological and the reflection is sequentially right exact.*

Proof. $\mathcal{L}$ is homological by Corollary 6. The reflection preserves regular epimorphisms and normal monomorphisms, thus preserves proper morphisms; one concludes by Proposition 15. □

Proposition 19. *Let $\iota, \lambda\colon \mathcal{L} \rightleftarrows \mathcal{C}$ be a protolocalisation of a homological category $\mathcal{C}$. The reflection λ preserves finite products, pullbacks along regular epimorphisms and exact forks.*

Proof. Consider the following commutative diagram in $\mathcal{C}$, where the horizontal sequences are exact.

$$\begin{array}{ccccccccc}
\mathbf{0} & \longrightarrow & S & \xrightarrow{\;s\;} & A & \xrightarrow{\;q\;} & Q & \longrightarrow & \mathbf{0} \\
 & & {\scriptstyle\alpha}\downarrow & (1) & {\scriptstyle\beta}\downarrow & (2) & {\scriptstyle\gamma}\downarrow & & \\
\mathbf{0} & \longrightarrow & T & \xrightarrow{\;t\;} & B & \xrightarrow{\;p\;} & P & \longrightarrow & \mathbf{0}
\end{array} \qquad \text{(Diagram A)}$$

The reflection λ transforms this in a diagram in $\mathcal{L}$ which is still commutative, with exact horizontal sequences.

In homological categories, the square (2) is a pullback if and only if α is an isomorphism (see [6], 4.2). This last condition is trivially preserved by λ, which thus preserves pullbacks along regular epimorphisms.

The zero object is trivially preserved by λ, while the product of two objects is their pullback over $\mathbf{0}$. But every morphism to $\mathbf{0}$ is a split, thus a regular epimorphism. One concludes by the first part of the proof.

Finally λ preserves coequalisers and, again by the first part of the proof, the kernel pair of a regular epimorphism. Thus λ preserves exact forks. □

In the semi-abelian case, additional properties are valid. First of all:

Proposition 20. *A protolocalisation of a semi-abelian category is again semi-abelian.*

Proof. This follows from Corollary 6 and Theorem 7, via Proposition 19. □

Proposition 21. *Let $\iota, \lambda\colon \mathcal{L} \rightleftarrows \mathcal{C}$ be a protolocalisation of a semi-abelian category $\mathcal{C}$. The reflection λ preserves finite intersections of normal subobjects.*

Proof. Let us refer again to (Diagram A). In homological categories, the square (1) is a pullback if and only if γ is a monomorphism (see again [6], 4.2). When β is a normal monomorphism and $\mathcal{C}$ is semi-abelian, then γ – the image of β along the regular epimorphism p – is again a normal monomorphism. This proves the result since normal monomorphisms are preserved by λ (see Lemma 16). □

The next proposition gives characterisations of protolocalisations preserving monomorphisms. Recall that a detailed treatment of reflector functors preserving monomorphisms was presented in [33]. In the abelian context any protolocalisation is, of course, sequentially exact: however, this is not the case in our general context (see Example 58).

Proposition 22. *Let $\iota, \lambda\colon \mathcal{L} \rightleftarrows \mathcal{C}$ be a protolocalisation of a homological category $\mathcal{C}$. The following conditions are equivalent:*

1. λ *preserves monomorphisms;*
2. λ *preserves image factorisations;*
3. λ *preserves kernels;*
4. λ *preserves left exact sequences;*
5. λ *preserves exact sequences;*
6. λ *preserves inverse images of normal monomorphisms;*
7. λ *preserves kernel pairs.*

Proof. $(1 \Rightarrow 2)$ because λ preserves regular epimorphisms. $(2 \Rightarrow 3)$ because the kernel of a morphism is the same as the kernel of the epi-part of its image factorisation and this last kernel is preserved by λ. $(3 \Rightarrow 1)$ because in a homological category, a monomorphism is characterised by having a zero kernel. And trivially $(3 \Leftrightarrow 4)$ since a sequence

$$\mathbf{0} \longrightarrow K \xrightarrow{\ k\ } A \xrightarrow{\ f\ } B$$

is left exact when $k = \mathsf{Ker}\, f$. $(5 \Leftrightarrow 2, 3)$ since preserving an exact sequence reduces to preserving kernels and images.

$(1 \Rightarrow 6)$ since considering again (Diagram A), the square (1) is a pullback if and only if γ is a monomorphism. $(6 \Rightarrow 3)$ because the kernel of a morphism is its pullback over zero, and every morphism with domain $\mathbf{0}$ is a normal monomorphism.

$(7 \Rightarrow 1)$ because being a monomorphism is characterised by the equality of the two projections of its kernel pair. $(1 \Rightarrow 7)$ because given a morphism $f\colon A \longrightarrow B$ in $\mathcal{C}$ and its image factorisation $f = s \circ p$, the pullback of f with itself can then be computed in four steps. The first step is the pullback of s with itself, which simply yields identities because s is a monomorphism. Since the reflection preserves monomorphisms by assumption, this pullback is trivially preserved. All other partial pullbacks involve regular epimorphisms, thus are preserved as well, by Proposition 19. □

Definition 23. A protolocalisation of a homological category is *sequentially exact* when it satisfies the conditions of Proposition 22.

5. The associated closure operator

In this section, we consider a protolocalisation $\iota, \lambda\colon \mathcal{L} \rightleftarrows \mathcal{C}$ of a homological category $\mathcal{C}$. We write $\eta\colon \mathsf{id}_{\mathcal{C}} \Rightarrow \iota\lambda$ for the unit of the adjunction and $(\mathcal{E}, \mathcal{M})$ for the corresponding prefactorisation system. We shall freely use that notation without recalling it any more.

Proposition 24. *Let $\iota, \lambda: \mathcal{L} \leftrightarrows \mathcal{C}$ be a protolocalisation of a homological category $\mathcal{C}$. The class $\mathcal{E}$ of the corresponding prefactorisation system is stable under pullbacks along regular epimorphisms.*

Proof. This follows at once from Proposition 19. □

Definition 25. Let $\iota, \lambda: \mathcal{L} \leftrightarrows \mathcal{C}$ be a protolocalisation of a homological category $\mathcal{C}$.

- A monomorphism is *stable* (with respect to the prefactorisation system) when it admits an $(\mathcal{E}, \mathcal{M})$-factorisation $s = \overline{s} \circ \widetilde{s}$ both of whose parts are still monomorphisms.
- The closure of the stable monomorphism $s: S \rightarrowtail A$ is the $\mathcal{M}$-part $\overline{s}: \overline{S} \rightarrowtail A$ of its $(\mathcal{E}, \mathcal{M})$-factorisation.

It should be noticed that the composite of two stable (resp. normal) monomorphisms has a priori no reason to be still stable (resp. normal). Moreover, in general stable monomorphisms are not pullback-stable, while normal monomorphisms are. In the homological case, with respect to the factorisation system (regular epi, mono), none of the two classes is stable under images; that is, given a stable (resp. normal) monomorphism $s: S \rightarrowtail A$ and a morphism $f: A \longrightarrow B$, in the (regular epi, mono)-factorisation $f \cdot s = m \cdot e$, m need not be stable (resp. normal). However, the closure defined above constitutes a closure operator in the sense of [19], Def. 5.2, and it makes perfect sense to define:

Definition 26. Let $\iota, \lambda: \mathcal{L} \leftrightarrows \mathcal{C}$ be a protolocalisation of a homological category $\mathcal{C}$. Given a stable subobject $s: S \rightarrowtail A$ and its closure $\overline{s}: \overline{S} \rightarrowtail A$:

1. the subobject $s: S \rightarrowtail A$ is dense when $\overline{s}: \overline{S} \rightarrowtail A$ is an isomorphism, that is, when $s \in \mathcal{E}$;
2. the subobject $s: S \rightarrowtail A$ is closed when $\widetilde{s}: S \rightarrowtail \overline{S}$ is an isomorphism, that is, when $s \in \mathcal{M}$.

Proposition 27. *Let $\iota, \lambda: \mathcal{L} \leftrightarrows \mathcal{C}$ be a protolocalisation of a homological category $\mathcal{C}$. Given a stable subobject $s: S \rightarrowtail A$ and its closure $\overline{s}: \overline{S} \rightarrowtail A$:*

1. *$\widetilde{s}: S \rightarrowtail \overline{S}$ is stable and dense;*
2. *$\overline{s}: \overline{S} \rightarrowtail A$ is stable and closed.*

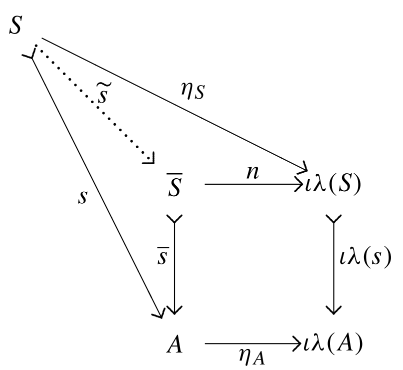
Proof. Given a stable monomorphism s and its $(\mathcal{E}, \mathcal{M})$-factorisation $s = \overline{s} \circ \widetilde{s}$, the $(\mathcal{E}, \mathcal{M})$-factorisations of $\overline{s}$ and $\widetilde{s}$ are respectively $\overline{s} \circ \mathrm{id}$ and $\mathrm{id} \circ \widetilde{s}$. □

The following result recaptures a well-known construction of the closure in the case of a localisation.

Proposition 28. *Let $\iota, \lambda: \mathcal{L} \leftrightarrows \mathcal{C}$ be a protolocalisation of a homological category $\mathcal{C}$.*

1. *Every normal monomorphism is stable and its closure is still normal.*
2. *The closure of a normal monomorphism $s: S \rightarrowtail A$ is the pullback of the monomorphism $\iota\lambda(s)$ along the unit $\eta_A: A \longrightarrow \iota\lambda(A)$ of the adjunction.*

Proof. If s is a normal monomorphism, the protolocalisation axiom implies that $\lambda(s)$ is a normal monomorphism. Thus the pullback of $\iota\lambda(s)$ along η_A is a normal monomorphism as well: let us denote it at once by $\overline{s}$. Consider then the following diagram



(Diagram B)

where the square is a pullback. The well-known properties of the prefactorisation system imply that $\iota\lambda(s) \in \mathcal{M}$, as a morphism of $\mathcal{L}$, and therefore $\overline{s} \in \mathcal{M}$, as the pullback of a morphism in $\mathcal{M}$. Next since η_S is mapped by λ to an isomorphism, n is mapped by λ to a regular epimorphism. But $\lambda(n)$ is also a monomorphism because so is $\lambda(\eta_A \circ \overline{s})$: indeed $\lambda(\eta_A)$ is an isomorphism and since $\overline{s}$ is normal, $\lambda(\overline{s})$ is a monomorphism as well. So $\lambda(n)$ is an isomorphism and both η_S and n are in $\mathcal{E}$, proving that $\widetilde{s} \in \mathcal{E}$. In particular $s = \overline{s} \circ \widetilde{s}$ is the $(\mathcal{E}, \mathcal{M})$-factorisation of s. Thus s is stable and its closure $\overline{s}$ is still normal. □

And in the case of a sequentially exact protolocalisation (see Definition 23):

Proposition 29. *Let $\iota, \lambda \colon \mathcal{L} \rightleftarrows \mathcal{C}$ be a sequentially exact protolocalisation of a homological category $\mathcal{C}$.*

1. *Every monomorphism is stable.*
2. *The closure of a monomorphism $s \colon S \rightarrowtail A$ is the pullback of $\iota\lambda(s)$ along the unit $\eta_A \colon A \longrightarrow \iota\lambda(A)$ of the adjunction, as in (Diagram B).*

Proof. The proof of Proposition 28 applies as such, simply omitting everywhere the word “normal”. □

Let us now exhibit some basic properties of the closure operator.

Proposition 30. *Let $\iota, \lambda \colon \mathcal{L} \rightleftarrows \mathcal{C}$ be a protolocalisation of a homological category $\mathcal{C}$. If $S \subseteq A$, $T \subseteq A$ are stable subobjects and $f \colon B \longrightarrow A$ is a morphism in $\mathcal{C}$:*

1. $S \subseteq \overline{S}$;
2. $\overline{\overline{S}} = \overline{S}$;
3. $S \subseteq T \Rightarrow \overline{S} \subseteq \overline{T}$;
4. *when f is a regular epimorphism,* $\overline{f^{-1}(S)} = f^{-1}(\overline{S})$.

   When $s \colon S \rightarrowtail A$ is a normal monomorphism and f is arbitrary,

5. $\overline{f^{-1}(S)} \subseteq f^{-1}(\overline{S})$.

   Moreover, in the semi-abelian case, for normal subobjects $S \subseteq A$, $T \subseteq A$ and a regular epimorphism $g \colon A \twoheadrightarrow C$:

6. $S \cap \overline{T} = \overline{S \cap T}$;
7. $g\left(\overline{S}\right) \subseteq \overline{g(S)}$.

Proof. (1) holds by definition and (2) follows from Proposition 27. (3) follows at once from the uniqueness of the $(\mathcal{E}, \mathcal{M})$-factorisation of s, which forces the $(\mathcal{E}, \mathcal{M})$-factorisation of $S \subseteq \overline{T}$ to be simply $\overline{S}$. (4) holds by Proposition 24 and the fact that morphisms in $\mathcal{M}$ are stable under pullbacks. (5) makes sense because the pullback $f^{-1}(s)$ of a normal monomorphism is normal and thus stable, by Proposition 28; the proof reduces then to a simple case based on (Diagram B).

In the semi-abelian case, (6) follows from Proposition 21. To prove (7), observe that, in the semi-abelian case, when $S \subseteq A$ is normal, so is its image $g(S) \subseteq C$ under the regular epimorphism g (see [6]). The inclusion follows from assertions 3 and 4. □

We point out that conditions 5 and 7 assert that, for normal monomorphisms, morphisms (resp. regular epimorphisms) are *continuous*, while condition 4 says that regular epimorphisms are *closure-preserving* (see [19]). This property will play a key role later (cf. Theorem 42).

Let us recall another well-known notion (see for example [13]).

Definition 31. A torsion theory on a homological category $\mathcal{C}$ consists of giving two full replete subcategories $\mathcal{T}$ (the torsion objects) and $\mathcal{F}$ (the torsion-free objects) of $\mathcal{C}$, with the two properties:

- every arrow $T \longrightarrow F$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$ is the zero arrow;
- for every object A in $\mathcal{C}$ there exists a (necessarily unique) short exact sequence

  $$\mathbf{0} \longrightarrow T \longrightarrow A \longrightarrow F \longrightarrow \mathbf{0}$$

  with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

The torsion theory is called $\mathcal{N}$-hereditary for a class $\mathcal{N}$ of monomorphisms when $\mathcal{T}$ is closed under $\mathcal{N}$-subobjects.

Example 32. Every protolocalisation of a homological category $\mathcal{C}$ induces a torsion theory on $\mathcal{C}$.

Proof. By Proposition 30 we get for normal monomorphisms what is called in [13] a *weakly hereditary closure operator*; the result follows then from Theorem 4.15 of that paper. The class $\mathcal{T}$ is that of objects in which $\mathbf{0}$ is dense, while $\mathcal{F}$ is the class of those objects in which $\mathbf{0}$ is closed. □

In [13] it is proved that torsion theories in a homological category are in bijection with fibered regular epireflections (see our Definition 35). It should be underlined that in general, such a regular epireflection is by no means a protolocalisation. Our Theorem 42 will investigate further this question.

Our main concern in this section is to show that the closure operator on stable monomorphisms, induced by a protolocalisation, characterises entirely that protolocalisation.

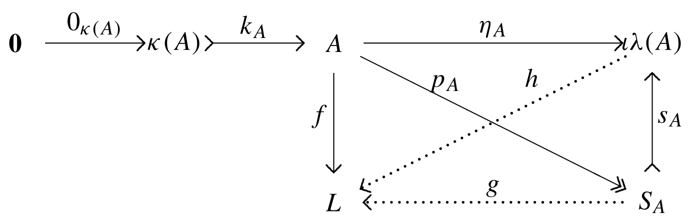
Lemma 33. *Consider a protolocalisation* $\iota, \lambda \colon \mathcal{L} \rightleftarrows \mathcal{C}$ *of a homological category* $\mathcal{C}$. *A monomorphism* $s \colon S \rightarrowtail A$ *in* $\mathcal{C}$ *is dense stable if and only if* $\lambda(s)$ *is an isomorphism.*

Proof. By definition of the closure operator, a dense stable monomorphism s is isomorphic to the $\mathcal{E}$-part of its $(\mathcal{E}, \mathcal{M})$-factorisation, thus $\lambda(s)$ is an isomorphism. Conversely if $\lambda(s)$ is an isomorphism, we have $s \in \mathcal{E}$ and thus its $(\mathcal{E}, \mathcal{M})$-factorisation is $\mathsf{id}_A \circ s$. □

Theorem 34. *Let* $\iota, \lambda \colon \mathcal{L} \rightleftarrows \mathcal{C}$ *be a protolocalisation of a homological category* $\mathcal{C}$. *The full subcategory* $\mathcal{L}$ *is that of those objects of* $\mathcal{C}$ *orthogonal to the dense stable monomorphisms.*

Proof. It is well known that each object L in $\mathcal{L}$ is orthogonal to every morphism $e \colon A \longrightarrow B \in \mathcal{E}$: that is, given $f \colon A \longrightarrow L$ there exists a unique $g \colon B \longrightarrow L$ such that $g \circ e = f$. In particular, L is orthogonal to each dense stable monomorphism (see Lemma 33).

Conversely, it is well known also that being in $\mathcal{L}$ is equivalent to being orthogonal to $\eta_A \colon A \longrightarrow \iota\lambda(A)$, the unit of the adjunction, for each $A \in \mathcal{C}$. Let us consider the following diagram



where $\eta_A = s_A \circ p_A$ is the image factorisation of η_A and $k_A = \mathsf{Ker}\, p_A$.

Since $\lambda(\eta_A)$ is an isomorphism, the regular epimorphism $\lambda(p_A)$ is also a monomorphism, thus an isomorphism. Thus $\lambda(s_A)$ is an isomorphism as well, proving that s_A is a dense stable monomorphism (Lemma 33).

On the other hand the protolocalisation λ preserves the short exact sequence (k_A, p_A). Thus $\lambda(k_A) = \mathsf{Ker}\, \lambda(p_A)$ and since $\lambda(p_A)$ is an isomorphism, $\lambda(\kappa(A)) \cong \mathbf{0}$. This proves that $\lambda(0_{\kappa(A)})$ is an isomorphism, thus $0_{\kappa(A)}$ is a dense stable monomorphism (Lemma 33).

Now consider an object $L \in \mathcal{C}$ orthogonal to every dense stable monomorphism and a morphism $f \colon A \longrightarrow L$. Since $f \circ k_A \circ 0_{\kappa(A)} = 0 = 0 \circ 0_{\kappa(A)}$, we obtain $f \circ k_A = 0$ by the uniqueness part of the orthogonality condition $0_{\kappa(A)} \perp L$. But $p_A = \mathsf{Coker}\,\mathsf{Ker}\, p_A = \mathsf{Coker}\, k_A$, from which there is a unique factorisation $g \colon S_A \longrightarrow L$ such that $g \circ p_A = f$. The orthogonality condition $s_A \perp L$ forces finally the existence of a unique morphism $h \colon \iota\lambda(A) \longrightarrow L$ such that $h \circ s_A = g$. □

When the unit η_A of the adjunction is proper for every $A \in \mathcal{C}$ (i.e. its image is a normal monomorphism), the proof of Theorem 34 shows at once that $L \in \mathcal{L}$ is equivalent to L being orthogonal to every dense normal monomorphism: indeed s_A, and of course $0_{\kappa(A)}$, are now normal monomorphisms. Then the closure operator on normal subobjects suffices already to characterise the reflection. This is in particular the case for regular epireflective protolocalisations, since then the image of η_A is an isomorphism.

6. Fibered protolocalisations

The following notion is borrowed from [8,13].

Definition 35. A protolocalisation $\iota, \lambda: \mathcal{L} \rightleftarrows \mathcal{C}$ of a homological category $\mathcal{C}$ is *fibered* when the functor $\lambda: \mathcal{C} \longrightarrow \mathcal{L}$ is a fibration (see [5], vol. 2).

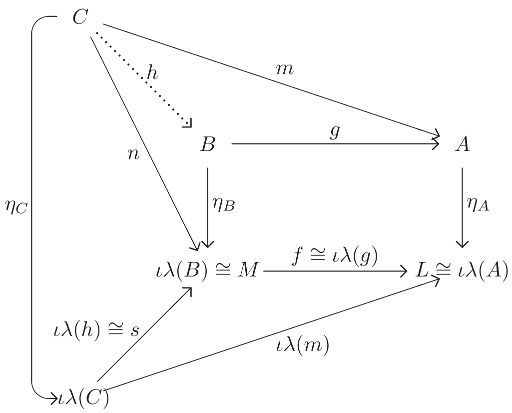
Writing $(\mathcal{E}, \mathcal{M})$ for the corresponding prefactorisation system we have the following result, various parts of which are known. To make our argument sufficiently self-contained, we give an explicit direct proof.

Proposition 36. *Let $\iota, \lambda: \mathcal{L} \rightleftarrows \mathcal{C}$ be a protolocalisation of a homological category $\mathcal{C}$. The following conditions are equivalent:*

1. *λ is fibered;*
2. *the pullback of a unit η_A of the adjunction along a morphism $f \in \mathcal{L}$ is again a unit;*
3. *the class $\mathcal{E}$ is stable under pullbacks along morphisms $f \in \mathcal{M}$;*
4. *the functor λ is semi-left-exact in the sense of* [16].

In these conditions the prefactorisation system is a factorisation system and a morphism m belongs to the class $\mathcal{M}$ if and only if the η-naturality diagram for m is a pullback.

Proof. ($1 \Rightarrow 2$). The object $A \in \mathcal{C}$ is in the fiber over $L \in \mathcal{L}$ when $L \cong \lambda(A)$. Consider $f: M \longrightarrow L$ in $\mathcal{L}$ and the corresponding cartesian morphism g. We have thus $\lambda(g) = f$; in particular, the rectangle in the following diagram is commutative and we are going to prove that it is a pullback.



Given $f \circ n = \eta_A \circ m$ in $\mathcal{C}$, n factors uniquely through η_C via a morphism s. From the equalities

$$\iota\lambda(m) \circ \eta_C = \eta_A \circ m = f \circ n = f \circ s \circ \eta_C$$

we deduce $f \circ s = \iota\lambda(m)$. Since g is cartesian over f, this forces the existence of a unique h such that $\lambda(h) = s$ and $g \circ h = m$. But $\lambda(h) = s$ is equivalent to $\eta_B \circ h = n$, the second condition needed to have a pullback. Indeed $\lambda(h) = s$ forces the equality

$$\eta_B \circ h = \iota\lambda(h) \circ \eta_C = s \circ \eta_C = n.$$

Conversely $\eta_B \circ h = n$ implies

$$\iota\lambda(h) \circ \eta_C = \eta_B \circ h = n = s \circ \eta_C$$

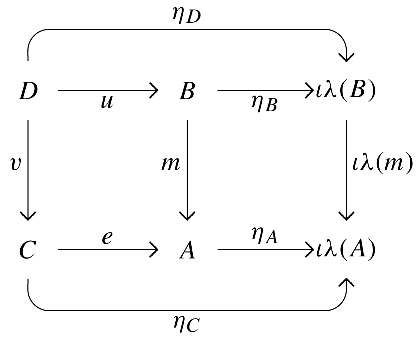
from which $\iota\lambda(h) \cong s$.

(2 ⇒ 1). Conversely when the square is a pullback and m is such that $\iota\lambda(m)$ factors as $f \circ s$, simply put $n = s \circ \eta_C$ to get the expected factorisation h.

Under assumptions 1, 2, let us now deduce the characterisation of the morphisms in $\mathcal{M}$. When the square of the statement is a pullback, $\iota\lambda(m) \in \mathcal{M}$ as a morphism in $\mathcal{L}$ and $m \in \mathcal{M}$ as the pullback of a morphism in $\mathcal{M}$. Conversely when $m \in \mathcal{M}$, choose $s = \mathsf{id}_{\iota\lambda(C)}$ in the diagram of this proof. Then $h \in \mathcal{E}$ since so do η_C and η_B. But $h \in \mathcal{M}$ because $g \circ h = m \in \mathcal{M}$ with $g \in \mathcal{M}$ as well (see [16]). Thus h is an isomorphism and the square of the statement is a pullback.

Next choosing $n = \eta_C$ and $f = \iota\lambda(m)$, with the square still a pullback, we have $g \in \mathcal{M}$ but also $h \in \mathcal{E}$, since η_B and $n = \eta_C$ are in $\mathcal{E}$. Thus $g \circ h$ is the $(\mathcal{E}, \mathcal{M})$- factorisation of m and the prefactorisation system is a factorisation system.

(2 ⇒ 3). Consider the following diagram, with $e \in \mathcal{E}$ and $m \in \mathcal{M}$.



The right-hand square is a pullback, by the characterisation of morphisms in $\mathcal{M}$ and the left-hand square is a pullback by definition. Since the bottom composite is in $\mathcal{E}$, it is isomorphic to η_C. But by condition 2 of the statement, the upper composite is then isomorphic to η_D. Since η_B and η_D are in $\mathcal{E}$, we obtain $u \in \mathcal{E}$.

(3 ⇒ 2) is obvious since every morphism of $\mathcal{L}$ is in $\mathcal{M}$.

(3 ⇔ 4) is just the definition of a semi-left-exact reflection (see [16]). □

The fibered case reinforces the role of stable monomorphisms (see Definition 25):

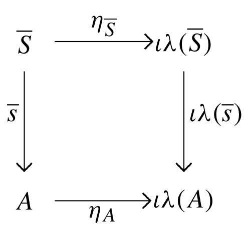
Proposition 37. *Consider a fibered protolocalisation $\iota, \lambda : \mathcal{L} \rightleftarrows \mathcal{C}$ of a homological category. For a monomorphism $s : S \rightarrowtail A$ in $\mathcal{C}$, the following conditions are equivalent:*

1. *s is stable;*
2. *$\lambda(s)$ is a monomorphism.*

Moreover, the closure of a stable monomorphism is computed via the pullback in (Diagram B).

Proof. Let us write $s = \overline{s} \circ \widetilde{s}$ for the $(\mathcal{E}, \mathcal{M})$-factorisation of an arbitrary morphism s (see Proposition 36). In any case, $\lambda(\widetilde{s})$ is an isomorphism. Thus $\lambda(s)$ is a monomorphism if and only if $\lambda(\overline{s})$ is a monomorphism.

Since $\overline{s} \in \mathcal{M}$, by fiberedness the following square is a pullback (see Proposition 36):



When s is a stable monomorphism, $\overline{s}$ is a monomorphism; by protomodularity, pullbacks reflect monomorphisms (see [9]), thus $\iota\lambda(\overline{s})$ is a monomorphism as well. The converse is trivial.

The proof of Proposition 28 applies as such to prove the last assertion: simply omit everywhere the word "normal". □

Our following result further underlines the important role of proper morphisms in the semi-abelian case.

Proposition 38. *Consider a protoregular reflection $\iota, \lambda \colon \mathcal{L} \rightleftarrows \mathcal{C}$ of a semi-abelian category $\mathcal{C}$. The following conditions are equivalent:*

1. *the class $\mathcal{E}$ is stable under pullbacks along the morphisms of $\mathcal{L}$, while the class of normal monomorphisms in $\mathcal{E}$ is stable under pullbacks along proper morphisms;*
2. *the reflection is a fibered protolocalisation.*

Proof. As usual we call *dense* a monomorphism belonging to the class $\mathcal{E}$.

(1 $\Rightarrow$ 2). Consider a short exact sequence (s, q) in $\mathcal{C}$, the morphism $\iota\lambda(q)$ and its kernel l in $\mathcal{L}$. Let us pay attention: of course $\lambda(q)$ is a regular epimorphism in $\mathcal{L}$, but $\iota\lambda(q)$ has no reason to be still a regular epimorphism in $\mathcal{C}$. We consider further the commutative square on the right and the corresponding vertical factorisation on the left.

$$\begin{array}{ccccccccc}
\mathbf{0} & \longrightarrow & S & \xrightarrow{s} & A & \xrightarrow{q} & Q & \longrightarrow & \mathbf{0} \\
 & & {\scriptstyle n}\,\vdots & & {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_Q} & & \\
\mathbf{0} & \longrightarrow & L & \xrightarrow[l]{} & \iota\lambda(A) & \xrightarrow[\iota\lambda(q)]{} & \iota\lambda(Q) & &
\end{array}$$

It suffices to prove that $n \in \mathcal{E}$: indeed since $L \in \mathcal{L}$, this will prove that $L \cong \iota\lambda(S)$ and finally $l \cong \iota\lambda(s)$. So in $\mathcal{L}$ we shall have the short exact sequence

$$\mathbf{0} \longrightarrow \lambda(S) \xrightarrow{\lambda(s)} \lambda(A) \xrightarrow{\lambda(q)} \lambda(Q) \longrightarrow \mathbf{0}$$

as expected, because $\lambda(q)$ is a regular epimorphism in $\mathcal{L}$.

Let us now consider the pullback P of l and η_A and the corresponding factorisation m:

$$\begin{array}{ccccc}
S & & & & \\
 & \overset{m}{\searrow} & & \overset{s}{\searrow} & \\
 {\scriptstyle n}\downarrow & & P & \xrightarrow{t} & A \\
 & & {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle \eta_A} \\
 & \to & L & \xrightarrow[l]{} & \iota\lambda(A)
\end{array}$$

We have $l \in \mathcal{L}$ and $\eta_A \in \mathcal{E}$, thus by assumption we get $r \in \mathcal{E}$. So to prove that $n \in \mathcal{E}$, it suffices to prove that $m \in \mathcal{E}$. Notice at once that since $l = \mathsf{Ker}\,\iota\lambda(q)$ and the square is a pullback, then $t = \mathsf{Ker}\,(\iota\lambda(q) \circ \eta_A)$. Notice also that $m = \mathsf{Ker}\,(q \circ t)$ since $s = \mathsf{Ker}\,q$.

To prove that $m \in \mathcal{E}$, we observe first that by assumption, the following pullback

$$\begin{array}{ccc}
\kappa(Q) & \xrightarrow{k_Q} & Q \\
\downarrow & & \downarrow{\scriptstyle \eta_Q} \\
\mathbf{0} & \longrightarrow & \iota\lambda(Q)
\end{array}$$

is preserved by λ: indeed, $\eta_Q \in \mathcal{E}$ while the bottom morphism is in $\mathcal{L}$. In other words, λ preserves the kernel of η_Q and since $\lambda(\eta_Q)$ is an isomorphism, its kernel is $\mathbf{0}$. This proves that the monomorphism $\mathbf{0} \rightarrowtail \kappa(Q)$ is inverted by λ, thus lies in $\mathcal{E}$.

Next the epimorphism q induces trivially a factorisation q' in the following diagram of short exact sequences

$$\begin{array}{ccccccccc}
\mathbf{0} & \longrightarrow & P & \stackrel{t}{\rightarrowtail} & A & \xrightarrow{\iota\lambda(q)\circ\eta_A} & \iota\lambda(Q) & \longrightarrow & \mathbf{0} \\
 & & {\scriptstyle q'}\vdots & (1) & {\scriptstyle q}\downarrow & (2) & \| & & \\
\mathbf{0} & \longrightarrow & \kappa(Q) & \underset{k_Q}{\rightarrowtail} & Q & \underset{\eta_Q}{\longrightarrow} & \iota\lambda(Q) & \longrightarrow & \mathbf{0}
\end{array}$$

Since the right-hand vertical morphism is a monomorphism, the square (1) is a pullback. Since q is a regular epimorphism, q' is a regular epimorphism as well. Moreover, still because the square (1) is a pullback, we get the isomorphism $\mathsf{Ker}\, q' \cong \mathsf{Ker}\, q = m$.

We have thus obtained the following pullback square

$$\begin{array}{ccc}
S & \longrightarrow & \mathbf{0} \\
{\scriptstyle m}\downarrow & & \downarrow \\
P & \underset{q'}{\longrightarrow} & \kappa(Q)
\end{array}$$

where the right-hand vertical arrow is a dense monomorphism and – of course – a normal one. Since q' is a regular epimorphism, $m \in \mathcal{E}$ by assumption.

Conversely, suppose that we have a fibered protolocalisation. By Proposition 36, the class $\mathcal{E}$ is stable under pullbacks along the morphisms of $\mathcal{L}$. By Proposition 19, the class of dense normal monomorphisms is closed under pullbacks along regular epimorphisms and by Proposition 21, it is also closed under pullbacks along normal monomorphisms. □

7. The case of regular epireflections

A reflection of a regular category having regular epimorphic units will be called *regular epireflection* (see Example 11).

Proposition 39. *Every regular epireflective protolocalisation* $\iota, \lambda : \mathcal{L} \rightleftarrows \mathcal{C}$ *of a homological category* $\mathcal{C}$ *has stable units in the sense of* [16] *and in particular, is fibered.*

Proof. The reflection λ preserves pullbacks along regular epimorphisms (see Proposition 19). Since the unit η_A of the adjunction is a regular epimorphism mapped by λ to an isomorphism, so is thus the pullback of η_A along an arbitrary morphism. This means that the reflection has stable units in the sense of [16]; in particular condition 2 in Proposition 36 is satisfied. □

Definition 40. Let $\mathcal{C}$ be a semi-abelian category.

- A *radical* is a normal subfunctor $\kappa : \mathcal{C} \longrightarrow \mathcal{C}$ of the identity functor satisfying, for every $A \in \mathcal{C}$, the property $\kappa\,(A/\kappa(A)) = \mathbf{0}$. We write $k_A : \kappa(A) \rightarrowtail A$ for the canonical normal inclusion.
- A radical κ is *short exact* when the functor $\kappa : \mathcal{C} \longrightarrow \mathcal{C}$ preserves short exact sequences.

Proposition 41. *Every short exact radical* κ *on a semi-abelian category is idempotent.*

Proof. Indeed, applying κ to the short exact sequence

$$\mathbf{0} \longrightarrow \kappa(A) \xrightarrow{k_A} A \xrightarrow{q_A} A/\kappa(A) \longrightarrow \mathbf{0}$$

yields

$$\mathbf{0} \longrightarrow \kappa\big(\kappa(A)\big) \xrightarrow{\kappa(k_A)} \kappa(A) \xrightarrow{\kappa(q_A)} \mathbf{0} \longrightarrow \mathbf{0}. \qquad \square$$

Given a semi-abelian category $\mathcal{C}$, we write $\mathcal{N}$ for the class of normal monomorphisms and use accordingly Definition 31. We refer also to Definition 12. Given a normal subobject $s: S \rightarrowtail A$, and a regular epimorphism $q: A \twoheadrightarrow Q$, we write $q(S)$ for the regular image of S along q.

Theorem 42. *Let $\mathcal{C}$ be a semi-abelian category. There are bijections between:*

1. *the regular epireflective protolocalisations $\iota, \lambda: \mathcal{L} \rightleftarrows \mathcal{C}$ of $\mathcal{C}$;*
2. *the torsion-free Birkhoff subcategories $\mathcal{L}$ of $\mathcal{C}$, for an $\mathcal{N}$-hereditary torsion theory $(\mathcal{T}, \mathcal{L})$;*
3. *the closure operators on normal subobjects satisfying the properties:*
   (a) $S \subseteq \overline{S}$;
   (b) $\overline{\overline{S}} = \overline{S}$;
   (c) $S \subseteq T$ *implies* $\overline{S} \subseteq \overline{T}$;
   (d) $\overline{f^{-1}(S)} = f^{-1}(\overline{S})$ *for a proper arrow* f;
   (e) $\overline{f^{-1}(S)} \subseteq f^{-1}(\overline{S})$ *for an arbitrary arrow* f;
   (f) $f(\overline{S}) = \overline{f(S)}$ *for a regular epimorphism* f;
4. *the short exact radicals κ on $\mathcal{C}$.*

Proof. First we remark that, with respect to the closure on normal subobjects, 3(e) means continuity of every morphism, while 3(f) says that regular epimorphisms are closure-preserving and 3(d) means that proper morphisms are open. (Here the reader should not confuse proper morphisms, in our algebraic sense, with (Bourbaki) proper maps, i.e. c-compact, or c-preserving morphisms, with respect to a closure operator c — see [18].)

It is shown in [13] that there are bijections between:

(2″) the Birkhoff subcategories of a semi-abelian category $\mathcal{C}$;
(3″) the closure operators on normal subobjects satisfying the properties (a), (b), (c), (d′), (e), (f), where (d′) is condition (d) restricted to the case of a regular epimorphism f;
(4″) the idempotent radicals in $\mathcal{C}$ preserving regular epimorphisms.

The bijections that we shall establish are just restrictions of those above. More precisely, we are going to show that for a regular epireflection $\lambda: \mathcal{C} \longrightarrow \mathcal{L}$ of a semi-abelian category, the following conditions are equivalent, which will immediately give the result:

(1′) the regular epireflection $\lambda: \mathcal{C} \longrightarrow \mathcal{L}$ preserves short exact sequences;
(2′) the regular epireflective subcategory $\mathcal{L}$ is Birkhoff and $\mathcal{N}$-hereditary torsion-free;
(3′) the corresponding closure operator satisfies axiom (d);
(4′) the corresponding radical is short exact.

($1' \Rightarrow 2'$). Of course, condition (1′) implies that λ preserves normal monomorphisms. Let us first prove that $\mathcal{L}$ is Birkhoff in $\mathcal{C}$. Let $q: L \twoheadrightarrow Q$ be a regular epimorphism in $\mathcal{C}$, with L in $\mathcal{L}$. Since $\mathcal{L}$ is a regular epireflective subcategory of $\mathcal{C}$, it is closed in $\mathcal{C}$ under subobjects, so that the kernel S of q belongs to $\mathcal{L}$ as well. We obtain then a commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
\mathbf{0} & \longrightarrow & S & \xrightarrow{s} & L & \xrightarrow{q} & Q & \longrightarrow & \mathbf{0} \\
 & & {\scriptstyle \eta_S}\downarrow{\scriptstyle \cong} & & {\scriptstyle \eta_L}\downarrow{\scriptstyle \cong} & & {\scriptstyle \eta_Q}\downarrow & & \\
\mathbf{0} & \longrightarrow & \iota\lambda(S) & \xrightarrow[\iota\lambda(s)]{} & \iota\lambda(L) & \xrightarrow[\iota\lambda(q)]{} & \iota\lambda(Q) & \longrightarrow & \mathbf{0}
\end{array}$$

where the vertical arrows are the various components of the unit η of the adjunction. Indeed, λ preserves the top exact sequence, while ι preserves further the kernel $\lambda(s) = \operatorname{Ker}\lambda(q)$; but since η_Q and q are regular epimorphisms in $\mathcal{C}$, so is $\iota\lambda(q)$ and thus it is the cokernel of its kernel $\iota\lambda(s)$. This proves that the bottom line is exact in $\mathcal{C}$. The fact that the unit η_S is an isomorphism implies that the right-hand square is a pullback, because the category $\mathcal{C}$ is semi-abelian. Since in $\mathcal{C}$ pullbacks reflect monomorphisms, it follows that the regular epimorphism η_Q is a monomorphism, hence an isomorphism, so that $Q \in \mathcal{L}$.

Let us prove that $\mathcal{L}$ is a torsion-free subcategory of $\mathcal{C}$. Given $A \in \mathcal{C}$, consider the canonical exact sequence (k_A, η_A) obtained by taking the kernel of the unit of the adjunction. Since λ preserves short exact sequences, applying the functor $\iota\lambda\colon \mathcal{C} \longrightarrow \mathcal{C}$ yields the following canonical commutative diagram.

$$\begin{array}{ccccccccc}
\mathbf{0} & \longrightarrow & \kappa(A) & \xrightarrow{k_A} & A & \xrightarrow{\eta_A} & \iota\lambda(A) & \longrightarrow & \mathbf{0} \\
 & & \downarrow{\scriptstyle \eta_{\kappa(A)}} & & \downarrow{\scriptstyle \eta_A} & & \| & & \\
\mathbf{0} & \longrightarrow & \iota\lambda(\kappa(A)) & \xrightarrow[\iota\lambda(k_A)]{} & \iota\lambda(A) & = & \iota\lambda(A) & \longrightarrow & \mathbf{0}
\end{array}$$

Since the lower row is left exact, it follows that $\iota\lambda(\kappa(A)) = \mathbf{0}$. Thus $\lambda(\kappa(A)) = \mathbf{0}$ for all $A \in \mathcal{C}$, proving that $\mathcal{L}$ is a torsion-free subcategory in $\mathcal{C}$.

The induced torsion theory $(\mathcal{L}, \mathcal{T})$ is $\mathcal{N}$-hereditary. Indeed given a normal monomorphism $s\colon S \rightarrowtail T$, with T in the torsion subcategory $\mathcal{T}$, its reflection $\lambda(s)\colon \lambda(S) \rightarrowtail \mathbf{0}$ is a normal monomorphism, thus $\lambda(S) \cong \mathbf{0}$.

$(2' \Rightarrow 3')$. First, let us prove that under assumption $(2')$, $\lambda(s)$ is a monomorphism for every normal monomorphism $s\colon S \rightarrowtail A$.

Consider for this the following diagram of short exact sequences

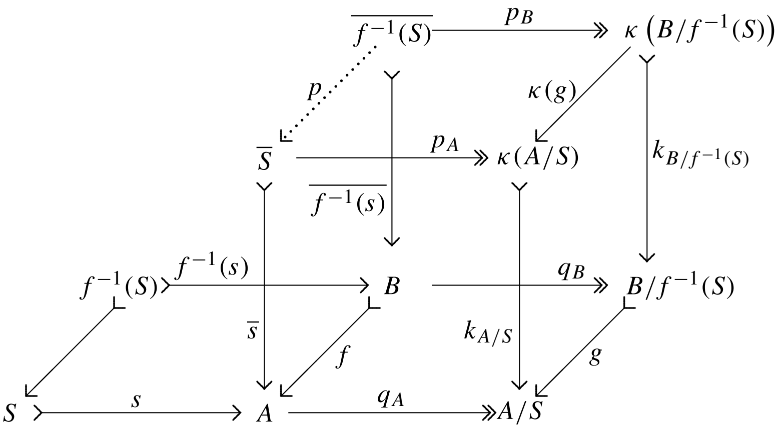
$$\begin{array}{ccccccccc}
\mathbf{0} & \longrightarrow & \kappa(S) & \xrightarrow{k_S} & S & \xrightarrow{\eta_S} & \iota\lambda(S) & \longrightarrow & \mathbf{0} \\
 & & \downarrow{\scriptstyle \kappa(s)} & (1) & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \iota\lambda(s)} & & \\
\mathbf{0} & \longrightarrow & \kappa(A) & \xrightarrow[k_A]{} & A & \xrightarrow[\eta_A]{} & \iota\lambda(A) & \longrightarrow & \mathbf{0}
\end{array}$$

and, computing the pullback P of k_A and s, the other diagram:

$$\begin{array}{ccccccccc}
\mathbf{0} & \longrightarrow & P & \xrightarrow{t} & S & \xrightarrow{q} & S/P & \longrightarrow & \mathbf{0} \\
 & & \downarrow{\scriptstyle s'} & (2) & \downarrow{\scriptstyle s} & & \vdots{\scriptstyle m} & & \\
\mathbf{0} & \longrightarrow & \kappa(A) & \xrightarrow[k_A]{} & A & \xrightarrow[\eta_A]{} & \iota\lambda(A) & \longrightarrow & \mathbf{0}
\end{array}$$

Since the square (2) is a pullback and s is a normal monomorphism, s' is a normal monomorphism as well. But $\kappa(A) \in \mathcal{T}$, thus by heredity, $P \in \mathcal{T}$. Again since (2) is a pullback, m is a monomorphism and thus $S/P \in \mathcal{L}$, by epireflectivity. By the uniqueness of the exact sequence in Definition 31, the two upper exact sequences are isomorphic, thus finally also the two diagrams. So $\iota\lambda(s) \cong m$ is a monomorphism and the square (1) is a pullback.

To show that axiom (d) holds, it is enough to show that $\overline{f^{-1}(S)} = f^{-1}\left(\overline{S}\right)$ for $f\colon B \longrightarrow A$ a normal monomorphism; indeed by axiom (d′), we already know that the same equality holds when f is a regular epimorphism. It is proved in [13] that under the bijections involved here, the closure of a normal subobject $s\colon S \rightarrowtail A$ is the pullback of the kernel $k_{A/S}$ of the unit $\eta_{A/S}$ along the quotient map $q_A\colon A \twoheadrightarrow A/S$. So, let f be a normal monomorphism and consider the following diagram, where thus the front and the back faces are pullbacks.

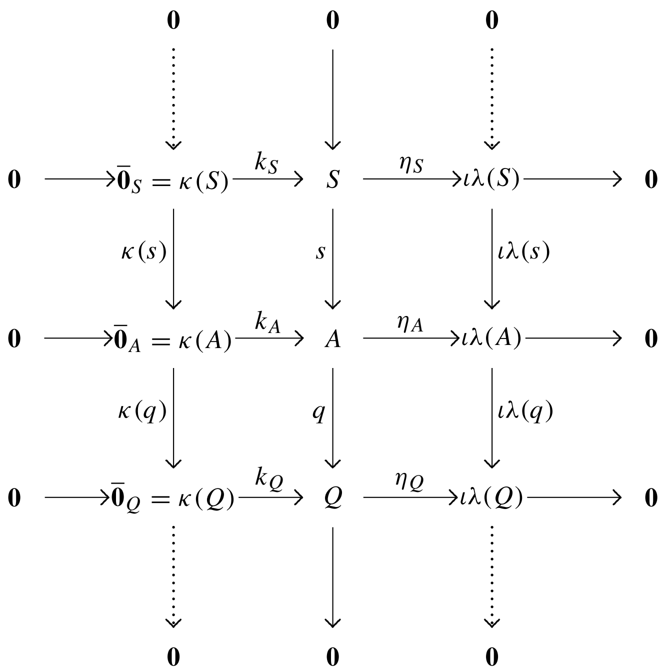


We are going to prove that the left-hand vertical square is a pullback. First remark that $\mathcal{C}$ semi-abelian implies that the induced arrow g is a monomorphism because the left-hand horizontal square is a pullback by construction (see [10]). On the other hand, since f is a normal monomorphism, so is g because in a semi-abelian category, the regular image of a normal monomorphism is normal (see [27]). As already observed, the right-hand vertical square is then a pullback as well. By associativity of pullbacks one concludes that the left-hand vertical square is a pullback, and $\overline{f^{-1}(S)} = f^{-1}(\overline{S})$ as desired.

($3' \Rightarrow 4'$). Given a short exact sequence in $\mathcal{C}$

$$\mathbf{0} \longrightarrow S \xrightarrow{\;s\;} A \xrightarrow{\;q\;} Q \longrightarrow \mathbf{0}$$

one considers the canonical commutative diagram



where $\overline{\mathbf{0}}_X$ indicates the closure of $\mathbf{0}$ in X.

Condition (d) implies that $s^{-1}\left(\bar{\mathbf{0}}_A\right) = \bar{\mathbf{0}}_S$: in other words, the upper left square is a pullback. Accordingly, the arrow $\iota\lambda(s)$ is a monomorphism in $\mathcal{C}$, thus a normal one as the image of the normal monomorphism s along the regular epimorphism η_A in the semi-abelian category $\mathcal{C}$. Thus $\iota\lambda(s) = \mathsf{Ker}\,\mathsf{Coker}\,\iota\lambda(s)$. But the bijections established in [13] and recalled at the beginning of this proof tell us in particular that $\mathcal{L}$ is Birkhoff in $\mathcal{C}$. Therefore $\mathsf{Coker}\,\iota\lambda(s) \in \mathcal{L}$ and thus is the cokernel of $\lambda(s)$ in $\mathcal{L}$. But trivially, $\lambda(q) = \mathsf{Coker}\,\lambda(s)$ in $\mathcal{L}$. So $\iota\lambda(q) = \mathsf{Coker}\,\iota\lambda(s)$ in $\mathcal{C}$ and the right-hand vertical sequence is exact. The (3×3)-Lemma (see [10]) now implies that the left-hand vertical sequence is exact as well.

($4' \Rightarrow 1'$). When κ is a short exact radical, for any exact sequence

$$\mathbf{0} \longrightarrow S \xrightarrow{\ s\ } A \xrightarrow{\ q\ } Q \longrightarrow \mathbf{0}$$

the left-hand and the central vertical sequences in the diagram above are exact. Consequently, the right-hand vertical sequence is exact as well, again by the (3×3)-Lemma. □

8. The case of monoreflections

We are now interested in a protolocalisation $\iota, \lambda \colon \mathcal{L} \rightleftarrows \mathcal{C}$ of a homological category $\mathcal{C}$, whose unit $\eta_A \colon A \longrightarrow \iota\lambda(A)$ is a monomorphism (and, then, a bimorphism) in each component. Our Example 67 is of that nature.

Theorem 43. *Consider a protolocalisation $\iota, \lambda \colon \mathcal{L} \rightleftarrows \mathcal{C}$ of a homological category $\mathcal{C}$. The following conditions are equivalent:*

1. *the protolocalisation is monoreflective;*
2. *every dense stable monomorphism is an epimorphism.*

In particular, the unit of the adjunction is both a monomorphism and an epimorphism.

Proof. ($1 \Rightarrow 2$). Consider a dense stable monomorphism $s \colon S \rightarrowtail A$. Given $f, g \colon A \rightrightarrows B$ such that $f \circ s = g \circ s$, we get $\lambda(f) = \lambda(g)$ since $\lambda(s)$ is an isomorphism. Then

$$\eta_B \circ f = \iota\lambda(f) \circ \eta_A = \iota\lambda(g) \circ \eta_A = \eta_B \circ g$$

and thus $f = g$ since η_B is a monomorphism.

($2 \Rightarrow 1$). Given an object $A \in \mathcal{C}$, consider the image factorisation $\eta_A = s_A \circ p_A$ of the unit. Consider further the kernel k_A of p_A, yielding thus the short exact sequence

$$\mathbf{0} \longrightarrow \kappa(A) \xrightarrow{\ k_A\ } A \xrightarrow{\ p_A\ } \!\!\!\rightarrow S_A \longrightarrow \mathbf{0}.$$

This short exact sequence is preserved by λ. But $\lambda(p_A)$ is an isomorphism, as observed in the proof of Theorem 34. Thus $\lambda(\kappa(A)) = \mathbf{0}$, proving that the monomorphism $\mathbf{0} \rightarrowtail \kappa(A)$ is dense. By assumption, this monomorphism is an epimorphism and since it admits trivially a retraction, it is an isomorphism. But since $\mathcal{C}$ is homological, $\kappa(A) \cong \mathbf{0}$ implies that p_A is a monomorphism. Therefore p_A is an isomorphism and $\eta_A \cong s_A$ is a monomorphism.

The unit of the adjunction is an $\mathcal{E}$-morphism for the corresponding factorisation system $(\mathcal{E}, \mathcal{M})$. Thus it is a dense stable monomorphism and therefore an epimorphism, as soon as it is a monomorphism. □

We exhibit now an interesting relation with another known notion.

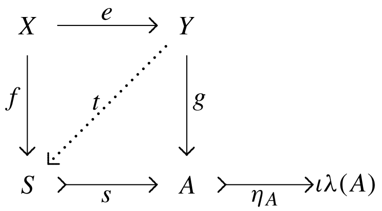
Definition 44. Consider a protolocalisation $\iota, \lambda \colon \mathcal{L} \rightleftarrows \mathcal{C}$ of a homological category $\mathcal{C}$. An object $S \in \mathcal{C}$ is *absolutely closed* when every stable monomorphism with domain S is closed.

As far as we know, the concept of "absolutely closed object" has been introduced in [25] and used later by various authors; see for example [34].

Proposition 45. *Consider a monoreflective protolocalisation $\iota, \lambda \colon \mathcal{L} \rightleftarrows \mathcal{C}$ of a homological category $\mathcal{C}$. Then $\mathcal{L}$ is (up to an equivalence) the full subcategory of absolutely closed objects.*

Proof. Assume that S is absolutely closed. The unit $\eta_S\colon S \rightarrowtail \iota\lambda(S)$ is a dense stable monomorphism, but is also closed by assumption on S; therefore it is an isomorphism.

Conversely consider a stable subobject $s\colon S \rightarrowtail A$ with $S \in \mathcal{L}$; we must prove that $s \in \mathcal{M}$ (see Definition 26). Given $s \circ f = g \circ e$ with $e \in \mathcal{E}$



we have $\eta_A \circ s \in \mathcal{L}$ thus $\eta_A \circ s \in \mathcal{M}$. This implies the existence of a unique t such that $t \circ e = f$ and $\eta_A \circ s \circ t = \eta_A \circ g$. Since η_A is a monomorphism, the second equality is equivalent to $s \circ t = g$, proving that $s \in \mathcal{M}$. $\square$

9. Algebraic examples

Of course, in view of Proposition 2, one would like to know if the category of groups admits non-trivial protolocalisations: this remains an open problem. But there are many other interesting examples.

Given a ring R, the category $\mathsf{Alg}(R)$ of R-algebras without necessarily a unit is semi-abelian because the corresponding theory contains a group operation (see [15]); $\mathsf{Alg}(R)$ is not abelian since it is not additive. Nevertheless, most examples of localisations in module theory carry over rather trivially to the case of algebras. Just to underline this fact, let us observe the result in the case which is at the origin of the name *localisation*.

Example 46. Let $\mathfrak{p}$ be a prime ideal in a ring R with unit. Consider the corresponding localised ring $R_\mathfrak{p}$. The functors

$$U\colon \mathsf{Alg}(R_\mathfrak{p}) \longrightarrow \mathsf{Alg}(R), \quad U(A) = A, \qquad - \otimes_R R_\mathfrak{p}\colon \mathsf{Alg}(R) \longrightarrow \mathsf{Alg}(R_\mathfrak{p})$$

constitute a localisation between the corresponding categories of algebras.

Proof. It is well known that we obtain a localisation

$$U\colon \mathsf{Mod}(R_\mathfrak{p}) \longrightarrow \mathsf{Mod}(R), \quad U(A) = A, \qquad - \otimes_R R_\mathfrak{p}\colon \mathsf{Mod}(R) \longrightarrow \mathsf{Mod}(R_\mathfrak{p})$$

for the corresponding categories of modules. This adjunction restricts to the categories of algebras: given an R-algebra A, it suffices to provide the tensor product $A \otimes_R R_\mathfrak{p}$ with the multiplication induced by $(a \otimes r) \cdot (a' \otimes r') = (a \cdot a') \otimes (r \cdot r')$. This is still a localisation since finite limits of algebras are computed as for modules (that is, as in the category of sets). $\square$

Here is another general result of interest. We recall that a monomorphism in an algebraic variety is *pure* (see [2]) when it is a filtered colimit of monomorphisms admitting a retraction. Notice that the retractions are not requested to be compatible, so that a pure monomorphism does not have a retraction in general. See [7] for examples of varieties where all monomorphisms are pure.

Proposition 47. *Let $\mathcal{C}$ be a semi-abelian algebraic variety where every normal monomorphism is pure. Then every subvariety $\mathcal{L} \subseteq \mathcal{C}$ is a regular epireflective protolocalisation of $\mathcal{C}$.*

Proof. A subvariety $\mathcal{L}$ is obtained by adding axioms to the algebraic theory defining $\mathcal{C}$: thus $\mathcal{L}$ is regular epireflective and Birkhoff (see Definition 12) in $\mathcal{C}$.

A normal monomorphism $s\colon A \rightarrowtail B$ in $\mathcal{C}$ is pure, thus is a filtered colimit of monomorphisms $s_j\colon A_j \rightarrowtail B_j$ admitting a retraction. Of course each $\iota\lambda(s_j)$ has a retraction, thus is a monomorphism. Therefore $\iota\lambda(s)$ is a filtered colimit of monomorphisms and so is a monomorphism.

The monomorphism $\iota\lambda(s)$ is the image of the normal monomorphism s along the regular epimorphism η_B (the unit of the adjunction), thus it is a normal monomorphism in $\mathcal{C}$, because $\mathcal{C}$ is semi-abelian (see [6]).

But then $\iota\lambda(s) = \mathsf{Ker}\,\mathsf{Coker}\,\iota\lambda(s)$ in $\mathcal{C}$, with $\mathsf{Coker}\,\iota\lambda(s) \in \mathcal{L}$ because $\mathcal{L}$ is Birkhoff in $\mathcal{C}$. Thus $\lambda(s)$ is indeed a kernel in $\mathcal{L}$ and the reflection λ preserves normal monomorphisms. One concludes by Lemma 16. $\square$

A ring is von Neumann regular (see [32]) when for every element x there exists an element x' such that $x \cdot x' \cdot x = x$. Putting $x^* = x' \cdot x \cdot x'$ one obtains both $x \cdot x^* \cdot x = x$ and $x^* \cdot x \cdot x^* = x^*$. In the commutative case, a straightforward computation shows that an element x^* with these two latter properties is necessarily unique. Thus the theory of commutative von Neumann regular rings is the algebraic theory obtained from that of rings by adding an operation $()^*$ satisfying the two axioms above. The uniqueness of x^* implies also that every ring homomorphism commutes with the $()^*$ operation. We write VNReg for the category of commutative von Neumann regular rings, not necessarily with unit. This is a semi-abelian category, since the theory is equipped with a group operation. Furthermore, it is an arithmetical category, as proved in [6], Example 2.9.15.

Lemma 48. *In the category* VNReg *of von Neumann regular rings, every normal monomorphism is pure.*

Proof. Let $R \in \mathsf{VNReg}$. For every element $a \in R$, the element $e_a = a \cdot a^*$ satisfies $e_a = e_a \cdot e_a$, $e_a = e_a^*$ and $a \cdot e_a = a$. So a belongs to the principal ideal $R \cdot e_a$ and this ideal is a retract of R: the retraction is simply the multiplication by e_a.

Given two elements a, b in R, the element $e = e_a + e_b - e_a \cdot e_b$ has the properties $e \cdot e = e$, $e = e^*$, $a \cdot e = a$, $b \cdot e = b$. This implies at once $R \cdot e_a + R \cdot e_b = R \cdot e$, proving that the family of principal ideals of the form $R \cdot e$, with $e = e \cdot e$ and $e^* = e$ is a filtered family of retracts of R. And as we have seen, every element $a \in R$ belongs to such an ideal.

Thus each monomorphism $s : I \rightarrowtail R$ is the filtered union of the monomorphisms

$$I \cdot e \rightarrowtail R, \quad e \in I, \quad e = e \cdot e, \quad e = e^*.$$

When s is normal, I is an ideal in R and each $I \cdot e$ is a retract of R, with the multiplication by e as a retraction. This proves that s is pure. □

Let us now denote by Boole the variety of Boolean rings: this is the subvariety of the category of rings determined by the identity: $x \cdot x = x$. In particular $x = x \cdot x \cdot x$, so that every Boolean ring is von Neumann regular, with $x^* = x$.

In view of Proposition 47 and Lemma 48, we obtain at once:

Example 49. The subvariety Boole of Boolean rings is a protolocalisation of the variety VNReg of von Neumann regular rings. □

It remains an open question to determine whether Boole is a localisation of VNReg.

10. Examples in terms of colimits

A whole bunch of examples are based on the following trivial fact:

Lemma 50. *Let $\mathcal{D}$ be a small category and $\mathcal{A}$ a $\mathcal{D}$-cocomplete category. When $\mathcal{D}$ is connected, we obtain a full reflective subcategory*

$$\Delta : \mathcal{A} \longrightarrow [\mathcal{D}, \mathcal{A}], \quad \mathsf{colim} : [\mathcal{D}, \mathcal{A}] \longrightarrow \mathcal{A}, \quad \mathsf{colim} \dashv \Delta$$

where $\Delta(A)$ is the constant functor on A and $\mathsf{colim} F$ *is the colimit object of F. Moreover when $\mathcal{A}$ is homological (resp. semi-abelian), so is the functor category* $[\mathcal{D}, \mathcal{A}]$.

Proof. The adjunction is just the rephrasing of the definition of a colimit. The functor Δ is full and faithful as soon as $\mathcal{D}$ is connected.

In a category $[\mathcal{D}, \mathcal{A}]$ of functors, all ingredients appearing in the definitions of a homological or a semi-abelian category are pointwise notions, so that $[\mathcal{D}, \mathcal{A}]$ is homological (resp. semi-abelian) as soon as $\mathcal{A}$ is homological (resp. semi-abelian). □

The first type of colimit that we consider is (see [29,1]):

Definition 51. A category $\mathcal{D}$ is *sifted* when $\mathcal{D}$-colimits commute in Set with finite products.

In particular, the commutation with the terminal object forces a sifted category to be connected. More precisely, a category is sifted when, for every pair of objects, the corresponding category of cospans is connected (see [29,1]).

Example 52. Let $\mathbb{T}$ be a semi-abelian algebraic theory (see [15]) and $\mathcal{D}$ a small sifted category. The reflection

$$\Delta\colon \mathsf{Set}^{\mathbb{T}} \longrightarrow [\mathcal{D}, \mathsf{Set}^{\mathbb{T}}], \quad \mathsf{colim}\colon [\mathcal{D}, \mathsf{Set}^{\mathbb{T}}] \longrightarrow \mathsf{Set}^{\mathbb{T}}, \quad \mathsf{colim} \dashv \Delta$$

is semi-left-exact and sequentially right exact.

Proof. In an algebraic variety, sifted colimits are computed as in the category of sets and so in particular, are universal. Thus in the following pullback square, where $F, G \in \left[\mathcal{D}, \mathsf{Set}^{\mathbb{T}}\right]$ and $A \in \mathsf{Set}^{\mathbb{T}}$:

$$\begin{array}{ccc} G & \longrightarrow & A \\ \downarrow & & \downarrow \\ F & \longrightarrow & \Delta\mathsf{colim}(F) \end{array}$$

we have also $A \cong \Delta\mathsf{colim} G$. This precisely means that the reflection is semi-left-exact (see Theorem 4.3 in [16]).

To prove the sequential right exactness, we must show that a $\mathcal{D}$-colimit of proper morphisms is still proper (see Proposition 15). Since a colimit of regular epimorphisms is a regular epimorphism, it suffices to prove that a $\mathcal{D}$-colimit of normal monomorphisms is a proper morphism. Considering as well the cokernels of these normal monomorphisms, we start thus with a $\mathcal{D}$-colimit of short exact sequences

$$\mathbf{0} \longrightarrow S_i \xrightarrow{\ s_i\ } A_i \xrightarrow{\ q_i\ } Q_i \longrightarrow \mathbf{0}$$

and consider its colimit

$$S \xrightarrow{\ s\ } A \xrightarrow{\ q\ } Q \longrightarrow \mathbf{0}.$$

Of course $q = \mathsf{Coker}\, s$ and it remains to prove that $\mathsf{Im}\, s = \mathsf{Ker}\, q$, that is, every element $a \in A$ such that $q(a) = 0$ has the form $s(x)$ for some $x \in S$; this is so when a is the equivalence class of some element $a_l \in A_l$ which belongs to S_l.

The element a is the equivalence class of some element a_i in some A_i. Since $q_i(a_i)$ is identified with 0 in the colimit Q, there exists a zigzag of arrows and elements b_j in the diagram of the Q_j's which connects $q_i(a_i)$ and 0.

If the zigzag starts with a morphism $i \longrightarrow j$, we can simply replace $a_i \in A_i$ by its image $a_j \in A_j$ and it suffices now to prove that a_j is equivalent to some element in some $S_{j'}$.

If the zigzag starts with a morphism $j \longrightarrow i$, consider the element $b_j \in Q_j$ of the zigzag which is mapped to $b_i = q_i(a_i)$. By surjectivity of q_j, we can choose $a_j \in A_j$ such that $q_j(a_j) = b_j$. Write a'_i for the image of a_j in A_i. Then $q_i(a_i) = q_i(a'_i)$.

Let us recall that the semi-abelian theory $\mathbb{T}$ contains a unique constant 0, a certain number n of binary operations α_k and a $(n+1)$-ary operation β such that

$$\alpha_1(t, t) = 0, \ldots, \alpha_n(t, t) = 0 \qquad \beta\,(\alpha_1(r, s), \ldots, \alpha_n(r, s), s) = r$$

(see [15]). Thus $q_i\left(\alpha_m(a_i, a'_i)\right) = 0$ for each index m, proving that $\alpha_m(a_i, a'_i) \in S_i$ for each m. And since

$$a_i = \beta\left(\alpha_1(a_i, a'_i), \ldots, \alpha_n(a_i, a'_i), a'_i\right)$$

with each $\alpha_m(a_i, a'_i)$ in S_i, we shall get that a_i is equivalent to some element in some $S_{i'}$ as soon as a'_i does. But for that, it suffices to prove that a_j itself is equivalent to some element in some $S_{j'}$.

Repeating these two steps along each leg of the zigzag, we reach the level l where the zigzag of elements becomes 0; and then the corresponding element a_l is in $S_l = \mathsf{Ker}\, s_l$. □

The second type of colimits that we consider is:

Definition 53. A category $\mathcal{D}$ is called *protofiltered* when it is connected and every span can be completed in a commutative square.

Of course filtered categories are protofiltered. In fact it is trivial to observe that:

Lemma 54. *A category $\mathcal{D}$ is protofiltered if and only if*

1. *$\mathcal{D}$ is non-empty;*
2. *there exists a cospan on every pair of objects;*
3. *given two arrows $u, v \colon A \rightrightarrows B$, there are arrows x, y such that $x \circ u = y \circ v$.* □

In other words, a protofiltered category is filtered as soon as in condition 3 of Lemma 54, one can choose $x = y$. The interest on protofiltered colimits lies in the fact that they are computed in the category of sets via the same well-known process as filtered colimits:

Lemma 55. *Let $(A_i)_{i\in\mathcal{D}}$ be a protofiltered diagram of sets. The colimit $\mathsf{colim}_{i\in\mathcal{D}} A_i$ is the quotient of the coproduct $\coprod_{i\in\mathcal{D}} A_i$ by the equivalence relation which identifies two elements $a_i \in A_i$, $a_j \in A_j$ when there exists a cospan on i, j along which a_i and a_j are already identified.*

Proof. The protofilteredness axiom forces the transitivity of the relation in the statement. □

Example 56. The monoid $(\mathbb{N}, +)$, viewed as a category with a single object, is protofiltered but not filtered.

Proof. Of course given $u, v \in \mathbb{N}$, there are $x, y \in \mathbb{N}$ such that $x + u = y + v$; but when $u \neq v$, it is impossible to choose $x = y$. □

We can then reinforce our Example 52:

Example 57. Let $\mathbb{T}$ be a semi-abelian algebraic theory (see [15]) and $\mathcal{D}$ a small sifted and protofiltered category. The reflection

$$\Delta \colon \mathsf{Set}^{\mathbb{T}} \longrightarrow [\mathcal{D}, \mathsf{Set}^{\mathbb{T}}], \quad \mathsf{colim} \colon [\mathcal{D}, \mathsf{Set}^{\mathbb{T}}] \longrightarrow \mathsf{Set}^{\mathbb{T}}, \quad \mathsf{colim} \dashv \Delta$$

is a sequentially exact fibered protolocalisation.

Proof. We observe first that a $\mathcal{D}$-colimit of monomorphisms is still a monomorphism. Choose thus a $\mathcal{D}$-diagram of monomorphisms $s_i \colon S_i \rightarrowtail A_i$ and their colimit $s \colon S \longrightarrow A$. Consider $x \in S$ such that $s(x) = 0$; by semi-abelianess, it suffices to prove that $x = 0$. But x is the equivalence class of some $x_i \in S_i$. Since $s_i(x_i)$ is identified with 0 in the colimit, it is already identified with 0 at some further level A_j of the diagram (see Lemma 55). But then the image x_j of x_i at the level j is mapped to 0 by the monomorphism s_j, thus $x_j = 0$ and $x = 0$ as required.

Going back to the proof of Example 52, we have now that s is a monomorphism with $\mathsf{Im}\, s = \mathsf{Ker}\, q$, that is, $s = \mathsf{Ker}\, q$. So the reflection is a protolocalisation (see Lemma 16). By Proposition 22, the protolocalisation is sequentially exact. □

Of course when $\mathcal{D}$ is filtered, the situation of the previous example becomes a localisation, since finite limits in $\mathsf{Set}^{\mathbb{T}}$ commute with filtered colimits. It remains an open problem to determine whether a sifted protofiltered category is filtered.

Our next example is of a rather different nature, even if it looks similar to the previous ones.

It is known that coequalisers of reflexive pairs are sifted colimits (see [1]), thus in particular quotients by equivalence relations are sifted colimits. But these colimits are not protofiltered and do not in general give rise to protolocalisations. For example, in the abelian case, the reflexive pair given by the discrete equivalence relation on an object A is a (normal) subobject of the one given by the indiscrete relation: and of course the factorisation $A \longrightarrow \mathbf{0}$ between the corresponding quotients is by no means a (normal) monomorphism. Thus the colimit functor does not preserve (normal) monomorphisms.

But given a category $\mathcal{C}$ with finite limits, write now $\mathsf{Eq}(\mathcal{C})$ for the category

- whose objects are the pairs (A, R), where $A \in \mathcal{C}$ and R is an equivalence relation on A;
- whose morphisms $f \colon (A, R) \longrightarrow (B, S)$ are the morphisms $f \colon A \longrightarrow B$ in $\mathcal{C}$ such that $f \times f$ restricts as a morphism from R to S.

In the presence of a zero object, the kernel of f in $\mathsf{Eq}(\mathcal{C})$ is its kernel in $\mathcal{C}$ *provided with the restriction of* R. This is a striking difference with considering equivalence relations as (particular) reflexive pairs.

Example 58. Let $\mathcal{C}$ be an arithmetical semi-abelian category (see [31]). Consider

$$\Delta\colon \mathcal{C} \longrightarrow \mathsf{Eq}(\mathcal{C}), \quad \chi\colon \mathsf{Eq}(\mathcal{C}) \longrightarrow \mathcal{C}$$

where $\Delta(A)$ is A equipped with the discrete equivalence relation on A, while $\chi(A, R)$ is the quotient of A by the equivalence relation R. This is a regular epireflective protolocalisation between semi-abelian categories, but not a localisation.

Proof. In [12], it is proved that a category $\mathcal{C}$ is exact protomodular if and only if the category $\mathsf{Grpd}(\mathcal{C})$ of internal groupoids in $\mathcal{C}$ is so.

In [31] it is proved that an exact Mal'tsev category $\mathcal{C}$ is arithmetical (i.e. the lattice of equivalence relations on each object is distributive) if and only if every groupoid is an equivalence relation, that is, $\mathsf{Grpd}(\mathcal{C}) \cong \mathsf{Eq}(\mathcal{C})$.

Thus for a semi-abelian (in particular, Mal'tsev) and arithmetical category $\mathcal{C}$, $\mathsf{Eq}(\mathcal{C}) \cong \mathsf{Grpd}(\mathcal{C})$ is semi-abelian as well.

The conclusion follows easily. The functor χ is trivially left adjoint to Δ and the unit of the adjunction is a regular epimorphism (the quotient map). Given a normal monomorphism $f\colon (A, R) \rightarrowtail (B, S)$, the factorisation $\overline{f}\colon A/R \longrightarrow B/S$ is still a monomorphism, because by normality $R = S \cap (A \times A)$. But $\overline{f}$ is then the image of the normal monomorphism f along the regular epimorphism $A \twoheadrightarrow A/R$; since $\mathcal{C}$ is semi-abelian, $\overline{f}$ is a normal monomorphism as well (see [6]).

To observe that we do not have a localisation, it suffices to prove that χ does not preserve monomorphisms. Indeed, $(A, \Delta_A) \rightarrowtail (A, A \times A)$ is a monomorphism mapped by χ to $A \longrightarrow \mathbf{0}$. □

Remark 59. The notions of "protolocalisation" and of "Mono-hereditary torsion theory", for the class Mono of all monomorphisms, are independent of each other.

Proof. On the one hand, we consider the example of Mono-hereditary torsion theory given in Section 5 of [13]. It is the one induced by the regular epireflection

$$\iota\colon \mathsf{Eq}(\mathcal{C}) \longrightarrow \mathsf{Grpd}(\mathcal{C}), \quad \sigma\colon \mathsf{Grpd}(\mathcal{C}) \longrightarrow \mathsf{Eq}(\mathcal{C}), \quad \sigma \dashv \iota$$

where $\mathcal{C}$ is semi-abelian, $\mathsf{Grpd}(\mathcal{C})$ is the category of internal groupoids in $\mathcal{C}$ and σ is the "support" functor: with a groupoid is associated the equivalence relation on its object of objects, which identifies two connected objects. This is *not* a protolocalisation, since $\mathsf{Eq}(\mathcal{C})$ is generally not closed under regular quotients in $\mathsf{Grpd}(\mathcal{C})$ ... unless $\mathcal{C}$ is arithmetical (see [11]).

On the other hand, the protolocalisation of Example 58 does *not* yield a Mono-hereditary torsion theory. Indeed the torsion part $\mathcal{T}$ is given by the indiscrete equivalence relations, and this category is not closed in $\mathsf{Eq}(\mathcal{C})$ under subobjects. □

11. Some topos theoretic examples

It is known (see [6]) that the dual of the category of pointed objects of a topos is semi-abelian. For simplicity, we work directly in the category of pointed objects and exhibit a coprotolocalisation.

Consider a topos $\mathcal{E}$ and write $\mathcal{E}_*$ for its category of pointed objects. Write further $\mathcal{E}_*^\sigma$ for the category of pointed objects of $\mathcal{E}$ provided with an endomorphism which respects the base point. This is a category of diagrams in $\mathcal{E}_*$, thus finite limits and finite colimits in $\mathcal{E}_*^\sigma$ are computed as in $\mathcal{E}_*$. Therefore the dual of $\mathcal{E}_*^\sigma$ is still semi-abelian, since so is the dual of $\mathcal{E}_*$.

There is an obvious full and faithful inclusion

$$\iota\colon \mathcal{E}_* \longrightarrow \mathcal{E}_*^\sigma, \quad (A, *) \mapsto (A, *, \mathsf{id}_A).$$

This inclusion admits a right adjoint which, in the internal logic of the topos $\mathcal{E}$, is simply given by

$$\mathsf{Fix}\colon \mathcal{E}_*^\sigma \longrightarrow \mathcal{E}_*, \quad (A, *, \sigma) \mapsto \big(\{a \in A | \sigma(a) = a\}, *\big).$$

Example 60. Given a topos $\mathcal{E}$, the functors

$$\iota^{\mathsf{op}}\colon (\mathcal{E}_*)^{\mathsf{op}} \longrightarrow (\mathcal{E}_*^\sigma)^{\mathsf{op}}, \quad \mathsf{Fix}^{\mathsf{op}}\colon (\mathcal{E}_*^\sigma)^{\mathsf{op}} \longrightarrow (\mathcal{E}_*)^{\mathsf{op}}$$

constitute a regular epireflective protolocalisation between semi-abelian categories. This protolocalisation is not a localisation.

Proof. We must prove that the functor Fix preserves normal epimorphisms. But $f\colon (A,*) \twoheadrightarrow (B,*)$ is a normal epimorphism in $\mathcal{E}_*$ precisely when, in the internal language of the topos, it is surjective and

$$\big(f(a) = f(a')\big) \Rightarrow \big(a = a' \text{ or } f(a) = * = f(a')\big).$$

But we have already noticed that finite colimits in $\mathcal{E}_*^\sigma$ are computed as in $\mathcal{E}$. Thus

$$f\colon (A,*,\sigma) \twoheadrightarrow (B,*,\tau)$$

is a normal epimorphism in $\mathcal{E}_*^\sigma$ precisely when f is a normal epimorphism in $\mathcal{E}_*$. We must prove that also $\mathsf{Fix}(f)$ is a normal epimorphism in $\mathcal{E}_*$.

Given a fixed point $b = \tau(b) \in B$, we have $b = f(a)$ for some $a \in A$. But

$$f(\sigma(a)) = \tau(f(a)) = \tau(b) = b = f(a)$$

from which we deduce, since f is a normal epimorphism in $\mathcal{E}_*$,

$$\sigma(a) = a \quad \text{or} \quad f(\sigma(a)) = * = f(a).$$

In the first case we get at once $b = f(a)$ with $a = \sigma(a)$ a fixed point; in the second case we deduce $b = f(a) = * = f(*)$ with of course $* \in A$ a fixed point. Thus in both cases, we have proved that b is the image of a fixed point of A, proving that $\mathsf{Fix}(f)$ is surjective.

It remains to verify that the epimorphism $\mathsf{Fix}(f)$ is normal, that is, it identifies two points when they are equal or both mapped to the base point: this is trivial since $\mathsf{Fix}(f)$ is the restriction of f, which has that property.

This coprotolocalisation is not a colocalisation, because it does not preserve epimorphisms. For example take $A = \mathbf{1} \amalg \mathbf{1} \amalg \mathbf{1}$ and $B = \mathbf{1} \amalg \mathbf{1}$, with each time the first term as base point. On A, choose the endomorphism σ which interchanges the last two terms and, on B, choose τ to be the identity. The morphism $f\colon A \longrightarrow B$ which identifies the last two terms is an epimorphism in Set_*^σ, but $\mathsf{Fix}(f)$ is the first inclusion $\mathbf{1} \rightarrowtail \mathbf{1} \amalg \mathbf{1}$, which is not surjective. □

Notice that Example 60 enters the considerations of the previous section, since $\mathcal{E}_*^\sigma$ is equivalent to the functor category $\left[(\mathbb{N},+), \mathcal{E}_*^{\mathsf{op}}\right]$, with $(\mathbb{N},+)$ the protofiltered category of Example 56, which is also the free monoid on one generator. Identifying id_A and σ is indeed equivalent to identifying id_A and all the powers of σ, thus applying the colimit functor. In the case of the topos of sets, we have a much more general result:

Example 61. Let $\mathsf{Set}_*^{\mathsf{op}}$ be the dual of the category of pointed sets (which is semi-abelian: see [6]). For every protofiltered category $\mathcal{D}$, the reflection

$$\Delta\colon \mathsf{Set}_*^{\mathsf{op}} \longrightarrow [\mathcal{D}, \mathsf{Set}_*^{\mathsf{op}}], \quad \mathsf{colim}\colon [\mathcal{D}, \mathsf{Set}_*^{\mathsf{op}}] \longrightarrow \mathsf{Set}_*^{\mathsf{op}}, \quad \mathsf{colim} \dashv \Delta$$

is a protolocalisation.

Proof. Again for the sake of clarity, we work in the category of pointed sets, proving thus that the limit functor

$$\Delta\colon \mathsf{Set}_* \longrightarrow [\mathcal{D}, \mathsf{Set}_*], \quad \mathsf{lim}\colon [\mathcal{D}, \mathsf{Set}_*] \longrightarrow \mathsf{Set}_*, \quad \Delta \dashv \mathsf{lim}$$

yields a coprotolocalisation when $\mathcal{D}$ is a small coprotofiltered category. By Lemma 16, we must prove that a $\mathcal{D}$-limit $p\colon A \longrightarrow B$ of normal epimorphisms $(p_D\colon A_D \twoheadrightarrow B_D)_{D\in\mathcal{D}}$ in Set_* is still a normal epimorphism.

First, we prove that p is surjective. Consider a compatible family of elements $(b_D \in B_D)_{D\in\mathcal{D}}$, that is, an element b of the limit B. If b is the base point, it is the image of the base point of A.

Next, for each b_D which is not the base point, then $b_D = p_D(a_D)$ for a unique element $a_D \in A_D$, by normality of p_D. The uniqueness condition forces at once the subfamily of all these a_D to be compatible along all the morphisms of $\mathcal{D}$ connecting two such levels. And of course if this situation holds for each $D \in \mathcal{D}$, we get so an element $a \in A$ such that $p(a) = b$.

Suppose now that b is not the base point, thus some $b_{\widetilde{D}}$ is not the base point, while some b_D is the base point. By Lemma 54 there exists a span

$$D \xleftarrow{f'} D' \xrightarrow{\widetilde{f}} \widetilde{D}$$

in $\mathcal{D}$. Since $b_{\widetilde{D}}$ is not the base point, $b_{D'}$ is not the base point. So there exists always $f'\colon D' \longrightarrow D$ in $\mathcal{D}$ such that $b_{D'}$ is not the base point and we know already that $b_{D'} = p_{D'}(a_{D'})$ for a unique $a_{D'} \in A_{D'}$. Define a_D to be the image of $a_{D'}$ along f'; by naturality, a_D is mapped by p_D to the base point. This definition is independent of the choice of (D', f'), since by coprotofilteredness, given another choice (D'', f''), the span (f, f') can be completed in a commutative square. Thus p is surjective.

To prove the normality of p, choose two compatible families $(a_D)_{D\in\mathcal{D}}$ and $(a'_D)_{D\in\mathcal{D}}$ in A which are identified by p. For each $D \in \mathcal{D}$, we get $p_D(a_D) = p_D(a'_D)$. If this is the base point of B_D for each D, we are done. And if $p_D(a_D) = p_D(a'_D)$ is not the base point for some fixed $D \in \mathcal{D}$, we must prove that $a_{\widetilde{D}} = a'_{\widetilde{D}}$ for all $\widetilde{D} \in \mathcal{D}$. But if $a_{\widetilde{D}} \neq a'_{\widetilde{D}}$ for some $\widetilde{D}$, choose a span $(f', \widetilde{f})$ as above. Then of course $a_{D'} \neq a'_{D'}$ since the images along $\widetilde{f}$ are distinct. Thus $p_{D'}(a_{D'}) = p_{D'}(a'_{D'})$ is the base point of $B_{D'}$. Taking the image along f', we get that $p_D(a_D) = p_D(a'_D)$ is the base point of B_D, which is a contradiction. □

A special case is worth being considered in more detail:

Example 62. Consider the poset $(\mathbb{N}, \leq)$ viewed as a protofiltered category and the corresponding protolocalisation.

$$\Delta\colon \mathsf{Set}_*^{\mathrm{op}} \longrightarrow \left[(\mathbb{N}, \leq), \mathsf{Set}_*^{\mathrm{op}}\right], \quad \mathsf{colim}\colon \left[(\mathbb{N}, \leq), \mathsf{Set}_*^{\mathrm{op}}\right] \longrightarrow \mathsf{Set}_*^{\mathrm{op}}.$$

Of course $(\mathbb{N}, \leq)$ is filtered, but the corresponding protolocalisation is neither a localisation, nor a regular epireflection nor a monoreflection.

Proof. Notice that the projections of a limit over $(\mathbb{N}, \geq)$ in Set_* are generally not injective nor surjective, thus the protolocalisation of the statement (see Example 61) is neither regular epireflective nor monoreflective.

To show that the protolocalisation is not a localisation, it suffices to show that it does not preserve monomorphisms. So we must prove that in Set_*, a $(\mathbb{N}, \geq)$-limit of surjections is no longer surjective. Simply define p_n to be

$$p_n\colon A_n = \mathbb{N} \longrightarrow \{0, 1, \ldots, n\} = B_n$$

where

- on both sides, 0 is the base point;
- the restriction mapping $A_{n+1} \longrightarrow A_n$ is the identity;
- the restriction mapping $B_{n+1} \longrightarrow B_n$ is the one identifying $n + 1$ and n;
- $p_n(m) = \min\{n, m\}$.

In $\mathsf{lim} B_n$ we have the compatible sequence $(n)_{n\in\mathbb{N}}$ while in $\mathsf{lim} A_n$ all compatible sequences are constant; thus none of them can be mapped to $(n)_{n\in\mathbb{N}}$ by $\mathsf{lim}\, p_n$. □

Coming back to Example 60 in the case of the topos of sets, we observed already that Set_*^{σ} is equivalent to the category of pointed objects of the topos of $(\mathbb{N}, +)$-sets. The following generalisation holds and can be internalised in a Boolean topos.

Example 63. Let M be a monoid. The dual of the category of pointed sets is a regular epireflective protolocalisation of the dual of the category of pointed M-sets. Both categories are semi-abelian and the reflection is generally not a localisation.

Proof. Let us work with pointed sets and M-sets, not the dual categories. With the pointed set $(A, *)$ is associated the pointed M-set $(A, *, \pi)$ where all elements of A are fixed: $m \cdot a = a$ for all $m \in M$ and $a \in A$. With a pointed M-set $(A, *, \chi)$ is associated the subobject $\mathsf{Fix}(A, *, \chi) \subseteq (A, *)$ of fixed points. Routine verifications show that this yields a coreflection.

The category of pointed M-sets is a functor category of pointed sets, thus its dual is semi-abelian and normal epimorphisms of pointed M-sets are those morphisms which are normal epimorphisms of pointed sets. Given a normal epimorphism $f\colon (A, *, \chi) \longrightarrow (B, *, \xi)$, we prove first that $\mathsf{Fix}(f)$ is still surjective. Given $b \in B$, there is $a \in A$ such that $f(a) = b$. Then for every $m \in M$,

$$f(m \cdot a) = m \cdot f(a) = m \cdot b = b = f(a).$$

By normality of f, this implies

$$\forall m \in M \ (m \cdot a = a \text{ or } b = *).$$

And since our logic of sets is Boolean, this is equivalent to

$$(\forall m \in M \, m \cdot a = a) \quad \text{or} \quad (b = *).$$

In the first case, $b = f(a)$ with $a \in \mathsf{Fix}(A, *, \chi)$ and in the second case, $b = f(*)$ with of course $* \in \mathsf{Fix}(A, *, \chi)$. One concludes as in Example 60. □

12. Homological categories of monomorphisms

This section will, among other interesting things, allow us to construct an example of a monoreflective protolocalisation.

Let $\mathcal{C}$ be a homological category, and $\mathcal{D}$ a small category. We denote by $\mathsf{Mono}_{\mathcal{D}}(\mathcal{C})$ the full subcategory of the homological category $[\mathcal{D}, \mathcal{C}]$ whose objects are the functors $F: \mathcal{D} \longrightarrow \mathcal{C}$ with the property that, for every $d: i \longrightarrow j$ in $\mathcal{D}$, the arrow $F(d): F(i) \longrightarrow F(j)$ is a monomorphism in $\mathcal{C}$.

Lemma 64. *Let $\mathcal{C}$ be a homological category, $\mathcal{D}$ a small category. Then $\mathsf{Mono}_{\mathcal{D}}(\mathcal{C})$ is a homological category and the inclusion $U: \mathsf{Mono}_{\mathcal{D}}(\mathcal{C}) \rightarrowtail [\mathcal{D}, \mathcal{C}]$ preserves finite limits and regular epimorphisms thus, in particular, short exact sequences.*

Proof. It is easy to see that $\mathsf{Mono}_{\mathcal{D}}(\mathcal{C})$ is closed under finite limits in the homological category of functors $[\mathcal{D}, \mathcal{C}]$. This implies that the full inclusion $U: \mathsf{Mono}_{\mathcal{D}}(\mathcal{C}) \rightarrowtail [\mathcal{D}, \mathcal{C}]$ preserves and reflects finite limits and, of course, isomorphisms. The protomodularity of $[\mathcal{D}, \mathcal{C}]$ can so be lifted to $\mathsf{Mono}_{\mathcal{D}}(\mathcal{C})$.

Now let us show that the category $\mathsf{Mono}_{\mathcal{D}}(\mathcal{C})$ is regular. Consider the regular epi–mono-factorisation $f = s \circ p$ in $[\mathcal{D}, \mathcal{C}]$ of an arrow $f: F \Rightarrow G$ of $\mathsf{Mono}_{\mathcal{D}}(\mathcal{C})$. The image object is still in $\mathsf{Mono}_{\mathcal{D}}(\mathcal{C})$: indeed, the commutativity of the diagram

$$\begin{array}{ccccc}
F(i) & \xrightarrow{p_i} & H(i) & \stackrel{s_i}{\rightarrowtail} & G(i) \\
{\scriptstyle F(d)}\downarrow & & \downarrow{\scriptstyle H(d)} & & \downarrow{\scriptstyle G(d)} \\
F(j) & \xrightarrow[p_j]{} & H(j) & \underset{s_j}{\rightarrowtail} & G(j)
\end{array}$$

for every $d: i \longrightarrow j$ in $\mathcal{D}$, tells us at once that $H(d)$ is a monomorphism. This yields thus a regular epi–mono-factorisation of f in $\mathsf{Mono}_{\mathcal{D}}(\mathcal{C})$ and proves at the same time that the inclusion U preserves regular epimorphisms. Since these factorisations are pullback stable, $\mathsf{Mono}_{\mathcal{D}}(\mathcal{C})$ is regular and thus homological. □

Remark 65. When $\mathcal{C}$ is semi-abelian, it is not true in general that the category $\mathsf{Mono}_{\mathcal{D}}(\mathcal{C})$ is semi-abelian.

Proof. Consider the category $\mathcal{D} = \{\bullet \to \bullet\}$ and choose $\mathcal{C}$ to be abelian. Then $\mathsf{Mono}_{\mathcal{D}}(\mathcal{C})$ is the category of monomorphisms in $\mathcal{C}$. This category is well known to be regular, but it is not exact. Indeed it is additive, thus being exact would imply being abelian. This is not the case, since not all monomorphisms are kernels: simply consider the monomorphism

$$(0, \mathsf{id}_A): (0: \mathbf{0} \rightarrowtail A) \longrightarrow (\mathsf{id}_A: A \rightarrowtail A)$$

which is not a kernel, except when $A = \mathbf{0}$. □

Proposition 66. *Let $\mathcal{C}$ be a homological category admitting $\mathcal{D}$-colimits, for some small category $\mathcal{D}$. Assume that $\Delta, \mathsf{colim}: \mathcal{C} \rightleftarrows [\mathcal{D}, \mathcal{C}]$ is a protolocalisation. Then the restriction*

$$\Delta, \mathsf{colim}: \mathcal{C} \rightleftarrows \mathsf{Mono}_{\mathcal{D}}(\mathcal{C})$$

is still a protolocalisation.

Proof. To prove this result, it suffices to know that the full inclusion $U : \mathsf{Mono}_{\mathcal{D}}(\mathcal{C}) \rightarrowtail [\mathcal{D}, \mathcal{C}]$ preserves normal monomorphisms, which is attested by Lemma 64. $\square$

Example 67. By Proposition 66, the protolocalisation of Example 62 restricts as a protolocalisation:

$$\Delta : \mathsf{Set}_*^{\mathsf{op}} \longrightarrow \mathsf{Mono}_{(\mathbb{N},\leq)}(\mathsf{Set}_*^{\mathsf{op}}), \quad \mathsf{colim} : \mathsf{Mono}_{(\mathbb{N},\leq)}(\mathsf{Set}_*^{\mathsf{op}}) \longrightarrow \mathsf{Set}_*^{\mathsf{op}}.$$

This protolocalisation is monoreflective and is not a localisation.

Proof. Working again in Set_* instead of its dual, the counit of the adjunction, given by the projections $(\eta_i : \mathsf{lim}_{i\in\mathbb{N}} A_i \longrightarrow A_i)_{i\in\mathbb{N}}$ of the limit, is now surjective in each component; thus the protolocalisation of the statement is monoreflective. Indeed, given an element $x_j \in A_j$ for some fixed index j, it is easy to extend it to a compatible family $(x_i \in A_i)_{i\in\mathbb{N}}$, that is, an element of $\mathsf{lim}_{i\in\mathbb{N}} A_i$. For $i \geq j$ simply choose the restriction of x_j in A_i. And since the restriction $a_j : A_{j+1} \twoheadrightarrow A_j$ is surjective, choose for $x_{j+1} \in A_{j+1}$ an element mapped to x_j and repeat the process inductively.

The counter-example in Example 62 applies to conclude that we still do not have a localisation. $\square$

13. Examples in functional analysis

In [23] it is proved that the category $\mathbb{C}^*$-Alg of commutative $\mathbb{C}^*$-algebras without necessarily a unit is semi-abelian. But these $\mathbb{C}^*$-algebras have nevertheless a so-called *approximate unit* (see [20]):

> In a $\mathbb{C}^*$-algebra A, there exists a net $(\varepsilon_\omega)_{\omega\in\Omega}$ of elements such that for every element $a \in A$, one has $a = \mathsf{lim}_{\omega\in\Omega} \varepsilon_\omega \cdot a$.

The existence of approximate units forces in particular the following known property (see [20]):

Lemma 68. *In the category $\mathbb{C}^*$-Alg of commutative $\mathbb{C}^*$-algebras, the composite of two normal monomorphisms is still a normal monomorphism.*

Proof. A normal monomorphism in $\mathbb{C}^*$-Alg is exactly a closed ideal. Consider thus the composite $I \rightarrowtail J \rightarrowtail A$ of two normal monomorphisms. Since I is closed in J which is itself closed in A, then I is closed in A.

Next choose elements $i \in I$ and $a \in A$ and write $(\varepsilon_\omega)_{\omega\in\Omega}$ for an approximate unit of J. Since $i \in J$ and J is an ideal of A, we get $a \cdot i \in J$ and thus $a \cdot i = \mathsf{lim}_{\omega\in\Omega} \varepsilon_\omega \cdot a \cdot i$. Since J is an ideal of A, we have also $\varepsilon_\omega \cdot a \in J$ and since I is an ideal in J, this forces $\varepsilon_\omega \cdot a \cdot i \in I$ for each $\omega \in \Omega$. Since I is closed in A, $a \cdot i = \mathsf{lim}_{\omega\in\Omega} \varepsilon_\omega \cdot a \cdot i \in I$. $\square$

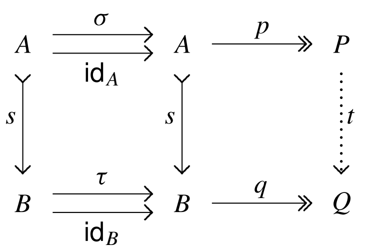
Let us now consider the category $\mathbb{C}^*$-Alg^σ of $\mathbb{C}^*$-algebras provided with an endomorphism σ, and the morphisms of $\mathbb{C}^*$-algebras commuting with the given endomorphisms. In other words, $\mathbb{C}^*$-Alg^σ is the functor category $\big[(\mathbb{N}, +), \mathbb{C}^*\text{-}\mathsf{Alg}\big]$ (see Example 56), which is thus semi-abelian since so is $\mathbb{C}^*$-Alg. Keeping in mind Lemma 50, let us now prove that:

Example 69. The functors

$$\Delta : \mathbb{C}^*\text{-}\mathsf{Alg} \longrightarrow \mathbb{C}^*\text{-}\mathsf{Alg}^\sigma \cong \big[(\mathbb{N}, +), \mathbb{C}^*\text{-}\mathsf{Alg}\big], \quad \mathsf{colim} : \mathbb{C}^*\text{-}\mathsf{Alg}^\sigma \cong \big[(\mathbb{N}, +), \mathbb{C}^*\text{-}\mathsf{Alg}\big] \longrightarrow \mathbb{C}^*\text{-}\mathsf{Alg}$$

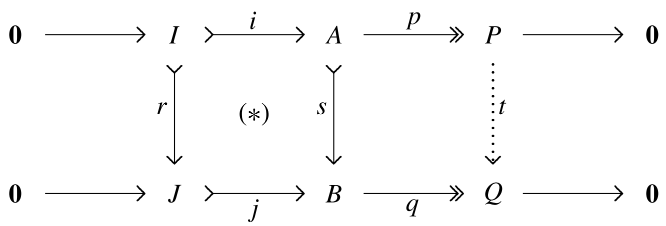
constitute a regular epireflective protolocalisation between semi-abelian categories.

Proof. Let us consider a normal monomorphism $s : (A, \sigma) \longrightarrow (B, \tau)$ in $\mathbb{C}^*$-Alg^σ. This is simply a normal monomorphism in $\mathbb{C}^*$-Alg such that σ is the restriction of τ. We consider the coequalisers p of (σ, id_A) and q of (τ, id_B): we must prove that the corresponding factorisation t is a normal monomorphism in $\mathbb{C}^*$-Alg.



If we prove that t is injective, it will be a normal monomorphism as image of the normal monomorphism s along the regular epimorphism q in the semi-abelian category $\mathbb{C}^*$-Alg.

The coequaliser p of σ and id_A is the quotient by the smallest closed ideal I of A which contains all the elements of the form $\sigma(a) - a$, for all elements $a \in A$. Analogously the coequaliser q of τ and id_B is the quotient by the smallest closed ideal J of B containing the elements of the form $\tau(b) - b$ with $b \in B$. Considering the diagram of short exact sequences



t will be a monomorphism as soon as the square (*) is a pullback.

Trivially, $I \subseteq J \cap A$ and it remains to prove that $J \cap A \subseteq I$. Write J' for the ideal generated by all the elements of the form $\tau(b) - b$: it suffices to prove that $J' \cap A \subseteq I$. Indeed if $J' \cap A \subseteq I$ and $x \in J \cap A$, write $x = \mathsf{lim}_{n\in\mathbb{N}} x_n$, with $x_n \in J'$. Writing $(\varepsilon_\omega)_{\omega\in\Omega}$ for an approximate unit of A, we have further $x = \mathsf{lim}_{\omega\in\Omega}\varepsilon_\omega \cdot x$. This yields

$$x = \lim_{\omega\in\Omega} \varepsilon_\omega \cdot x = \lim_{\omega\in\Omega} \left(\varepsilon_\omega \cdot \lim_{n\in\mathbb{N}} x_n\right) = \lim_{\omega\in\Omega} \lim_{n\in\mathbb{N}} \varepsilon_\omega \cdot x_n.$$

Since $\varepsilon_\omega \in A$ and $x_n \in J'$, we have $\varepsilon_\omega \cdot x_n \in J' \cap A \subseteq I$, thus the limit lies still in the closed ideal I.

To prove that $J' \cap A \subseteq I$, consider an element $x \in J' \cap A$. As an element of J', it has the form

$$x = \sum_{n=1}^{m} b'_n \cdot (\tau(b_n) - b_n), \quad b'_n, b_n \in B.$$

We further get, since $x \in A$

$$x = \lim_{\omega\in\Omega} \varepsilon_\omega \cdot x = \lim_{\omega\in\Omega} \varepsilon_\omega \cdot \left(\sum_{n=1}^{m} b'_n \cdot (\tau(b_n) - b_n)\right) = \lim_{\omega\in\Omega} \sum_{n=1}^{m} \left(\varepsilon_\omega \cdot b'_n \cdot \tau(b_n) - \varepsilon_\omega \cdot b'_n \cdot b_n\right).$$

To prove that this limit is in the closed ideal I, it suffices to prove that each term appearing in this limit is in I. But since σ is the restriction of τ on A, we have

$$\begin{aligned}\varepsilon_\omega \cdot b'_n \cdot \tau(b_n) - \varepsilon_\omega \cdot b'_n \cdot b_n &= \left(\varepsilon_\omega \cdot b'_n - \tau(\varepsilon_\omega \cdot b'_n)\right)\tau(b_n) + \left(\tau(\varepsilon_\omega \cdot b'_n \cdot b_n) - \varepsilon_\omega \cdot b'_n \cdot b_n\right)\\ &= \left(\varepsilon_\omega \cdot b'_n - \sigma(\varepsilon_\omega \cdot b'_n)\right)\tau(b_n) + \left(\sigma(\varepsilon_\omega \cdot b'_n \cdot b_n) - \varepsilon_\omega \cdot b'_n \cdot b_n\right).\end{aligned}$$

This last expression lies in I since so does every element of the form $\sigma(a) - a$, with $a \in A$, while I is an ideal in B, by Lemma 68.

And the unit $p = \eta_{(A,\sigma)}$ of the adjunction is a regular epimorphism for each $(A, \sigma) \in \mathbb{C}^*$-$\mathsf{Alg}^\sigma$. □

A careful analysis of the proof of Example 69 shows that the conclusion still holds true when $\mathbb{C}^*$-Alg^σ is replaced by some adequate full subcategory of it: for example, that of pairs (A, σ) for an idempotent σ (i.e. $\sigma^2 = \sigma$) or an involutive one (i.e. $\sigma^2 = \mathsf{id}_A$).

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