Stability of finite difference schemes for nonlinear complex reaction–diffusion processes

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In this paper we consider explicit, implicit and semiimplicit finite difference schemes for a general nonlinear reaction–diffusion equation. The stability condition for each method is established and several particular cases are highlighted. To illustrate the theoretical results we present some numerical examples.

Keywords: finite differences; complex reaction–diffusion; stability.

1. Introduction

Complex diffusion is a commonly used denoising procedure in image processing (Gilboa *et al.*, 2004). In particular, nonlinear complex diffusion proved to be a numerically well-conditioned technique that has been successfully applied in medical imaging despeckling (Bernardes *et al.*, 2010). The stability condition for finite difference methods applied to the linear diffusion equation has been investigated extensively and it is widely documented in the literature (see, e.g., Thomas, 1995; Jovanović & Süli, 2014). A stability result for the linear complex case was derived in Chan & Shen (1987).

The stability properties of a class of finite difference schemes for the nonlinear complex diffusion equation were studied in Araújo *et al.* (2012), where only the explicit and implicit schemes were investigated and no reaction term was considered. In this paper we achieved a twofold extension of the previous work, namely (a) by considering nonlinear complex reaction–diffusion equations through the introduction of a reactive source term and (b) by considering also the semiimplicit finite difference scheme (in addition to the previously considered explicit and implicit schemes). Applications of interest include diffusion processes which are commonly used in image processing, as, for example, in noise removal, inpainting, stereo vision or optical flow (see, e.g., Perona & Malik, 1990; Weickert, 1994, 1997; Grossauer & Scherzer, 2003; Brox *et al.*, 2004; Gilboa *et al.*, 2004; Salinas & Fernández, 2007; Zimmer *et al.*, 2008; Bernardes *et al.*, 2010). Complex diffusion with a reactive term appears also in the well-known Schrödinger equation, though conservative numerical methods are usually used instead of the finite difference approach (Sanz-Serna & Verwer, 1986; Matsuoa & Furihata, 2001). In the theory of heat conduction and chemical diffusion processes, if thermal conductivity depends on the unknown function, the temperature distribution in a bounded medium is governed by a reaction–diffusion process (Wang & Pao, 2000).

The complex reaction–diffusion equation considered in the present paper can be written as a system in the real and complex parts of the dependent complex variable, resulting in a particular strongly coupled system of reaction–diffusion equations of real variables. Though this doubles the dimensions, the entries become real instead of complex, and therefore the computational complexity of the method is not changed.

The stability and convergence of finite difference methods for systems of nonlinear reaction–diffusion equations with real variables were studied in Hoff (1978). For the complex case, we mention Wang *et al.* (2010) where the authors consider the analysis of conservative schemes for a coupled nonlinear Schrödinger system. To the best of our knowledge there is no rigorous proof for the stability of finite difference schemes for the general equation that we are considering in this paper. This paper fills this gap in the theory of finite difference schemes and it is complemented by the work of Araújo *et al.* (2013) which establishes a convergence result for both implicit and semiimplicit finite different schemes in the context of complex diffusion with reactive terms.

Let Ω be a bounded open set in $\mathbb{R}^d$, $d \geqslant 1$, with boundary $\Gamma = \partial\Omega$. For the sake of clearness, we consider that Ω is the Cartesian product of open intervals in $\mathbb{R}$, that is

$$\Omega = \prod_{j=1}^{d} (a_j, b_j), \tag{1.1}$$

with $a_j, b_j \in \mathbb{R}$. Let $Q = \Omega \times (0, T]$, with $T > 0$, and $v : \bar{Q} = \bar{\Omega} \times [0, T] \longrightarrow \mathbb{C}$. We consider a reaction–diffusion process with a nonconstant complex coefficient $D(x,t,v) = D_{\mathrm{R}}(x,t,v) + \mathrm{i}D_{\mathrm{I}}(x,t,v)$ and nonconstant complex reaction term $F(x,t,v) = F_{\mathrm{R}}(x,t,v) + \mathrm{i}F_{\mathrm{I}}(x,t,v)$, where $D_{\mathrm{R}}(x,t,v)$, $D_{\mathrm{I}}(x,t,v)$, $F_{\mathrm{R}}(x,t,v)$, $F_{\mathrm{I}}(x,t,v)$ are real functions dependent on v. We assume that

$$D_{\mathrm{R}}(x,t,v) \geqslant 0, \quad (x,t) \in \bar{Q}, \tag{1.2}$$

and that there exists a constant $L > 0$ such that

$$|D(x,t,v)| \leqslant L, \quad (x,t) \in \bar{Q}. \tag{1.3}$$

The inequalities (1.2) and (1.3) can easily be shown to hold for the diffusion coefficient in Gilboa *et al.* (2004) and Bernardes *et al.* (2010).

We define the initial boundary value problem for the unknown complex function u:

$$\frac{\partial u}{\partial t}(x,t) = \nabla \cdot (D(x,t,u)\nabla u(x,t)) + F(x,t,u), \quad (x,t) \in Q, \tag{1.4}$$

under the initial condition

$$u(x,0) = u^0(x), \quad x \in \bar{\Omega}, \tag{1.5}$$

and with either the Dirichlet boundary condition

$$u(x,t) = 0, \quad x \in \Gamma, \ \ t \in [0,T], \tag{1.6}$$

or the Neumann boundary condition

$$\frac{\partial u}{\partial \nu}(x,t) = 0, \quad x \in \Gamma, \ \ t \in [0,T], \tag{1.7}$$

where $\partial u/\partial \nu$ denotes the derivative in the direction of the exterior normal to Γ.

For the reaction term we will consider the following decomposition:

$$F(x,t,v) = F_0(x,t) + F_{\mathrm{L}}(x,t)v + F_{\mathrm{NL}}(x,t,v), \tag{1.8}$$

with $F_0(x,t) = F_{0R}(x,t) + \mathrm{i}F_{0I}(x,t)$, $F_{\mathrm{L}}(x,t) = F_{LR}(x,t) + \mathrm{i}F_{LI}(x,t)$ and $F_{\mathrm{NL}}(x,t,v) = F_{NLR}(x,t,v) + \mathrm{i}F_{NLI}(x,t,v)$, where $F_{0R}(x,t)$, $F_{0I}(x,t)$, $F_{LR}(x,t)$, $F_{LI}(x,t)$, $F_{NLR}(x,t,v)$ and $F_{NLI}(x,t,v)$ are real functions. For the nonlinear term, we consider that there exists a complex function χ such that

$$F_{\mathrm{NL}}(x,t,v) = F_{\mathrm{NL}}(x,t,0) + J(x,t,v)v, \tag{1.9}$$

with

$$J(x,t,v) = F'_{\mathrm{NL}}(x,t,v) + \chi(v), \tag{1.10}$$

and $|\chi(r)| \longrightarrow 0$ as $|r| \longrightarrow 0$, where F'_{NL} is the Fréchet derivative of F_{NL} with respect to the third component.

We assume that the problem is well posed, in the sense that it admits a unique solution and it depends continuously on the data. We note that expression (1.4) involves both Schrödinger-type equations and parabolic equations and includes the possibility of having a source term, a linear reaction term, a nonlinear reaction term or none of them (see (1.8)).

The paper is organized as follows: in Section 2 we describe the implicit and semiimplicit numerical methods simultaneously by embedding them into a two-parameter family of finite difference schemes. In Section 3 we derive a stability result for the numerical methods considered in the previous section. In Section 4 some numerical experiments are shown to confirm the theoretical analysis. In Section 5 we gather some final conclusions. Several technical results needed throughout the paper are presented in the Appendix.

2. Numerical method

Let us construct a nonequidistant rectangular grid on $\bar{Q}$. Let $(h_{k,j_k})_{0\leqslant j_k\leqslant N_k-1}$ be a vector of mesh sizes (i.e., positive numbers) in the kth spatial coordinate direction, $k=1,\ldots,d$, with $N_k \geqslant 2$ an integer. We denote by h the maximal mesh size. We define the space grid by

$$\bar{\Omega}_h = \prod_{k=1}^{d} \bar{\Omega}_{h_k}, \tag{2.1}$$

where, for $k=1,\ldots,d$,

$$\bar{\Omega}_{h_k} = \{x_{k,j_k} \in \mathbb{R} : x_{k,0} = a_k,\ x_{k,j_k+1} = x_{k,j_k} + h_{k,j_k},\ j_k = 1,\ldots,N_k-1\}.$$

The set of grid points is denoted by x_j, where $j=(j_1,\ldots,j_d)$, $0 \leqslant j_1 \leqslant N_k$. Points halfway between two adjacent grid points are denoted by $x_{j+(1/2)e_k} = x_j + \frac{1}{2}h_{k,j_k}e_k$ and $x_{j-(1/2)e_k} = x_j - \frac{1}{2}h_{k,j_k-1}e_k$, where e_k denotes the kth element of the natural basis in $\mathbb{R}^d$. We will also use the notation $h_{k,j_k-1/2} = (h_{k,j_k-1} + h_{k,j_k})/2$, $j_k = 1,\ldots,N_k-1$. For the temporal interval we consider the mesh

$$0 = t^0 < t^1 < \cdots < t^{M-1} < t^M = T,$$

where $M \geqslant 1$ is an integer and $\Delta t^m = t^{m+1} - t^m$, $m = 0, \ldots, M-1$. Let $\Delta t = \max \Delta t^m$. We denote by $\bar{Q}_h^{\Delta t}$ the mesh in $\bar{Q}$ defined by the Cartesian product of the space grid $\bar{\Omega}_h$ and a grid in the temporal domain. Let $Q_h^{\Delta t} = \bar{Q}_h^{\Delta t} \cap Q$ and $\Gamma_h^{\Delta t} = \bar{Q}_h^{\Delta t} \cap \Gamma \times [0, T]$.

We associate the coordinate $(j, m) = (j_1, \ldots, j_d, m)$ to the point $(x_j, t^m) \in \bar{Q}_h^{\Delta t}$ and $(j + \frac{1}{2}e_k, m)$ and $(j - \frac{1}{2}e_k, m)$ to the midpoints $(x_{j+(1/2)e_k}, t^m)$ and $(x_{j-(1/2)e_k}, t^m)$, respectively. We consider the notation $V_j^m = V(x_j, t^m)$, $V_{j+(1/2)e_k}^m = V(x_{j+(1/2)e_k}, t^m)$ and $V_{j-(1/2)e_k}^m = V(x_{j-(1/2)e_k}, t^m)$. For the formulation of the finite difference approximations, we use the centred finite difference quotients in the kth spatial direction

$$\delta_k V_j^m = \frac{V_{j+(1/2)e_k}^m - V_{j-(1/2)e_k}^m}{h_{k,j_k-1/2}}, \quad \delta_k V_{j-(1/2)e_k}^m = \frac{V_j^m - V_{j-e_k}^m}{h_{k,j_k-1}}, \quad k = 1, \ldots, d.$$

On $\bar{Q}_h^{\Delta t}$ we approximate (1.4–1.5) by the one-parameter family of finite difference schemes

$$\frac{U_j^{m+1} - U_j^m}{\Delta t^m} = \sum_{k=1}^{d} \delta_k (D_j^{m,\mu,\theta} \delta_k U_j^{m+\theta}) + F_j^{m,\mu,\theta} \quad \text{in } \tilde{Q}_h^{\Delta t}, \tag{2.2}$$

with

$$U_j^0 = u^0(x_j) \quad \text{in } \bar{\Omega}_h, \tag{2.3}$$

and either

$$U_j^m = 0 \quad \text{in } \Gamma_h^{\Delta t} \tag{2.4}$$

in the case of homogeneous Dirichlet boundary conditions (1.6), or

$$\sum_{k=1}^{d} (\delta_k U_{j+(1/2)e_k}^m + \delta_k U_{j-(1/2)e_k}^m) \nu_k = 0 \quad \text{in } \Gamma_h^{\Delta t} \tag{2.5}$$

in the case of homogeneous Neumann boundary conditions (1.7), where U_j^m represents the approximation of $u(x_j, t^m)$. In (2.2) we consider, for $\mu \in \{0, 1\}$ and $\theta \in [0, 1]$,

$$D_j^{m,\mu,\theta} = D(x_j, t^{m+\theta}, U_j^{m+\mu\theta}) = D_{Rj}^{m,\mu,\theta} + \mathrm{i} D_{Ij}^{m,\mu,\theta},$$

$$D_{j+(1/2)e_k}^{m,\mu,\theta} = \frac{D(x_j, t^{m+\theta}, U_j^{m+\mu\theta}) + D(x_{j+e_k}, t^{m+\theta}, U_{j+e_k}^{m+\mu\theta})}{2}$$

and

$$F_j^{m,\mu,\theta} = F_0(x_j, t^{m+\theta}) + F_{\mathrm{L}}(x_j, t^{m+\theta}) U_j^{m+\theta} + F_{\mathrm{NL}}(x_j, t^{m+\theta}, U_j^{m+\mu\theta}),$$

where

$$U_j^{m+\mu\theta} = \mu U_j^{m+1} + (1 - \mu\theta) U_j^m. \tag{2.6}$$

We use the notation $\tilde{Q}_h^{\Delta t}$ for the set $Q_h^{\Delta t}$ or $\bar{Q}_h^{\Delta t}$ in the case of Dirichlet or Neumann boundary conditions, respectively, and ν_k represents the kth component of the normal vector ν.

Note that, when $\mu = 1$, the cases $\theta = 0$, $\theta = \frac{1}{2}$ and $\theta = 1$ correspond, respectively, to the explicit Euler, Crank–Nicolson and implicit Euler schemes. When $\mu = 0$, we have the semiimplicit case (semi-implicit Euler method when $\theta = 1$), that is, the diffusion coefficient and the nonlinear part of the reaction term are treated explicitly.

In this paper we will consider two cases: the case when $\mu = 1$, which corresponds to the usual θ method, and the case where $\mu = 0$ and $\theta = 1$, that is the semiimplicit Euler scheme. For all cases we suppose that

$$F_{LR}(x_j, t^{m+1}) \leqslant F_{LR\max} \tag{2.7}$$

and

$$J_{\mathrm{R}}(x_j, t^{m+1}, U_j^{m+\theta}) \leqslant J_{R\max} \tag{2.8}$$

for all $(x_j, t^{m+1}) \in \tilde{Q}_h^{\Delta t}$, where $J_{\mathrm{R}}(x,t,v)$ is the real part of $J(x,t,v)$ given by (1.10). For $\mu = 1$ and $\theta \in [0, \frac{1}{2})$ or $\mu = 0$ and $\theta = 1$ we also consider

$$J_{\mathrm{I}}(x_j, t^{m+1}, U_j^{m+\theta}) \leqslant J_{I\max} \tag{2.9}$$

for all $(x_j, t^{m+1}) \in \tilde{Q}_h^{\Delta t}$, where $J_{\mathrm{I}}(x,t,v)$ is the imaginary part of $J(x,t,v)$ given by (1.10). In addition, for $\mu = 1$ and $\theta \in [0, \frac{1}{2})$ we also need to assume that

$$F_{LI}(x_j, t^{m+1}) \leqslant F_{LI\max} \tag{2.10}$$

for all $(x_j, t^{m+1}) \in \tilde{Q}_h^{\Delta t}$. We need the notation

$$|F_{L\max}|^2 = F_{LR\max}^2 + F_{LI\max}^2, \quad |J_{\max}|^2 = J_{R\max}^2 + J_{I\max}^2. \tag{2.11}$$

In what follows, $\|\cdot\|_h$ denotes the discrete L^2 norm, which will be specified in the next section.

3. Stability

In this section we derive the continuous dependence of the numerical solution on the initial data and on the right-hand side.

3.1 *Implicit and explicit case*

Let us first consider the case where $\mu = 1$. In this case we have the usual θ-method.

THEOREM 3.1 Let us consider $\mu = 1$ in the numerical method (2.2–2.3) with (2.4) or (2.5) and suppose that (2.7) and (2.8) hold for all $(x_j, t^{m+1}) \in \tilde{Q}_h^{\Delta t}$. If $\theta \in [\frac{1}{2}, 1]$ the method is stable if, for some $\zeta \in \mathbb{R}^+$ independent of discretization parameters,

$$0 < \zeta \leqslant 1 - 4\theta^2 \Delta t^m K_\epsilon, \tag{3.1}$$

with

$$K_\epsilon = F_{LR\max} + J_{R\max} + \epsilon^2, \tag{3.2}$$

where $\epsilon \neq 0$ is a constant arbitrarily chosen. If $\theta \in [0, \frac{1}{2})$ then the method is stable under the condition (3.1), for some $\zeta \in \mathbb{R}^+$ independent of discretization parameters, with

$$\begin{aligned} K_\epsilon = {} & F_{LR\max} + J_{R\max} + \epsilon^2 + \Delta t^m \left(\tfrac{1}{2} - \theta\right) (1 + \epsilon^{-2})(1 + \epsilon^2) \\ & \times ((1 + \epsilon^2)|F_{L\max}|^2 + (1 + \epsilon^{-2})|J_{\max}|^2), \end{aligned} \tag{3.3}$$

where $\epsilon \neq 0$ is a constant arbitrarily chosen and

$$1 - \Delta t^m \left(\frac{1}{2} - \theta\right)(1+\epsilon^2)\frac{4}{(\min h_{k,j_k})^2} \max_{x_j \in \bar{\Omega}_h} \frac{|D_j^{m+\theta}|^2}{D_{Rj}^{m+\theta}} \geqslant 0, \tag{3.4}$$

provided that (2.9) and (2.10) hold, for all $(x_j, t^{m+1}) \in \tilde{Q}_h^{\Delta t}$, $|D_j^{m,1,\theta}|$ is bounded and there exists some constant ξ, independent of discretization parameters, such that

$$0 < \xi \leqslant D_{Rj}^{m,1,\theta} \quad \forall j, m. \tag{3.5}$$

Proof. To prove this result we will consider the unidimensional case and Neumann boundary conditions. For higher dimension or Dirichlet boundary conditions, the proof follows the same steps.

We rewrite (2.2–2.3) and (2.5) as a system by separating the real and imaginary parts, U_R and U_I, respectively, of the main variable $U = (U_0, \dots, U_N)$. We shall then study the stability of the family of finite difference schemes: find $U_j^m \approx u(x_j, t^m)$, $j = 0, \dots, N$, $m = 0, \dots, M$, such that

$$\begin{cases} \dfrac{U_{Rj}^{m+1} - U_{Rj}^m}{\Delta t^m} = \delta_x(D_{Rj}^{m+\theta}\delta_x U_{Rj}^{m+\theta}) - \delta_x(D_{Ij}^{m+\theta}\delta_x U_{Ij}^{m+\theta}) + F_{Rj}^{m+\theta}, \\ j = 0, \dots, N, \quad m = 0, \dots, M-1, \\ \dfrac{U_{Ij}^{m+1} - U_{Ij}^m}{\Delta t^m} = \delta_x(D_{Ij}^{m+\theta}\delta_x U_{Rj}^{m+\theta}) + \delta_x(D_{Rj}^{m+\theta}\delta_x U_{Ij}^{m+\theta}) + F_{Ij}^{m+\theta}, \\ j = 0, \dots, N, \quad m = 0, \dots, M-1, \end{cases} \tag{3.6}$$

with initial condition

$$U_{Rj}^0 = u_R^0(x_j), \quad U_{Ij}^0 = u_I^0(x_j), \ j = 0, \dots, N, \tag{3.7}$$

and homogeneous Neumann boundary conditions

$$U_{R-1}^m = U_{R1}^m, \quad U_{RN-1}^m = U_{RN+1}^m, \quad U_{I-1}^m = U_{I1}^m, \quad U_{IN-1}^m = U_{IN+1}^m, \quad m = 0, \dots, M, \tag{3.8}$$

where, for simplicity, we write $D_j^{m,1,\theta} = D_j^{m+\theta} = D_{Rj}^{m+\theta} + \mathrm{i}D_{Ij}^{m+\theta}$ and $F_j^{m+\theta} = F_j^{m,1,\theta} = F_{Rj}^{m+\theta} + \mathrm{i}F_{Ij}^{m+\theta}$, $j = 0, \dots, N$, $m = 0, \dots, M-1$. We also use the notation $D_{j^+}^{m+\theta} = D_{j+(1/2)e_1}^{m,1,\theta} = D_{Rj^+}^{m+\theta} + \mathrm{i}D_{Ij^+}^{m+\theta}$. In (3.6) and (3.8) we need the extra points $x_{-1} = x_0 - h_0$ and $x_{N+1} = x_N + h_{N-1}$ and we define $D_{-1^+}^{m+\theta} = D_{0^+}^{m+\theta}$, $D_{N^+}^{m+\theta} = D_{N-1^+}^{m+\theta}$.

We consider the discrete L^2 inner products

$$(U, V)_h = \sum_{j=0}^{N-1} \frac{h_j}{2}(U_j \bar{V}_j + U_{j+1}\bar{V}_{j+1}) \tag{3.9}$$

and

$$(U, V)_{h^*} = \sum_{j=0}^{N-1} h_j U_{j+1/2} \bar{V}_{j+1/2}, \tag{3.10}$$

and their corresponding norms

$$\|U\|_h = (U, U)_h^{1/2} \quad \text{and} \quad \|U\|_{h^*} = (U, U)_{h^*}^{1/2}. \tag{3.11}$$

Multiplying both members of the first and second equations of (3.6) by $U_{\mathrm{R}}^{m+\theta}$ and $U_{\mathrm{I}}^{m+\theta}$, respectively, according to the discrete inner product $(\cdot,\cdot)_h$ and using summation by parts we obtain

$$\left(\frac{U_{\mathrm{R}}^{m+1}-U_{\mathrm{R}}^{m}}{\Delta t^m}, U_{\mathrm{R}}^{m+\theta}\right)_h + \left(\frac{U_{\mathrm{I}}^{m+1}-U_{\mathrm{I}}^{m}}{\Delta t^m}, U_{\mathrm{I}}^{m+\theta}\right)_h + \|(D_{R^+}^{m+\theta})^{1/2}\delta_x U^{m+\theta}\|_{h^*}^2$$
$$= (F_{\mathrm{R}}^{m+\theta}, U_{\mathrm{R}}^{m+\theta})_h + (F_{\mathrm{I}}^{m+\theta}, U_{\mathrm{I}}^{m+\theta})_h.$$

Since we can write

$$U^{m+\theta} = \Delta t^m \left(\theta - \frac{1}{2}\right)\frac{U^{m+1}-U^m}{\Delta t^m} + \frac{U^{m+1}+U^m}{2}, \tag{3.12}$$

we get

$$\Delta t^m \left(\theta - \frac{1}{2}\right)\left\|\frac{U^{m+1}-U^m}{\Delta t^m}\right\|_h^2 + \frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} + \|(D_{R^+}^{m+\theta})^{1/2}\delta_x U^{m+\theta}\|_{h^*}^2$$
$$= (F_{\mathrm{R}}^{m+\theta}, U_{\mathrm{R}}^{m+\theta})_h + (F_{\mathrm{I}}^{m+\theta}, U_{\mathrm{I}}^{m+\theta})_h.$$

If $\theta \in [\frac{1}{2}, 1]$ we immediately obtain

$$\frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} + \|(D_{R^+}^{m+\theta})^{1/2}\delta_x U^{m+\theta}\|_{h^*}^2 \leqslant (F_{\mathrm{R}}^{m+\theta}, U_{\mathrm{R}}^{m+\theta})_h + (F_{\mathrm{I}}^{m+\theta}, U_{\mathrm{I}}^{m+\theta})_h. \tag{3.13}$$

Let us now look at the right-hand side of (3.13). Considering the decomposition (1.8–1.9) we can write

$$\begin{aligned}(F_{\mathrm{R}}^{m+\theta}, U_{\mathrm{R}}^{m+\theta})_h + (F_{\mathrm{I}}^{m+\theta}, U_{\mathrm{I}}^{m+\theta})_h &= (F_{\mathrm{R}}(\cdot, t^{m+\theta}, 0), U_{\mathrm{R}}^{m+\theta})_h + (F_{\mathrm{I}}(\cdot, t^{m+\theta}, 0), U_{\mathrm{I}}^{m+\theta})_h \\ &\quad + (F_{LR}(\cdot, t^{m+\theta})U_{\mathrm{R}}^{m+\theta}, U_{\mathrm{R}}^{m+\theta})_h + (F_{LR}(\cdot, t^{m+\theta})U_{\mathrm{I}}^{m+\theta}, U_{\mathrm{I}}^{m+\theta})_h \\ &\quad + (J_{\mathrm{R}}(\cdot, t^{m+\theta}, U^{m+\theta})U_{\mathrm{R}}^{m+\theta}, U_{\mathrm{R}}^{m+\theta})_h \\ &\quad + (J_{\mathrm{R}}(\cdot, t^{m+\theta}, U^{m+\theta})U_{\mathrm{I}}^{m+\theta}, U_{\mathrm{I}}^{m+\theta})_h.\end{aligned}$$

Since

$$(J_{\mathrm{R}}(\cdot, t^{m+\theta}, U^{m+\theta})U_{\mathrm{R}}^{m+\theta}, U_{\mathrm{R}}^{m+\theta})_h \leqslant J_{R\max}\|U_{\mathrm{R}}^{m+\theta}\|_h^2,$$

and, with the necessary modifications, we obtain a correspondent inequality for $(J_{\mathrm{R}}(\cdot, t^{m+1}, U^{m+\theta})U_{\mathrm{I}}^{m+\theta}, U_{\mathrm{I}}^{m+\theta})_h$, then using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}(F_{\mathrm{R}}^{m+\theta}, U_{\mathrm{R}}^{m+\theta})_h + (F_{\mathrm{I}}^{m+\theta}, U_{\mathrm{I}}^{m+\theta})_h &\leqslant \|F_{\mathrm{R}}(\cdot, t^{m+\theta}, 0)\|_h\|U_{\mathrm{R}}^{m+\theta}\|_h + \|F_{\mathrm{I}}(\cdot, t^{m+\theta}, 0)\|_h\|U_{\mathrm{I}}^{m+\theta}\|_h \\ &\quad + F_{LR\max}\|U^{m+\theta}\|_h^2 + J_{R\max}\|U^{m+\theta}\|_h^2,\end{aligned}$$

which leads to

$$(F_{\mathrm{R}}^{m+\theta}, U_{\mathrm{R}}^{m+\theta})_h + (F_{\mathrm{I}}^{m+\theta}, U_{\mathrm{I}}^{m+\theta})_h \leqslant \frac{1}{4\epsilon^2}\|F(\cdot, t^{m+\theta}, 0)\|_h^2 + \epsilon^2\|U^{m+\theta}\|_h^2 + (F_{LR\max} + J_{R\max})\|U^{m+\theta}\|_h^2,$$

where $\epsilon \neq 0$. Then, from (3.13),

$$\frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} + \|(D_{R^+}^{m+\theta})^{1/2}\delta_x U^{m+\theta}\|_{h^*}^2$$
$$\leqslant \frac{1}{4\epsilon^2}\|F(\cdot, t^{m+\theta}, 0)\|_h^2 + \epsilon^2\|U^{m+\theta}\|_h^2 + (F_{LR\max} + J_{R\max})\|U^{m+\theta}\|_h^2, \tag{3.14}$$

and so

$$\frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} \leqslant \frac{1}{4\epsilon^2}\|F(\cdot, t^{m+\theta}, 0)\|_h^2 + \epsilon^2\|U^{m+\theta}\|_h^2 + (F_{LR\max} + J_{R\max})\,\|U^{m+\theta}\|_h^2. \tag{3.15}$$

Using the definition of $U^{m+\theta}$ we get

$$(1 - 4\theta^2\Delta t^m K_\epsilon)\|U^{m+1}\|_h^2 \leqslant (1 + 4(1-\theta)^2\Delta t^m K_\epsilon)\|U^m\|_h^2 + \frac{\Delta t^m}{2\epsilon^2}\|F(\cdot, t^{m+\theta}, 0)\|_h^2,$$

for $m = 0, \ldots, M-1$, with K_ϵ given by (3.2). If (3.1) holds we get

$$\|U^{m+1}\|_h^2 \leqslant \frac{1 + 4(1-\theta)^2\Delta t^m K_\epsilon}{1 - 4\theta^2\Delta t^m K_\epsilon}\|U^m\|_h^2 + \frac{\Delta t^m}{2\epsilon^2(1 - 4\theta^2\Delta t^m K_\epsilon)}\|F(\cdot, t^{m+\theta}, 0)\|_h^2$$
$$\leqslant (1 + 4(\theta^2 + (1-\theta)^2)\zeta^{-1}\Delta t^m K_\epsilon)\|U^m\|_h^2 + \frac{\Delta t^m}{2\epsilon^2\zeta}\|F(\cdot, t^{m+\theta}, 0)\|_h^2.$$

Summing over m and using the discrete Duhamel principle (Chan & Shen, 1987, Lemma 4.1 in Appendix B) we get

$$\|U^k\|_h^2 \leqslant e^{4(\theta^2 + (1-\theta)^2)\zeta^{-1}K_\epsilon t^k}\left(\|U^0\|_h^2 + \frac{1}{2\epsilon^2\zeta}\sum_{m=0}^{k-1}\|F(\cdot, t^{m+\theta}, 0)\|_h^2\Delta t^m\right),$$

which proves the stability.

We now consider the case where $\theta \in [0, \frac{1}{2})$. In this case we have

$$\frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} + \|(D_{R^+}^{m+\theta})^{1/2}\delta_x^- U^{m+\theta}\|_{h^*}^2$$
$$\leqslant \frac{1}{4\epsilon^2}\|F(\cdot, t^{m+\theta}, 0)\|_h^2 + \epsilon^2\|U^{m+\theta}\|_h^2 + (F_{LR\max} + J_{R\max})\|U^{m+\theta}\|_h^2$$
$$+ \Delta t^m\left(\frac{1}{2} - \theta\right)\left\|\frac{U^{m+1} - U^m}{\Delta t^m}\right\|_h^2. \tag{3.16}$$

Since

$$\left\|\frac{U^{m+1} - U^m}{\Delta t^m}\right\|_h^2 = \left\|\frac{U_{\mathrm{R}}^{m+1} - U_{\mathrm{R}}^m}{\Delta t^m}\right\|_h^2 + \left\|\frac{U_{\mathrm{I}}^{m+1} - U_{\mathrm{I}}^m}{\Delta t^m}\right\|_h^2, \tag{3.17}$$

and, following Araújo *et al.* (2012), we deduce that

$$\left\|\frac{U^{m+1}-U^m}{\Delta t^m}\right\|_h^2 \leqslant (1+\eta_1^2)\frac{4}{(\min h_j)^2}\max_{x_j\in\bar{\Omega}_h}\frac{|D_j^{m+\theta}|^2}{D_{Rj}^{m+\theta}}\|(D_{R^+}^{m+\theta})^{1/2}\delta_x U^{m+\theta}\|_{h^*}^2 + (1+\eta_1^{-2})(\|F_{\mathrm{R}}^{m+\theta}\|_h^2+\|F_{\mathrm{I}}^{m+\theta}\|_h^2),$$

where $\eta_1 \neq 0$. Using (1.8–1.9) we get

$$\begin{aligned}\left\|\frac{U^{m+1}-U^m}{\Delta t^m}\right\|_h^2 \leqslant\ & (1+\eta_1^2)\frac{4}{(\min h_j)^2}\max_{x_j\in\bar{\Omega}_h}\frac{|D_j^{m+\theta}|^2}{D_{Rj}^{m+\theta}}\|(D_{R^+}^{m+\theta})^{1/2}\delta_x U^{m+\theta}\|_{h^*}^2\\ &+(1+\eta_1^{-2})(1+\eta_2^{-2})\|F(\cdot,t^{m+\theta},0)\|_h^2\\ &+(1+\eta_1^{-2})(1+\eta_2^2)(1+\eta_3^2)(F_{LR\max}^2+F_{LI\max}^2)\|U^{m+\theta}\|_h^2\\ &+(1+\eta_1^{-2})(1+\eta_2^2)(1+\eta_3^{-2})(J_{R\max}^2+J_{I\max}^2)\|U^{m+\theta}\|_h^2,\end{aligned}$$

where $\eta_2,\eta_3 \neq 0$. Using the definition of $U^{m+\theta}$ and setting $\eta_1=\eta_2=\eta_3=\epsilon$ we get

$$\begin{aligned}\left\|\frac{U^{m+1}-U^m}{\Delta t^m}\right\|_h^2 \leqslant\ & (1+\epsilon^2)\frac{4}{(\min h_j)^2}\max_{x_j\in\bar{\Omega}_h}\frac{|D_j^{m+\theta}|^2}{D_{Rj}^{m+\theta}}\|(D_{R^+}^{m+\theta})^{1/2}\delta_x U^{m+\theta}\|_{h^*}^2\\ &+(1+\epsilon^{-2})^2\|F(\cdot,t^{m+\theta},0)\|_h^2\\ &+2\theta^2(1+\epsilon^{-2})(1+\epsilon^2)((1+\epsilon^2)|F_{L\max}|^2+(1+\epsilon^{-2})|J_{\max}|^2)\|U^{m+1}\|_h^2\\ &+2(1-\theta)^2(1+\epsilon^{-2})(1+\epsilon^2)((1+\epsilon^2)|F_{R\max}|^2+(1+\epsilon^{-2})|J_{\max}|^2)\|U^m\|_h^2.\end{aligned}$$

Then, considering the previous inequality in (3.16) and if (3.4) holds, we get

$$\begin{aligned}(1-4\theta^2\Delta t^m K_\epsilon)\|U^{m+1}\|_h^2 &\leqslant (1+4(1-\theta)^2\Delta t^m K_\epsilon)\|U^m\|_h^2\\ &+2\Delta t^m\left(\frac{1}{4\epsilon^2}+\Delta t^m\left(\frac{1}{2}-\theta\right)(1+\epsilon^{-2})^2\right)\|F(\cdot,t^{m+1},0)\|_h^2,\end{aligned}$$

for $m=0,\ldots,M-1$, with K_ϵ given by (3.3). If (3.1) holds, summing over m and using the discrete Duhamel principle we get

$$\begin{aligned}\|U^k\|_h^2 &\leqslant e^{4(\theta^2+(1-\theta)^2)\zeta^{-1}K_\epsilon t^k}\\ &\times\left(\|U^0\|_h^2+2\left(\frac{1}{4\epsilon^2}+T\left(\frac{1}{2}-\theta\right)(1+\epsilon^{-2})^2\right)\sum_{m=0}^{k-1}\|F(\cdot,t^{m+\theta},0)\|_h^2\Delta t^m\right),\end{aligned}$$

which concludes the proof. □

REMARK 3.2 If $F(x,t,0)=0$, we may prove that, for $\theta\in[\frac{1}{2},1]$, if

$$0<\zeta\leqslant 1-4\theta^2\Delta t^m K,$$

for some $\zeta \in \mathbb{R}^+$, with

$$K = F_{LR\max} + J_{R\max},$$

we get

$$\|U^{m+1}\|_h^2 \leqslant (1 + 4(\theta^2 + (1-\theta)^2)\zeta^{-1}\Delta t^m K)\|U^m\|_h^2.$$

Summing over m and using the discrete Duhamel principle we obtain

$$\|U^k\|_h^2 \leqslant e^{4(\theta^2+(1-\theta)^2)\zeta^{-1}Kt^k}\|U^0\|_h^2.$$

If, in addition, $F_{LR\max}$ and $J_{R\max}$ are nonpositive, the method is unconditionally stable.

REMARK 3.3 For $\theta \in [0, \frac{1}{2})$, the following particular cases are easily deduced from the previous theorem.

1. If $F(x, t, 0) = 0$, the stability conditions are (3.1) and (3.4) with

$$K_\epsilon = F_{LR\max} + J_{R\max} + \Delta t^m \left(\tfrac{1}{2} - \theta\right)(1 + \epsilon^{-1})((1 + \epsilon^2)|F_{L\max}|^2 + (1 + \epsilon^{-2})|J_{\max}|^2).$$

2. If $F_{\mathrm{L}}(x, t) = 0$, the stability conditions are (3.1) and (3.4) with

$$K_\epsilon = J_{R\max} + \epsilon^2 + \Delta t^m \left(\tfrac{1}{2} - \theta\right)(1 + \epsilon^{-2})(1 + \epsilon^2)|J_{\max}|^2.$$

3. If $J(x, t, U) = 0$, the stability conditions are (3.1) and (3.4) with

$$K_\epsilon = F_{LR\max} + \epsilon^2 + \Delta t^m \left(\tfrac{1}{2} - \theta\right)(1 + \epsilon^{-2})(1 + \epsilon^2)|F_{L\max}|^2.$$

COROLLARY 3.4 If Dirichlet boundary conditions and (3.5) hold, then for $\theta \in [\frac{1}{2}, 1]$ the stability condition is (3.1) with $K_\epsilon = F_{LR\max} + J_{R\max}$ (does not depend on ϵ). In addition, if both $F_{LR\max}$ and $J_{R\max}$ are nonpositive, the method is unconditionally stable. For $\theta \in [0, \frac{1}{2})$ the stability conditions are (3.1) and (3.4) with

$$K_\epsilon = F_{LR\max} + J_{R\max} + \Delta t^m \left(\tfrac{1}{2} - \theta\right)(1 + \epsilon^{-2})(1 + \epsilon^2)((1 + \epsilon^2)|F_{L\max}|^2 + (1 + \epsilon^{-2})|J_{\max}|^2).$$

Proof. According to the discrete Poincaré–Friedrichs inequality (Lemma A.1), there exists a constant $C(\Omega)$, depending on Ω, such that

$$C(\Omega)\|U^m\|_h^2 \leqslant \|\delta_x U^m\|_{h^*}^2.$$

So, for $\theta \in [\frac{1}{2}, 1]$, inequalities (3.5) and (3.14) imply

$$\begin{aligned}&\frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} + \xi C(\Omega)\|U^{m+\theta}\|_h^2 \\ &\quad \leqslant \frac{1}{4\epsilon^2}\|F(\cdot, t^{m+\theta}, 0)\|_h^2 + \epsilon^2\|U^{m+\theta}\|_h^2 + (F_{LR\max} + J_{R\max})\|U^{m+\theta}\|_h^2. \qquad (3.18)\end{aligned}$$

Considering $\epsilon^2 = \frac{1}{2}\xi C(\Omega)$, then $\xi C(\Omega) - \epsilon^2 > 0$ and we obtain

$$(1 - 4\theta^2\Delta t^m K)\|U^{m+1}\|_h^2 \leqslant (1 + 4(1-\theta)^2\Delta t^m K)\|U^m\|_h^2 + \frac{\Delta t^m}{\xi C(\Omega)}\|F(\cdot, t^{m+\theta}, 0)\|_h^2,$$

for $m = 0, \ldots, M-1$, with

$$K = F_{LR\max} + J_{R\max}.$$

Then, the stability condition is (3.1) with $K_\epsilon = K$.

With the same arguments, for $\theta \in [0, \frac{1}{2})$ and Dirichlet boundary conditions, we may prove that, if (3.5) holds, the stability conditions are (3.1) and (3.4) with

$$K_\epsilon = F_{LR\max} + J_{R\max} + \Delta t^m \left(\tfrac{1}{2} - \theta\right)(1+\epsilon^{-2})(1+\epsilon^2)((1+\epsilon^2)|F_{L\max}|^2 + (1+\epsilon^{-2})|J_{\max}|^2). \quad \square$$

COROLLARY 3.5 If $F(x,t,v) = F_0(x,t)$ and (3.5) hold then, for $\theta \in [\frac{1}{2}, 1]$, the method is unconditionally stable and for $\theta \in [0, \frac{1}{2})$ the stability condition is (3.4).

Proof. If we consider Dirichlet boundary conditions, the result is included in the previous corollary. Let us consider Neumann boundary conditions. According to the discrete Poincaré inequality (Lemma A.2), there exists a constant $C(\Omega)$, depending on Ω, such that

$$C(\Omega)\|U^m - \bar{U}^m\|_h^2 \leqslant \|\delta_x U^m\|_{h^*}^2,$$

where

$$\bar{U}^m = \frac{1}{|\Omega|}(U^m, \mathbf{1})_h,$$

and $\mathbf{1}$ is a vector with all entries equal to 1. Then

$$\frac{C(\Omega)}{2}\|U^m\|_h^2 - C(\Omega)\|\bar{U}^m\|_h^2 \leqslant \|\delta_x U^m\|_{h^*}^2.$$

So, for $\theta \in [\frac{1}{2}, 1]$, inequalities (3.5) and (3.14) imply

$$\frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} + \xi\frac{C(\Omega)}{2}\|U^{m+\theta}\|_h^2 \leqslant \frac{1}{4\epsilon^2}\|F_0(\cdot, t^{m+\theta})\|_h^2 + \epsilon^2\|U^{m+\theta}\|_h^2 + C(\Omega)\|\bar{U}^{m+\theta}\|_h^2.$$

Considering $\epsilon^2 = \frac{1}{4}\xi C(\Omega)$, then $\xi\frac{1}{2}C(\Omega) - \epsilon^2 > 0$ and we obtain

$$\|U^{m+1}\|_h^2 \leqslant \|U^m\|_h^2 + \frac{\Delta t^m}{\xi C(\Omega)}\|F_0(\cdot, t^{m+\theta})\|_h^2 + C(\Omega)\|\bar{U}^{m+\theta}\|_h^2.$$

By Lemma A.3 we conclude that

$$\|U^{m+1}\|_h^2 \leqslant \|U^m\|_h^2 + \frac{\Delta t^m}{\xi C(\Omega)}\|F_0(\cdot, t^{m+\theta})\|_h^2 + \frac{C(\Omega)}{|\Omega|^{1/2}}\left(\|\bar{U}^0\|_h + \sum_{k=0}^{m}\Delta t^k\|F_0(\cdot, t^{k+\theta})\|_h\right)^2.$$

Then, the method is unconditionally stable. $\square$

3.2 *Semiimplicit case*

Let us now consider the case where $\mu = 0$ and $\theta = 1$, that is the semiimplicit Euler method.

THEOREM 3.6 Let us consider $\mu = 0$, $\theta = 1$ in the numerical method (2.2–2.3) with (2.4) or (2.5) and suppose that (2.7–2.9) hold for all $(x_j, t^{m+1}) \in \tilde{Q}_h^{\Delta t}$. The numerical method is stable if, for some $\zeta \in \mathbb{R}^+$ independent of discretization parameters,

$$0 < \zeta \leqslant 1 - 2\Delta t^m K_\epsilon, \tag{3.19}$$

with

$$K_\epsilon = F_{LR\max} + \tfrac{1}{2}|J_{\max}|^2 + \epsilon^2, \tag{3.20}$$

where $\epsilon \neq 0$ is a constant arbitrarily chosen.

Proof. As for the previous theorem, to prove this result we will consider the unidimensional case and Neumann boundary conditions. For higher dimensions or Dirichlet boundary conditions, the proof follows the same steps. We shall study the stability of the semiimplicit finite difference scheme: find $U_j^m \approx u(x_j, t^m)$, $j = 0, \ldots, N$, $m = 0, \ldots, M$, such that

$$\begin{cases} \dfrac{U_{Rj}^{m+1} - U_{Rj}^m}{\Delta t^m} = \delta_x(D_{Rj}^{m,0,1}\delta_x U_{Rj}^{m+1}) - \delta_x(D_{Ij}^{m,0,1}\delta_x U_{Ij}^{m+1}) + F_{Rj}^{m,0,1}, \\ \quad j = 0, \ldots, N, m = 0, \ldots, M-1, \\ \dfrac{U_{Ij}^{m+1} - U_{Ij}^m}{\Delta t^m} = \delta_x(D_{Ij}^{m,0,1}\delta_x U_{Rj}^{m+1}) + \delta_x(D_{Rj}^{m,0,1}\delta_x U_{Ij}^{m+1}) + F_{Ij}^{m,0,1}, \\ \quad j = 0, \ldots, N, m = 0, \ldots, M-1, \end{cases} \tag{3.21}$$

with initial condition (3.7) and homogeneous Neumann boundary conditions (3.8), where, for simplicity, we write $D_{j^+}^{m,0,1} = D_{j+(1/2)e_1}^{m,0,1} = D_{Rj^+}^{m,0,1} + \mathrm{i}D_{Ij^+}^{m,0,1}$. In (3.21) we need the extra points $x_{-1} = x_0 - h_0$ and $x_{N+1} = x_N + h_{N-1}$ and we define $D_{-1^+}^{m,0,1} = D_{0^+}^{m,0,1}$, $D_{N^+}^{m,0,1} = D_{N-1^+}^{m,0,1}$.

We consider the discrete L^2 inner products defined by (3.9–3.10) and their corresponding norms.

Multiplying both members of the first and second equations of (3.21) by, respectively, U_{R}^{m+1} and U_{I}^{m+1}, according to the discrete inner product $(\cdot,\cdot)_h$, and using summation by parts we obtain, as for (3.13),

$$\frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} + \|(D_{R^+}^{m,0,1})^{1/2}\delta_x U^{m+1}\|_{h^*}^2 \leqslant (F_{\mathrm{R}}^{m,0,1}, U_{\mathrm{R}}^{m+1})_h + (F_{\mathrm{I}}^{m,0,1}, U_{\mathrm{I}}^{m+1})_h. \tag{3.22}$$

Let us now look to the right-hand side of (3.22). Considering (1.8–1.9) we obtain

$$\begin{aligned}(F_{\mathrm{R}}^{m,0,1}, U_{\mathrm{R}}^{m+1})_h + (F_{\mathrm{I}}^{m,0,1}, U_{\mathrm{I}}^{m+1})_h &= (F_{\mathrm{R}}(\cdot, t^{m+1}, 0), U_{\mathrm{R}}^{m+1})_h + (F_{\mathrm{I}}(\cdot, t^{m+1}, 0), U_{\mathrm{I}}^{m+1})_h \\ &+ (F_{LR}(\cdot, t^{m+1})U_{\mathrm{R}}^{m+1}, U_{\mathrm{R}}^{m+1})_h + (F_{LR}(\cdot, t^{m+1})U_{\mathrm{I}}^{m+1}, U_{\mathrm{I}}^{m+1})_h \\ &+ (J_{\mathrm{R}}(\cdot, t^{m+1}, U^m)U_{\mathrm{R}}^m, U_{\mathrm{R}}^{m+1})_h + (J_{\mathrm{R}}(\cdot, t^{m+1}, U^m)U_{\mathrm{I}}^m, U_{\mathrm{I}}^{m+1})_h \\ &- (J_{\mathrm{I}}(\cdot, t^{m+1}, U^m)U_{\mathrm{I}}^m, U_{\mathrm{R}}^{m+1})_h + (J_{\mathrm{I}}(\cdot, t^{m+1}, U^m)U_{\mathrm{R}}^m, U_{\mathrm{I}}^{m+1})_h.\end{aligned}$$

So, using the Cauchy–Schwarz inequality, we have

$$(J_{\mathrm{R}}(\cdot, t^{m+1}, U^m)U_{\mathrm{R}}^m, U_{\mathrm{R}}^{m+1})_h \leqslant J_{R\max}^2 \|U_{\mathrm{R}}^{m+1}\|_h \|U_{\mathrm{R}}^m\|_h,$$

and so

$$(J_{\rm R}(\cdot, t^{m+1}, U^m)U_{\rm R}^m, U_{\rm R}^{m+1})_h \leqslant \tfrac{1}{2}\left(J_{R\max}^2 \|U_{\rm R}^{m+1}\|_h^2 + \|U_{\rm R}^m\|_h^2\right),$$

and, with the necessary modifications, we obtain a corresponding inequality for $(J_{\rm R}(\cdot, t^{m+1}, U^m)$ $U_{\rm I}^m, U_{\rm I}^{m+1})_h$. We also have, considering the Cauchy–Schwarz inequality,

$$-(J_{\rm I}(\cdot, t^{m+1}, U^m)U_{\rm I}^m, U_{\rm R}^{m+1})_h + (J_{\rm I}(\cdot, t^{m+1}, U^m)U_{\rm R}^m, U_{\rm I}^{m+1})_h \leqslant \tfrac{1}{2}\left(J_{I\max}^2 \|U^{m+1}\|_h^2 + \|U^m\|_h^2\right).$$

Then, for the right-hand side of (3.22), we have

$$\begin{aligned}(F_{\rm R}^{m,0,1}, U_{\rm R}^{m+1})_h + (F_{\rm I}^{m,0,1}, U_{\rm I}^{m+1})_h \leqslant{}& \|F_{\rm R}(\cdot, t^{m+1}, 0)\|_h \|U_{\rm R}^{m+1}\|_h + \|F_{\rm I}(\cdot, t^{m+1}, 0)\|_h \|U_{\rm I}^{m+1}\|_h \\ &+ F_{LR\max}\|U^{m+1}\|_h^2 + \tfrac{1}{2}\left(J_{R\max}^2 + J_{I\max}^2\right)\|U^{m+1}\|_h^2 + \|U^m\|_h^2,\end{aligned}$$

which leads to

$$\begin{aligned}(F_{\rm R}^{m,0,1}, U_{\rm R}^{m+1})_h + (F_{\rm I}^{m,0,1}, U_{\rm I}^{m+1})_h \leqslant{}& \frac{1}{4\epsilon^2}\|F(\cdot, t^{m+1}, 0)\|_h^2 + \epsilon^2\|U^{m+1}\|_h^2 \\ &+ \left(F_{LR\max} + \frac{1}{2}|J_{\max}|^2\right)\|U^{m+1}\|_h^2 + \|U^m\|_h^2,\end{aligned}$$

where $\epsilon \neq 0$. Then, from (3.22),

$$\begin{aligned}\frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} + \|(D_{R^+}^{m,0,1})^{1/2}\delta_x U^{m+1}\|_{h^*}^2 \leqslant{}& \frac{1}{4\epsilon^2}\|F(\cdot, t^{m+1}, 0)\|_h^2 + \epsilon^2\|U^{m+1}\|_h^2 \\ &+ \left(F_{LR\max} + \frac{1}{2}|J_{\max}|^2\right)\|U^{m+1}\|_h^2 + \|U^m\|_h^2, \quad (3.23)\end{aligned}$$

and so

$$\begin{aligned}\frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} \leqslant{}& \frac{1}{4\epsilon^2}\|F(\cdot, t^{m+1}, 0)\|_h^2 + \epsilon^2\|U^{m+1}\|_h^2 \\ &+ \left(F_{LR\max} + \frac{1}{2}|J_{\max}|^2\right)\|U^{m+1}\|_h^2 + \|U^m\|_h^2. \quad (3.24)\end{aligned}$$

Using the definition of $U^{m+\theta}$ we get

$$(1 - 2\Delta t^m K_\epsilon)\|U^{m+1}\|_h^2 \leqslant (1 + 2\Delta t^m)\|U^m\|_h^2 + \frac{\Delta t^m}{2\epsilon^2}\|F(\cdot, t^{m+1}, 0)\|_h^2, \quad (3.25)$$

for $m = 0, \ldots, M-1$, with K_ϵ given by (3.20). If (3.19) holds, summing over m and using the discrete Duhamel principle we get

$$\|U^k\|_h^2 \leqslant e^{2(1+K_\epsilon)\zeta^{-1}t^k}\left(\|U^0\|_h^2 + \frac{1}{2\epsilon^2\zeta}\sum_{m=0}^{k-1}\|F(\cdot, t^{m+1}, 0)\|_h^2 \Delta t^m\right),$$

which concludes the proof. □

REMARK 3.7 If $F(x,t,0)=0$, we may prove that if

$$0<\zeta\leqslant 1-2\Delta t^m K, \tag{3.26}$$

for some $\zeta\in\mathbb{R}^+$, with

$$K=F_{LR\max}+\tfrac{1}{2}|J_{\max}|^2, \tag{3.27}$$

we get

$$\|U^{m+1}\|_h^2\leqslant(1+2\Delta t^m(1+K)\zeta^{-1})\|U^m\|_h^2,$$

for $m=0,\ldots,M-1$, and so

$$\|U^k\|_h^2\leqslant e^{2(1+K)\zeta^{-1}t^k}\|U^0\|_h^2.$$

If, in addition, $K\leqslant 0$, then the method is unconditionally stable.

REMARK 3.8 If we consider the Dirichlet boundary conditions and (3.5) holds, the stability condition is (3.26) with K given by (3.27). In addition, if $F_{\mathrm{NL}}\equiv 0$ and $F_{LR\max}$ is nonpositive, the method is unconditionally stable. We may conclude this result with the same arguments as in Corollary 3.4.

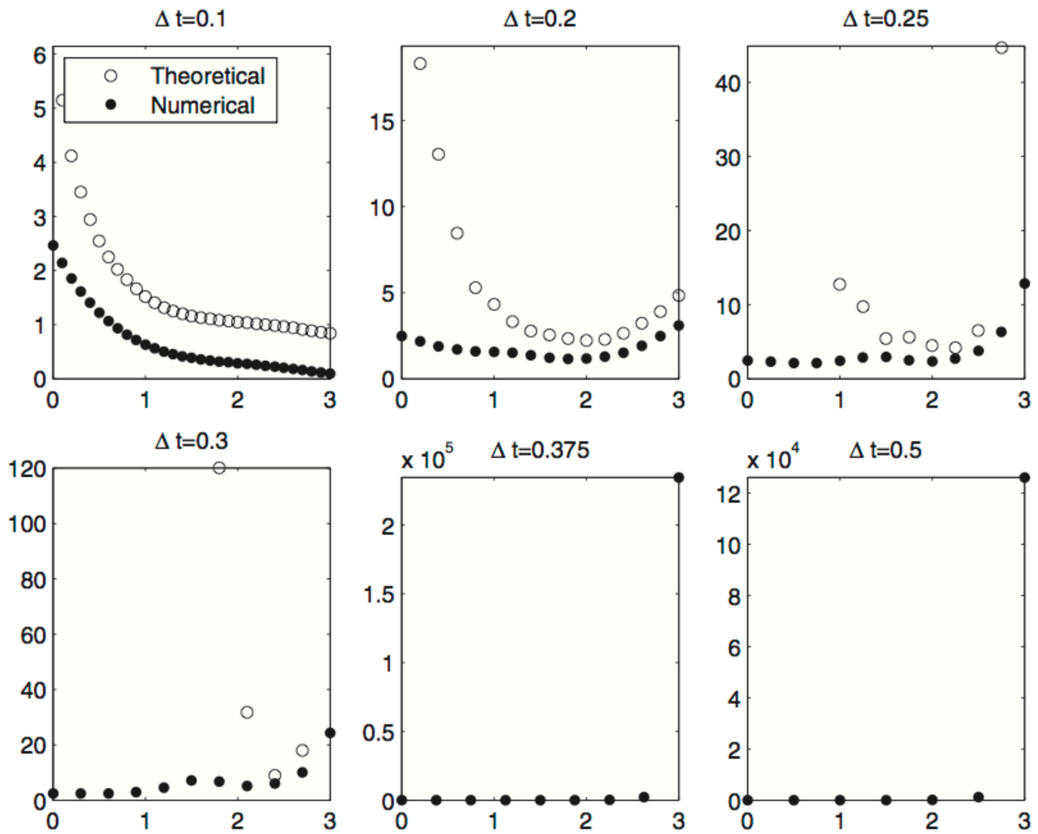

FIG. 1. Case 1: evolution in time of numerical norm $\|U^m\|_h^2$ and the theoretical upper bound (3.25) for several time steps Δt. No plot on the theoretical upper bound means there exists no ξ that satisfies (3.19).

REMARK 3.9 If $F(x,t,v) = F_0(x,t)$, the method is unconditionally stable. We may conclude this result with the same arguments as in Corollary 3.5.

4. Numerical examples

In this section we will illustrate the stability results using appropriate numerical examples. We start by noticing that the stability condition for the explicit method has already been illustrated in Araújo *et al.* (2012), though without a reactive term. Since the numerical results are very similar, we will leave the explicit scheme out of this illustration, referring the reader to Araújo *et al.* (2012) for details. We will also leave out of this section the illustration of the stability of the implicit scheme, since we expect that the choice of linearization method may further influence the results. In this way, we will focus the numerical illustrations on the stability of the semiimplicit scheme with Neumann boundary condition, since the stability condition (though similar to the Dirichlet case) is slightly more complex.

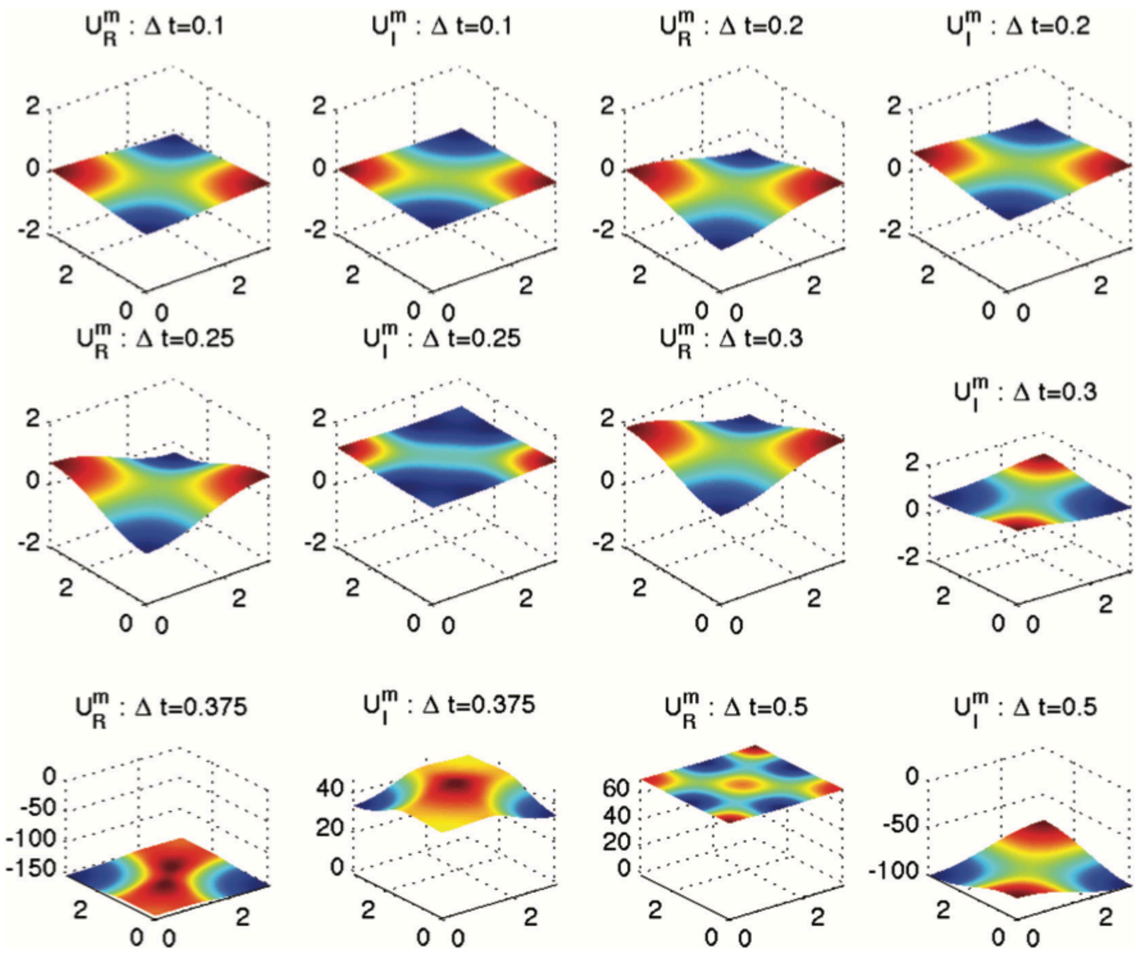

FIG. 2. Case 1: real and imaginary parts of the approximation U^m for the final time $T = 3$ for several time steps Δt.

Let us consider equation (1.4) with

$$(x_1, x_2) \in (0, \pi) \times (0, \pi), \quad t \in (0, T],$$

with initial and Neumann boundary conditions given, respectively, by

$$u(x_1, x_2, 0) = \cos(x_1)\cos(x_2)$$

and

$$\frac{\partial u}{\partial \nu}(0, x_2, t) = \frac{\partial u}{\partial \nu}(\pi, x_2, t) = \frac{\partial u}{\partial \nu}(x_1, 0, t) = \frac{\partial u}{\partial \nu}(x_1, \pi, t) = 0.$$

Given a constant $A \in \mathbb{C}$, for

$$F(x_1, x_2, t, v) = (A + 2\mathrm{i})v + 2v^2 - (\sin^2(x_1)\cos^2(x_2) + \cos^2(x_1)\sin^2(x_2))\, e^{2At}$$

and

$$D(x_1, x_2, t, v) = \mathrm{i} + v,$$

the exact solution is given by

$$u(x_1, x_2, t) = \cos(x_1)\cos(x_2)\, e^{At}.$$

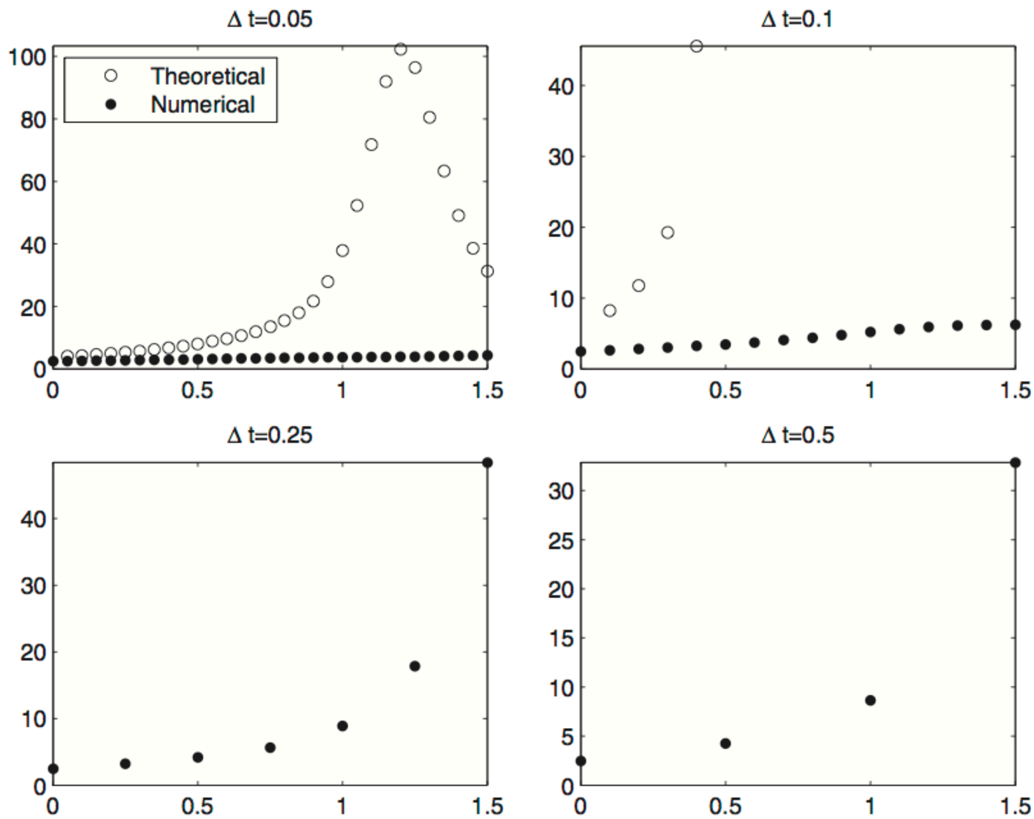

FIG. 3. Case 2: evolution in time of numerical norm $\|U^m\|_h^2$ and the theoretical upper bound (3.25) for several time steps Δt. No plot on the theoretical upper bound means there exists no ξ that satisfies (3.19).

We also note that with this choice of reactive term F we have

$$
\begin{aligned}
F_0(x_1,x_2,t) &= -(\sin^2(x_1)\cos^2(x_2)+\cos^2(x_1)\sin^2(x_2))\,e^{2At},\\
F_{\mathrm{L}}(x,t) &= A+2\mathrm{i},\\
F_{\mathrm{NL}}(x,t,v) &= 2v^2 \quad (\text{and } F_{\mathrm{NL}}(x,t,0)=0),\\
J(x,t,v) &= 2v^2.
\end{aligned}
$$

We will now consider two different possibilities for the value of the constant A that will induce different behaviours on the solution and therefore on the stability condition.

4.1 *Case* 1: $\mathrm{F}_{\mathrm{LR}} \leqslant 0$

For $A=-1+\mathrm{i}$, we have that $F_{LR}=-1<0$. We will now consider the upper bound (3.25) (taking $\epsilon=1$) and compare it with the actual norm $\|U^m\|_h^2$. We also note that if the time step Δt is such that there exists no $\xi>0$ so that (3.19) is satisfied, then no theoretical upper bound is known and the numerical solution might become unbounded in time (even in cases where the solution is bounded).

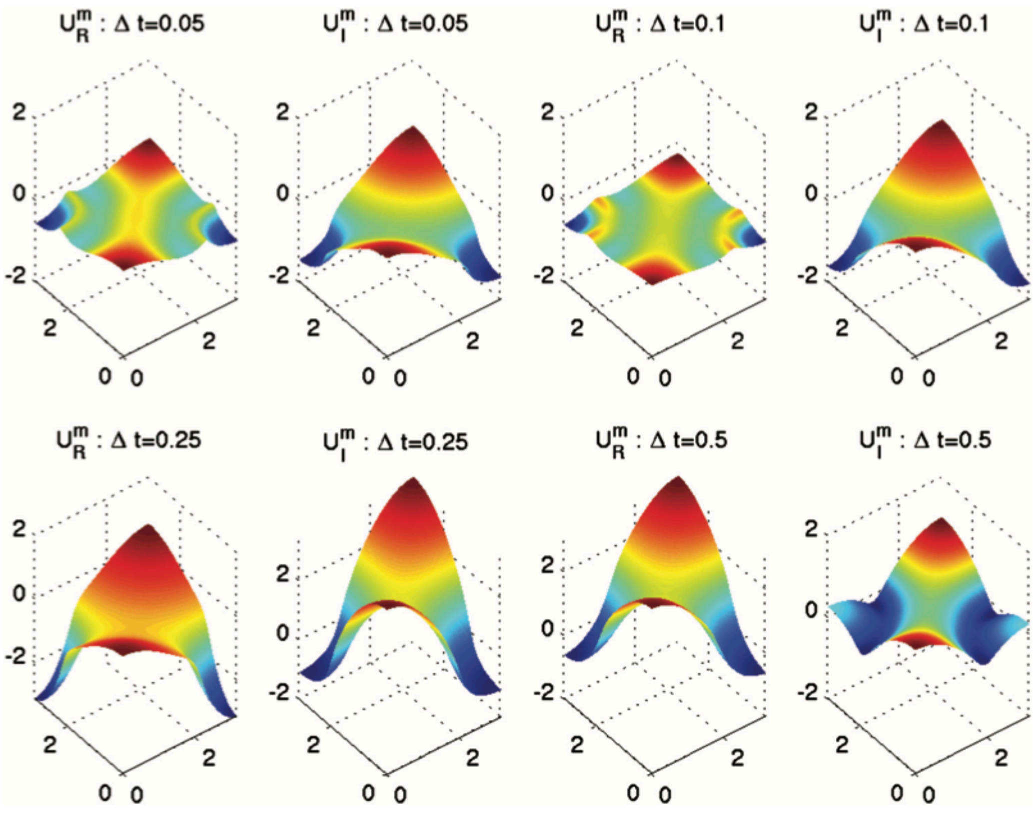

FIG. 4. Case 2: real and imaginary parts of the approximation U^m for final time $T=1.5$ for several time steps Δt.

The numerical results are shown in Figs 1 and 2. It can be seen that for smaller steps in time, the ratio stays bounded by the theoretical upper bound. For higher time steps (namely for time steps that do not satisfy the stability condition), there is no theoretical upper bound and the norm of the approximation increases rapidly.

4.2 *Case* 2*:* $F_{LR} > 0$

For $A = 0.1 + \mathrm{i}$, we have $F_{LR} = 0.1 > 0$. In this way, the condition (3.19) is harder to satisfy, since now $F_{LR\max}$ is positive. Again we compare the theoretical upper bound (3.25) and the actual norm $\|U^m\|_h^2$.

The numerical results are shown in Figs 3 and 4. It can be seen that though in some cases the theoretical bound increases, the numerical results might stay bounded. Similarly to the previous case, for higher steps in time, the approximation's norm increases rapidly.

To better illustrate this phenomenon we also considered nonuniform meshes. To this end, we considered 50 points in each spatial direction randomly distributed (by a uniform distribution) to define the spatial mesh. Moreover we considered 30 steps in time, corresponding to instants randomly chosen in the interval [0,1]. The evolution in time of numerical norm $\|U^m\|_h^2$ and the theoretical upper bound (3.25) is given in Fig. 5 for four different cases. Again, similar behaviour is observed.

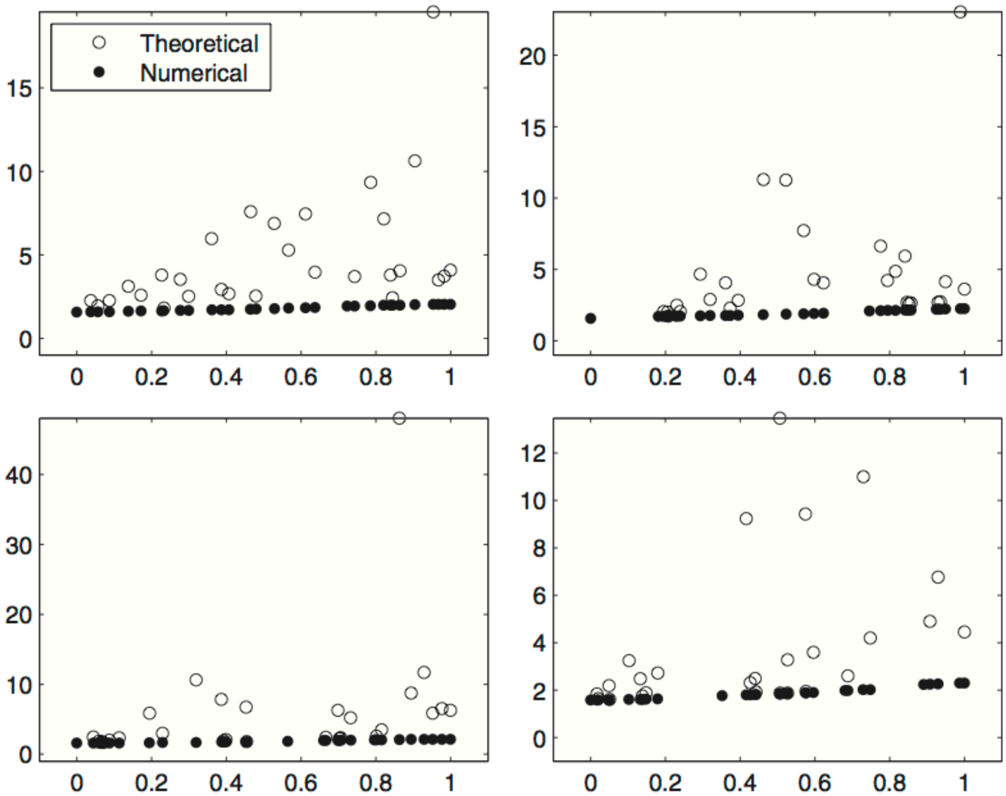

FIG. 5. Case 2: evolution in time of numerical norm $\|U^m\|_h^2$ and the theoretical upper bound (3.25) for nonuniform time steps Δt. No plot on the theoretical upper bound means there exists no ξ that satisfies (3.19).

5. Conclusions

In this paper we have established the stability conditions for finite difference schemes in the context of complex diffusion with reactive terms. We have extended a previous stability result (Araújo *et al.*, 2012) to the semiimplicit scheme and to the presence of reactive terms in complex diffusion. In this way we have shown that both the implicit and semiimplicit schemes are stable under some conditions on the time step. We note that at a fixed time, there is always a small enough time step for which the method is stable, since the stability condition is an upper bound for the time step. As usual, for the explicit scheme, a stability condition that relates the magnitude of the time step and the spatial step size needs to be satisfied. Finally we have illustrated the theoretical results with numerical examples.

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Appendix A. Technical results

LEMMA A.1 (Discrete Poincaré–Friedrichs inequality) Let U be a discrete function defined on $\bar{\Omega}_h$ given by (2.1) such that $U = 0$ on $\Gamma \cap \bar{\Omega}_h$, with Γ the boundary of Ω given by (1.1). There exists a positive constant $C(\Omega)$ independent of U and h such that

$$C(\Omega)\|U\|_h^2 \leqslant \|\delta_x U\|_{h^*}^2.$$

Proof. Since $\|U\|_h^2 = \|U_\mathrm{R}\|_h^2 + \|U_\mathrm{I}\|_h^2$ and $\|\delta_x U\|_{h^*}^2 = \|\delta_x U_\mathrm{R}\|_{h^*}^2 + \|\delta_x U_\mathrm{I}\|_{h^*}^2$ the proof follows from the equivalent result for the real case (see, e.g., Jovanović & Süli, 2014). □

LEMMA A.2 (Discrete Poincaré inequality) Let U be a discrete function defined on $\bar{\Omega}_h$ given by (2.1), with Γ the boundary of Ω given by (1.1). There exists a positive constant $C(\Omega)$ independent of U and h such that

$$C(\Omega)\|U - \bar{U}\|_h^2 \leqslant \|\delta_x U\|_{h^*}^2,$$

where

$$\bar{U} = \frac{1}{|\Omega|}(U, \mathbf{1})_h,$$

and $\mathbf{1}$ is a vector with all entries equal to 1.

Proof. Let us consider $\bar{U} = \bar{U}_\mathrm{R} + \mathrm{i}\bar{U}_\mathrm{I}$. We start by proving that

$$C(\Omega)\|U_\mathrm{R} - \bar{U}_\mathrm{R}\|_h^2 \leqslant \|\delta_x U_\mathrm{R}\|_{h^*}^2.$$

To prove the result we will just consider the unidimensional case. The proof is similar for higher dimensions. Since

$$\bar{U}_\mathrm{R} = \frac{1}{|\Omega|}(U_\mathrm{R}, \mathbf{1})_h,$$

there exists some index $j_{\max}$ such that

$$|U_{Rj} - \bar{U}_{\mathrm{R}}| \leqslant |U_{Rj} - U_{Rj_{\max}}|, \quad j = 0, \ldots, N.$$

Then, using the Cauchy–Schwarz inequality,

$$(U_{Rj} - \bar{U}_{\mathrm{R}})^2 \leqslant \left(\sum_{\ell=\min\{j,j_{\max}\}}^{\max\{j,j_{\max}\}} h_\ell \delta_x U_{R\ell-1/2} \right)^2 \leqslant |\Omega| \sum_{\ell=1}^{N} h_\ell (\delta_x U_{R\ell-1/2})^2, \quad j = 0, \ldots, N.$$

Summing over j we get

$$\sum_{j=0}^{N-1} \frac{h_j}{2} ((U_{Rj} - \bar{U}_{\mathrm{R}})^2 + (U_{Rj+1} - \bar{U}_{\mathrm{R}})^2) \leqslant |\Omega|^2 \sum_{\ell=1}^{N} h_\ell (\delta_x U_{R\ell-1/2})^2.$$

In the same way we have

$$\sum_{j=0}^{N-1} \frac{h_j}{2} ((U_{Ij} - \bar{U}_{\mathrm{I}})^2 + (U_{Ij+1} - \bar{U}_{\mathrm{I}})^2) \leqslant |\Omega|^2 \sum_{\ell=1}^{N} h_\ell (\delta_x U_{I\ell-1/2})^2,$$

which concludes the proof. □

LEMMA A.3 (Discrete conservation property) Let U^m be the solution of (2.2–2.3) with (2.4) or (2.5). If $F(x,t,v) = F_0(x,t)$ the following discrete conservation property holds:

$$(U^m, \mathbf{1})_h = (U^0, \mathbf{1})_h + \sum_{k=0}^{m} \Delta t^k (F_0(\cdot, t^{k+\theta}), \mathbf{1})_h.$$

Proof. To prove the result we will just consider the unidimensional case. For higher dimensions, the proof follows the same steps.

Note that we have

$$U_{\mathrm{R}}^{m+1} = U_{\mathrm{R}}^m + \Delta t^m (A_1 U_{\mathrm{R}}^{m+\theta} + A_2 U_{\mathrm{I}}^{m+\theta} + F_{0R}^{m+\theta})$$

and

$$U_{\mathrm{I}}^{m+1} = U_{\mathrm{I}}^m + \Delta t^m (A_3 U_{\mathrm{R}}^{m+\theta} + A_4 U_{\mathrm{I}}^{m+\theta} + F_{0I}^{m+\theta}),$$

where A_ℓ, $\ell = 1,2,3,4$ are matrices that depend on D, U and on the spatial step sizes. Then, summing according to the discrete inner product, and taking into account that $(A_1 U_{\mathrm{R}}^{m+\theta}, \mathbf{1})_h = (A_2 U_{\mathrm{I}}^{m+\theta}, \mathbf{1})_h = (A_3 U_{\mathrm{R}}^{m+\theta}, \mathbf{1})_h = (A_4 U_{\mathrm{I}}^{m+\theta}, \mathbf{1})_h = 0$, we get

$$(U_{\mathrm{R}}^m, \mathbf{1})_h = (U_{\mathrm{R}}^0, \mathbf{1})_h + \sum_{k=0}^{m} \Delta t^k (F_{0R}(\cdot, t^{k+\theta}), \mathbf{1})_h$$

and

$$(U_{\mathrm{I}}^m, \mathbf{1})_h = (U_{\mathrm{I}}^0, \mathbf{1})_h + \sum_{k=0}^{m} \Delta t^k (F_{0I}(\cdot, t^{k+\theta}), \mathbf{1})_h,$$

which concludes the proof. □