Proceedings of the 14th International Conference on Computational and Mathematical Methods in Science and Engineering, CMMSE 2014 3–7July, 2014.

Decay of solutions of wave equations with memory

J. A. Ferreira[1], **P. de Oliveira**[1] **and G. Pena**[1]

[1] *CMUC, Department of Mathematics, University of Coimbra*

emails: `ferreira@mat.uc.pt`, `poliveir@mat.uc.pt`, `gpena@mat.uc.pt`

Abstract

In this paper we consider a linear damping wave equation with a memory effect using exponential kernels. We establish a bound for an energy function that is shown to converge to zero. Numerical waves that mimic their continuous counterpart are also introduced using the finite element approach and the qualitative behaviour of the solutions is explored.

Key words: wave equation, memory, viscoelasticity, energy estimates

1 Introduction

In this paper we consider the following wave equation with memory

$$\frac{d^2u}{dt^2}(t) + c\frac{du}{dt}(t) - D_1\Delta u(t) = -D_2\int_0^t K_{er}(t-s)\Delta u(s)ds + f(t),\ t \in \mathbb{R}^+, \tag{1}$$

where $K_{er}(s) = \dfrac{1}{\tau}e^{-\frac{s}{\tau}}, \tau > 0$, $u(t)$ denotes a function defined from $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}$, c is a function depending only on spatial variables and accounts for the damping of the wave, D_1, D_2 and τ are positive constants and f denotes a source term.

Equation (1) can be used to model the displacement of a viscoelastic material under the action of an external force when the stress tensor $\sigma(t)$ and the strain tensor $\epsilon(t)$ are related by the following constitutive equation

$$\sigma(t) = E(0)D\epsilon(t) - \int_0^t \frac{\partial}{\partial s}E(t-s)D\epsilon(s)\,ds, \tag{2}$$

where D is an elastic tensor and the stress relaxation function, E, is nonnegative and monotone decreasing. Assuming that the viscoelastic behaviour is described by Maxwell-Wiechert model (with only one Maxwell arm), then $E(t) = E_0 + E_1 e^{-\alpha_1 t}$, where E_0 is the

 ISBN: 978-84-616-9216-3

Young modulus of the spring arm, E_1 is the Young modulus of the Maxwell arm and $\alpha_1 = \frac{E_1}{\mu_1}$ being μ_1 the associated viscosity. The relation between the displacement u, the stress σ and the external force f is given by Newton's second law

$$\rho \frac{d^2u}{dt^2} = \nabla \cdot \sigma + f, \tag{3}$$

where ρ is the mass density of the body and $\nabla \cdot \sigma = \left[\sum_{j=1}^{n} \frac{\partial \sigma_{ij}}{\partial x_j} \right]_{i,j=1}^{n}$. Assuming that the strain and the displacement is given by $\epsilon(t) = \frac{1}{2}\left(\nabla u(t) + \nabla u(t)^t\right)$, from (3) we obtain for the displacement the following second order integro-differential equation

$$\rho \frac{d^2u}{dt^2}(t) - D_1 \Delta u(t) = -D_2 \int_0^t K_{er}(t-s) \Delta u(s)\; ds\, ds + f, \tag{4}$$

with $D_1 = D(E(0) + E_1)$, $D_2 = \frac{E_1}{2}$ and $\tau = \alpha_1^{-1}$.

In what follows we consider homogeneous Dirichlet boundary conditions and the following initial conditions

$$u(0) = u_0, \frac{du}{dt}(0) = u_1. \tag{5}$$

A quasilinear problem of the type of (1) was also introduced for instance in [5], [9] and [11] to describe a viscoleasticity physical problem. Without being exhaustive we mention [1], [2], [], [7], [8],[10] and [12] for the study of qualitative properties of partial differential problems defined by equations of the type of (1).

The initial boundary value problem (IBVP) (1), (5) with homogeneous Dirichlet boundary conditions is now replaced by its weak formulation. To define such formulation we introduce the functional context needed. Let $L^2(\Omega), L^\infty(\Omega)$ and $H_0^1(\Omega)$ be the usual Sobolev spaces. In $L^2(\Omega)$ we consider the usual inner product $(\cdot,\cdot)$ and the norm induced by this inner product is denoted by $\|\cdot\|_0$. In $H_0^1(\Omega)$ we consider the usual norm $\|\cdot\|_1$. Let $L^2(\mathbb{R}^+; H_0^1(\Omega))$ be the space of functions $v : \mathbb{R}^+ \to H_0^1(\Omega)$ such that $\int_0^T \|v(t)\|_1^2\, dt < \infty,\ \forall T > 0$. Let $H^1(\mathbb{R}^+; H_0^1(\Omega))$ be the subspace of $L^2(\mathbb{R}^+; H_0^1(\Omega))$ of all functions v such that its weak derivative $\frac{dv}{dt} : \mathbb{R}^+ \to H_0^1(\Omega)$ belongs to $L^2(0,\infty; H_0^1(\Omega))$. By $L^\infty(\mathbb{R}^+; L^2(\Omega))$ we represent the space of all functions $v : \mathbb{R}^+ \to L^2(\Omega)$ such that

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|v(t)\|_0 < \infty, \forall T > 0.$$

Let $V = L^2(\Omega)$ or $V = H_0^1(\Omega)$. By $C^m(\mathbb{R}^+; V)$, $m \in \mathbb{N}$, we represent the space of function $v : \mathbb{R}_0^+ \to V$ with continuous derivatives $\frac{d^jv}{dt^j} : \mathbb{R}_0^+ \to V$, for $j = 0, \ldots, m$.

 ISBN: 978-84-616-9216-3

Let $u \in H^1(\mathbb{R}^+; H_0^1(\Omega))$ be such that $\dfrac{d^2u}{dt^2} \in L^\infty(\mathbb{R}^+; L^2(\Omega))$ and, for all $T > 0$, holds the following

$$\begin{cases} \left(\dfrac{d^2u}{dt^2}(t) + c\dfrac{du}{dt}(t), w\right) + D_1(\nabla u(t), \nabla w) = \dfrac{D_2}{\tau}\displaystyle\int_0^t e^{-\frac{t-s}{\tau}}(\nabla u(s), \nabla w)\, ds + (f(t), w), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a. e. in } (0,T),\ \forall w \in H_0^1(\Omega), \\ \dfrac{du}{dt}(0) = u_1, \\ u(0) = u_0. \end{cases} \tag{6}$$

In (6) the inner products in $L^2(\Omega)$ and $(L^2(\Omega))^n$ are denoted indifferently by $(.,.)$. Their norms will be also represented indifferently by $\|.\|$.

The main objective of this paper is the analysis of an energy functional under general assumptions and the illustration of the qualitative behaviour of numerical solutions under several different choices for the parameters. We shall prove that a suitable energy functional converges to zero as $t \to \infty$. Numerical wave equations that mimic their continuous counterpart will be also considered and their behaviour will be explored.

The paper is organized as follows. In Section 2 we introduce the new energy functional and we prove that under convenient assumptions we have

$$\lim_{t\to\infty} \frac{du}{dt} = 0 \text{ in } L^2(\Omega), \qquad \lim_{t\to\infty} u(t) = 0 \text{ in } H^1(\Omega),$$

and

$$\lim_{t\to\infty} \int_0^t K_{er}(t-s)\nabla u(s) ds = 0 \text{ in } L^2(\Omega).$$

A finite element method is introduced in Section 3 that mimics energy behavior of the IBVP studied in this paper. The behavior of the IBVP (1), (5) with homogeneous Dirichlet boundary conditions and in Section 5 we summarize some conclusions.

2 Energy behaviour

The energy functional that we introduce here extend several definitions introduced before in the literature. For instance, in [4] and [16] the authors considered the classical energy functional

$$E(u)(t) = \|u(t)\|_0 + \|\nabla u(t)\|_0^2,$$

while in ([10]) a term was added to the last energy functional induced by the boundary conditions. Also in ([3]), for a quasilinear problem, a term related with the reaction term

 ISBN: 978-84-616-9216-3

was also added. In [12] the energy functional

$$E(u)(t) = \frac{1}{2}\|\frac{du}{dt}\|_0^2 + \frac{1}{2}\Big(1 - \int_0^t K_{er}(t-s)ds\Big)\|\nabla u(t)\|_0^2 + \int_0^t K_{er}(t-s)\|\nabla u(t) - \nabla u(s)\|_0^2 ds, \tag{7}$$

was introduced. A similar definition to (7) was considered in [16] but with the last term of (7) replaced by $\displaystyle\int_0^t K_{er}(t-s)\|u(t)-u(s)\|_0^2 ds$.

In the first result that we present we establish an estimate for the usual energy for the wave equation

$$\begin{aligned} E(u)(t) \;\; = \;\; & \left\|\frac{du}{dt}(t)\right\|_0^2 + \|u(t)\|_1^2 + \left\|\int_0^t \frac{e^{-\frac{t-s}{\tau}}}{\tau}\nabla u(s)\,ds - \nabla u(t)\right\|_0^2 \\ & + \left\|\int_0^t \frac{e^{-\frac{t-s}{\tau}}}{\tau}\nabla u(s)\,ds\right\|_0^2, \end{aligned} \tag{8}$$

for $t > 0$, where u is a solution of (6). Under suitable regularity conditions, it can be shown the following result.

Theorem 1. *Let $u \in C^2(0,\infty, L^2(\Omega)) \bigcap C^1(0,\infty, H_0^1(\Omega))$ be a solution of (6) for $D_1 > D_2$, $K_{er}(s) = Ke^{-\beta s}$, $c \in L^\infty(\Omega)$ satisfying*

$$c \geq c_0 > 0 \;\; on \; \overline{\Omega}. \tag{9}$$

and $f = 0$. If there exists a positive constant γ such that $\gamma > \min\{\|c\|_\infty, \beta + K\}$, and

$$\frac{\max\left\{2(2\gamma - c_0), 2D_2\frac{(\gamma-\beta)^2}{K}\right\}}{\min\left\{1, \gamma^2 - \gamma\|c\|_\infty, D_1 - D_2, D_2, D_2\frac{\gamma-\beta-K}{K}\right\}} - 2\gamma < 0, \tag{10}$$

then

$$\begin{aligned} \lim_{t\to\infty} \Bigg(& \left\|\frac{du}{dt}(t)\right\|_0^2 + \|u(t)\|_1^2 + \left\|\int_0^t K_{er}(t-s)\nabla u(s)\,ds - \nabla u(t)\right\|_0^2 \Bigg) \\ & + \left\|\int_0^t K_{er}(t-s)\nabla u(s)\,ds\right\|_0^2 + \int_0^t e^{-2\gamma(t-s)}\|\nabla u(s)\|_0^2\,ds = 0. \end{aligned} \tag{11}$$

3 Decay decreasing of numerical waves

The study of numerical methods to solve numerically the IBVP (1), (5) with homogeneous Dirichlet boundary conditions was presented for instance in [6], [14], [13], [15] and some of the references of these papers.

 ISBN: 978-84-616-9216-3

In this section we establish that numerical approximations for the solution of the IBVP(1), (5) with homogeneous Dirichlet boundary conditions presents the same qualitative behaviour of the solution of this problem. Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and let $h > 0$ be a fixed parameter and let $\mathcal{T}_h$ be an admissible triangulation of Ω with diameter h, that is,

$$h = \max_{\Delta \in \mathcal{T}_h} \mathrm{diam}(\Delta),$$

where $\mathrm{diam}(\Delta)$ denotes the diameter of Δ. Let $\mathcal{V}_h$ be the space of piecewise polynomials of degree m defined in $\mathcal{T}_h$, that is

$$\mathcal{V}_h = \{v \in C^0(\overline{\Omega}) :\ v = 0 \text{ on } \partial\Omega,\ v = p_m \text{ in } \Delta,\ \Delta \in \mathcal{T}_h\},$$

where p_m denotes a polynomial of degree at most m. By $\mathcal{P}_{\partial\Omega}$ and $\mathcal{P}_\Omega$ we represent the set of nodes of $\mathcal{T}_h$ on $\partial\Omega$ and Ω, respectively. Let $\{\phi_P, P \in \mathcal{P}_\Omega\}$ be a basis of $\mathcal{V}_h$. The finite element approximation for the solution of the IBVP (1), (5) with homogeneous Dirichlet boundary conditions is $u_h(x,t) = \sum_{P \in \mathcal{P}_\Omega} \alpha_P(t)\phi_P(x)$ that satisfies the following

$$\begin{cases} \left(\dfrac{d^2 u_h}{dt^2}(t) + c\dfrac{du_h}{dt}(t), w_h\right) + (\nabla u_h(t), \nabla w_h) = \dfrac{D_2}{\tau}\displaystyle\int_0^t e^{-\frac{t-s}{\tau}}(\nabla u_h(s), \nabla w_h)\, ds + (f(t), w_h), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a. e. in } (0,T),\ \forall w_h \in \mathcal{V}_h, \\ \dfrac{du_h}{dt}(0) = u_{1,h}, \\ u_h(0) = u_{0,h}. \end{cases} \tag{12}$$

In (12) $u_{1,h}$ and $u_{0,h}$ are approximations of u_1 and u_0 in $\mathcal{V}_h$. To compute $u_h(t)$ we need to solve the following system of second order integro-differential equations

$$\begin{cases} M_h\alpha''(t) + C_h\alpha'(t) + A_h\alpha(t) = \displaystyle\int_0^t e^{-\frac{t-s}{\tau}} B_h\alpha(s)\, ds + F_h(t),\ t > 0, \\ \alpha'(0) = \mathcal{U}_{1,h}, \\ \alpha(0) = \mathcal{U}_{0,h}, \end{cases} \tag{13}$$

where $\alpha(t) = \left[(\alpha_P(t))_{P\in\mathcal{P}_\Omega}\right], \mathcal{U}_{i,h}, i = 0,1$, are the vectors whose components are the coordinates of $u_{i,h}, i = 0, 1$, with respect to the basis $\{\phi_P, P \in \mathcal{P}_\Omega\}$, and

$$M_h = \left[((\phi_P, \phi_Q))_{P,Q\in\mathcal{P}_\Omega}\right],\ C_h = \left[((c\phi_P, \phi_Q))_{P,Q\in\mathcal{P}_\Omega}\right], A_h = \left[((\nabla\phi_P, \nabla\phi_Q))_{P,Q\in\mathcal{P}_\Omega}\right],$$

$$B_h = \left[\frac{D_2}{\tau}((\nabla\phi_P, \nabla\phi_Q))_{P,Q\in\mathcal{P}_\Omega}\right], F_h(t) = [((f(t), \phi_Q))_{Q\in\mathcal{P}_\Omega}].$$

 ISBN: 978-84-616-9216-3

Introducing the new variable $Z(t) = (z_1(t), z_2(t))$ where $z_1(t) = \alpha(t), z_2(t) = \alpha'(t)$, then the initial value problem (13) of second order is equivalent to

$$\begin{cases} Z'(t) = \mathcal{A}_h Z(t) + \displaystyle\int_0^t K_{er}(t-s)\mathcal{B}_h Z(s)\, ds + \mathcal{F}_h(t),\ t > 0, \\ Z(0) = \mathcal{U}_h, \end{cases} \tag{14}$$

where

$$\mathcal{A}_h = \begin{bmatrix} 0 & I \\ -M_h^{-1}A_h & -M_h^{-1}C_h \end{bmatrix}, \mathcal{B}_h = \begin{bmatrix} M_h^{-1}B_h & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{F}_h(t) = \begin{bmatrix} 0 \\ M_h^{-1}F_h \end{bmatrix}, \mathcal{U}_h = \begin{bmatrix} U_{0,h} \\ U_{1,h} \end{bmatrix}.$$

As the unique solution of the IVP (14) is smooth enough, then for the unique solution $u_h(t) \in \mathcal{V}_h$ of (12) it can be shown the following results:

Theorem 2. *Under the assumptions of Theorem 1 we have*

$$\begin{aligned} \lim_{t\to\infty} \Bigg(\left\| \frac{du_h}{dt}(t) \right\|_0^2 + \|u_h(t)\|_1^2 + \left\| \int_0^t K_{er}(t-s)\nabla u_h(s)\, ds - \nabla u_h(t) \right\|_0^2 \Bigg) \\ + \left\| \int_0^t K_{er}(t-s)\nabla u_h(s)\, ds \right\|_0^2 + \int_0^t e^{-2\gamma(t-s)} \|\nabla u_h(s)\|_0^2\, ds = 0. \end{aligned} \tag{15}$$

4 Numerical results

In this section we illustrate the qualitative behaviour of numerical solutions of (13). We now introduce the specifics of our test problems. Given the weak formulation (6), we specify the domain $\Omega = (-1,1)^2$ and the initial data $u(x,y,0) = e^{-\frac{x^2+y^2}{0.1}}$, $\dfrac{du}{dt}(x,y,0) = 0$ for $(x,y) \in \Omega$.

Following the spatial discretisation in (12), we introduce the time step Δt and a uniform partition $t_j = j\Delta t, j = 0,1,2,\ldots,N = [\frac{T}{\Delta t}]$. Applying standard centered finite differences schemes in time and the composite trapezoidal rule to the formulation (13), the following second order in time method is obtained:

$$\begin{aligned} &\left(\frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2}, v \right) + c\left(\frac{u_h^{n+1} - u_h^{n-1}}{2\Delta t}, v \right) + D_1\left(\nabla u_h^{n+1}, \nabla v\right) = \\ &= \frac{D_2 \Delta t}{2\tau} \sum_{j=0}^{n} \left(e^{-\frac{t_{n+1}-t_{j+1}}{\tau}} \nabla u_h^{j+1} + e^{-\frac{t_{n+1}-t_j}{\tau}} \nabla u_h^j, \nabla v \right) \end{aligned} \tag{16}$$

where u_h^j is an approximation for $u(t_j)$, $j = 0,1,\ldots,N$.

 ISBN: 978-84-616-9216-3

Let $I_{n+1} = \dfrac{D_2 \Delta t}{2\tau} \displaystyle\sum_{j=0}^{n} \left(e^{-\frac{t_{n+1}-t_{j+1}}{\tau}} \nabla u_h^{j+1} + e^{-\frac{t_{n+1}-t_j}{\tau}} \nabla u_h^j \right)$. It is straightforward that I_n satisfies

$$\begin{cases} I_{n+1} = e^{-\frac{\Delta t}{\tau}} I_n + \dfrac{D_2 \Delta t}{2\tau} \left(e^{-\frac{\Delta t}{\tau}} \nabla u_h^n + \nabla u_h^{n+1} \right), \quad n > 1 \\ I_1 = \dfrac{D_2 \Delta t}{2\tau} \left(e^{-\frac{\Delta t}{\tau}} \nabla u_h^0 + \nabla u_h^1 \right). \end{cases} \tag{17}$$

With this new notation, method (16) can be rewritten as

$$\begin{aligned} &\left(\left(\frac{1}{\Delta t^2} + \frac{c}{2\Delta t} \right) u_h^{n+1}, v \right) + \left(D_1 - \frac{D_2 \Delta t}{2\tau} \right) \left(\nabla u_h^{n+1}, \nabla v \right) \\ &= \left(\frac{2}{\Delta t^2} u_h^n + \left(\frac{c}{2\Delta t} - \frac{1}{\Delta t^2} \right) u_h^{n-1}, v \right) + e^{-\frac{\Delta t}{\tau}} \left(I_n, \nabla v \right). \end{aligned} \tag{18}$$

Remark 1. *For efficiency reasons, the right hand side term should not be computed as it is in formula* (16) *but rather as recursion formula* (17) *to avoid the computational cost induced by the sum.*

The discrete energy obtained from $E_{h,n}$ by applying the same integration schemes as in (16), for a selection of parameters τ and D_2. For clarity, the discrete energy is calculated as $E_{h,n} = \left\| \dfrac{u_h^n - u_h^{n-2}}{2\Delta t} \right\|_0^2 + \|u_h^n\|_1^2 + \|I_n - \nabla u_n^n\|_0^2,\ n \geqslant 2.$

4.1 Solution's behaviour with damping

In the presence of a positive damping factor c, the numerical solutions tend to zero over time. This behaviour is clearly illustrated by Figure **??**, which plots the discrete energy function, for different values of D_2 and τ. It can be observed also that the larger the damping factor, the faster the energy approximates zero.

A similar result is observed when analysing the numerical solution at the central point $(0,0)$ of the square $[-1,1]^2$. As expected from the previous results, the solution at this point approximates zero. In Figure **??** we plot the numerical solution at this point, for the same profiles as in Figure **??**. It is observed that the smaller the value D_2 is, the closer the solutions are, for different values of τ, to the solution of the limit case $D_1 = 1$ and $D_2 = 0$.

4.2 Limiting case as τ tends to zero

For a fixed value of D_2, the variation of τ appears to induce a different time scale on the oscillations of the solutions (for smaller values of D_2 such difference is reduced due to the previous conclusions). To further investigate the behaviour of the numerical solution for varying τ, we calculated the restriction of the numerical solutions, at time $t = 4$, in the set

 ISBN: 978-84-616-9216-3

$[-1,1]\times\{0\}$. These results are plotted in Figure **??**. Combining the information from Figures **??**, **??** and **??** it seems apparent that as τ approaches zero, the corresponding numerical solution approximates the numerical solution obtained taking the pure wave equation (with damping effect included) with wave coefficient $D_1 - D_2$. In fact, if we consider the differential term $I(t;\tau) := \dfrac{D_2}{\tau}\displaystyle\int_0^t e^{-\frac{t-s}{\tau}}\Delta u(s)\,ds$ it can be shown that for a sufficiently smooth function u and fixed $t>0$, $\lim\limits_{\tau\to 0^+} I(t;\tau) = D_2\Delta u(t)$ which sheds some light into the observed behaviour. However, we do not have, at this point, a rigorous proof to analytically support this statement.

5 Conclusions

This paper establishes bounds for an energy function associated with the solution of a linear wave equation with a memory term. Under certain conditions, the energy converges to zero as $t \longrightarrow \infty$. It was also shown that a semi-discrete counterpart of the equation (obtained by discretisation in space with finite elements) inherits the same property.

The numerical waves studied also exhibit the same convergence to zero of a discretised energy function. It is moreover noticeable that as the coefficient D_2 approximates zero, the solutions approximate the solution of a pure wave equation (with no memory). Also, as τ approximates zero, a similar behaviour is observed.

Acknowledgements

This work was partially supported by the Centro de Matemática da Universidade de Coimbra (CMUC), funded by the European Regional Development Fund through the program COMPETE and by the Portuguese Government through the FCT - Fundação para a Ciência e a Tecnologia under the project PEst-C/MAT/UI0324/2013.

References

[1] R. G. C. Almeida, M. L. Santos, *Lack of exponential decay of a coupled system of wave equations with memory*, Nonlinear Anal Real World Appl **12** (2011) 1023–1032.

[2] J.R. Branco, J.A. Ferreira, *A singular perturbation of the heat equation with memory*, J. Comp. Appl. Math. **218** (2008) 376–394.

[3] M. M. Cavalcanti, V.N.D. Cavalcanti, J.A. Soriano, *Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping*, Electron. J. Diff. Equat. **44** (2002) 1–14.

 ISBN: 978-84-616-9216-3

[4] M. M. Cavalcanti, V.N.D. Cavalcanti,P. Martinez, *General decay rate for viscoelastic diisipative systems*, Nonlinear Anal. Theor. Meth. Appl. **68** 177–193.

[5] M. Fabrizio, A. Morro, *Mathematical Problems in Linear Viscoelasticity*, SIAM, Philadelphia, 1992.

[6] V. Janovský, S. Shaw, M.K. Warby, J.R. Whiteman, *Numerical methods for treating problems of viscoelastic isotropic solid deformation*, J. Comput. Appl. Math. **63** (1995) 91–107.

[7] M. Kafini, N. Tatar, *A decay result to a viscoelastic problem in $\mathbb{R}^n$ with an oscillating kernel*, J. Math. Phys. **51** (2010) 073506.

[8] S. Messaoudi, M. Mustafa, *A general stability result for a quasilinear wave equation with memory*, Nonlinear Anal. R. World Appl. **14** (2013) 1854–1864.

[9] J. A. Nohel, *A nonlinear hyperbolic Volterra equation*, in *Volterra Equations*, S-O Londen and O. J. Staffans (edts), Lect. Notes Math. **737** (1979) 220–235.

[10] J. Y. Park, S. H. Park, *Decay rate estimates for wave equations with memory type with acoustic boundary conditions*, Nonlinear Anal. Theor. Meth. Appl. **74** (2011) 993–998.

[11] M. Renardy, W. Hrusa, J. Nohel, *Mathematical Problems in Viscoelasticity*, Longman Scientific and Technical, 1987.

[12] M. L. Santos, *Decay rates for solutions of a system of wave equations with memory*, Electron. J. Differ. Equat. **38** (2002) 1-14.

[13] S. Shaw, J. Whiteman, *Towards adaptive finite element schemes for partial differential Volterra equation solvers*, Adv. Comput. Math. **6** (1996) 309–323.

[14] S. Shaw,J. Whiteman, *A posteriori error estimates for spacetime finite element approximation of quasistatic hereditary linear viscoelasticity problems*, Comput. Meth. Appl. Mech. Eng. **193** (2004) 5551–5572.

[15] S. Shaw, M.K. Warby, J.R. Whiteman, *Discrete schemes for treating hereditary problems of viscoelasticity and applications*, J. Compt. Appl. Math. **74** (1996) 313–329.

[16] N. Tatar, *Arbitrary decays in linear viscoelasticity*, J. Math. Phys. **52** (2011) 013502.

 ISBN: 978-84-616-9216-3