Supraconvergence and supercloseness in quasilinear coupled problems

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Abstract
The aim of this paper is to study a finite difference method for quasilinear coupled problems of partial differential equations that presents numerically an unexpected second order convergence rate. The error analysis presented allow us to conclude that the finite difference method is supraconvergent. As the method studied in this paper can be seen as a fully discrete piecewise linear finite element method, we conclude the supercloseness of our approximations.

Key words: Finite difference methods, piecewise linear finite element method, supraconvergence, supercloseness, pressure, velocity, concentration, porous media.

Mathematics Subject Classification (2000): 65M06, 65M20, 65M15

1 Introduction
In this paper we study finite difference approximations for the solution of the coupled system

\[-(a(c)p_x)_x = q_1 \text{ in } (0, 1) \times (0, T],\]
\[c_t + (b(c, p_x)c)_x - (d(c, p_x)c_x)_x = q_2 \text{ in } (0, 1) \times (0, T],\]

with the following boundary conditions

\[p(0, t) = p_{\ell}(t), \quad p(1, t) = p_r(t), \quad t \in (0, T],\]
\[c(0, t) = c_{\ell}(t), \quad c(1, t) = c_r(t), \quad t \in (0, T],\]

and initial conditions

\[c(x, 0) = c_0(x), \quad x \in (0, 1), \quad p(x, 0) = p_0(x), \quad x \in (0, 1).\]

The initial boundary value problem (IBVP) (1)-(5) can be used to describe miscible displacement of one incompressible fluid (resident fluid) by another (injected fluid) in one dimensional porous media. In this case, \(a(c) = K\mu(c)^{-1}, \quad b(c, p_x) = \frac{1}{\sigma}v, \quad d(c, p_x) = D_m + D_d\frac{1}{2}|v|, \) and \(v = -K\mu(c)^{-1}p_x\) denotes the Darcy velocity of the fluid mixture, \(p\) the pressure of the fluid

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mixture, $c$ the concentration of the injected fluid, $K$ the permeability of the medium, $D_m$ the molecular diffusion coefficient, $D_d$ the dispersion coefficient and $\phi$ represents the porosity. The viscosity of the mixture $\mu(c)$ is determined by the commonly used rule $\mu(c) = \mu_0((1-c)+M^4c)^{-4}$, where $M$ denotes the mobility ratio and $\mu_0$ represents the viscosity of the resident fluid. The two-dimensional or three-dimensional versions of this problem with Dirichlet boundary conditions or with Neumann or Robin boundary conditions were largely considered in the literature to study the miscible displacement of one incompressible fluid by another in a porous medium (see for instance [10], [17], [18], [20]).

Piecewise linear finite element method for (1) leads to a first order approximation for the space derivative of $p$ in the $L^2$-norm. This accuracy deteriorates the numerical approximation for $c$ obtained from (2) if the same method is considered. Several approaches have been considered in the literature to increase the convergence order of the numerical approximation for the velocity. Without be exhaustive we mention the use of cell centered schemes ([21]), mixed finite element methods ([2], [5], [12], [19]), gradient recovery technique ([7] and [16]) and mimetic finite difference approximations which can be seen as a mixed finite element methods with convenient quadrature rules ([4]).

Finite difference methods that can be seen as fully discrete piecewise linear Galerkin methods that allow to obtain a second order approximation for the gradient of the solution of elliptic problems have been studied in [3], [8], [9], [13] and [14].

In the present paper we introduce for the IBVP (1)-(5) a finite difference method belonging to the class of methods analysed in the last mentioned works that enable us to compute second order approximations for the pressure, for its gradient and for the concentration. As such finite difference scheme can be seen as a fully discrete Galerkin method based on piecewise linear approximation and convenient quadrature rules, our results can be also seen as supercloseness results.

In the convergence analysis we do not follow the approach introduced by Wheeler in [22] and largely followed by a huge number of authors in the study of numerical methods for parabolic problems (finite difference methods or Ritz-Galerkin methods). In the present paper we treat in an adequately way the error considering the error equation. We point out that our approach avoids the smoothness requirements imposed when Wheeler’s approach is used.

The paper is organized as follows. In Section 2 we introduce the semi-discretization of problem (1)-(5) and its convergence analysis is presented in Section 3. In the main result of this paper - Theorem 1-presented in this section we establish that the semi-discrete approximations introduced for the pressure, velocity and concentration are second order accurate. This result is illustrated numerically in Section 4. Finally in Section 5 we draw some conclusion. We remark that for the implicit-explicit method used in the numerical illustration we can show that a fully discrete version of Theorem 1 holds.

2 The semi-discrete approximation

In what follows we introduce the variational formulation of the IBVP (1)-(5). To simplify we assume homogeneous boundary conditions. By $L^2(0,1)$, $H^1(0,1)$ and $H^1_0(0,1)$ we denote the usual Sobolev spaces where we consider the usual inner products $(\cdot, \cdot)_0$, $(\cdot, \cdot)_1$ and the corresponding norms $\|\cdot\|_0$, $\|\cdot\|_1$. respectively. Let $\Omega \subseteq \mathbb{R}^n$ and $r \in \mathbb{N}$. For $p \in [1, \infty)$ we represent by
$W^r_p(\Omega)$ the space of functions $v : \Omega \to \mathbb{R}$ such that $D^\alpha v \in L^p(\Omega)$ for $|\alpha| \leq r$ and in this space we consider the following norm $\|v\|_{W^r_p(\Omega)} = \left( \int_\Omega |D^\alpha v(x)|^p \, dx \right)^{1/p}$. In this definition we use the notation $D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \ldots x_n^{\alpha_n}}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha_i \in \mathbb{N}_0$, $i = 1, \ldots, n$. As usual, for $p = 2$ we use the notation $W^{r,2}(\Omega) = H^r(\Omega)$.

By $W^{r,\infty}(\Omega)$ we represent the space of functions $v : \Omega \to \mathbb{R}$ such that $\|v\|_{W^{r,\infty}(\Omega)} = \max_{|\alpha| \leq r} \|D^\alpha v\|_{\Omega}$ is finite. Let $V$ be a Banach space. By $L^2(0,T;V)$ we denote the space of functions $v : (0,T) \to V$ such that $\|v\|_{L^2(0,T;V)} = \left( \int_0^T \|v(t)\|^2_V \, dt \right)^{1/2}$ is finite. Let $L^\infty(0,T;V)$ be the space of functions $v : (0,T) \to V$ such that $\|v\|_{L^\infty(0,T;V)} = \esssup_{[0,T]} \|v(t)\|_V$ is finite. The space of function $v : (0,T) \to V$ such that its derivatives $v^{(j)} : (0,T) \to V$, $j = 0, \ldots, r, r \in \mathbb{N}$, with $v(0) = v$, defined in distributional sense satisfy

$$\|v\|_{W^{r,\infty}(0,T;V)} = \max_{j=0,\ldots,r} \esssup_{[0,T]} \|v^{(j)}(t)\|_V < \infty,$$

is denoted by $W^{r,\infty}(0,T;V)$.

We replace the IBVP (1)-(5) by the following variational problem: find $u : \Omega \to \mathbb{R}$ such that for $p \in L^\infty(0,T;H^1(0,1))$, $c \in L^2(0,T;H^1(0,1))$ such that $c' \in L^2(0,T;L^2(0,1))$, conditions (3), (4) hold a.e. and

$$\begin{align*}
(a(c(t))p_x(t),w')_0 &= (q_1(t),w)_0 \text{ a.e. in } (0,T), \forall w \in H^1_0(0,1), \\
(c'(t),w)_0 + & (d(c(t),p_x(t))c_x(t),w_x)_0 - (b(c(t),p_x(t))c(t),w_x)_0 \\
&= (q_2(t),w)_0 \text{ a.e. in } (0,T), \forall w \in H^1_0(0,1).
\end{align*}$$

Let $H$ be a sequence of vectors $h = (h_1, \ldots, h_N)$ such that $\sum_{i=1}^N h_i = 1$ and $h_{\max} = \max_i h_i \to 0$.

Let $\mathbb{N}_0 = \{x_i, i = 0, \ldots, N, x_0 = 0, x_N = 1, x_i - x_{i-1} = h_i, i = 1, \ldots, N\}$ be a nonuniform partition of $[0,1]$. By $\mathcal{W}_h$ we represent the space of grid functions defined on $\mathbb{N}_0$ and by $\mathcal{W}_{h,0}$ we represent the subspace of $\mathcal{W}_h$ of functions null on the boundary points. Let $\mathbb{P}_{h} u_h$ be the piecewise linear interpolator of a grid function $u_h \in \mathcal{W}_h$. The space of piecewise linear functions induced by the partition $\mathbb{N}_0$ is denoted by $S_h$.

The piecewise linear approximations for the pressure and for the concentration are solutions of the finite dimensional coupled variational problem: find $\mathbb{P}_{h} p_h \in L^\infty(0,T;S_h)$ and $\mathbb{P}_{h} c_h \in L^2(0,T;S_h)$ such that $\mathbb{P}_{h} c'_h \in L^2(0,T;S_h)$, boundary conditions (3), (4) hold a.e. and

$$\begin{align*}
(a(\mathbb{P}_{h} c_h(t))(\mathbb{P}_{h} p_h)_x(t),\mathbb{P}_{h} w'_h)_0 &= (q_1(t),\mathbb{P}_{h} w_h)_0 \text{ a.e. in } (0,T), \forall w_h \in \mathcal{W}_{h,0}, \\
((\mathbb{P}_{h} c_h(t),\mathbb{P}_{h} w_h)_0) + & (d(\mathbb{P}_{h} c_h(t), (\mathbb{P}_{h} p_h)_x(t))(\mathbb{P}_{h} c_h)_x(t),\mathbb{P}_{h} w'_h)_0 \\
&= (q_2(t),\mathbb{P}_{h} w_h)_0 \text{ a.e. in } (0,T), \forall w_h \in \mathcal{W}_{h,0}.
\end{align*}$$

In the space $\mathcal{W}_h$ we consider the norm

$$\|u_h\|_{h}^2 = \|u_h\|^2_h + \|D^- u_h\|^2_{h,+},$$

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where $D_{-x}$ denotes the backward finite difference operator with respect to the space variable, $\|\cdot\|_h$ is the norm induced by the inner product,

$$\langle w_h, v_h \rangle_h = \sum_{i=1}^{N} \frac{h_i}{2} \left( w_h(x_{i-1})v_h(x_{i-1}) + w_h(x_i)v_h(x_i) \right), \quad w_h, v_h \in \mathcal{W}_h,$$

(11)

and $\|w_h\|_{h,+} = \left( \sum_{i=1}^{N} h_i w_h(x_i)^2 \right)^{1/2}$. In what follows we use the notation

$$\langle w_h, v_h \rangle_{h,+} = \sum_{i=1}^{N} h_i w_h(x_i)v_h(x_i), \quad w_h, v_h \in \mathcal{W}_h.$$

The fully discrete (in space) approximations for the pressure and for the concentration are solutions of the following coupled variational problem: find $p_h \in L^\infty(0,T;\mathcal{W}_h)$, $c_h \in L^2(0,T;\mathcal{W}_h)$ such that $c_h \in L^2(0,T;\mathcal{W}_h)$, and

\begin{align}
(a_h(t)D_{-x}p_h(t), D_{-x}w_h)_{h,+} &= (q_{1,h}(t), w_h)_h \text{ a.e. in } (0,T), \forall w_h \in \mathcal{W}_{h,0}, \\
(c_h(t), w_h)_{h,+} + (d_h(t)D_{-x}c_h(t), D_{-x}w_h)_{h,+} &- (M_h(b(t)c_h(t)), D_{-x}w_h)_{h,+} \\
&= (q_{2,h}(t), w_h)_h \text{ a.e. in } (0,T), \forall w_h \in \mathcal{W}_{h,0}, \\
p_h(x_0, t) = p_t(t), p_h(x_N, t) = p_r(t) \text{ a.e. in } (0,T), \\
c_h(x_0, t) = c_t(t), c_h(x_N, t) = c_r(t) \text{ a.e. in } (0,T), \\
c_h(x_i, 0) = c_{0,h}(x_i), p_h(x_i, 0) = p_{0,h}(x_i), i = 1, \ldots, N - 1.
\end{align}

In (12), (13) the following notations were used

$$q_{\ell,h}(x_i, t) = \frac{1}{h_{i+\ell/2}} \int_{x_{i-1/2}}^{x_{i+1/2}} q_{\ell}(x, t) \, dx, \quad i = 1, \ldots, N - 1, \ell = 1, 2,$$

(17)

\begin{align}
h_{i+1/2} &= \frac{1}{2} (h_i + h_{i+1}), \quad M_h(w_h)(x_i) = \frac{1}{2} (w_h(x_{i-1}) + w_h(x_i)), i = 1, \ldots, N. \quad \text{The coefficient functions } a_h(t) \text{ and } d_h(t) \text{ are defined by} \\
a_h(x_i, t) &= a(M_h(c_h(t))(x_i)), \\
d_h(x_i, t) &= d(M_h(c_h(t))(x_i), D_{-x}p_h(x_i, t))
\end{align}

(18) (19)

and the grid function $b_h(t)$ is given by

$$b_h(x_i, t) = \begin{cases} 
  b(c_h(x_0, t), D_{-x}p_h(x_0, t)), & i = 0, \\
  b(c_h(x_i, t), D_{-x}p_h(x_i, t)), & i = 1, \ldots, N - 1, \\
  b(c_h(x_N, t), D_{-x}p_h(x_N, t)), & i = N,
\end{cases}$$

(20)

with

$$D_{-x}p_h(x_i, t) = \frac{1}{h_i + h_{i+1}} (h_i D_{-x}p_h(x_{i+1}, t) + h_{i+1} D_{-x}p_h(x_i, t)).$$

(21)
In what follows we establish an ordinary differential algebraic coupled system equivalent to the variational problem (12)-(16). In order to do that we introduce the following finite difference operators

\[(D_c w_h)_i = \frac{w_{i+1} - w_{i-1}}{h_i + h_{i+1}}, \quad (D_x w_h)_{i+1/2} = \frac{w_{i+1} - w_i}{h_{i+1}}, \quad (D^{1/2}_x w_h)_i = \frac{w_{i+1/2} - w_{i-1/2}}{h_{i+1/2}},\]

where \(w_j := w_h(x_j)\) and \(w_{j+1/2}\) are used as far as it makes sense. In order to simplify the presentation we also consider that \(a_h(x_{i+1/2}, t) = a_h(x_{i+1}, t)\), \(d_h(x_{i+1/2}, t) = d_h(x_{i+1}, t)\).

It can be shown that the approximations \(p_h(t)\) and \(c_h(t)\) are solutions of the following discrete problem:

\[-D_x^{1/2}(a_h(t)D_x p_h(t)) = q_{1,h}(t)\text{ in } \mathbb{I}_h - \{0,1\} \quad \text{a. e. in } (0,T) \quad (22)\]

\[c'_h(t) - D_x^{1/2}(dh(t)D_x p_h(t)) + D_c(b_h(t)c_h(t)) = q_{2,h}(t)\text{ in } \mathbb{I}_h - \{0,1\} \quad \text{a. e. in } (0,T), \quad (23)\]

with the conditions (14), (15) and (16).

### 3 Supraconvergent result

#### 3.1 Auxiliary results

The stability analysis the coupled variational problem (12), (13), or equivalently the stability of the coupled finite difference problem (22), (23), under homogeneous Dirichlet boundary conditions, that is, \(p(t) = p_r(t) = c(t) = c_r(t) = 0\), was presented in [15]. In the analysis that we present in what follows we need to assume that the semi-discrete approximation for the pressure satisfies the following

\[\max_{i=1,\ldots,N} |D_{-x} p_h(x_i, t)| \leq C_p, \quad (24)\]

for some positive constant \(C_p\). We remark that this assumption can be assumed provided that \(q_1\) satisfies

\[\|q_1(t)\|_0 \leq C_{q_1}, t \in [0,T]. \quad (25)\]

In fact, as we have

\[a(M_h(c(t))(x_{i+1}))D_{-x} p_h(x_{i+1}, t) = \sum_{j=1}^{i} h_{j+1/2} D^{(1/2)}_x(a_h(t)D_{-x} p_h(t))(x_j)\]

\[+a(M_h(c(t))(x_1))D_{-x} p_h(x_1, t) = - \sum_{j=1}^{i} h_{j+1/2} q_{1,h}(x_j, t) + a(M_h(c(t))(x_1))D_{-x} p_h(x_1, t),\]

for \(i = 1, \ldots, N-1\), using (25) we deduce

\[\max_{i=2,\ldots,N} |a(M_h(c(t))(x_i))D_{-x} p_h(x_i, t)| \leq C_{q_1} + |a(M_h(c(t))(x_1))|D_{-x} p_h(x_1, t)|.\]

It is then effectively plausible to admit that (24) holds for some positive constant \(C_p\).

We start by introducing two auxiliary problems. We assume that \(a \in W^{1,\infty}(\mathbb{R})\), \(d \in W^{1,\infty}(\mathbb{R}^2)\) and \(b \in W^{2,\infty}(\mathbb{R}^2)\). Let \(\bar{p}_h(t), \bar{c}_h(t) \in \mathbb{W}_{h,0}\) be solutions of the discrete variational problems

\[(\bar{a}_h(t)D_{-x} \bar{p}_h(t), D_{-x} w_h)_{h,+} = (q_{1,h}(t), w_h)_{h}, \quad w_h \in \mathbb{W}_{h,0}, \quad (26)\]
\begin{align}
(\tilde{d}_h(t)D_{-x}\tilde{c}_h(t), D_{-x}w_h)_{h,+} - (M_h(\tilde{b}_h(t)\tilde{c}_h(t)), D_{-x}w_h)_{h,+} = (\tilde{q}_2,h(t), w_h)_{h}, w_h \in \mathbb{W}_{h,0},
\end{align}

with \(\tilde{q}_2,h(t)\) defined by (17) with \(q_2(t)\) replaced by \(q_2(t) - c'(t)\). In (26) and (27) the coefficient functions \(\tilde{a}_h\) and \(\tilde{d}_h\) are defined by

\[\tilde{a}_h(x_i, t) = a(c(x_{i-1/2}, t)), \quad \tilde{d}_h(x_i, t) = d(c(x_{i-1/2}, t), p_x(x_{i-1/2}, t)), \quad i = 1, \ldots, N,\]

and \(\tilde{b}_h(x_i, t)\tilde{c}_h(x_i, t) = b(c(x_i, t), p_x(x_i), t)\tilde{c}_h(x_i, t), \quad i = 1, \ldots, N - 1, \quad \tilde{b}_h(x_1, t)\tilde{c}_h(x_1, t) = 0, \quad i = 0, N.\]

It can be shown that \(\tilde{p}_h(t)\) and \(\tilde{c}_h(t)\) are solutions of a coupled finite difference problem analogous to system (22), (23).

An error bound for \(\tilde{p}_h(t)\) is established now considering Theorem 3.1 of [3]. By \(R_h\) we denote the restriction operator \(R_h: C[0, 1] \to \mathbb{W}_h, R_h v(x) = v(x), x \in I_h.\)

**Proposition 1** If \(0 < a_0 \leq a\) then, for \(\tilde{p}_h(t)\) defined by (26) and for \(h \in H\) with \(h_{\text{max}}\) small enough, holds the following error estimate

\[
\|\tilde{p}_h(t) - R_h p(t)\|_2^2 \leq C_p \sum_{i=1}^{N} h_i^{2s} \|p(t)\|_{H^{s+1}(I_i)}^2
\]

provided that \(p(t) \in H^{s+1}(0, 1) \cap H_0^1(0, 1), s \in \{1, 2\}.\) In (28) \(I_i = (x_{i-1}, x_i)\) and \(C_p\) denotes a positive constant which does not depend on \(h.\)

As a consequence of this result, we conclude that, for \(h \in H\) with \(h_{\text{max}}\) small enough, we have

\[
\max_{i=1, \ldots, N} |D_{-x}\tilde{p}_h(x_i, t)| \leq C_p,
\]

for some positive constant \(C_p.\) In fact, from (28) we obtain \(|D_{-x}(\tilde{p}(x_i, t) - p(x_i, t))| \leq C h_{\text{max}}^{s-1/2},\) for some positive constant \(C.\) Then

\[
|D_{-x}\tilde{p}_h(x_i, t)| \leq |D_{-x}(\tilde{p}(x_i, t) - p(x_i, t))| + \frac{1}{h_j} \int_{x_{i-1}}^{x_j} p_x(x, t) dx \leq C h_{\text{max}}^{s-1/2} + \|p_x(t)\|_\infty,
\]

that leads to (29) provided that \(p \in L^\infty(0, T; H^{s+1}(0, 1) \cap H_0^1(0, 1)), s \in \{1, 2\}.\)

In order to obtain an upper bound for the error of \(\tilde{c}_h(t)\) we need to guarantee the stability of the bilinear form

\[
a_h(v_h, w_h) = (\tilde{d}_h(t)D_{-x}v_h, D_{-x}w_h)_{h,+} - (M_h(\tilde{b}_h(t)v_h), D_{-x}w_h)_{h,+}, v_h, w_h \in \mathbb{W}_{h,0}.
\]

In the next proposition we specify conditions that allow us to conclude such stability (see Proposition 3.1 of [3]).

**Proposition 2** Let \(\tilde{d}(t)\) and \(\tilde{b}(t)\) be defined by \(\tilde{d}(t) = d(c(t), p_x(t)), \tilde{b}(t) = b(c(t), p_x(t)),\) where \(p, c\) are the solutions of the coupled variational problem (6), (7) with homogeneous Dirichlet boundary conditions. If the variational problem: find \(u \in H_0^1(0, 1)\) such that \((\tilde{d}(t)v_x, w_x)_0 - (\tilde{b}(t)v, w_x)_0 = 0\) for \(w \in H_0^1(0, 1),\) has only the null solution, then there exists a positive constant
\( \alpha_{e,c} \) which does not depend on \( h \) such that, for \( h \in H \) with \( h_{\text{max}} \) small enough, holds the following

stability inequality

\[
\| \mathbb{P}_h v_h \| \leq \alpha_{e,c} \sup_{0 \neq w_h \in \mathbb{W}_{h,0}} \frac{|\alpha_{c_h}(v_h, w_h)|}{\| \mathbb{P}_h w_h \|}, \quad v_h \in \mathbb{W}_{h,0}. \tag{30}
\]

Using now Theorem 3.1 of [3] we can state the error estimate for \( \tilde{T} \). Considering this result, it suffices to estimate

\[
|T_d| = \sum_{i=1}^{N} h_i d_{i-1/2} \left( D_{-x} c(x_i, t) - c_{x}(x_{i-1/2}, t) \right) D_{-x} w_h(x_i), \tag{31}
\]

\[
T_b = \sum_{i=1}^{N} h_i \left( b(x_{i-1/2}, t) - \frac{b(x_{i-1}, t) + b(x_i, t)}{2} \right) D_{-x} w_h(x_j) \tag{32}
\]

with

\[
d_{i-1/2} = (c(x_i-1/2, t), p_x(x_{i-1/2}, t)), \quad \text{and} \quad b(x_{\ell}, t) = b(c(x_{\ell}, t), p_x(x_{\ell}, t)), \quad \ell = i - 1, i - 1/2, i.
\]

Using Bramble-Hilbert Lemma in \( T_d \) we get

\[
|T_d| \leq C \| d(c(t), p_x(t)) \|_{\infty} \left( \sum_{i=1}^{N} h_i^{2s} \| c(t) \|_{H^{s+1}(I_i)}^2 \right)^{1/2} \| D_{-x} w_h \|_{h,+}. \tag{33}
\]

provided that \( c(t) \in H^{s+1}(0, 1) \cap H^1_{0}(0, 1), \) for \( s \in \{1, 2\} \).

To estimate \( T_b \) we apply Bramble-Hilbert Lemma again. In this case we obtain, for \( s \in \{1, 2\} \),

\[
|T_b| \leq C \left( \sum_{i=1}^{N} h_i^{2s} |b(c(t), p_x(t))c(t)|_{H^{s}(I_i)}^2 \right)^{1/2} \| D_{-x} w_h \|_{h,+}. \tag{34}
\]

As the imbedding of \( H^{s+1}(0, 1) \) into \( C^s_B(0, 1) \) is continuous, where \( C^s_B(0, 1) \) denotes the space of functions having bounded, continuous derivatives up to order \( j \) on \( (0, 1) \) (Theorem 4.12 of [1]),

we deduce for \( s = 1 \)

\[
|T_b| \leq C \left( \sum_{i=1}^{N} h_i^2 \| c(t) \|_{\infty}^2 (\| c(t) \|_{H^1(I_i)}^2 + \| p(t) \|_{H^2(I_i)}^2) \right)^{1/2} \| D_{-x} w_h \|_{h,+} \tag{35}
\]

and for \( s = 2 \)

\[
|T_b| \leq C \left( \sum_{i=1}^{N} h_i \left( \| c_x(t) \|_{\infty}^2 (\| c(t) \|_{\infty}^2 + 1) (\| c_x(t) \|_{L^2(I_i)}^2 + \| p_{x^2}(t) \|_{L^2(I_i)}^2) \right.ight.
\]

\[
\left. + \| c(t) \|_{\infty}^2 (\| p_{x^2}(t) \|_{\infty}^2 \| p_{x^2} \|_{L^2(I_i)}^2 + \| p_{x^3} \|_{L^2(I_i)}^2) + \| c_x \|_{L^2(I_i)}^2 \right) \right)^{1/2} \| D_{-x} w_h \|_{h,+}. \tag{36}
\]

We summarize the previous error estimates in the following proposition.
Proposition 3  Under the assumptions of Proposition 2, for $\tilde{c}_h(t)$ defined by (27) and for $h \in H$ with $h_{\text{max}}$ small enough, holds the following error estimate

$$\|\mathbb{P}_h(\tilde{c}_h(t) - R_h c(t))\|_1^2 \leq C_{\tilde{c}} \sum_{i=1}^{N} h_i^{2s} \left( \|c(t)\|^2_{H^{s+1}(I_i)} + \|p(t)\|^2_{H^{s+1}(I_i)} \right),$$  

(37)

provided that $c(t), p(t) \in H^{s+1}(0,1) \cap H_0^1(0,1)$. In (37), $s \in \{1, 2\}$ and $C_{\tilde{c}}$ denotes a positive constant which does not depend on $h$.

Under the assumptions of Proposition 2, it is clear that $\|\tilde{c}_h(t)\|_{1,h} \leq C_{\tilde{c}}$, for some positive $C_{\tilde{c}}$, which implies that

$$\|\tilde{c}_h(t)\|_{\infty} \leq C_{\tilde{c}},$$  

(38)

provided that $c, p \in L^\infty(0,T; H^2(0,1) \cap H_0^1(0,1))$, for some positive constant $C_{\tilde{c}}$ and for $h \in H$ with $h_{\text{max}}$ small enough.

As for $\tilde{p}_h(t)$, it is plausible to assume that

$$\max_{i=1, \ldots, N} |D_{-x} \tilde{c}_h(x_i, t)| \leq C_{\tilde{c}},$$  

(39)

for $h \in H$ with $h_{\text{max}}$ small enough.

In the next proposition we establish an upper bound for $\|\mathbb{P}_h(p_h(t) - \tilde{p}_h(t))\|_1$.

Proposition 4  If $0 < a_0 \leq a$, then, for $h \in H$ with $h_{\text{max}}$ small enough, we have

$$\|\mathbb{P}_h(p_h(t) - \tilde{p}_h(t))\|_1 \leq C_{p,\tilde{p}} \left( \|c_h(t) - R_h c(t)\|_h + \left( \sum_{i=1}^{N} h_i^{2s} \|c(t)\|^2_{H^{s}(I_i)} \right)^{1/2} \right),$$  

(40)

provided that $c(t) \in H^s(0,1) \cap H_0^1(0,1)$. In (40), $s \in \{1, 2\}$ and $C_{p,\tilde{p}}$ denotes a positive constant which does not depend on $h$.

Proof: From (12) and (26) it can be shown that, for $w_h \in \mathbb{W}_{h,0}$, holds the following

$$(a_h(t)D_{-x}(p_h(t) - \tilde{p}_h(t)), D_{-x}w_h)_{h,+}$$

$$= ((\tilde{a}_h(t) - a_h^*(t))D_{-x}\tilde{p}_h(t), D_{-x}w_h)_{h,+} + ((a_h^*(t) - a_h(t))D_{-x}\tilde{p}_h(t), D_{-x}w_h)_{h,+},$$  

(41)

where $a_h^*(t)$ is defined as $a_h(t)$ but with $c_h(t)$ replaced by $R_h c(t)$.

For the second term of the second member of (41) we have

$$\|(a_h^*(t) - a_h(t))D_{-x}\tilde{p}_h(t), D_{-x}w_h)_{h,+}\| \leq C\|c_h(t) - R_h c(t)\|_h\|D_{-x}w_h\|_{h,+},$$  

(42)

for $w_h \in \mathbb{W}_{h,0}$.

Considering now the Bramble-Hilbert Lemma in the first term of the second member of (41) we deduce

$$\|(\tilde{a}_h(t) - a_h^*(t))D_{-x}\tilde{p}_h(t), D_{-x}w_h)_{h,+}\| \leq C\left( \sum_{i=1}^{N} h_i^{2s}\|c(t)\|^2_{H^{s}(I_i)} \right)^{1/2}\|D_{-x}w_h\|_{h,+},$$  

(43)

for $w_h \in \mathbb{W}_{h,0}$.

Taking (42) and (43) in (41), we conclude the proof of (40) choosing $w_h = p_h(t) - \tilde{p}_h(t)$. □
Corollary 1 If $0 < a_0 \leq a$, then for $p_h(t)$ and $c_h(t)$ defined by (12), (13) and for $h \in H$ with $h_{\text{max}}$ small enough, holds the following

$$\|\mathbb{P}_h(p_h(t) - R_h p(t))\|_1 \leq C\left(\|c_h(t) - R_h c(t)\|_h + \left(\sum_{i=1}^{N} h_i^{2s} \|c(t)\|_{H^s(I_i)}^2\right)^{1/2} + \left(\sum_{i=1}^{N} h_i^{2s} \|p(t)\|_{H^{s+1}(I_i)}^2\right)^{1/2}\right),$$

(44)

provided that $c(t) \in H^s(0,1) \cap H_0^1(0,1)$, $p(t) \in H^{s+1}(0,1) \cap H_0^1(0,1)$, $s \in \{1, 2\}$.

Lemma 1 Let $\tilde{c}_h(t)$ be defined by (27) and $p(t), c(t) \in H^{s+1}(0,1) \cap H_0^1(0,1)$, $s \in \{1, 2\}$. Under the assumptions of Proposition 2 and Corollary 1, for the functional

$$\tau_d(t, w_h) = (\tilde{d}_h(t) D_x \tilde{c}_h(t), D_x w_h)_{h,+} - (d_h(t) D_x c_h(t), D_x w_h)_{h,+},$$

defined on $\mathbb{W}_{h,0}$ and for $h \in H$ with $h_{\text{max}}$ small enough, holds the following

$$\tau_d(t, w_h) = (d_h(t) D_x (R_h c(t) - c_h(t)), D_x w_h)_{h,+} + \tau_d,h(t, w_h),$$

(45)

where

$$|\tau_{d,h}(t, w_h)| \leq C_d\left(\|c_h(t) - R_h c(t)\|_h + \left(\sum_{i=1}^{N} h_i^{2s} \|p(t)\|_{H^{s+1}(I_i)}^2\right)^{1/2} + \left(\sum_{i=1}^{N} h_i^{2s} \|c(t)\|_{H^s(I_i)}^2\right)^{1/2}\right)\|D_x w_h\|_{h,+}, \ w_h \in \mathbb{W}_{h,0}.$$

(46)

Proof: For $\tau_d(t, w_h)$ holds the representation (45) with $\tau_d,h(t, w_h)$ given by

$$\tau_{d,h}(t, w_h) = \tau_{d,h}^{(1)}(t, w_h) + \tau_{d,h}^{(2)}(t, w_h) + \tau_{d,h}^{(3)}(t, w_h),$$

(47)

where

$$\tau_{d,h}^{(1)}(t, w_h) = ((\tilde{d}_h(t) - d_h(t)) D_x \tilde{c}_h(t), D_x w_h)_{h,+},$$

$$\tau_{d,h}^{(2)}(t, w_h) = ((d_h(t) - d_h(t)) D_x c_h(t), D_x w_h)_{h,+},$$

$$\tau_{d,h}^{(3)}(t, w_h) = (d_h(t) D_x (\tilde{c}_h(t) - R_h c(t)), D_x w_h)_{h,+},$$

and $d_h^*$ is defined as $d_h$ with $c_h$ and $p_h$ replaced by $R_h c$ and $R_h p$, respectively. Using the Bramble-Hilbert Lemma it can be shown that for $\tau_{d,h}^{(1)}(t, w_h)$, for $w_h \in \mathbb{W}_{h,0}$ and for $h \in H$ with $h_{\text{max}}$ small enough, holds the following

$$|\tau_{d,h}^{(1)}(t, w_h)| \leq C\left(\left(\sum_{i=1}^{N} h_i^{2s} \|c(t)\|_{H^s(I_i)}^2\right)^{1/2} + \left(\sum_{i=1}^{N} h_i^{2s} \|p(t)\|_{H^{s+1}(I_i)}^2\right)^{1/2}\right)\|D_x w_h\|_{h,+}, \ w_h \in \mathbb{W}_{h,0}.$$

For $\tau_{d,h}^{(2)}(t, w_h)$ we have, for $w_h \in \mathbb{W}_{h,0},$

$$|\tau_{d,h}^{(2)}(t, w_h)| \leq C\left(\|R_h c(t) - c_h(t)\|_h + \|D_x (p_h(t) - R_h p(t))\|_{h,+}\right)\|D_x w_h\|_{h,+}. $$
Considering Corollary 1 we get
\[ |\tau_{d,h}^{(2)}(t, w_h)| \leq C \left( \|c_h(t) - R_h c(t)\|_h + \left( \sum_{i=1}^{N} h_i^{2s}\|c(t)\|_{H^{s+1}(I_i)}^2 \right)^{1/2} \right.\]
\[ + \left. \left( \sum_{i=1}^{N} h_i^{2s}\|p(t)\|_{H^{s+1}(I_i)}^2 \right)^{1/2} \right) \|D_{-x}w_h\|_{h, +}, \quad w_h \in \mathbb{W}_{h,0}. \]

Taking into account Proposition 3, for \( \tau_{d,h}^{(3)}(t, w_h) \) we deduce, for \( w_h \in \mathbb{W}_{h,0} \) and for \( h \in H \) with \( h_{max} \) small enough,
\[ |\tau_{d,h}^{(3)}(t, w_h)| \leq C \left( \left( \sum_{i=1}^{N} h_i^{2s}\|c(t)\|_{H^{s+1}(I_i)}^2 \right)^{1/2} + \left( \sum_{i=1}^{N} h_i^{2s}\|p(t)\|_{H^{s+1}(I_i)}^2 \right)^{1/2} \right) \|D_{-x}w_h\|_{h, +}. \]

From the estimates established for \( \tau_{d,h}^{(l)}(t, w_h), l = 1, 2, 3 \), we conclude (46).

**Lemma 2** Let \( \tilde{c}_h(t) \) be defined by (27) and \( c(t), p(t) \in H^{s+1}(0, 1) \cap H_0^2(0, 1), s \in \{1, 2\} \). If \( 0 < a_0 \leq a \), condition (24) holds and the coefficient function \( b \) satisfies
\[ |b(x, y)| \leq C_b |y|, (x, y) \in \mathbb{R}^2, \]
then, under the assumptions of Proposition 2, for the functional
\[ \tau_b(t, w_h) = (M_h(b_h(t)c_h(t)), D_{-x}w_h)_{h, +} - (M_h(\tilde{b}_h(t)\tilde{c}_h(t)), D_{-x}w_h)_{h, +}, \]
defined on \( \mathbb{W}_{h,0} \) and for \( h \in H \) with \( h_{max} \) small enough, holds the following
\[ \tau_b(t, w_h) = (M_h(b_h(t)(c_h(t) - R_h c(t))), D_{-x}w_h)_{h, +} + \tau_{b,h}(t, w_h), \]
where
\[ |\tau_{b,h}(t, w_h)| \leq C_{b,2} \left( \|c_h(t) - R_h c(t)\|_h + \left( \sum_{i=1}^{N} h_i^{2s}\|c(t)\|_{H^{s+1}(I_i)}^2 \right)^{1/2} \right.\]
\[ + \left. \left( \sum_{i=1}^{N} h_i^{2s}\|p(t)\|_{H^{s+1}(I_i)}^2 \right)^{1/2} \right) \|D_{-x}w_h\|_{h, +}, \quad w_h \in \mathbb{W}_{h,0}. \]

**Proof:** For \( \tau_b(t, w_h) \) holds the representation (49) with
\[ \tau_{b,h}(t, w_h) = \tau_{b,h}^{(1)}(t, w_h) + \tau_{b,h}^{(2)}(t, w_h) + \tau_{b,h}^{(3)}(t, w_h), \]
\[ \tau_{b,h}^{(1)}(t, w_h) = (M_h(b_h(t)(R_h c(t) - \tilde{c}_h(t))), D_{-x}w_h)_{h, +}, \]
\[ \tau_{b,h}^{(2)}(t, w_h) = (M_h((b_h(t) - \tilde{b}_h(t))\tilde{c}_h(t)), D_{-x}w_h)_{h, +}, \]
To estimate $\tau_{b,h}^{(3)}(t, w_h) = (M_h((b_h(t) - \hat{b}_h(t))\tilde{c}_h(t)), D_{-x}w_h)_{h,+},$

being $b_h^*$ defined as $b_h$ with $c_h$ and $p_h$ replaced by $R_hc$ and $R_hp$, respectively.

Considering Proposition 3 and condition (24), under the assumptions (48) for $b$ it can be shown that for $\tau_{b,h}^{(3)}(t, w_h)$ and for $h \in H$ with $h_{\text{max}}$ small enough, holds the following

$$|\tau_{b,h}^{(1)}(t, w_h)| \leq C \left( \sum_{i=1}^{N} h_i^{2s} \|c(t)\|_{H^{s+1}(I_i)}^{2} \right)^{1/2} + \left( \sum_{i=1}^{N} h_i^{2s} \|p(t)\|_{H^{s+1}(I_i)}^{2} \right)^{1/2} \|D_{-x}w_h\|_{h,+},$$

provided that $c(t), p(t) \in H^{s+1}(0,1) \cap H_0^1(0,1)$, for $s \in \{1,2\}$.

As $\tilde{c}_h(t)$ satisfies (38), we can establish for $\tau_{b,h}^{(2)}(t, w_h)$ the upper bound

$$|\tau_{b,h}^{(2)}(t, w_h)| \leq C \left( \|c_h - R_h c\|_h + \|D_{-x}(p_h(t) - R_h p(t))\|_{h,+} \right) \|D_{-x}w_h\|_{h,+}. $$

Considering now Corollary 1, for $h \in H$ with $h_{\text{max}}$ small enough, we conclude

$$|\tau_{b,h}^{(2)}(t, w_h)| \leq C \left( \|c_h(t) - R_h c(t)\|_h + \left( \sum_{i=1}^{N} h_i^{2s} \|c(t)\|_{H^{s+1}(I_i)}^{2} \right)^{1/2} \right)^{1/2} \|D_{-x}w_h\|_{h,+},$$

provided that $c(t) \in H^s(0,1) \cap H_0^1(0,1), p(t) \in H^{s+1}(0,1) \cap H_0^1(0,1)$, for $s \in \{1,2\}$.

To estimate $\tau_{b,h}^{(3)}(t, w_h)$ we start by remarking that $p_{x}(x_{i}, t) - D_{h}p_{x}(x_{i}, t) = \frac{1}{h_{i} + h_{i+1}} \lambda(v)$, with

$$\lambda(v) = v_{x}(\rho) - \hat{\rho}(v(1) - v(0)) - \frac{1}{\hat{\rho}} (v(\rho) - v(0)), \quad \hat{\rho}(\xi) = p(x_{i-1} + \xi(h_{i+1} + h_{i+1}, t)), $$

$\rho = \frac{h_{i}}{h_{i} + h_{i+1}}, \quad \hat{\rho} = \frac{h_{i}}{h_{i+1}}$. Applying Bramble-Hilbert Lemma to $\lambda(v)$ we obtain, for $s \in \{1,2\}$,

$$|\lambda(v)| \leq C \int_{0}^{1} |v_{x}(\xi)| \, dx \leq C(h_{i+1} + h_{i+1})^{s-1} \int_{x_{i-1/2}}^{x_{i+1/2}} |p_{x}(x, t)| \, dx.$$ 

Then, for $h \in H$ with $h_{\text{max}}$ small enough, we have

$$|\tau_{b,h}^{(3)}(t, w_h)| \leq C \left( \sum_{i=1}^{N} h_i^{2s} \|p(t)\|_{H^{s+1}(I_i)}^{2} \right)^{1/2} \|D_{-x}w_h\|_{h,+},$$

provided that $p(t) \in H^{s+1}(0,1) \cap H_0^1(0,1)$, for $s \in \{1,2\}$.

From the upper bounds obtained for $\tau_{b,h}^{(l)}(t, w_h), \ell = 1, 2, 3$, we conclude the proof.

The following result was proved in [3] and has an important role in the proof of the main result of this paper - Theorem 1.
Lemma 3 If $g \in H^2(0,1)$ and $g_h$ is defined by (17) with $q_k$ replaced by $g$, then there exists a positive constant $C_{in}$ which does not depend on $h$ such that
\[
|(g_h - R_h g, w_h)_h| \leq C_{in} \left( \sum_{i=1}^{N} h_i^4 \|g\|_{H^2(I_i)}^2 \right)^{1/2} \|w_h\|_{1,h}, \quad w_h \in W_{h,0},
\]
(51)
for $h \in H$ with $H_{\max}$ small enough.

3.2 Main convergence result
Let $e_{c,h}(t) = c_h(t) - R_h c(t)$, $e_{p,h}(t) = p_h(t) - R_h p(t)$ be the semi-discretization error induced by the discretization (12), (13), (14) and (15). An estimate for $\|P_h e_{p,h}(t)\|_1$ depending on $\|e_{c,h}(t)\|_h$ was established in Corollary 1. In the next result we establish an estimate for $\|e_{c,h}(t)\|_h$ that allow us to obtain with Corollary 1 an estimate for $\|P_h e_{p,h}(t)\|_1$.

Theorem 1 Let $c$ and $p$ be the solutions of the coupled quasi-linear problem (6), (7), $c \in L^2(0,T;H^{s+1}(0,1) \cap H^3_0(0,1)) \cap H^1(0,T;H^2(0,1))$, $p \in L^\infty(0,T;H^{s+1}(0,1) \cap H^3_0(0,1))$, $s \in \{1, 2\}$, and let $c_h$ and $p_h$ be their approximations defined by (12), (13). We assume that the variational problem: find $v \in H^1_0(0,1)$ such that $(\tilde{d}(t)v_x, w_x)_0 - (\tilde{b}(t)v, w)_0 = 0$ for $w \in H^1_0(0,1)$, has only the null solution, where $\tilde{d}(t) = d(c(t), p_x(t))$ and $\tilde{b}(t) = b(c(t), p_x(t))$.

If $0 < a_0 \leq a, 0 < d_0 \leq d$, $b$ satisfies (48), then, under the assumption (24), there exists positive constant $C_c$ such that, for $h \in H$ with $h_{\max}$ small enough, holds the following
\[
\begin{align*}
\|e_{c,h}(t)\|^2_h &+ \int_0^t \|D_x e_{c,h}(\mu)\|^2_{h,+} d\mu \leq \frac{1}{\min\{1, 2(d_0 - 4\epsilon^2)\}} e^{\omega t} \left( \|c_{c,h}(0)\|_h^2 \right) \\
&+ C_e \sum_{i=1}^{N} \int_0^t \left( h_i^{2s} \left( \|p(\mu)\|_{H^{s+1}(I_i)}^2 + \|c(\mu)\|_{H^{s+1}(I_i)}^2 \right) + h_i^4 \|c'(\mu)\|_{H^2(I_i)}^2 \right) d\mu \\
&\leq \frac{1}{\min\{1, 2(d_0 - 4\epsilon^2)\}} e^{\omega t} \left( \|c_{c,h}(0)\|_h^2 \right) + C_e \left( h_{\max}^2 \left( \|c\|_{L^2(0,T;H^{s+1}(0,1))}^2 \right) \\
&+ \|p\|_{L^2(0,T;H^{s+1}(0,1))}^2 + h_{\max}^4 \|c\|_{H^1(0,T;H^2(0,1))}^2 \right),
\end{align*}
\]
(52)
where $\epsilon$ is nonzero constant such that $d_0 - 4\epsilon^2 > 0$, $\omega$ is given by
\[
\omega = \frac{1}{\epsilon^2} \left( C_d^2 + C_{h,2}^2 + \frac{1}{2} C_b^2 C_p^2 \right) + 2\epsilon^2
\]
(53)
and $C_d, C_b, C_{h,2}, C_{in}$ were introduced before.

Proof: It can be shown that $e_{c,h}(t)$ is solution of the variational problem
\[
(e_{c,h}'(t), w_h)_h = -(d_h(t)D_x c_h(t), D_x w_h)_{h,+} + (M_h(b_h(t)c_h(t)), D_x w_h)_{h,+} \\
+ (q_{2,h}(t), w_h)_h - (R_h c'(t), v_h)_h.
\]
As \( \tilde{c}_h(t) \) satisfies (27) we obtain
\[
(c'_c(t),w_h)_h = (\tilde{d}_h(t)D_{-x}\tilde{c}_h(t),D_{-x}w_h)_{h,+} - (d_h(t)D_{-x}c_h(t),D_{-x}w_h)_{h,+} \\
+ (M_h(b_h(t)c_h(t)),D_{-x}w_h)_{h,+} - (M_h(b_h(t)\tilde{c}_h(t)),D_{-x}w_h)_{h,+} + (\tilde{c}'_h(t),w_h)_h - (R_h \tilde{c}'(t),w_h)_h, 
\]
where \( \tilde{c}'_h(t) \) is given by (17) with \( q_6 \) replaced by \( c'(t) \).

From (54) with \( w_h = e_{c,h}(t) \), taking into account Lemmas 1 and 2, we deduce the inequality
\[
(c'_c(t),e_{c,h}(t))_h \leq -(d_h(t)D_{-x}e_{c,h}(t),D_{-x}e_{c,h}(t))_{h,+} + (M_h(b_h(t)e_{c,h}(t)),D_{-x}e_{c,h}(t))_{h,+} \\
+ (\tilde{c}'_h(t) - R_h c'(t),e_{c,h}(t))_h + \tau_d,h(t,e_{c,h}(t)) + \tau_b,h(t,e_{c,h}(t)),
\]
(55)

We estimate in what follows the quantities \( (\tilde{c}_h(t) - R_h c(t),e_{c,h}(t))_h, \tau_d,h(t,e_{c,h}(t)) \) and \( \tau_b,h(t,e_{c,h}(t)) \). From Lemma 3 we have
\[
|\tilde{c}'_h(t) - R_h c'(t),e_{c,h}(t))_h| \leq \frac{1}{4\sigma^2} C_{in}^2 \sum_{i=1}^{N} h_i^4 |c'(t)|^2_{H^2(I_i)} + \sigma^2 |e_{c,h}(t)|^2_{1,h},
\]
provided that \( c'(t) \in H^2(0,1) \). In the previous inequality \( \sigma \neq 0 \) is an arbitrary constant.

We remark that for \( \tau_d,h(t,e_{c,h}(t)) \) and \( \tau_b,h(t,e_{c,h}(t)) \) hold the estimates (46) and (50), respectively. Consequently
\[
|\tau_d,h(t,e_{c,h}(t))| \leq \frac{1}{2\epsilon^2} C_d^2 |e_{c,h}(t)|^2_{h} + \epsilon^2 |D_{-x}e_{c,h}(t)|^2_{h,+} \\
+ \frac{1}{2\epsilon^2} C_d^2 \sum_{i=1}^{N} h_i^2 \left( \|p(t)\|_{H^{s+1}(I_i)}^2 + \|c(t)\|_{H^{s+1}(I_i)}^2 \right),
\]
(57)

and
\[
|\tau_b,h(t,e_{c,h}(t))| \leq \frac{1}{2\eta^2} C_{b,2}^2 |e_{c,h}(t)|^2_{h} + \eta^2 |D_{-x}e_{c,h}(t)|^2_{h,+} \\
+ \frac{1}{2\eta^2} C_{b,2}^2 \sum_{i=1}^{N} h_i^2 \left( \|p(t)\|_{H^{s+1}(I_i)}^2 + \|c(t)\|_{H^{s+1}(I_i)}^2 \right),
\]
(58)

where \( \epsilon \neq 0, \eta \neq 0 \) are arbitrary constants.

Considering estimates (56), (57) and (58) in (55) we obtain
\[
\frac{1}{2} \frac{d}{dt} |e_{c,h}(t)|^2_{h} + (d_h(t)D_{-x}e_{c,h}(t),D_{-x}e_{c,h}(t))_{h,+} - (M_h(b_h(t)e_{c,h}(t)),D_{-x}e_{c,h}(t))_{h,+} \\
- \left( \frac{1}{2\epsilon^2} C_d^2 + \frac{1}{2\eta^2} C_{b,2}^2 + \sigma^2 \right) |e_{c,h}(t)|^2_{h} - (\epsilon^2 + \eta^2 + \sigma^2) |D_{-x}e_{c,h}(t)|^2_{h,+} \leq \tau_h(t)^2,
\]
(59)

where
\[
\tau_h(t)^2 \leq \left( \frac{1}{2\epsilon^2} C_d^2 + \frac{1}{2\eta^2} C_{b,2}^2 \right) \left( \sum_{i=1}^{N} h_i^2 \left( \|p(t)\|_{H^{s+1}(I_i)}^2 + \|c(t)\|_{H^{s+1}(I_i)}^2 \right) \right) \\
+ \frac{1}{4\sigma^2} C_{in}^2 \sum_{i=1}^{N} h_i^4 |c'(t)|^2_{H^2(I_i)}. 
\]
In what concerns \((d_h(t)(D_{-x}e_{c,h}(t)), D_{-x}e_{c,h}(t))_{h,+}\) and \((M_h(b_h(t)e_{c,h}(t)), D_{-x}e_{c,h}(t))_{h,+}\), we have
\[
(d_h(t)(D_{-x}e_{c,h}(t), D_{-x}e_{c,h}(t))_{h,+} \geq d_0 \|D_{-x}e_{c,h}(t)\|_{h,+}^2, \tag{60}
\]
and
\[
|(M_h(b_h(t)e_{c,h}(t)), D_{-x}e_{c,h}(t))_{h,+}| \leq \frac{1}{4\gamma^2} C_b^2 C_p^2 \|e_{c,h}(t)\|_{h}^2 + \gamma^2 \|D_{-x}e_{c,h}(t)\|_{h,+}^2, \tag{61}
\]
where \(\gamma \neq 0\) is an arbitrary constants.

Considering now in (59) the estimates (60) and (61) for \(\epsilon = \eta = \gamma = \sigma\), we conclude
\[
\frac{d}{dt} \|e_{c,h}(t)\|_{h}^2 + 2(d_0 - 4\epsilon^2) \|D_{-x}e_{c,h}(t)\|_{h,+} \leq \omega \|e_{c,h}(t)\|_{h}^2 + \tau_h(t)^2 \tag{62}
\]
with \(\omega\) defined by (53).

Inequality (62) implies
\[
\|e_{c,h}(t)\|_{h}^2 + 2(d_0 - 4\epsilon^2) \int_0^t \|D_{-x}e_{c,h}(s)\|_{h,+}^2 ds \leq \|e_{c,h}(0)\|_{h}^2 + \omega \int_0^t \|e_{c,h}(\mu)\|_{h}^2 d\mu + \int_0^t \tau_h(\mu)^2 d\mu
\]
that leads to (52).

Theorem 1 and Corollary 1 imply the error estimate for the pressure.

**Corollary 2** Under the assumption of Theorem 1, for the pressure we have
\[
\|P_{h} e_{p,h}(t)\|_{h}^2 \leq C_{p,n} \left( \|c_h(0) - c(0)\|_{h}^2 + C_e \sum_{i=1}^{N} \int_0^t \left( h^{2e}_{i} \|p(\mu)\|_{H^{1+1}(I_i)} \right. \right. \\
+ \|c(\mu)\|_{H^{2+1}(I_i)}^2 \left. \right) d\mu \right) \leq C_{p,n} \left( \|c_h(0) - c(0)\|_{h}^2 + C_e \left( h^{2e}_{i} \|c(\mu)\|_{H^{2}(0,T;H^{2+1}(0,1))} \right. \right. \\
+ \|p\|_{L^2(0,T;H^{2+1}(0,1))} \right. \right) + h^{4}_{i} \|c\|_{H^1(0,T;H^2(0,1))}.
\] (63)

for some positive constants \(C_{p,n}\) and \(C_e\) which do not depend on \(h\) and for \(h \in H\) with \(h_{max}\) small enough.

**4 Numerical illustration**

We illustrate in what follows the estimates (52) and (63). To do that we next introduce an implicit-explicit method for the IBVP (1)-(5) defining in \([0,T]\) a uniform grid \(\{t_n\}\) with \(t_0 = 0, t_M = T\) and \(t_j - t_{j-1} = \Delta t\). By \(D_{-t}\) we denote the backward finite difference operator with respect to \(t\). Let us suppose that the numerical approximations \(p^n_h(x_i)\) and \(c^n_h(x_i)\) for \(p(x_i, t_n)\) and \(c(x_i, t_n)\), respectively, are known. By \(p^{n+1}_h(x_i)\) and \(c^{n+1}_h(x_i)\) we represent the numerical approximations for \(p(x_i, t_{n+1})\) and \(c(x_i, t_{n+1})\), respectively, defined by the following system
\[
(a^n_h D_{-x}p^{n+1}_h, D_{-x}w_h)_{h,+} = (q^n_{1,h}, w_h)_{h,+} \quad w_h \in \mathbb{W}_{h,0},
\] (64)
\[ (D_x c^{n+1}_h, w_h)_h + (q^{n+1}_h, D_x w_h)_h - (M_h(b^{n+1}_h c^{n+1}_h), D_x w_h)_h, \]

\[ = (q^{n+1}_{2,h}, w_h)_h, \quad w_h \in \mathbb{W}_{h,0}, \]

with the boundary conditions \( p^{n+1}_h(x_0) = p(t_{n+1}), \quad p^{n+1}_h(x_N) = p_r(t_{n+1}), \quad c^{n+1}_h(x_0) = c(t_{n+1}), \quad c^{n+1}_h(x_N) = c_r(t_{n+1}), \) and with the initial conditions \( c^{0}_h(x_i) = c_0(x_i), \quad p^{0}_h(x_i) = p_0(x_i), i = 1, \ldots, N - 1. \)

In (64) and (65), \( q^{n+1}_{\ell, h} \) is obtained from \( q_{\ell,h}(t) \) taking \( t = t_{n+1}, (\ell = 1, 2) \), the coefficient \( a^n_h \) is obtained from \( a_h(t) \) replacing \( c_h(t) \) by \( c^n_h \) and \( b^{n+1}_h \) are obtained from \( d_h(t) \) and \( b_h(t) \), respectively, replacing \( c_h(t) \) and \( p_h(t) \) by \( c^n_h \) and \( p^{n+1}_h \), respectively.

Let us consider (1)-(5) with \( a(c) = 1 + c, b(c, p_x) = (cp_x)^2, d(c, p_x) = c + p_x + 2 \), where \( q_1, q_2 \), the initial and boundary conditions are such that this IBVP has the following solution:

\[ p(x, t) = e^{t}x(x - 1), \quad c(x, t) = e^{t}(1 - \cos(2\pi x))\sin(x), \quad x \in [0, 1], \quad t \in [0, T]. \]

The numerical approximations \( c^n_h \) and \( p^n_h \) were obtained with the IMEX method (64)-(65) with nonuniform grids in \([0, 1]\) and with \( T = 0.1 \) and \( \Delta t = 10^{-6} \). The first spatial grid is arbitrary and the new grid is obtained from the previous one introducing in \([x_i, x_{i+1}]\) the midpoint. In Table 1 we present the errors

\[ \text{Error}_c = \max_{n=1, \ldots, M} \left( \|e^n_c\|_h^2 + \Delta t \sum_{j=0}^n \|D_x e^n_{c,j}\|_{h,+}^2 \right)^{1/2}, \quad \text{Error}_p = \max_{n=1, \ldots, M} \|D_x e^n_{p,h}\|_{h,+} \]

and the rates \( \text{Rate}_c, \text{Rate}_p \) that were computed by the formula \( \text{Rate} = \frac{\ln \left( \frac{\text{Error}_{\text{max},1}}{\text{Error}_{\text{max},2}} \right)}{\ln \left( \frac{h_{\text{max},1}}{h_{\text{max},2}} \right)} \), where \( h_{\text{max},1} \) and \( h_{\text{max},2} \) are the maximum step sizes of two consecutive partitions.

<table>
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<th>( h_{\text{max}} )</th>
<th>( \text{Error}_c )</th>
<th>( \text{Error}_p )</th>
<th>( \text{Rate}_c )</th>
<th>( \text{Rate}_p )</th>
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Table 1: Convergence rates for the numerical approximations defined by the IMEX method (64)-(65).

The numerical results presented in Table 1 show that \( \text{Error}_p = O(h_{\text{max}}^2) \) and \( \text{Error}_c = O(h_{\text{max}}^2) \).

## 5 Conclusions

The behavior of the pressure and concentration of an incompressible fluid in a one dimensional porous media is described by an elliptic equation for the pressure and a parabolic equation for the concentration linked by the Darcy’s law for the velocity. Quasilinear coupled problems that have as a particular case the previous problem were considered in this paper.
The use of piecewise linear finite element method for the pressure and concentration of an incompressible fluid in a porous media leads to a first order approximation to the velocity. Consequently, the concentration is of first order in the $L^2$-norm. This behavior is observed for uniform and nonuniform partitions of the spatial domain. Semi-discretizations based on the piecewise linear finite element method with special quadrature formulas were studied in this paper. For such semi-discrete approximations error estimates were established that allow us to conclude second order accuracy for the pressure and its gradient and for the concentration.

A common approach in the convergence analysis of the spatial discretization of parabolic equations is the split of the semi-discretization error into two terms ([22]) considering the corresponding discretization of an auxiliary elliptic problem. Such approach was largely followed in the literature and implies an increasing in the smoothness requirements of the solution for the parabolic problem. In this paper a different approach was followed that avoids such smoothness requirements.

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References


